

# Optimal Portfolio Choice over the Life Cycle with Epstein-Zin-Weil Preferences and G-and-H Distribution

Jialun Li      Kent Smetters

April 2010

## Abstract

In this paper we develop a lifecycle model to solve numerically for the optimal consumption and portfolio rules of households who face uninsurable labor income uncertainty, mortality risk, and borrowing constraints. We incorporate generalized utility forms (Epstein-Zin-Weil utility) and generalized stock return shock distribution (g-and-h distribution). The flexibility of our model enables us to match the empirical stock/bond ratio and wealth/income ratio.

## 1 Introduction

The decision of consumption and portfolio allocation over the lifecycle is encountered by every investor. There is a rich literature exploring consumption and saving behavior in a lifecycle model in which individual maximizes expected discounted utility. The early works by Samuelson (1969) and Merton (1971) implied that if markets are complete, then labor income can be capitalized and traded using the financial markets securities so that idiosyncratic labor income risks can be fully insured. However, human capital is a non-tradable asset due to the moral hazard problem. An individual who has sold a claim against future labor income can always stop working (Campbell and Viceira 2002, Heaton and Lucas 1996). Therefore human capital risk is uninsurable under incomplete markets.

Many papers studied the effects of uninsurable human capital risk on consumption and portfolio choice. Bodie, Merton and Samuelson (1992) allowed endogenous labor supply and showed that labor flexibility induces the individual to assume greater risks. Deaton (1991) developed an infinite horizon model with borrowing constraints and concluded that assets act like a buffer stock protecting consumption against income uncertainty. Viceira (2001) allows risky labor income to be correlated with stock returns. Similar to Deaton (1991), Carroll (1992, 1997) also looked at buffer stock saving behavior and introduced both permanent and transitory labor income shocks in the model. Koo (1998) considered risky labor income in a continuous approach. Cocco, Gomes and Maenhout (2005) developed a quantitative and realistically calibrated model. Campbell, Cocco, Gomes and Maenhout (2000) extended the work by CGM (2005) to explore alternative retirement savings systems and fixed costs of equity market participation.

However, it is well known that the standard expected-utility model cannot closely match the empirical evidence on portfolio allocation. Several papers turn to alternative utility forms such as Epstein-Zin-Weil (EZW) utility functions (Epstein and Zin (1989), Weil (1990)), which disentangles risk aversion and elasticity of intertemporal substitution. Gomes and Michaelides (2005) show that with an EZW preference, a fixed stock market entry cost, and moderate heterogeneity in risk aversion, their model can match stock market participation rates and asset allocations. Michaelides, Gomes and Polkovnichenko (2006) solve for optimal savings in the presence of tax-deferred retirement accounts using EZW utility.

All the previous literature, however, assume a normal distribution when they model risky assets. This conventional assumption in the finance literature has been increasingly challenged in recent years. A variety of empirical studies show that financial assets are skewed and fat-tailed (Mandelbrot (1963), Fama (1965), Hols et al. (1991)). Several other distributional forms have been explored (Mandelbrot (1963), Fama (1965), Praetz (1972), Blatberg and Gonedes (1974), Kon (1984)). Tukey (1977) introduced the g-and-h distribution, which is a transformation of the standard normal distribution. The power of this distribution is its ability to fit almost all parametric distributions up to fourth moments. Therefore, g-and-h distribution seems to be superior to the conventional normal distribution in modelling equity returns.

The main contribution of this paper is to combine both generalized utility forms (Epstein-Zin-Weil) and generalized return distribution (g-and-h distribution) and solve for the optimal consumption and portfolio choice in

the presence of uninsurable labor income risk and liquidity constraint. We present several sensitivity analysis to measure the importance of the model parameters. We also match our model with empirical mean stock/bond ratio and median wealth/income ratio and show that with the introduction of skewness and elongation parameters we are able to match the empirical data fairly closely.

The paper is organized as follows. Section 2 describes the model setup. Section 3 discusses the calibration of the model parameters. Section 4 presents the simulated optimal consumption and portfolio allocation rules and several sensitivity analysis. Section 5 shows the estimation results that match the empirical evidence. Section 6 concludes.

## 2 Model

### 2.1 Preferences

The individual enters the job market at age  $\tau$ , works for  $K - \tau$  years, retires at time  $K$ , and lives a maximum of  $T$  periods. For simplicity, we assume exogenous labor supply so that  $K$  is exogenous, and the individual has no bequest motive. Mortality is captured by  $p_t$ , the probability that the individual is alive at time  $t + 1$ , conditional on being alive at time  $t$ . Households have Epstein-Zin-Weil utility functions (Epstein and Zin (1989), Weil (1990)), which allow us to disentangle risk aversion and elasticity of intertemporal substitution. Therefore at time  $t$ , the household  $i$ 's preferences are given by the following recursion:

$$V_{i,t} = \{(1 - \beta)C_t^{1-1/\psi} + \beta p_t(E_t[V_{t+1}^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}}\}^{\frac{1}{1-1/\psi}} \quad (1)$$

where  $C_t$  is consumption at time  $t$ ,  $\beta$  is the discount rate,  $\gamma$  is the coefficient of relative risk aversion,  $\psi$  is the elasticity of intertemporal substitution. This utility reduces to the standard power utility when  $\gamma = 1/\psi$ .

### 2.2 Labor Income

Let  $Y_{i,t}$  denote the individual  $i$ 's labor income at time  $t$ . We assume a markov transition matrix for individual's labor income process. Let the number of discrete states of labor income be  $n$  and the level of labor income at state  $j$  and time  $t$  be  $L_{j,t}$ . Then

$$\Pr(Y_{i,t} = L_{j,t} | Y_{i,t-1} = L_{k,t-1}) = p_{kj,t} \text{ for } j = 1, \dots, n, k = 1, \dots, n, t < K \quad (2)$$

After retirement, the individual has no labor income but instead receives social security benefit as a progressive function of his final working year labor income.

### 2.3 Financial Assets

The assumption that risky assets are lognormally distributed has been increasingly criticized in recent years as it cannot closely depict a realistic financial market. A variety of empirical studies show that financial assets are skewed and fat-tailed (Mandelbrot (1963), Fama (1965), Hols et al. (1991)). To reconcile the problem, researchers have been working on two main areas of extension. The first one is alternative continuous distribution such as stable Peretian (Mandelbrot 1963) and g-and-h distribution (Tukey 1977). The second one is discontinuous process such as jumps (Merton 1976, Ball and Torous 1985, Jorion 1988). We assume the equity shock follows g-and-h distribution. The characteristics of g-and-h will be explained later.

At each time  $t$  the individual can invest a portion  $\alpha_t$  of his liquid wealth in the following two assets:

- 1) A risk-free asset (bond) with gross real return  $R$ ;
- 2) A risky asset (stock) with return:

$$R_t - R = \mu + \eta_t \quad (3)$$

where  $\mu$  is the deterministic equity premium and  $\eta_t$  is an independently and identically distributed (i.i.d) equity return shock which follows a g-and-h distribution.

