Asset Prices When Agents are Marked-to-Market*

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Abstract

"Risk management" in securities markets refers to the oversight of portfolio managers and professional traders when they trade on behalf of investors in security markets. Monitoring of their trading performance is usually based on "marking-to-market," that is, the agents' performance, profit and loss, and risk-taking behavior, is measured by principals using market security prices. We study the optimality of the practice of marking-to-market and provide conditions under which investing principals should optimally monitor their agent traders using market prices to measure the traders' performance. Asset prices, however, can be affected by mark-to-market contracts. We show that such contracts introduce an externality when there are many traders. Traders may rationally herd, trading on irrelevant information. Ironically, this causes asset prices to be less informative than they would be without mark-to-market contracts.

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1 Introduction

"Marking-to-market" refers to the widespread practice of monitoring portfolio managers' and security traders' performance and risk-taking propensity based on market security prices. As a risk management practice, marking-to-market of professional traders' positions received fresh impetus after the collapse of Enron, and other derivatives scandals.[1] Moreover, accounting measurements of profit and loss are also increasingly based on security market prices, as the best measure of "fair value".[2] Marking-to-market follows from the "efficient market" paradigm of security prices. In this paper, we examine the issue of "marking-to-market" in the context of a security market in which some traders are trading on behalf of investing principals. The principals must design compensation contracts to try to control the trading behavior of the professional traders (or money managers) they hire. Since market security prices are observable and contractible, these prices may be used to value the traders' positions as part of the optimal contract. We provide conditions under which "marking-to-market" is part of an optimal contract. This confirms the logic of using market prices as the best measure of a trader's performance. This is the basis for the accounting professions' equating of "fair value" with market prices and also for the widespread risk management practice of using security prices to assess the risk-taking propensity of traders.

The notion of efficient markets views security prices as exogenous. However, we go on to show that the security prices themselves may not be exogenous indicators of value when traders are marked-to-market. Equating "fair value" with market prices is problematic because marking-to-market introduces an externality among the professional traders. Their trading behavior affects the security prices, which in turn, affects the mark-to-market value of their positions. This feedback occurs for each trader because of his beliefs about what other traders are doing. As a result, traders may rationally trade on irrelevant information, herding with other traders, and making the security prices used to mark their positions, and control their risk-taking, less efficient. Ironically, the very act of marking-to-market can make the security prices less informative than they would be otherwise.

Trading in security markets is conducted by professional portfolio managers and traders. In over-the-counter markets, like those for corporate bonds, residential and commercial mortgaged-backed securities, other asset-backed securities, and derivative securities (comprising interest rate,

[1] Derivatives-related scandals include the failure of Organge County, California in the early 1990s, the failure of Barings PLC in 1995, the failure of Mettalgesellschaft, the collapse of Long Term Capital Management, Enron, and the recent failure of China Aviation.

[2] Indeed, the sudden collapse of Enron in late 2001 gave further impetus to the confluence of the efficient markets theory of security prices and the accounting profession's notion of "fair value." Accountants are increasingly embracing the notion that "fair value" for accounting purposes is best measured using security market prices. Enron was accused of mis-marking complex derivatives using models. The issue of whether "fair value" is best represented by market prices is most controversial and problematic in derivatives markets, but is present in all security markets.

equity, foreign currency, credit, commodity, energy and other derivatives), the entire market works this way. These are very large and important markets. For example,the notional amount of interest rate derivatives in June 2005 was $204.4 trillion (see Bank for International Settlements (2005)). In the case of credit derivatives, the British Bankers' Association predicts that the market will reach $8.2 trillion by the end of 2006.[3]

Since portfolio managers or professional traders (sometimes called "traders" when the meaning is clear) do not manage their own money, there is conflict of interest between the such traders and the investors who hire them. In the model we study, investors hire traders to trade for them because traders have the ability to become informed at a cost. Investors need to design contracts to motivate their traders to collect information and to exploit that information to generate the highest possible return by trading. If a trader is hired, he decides whether to exert effort to collect information. Then trading occurs and an interim security price is determined based on the market maker's rational expectation, as in Kyle (1985). Later, all information becomes public and the security price converges to the final value or realized payoff. Because the information collected by the trader is noisy, it is impossible to infer from the realized payoff whether he has collected information and traded in the best interests of the investor or not

Compensation contracts for traders can only depend on observable and verifiable variables, but efforts made by the traders and any information they may collect are unobservable. Security prices are observable and contractible information. There is a final price which is realized after all information is public, uncertainty is resolved. More importantly, there is a price observed earlier, an interim price, just after security trading when there is still asymmetric information. In order to better motivate and monitor the trader, the optimal contract may include the interim price–a noisy signal of the interim private information–in addition to the final price and the manager's trading decision. In addition to security prices, each trader's portfolio is observable to the investor who hired him. The manager only gets paid when his trading decision is consistent with the realized return and can be justified by the interim price. Contracts based on the final fair price are the performance-based contracts, which are widely studied in the literature on principal-agent problems. Contracting on the interim price, and trading position, when there is residual asymmetric information is useful because it enables investors to better monitor their traders. This type of contract "marks-to-market." Since the interim price itself depends on the traders' actions, "mark-to-market" contracts can have feedback impacts on asset prices.

[3]Even in the public equity market, most trading is delegated to profesional portfolio managers. In 2004, U.S. households directly held less than 40% of corporate equities, while they held about 90% in 1950 and 70% in 1970 (see "Flow of Funds" issued by the Federal Reserve Board). Also according to the survey results released by the Investment Company Institute (ICI) and the Securities Industry Association (SIA), in 2002, 89% of the investors invested in mutual funds and 58% of the investors relied on professional financial advisers when making investment decisions.

We first analyze the case where there is one single professional trader, hired by an investor. We find that the optimal contract consists of a step function of the interim price. Holding the security position, the trader gets paid only if the interim price is high enough. Selling the shares, the trader gets paid only if the interim price is low enough. The cut-off prices depends on the realized state. Intuitively, the trader should hold the shares if and only if he privately receives good news. When the interim price is low, it indicates that bad news has arrived to the market. If the trader has made an effort to collect information, he should be informed of the bad news and sell the securities. On the other hand, a high interim price suggests good news and the hard-working trader should hold his position. Because the signals are noisy and there are noise traders in the market, the final fair price is included in the contract as a complementary meanings of monitoring.

Once we solve the case of a single professional trader, we then extend the model to the case with multiple principal-trader pairs. We find that existence of multiple professional traders introduces additional equilibria. It is possible that traders hold security positions on bad news, and it is also possible that managers sell their positions upon good news arriving. This is caused by the externality referred to above. When other professional traders sell on good news, the security price goes down. If a trader holds his position, then he will be in trouble when the realized state is bad. In that case, the investor who hires him would ask: "Why didn't you sell the securities at the same time the other traders did? Did you work to collect the information?" Under suspicion of shirking, the trader would get punished under the optimal contract. Therefore, the trader ignores his information and sells the securities. The seemingly irrational equilibria are generated by the agency problem and the externality. Combined, these two factors can alter incentives of the traders and cause mispricing of the securities in the market.

Our paper belongs to the strand of the literature that examines the impact of agency problems on asset pricing. For example, Allen and Gorton (1993) show that when there is asymmetric information between investors and portfolio managers, portfolio managers have an incentive to churn; their trades are not motivated by changes in information liquidity needs or risk sharing but rather by a desire to profit at the expense of the investors that hire them. As a result, assets can trade at prices that do not reflect their fundamentals and bubbles can exist. Dow and Gorton (1994) and Goldman and Slezak (2003) examine the pricing impact of an interaction between agency problems and time-horizon and show that managers' incentives can be distorted by an information externality. In our paper, while the principal only cares about the final price, a trader's horizon is endogenously shorter since the optimal contract links his payoff to the interim price. Dow and Gorton (1997) study a model in which investors optimally contract with portfolio managers who may have stock-picking abilities, and portfolio managers trade optimally given the incentives provided by this contract. Because investors cannot distinguish "actively doing nothing" from "simply doing nothing," some managers trade with no proper reason (noise trade). Noise trade causes high levels and turnover.[4]

The paper proceeds as follows. In Section 2 we set up the model and state some assumptions about the security payoffs and the noisy signals. In Section 3 we solve for the optimal contract in the case of one trader. We show that the optimal contract consists of a step function based on the interim security price, which is the proxy for the trader's private information. In order to prevent the trader from shirking, the optimal contract requires that the trader's trading decision (i.e., buy or sell) be consistent with the change in the security price. We introduce the irrelevant signal - the noise - in Section 4 and show that it does not generate an additional equilibrium when there is only one trader in the market. The case of multiple traders is studied in Section 5. We find that when there is more than one trader in the market, additional equilibria emerge. In these equilibria, traders either sell the security upon receiving private good news or hold the stock upon receiving bad news. The interaction of the agency problem and the externality give rise to these seemingly irrational equilibria. The impact of this behavior on security prices and social welfare are also analyzed. We conclude in Section 6.

2 Model Setup

The setting is a security market in which there are five types of (risk neutral) participants: a direct investor, an indirect investor, a professional securities trader, noise traders, and a market maker.[5] Both the professional trader and the direct investor have the technology to become imperfectly informed about the value of the security at a final date. The direct investor has money to invest in the securities market, but the trader does not. However, the trader can be hired by the indirect investor, who has the money but no access to the technology to become (privately) informed. After the trader is hired, he can choose not to make an effort to acquire information; not making an effort yields a shirking benefit of $\kappa > 0$. For ease of exposition, we will refer to the security as "stock."

There are three dates in this economy.

- Date 1: The direct investor enters the market holding x shares of the stock. The professional trader and the indirect investor sign a contract to authorizes the trader to trade on behalf of the indirect investor and which specifies the payoffs to the trader under each contingency

[4] There is a large literature on delegated portfolio management, which focuses on issues of the manager's compensation structure. E.g., Bhattacharya and Pfleiderer (1985), Brennan (1993), Admati and Pfleiderer (1997), Cuoco and Kaniel (2000), and Ou-Yang (2003). For the most part, the issue in these studies concerns the choice of a "benchmark" to use to evaluate the manager's performance.

[5] The role of the direct investor is to make the price informative even when the trader shirks.

observable to both parties to the contract. The trader then enters the market holding x shares of the stock.

- Date 2: Both the direct investor and the professional trader (if he does not shirk) receive a signal about the terminal value of the stock. Then each decides whether to liquidate his position (i.e., sell his shares) or not. The noise traders trade $\delta \sim N(0, \sigma^2)$. The market maker sets the price based on the total order flow.

- Date 3: The liquidation value of the stock is realized, and the indirect investor pays his trader according to the contract.

The liquidation value of the stock has the following distribution:

$$v = \begin{cases} v_H & \text{with probability } \pi_v \\ v_L & \text{with probability } 1 - \pi_v \end{cases},$$

and assume $v_H > v_L$.