The g-and-h distribution, introduced by Tukey (1977) is a transformation of the standard normal distribution. In particular, if  $Z$  is a random variable that follows the standard normal distribution, then  $X$  follows g-and-h distribution if it can be written as:

$$X_{g,h}(Z) = a + b \frac{e^{gZ} - 1}{g} e^{hZ^2/2} \quad (4)$$

where a,b,g,h represent location, scale, skewness, and elongation respectively. When  $g=0$ , the g-and-h distribution reduces to  $X_{0,h}(Z) = a +$

$bZe^{hZ^2/2}$ , known as the h-distribution. Similarly, when  $h=0$ , the g-and-h distribution reduces to  $X_{g,0}(Z) = a + b\frac{e^{gZ} - 1}{g}$ , known as the g-distribution. When  $g=h=0$ , g-and-h distribution reduces to normal distribution with mean  $a$  and standard deviation  $b$ . The g-and-h distribution has been studied by Martinez and Iglewicz (1984) and Hoaglin et al. (1985) and used by Badri-nath and Chatterjee (1988, 1991), Mills (1995), Dutta and Babbel (2002, 2005), Dutta and Perry (2007) to model equity returns, interest rates and interest rates options and operational risks. The power of this distribution is its ability to fit almost all parametric distributions up to fourth moments. Therefore, g-and-h distribution seems to be superior to the conventional normal distribution in modelling equity returns that are known to be skewed and heavy-tailed.

We will also compare our results with an alternative jump process:

$$R_t - R = \mu + \eta_t + \sum_{i=1}^{n_t} \ln Y_i \quad (5)$$

where  $Y$  is a jump size which is assumed independently lognormally distributed  $\ln Y \sim N(\theta, \delta^2)$ ,  $n_t$  is the actual number of jumps during the interval. Jump arrival is assumed to follow a poisson distribution with intensity  $\lambda$ .

## 2.4 Optimization problem

In each period  $t$ , the individual  $i$  has initial wealth  $W_{i,t}$ . He receives a labor income  $Y_{i,t}$  if he is employed and no income if he is unemployed or retired. He spends  $h_t$  of his labor income for housing. He then has a "cash-on-hand"

$$X_{i,t} = W_{i,t} + (1 - h_t)Y_{i,t} \quad (6)$$

where he chooses consumption  $C_{i,t}$  and portfolio share in equity  $\alpha_{i,t}$ . The recursive relation of wealth is given by:

$$W_{i,t} = R_{i,t}(X_{i,t-1} - C_{i,t-1}) \quad (7)$$

where  $R_{i,t}$  is the gross return on the individual's portfolio from  $t - 1$  to  $t$ :

$$R_{i,t} = \alpha_{i,t-1}R_t + (1 - \alpha_{i,t-1})R = R + \alpha_{i,t-1}(\mu + \eta_t) \quad (8)$$

We use subscript  $t$  for the portfolio return because it is not in the information set when the individual makes decision at  $t - 1$ .

The two control variables are  $\{C_{i,t}, \alpha_{i,t}\}$ . The two state variables are  $\{t, X_{i,t}\}$ .

Throughout the individual's lifetime, we assume that consumption and portfolio weight in risk-free and risky assets are all nonnegative. In other words, we prohibit the individual from borrowing and short-selling:

$$0 \leq C_{i,t} \leq X_{i,t} \tag{9}$$

$$0 \leq \alpha_{i,t} \leq 1 \tag{10}$$

In summary, the individual maximizes equation (1) subject to constraints (6) to (10).

## 2.5 Model Solution

The optimization problem cannot be solved analytically. Therefore we use backward induction to derive the optimal decision rules for this dynamic program. In other words, the Bellman equation is solved from year  $T$  back to year  $\tau$ . The continuous state space is discretized using unevenly spaced grids, with more points used at lower levels of wealth. The value function  $V_{i,t}(X_{i,t})$  is interpolated between grid points of the state variable using Schumaker shape-preserving quadratic splines. The expectation over the g-and-h distributed equity shock  $\eta_t$  is approximated using a transformation of Gauss-Hermite quadrature numerical integration by discretizing the shock into several nodes. Maximization at each grid point is performed using the Nelder-Mead simplex optimization.

## 3 Calibration

### 3.1 Preferences

We follow the calibration by Campbell et al. (2000) and assume that the individual with a college degree enters the job market at age 22. The age of retirement is 65. The individual dies with probability one at age 100. The discount factor  $\beta = 0.95$ . In the benchmark case, the coefficient of

relative risk aversion is set to  $\gamma = 4$  and the elasticity of intertemporal substitution (IES) is set to  $\psi = 0.25$ , which is the reciprocal of risk aversion. In this case, the utility function is equivalent to a standard power utility. We will report results for different values of RRA and IES as well. We use the mortality tables in the National Vital Statistics Reports from National Center for Health Statistics to determine the conditional survival probabilities.

### 3.2 Labor Income

Nishiyama and Smetters (2005) estimated eight discrete levels of working abilities and the corresponding Markov transition matrices from the hourly wage of each household in the 2001 SCF and PSID. The estimated working abilities are in five-year age cohorts from age 20-24 to age 75-79. Markov transition matrices are constructed for ten-year age windows: 20-29, 30-39, 40-49 and 50-59. We follow their results and calibrate the annual labor income by multiplying the hourly wage levels by 2080, which is the number of hours that an individual would typically work in a year under a 40-hour workweek assumption. We use a shape-preserving cubic spline interpolation between each five-year age cohort to obtain the annual labor income levels for each age cohort. We omit the working abilities for 65 and older as individual receives no labor income after 65. We use their Markov transition matrices and use the matrix for 50-59 for individuals aged 60-64. After 65, individuals remain in their income state at age 64, so there is no transition matrices for those who are retired.

Social security benefit received by an individual after retirement is calculated using a realistic progressive formula as a function of the individual's final working year income. Detailed formula is presented in the Appendix.

### 3.3 Housing Expenditures

The share of labor income for housing expenditures  $h_t$  is taken from Gomes and Michaelides (2005), who compute the ratio of annual mortgage payments and rent payments to annual labor income using data from the Panel Study of Income Dynamics (PSID) from 1976 to 1993. The age effects are identified by regressing the ratio on a constant, a cubic polynomial of age and time dummies. We assume no housing expenditures after age 65. The regression coefficients are:

$$\widehat{h}_t = 0.703998 - 0.0352276 \times age + 0.0007205 \times age^2 - 0.0000049 \times age^3 \quad (11)$$

### 3.4 Financial Assets

We use a risk-free rate of 2% and a risky equity premium of  $\mu$  equal to 7%. The standard deviation of risky asset return shock  $\sigma_\eta$  is set to be its historical value of 0.2059. In the benchmark case, we set  $g = h = 0$  to show results for normal distribution. We consider different values of  $g$  and  $h$  later on in the paper to account for skewness and kurtosis in the observed data on US equity. Badrinath and Chatterjee (1988) estimated  $g$  and  $h$  parameters for daily and monthly returns on the CRSP equal-weighted and value-weighted market portfolios. The median  $g$  ranges from -0.034 to 0.042, while the median  $h$  ranges from 0.091 to 0.146. We follow their approach and use the quantile-based method to estimate  $g$  and  $h$  parameters for yearly, quarterly and monthly returns on S&P 500, CRSP value-weighted and equal-weighted market portfolios from 1925-2009. We find monthly and quarterly indices are highly negatively skewed and fat-tailed. Yearly market portfolios show even higher negative skewness. However, we do not find fat-tailness in these portfolios, probably because we have too few yearly observations. We report the estimation results and the bootstrapping standard errors in Table 1. We will report model results for both positive and negative  $g$  values and positive  $h$  values in section 4.

Table 1: Estimation Results for market indices

	a	b	g	h
Panel A: Yearly returns				
S&P 500	0.0906 (0.0290)	0.2059 (0.0234)	-0.1697 (0.1003)	-0.0376 (0.0396)
Value Weighted	0.1085 (0.0279)	0.2006 (0.0226)	-0.2852 (0.1100)	-0.0384 (0.0424)
Equal Weighted	0.1522 (0.0407)	0.2788 (0.0328)	-0.0444 (0.1478)	0.0629 (0.0688)
Panel B: Quarterly returns				
S&P 500	0.0262 (0.0045)	0.0680 (0.0059)	-0.1637 (0.1333)	0.2456 (0.0620)
Value Weighted	0.0277 (0.0051)	0.0687 (0.0060)	-0.1943 (0.1358)	0.2363 (0.0637)
Equal Weighted	0.0273 (0.0069)	0.0899 (0.0103)	0.0896 (0.1214)	0.3079 (0.0863)
Panel C: Monthly returns				
S&P 500	0.0090 (0.0013)	0.0370 (0.0017)	-0.1104 (0.0616)	0.2144 (0.0272)
Value Weighted	0.0090 (0.0014)	0.0366 (0.0016)	-0.1269 (0.0632)	0.2047 (0.0243)
Equal Weighted	0.0102 (0.0019)	0.0435 (0.0021)	0.0134 (0.0768)	0.2592 (0.0235)

We follow Jorion (1988) and use maximum-likelihood estimation method to estimate the mixed jump-diffusion process parameters  $\mu$ ,  $\sigma$ ,  $\theta$ ,  $\delta$ , and  $\lambda$ . We use S&P 500 returns from 1925-2009. The estimation results are presented in Table 2. Detailed estimation procedure can be found in the Appendix of Jorion (1988).

Table 2: Estimation Results for jump process

$\mu$	0.3775
$\sigma$	0.04
$\theta$	-0.1304
$\delta$	0.0001
$\lambda$	2.4742

Table 3 summarizes the parameter values in the benchmark case.

Table 3: parameter values in the benchmark case

Parameter	Value
Beginning of career ( $\tau$ )	22
Retirement age ( $K$ )	65
Maximum age ( $T$ )	100
Discount rate ( $\beta$ )	0.95
Risk aversion ( $\gamma$ )	4
Elasticity of intertemporal substitution ( $\psi$ )	0.25
Riskfree rate ( $R$ )	1.02
Risk premium ( $\mu$ )	0.0706
Std of asset return shock ( $\sigma_\eta$ )	0.2059
Skewness parameter ( $g$ )	0
Elongation parameter ( $h$ )	0

## 4 Results

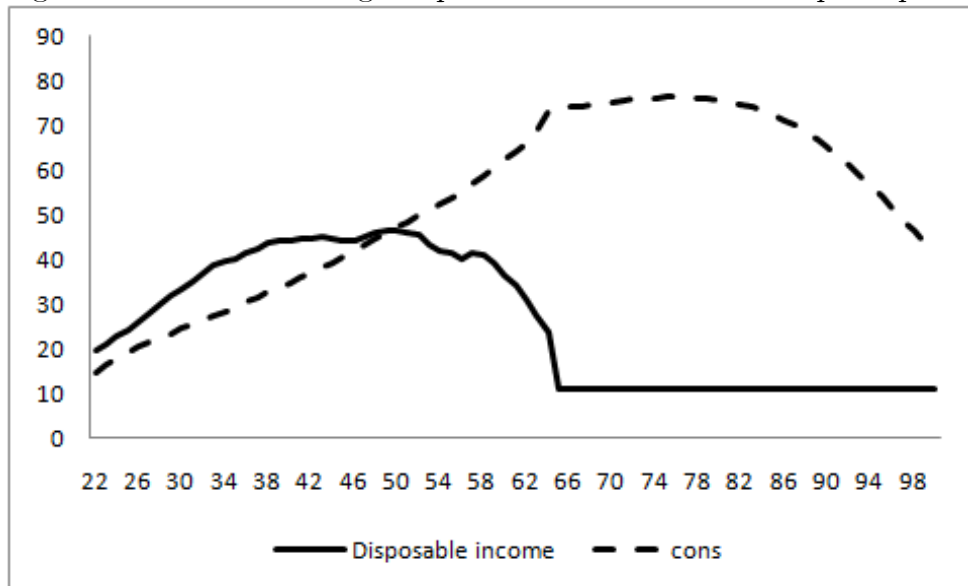
After solving for the optimal policy functions, we simulate the model over 10,000 households and compute the average household behavior. We first report the average outcomes using the parameters in the benchmark case. Then we discuss the importance of model parameters for these results.

### 4.1 Benchmark case

Figure 1 plots the simulated disposable income (after housing expenditures) and consumption profiles. The results are generally similar to the previous papers on lifecycle models (Coco, Gomes and Maenhout (CGM) (2005), Gomes and Michaelides (2005)). Consumption displays hump-shape over the life time as a result of precautionary saving and impatience. Households are liquidity constrained during the first decade of their working lives. As disposable income increases, households start to accumulate wealth and consumption becomes smooth during middle age. As households get older after retirement, the effective impatience increases due to increasing mortality risk. Therefore consumption decreases gradually toward the end of life.

Figure 2 presents the simulated wealth profile. Households accumulate wealth to save for retirement as well as precautionary motives. Wealth accumulation remains very low during the early years and increases steeply as households receive more income. Households start to decumulate wealth

Figure 1: Simulated average disposable income and consumption profile



after retirement to smooth consumption due to decreased retirement income.

Figure 3 describes the simulated stock allocation (fraction of liquid wealth invested in risky assets). Overall, households' stock allocation displays "hump-shape" during their working life and remains stable after retirement. Households invest almost fully in stock during their young ages. Starting from around age 50, equity share gradually falls as households approach their retirement. It goes up a little right before the age of retirement as income uncertainty goes away. After retirement, stock allocation becomes quite stable at around 70% of their financial assets in stock. We show here that in the benchmark case where standard power utility and lognormal stock returns are assumed, stock allocation is much higher than the empirical evidence.

## 4.2 Sensitivity Analysis

In this section we solve for the optimal consumption and portfolio rules for different values of risk aversion, elasticity of intertemporal substitution, skewness and elongation and analyze the sensitivity of our results to these parameters.