For simplicity, the direct investor and the professional trader (if he does not shirk) receive the same private signal, which can be either s_H or s_L, and the signal is correlated with the true liquidate value of the stock as follows:

$$s = \begin{cases} s_H \text{ with probability } \theta > 1/2 \text{ if } v = v_H \\ s_L \text{ with probability } \theta > 1/2 \text{ if } v = v_L \end{cases}.$$

Let π_s denote the unconditional probability of receiving a high signal, then we have:

$$\pi_s = \pi_v \theta + (1 - \pi_v)(1 - \theta).$$

3 The Optimal Contract

In this section we are interested in equilibria in which, given the optimal contract, the professional trader does not shirk, and trades in the best interests of the indirect investor. We first define:

Definition 1 ***(Regular Equilibrium)*** *A Regular Equilibrium is an equilibrium in which: (i) the professional trader does not shirk; and (ii) both the professional trader and the direct investor trade truthfully based on the information they acquire, i.e., they hold their position when a good signal is received and they liquidate the position when a bad signal is received.*

In a Regular Equilibrium, the distribution of the market order flow z is a mixture of two Normal distributions:

$$z \sim \pi_s N(0, \sigma^2) + (1 - \pi_s) N(-2x, \sigma^2).$$

Let $\phi^+(z)$ denote the probability density function for distribution $N(0, \sigma^2)$, and $\phi^-(z)$ for $N(-2x, \sigma^2)$. We also denote $\phi^x(z)$ as the probability density function for distribution $N(-x, \sigma^2)$. We have:

$$\begin{aligned}
\phi^+(z) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\{\frac{-z^2}{2\sigma^2}\}, \qquad (1) \\
\phi^-(z) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\{\frac{-(z+2x)^2}{2\sigma^2}\}, \\
\text{and } \phi^x(z) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\{\frac{-(z+x)^2}{2\sigma^2}\}.
\end{aligned}$$

Lemma 1 *$\frac{\phi^+(z)}{\phi^x(z)}$ is increasing in z and $\frac{\phi^-(z)}{\phi^x(z)}$ is decreasing in z.*

Proof. Algebra. ■

When the professional trader holds his position, it could either be the case that he received a good signal or, alternatively, that he received a bad signal. The former case will lead to a zero informed order flow; in the latter case, the informed order flow will be $-x$. The monotonicity of the likelihood ratio $\frac{\phi^+(z)}{\phi^x(z)}$ tells us that the higher the order flow z (equivalently, a higher p), the more likely that it is the former case. A similar argument applies for the likelihood ratio $\frac{\phi^-(z)}{\phi^x(z)}$, when the trader liquidates his position. This property of the monotone likelihood ratio is important for us to characterize the optimal contract.

Let p_H denote the expected liquidation value at date 3 conditional on that signal s_H being received at date 2 by the trader and the direct investor; p_L is defined similarly. Then:

$$\begin{aligned}
p_H &= E[v|s_H] = \frac{\pi_v\theta}{\pi_v\theta + (1-\pi_v)(1-\theta)} v_H + \frac{(1-\pi_v)(1-\theta)}{\pi_v\theta + (1-\pi_v)(1-\theta)} v_L \\
\text{and } p_L &= E[v|s_L] = \frac{\pi_v(1-\theta)}{\pi_v(1-\theta) + (1-\pi_v)\theta} v_H + \frac{(1-\pi_v)\theta}{\pi_v(1-\theta) + (1-\pi_v)\theta} v_L.
\end{aligned}$$

The market maker will set the price to be the expected liquidation value conditional on the order flow. In a Regular Equilibrium the price can be expressed as a function of the order flow z as follows:

$$p(z) = \frac{\pi_s\phi^+(z)}{\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z)} p_H + \frac{(1-\pi_s)\phi^-(z)}{\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z)} p_L. \qquad (2)$$

Lemma 2 *In a Regular Equilibrium, we have:*

(i) $p(z)$ is strictly increasing in z; and

(ii) $\lim_{z\to\infty} p(z) = p_H$ and $\lim_{z\to-\infty} p(z) = p_L$.

Proof. Algebra. ■

The market maker can only observe the total order, z, but not the identity of traders or their positions. From the magnitude of the total order flow, he infers the news received by the informed agents. The larger the order size, the more likely that the good news was received, and thus the market maker sets a higher price. In the limit, when the market maker receives a huge buy order, he is pretty sure that the good news was received by the informed agents and the price is set to p_H, the expected value conditional on the good news; when the market maker receives a huge sell order, he is pretty sure that the bad news was received by the informed agents and the price is set to p_L, the expected value conditional on the bad news.

As we can see from equation (2), the market price in a Regular Equilibrium will be between p_H and p_L. Therefore, it will be in the best interests of the indirect investor and the direct investor to hold their positions given a good signal and liquidate their positions given a bad signal.

The contract between the indirect investor and the professional trader stipulates a payoff to the trader of w, which depends on all the publicly available information, including the price at date 2, p, the trader's security position at date 2, λ, and the terminal value of the security, v. In a Regular Equilibrium, an optimal contract gives the trader an incentive to produce information and trade in the best interests of the indirect investor.

To simplify the notation, define the following:

$$\begin{aligned} A^{\chi}_{\lambda\eta} &= \int_{-\infty}^{\infty} w(p(z), \lambda, v_\eta)\phi^{\chi}(z)dz, \qquad (3) \\ \text{where } \chi &= +,\ -\ , \text{ or } x \\ \lambda &= x \text{ or } 0 \\ \text{and } \eta &= H \text{ or } L. \end{aligned}$$

Basically, $A^{\chi}_{\lambda\eta}$ is the expected payoff to the trader when his security position is λ, the informed order flow is χ, and the realized liquidation value is v_η.

For a Regular Equilibrium, a contract $w(p, \lambda, v)$ is incentive compatible if it satisfies:

$$\begin{aligned} \text{(IC1.1)}\ \frac{\pi_v\theta}{\pi_s}A^{+}_{xH} + \frac{(1-\pi_v)(1-\theta)}{\pi_s}A^{+}_{xL} &\geq \frac{\pi_v\theta}{\pi_s}A^{x}_{0H} + \frac{(1-\pi_v)(1-\theta)}{\pi_s}A^{x}_{0L} \qquad (4) \\ \text{and } \frac{\pi_v(1-\theta)}{1-\pi_s}A^{-}_{0H} + \frac{(1-\pi_v)\theta}{1-\pi_s}A^{-}_{0L} &\geq \frac{\pi_v(1-\theta)}{1-\pi_s}A^{x}_{xH} + \frac{(1-\pi_v)\theta}{1-\pi_s}A^{x}_{xL}, \end{aligned}$$

and

$$
\begin{aligned}
& \text{(IC1.2) } \pi_s[\frac{\pi_v\theta}{\pi_s}A^+_{xH} + \frac{(1-\pi_v)(1-\theta)}{\pi_s}A^+_{xL}] \\
& +(1-\pi_s)[\frac{\pi_v(1-\theta)}{1-\pi_s}A^-_{0H} + \frac{(1-\pi_v)\theta}{1-\pi_s}A^-_{0L}] \\
\geq\ & \max\{\pi_v[\theta A^+_{xH} + (1-\theta)A^x_{xH}] + (1-\pi_v)[(1-\theta)A^+_{xL} + \theta A^x_{xL}] + \kappa, \\
& \pi_v[\theta A^x_{0H} + (1-\theta)A^-_{0H}] + (1-\pi_v)[(1-\theta)A^x_{0L} + \theta A^-_{0L}] + \kappa\}.
\end{aligned} \tag{5}
$$

(IC1.1) says that, when the trader receives a good signal, he will be better off holding the position, and when he receives a bad signal, he will be better off liquidating his position. (IC1.2) says that the trader will be better off by acquiring the information and trading in the best interests of the indirect investor instead of shirking (getting κ) and blindly holding or liquidating the position.

(IC1.2) can be rewritten as:

$$
\begin{aligned}
\pi_v\theta A^+_{xH} + (1-\pi_v)(1-\theta)A^+_{xL} &\geq \pi_v\theta A^x_{0H} + (1-\pi_v)(1-\theta)A^x_{0L} + \kappa \\
\text{and } \pi_v(1-\theta)A^-_{0H} + (1-\pi_v)\theta A^-_{0L} &\geq \pi_v(1-\theta)A^x_{xH} + (1-\pi_v)\theta A^x_{xL} + \kappa,
\end{aligned} \tag{6}
$$

so it is obvious that (IC1.2) implies (IC1.1). Intuitively, if a trader collects information at a cost ex-ante, then he will make a full use of it ex-post.

- *Assumption: There is an upper bound $\bar{w}$ for $w(p, \lambda, v)$.*

The role of this assumption will become clear as we proceed; the interpretation of $\bar{w}$ will also be discussed later.

Define W as follows:

$$W \equiv \{w(p, \lambda, v) | 0 \leq w(p, \lambda, v) \leq \bar{w} \text{ for any } p,\ \lambda \text{ and } v\}.$$

W is the set of all feasible contracts.

The optimal contract $w^*(p, \lambda, v)$ solves the following programming problem:

$$
\begin{aligned}
& \min_{w \in W} \pi_v\theta A^+_{xH} + \pi_v(1-\theta)A^-_{0H} + (1-\pi_v)\theta A^-_{0L} + (1-\pi_v)(1-\theta)A^+_{xL} \\
& \text{subject to the IC conditions in (6).}
\end{aligned} \tag{7}
$$

Before we proceed to study the properties of the optimal contract $w^*(p, \lambda, v)$, we state the following lemma.

Lemma 3 *In any optimal contract in a Regular Equilibrium, the constraints in (6) are binding.*

Proof. If the first inequality is not binding, then we can lower $w(p, x, v)$ a bit, and both inequalities will still hold as long as the change in $w(p, x, v)$ is small. Similarly, if the second inequality is not binding, then we can lower $w(p, 0, v)$ a bit while not violating the inequalities. ■

Proposition 1 ***(Optimal Contract)*** *In a Regular Equilibrium, the optimal contract $w^*(p, \lambda, v)$ takes the following form: (i) when $\lambda = x$, there exists $z^*_{x\eta}$ with $\eta = H$ or L such that, for any $p < p(z^*_{x\eta})$, $w^*(p, x, v_\eta) = 0$, and for any $p \geq p(z^*_{x\eta})$, $w^*(p, x, v_\eta) = \bar{w}$; (ii) when $\lambda = 0$, there exists $z^*_{0\eta}$ with $\eta = H$ or L such that, for any $p > p(z^*_{0\eta})$, $w^*(p, 0, v_\eta) = 0$, and for any $p \leq p(z^*_{0\eta})$, $w^*(p, 0, v_\eta) = \bar{w}$.*

Proof. We will only show the proof for (i); the proof for part (ii) follows a similar argument. We will prove (i) by contradiction. Suppose that there exist two values of z, $z_1 < z_2$, such that $w^*(p(z), x, v_\eta) \geq \varepsilon_1$ for any $z \in [z_1 - \frac{\delta_1}{2}, z_1 + \frac{\delta_1}{2}]$, and $w^*(p(z), x, v_\eta) \leq \bar{w} - \varepsilon_2$ for any $z \in [z_2 - \frac{\delta_2}{2}, z_2 + \frac{\delta_2}{2}]$. Let $R^+ = \frac{\phi^+(z_1)}{\phi^+(z_2)}$, and, without loss of generality, assume that $\delta_1 = \delta_2 = \delta$, $\varepsilon_1 \geq \varepsilon$ and $\varepsilon_2 \geq R^+\varepsilon$ for some ε, and $z_2 > z_1 + \delta$. Now construct a new wage schedule $\widehat{w}^*(p(z), x, v_\eta)$ as follows:

$$\widehat{w}^*(p(z), x, v_\eta) = \begin{cases} w^*(p(z), x, v_\eta) - \varepsilon & \text{if } z \in [z_1 - \frac{\delta}{2}, z_1 + \frac{\delta}{2}] \\ w^*(p(z), x, v_\eta) + R^+\varepsilon & \text{if } z \in [z_2 - \frac{\delta}{2}, z_2 + \frac{\delta}{2}] \\ w^*(p(z), x, v_\eta) & \text{otherwise} \end{cases}.$$

Basically, we cut a piece of wage from the neighborhood of z_2 and paste another piece into the neighborhood of z_1. With the new constructed wage schedule $\widehat{w}^*(p(z), x, v_\eta)$, the resulting $\widehat{A}^+_{x\eta}$ remains approximately the same, while the resulting $\widehat{A}^x_{x\eta}$ will be strictly smaller since $\frac{\phi^x(z_1)}{\phi^x(z_2)} > \frac{\phi^+(z_1)}{\phi^+(z_2)}$, according to Lemma 1. Therefore, the second inequality of the IC conditions in (6) will become a strict inequality, which means that the contract can be strictly improved according to Lemma 3, a contradiction. ■

Intuitively, the price at date 2 reveals information about the true liquidation value of the stock no matter what the strategy of the professional trader because of the existence of the direct investor. In order to provide the professional trader with an incentive to acquire information and liquidate his position given a bad signal, the optimal contract punishes him when the price at date 2 is low and he chose to hold his position. Similarly, the optimal contract punishes him when the price at date 2 is high and the chose to liquidate his position.

If there were no upper bound on the contract payment, the optimal contract would be to pay an infinitely high amount to the trader but only when the price was close to p_H (order flow is infinitely large). The existence of $\bar{w}$ guarantees the boundedness of the optimal contract. One way to interpret the upper bound $\bar{w}$ is to imagine that the trader is risk averse instead of risk neutral.

Then paying a very high wage but only with a small probability is not optimal for the principal (indirect investor) since the risk premium required by the trader will make the expected payment very high. With a risk averse trader, the existence of $\bar{w}$ could be justified and endogenized. We do not pursue that here.

Proposition 1 tells us that the optimal contract for a Regular Equilibrium consist a of step function (or equivalent in measure) in terms of the security price p for each pair of λ and v. If the trader holds his position and the price at date 2, p, is lower than a certain level, $\min\{p(z^*_{xH}), p(z^*_{xL})\}$, he will receive a zero payoff for sure, no matter what v is realized at date 3. Similarly, if the trader liquidates his position and the security price at date 2, p, is higher than a certain level, $\max\{p(z^*_{0H}), p(z^*_{0L})\}$, he will receive a zero payoff for sure, no matter what v is realized at date 3. We can interpret these results as a general "mark-to-market" practice, i.e., if the market price has moved against what the trader's position, then he will be fired and receive no payment.

Now we show some properties of the optimal cutoff $z^*_{\lambda\eta}$ for the step function $w(p, \lambda, v_\eta)$.

Proposition 2 ***(Properties of the Optimal Contract)*** *In an optimal contract of a Regular Equilibrium, we have:*

$$z^*_{xH} - z^*_{xL} = z^*_{0H} - z^*_{0L} = \frac{2\sigma^2}{x}\ln(\frac{1-\theta}{\theta}), \tag{8}$$

and

$$z^*_{xH} - z^*_{0H} = z^*_{xL} - z^*_{0L} > x. \tag{9}$$

Proof. We only need to show:

$$\begin{aligned}
\frac{\phi^+(z^*_{xH})}{\phi^x(z^*_{xH})} &= \frac{(1-\theta)^2}{\theta^2}\frac{\phi^+(z^*_{xL})}{\phi^x(z^*_{xL})}, \\
\frac{\phi^-(z^*_{0L})}{\phi^x(z^*_{0L})} &= \frac{(1-\theta)^2}{\theta^2}\frac{\phi^-(z^*_{0H})}{\phi^x(z^*_{0H})}, \\
\frac{\phi^+(z^*_{xH})}{\phi^x(z^*_{xH})} &> \frac{\phi^x(z^*_{0H})}{\phi^-(z^*_{0H})}, \\
\text{and } \frac{\phi^+(z^*_{xL})}{\phi^x(z^*_{xL})} &> \frac{\phi^x(z^*_{0L})}{\phi^-(z^*_{0L})}.
\end{aligned}$$

See the Appendix for the proof of the above conditions. ■

Since $\theta > \frac{1}{2}$, Proposition 2 implies $z^*_{xH} < z^*_{xL}$ and $z^*_{0L} > z^*_{0H}$, which implies $A^+_{xH} > A^+_{xL}$ and $A^-_{0L} > A^-_{0H}$. Thus, in a Regular Equilibrium, the professional trader receives a lower payoff if the realized liquidation value contradicts the trader's trading decision. Notice that the difference between z^*_{xH} and z^*_{xL} (or z^*_{0H} and z^*_{0L}) is greater when the signal is more accurate, i.e., when the value of θ is greater. Intuitively, the trader gets harsher punishment if the probability of making a mistake is lower. Therefore, Proposition 2 characterizes how the trader's payoff will differ depending on how the realized liquidation value of the stock is compared to his position of stockholding.[6]

Using the results from Proposition 2 and Lemma 3, we can solve for the optimal contractual cut-off points $z^* = \{z^*_{xH},\ z^*_{xL},\ z^*_{0H},\ z^*_{0L}\}$ from the following two equations derived from the IC conditions in (6):

$$\begin{aligned} & \pi_v \theta[1-\Phi^+(z^*_{xH})]\bar{w} + (1-\pi_v)(1-\theta)[1-\Phi^+(z^*_{xL})]\bar{w} \\ = \quad & \pi_v \theta \Phi^x(z^*_{0H})\bar{w} + (1-\pi_v)(1-\theta)\Phi^x(z^*_{0L})\bar{w} + \kappa, \end{aligned} \tag{10}$$

and

$$\begin{aligned} & \pi_v(1-\theta)\Phi^-(z^*_{0H})\bar{w} + (1-\pi_v)\theta\Phi^-(z^*_{0L})\bar{w} \\ = \quad & \pi_v(1-\theta)[1-\Phi^x(z^*_{xH})]\bar{w} + (1-\pi_v)\theta[1-\Phi^x(z^*_{xL})]\bar{w} + \kappa, \end{aligned} \tag{11}$$

where $\Phi^\sim(.)$ is the cdf for the corresponding to the pdf $\phi^\sim(.)$.

The next Lemma shows how the optimal cutoff value $z^* = \{z^*_{xH}, z^*_{xL}, z^*_{0H}, z^*_{0L}\}$ changes with the cost of information production.

Lemma 4 *For the optimal contract in a Regular Equilibrium,* $\frac{\partial z^*_{xH}}{\partial \kappa} = \frac{\partial z^*_{xL}}{\partial \kappa} < 0$ *and* $\frac{\partial z^*_{0H}}{\partial \kappa} = \frac{\partial z^*_{0L}}{\partial \kappa} > 0$*. Define*

$$A \equiv \pi_v \theta A^+_{xH} + \pi_v(1-\theta)A^-_{0H} + (1-\pi_v)\theta A^-_{0L} + (1-\pi_v)(1-\theta)A^+_{xL}.$$

In a Regular Equilibrium $\frac{\partial A}{\partial \kappa} > 1$*.*

Proof. See Appendix. ■

The expected wage payoff to the professional trader is decreasing with z^*_{xH} and $z^*_{xL}$and increasing with z^*_{0H} and z^*_{0L}. Therefore, the indirect investor has to pay the trader more when the cost of information production (which can also be interpreted as the opportunity cost of the trader) increases. The second half of Lemma 4 shows that the expected payoff to the trader net of the cost of information production is also increase. The next corollary demonstrates the effect of $\bar{w}$ on z^*.

Corollary 1 *For the optimal contract in a Regular Equilibrium,* $\frac{\partial z^*_{xH}}{\partial \bar{w}} = \frac{\partial z^*_{xL}}{\partial \bar{w}} > 0$ *and* $\frac{\partial z^*_{0H}}{\partial \bar{w}} = \frac{\partial z^*_{0L}}{\partial \bar{w}} > 0$*.*

[6] The results in Proposition 2 can be obtained using the first order conditions with respect to z^*_{xH}, z^*_{xL}, z^*_{0H} and z^*_{0L}. The method that we employed in Appendix is different, but is very useful to characterize the equilibrium for the case with multiple principal-agent pairs as will be seen soon.

Proof. Similar to the proof of Lemma 4, and omitted. ■

Intuitively, a higher $\bar{w}$ allows the indirect investor to better exploit the information contained in the price to discipline the behavior of the trader.

Now we discuss the equilibrium payoffs to all the agents in the economy. In a Regular Equilibrium, the expected value of the investment under informed trading is:

$$V = x[\pi_s p_H + (1-\pi_s)\int_{-\infty}^{\infty} p(z)\phi^-(z)dz]. \tag{12}$$

As we can see, V is the joint expected payoff to the indirect investor and the trader, or the expected payoff to the direct investor.

The expected payoff to the noise traders is:

$$\begin{aligned} V_N &= -\int_{-\infty}^{\infty} z[\pi_s p(z) + (1-\pi_s)p(z-2x)]\phi^+(z)dz \\ &\quad + \int_{-\infty}^{\infty} z[\pi_s p_H + (1-\pi_s)p_L]\phi^+(z)dz \\ &= 2x(1-\pi_s)[p_L - \int_{-\infty}^{\infty} p(z)\phi^-(z)dz] < 0. \end{aligned}$$

As we can see $2V + V_N = 2x[\pi_s p_H + (1-\pi_s)p_L]$.

Lemma 5 *In a Regular Equilibrium,* $\frac{\partial V}{\partial \sigma} > 0$ *and* $\frac{\partial V_N}{\partial \sigma} < 0$.

Proof. We only need to show how $\int_{-\infty}^{\infty} p(z)\phi^-(z)dz$ changes with σ. We have:

$$\begin{aligned} &\int_{-\infty}^{\infty} p(z)\phi^-(z)dz \\ &= p_L + \pi_s(p_H - p_L)\int_{-\infty}^{\infty} \frac{\phi^+(z)\phi^-(z)}{\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z)}dz \\ &\overset{z'=z/\sigma}{=} p_L + \frac{1}{\sqrt{2\pi}}\pi_s(p_H - p_L)\int_{-\infty}^{\infty} \frac{\exp(-z^2/2)}{\pi_s \exp[(2x/\sigma)(z + x/\sigma)] + (1-\pi_s)}dz. \end{aligned}$$

Taking the derivative with respect to σ for the integral part, we have:

$$
\begin{aligned}
& \frac{\partial}{\partial\sigma}\int_{-\infty}^{\infty}\frac{\exp(-z^2/2)}{\pi_s\exp[(2x/\sigma)(z+x/\sigma)]+(1-\pi_s)}dz \\
= \; & \pi_s(2x/\sigma^2)\int_{-\infty}^{\infty}\frac{\exp[-(z+2x/\sigma)^2/2)]\exp[(4x/\sigma)(z+x/\sigma)](z+2x/\sigma)}{\{\pi_s\exp[(2x/\sigma)(z+x/\sigma)]+(1-\pi_s)\}^2}dz \\
= \; & \pi_s(2x/\sigma^2)\int_{-\infty}^{\infty}\frac{\exp[-(z+2x/\sigma)^2/2)](z+2x/\sigma)}{\{\pi_s+(1-\pi_s)\exp[-(2x/\sigma)(z+x/\sigma)]\}^2}dz \\
& \overset{z'=z+2x/\sigma}{=} \pi_s(2x/\sigma^2)\int_{-\infty}^{\infty}\frac{\exp(-z^2/2)z}{\{\pi_s+(1-\pi_s)\exp[-(2x/\sigma)(z-x/\sigma)]\}^2}dz \\
> \; & \pi_s(2x/\sigma^2)\int_{-\infty}^{\infty}\exp(-z^2/2)zdz.
\end{aligned}
$$

The last inequality comes from the fact that $\{\pi_s+(1-\pi_s)\exp[-(2x/\sigma)(z-x/\sigma)]\}^2$ is a positive function decreasing in z. ■

The next lemma studies the effect of σ on the optimal contract. It shows that, when the price becomes more informative (or the liquidity trading is less noisy), the optimal contract can be improved. Our results on the payoff dependence on σ are consistent with Kyle (1985), in which an informed trader always benefits from a higher σ.