Figure 2: Simulated average wealth accumulation

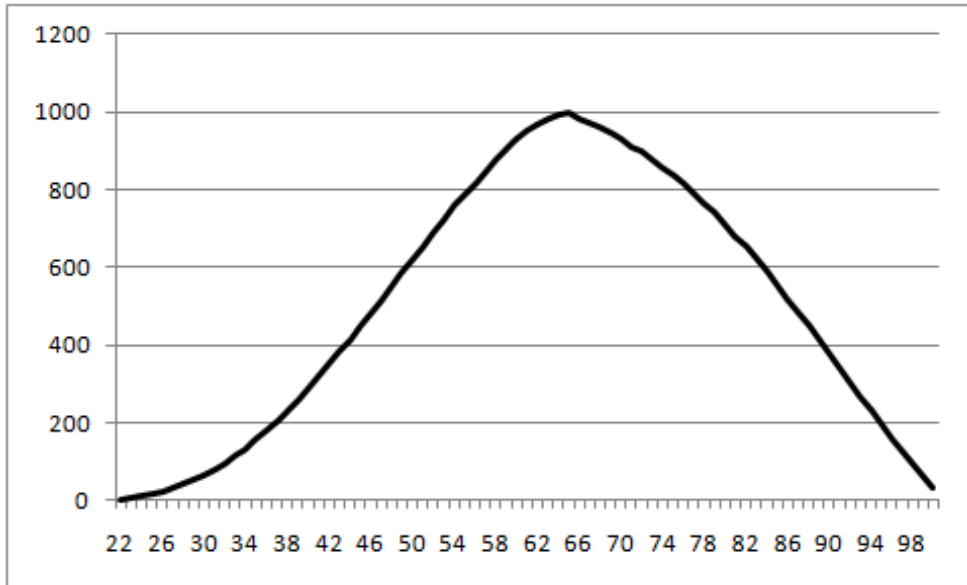


Figure 3: Simulated average stock allocation

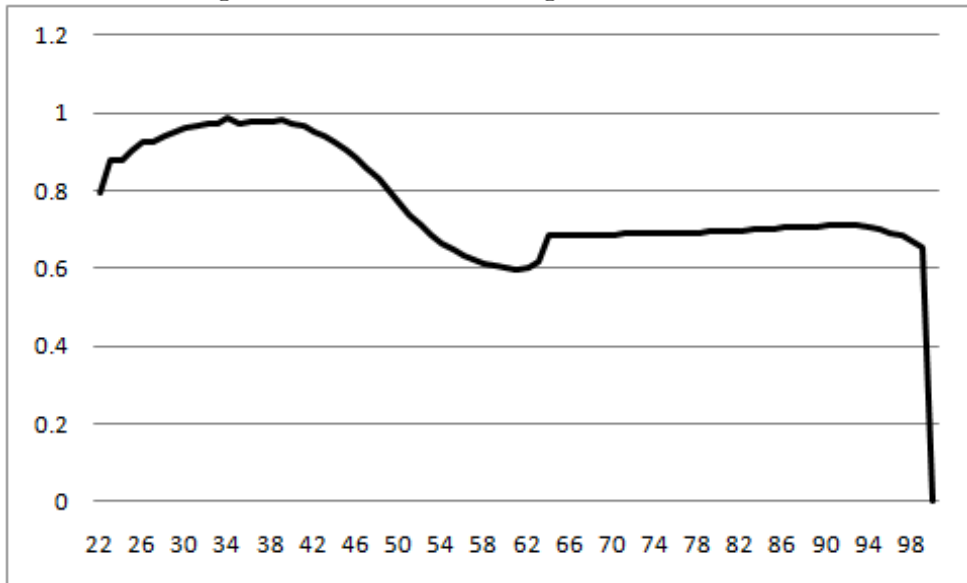
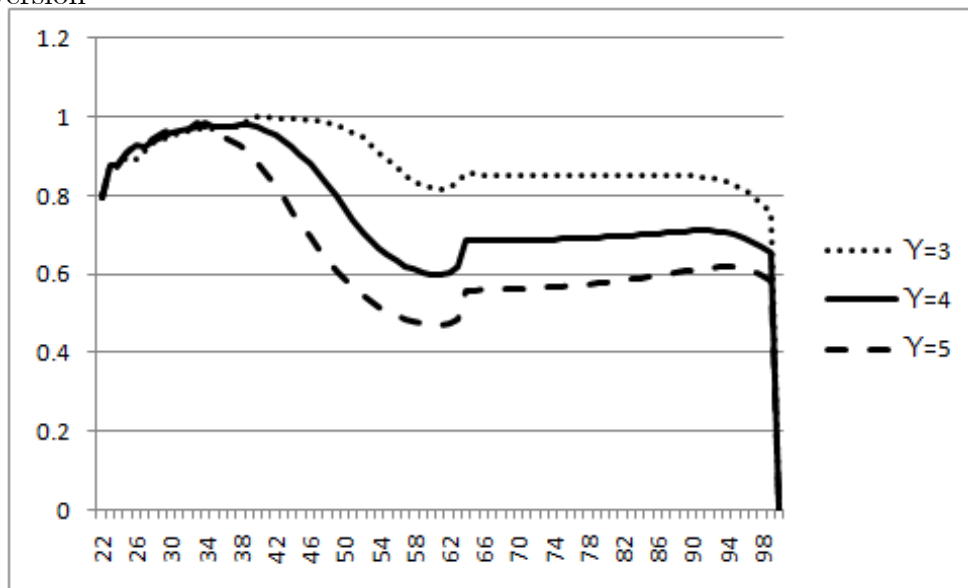


Figure 4: Simulated stock allocation for different coefficients of relative risk aversion



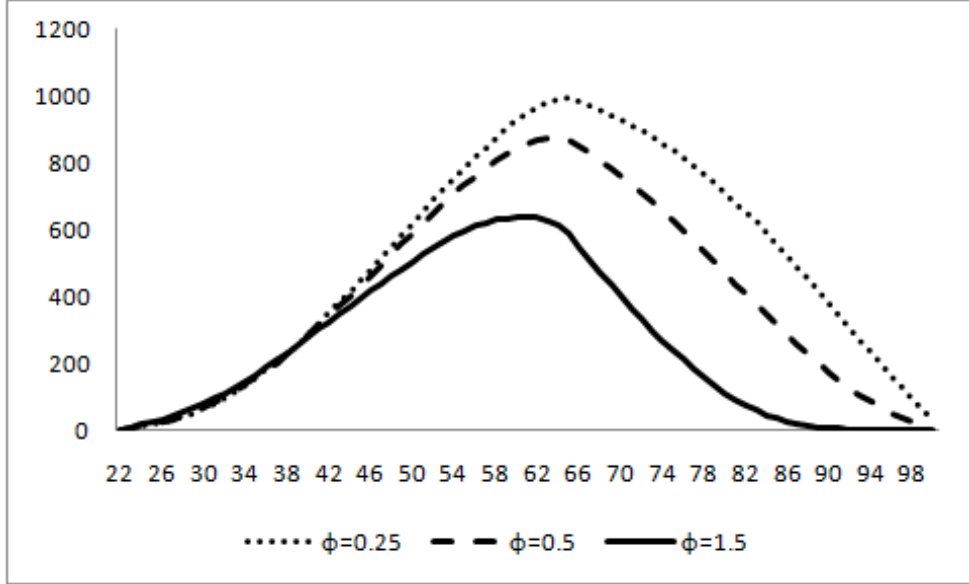
#### 4.2.1 Risk Aversion

Figure 4 shows the simulated stock allocation for coefficient of relative risk aversion of 3 and 5, compared with the benchmark case 4. In early years, portfolio shares in equity are still very close to 1. However, the stock allocation for more risk-averse households ( $\gamma = 5$ ) lowers substantially starting from middle age and remains at around 60% after retirement. This is intuitive because more risk-averse households invest more cautiously in the stock market. In contrast, less risk-averse households ( $\gamma = 3$ ) invest substantially more in equity. The portfolio share in equity remains higher than 80% even after retirement, compared to 70% in our benchmark case.