Lemma 6 *If there exists an optimal wage step function $w(p(z),\lambda,v_\eta)$ (as in Proposition 1) satisfying the IC conditions in (6) under σ, then for any $0<\hat{\sigma}<\sigma$, there exists an optimal contract $\hat{w}(p(z),\lambda,v_\eta)$ that pays less to the trader.*

Proof. See Appendix. ■

In a Regular Equilibrium, the expected payoffs to the direct investor and the noise traders are not affected by the parameter values of κ or $\bar{w}$, but are affected by σ. At the same time, κ and $\bar{w}$ do affect the distribution of the payoff between the indirect investor and his trader. We summarize the payoff dependency in the proposition below.

Proposition 3 *(i) The expected payoff to the direct investor is increasing with σ, and the expected payoff to the noise traders is decreasing with σ.*

(ii) The expected wage payment to the trader is increasing with σ, increasing with κ, and decreasing with $\bar{w}$; the trader's expected payoff net of the cost of information production is increasing with κ.

(iii) The expected payoff to the indirect investor is decreasing with κ, and increasing with $\bar{w}$.

Proof. The results are implied by Lemma 5, Lemma 6, Lemma 4 and Corollary 1. ■

A trader can benefit from a higher σ because he is monitored less when the price is less informative; when σ is smaller, the principal (indirect investor) can offer a contract more sensitive to price and reduce the payoff to his trader. However, it is not clear how the value of σ affects the expected payoff to the indirect investor. On the one hand, a higher σ leads to a higher expected wage payment, on the other hand, the joint payoff, V, also increases with σ. The results on κ and $\bar{w}$ are easier to interpret. When the (opportunity) cost of information production, κ, is higher, the principal has to pay the trader more. When we increase $\bar{w}$, the optimal contract with the original lower $\bar{w}$ is still feasible, but we know from Proposition 1 that it does not satisfy the conditions for the optimal contract with the higher $\bar{w}$, i.e., it can be improved upon. In other words, when $\bar{w}$ is higher, the contract in the form of "$\bar{w}$ or nothing" becomes more effective in generating an incentive for the trader to produce information.

The above results are derived under the assumption that the indirect investor has decided to hire a professional trader. We need to check whether the indirect investor has such an incentive. If no trader is hired, then the expected value of the investment from simply selling is:

$$V_s = x[\pi_s \int_{-\infty}^{\infty} p(z)\phi^x(z)dz + (1-\pi_s)\int_{-\infty}^{\infty} p(z)\phi^-(z)dz].$$

If no trader is hired, then the expected value of the investment from holding is:

$$V_h = x[\pi_s p_H + (1-\pi_s)p_L].$$

In order for the indirect investor to hire a professional trader, the condition $V > \max\{V_s, V_h\}$ has to be satisfied.

Lemma 7 *$V_h > V_s$.*

Proof. We have:

$$\begin{aligned} V_s &= \pi_s x \int_{-\infty}^{\infty} p(z)\phi^x(z)dz + (1-\pi_s)x\int_{-\infty}^{\infty} p(z)\phi^-(z)dz \\ &\overset{z'=z+x}{=} \pi_s x \int_{-\infty}^{\infty} p(z-x)\phi^+(z)dz + (1-\pi_s)x\int_{-\infty}^{\infty} p(z)\phi^-(z)dz \\ &< \pi_s x \int_{-\infty}^{\infty} p(z)\phi^+(z)dz + (1-\pi_s)x\int_{-\infty}^{\infty} p(z)\phi^-(z)dz \\ &= x\int_{-\infty}^{\infty} p(z)[\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z)]dz = V_h. \end{aligned}$$

■

Intuitively, because there is a direct investor in the market, when the uninformed indirect investor sells his position in the market, he will likely be taken advantage of by the informed direct investor. This will not happen if he holds his position, which guarantees a payoff of V_h.

The following proposition establishes the existence and uniqueness of the optimal contract.

Proposition 4 ***(Existence and Uniqueness)*** *When κ is small enough, there exists a Regular Equilibrium, and it is unique.*

Proof. First, existence. It is easy to show that $\lim_{\kappa\to 0} z^*_{xH} = \lim_{\kappa\to 0} z^*_{xL} = \infty$ and $\lim_{\kappa\to 0} z^*_{0H} = \lim_{\kappa\to 0} z^*_{0L} = -\infty$. Thus $z^*_{xH} > z^*_{0H} + x$ and $z^*_{xL} > z^*_{0L} + x$ are satisfied if κ is small enough. In addition, the expected payment to a trader under an optimal contract is very low when κ is small enough, so the principal will have incentive to hire a professional trader instead of trading by himself.

Next, uniqueness. Assume that there exist two optimal contracts characterized by z^* and $\widehat{z}^*$, and without loss of generality assume $z^*_{xH} > \widehat{z}^*_{xH}$. From Equation (10) and the fact that $z^*_{xH} - z^*_{xL} = z^*_{0H} - z^*_{0L} = \frac{2\sigma^2}{x}\ln(\frac{1-\theta}{\theta})$ in an optimal contract, we know that: $z^*_{xL} > \widehat{z}^*_{xL}$, $z^*_{0H} < \widehat{z}^*_{0H}$, and $z^*_{0L} < \widehat{z}^*_{0L}$. However, this implies that w^* yields a strictly smaller expected payoff to the trader as measured in (7), which is a contradiction. ■

4 Irrelevant Noise Signals

Now we introduce an irrelevant noise signal r that is received by the professional trader after he makes his effort to acquire information. This noise signal is independent of the fundamentals: the relevant signal s and the liquidation value v. Assume $r \sim U[0,1]$.

Define $U_{\chi_h\chi_l} = \{r | r \in [0,1]$ and given r, the professional trader has a position χ_h when $s = s_h$ and has a position χ_l when $s = s_l\}$. Let $q_{\chi_h\chi_l}$ be the probability measure of $U_{\chi_h\chi_l}$. We now define:

Definition 2 ***(Irregular Equilibrium)*** *Denote $q = \{q_{xx}, q_{x0}, q_{0x}, q_{00}\}$. A q-Irregular Equilibrium is an equilibrium in which the trader produces information, and after the trader receives his signals, with probability $q_{\chi_h\chi_l}$ he holds the position χ_h when he receives s_h and holds the position χ_l when he receives s_l.*

Note that from the above definition, when $q_{xx} = q_{0x} = q_{00} = 0$ (or $q_{x0} = 1$), a q-Irregular Equilibrium becomes a Regular Equilibrium.

Before we proceed to characterize a q-Irregular Equilibrium, we prove the following lemma to simplify our analysis.

Lemma 8 ***(Isomorphism)*** *(i) Any q-Irregular Equilibrium with* $q_{x0} \geq q_{0x}$ *yields the same equilibrium outcome as that of a* q'*-Irregular Equilibrium with:* $q'_{x0} = q_{x0} - q_{0x}$*,* $q'_{xx} = q_{xx} + q_{0x}$*,* $q'_{00} = q_{00} + q_{0x}$*, and* $q'_{0x} = q_{0x} - q_{0x} = 0$*;*

(ii) Any q-Irregular Equilibrium with $q_{x0} \leq q_{0x}$ *yields the same equilibrium outcome as that of a* q''*-Irregular Equilibrium with:* $q''_{x0} = q_{x0} - q_{x0} = 0$*,* $q''_{xx} = q_{xx} + q_{x0}$*,* $q''_{00} = q_{00} + q_{x0}$*, and* $q''_{0x} = q_{0x} - q_{x0}$*.*

Proof. We only prove the case with $q_{x0} \geq q_{0x}$. The proof for the case $q_{x0} \leq q_{0x}$ is similar. We check that the trader's actions conditional on the signal have the same distributions in a q-Irregular Equilibrium as in the corresponding q'-Irregular Equilibrium. In a q-Irregular Equilibrium, we have:

$$\begin{aligned}
\Pr(\chi &= x|s_h) = q_{x0} + q_{xx} \\
\Pr(\chi &= 0|s_h) = q_{00} + q_{0x} \\
\Pr(\chi &= x|s_l) = q_{xx} + q_{0x} \\
\Pr(\chi &= 0|s_l) = q_{x0} + q_{00}.
\end{aligned}$$

It is easy to check that the corresponding q'-Irregular Equilibrium has the same conditional distributions. ■

Lemma 8 tells us that when we study a q-Irregular Equilibrium with q_{x0} large enough, we can, without loss of generality, assume $q_{0x} = 0$. As we will see later, a q-Irregular Equilibrium exists only when q_{x0} is large enough, otherwise the principal (i.e., the indirect investor) will not have an incentive to hire the trader at the first place. In other words, if the professional trader does not act in the principal's best interests with a high probability, it is not optimal to hire him.

Proposition 5 ***(Non-Existence)*** *There does not exist a q-Irregular Equilibrium with* $q_{x0} < 1$ *in which there is one trader (working for the indirect investor) and one direct investor trading in the market.*

Proof. See Appendix. ■

Intuitively, if $q_{x0} = 0$, then the professional trader is not behaving in the best interests of the principal (indirect investor) at all, so the principal will not hire the trader in the first place. If $q_{x0} > 0$, i.e., the trader produces information and at least sometimes behaves in the best interests of the principal, then he will fully use the information that he gets at a cost, i.e. $q_{x0} = 1$.

However, when there are more principal-trader pairs in the economy, the q-Irregular Equilibrium might exist, as we will discuss in the next section.

5 Multiple Professional Traders

In this section, we assume there are two professional traders, working for two different principals, trading in the security market. For simplicity we remove the direct investor from the security market; this assumption is not crucial for the main result about the existence of q-Irregular Equilibrium. We will only study symmetric equilibria between the principal-trader pairs. The existence of a Regular Equilibrium will be the same as in the case with one direct investor, so we omit the analysis of that case.

The key point we will develop concerns the existence of an externality that will now arise when there are multiple delegated traders trading in the security market. The externality arises because the traders' payoffs are linked to the market price. In the optimal contract, a trader will be punished if the market price goes against his position. Thus, when there are multiple traders in the market, each professional trader will have beliefs about the other professional trader. If a trader believes that the other trader is going to sell (hold) at date 2, his best strategy might be to sell (hold) regardless of his signal about the fundamentals.

Note that the existence or not nonexistence of a direct investor is unrelated to the externality because the direct investor's payoff is independent of the market price and his strategy will always be to "hold with a good signal and sell with a bad signal."

Lemma 9 *There does not exist a q-Irregular Equilibrium with $q_{x0} < q_{0x}$.*

Proof. The case $q_{0x} > q_{x0}$ is equivalent to the case $q''_{0x} > q''_{x0} = 0$ according to Lemma 8. So, without loss of generality, we assume $q_{x0} = 0$. If $q_{x0} = 0$, notice that the strategy "hold with a bad signal and sell with a good signal" is always strictly dominated either by the strategy of "hold no matter what" or by "sell no matter what." Therefore, instead of paying a non-negative wage to the trader, the principal can just choose not to hire the trader and then choose to hold or liquidate the position, whichever is better in expected payoff. ■

Lemma 9 also rules out any q-Irregular Equilibrium with $q_{x0} = 0$. In the rest of this section, we will study the q-Irregular Equilibrium with $q_{x0} > 0$ and $q_{0x} = 0$. This is without loss of generality according to Lemma 8.

In a q-Irregular Equilibrium, the price can be written as:

$$p(z) = \frac{\pi_s[q_{x0}\phi^+(z) + q_{xx}\phi^+(z) + q_{00}\phi^-(z)]}{q_{x0}[\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z)] + q_{xx}\phi^+(z) + q_{00}\phi^-(z)} p_H + \frac{(1-\pi_s)[q_{x0}\phi^-(z) + q_{xx}\phi^+(z) + q_{00}\phi^-(z)]}{q_{x0}[\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z)] + q_{xx}\phi^+(z) + q_{00}\phi^-(z)} p_L. \tag{13}$$

The optimal contract $w^*(p, \lambda, v)$ must satisfy the following incentive constraints:

$$\text{(IC2.1) } \pi_v\theta A^+_{xH} + (1-\pi_v)(1-\theta)A^+_{xL} \geq \pi_v\theta A^x_{0H} + (1-\pi_v)(1-\theta)A^x_{0L} \tag{14}$$
$$\text{and } \pi_v(1-\theta)A^-_{0H} + (1-\pi_v)\theta A^-_{0L} \geq \pi_v(1-\theta)A^x_{xH} + (1-\pi_v)\theta A^x_{xL}, \tag{15}$$

$$\text{(IC2.2) } \pi_v(1-\theta)A^+_{xH} + (1-\pi_v)\theta A^+_{xL} \geq \pi_v(1-\theta)A^x_{0H} + (1-\pi_v)\theta A^x_{0L} \tag{16}$$
$$\text{and } \pi_v\theta A^-_{0H} + (1-\pi_v)(1-\theta)A^-_{0L} \geq \pi_v\theta A^x_{xH} + (1-\pi_v)(1-\theta)A^x_{xL}, \tag{17}$$

and

$$\text{(IC2.3) } [q_{x0}\pi_v\theta + q_{xx}\pi_v]A^+_{xH} + [q_{x0}(1-\pi_v)(1-\theta) + q_{xx}(1-\pi_v)]A^+_{xL} \tag{18}$$
$$\geq [q_{x0}\pi_v\theta + q_{xx}\pi_v]A^x_{0H} + [q_{x0}(1-\pi_v)(1-\theta) + q_{xx}(1-\pi_v)]A^x_{0L} + \kappa$$
$$\text{and } [q_{x0}\pi_v(1-\theta) + q_{00}\pi_v]A^-_{0H} + [q_{x0}(1-\pi_v)\theta + q_{00}(1-\pi_v)]A^-_{0L} \tag{19}$$
$$\geq [q_{x0}\pi_v(1-\theta) + q_{00}\pi_v]A^x_{xH} + [q_{x0}(1-\pi_v)\theta + q_{00}(1-\pi_v)]A^x_{xL} + \kappa.$$

(IC2.1) covers the case where $r \in U_{x0}$, the traders have an incentive to hold given a good signal and sell given a bad signal. (IC2.2) covers the case where $r \in U_{xx}$, the traders have an incentive to hold given a bad signal, and when $r \in U_{00}$, the traders have incentive to sell given a good signal. (IC2.3) covers the case where the traders have an incentive to produce information.

The optimal contracting problem can be written as:

$$\min_{w \in W} [q_{x0}\pi_v\theta + q_{xx}\pi_v]A^+_{xH} + [q_{x0}(1-\pi_v)(1-\theta) + q_{xx}(1-\pi_v)]A^+_{xL} \tag{20}$$
$$+[q_{x0}\pi_v(1-\theta) + q_{00}\pi_v]A^-_{0H} + [q_{x0}(1-\pi_v)\theta + q_{00}(1-\pi_v)]A^-_{0L}$$
$$\text{subject to the IC conditions in (14)-(19).}$$

Following Proposition 1, we can show that the optimal contract is in the form of a step function $w(p(z), \lambda, v_\eta)$ characterized by the cutoff values $z^*_{\lambda\eta}$ with $\lambda = x$ or 0 and $v_\eta = v_H$ or v_L.

Lemma 10 *For any solution to the programming in (20), the IC constraints in (18) and (19) are binding.*

Proof. See Appendix. ■

By dropping (IC2.1) and (IC2.2) from the full programming problem in (20), we can write an alternative simplified programming problem:

$$\begin{aligned}
&\min_{w\in W}[q_{x0}\pi_v\theta+q_{xx}\pi_v]A^+_{xH}+[q_{x0}(1-\pi_v)(1-\theta)+q_{xx}(1-\pi_v)]A^+_{xL} \\
&+[q_{x0}\pi_v(1-\theta)+q_{00}\pi_v]A^-_{0H}+[q_{x0}(1-\pi_v)\theta+q_{00}(1-\pi_v)]A^-_{0L} \\
&\text{subject to the IC conditions in (18) and (19).}
\end{aligned} \tag{21}$$

This problem is similar to the one in (7), and can be characterized in a similar way.

The characterization of (20) is tedious. Therefore, our strategy for establishing the existence of a solution for (20) is to first find a solution to (21), and then find conditions under which (IC2.3) implies (IC2.1) and (IC2.2).

We first characterize the necessary conditions for the solution to (21). Denote $z^*_{\lambda\eta}$ as the optimal cutoffs points for the step function $w(p(z),\lambda,v_\eta)$.

Corollary 2 *In an optimal contract for the programming problem in (21), we have:*

$$z^*_{xH}-z^*_{xL}=z^*_{0H}-z^*_{0L}=\frac{\sigma^2}{x}\ln\frac{[q_{x0}(1-\theta)+q_{xx}][q_{x0}(1-\theta)+q_{00}]}{(q_{x0}\theta+q_{xx})(q_{x0}\theta+q_{00})} \tag{22}$$

and

$$z^*_{xH}-z^*_{0H}=z^*_{xL}-z^*_{0L}>x. \tag{23}$$

Proof. This is similar to the proof of Proposition 2, and is therefore omitted. ■

Using the results from Corollary 2 and Lemma 10, we can solve for the optimal cut-off values $z^*=\{z^*_{xH},\, z^*_{xL},\, z^*_{0H},\, z^*_{0L}\}$ from the following two equations derived from the IC conditions in (18) and (19):

$$\begin{aligned}
&[q_{x0}\pi_v\theta+q_{xx}\pi_v][1-\Phi^+(z^*_{xH})]\bar{w}+[q_{x0}(1-\pi_v)(1-\theta)+q_{xx}(1-\pi_v)][1-\Phi^+(z^*_{xL})]\bar{w} \\
=\;&[q_{x0}\pi_v\theta+q_{xx}\pi_v]\Phi^x(z^*_{0H})\bar{w}+[q_{x0}(1-\pi_v)(1-\theta)+q_{xx}(1-\pi_v)]\Phi^x(z^*_{0L})\bar{w}+\kappa,
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
&[q_{x0}\pi_v(1-\theta)+q_{00}\pi_v]\Phi^-(z^*_{0H})\bar{w}+[q_{x0}(1-\pi_v)\theta+q_{00}(1-\pi_v)]\Phi^-(z^*_{0L})\bar{w} \\
=\;&[q_{x0}\pi_v(1-\theta)+q_{00}\pi_v][1-\Phi^x(z^*_{xH})]\bar{w}+[q_{x0}(1-\pi_v)\theta+q_{00}(1-\pi_v)][1-\Phi^x(z^*_{xL})]\bar{w}+\kappa,
\end{aligned} \tag{25}$$

where $\Phi^\sim(.)$ is the cdf for the corresponding $\phi^\sim(.)$, the pdf.

Notice that when $q_{xx}=0$ and $q_{00}=0$, (IC2.3) is reduced to (IC1) in (6). Therefore, when q_{xx} and q_{00} are small enough, (IC2.3) in (18) and (19) implies (IC2.1) in (14) and (15).

The next two lemmas establish a sufficient condition for (IC2.2) to be implied by (IC2.3).

Lemma 11 *In a q-Irregular Equilibrium,* $\lim_{\sigma\to 0} A^{\chi}_{xH} - A^{\chi}_{xL} = 0$ *for* $\chi = +$ *or* x *and* $\lim_{\sigma\to 0} A^{\chi}_{0H} - A^{\chi}_{0L} = 0$ *for* $\chi = -$ *or* x.

Proof. We will just show the result for the case of $A^{+}_{xH} - A^{+}_{xL}$. The other cases are similar. According to Corollary 2, define $\Delta = -\frac{\sigma^2}{x}\ln\frac{[q_{x0}(1-\theta)+q_{xx}][q_{x0}(1-\theta)+q_{00}]}{(q_{x0}\theta+q_{xx})(q_{x0}\theta+q_{00})}$, and we have:

$$\begin{aligned}
A^{+}_{xH} - A^{+}_{xL} &= \int_{z^*_{xH}}^{z^*_{xL}} \bar{w}\phi^{+}(z)dz = \int_{z^*_{xH}}^{z^*_{xH}+\Delta} \bar{w}\phi^{+}(z)dz \\
&\leq \int_{-\Delta/2}^{\Delta/2} \bar{w}\phi^{+}(z)dz,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\sigma\to 0}\int_{-\Delta/2}^{\Delta/2} \bar{w}\phi^{+}(z)dz &= \lim_{\sigma\to 0} \Delta\bar{w}\phi^{+}(0) \\
&= \lim_{\sigma\to 0} -\frac{\sigma^2}{x}\ln\frac{[q_{x0}(1-\theta)+q_{xx}][q_{x0}(1-\theta)+q_{00}]}{(q_{x0}\theta+q_{xx})(q_{x0}\theta+q_{00})}\bar{w}\frac{1}{\sqrt{2\pi}\sigma} \\
&= 0,
\end{aligned}$$

which implies: $\lim_{\sigma\to 0} A^{+}_{xH} - A^{+}_{xL} = 0$. ■

Intuitively, when σ is small, the security price is not very noisy. Therefore, the optimal contract will mainly rely on the security price to generate incentives for the professional traders to produce information; the contractual dependence on realized v will be weak, i.e., the expected payoffs to the traders will not differ much for different realized values of the stock.