#### 4.2.2 Elasticity of Intertemporal Substitution

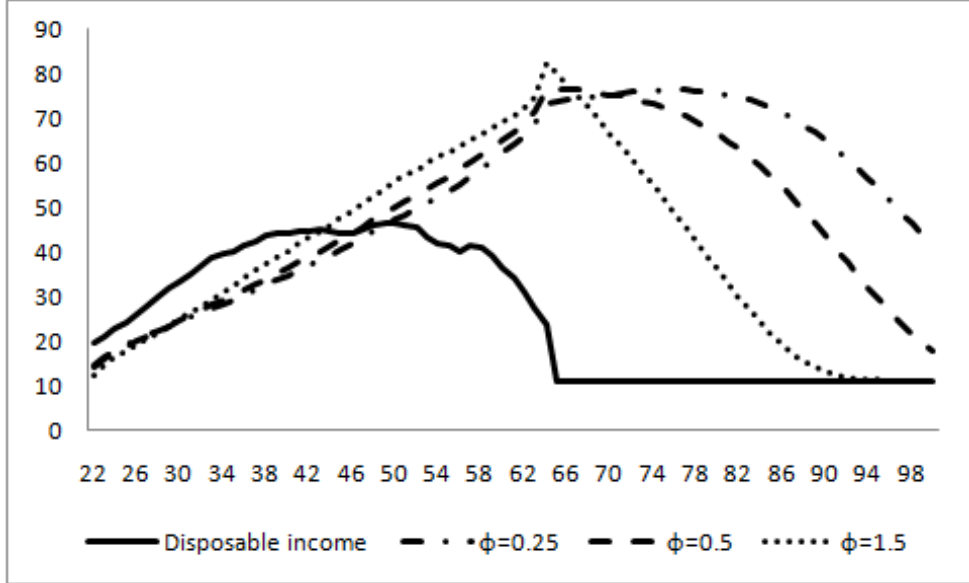
A large literature has attempted to estimate the elasticity of intertemporal substitution based on both macroeconomic and microeconomic evidence. This literature, however, has produced mixed results ranging from 0.1 to 2 (Hall (1988), Mulligan (2002), Dynan (1993), Blundell, Browning and Meghir's (1994), Gruber (2006)). Therefore we consider a fairly wide range of IES

Figure 5: Simulated wealth accumulation for different elasticities of intertemporal substitution



from the benchmark  $\psi = 0.25$  to a moderate  $\psi = 0.5$  and a high  $\psi = 1.5$ . IES controls the agent's willingness to substitute consumption intertemporally. By its nature, we should expect a change in IES to have more impact on households' consumption profile than their portfolio allocation. We therefore focus on wealth accumulation and consumption profile. Figure 5 shows the simulated wealth accumulation for different values of elasticity of intertemporal substitution (IES). Wealth accumulation is substantially lower when  $\psi = 1.5$  and higher when  $\psi = 0.25$ . This is intuitive because a high IES makes the household more willing to substitute intertemporally. In other words, he cares less about consumption smoothing, and therefore saves less for retirement. Similarly, a low IES produces higher wealth accumulation as the household is more concerned with consumption smoothing. This can also be seen in Figure 6, which describes the consumption profile for different values of IES. Consumption traces the income profile more closely with a high IES  $\psi = 1.5$ , while it has less curvature with a low IES  $\psi = 0.25$ .

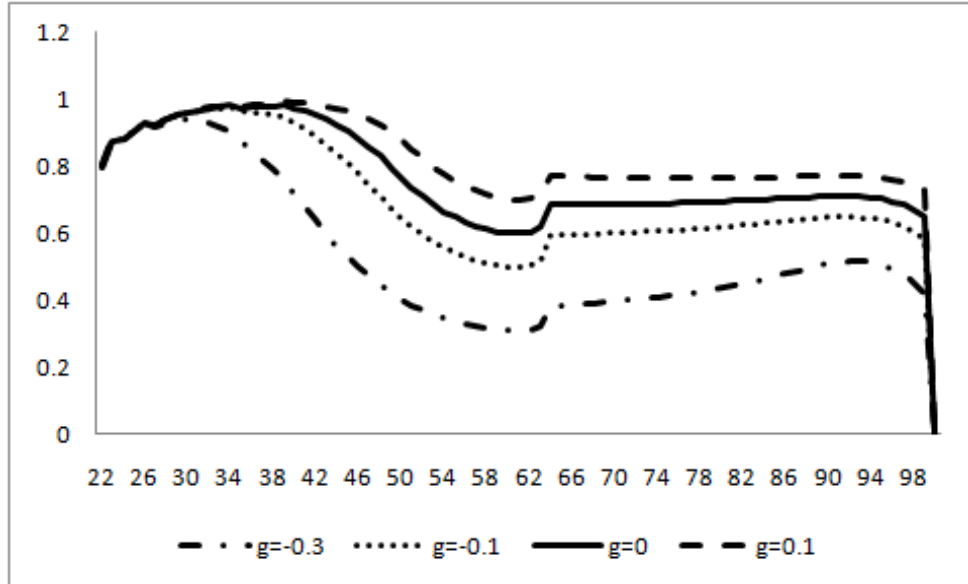
Figure 6: Simulated consumption for different elasticities of intertemporal substitution



### 4.2.3 Skewness

An interesting point of this paper is to introduce a non-normal equity shock distribution, namely g-and-h distribution, in the lifecycle model. Thus it is important to analyze how g-and-h distribution affects the optimal consumption and portfolio rules. We first examine the sensitivity of our results to different g values, which govern the skewness of the distribution. Figure 7 presents the simulated stock allocation for a positive  $g=0.1$ , a moderately negative  $g=-0.1$  and a highly negative  $g=-0.3$ . When  $g=0.1$ , i.e. the distribution is skewed to the right, households' portfolio share in equity increases compared to what we obtained in the benchmark case. This makes perfect sense because households are willing to invest more in stocks if high stock returns are more likely to realize. When  $g=-0.1$ , i.e. the equity shock distribution is skewed to the left, households' portfolio share in equity still approaches 1 in early years but is lower later on in life. When  $g=-0.3$ , i.e., the equity shock distribution is heavily skewed to the left, stock allocation shifts down even more. We conclude that a negative g value helps bringing down equity share effectively.

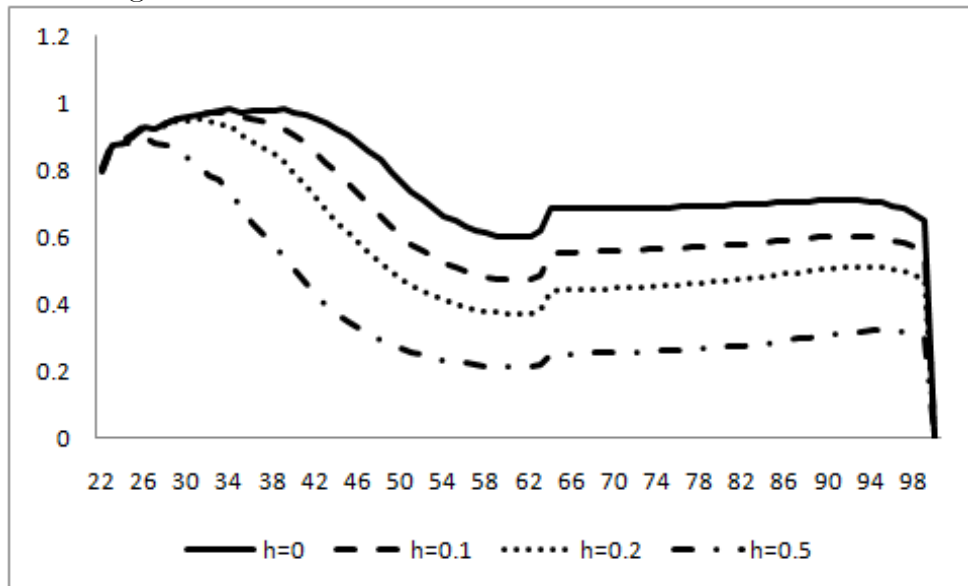
Figure 7: Simulated stock allocation for different  $g$  values



#### 4.2.4 Elongation

Now we examine the sensitivity of our results to different  $h$  values, which govern the elongation of the equity shock distribution. A higher  $h$  value means the distribution is more fat-tailed. In other words, extreme equity returns are more likely to happen. We consider a moderate  $h$  value of 0.1, which is close to the estimates obtained by Badrinath and Chatterjee (1988) on market index, as well as a medium  $h$  value of 0.2, which is close to our estimates, and a high  $h$  value of 0.5. Figure 8 shows the simulated stock allocation for these  $h$  values. When  $h = 0.1$ , portfolio share in equity falls over the lifecycle. The equity share after retirement is around 60% compared to 53% when there is no elongation. Stock allocation follows shifts down with a similar pattern when  $h = 0.2$ . It falls even more dramatically for  $h = 0.5$ . In this case, households never invest fully in stock and the lowest equity share over the life time is around 20%. We conclude that elongation has a substantial effect on optimal portfolio share in our lifecycle model.

Figure 8: Simulated stock allocation for different h values



## 5 Estimation

After we derive the optimal consumption and portfolio rules, we can estimate the parameter values that match the empirical evidence. Specifically, we show matching results in three scenarios: 1) lognormal stock returns, 2) lognormal stock returns with jumps, 3) g-and-h stock returns. For scenario 1) and 2), we fix  $g$  and  $h$  to be 0 and pick the risk aversion  $\gamma$  and elasticity of intertemporal substitution  $\psi$  that match the empirical mean stock/bond ratio and median wealth/income ratio. For scenario 3), we hold the coefficient of relative risk aversion ( $\gamma$ ) and discount rate ( $\beta$ ) constant, and given these values, we pick the elasticity of intertemporal substitution ( $\psi$ ) and elongation ( $h$ ) that match the empirical mean stock/bond ratio and median wealth/income ratio for a given skewness ( $g$ ). We can then, for each level of  $\gamma$ , draw a line that matches the two empirical data on a skewness-kurtosis ( $g$ - $h$ ) space. We then ask if there exist any combinations of  $g$  and  $h$  that we cannot reject empirically.

Bucks, Kennickell Mach, Moore (2009) report an average stock holdings as share of financial assets from the 2007 Survey of Consumer Finances (SCF) of 53.3%. This number converts to a stock/bond ratio of 1.14, which we use as the target mean stock/bond ratio. In our model, the stock/bond ratio is calculated as the simulated mean of the ratio of total amount of assets

held in equity to total amount of assets held in riskfree bond over the lifetime.

A variety of studies have estimated wealth/income ratio from macroeconomic and microeconomic data. Nishiyama and Smetters use the capital-GDP ratio of 2.74 from 2000 data from the Bureau of Economic Analysis (BEA). Gomes and Michaelides (2005)'s estimates of the median wealth-to-labor-income ratio from the 2001 SCF for different age groups range from 0.287 for the young group to 2.17 for the middle age group to 7.93 for the old group. We therefore use a reasonable median wealth/income ratio of 3.0 as a target. To calculate the wealth/income ratio, we first obtain the simulated median wealth and income.<sup>1</sup> Then we weight the simulated median wealth and income for each age by population and get the population weighted median wealth/income ratio.

Mehra and Prescott (1985) argue that risk aversion of 10 or below seems reasonable, although many believe it is much lower than 10 and between 1 and 2. Therefore we compare three scenarios and see which one produces the lowest risk aversion. Although macroeconomic estimates of the IES such as Hall (1988) are well below one, Bansal and Yaron (2004) and Barro (2005) point out that those estimates tend to be biased sharply toward zero and argue that IES greater than one seems more plausible. Moreover, IES estimation results from Mulligan (2002), Gruber (2006), and Vissing-Jorgensen and Attanasio (2003) range from 1 to 2. Therefore we believe that any IES between 1 and 2 is reasonable.

Table 2 presents estimation results for scenario 1) and 2). In scenario 1, the required risk aversion and IES in order to match mean stock/bond ratio of 1.14 and median wealth/income ratio is 6.43 and 2.88 respectively. Both are considered fairly high and inconsistent with many empirical estimates. In scenario 2, as the jump process introduces an additional risk in equity in the portfolio choice decision, investors naturally put less weight in stock relative to bond, leading to a lower required risk aversion of 5.1 to match the stock/bond ratio. The estimated IES is also lowered to 2.04. However, they are still high.

---

<sup>1</sup>Wealth is defined as household's liquid wealth at the beginning of the period. Income is defined as household's labor income plus interest income. We exclude home value from wealth, which lowers the wealth/income ratio. But notice that we do not model social security, which increases the wealth/income ratio after retirement. We argue that the two effects potentially cancel with each other overall.

Table 4: Estimation results for scenario 1 and 2

	Scenario 1: Lognormal	Scenario 2: Lognormal with jumps
Risk Aversion	6.43	5.1
IES	2.88	2.04

The estimation results for scenario 3) are presented in Figure 9 and Figure 10. Figure 9 plots the combinations of  $g$  and  $h$  values for different levels of risk aversion that simultaneously match the stock/bond ratio and wealth/income ratio. Each line represents a given level of risk aversion. Along each line, the corresponding  $g$  and  $h$  value produce the same mean stock/bond ratio and median wealth/income ratio that match the empirical evidence. The lines are upward-sloping, meaning that a higher  $g$  requires a higher  $h$ . This is intuitive because as stock returns become more skewed to the right, they must be more fat-tailed to prevent investors from over-investing in stocks. It is also shown in Figure 9 that as risk aversion goes up, the line shifts down. This is also intuitive as more risk averse investors are less willing to hold stocks. As a result a lower  $h$  value is required to maintain the same stock/bond ratio. We can conclude in general that a fairly high level of elongation is required in our model to match the empirical evidence. To see this, for our benchmark value of risk aversion  $\gamma = 4$  as an example, the required  $h$  value ranges from  $-0.06$  for  $g=-0.4$  to  $0.2$  for  $g=0$ . Even for a higher level of risk aversion  $\gamma = 5$ , an  $h$  value of  $0.1$  is still required when  $g=0$ . This explains why traditional models assuming normal distribution fail to produce realistic results.

In Figure 9 we also graph the 95% confidence region of the  $g$  and  $h$  estimation calculated using asymptotic properties of maximum likelihood estimation. We see that the lines that represent risk aversion level of 4 and 5 in figure 9 intersect with the confidence region. This means that we cannot empirically reject the corresponding  $g$  and  $h$  values on the part of the lines that intersect with the confidence region, in other words, the corresponding coefficients of relative risk aversion are feasible and achievable. Therefore, scenario 3) performs better than scenario 1) and 2) in a sense that it allows a risk aversion lower than 4.