Lemma 12 *When* σ *is small enough, the solution* z^* *to the programming problem in (21) also satisfies (IC2.2) in (16) and (17).*

Proof. We just need to show that when σ is small, (IC2.3) in (18) and (19) implies (IC2.2) in (16) and (17).

The inequality in (16) can be written as:

$$\alpha_1 A^{+}_{xH} + (1-\alpha_1)A^{+}_{xL} \geq \alpha_1 A^{x}_{0H} + (1-\alpha_1)A^{x}_{0L} \tag{26}$$

where $\alpha_1 = \frac{\pi_v(1-\theta)}{\pi_v(1-\theta)+(1-\pi_v)\theta}$.

The inequality in (18) can be written as:

$$\alpha_2 A^{+}_{xH} + (1-\alpha_2)A^{+}_{xL} \geq \alpha_2 A^{x}_{0H} + (1-\alpha_2)A^{x}_{0L} + \kappa' \tag{27}$$

where $\alpha_2 = \frac{q_{x0}\pi_v\theta + q_{xx}\pi_v}{q_{x0}[\pi_v\theta+(1-\pi_v)(1-\theta)]+q_{xx}}$ and $\kappa' = \frac{\kappa}{q_{x0}[\pi_v\theta+(1-\pi_v)(1-\theta)]+q_{xx}}$.

It is easy to show $\alpha_2 > \alpha_1$. Inequality (27) can be written as:

$$\alpha_1 A^+_{xH} + (1-\alpha_1)A^+_{xL} \geq \alpha_1 A^x_{0H} + (1-\alpha_1)A^x_{0L} - (\alpha_2 - \alpha_1)[(A^+_{xH} - A^+_{xL}) + (A^x_{0L} - A^x_{0H})] + \kappa'$$

which implies inequality (26) when σ is small since $A^+_{xH} - A^+_{xL}$ and $A^x_{0L} - A^x_{0H}$ go to zero as σ goes to zero according to Lemma 11.

Similarly, we can show that (19) implies (17) when σ is small. ■

In words, the externality effect is stronger to the extent that the liquidity trading is less very noisy, i.e., when σ is small. The economic interpretation of this is that there are fewer "noise traders" in the market, as would be the case for example in markets where professional traders predominate. In such a setting, the professional traders' contractual payoffs will mainly depend on the security price, i.e., if a trader expects the other trader to sell (hold), then his best strategy is also to sell (hold). The question is: will the existence of a solution to (21) be affected when σ gets smaller? The next corollary shows that when the price is more informative (i.e., liquidity trading is less noisy), the set of feasible contracts, in the form of step functions, is not smaller.

Corollary 3 *Given σ, if the set of wage step functions $w(p(z), \lambda, v_\eta)$ (as in Proposition 1) satisfying the IC conditions given in (18) and (19) is not empty, then for any $0 < \hat{\sigma} < \sigma$, the set is also not empty.*

Proof. Same as the proof for Lemma 6, and thus omitted. ■

The next corollary demonstrates how the trader's payoff varies as κ and $\bar{w}$ changes.

Corollary 4 *Let $z^* = \{z^*_{xH}, z^*_{xL}, z^*_{0H}, z^*_{0L}\}$ be the solution for (21) in a q-Irregular Equilibrium. We have:*

(i) $\frac{\partial z^*_{xH}}{\partial \kappa} = \frac{\partial z^*_{xL}}{\partial \kappa} < 0$, $\frac{\partial z^*_{0H}}{\partial \kappa} = \frac{\partial z^*_{0L}}{\partial \kappa} > 0$, *and* $\frac{\partial A}{\partial \kappa} > 1$, *where*

$$\begin{aligned} A \equiv\ & [q_{x0}\pi_v\theta + q_{xx}\pi_v]A^+_{xH} + [q_{x0}(1-\pi_v)(1-\theta) + q_{xx}(1-\pi_v)]A^+_{xL} \\ & +[q_{x0}\pi_v(1-\theta) + q_{00}\pi_v]A^-_{0H} + [q_{x0}(1-\pi_v)\theta + q_{00}(1-\pi_v)]A^-_{0L}. \end{aligned}$$

(ii) $\frac{\partial z^*_{xH}}{\partial \bar{w}} = \frac{\partial z^*_{xL}}{\partial \bar{w}} > 0$ *and* $\frac{\partial z^*_{0H}}{\partial \bar{w}} = \frac{\partial z^*_{0L}}{\partial \bar{w}} > 0$.

Proof. Similar to the proof of Lemma 4 and Corollary 1, and thus omitted. ■

Before we show the existence of q-Irregular Equilibria, we calculate the payoff for the agents in the economy. In a q-Irregular Equilibrium, the expected value of the investment from hiring a professional trader is:

$$\begin{aligned} V &= q_{x0}x[\pi_s p_H + (1-\pi_s)\int_{-\infty}^{\infty} p(z)\phi^-(z)dz] \\ &\quad + q_{xx}x[\pi_s p_H + (1-\pi_s)p_L] + q_{00}x\int_{-\infty}^{\infty} p(z)\phi^-(z)dz, \end{aligned} \tag{28}$$

where $p(z)$ is defined as in (13). Again, V is the joint payoff to the indirect investor and the professional trader. The expected payoff to the noise traders is:

$$\begin{aligned} V_N &= -\int_{-\infty}^{\infty} z[(q_{x0}\pi_s + q_{xx})p(z) + (q_{x0}(1-\pi_s) + q_{00})p(z-2x)]\phi^+(z)dz \\ &\quad + \int_{-\infty}^{\infty} z[\pi_s p_H + (1-\pi_s)p_L]\phi^+(z)dz \\ &= 2x[\pi_s q_{00} p_H + (1-\pi_s)(q_{x0}+q_{00})p_L - (q_{x0}(1-\pi_s)+q_{00})\int_{-\infty}^{\infty} p(z)\phi^-(z)dz]. \end{aligned}$$

It is easy to show that:

$$2V + V_N = 2x[\pi_s p_H + (1-\pi_s)p_L].$$

If no professional trader is hired, then the expected value of selling the investment is:

$$\begin{aligned} V_s &= q_{x0}x[\pi_s\int_{-\infty}^{\infty} p(z)\phi^x(z)dz + (1-\pi_s)\int_{-\infty}^{\infty} p(z)\phi^-(z)dz] \\ &\quad + q_{xx}x\int_{-\infty}^{\infty} p(z)\phi^x(z)dz + q_{00}x\int_{-\infty}^{\infty} p(z)\phi^-(z)dz. \end{aligned}$$

If no trader is hired, the expected value of holding the investment is:

$$V_h = x[\pi_s p_H + (1-\pi_s)p_L].$$

Following the proof for Lemma 7, we can show $V_h > V_s$. It is easy to show that:

$$\begin{aligned} V - V_h &= (1-q_{xx})x(1-\pi_s)[\int_{-\infty}^{\infty} p(z)\phi^-(z)dz - p_L] \\ &\quad - q_{00}x\pi_s[p_H - \int_{-\infty}^{\infty} p(z)\phi^-(z)dz]. \end{aligned} \tag{29}$$

Therefore, $V > V_h$ only if $q_{xx} < 1$ and q_{00} is small enough. Also, it is easy to show that $V > V_h$ implies $V_N < 0$, i.e., as long as a professional trader is hired (there is informed trading), the noise traders will get a negative expected payoff.

In the next corollary, we show some results similar to Lemma 5.

Corollary 5 *In a q-Irregular Equilibrium, when* ***either*** q_{xx} ***or*** q_{00} *is small enough,* $\frac{\partial V}{\partial \sigma} > 0$ *and* $\frac{\partial V_N}{\partial \sigma} < 0$.

Proof. We only need to study the sign of $\frac{\partial}{\partial \sigma}\int_{-\infty}^{\infty} p(z)dz$. We can write $p(z)$ defined in (13) in two alternative ways:

$$\begin{aligned} p(z) &= p_L + \frac{(p_H - p_L)\pi_s[(1-q_{00})\phi^+(z) + q_{00}\phi^-(z)]}{q_{x0}[\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z)] + q_{xx}\phi^+(z) + q_{00}\phi^-(z)} \\ &= p_H - \frac{(p_H - p_L)(1-\pi_s)[q_{xx}\phi^+(z) + (1-q_{xx})\phi^-(z)]}{q_{x0}[\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z)] + q_{xx}\phi^+(z) + q_{00}\phi^-(z)}. \end{aligned}$$

We have shown in the proof for Lemma 5 $\int_{-\infty}^{\infty} \frac{\phi^+(z)\phi^-(z)}{\pi_s\phi^+(z)+(1-\pi_s)\phi^-(z)}dz$ is increasing with σ, and similarly we can show $\int_{-\infty}^{\infty} \frac{\phi^-(z)\phi^-(z)}{\pi_s\phi^+(z)+(1-\pi_s)\phi^-(z)}dz$ is decreasing with σ. Setting either $q_{00} = 0$ or $q_{xx} = 0$ in either one of the expressions for $p(z)$, we can show that $\int_{-\infty}^{\infty} p(z)\phi^-(z)dz$ is increasing with σ. Our results are immediate. ■

When both q_{xx} and q_{00} are large, it is possible that $\frac{\partial V}{\partial \sigma} < 0$. There are two effects on V when σ increases: the effect on the payoff from informed trading and the effect on the payoff from "herding." A higher σ always benefits informed traders, but hurts the noise traders. As the values of q_{xx} and q_{00} increase, the professional trader's behavior becomes closer to that of a noise trader, whose payoff is decreasing with σ.

Denote π^+ as the probability that both professional traders hold their position in a q-Irregular Equilibrium, and let π^- be the probability that they both sell their position. We can see that $\pi^+ = q_{x0}\pi_s + q_{xx}$, $\pi^- = q_{x0}(1-\pi_s) + q_{00}$, and $\pi^+ + \pi^- = 1$. The next lemma demonstrates how V and V_N vary with q_{xx} or q_{00}.

Lemma 13 *In a q-Irregular Equilibrium, the joint payoff to the indirect investor and his professional trader is decreasing with* q_{xx} *or* q_{00}*, while the payoff to the noise traders is increasing with* q_{xx} *or* q_{00}.