Figure 10 presents our estimates of IES for different levels of risk aversion and skewness. It is interesting to notice that for most levels of risk aversion and skewness, IES estimates are between 1 and 2, which is consistent with many empirical estimates. For each level of risk aversion, IES increases with skewness parameter  $g$ . This is intuitive because a more positive skewness induce the investor to save more and invest more in stocks. Therefore IES

Figure 9: Estimation results on g-h space

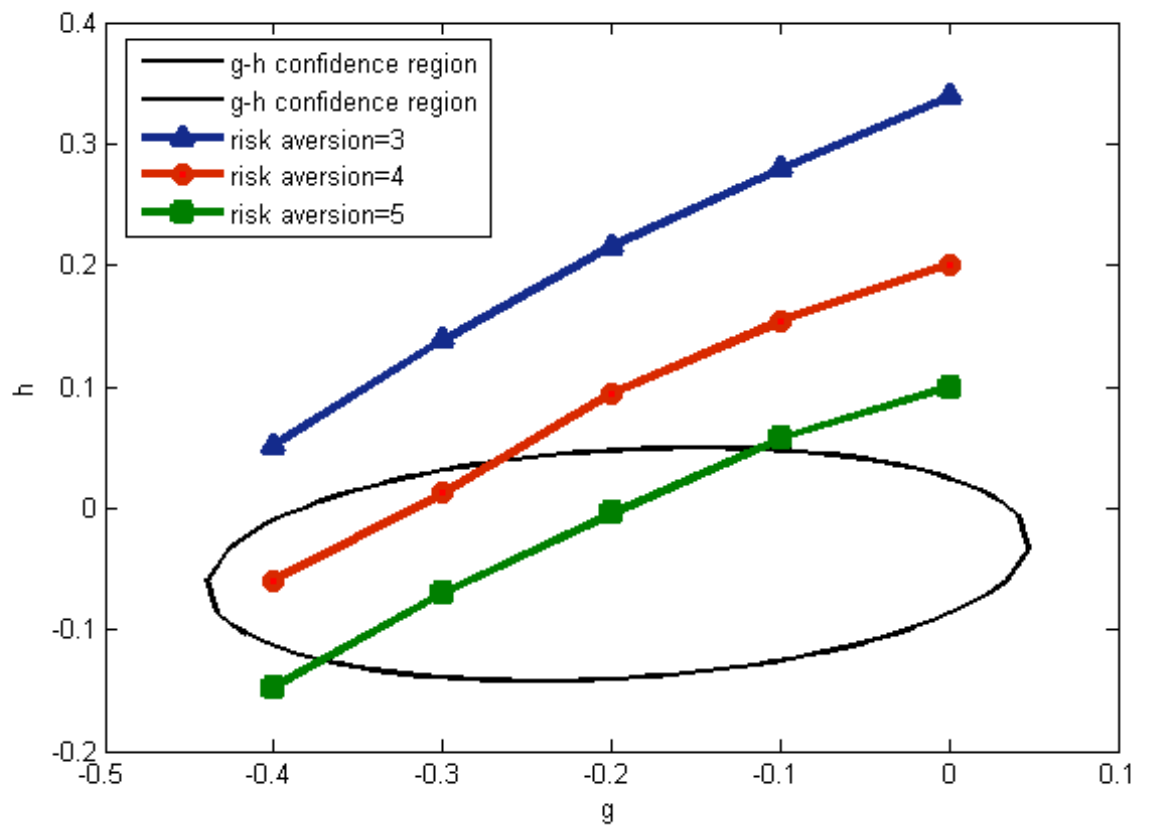
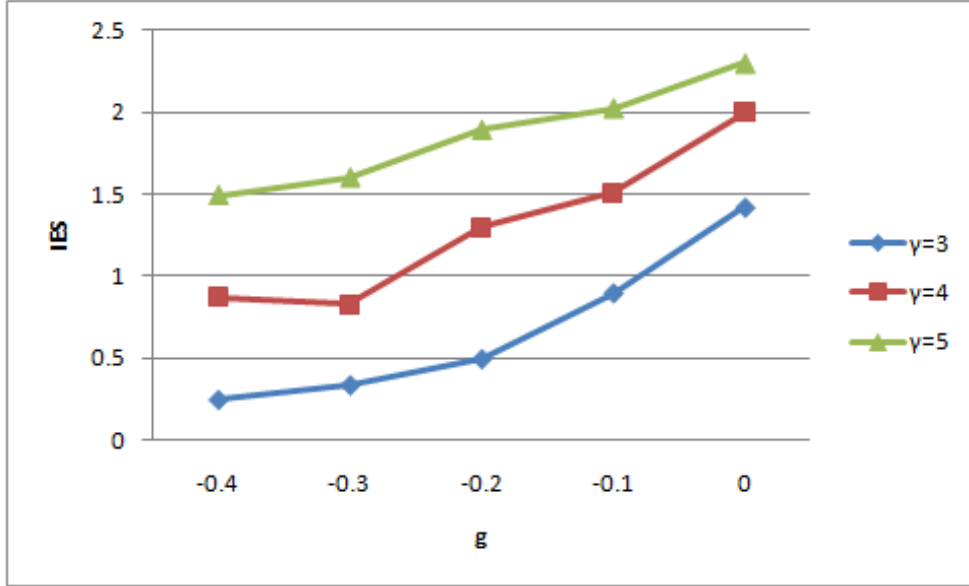


Figure 10: Estimation results on g-IES space



has to be higher to increase investor’s willingness to substitute consumption intertemporally and push down savings. For each level of skewness, IES decreases with risk aversion.

## 6 Conclusion

In this paper we develop a lifecycle model to solve numerically for the optimal consumption and portfolio rules of households who face uninsurable labor income uncertainty, mortality risk, and borrowing constraints. We incorporate generalized utility forms (Epstein-Zin-Weil utility) and generalized stock return shock distribution (g-and-h distribution). The model generates plausible consumption, wealth accumulation and portfolio choice profiles. We also perform several sensitivity analysis to see the importance of our model parameters to the results. We show that the intertemporal elasticity of substitution (IES) parameter is an important factor when individuals make consumption decisions. An individual with low IES cares less about consumption smoothing and therefore accumulates more wealth over the lifetime. Skewness and elongation, governed by g and h parameters respectively, are crucial when individuals make portfolio choices. An individual tend to

invest more in risky assets when risky assets returns are positively skewed and fat-tailed. Finally, we estimate the model parameters that allows the model results to match with empirical mean stock/bond ratio and median wealth/income ratio. We show that with the flexibility of Epstein-Zin-Weil preferences and g-and-h stock return distribution, we are able to bring the risk aversion down to lower than 4.

## 7 Appendix

### Calculation of Social Security Benefit

Individual's annual social security is calculated as the summation of:

- (a) 90 percent of the first  $\$761 \times 12$  of his final working year income, plus
- (b) 32 percent of his final working year income over  $\$761 \times 12$  and through  $\$4,586 \times 12$ , plus
- (c) 15 percent of his final working year income over  $\$4,586 \times 12$ .

## References

- [1] Ameriks, John and Zeldes, Stephen P., (2000), "How Do Household Portfolio Shares Vary With Age?". TIAA-CREF Working Paper
- [2] Badrinath, S. G., and Chatterjee, S., (1988), "On Measuring Skewness and Elongation in Common Stock Return Distributions: The Case of the Market Index," *Journal of Business*, 61(4), 451-472
- [3] Badrinath, S. G., and Chatterjee, S., (1991), "A Data-Analytic Look at Skewness and Elongation in Common-Stock-Return Distributions," *Journal of Business and Economic Statistics*, 9(9), 223-233.
- [4] Ball, C., and W. Torous, (1985), "On Jumps in Common Stock Prices and Their Impact on Call Option Pricing." *Journal of Finance*, 40, 155-173.
- [5] Bansal, Ravi, and Amir Yaron. (2004). "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *Journal of Finance*, 59(4): 1481–1509.
- [6] Barro, Robert J. (2005). "Rare Events and the Equity Premium." National Bureau of Economic Research Working Paper 11310.