Proof. With V defined in (28), we only need to show $\frac{\partial V}{\partial q_{xx}} < 0$ and $\frac{\partial V}{\partial q_{00}} < 0$, and we have:

$$\frac{\partial V}{\partial q_{xx}} = (1-\pi_s)[p_L - \int_{-\infty}^{+\infty} p(z)\phi^-(z)dz] + \pi^- \int_{-\infty}^{+\infty} \frac{\partial p(z)}{\partial q_{xx}}\phi^-(z)dz].$$

Write $p(z) = p_L + \frac{\pi_s[(1-q_{00})\phi^+(z)+q_{00}\phi^-(z)]}{\pi^+\phi^+(z)+\pi^-\phi^-(z)}(p_H - p_L)$, and we have:

$$\begin{aligned}\frac{\partial p(z)}{\partial q_{xx}} &= -\frac{\pi_s[(1-q_{00})\phi^+(z)+q_{00}\phi^-(z)]}{[\pi^+\phi^+(z)+\pi^-\phi^-(z)]^2}(1-\pi_s)[\phi^+(z)-\phi^-(z)](p_H-p_L) \\ &= -[p(z)-p_L]\frac{(1-\pi_s)[\phi^+(z)-\phi^-(z)]}{\pi^+\phi^+(z)+\pi^-\phi^-(z)}.\end{aligned}$$

Therefore, we have:

$$\begin{aligned}\frac{\partial V}{\partial q_{xx}} &= (1-\pi_s)[\int_{-\infty}^{+\infty}[p_L-p(z)]\{1+\frac{\pi^-[\phi^+(z)-\phi^-(z)]}{\pi^+\phi^+(z)+\pi^-\phi^-(z)}\}\phi^-(z)dz] \\ &= (1-\pi_s)[\int_{-\infty}^{+\infty}[p_L-p(z)]\phi^+(z)\phi^-(z)dz] < 0.\end{aligned}$$

Similarly, we can prove $\frac{\partial V}{\partial q_{00}} < 0$. ∎

The "herding" behavior of the professional traders results from neglecting the informative signal collected by each trader. Thus it reduces the joint payoff to the indirect investor and the trader while benefiting the noise traders. When $q_{xx} = 1$, $V = V_h$, and when $q_{00} = 1$, $V = V_s < V_h$; then the indirect investor does not have any incentive to hire a professional trader no matter how small κ.

Now, we are ready to present the proposition for the existence of q-Irregular Equilibria.

Proposition 6 ***(Existence)*** *Given q_{xx} and q_{00} small enough, there exists some small σ and κ, such that a q-Irregular Equilibrium exists.*

Proof. As we can see from (29), a necessary condition for the joint payoff to the principal and the trader, V, to be greater than the best alternative for the principals without hiring a trader, V_h, is to have $q_{xx} < 1$ and q_{00} small enough.

Small q_{xx} and q_{00} guarantees that (18) and (19) imply (14) and (15).

Small σ (according Corollary 3, we know that smaller σ will not lead to an empty set of feasible contracts) guarantees the solution to the programming problem (21) is also the solution for the programming problem with extra incentive constraints (IC2.2) in (16) and (17).

Small κ guarantees that the principals have an incentive to hire a professional trader even though with a small probability the traders will ignore their private signals. Small κ also guarantees that there exists a solution to the programming problem (21). ∎

5.1 Some Results on Asset Prices

Let $p(z)$ be the pricing function for a Regular Equilibrium, as defined in (2). Let $p_q(z)$ be the pricing function for a q-Irregular Equilibrium, as defined in (13).

Lemma 14 *In a q-Irregular Equilibrium with non-zero q_{xx} or q_{00}, we have:*

(i) $E_q[p_q(z)] = \pi_s p_H + (1-\pi_s)p_L$;

(ii) $\lim_{z\to\infty} p_q(z) = \frac{\pi_s(q_{x0}+q_{xx})}{\pi_s q_{x0}+q_{xx}} p_H + \frac{(1-\pi_s)q_{xx}}{\pi_s q_{x0}+q_{xx}} p_L$ *and* $\lim_{z\to-\infty} p_q(z) = \frac{\pi_s q_{00}}{(1-\pi_s)q_{x0}+q_{00}} p_H + \frac{(1-\pi_s)(q_{x0}+q_{00})}{(1-\pi_s)q_{x0}+q_{00}} p_L$;

(iii) When $z < x$, $p_q(z) > p(z)$; when $z > x$, $p_q(z) < p(z)$.

Proof. For parts (i) & (ii), the proofs are simple algebra, and are omitted.

For part (iii), we can show:

$$\begin{aligned}
p_q(z) - p(z) &= \Big[\frac{\pi_s\phi^+(z) + \pi_s q_{00}[\phi^-(z) - \phi^+(z)]}{\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z) + [\pi_s q_{00} - (1-\pi_s)q_{xx}][\phi^-(z) - \phi^+(z)]} \\
&\quad - \frac{\pi_s\phi^+(z)}{\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z)}\Big](p_H - p_L) \\
&\propto -\pi_s(1-\pi_s)[\phi^+(z) - \phi^-(z)][q_{xx}\phi^+(z) + q_{00}\phi^-(z)](p_H - p_L).
\end{aligned}$$

Notice that $\phi^+(z) \gtrless \phi^-(z)$ when $z \gtrless x$, and the result is immediate. ■

Proposition 7 *In a q-Irregular Equilibrium with non-zero q_{xx} or q_{00}, the price volatility is lower than that in a Regular Equilibrium.*

Proof. In a Regular Equilibrium, we have:

$$E[(p(z) - E[p(z)])^2] = \int_{-\infty}^{\infty} \frac{[\pi_s(1-\pi_s)(p_H - p_L)(\phi^+(z) - \phi^-(z))]^2}{\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z)} dz.$$

In a q-Irregular Equilibrium, we have:

$$E_q[(p_q(z) - E_q[p_q(z)])^2] = q_{x0}\int_{-\infty}^{\infty} \frac{[\pi_s(1-\pi_s)(p_H - p_L)(\phi^+(z) - \phi^-(z))]^2}{\pi_s\phi^+(z) + (1-\pi_s)\phi^-(z) + \frac{q_{xx}\phi^+(z) + q_{00}\phi^-(z)}{q_{x0}}} dz.$$

The result is immediate. ■

Intuitively, in a q-Irregular Equilibrium, because the professional traders sometimes herd with irrelevant signals, the order flow is less informative about the fundamental value of the security, and thus the price is less volatile compared to the price volatility in a Regular Equilibrium.

6 Conclusion

Market prices are widely viewed as the most accurate measure of value, as security markets are usually taken to be "efficient" in the sense that the prices aggregate diverse information. As exogenous signals of value, these prices are contractible. This view is at the root of the idea of using market prices to control the behavior of portfolio managers and traders, through the practice of marking-to-market; in other words, the price being informative is a key to justify the practice of marking-to-market. We show that "marking-to-market" is part of an optimal contract, confirming this logic. While the level of market efficiency affects how effective the market price can be used to measure the performance of portfolio managers and traders, the marking-to-market practice itself can weaken the market efficiency. The externality introduced by the marked-to-market contracts causes traders to possibly rationally trade on irrelevant information, making the prices used to mark their positions less informative and reducing price volatility.

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Appendix

Proof of Proposition 2

Proof. The equilibrium conditions are implied by:

$$\begin{aligned}
\frac{\phi^{+}(z^{*}_{xH})}{\phi^{x}(z^{*}_{xH})} &= \frac{(1-\theta)^2}{\theta^2}\frac{\phi^{+}(z^{*}_{xL})}{\phi^{x}(z^{*}_{xL})}, \ \frac{\phi^{-}(z^{*}_{0L})}{\phi^{x}(z^{*}_{0L})} = \frac{(1-\theta)^2}{\theta^2}\frac{\phi^{-}(z^{*}_{0H})}{\phi^{x}(z^{*}_{0H})} \\
\frac{\phi^{+}(z^{*}_{xH})}{\phi^{x}(z^{*}_{xH})} &> \frac{\phi^{x}(z^{*}_{0H})}{\phi^{-}(z^{*}_{0H})}, \text{ and } \frac{\phi^{+}(z^{*}_{xL})}{\phi^{x}(z^{*}_{xL})} > \frac{\phi^{x}(z^{*}_{0L})}{\phi^{-}(z^{*}_{0L})}.
\end{aligned}$$

We will only show the proof for the first equality or inequality; the proof for the second one is similar.

First, $\frac{\phi^{+}(z^{*}_{xH})}{\phi^{x}(z^{*}_{xH})} = \frac{(1-\theta)^2}{\theta^2}\frac{\phi^{+}(z^{*}_{xL})}{\phi^{x}(z^{*}_{xL})}$. Define $R^{+} = \frac{\pi_v\theta}{(1-\pi_v)(1-\theta)}\frac{\phi^{+}(z^{*}_{xH})}{\phi^{+}(z^{*}_{xL})}$, and construct $\widehat{w}^{*}(p,x,H)$ and $\widehat{w}^{*}(p,x,L)$ as follows:

$$\begin{aligned}
\widehat{w}^{*}(p(z),x,v_H) &= \begin{cases} 0 & \text{if } z \in [z^{*}_{xH}, z^{*}_{xH}+\delta] \\ w^{*}(p(z),x,v_H) & \text{otherwise} \end{cases} \\
\text{and } \widehat{w}^{*}(p(z),x,v_L) &= \begin{cases} \bar{w} & \text{if } z \in [z^{*}_{xL}-R^{+}\delta, z^{*}_{xL}] \\ w^{*}(p(z),x,v_L) & \text{otherwise} \end{cases}.
\end{aligned}$$

Basically we cut a piece of wage from $w^{*}(p(z),x,v_H)$ and add a piece (of difference size) of wage to $w^{*}(p(z),x,v_L)$. When δ is small enough, the value of the left hand side of the first inequality of the IC conditions in (6), $\pi_v\theta A^{+}_{xH}+(1-\pi_v)(1-\theta)A^{+}_{xL}$, does not change with the newly constructed wage function, nor does the value of the objective function in (7). The optimality of w^{*} means that the value of the right hand side of the second inequality of the IC conditions in (6) does not become smaller with the new wage function (otherwise the resulting strict inequality will have space for improvement), i.e.

$$-\pi_v(1-\theta)\delta\phi^{x}(z^{*}_{xH})+(1-\pi_v)\theta R^{+}\phi^{x}(z^{*}_{xL}) \geq 0,$$

or

$$\frac{\phi^{+}(z^{*}_{xH})}{\phi^{x}(z^{*}_{xH})} \geq \frac{(1-\theta)^2}{\theta^2}\frac{\phi^{+}(z^{*}_{xL})}{\phi^{x}(z^{*}_{xL})}.$$

Similarly, if we reverse the construction of $\widehat{w}$ by adding a piece of wage to $w^{*}(p(z),x,v_H)$ and cutting a piece (difference size) of wage from $w^{*}(p(z),x,v_L)$, we will get:

$$\frac{\phi^{+}(z^{*}_{xH})}{\phi^{x}(z^{*}_{xH})} \leq \frac{(1-\theta)^2}{\theta^2}\frac{\phi^{+}(z^{*}_{xL})}{\phi^{x}(z^{*}_{xL})}.$$

Therefore, we must have:

$$\frac{\phi^+(z^*_{xH})}{\phi^x(z^*_{xH})} = \frac{(1-\theta)^2}{\theta^2}\frac{\phi^+(z^*_{xL})}{\phi^x(z^*_{xL})}.$$

Second, $\frac{\phi^+(z^*_{xH})}{\phi^x(z^*_{xH})} > \frac{\phi^x(z^*_{0H})}{\phi^-(z^*_{0H})}$. Define $R^H = \frac{\phi^-(z^*_{0H})}{\phi^x(z^*_{xH})}$, and construct $\widehat{w}^*(p,0,v_H)$ and $\widehat{w}^*(p,x,v_H)$ as follows:

$$\begin{aligned}
\widehat{w}^*(p(z),0,v_H) &= \begin{cases} 0 & \text{if } z \in [z^*_{0H}-\delta, z^*_{0H}] \\ w^*(p(z),0,v_H) & \text{otherwise} \end{cases} \\
\text{and } \widehat{w}^*(p(z),x,v_H) &= \begin{cases} 0 & \text{if } z \in [z^*_{xH}, z^*_{xH}+R^H\delta] \\ w^*(p(z),x,v_H) & \text{otherwise} \end{cases} .
\end{aligned}$$