- [7] Blattberg, R., and Gonedes, N. (1974). A comparison of stable and student distributions as statistical models for stock prices. *Journal of Business* 47 (April): 244-80.
- [8] Blundell, Richard, Martin Browning and Costas Meghir (1994). "Consumer Demand and the Life-Cycle Allocation of Household Expenditures," *Review of Economic Studies*, 61, 57-80.
- [9] Bodie, Z., R. C. Merton, and W. Samuelson. (1991). "Labor supply flexibility and portfolio choice in a life cycle model". *Journal of Economic Dynamics and Control*, 16,427-449.
- [10] Bucks, Brian K; Kennickell, Arthur B; Mach, Traci L; Moore, Kevin B.(2009). "Changes in U.S. Family Finances from 2004 to 2007: Evidence from the Survey of Consumer Finances." *Federal Reserve Bulletin*, vol. 95, 1st Quarter, pp. A1-55.
- [11] Campbell, J. Y., Joao F. Cocco, Francisco J. Gomes, and Pascal J. Maenhout. (2000) "Investing retirement wealth: a life cycle model". In Feldstein, M. and Campbell, J. Y. (eds), *Risk Aspects of Investment-Based Social Security Reform*. Chicago: NBER, University of Chicago Press.
- [12] Campbell, John Y. and Luis M. Viceira, (2002). "Strategic Asset Allocation, Portfolio Choice for Long-Term Investors". New York: Oxford University Press.
- [13] Carroll, C. (1992) "Buffer stock saving: some macroeconomic evidence". *Brookings Papers on Economic Activity*, 2:61-135.
- [14] Carroll, C. (1997) "Buffer stock saving and the life cycle/permanent income hypothesis". *Quarterly Journal of Economics*, 112(1):1-56.
- [15] Cocco, Joao F., Francisco J. Gomes, and Pascal J. Maenhout (2005), "Consumption and Portfolio Choice over the Life Cycle." *The Review of Financial Studies*, Vol. 18, No. 2
- [16] Deaton, A. (1991) "Saving and liquidity constraints". *Econometrica*, 59(5): 1221-1248.

- [17] Dutta, K., and Babbel, D., (2002), "On Measuring Skewness and Kurtosis in Short Rate Distributions: The Case of the US Dollar London Inter Bank Offer Rates," Wharton Financial Institutions Center Working Paper.
- [18] Dutta, K., and Babbel, D., (2005), "Extracting Probabilistic Information from the Prices of Interest Rate Options: Tests of Distributional Assumptions," *Journal of Business*, 78 (3), 841-870.
- [19] Dutta, K., and Perry, J., (2007), "A Tale of Tails: An Empirical Analysis of Loss Distribution Models for Estimating Operational Risk Capital," FRB of Boston Working Paper.
- [20] Dynan, Karen (1993). "How Prudent Are Consumers?," *Journal of Political Economy*, 101, 1104-1113.
- [21] Epstein, Larry G., and Stanley E. Zin. (1989). "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework." *Econometrica*, 57(4): 937-69.
- [22] Fama, E. F. (1965). The behavior of stock prices. *Journal of Business* 38 (January): 34- 105.
- [23] Gomes, Francisco J., and Alexander Michaelides, 2005. "Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence," *Journal of Finance*, American Finance Association, vol. 60(2): 869-904.
- [24] Gruber, Jonathan. 2006. "A Tax-Based Estimate of the Elasticity of Intertemporal Substitution." National Bureau of Economic Research Working Paper 11945.
- [25] Hall, Robert (1988). "Intertemporal Substitution in Consumption," *Journal of Political Economy*, 96, 339-57.
- [26] Heaton, J. and Lucas, D., (1996). "Evaluating the effects of incomplete markets on risk sharing and asset pricing". *Journal of Political Economy*, 103(2): 360-399.
- [27] Hoaglin, D.C., Mosteller, F., and Tukey, J.W., (1985), "Exploring Data Tables, Trends, and Shapes," Wiley. New York.

- [28] Jorion, Philippe (1988). "On Jump Processes in the Foreign Exchange and Stock Markets," *The Review of Financial Studies*, Vol. 1, No. 4, pp 427-445.
- [29] Kon, S. J. (1984). Models of stock returns a comparison. *Journal of Finance* 36 (March): 149-65.
- [30] Koo, H. K. (1998) "Consumption and portfolio selection with labor income: a continuous time approach". *Mathematical Finance*, 8, 49-65.
- [31] Mandelbrot, B. B. (1963). The variation of certain speculative prices. *Journal of Business* 36 (October): 394-419.
- [32] Martinez, J., and Iglewicz, B., (1984), "Some Properties of the Tukey g-and-h Family of Distributions," *Communications in Statistics-Theory and Methods*, 13(3), 353-369.
- [33] Merton, R. C. (1971). "Optimum consumption and portfolio rules in a continuous-time model". *Journal of Economic Theory*, 3, 373-413.
- [34] Merton, R. C. (1976). "Option Pricing When Underlying Stock Returns Are Discontinuous." *Journal of Financial Economics*, Vol. 3, (January/March 1976), pp 125-144.
- [35] Michaelides, Alex, Francisco Gomes and Valery Polkovnichenko, (2006). "Wealth Accumulation and Portfolio Choice with Taxable and Tax-Deferred Accounts," *Computing in Economics and Finance* 2006 23, Society for Computational Economics.
- [36] Mills, T.C., (1995), "Modeling Skewness and Kurtosis in the London Stock Exchange FT-SE Index Return Distributions," *The Statistician* 44 (3), 323-332.
- [37] Mulligan, Casey (2002). "Capital, Interest, and Aggregate Intertemporal Substitution," NBER Working Paper #9373.
- [38] Nishiyama Shinichi and Kent Smetters, (2005). "Consumption Taxes and Economic Efficiency with Idiosyncratic Wage Shocks", *The Quarterly Journal of Economics*, MIT Press, vol. 122(4), pages 1677-1719, November.

- [39] Nishiyama Shinichi and Kent Smetters, (2007). "Does Social Security Privatization Produce Efficiency Gains?," *The Quarterly Journal of Economics*, MIT Press, vol. 122(4), pages 1677-1719, November.
- [40] Poterba, James M. and Andrew Samwick, (2001). "Household Portfolio Allocation over the Life Cycle," NBER Chapters, in: *Aging Issues in the United States and Japan*, pages 65-104 National Bureau of Economic Research, Inc.
- [41] Samuelson, P. A. (1969). "Lifetime portfolio selection by dynamic stochastic programming". *Review of Economics and Statistics*, 51, 239-246.
- [42] Tukey, J. W., (1977), "Modern Techniques in Data Analysis," NSF-sponsored regional research conference at Southeastern Massachusetts University, North Dartmouth, MA.
- [43] Weil, Philippe. (1990). "Nonexpected Utility in Macroeconomics." *Quarterly Journal of Economics*, 105(1): 29–42.