Basically we cut a piece of wage from $w^*(p(z),0,v_H)$ and cut another piece (difference size) from $w^*(p(z),x,v_H)$. When δ is small enough, the second inequality of the IC conditions in (6) remains balanced. With the newly constructed wage functions, the value of the objective function in (7) is reduced. The optimality of w^* means that the first inequality of the IC conditions in (6) must be broken, which means:

$$-\pi_v\theta R^H\delta\phi^+(z^*_{xH})\bar{w} < -\pi_v\theta\delta\phi^x(z^*_{0H})\bar{w},$$

or

$$\frac{\phi^+(z^*_{xH})}{\phi^x(z^*_{xH})} > \frac{\phi^x(z^*_{0H})}{\phi^-(z^*_{0H})}.$$

■

Proof of Lemma 4

Proof. We prove it using the perturbation method. Define:

$$\begin{aligned}
\beta_{11} &\equiv \pi_v\theta\phi^+(z^*_{xH}) + (1-\pi_v)(1-\theta)\phi^+(z^*_{xL}) \\
\beta_{12} &\equiv \pi_v\theta\phi^x(z^*_{0H}) + (1-\pi_v)(1-\theta)\phi^x(z^*_{0L}) \\
\beta_{21} &\equiv \pi_v(1-\theta)\phi^-(z^*_{0H}) + (1-\pi_v)\theta\phi^-(z^*_{0L}) \\
\beta_{22} &\equiv \pi_v(1-\theta)\phi^x(z^*_{xH}) + (1-\pi_v)\theta\phi^x(z^*_{xL}).
\end{aligned}$$

By Proposition 2, since the difference between z^*_{xH} and z^*_{xL} or z^*_{0H} and z^*_{0L} does not change with κ, we know that $\Delta z^*_{xL} = \Delta z^*_{xH}$ and $\Delta z^*_{0L} = \Delta z^*_{0H}$, and we can derive the following perturbation equations from Equation (10) and (11):

$$\begin{aligned}
-\beta_{11}\Delta z^*_{xH} - \beta_{12}\Delta z^*_{0H} &= \frac{\Delta\kappa}{\bar{w}} \\
\beta_{21}\Delta z^*_{0H} + \beta_{22}\Delta z^*_{xH} &= \frac{\Delta\kappa}{\bar{w}}.
\end{aligned}$$

We can show:

$$
\begin{aligned}
\frac{\Delta z_{xH}^*}{\Delta \kappa} &= \frac{\beta_{21}+\beta_{12}}{\bar{w}(-\beta_{11}\beta_{21}+\beta_{12}\beta_{22})} \\
\frac{\Delta z_{0H}^*}{\Delta \kappa} &= \frac{\beta_{11}+\beta_{22}}{\bar{w}(\beta_{11}\beta_{21}-\beta_{12}\beta_{22})}.
\end{aligned}
$$

We only need to show $\beta_{11}\beta_{21}-\beta_{12}\beta_{22}>0$ to prove our results. We have:

$$
\begin{aligned}
& \beta_{11}\beta_{21}-\beta_{12}\beta_{22} \\
= & \left[\pi_v\theta\phi^+(z_{xH}^*)+(1-\pi_v)(1-\theta)\phi^+(z_{xL}^*)\right]\left[\pi_v(1-\theta)\phi^-(z_{0H}^*)+(1-\pi_v)\theta\phi^-(z_{0L}^*)\right] \\
& -\left[\pi_v\theta\phi^x(z_{0H}^*)+(1-\pi_v)(1-\theta)\phi^x(z_{0L}^*)\right]\left[\pi_v(1-\theta)\phi^x(z_{xH}^*)+(1-\pi_v)\theta\phi^x(z_{xL}^*)\right] \\
= & \pi_v^2\theta(1-\theta)\left[\phi^+(z_{xH}^*)\phi^-(z_{0H}^*)-\phi^x(z_{xH}^*)\phi^x(z_{0H}^*)\right] \\
& +(1-\pi_v)^2\theta(1-\theta)\left[\phi^+(z_{xL}^*)\phi^-(z_{0L}^*)-\phi^x(z_{xL}^*)\phi^x(z_{0L}^*)\right] \\
& +\pi_v(1-\pi_v)\theta^2\left[\phi^+(z_{xH}^*)\phi^-(z_{0L}^*)-\phi^x(z_{xL}^*)\phi^x(z_{0H}^*)\right] \\
& +\pi_v(1-\pi_v)(1-\theta)^2\left[\phi^+(z_{xL}^*)\phi^-(z_{0H}^*)-\phi^x(z_{xH}^*)\phi^x(z_{0L}^*)\right].
\end{aligned}
$$

The first two items are non-negative since we know $\frac{\phi^+(z_{xH}^*)}{\phi^x(z_{xH}^*)} > \frac{\phi^x(z_{0H}^*)}{\phi^-(z_{0H}^*)}$ and $\frac{\phi^+(z_{xL}^*)}{\phi^x(z_{xL}^*)} > \frac{\phi^x(z_{0L}^*)}{\phi^-(z_{0L}^*)}$ according to Proposition 2. For the third item (the fourth item can be shown non-negative in a similar way), replace $\phi^+(z_{xH}^*) = \frac{(1-\theta)^2}{\theta^2}\frac{\phi^+(z_{xL}^*)\phi^x(z_{xH}^*)}{\phi^x(z_{xL}^*)}$ and $\phi^x(z_{0H}^*) = \frac{(1-\theta)^2}{\theta^2}\frac{\phi^-(z_{0H}^*)\phi^x(z_{0L}^*)}{\phi^-(z_{0L}^*)}$, and it becomes:

$$
\begin{aligned}
& \pi_v(1-\pi_v)(1-\theta)^2\left[\frac{\phi^+(z_{xL}^*)\phi^x(z_{xH}^*)}{\phi^x(z_{xL}^*)}\phi^-(z_{0L}^*)-\phi^x(z_{xL}^*)\frac{\phi^-(z_{0H}^*)\phi^x(z_{0L}^*)}{\phi^-(z_{0L}^*)}\right] \\
> & \pi_v(1-\pi_v)(1-\theta)^2\phi^x(z_{xL}^*)\phi^x(z_{0L}^*)\left[\frac{\phi^x(z_{xH}^*)}{\phi^x(z_{xL}^*)}-\frac{\phi^-(z_{0H}^*)}{\phi^-(z_{0L}^*)}\right]>0.
\end{aligned}
$$

To get the first inequality, we use $\frac{\phi^+(z_{xL}^*)}{\phi^x(z_{xL}^*)} > \frac{\phi^x(z_{0L}^*)}{\phi^-(z_{0L}^*)}$; to get the second inequality, we use $z_{xH}^*-z_{xL}^* = z_{0H}^*-z_{0L}^*<0$ and $z_{xH}^*-z_{0H}^*=z_{xL}^*-z_{0L}^*>x$ to show $\frac{\phi^x(z_{xH}^*)}{\phi^x(z_{xL}^*)}-\frac{\phi^-(z_{0H}^*)}{\phi^-(z_{0L}^*)}>0$.

For the second half of the lemma, we can show:

$$
\begin{aligned}
\frac{\Delta A}{\Delta \kappa} &= -\beta_{11}\bar{w}\frac{\Delta z_{xH}^*}{\Delta \kappa}+\beta_{21}\bar{w}\frac{\Delta z_{xH}^*}{\Delta \kappa} \\
&= \frac{\beta_{11}(\beta_{21}+\beta_{12})+\beta_{21}(\beta_{11}+\beta_{22})}{\beta_{11}\beta_{21}-\beta_{12}\beta_{22}}>1.
\end{aligned}
$$

■

Proof of Lemma 6

Proof. We prove this by construction. Given σ, pick an optimal contract $w(p(z), \lambda, v_\eta)$, characterized by the cutoff points $\{z_{xH}, z_{xL}, z_{0H}, z_{0L}\}$, satisfying the IC conditions in (6). For $\hat{\sigma} < \sigma$, we construct the new wage functions $\hat{w}(p(z), \lambda, v_\eta)$, characterized by the cutoff points $\{\hat{z}_{xH}, \hat{z}_{xL}, \hat{z}_{0H}, \hat{z}_{0L}\}$, as follows:

$$\frac{\hat{z}_{xH}}{\hat{\sigma}} = \frac{z_{xH}}{\sigma}, \ \frac{\hat{z}_{xL}}{\hat{\sigma}} = \frac{z_{xL}}{\sigma}, \ \frac{\hat{z}_{0H}+2x}{\hat{\sigma}} = \frac{z_{0H}+2x}{\sigma}, \text{ and } \frac{\hat{z}_{0L}+2x}{\hat{\sigma}} = \frac{z_{0L}+2x}{\sigma}.$$

We can show that, with the constructed $\hat{w}(p(z), \lambda, v_\eta)$,

$$\hat{A}^+_{xH} = A^+_{xH}, \ \hat{A}^+_{xL} = A^+_{xL}, \ \hat{A}^-_{0H} = A^-_{0H}, \text{ and } \hat{A}^-_{0L} = A^-_{0L}.$$

Therefore, the left hand side values of the IC conditions in (18) and (19) remain the same. At the same time, we can show that,

$$\hat{A}^x_{xH} < A^x_{xH}, \ \hat{A}^x_{xL} < A^x_{xL}, \ \hat{A}^x_{0H} < A^x_{0H}, \text{ and } \hat{A}^x_{0L} < A^x_{0L},$$

i.e. the right hand side values of of the IC conditions in (18) and (19) get smaller. We will only show $\hat{A}^x_{xH} < A^x_{xH}$ and $\hat{A}^x_{0H} < A^x_{0H}$ as examples, the proofs for the other inequalities follow the same logic. We have:

$$\begin{aligned}
\hat{A}^x_{xH} &= [1-\hat{\Phi}^x(\hat{z}_{xH})]\bar{w} = [1-\int_0^{\hat{z}_{xH}} \frac{1}{\sqrt{2\pi}\hat{\sigma}} \exp\left(-\frac{(\hat{z}+x)^2}{2\hat{\sigma}^2}\right) d\hat{z}]\bar{w} \\
&= [1-\int_0^{z_{xH}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(z+x\sigma/\hat{\sigma})^2}{2\sigma^2}\right) dz]\bar{w} \text{ (change of variable: } \frac{\hat{z}}{\hat{\sigma}} = \frac{z}{\sigma}) \\
&< [1-\int_0^{z_{xH}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(z+x)^2}{2\sigma^2}\right) dz]\bar{w} = A^x_{xH}.
\end{aligned}$$

$$\begin{aligned}
\hat{A}^x_{0H} &= \hat{\Phi}^x(\hat{z}_{0H})\bar{w} = [\int_0^{\hat{z}_{0H}} \frac{1}{\sqrt{2\pi}\hat{\sigma}} \exp\left(-\frac{(\hat{z}+x)^2}{2\hat{\sigma}^2}\right) d\hat{z}]\bar{w} \\
&= [\int_0^{z_{0H}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(z+2x-x\sigma/\hat{\sigma})^2}{2\sigma^2}\right) dz]\bar{w} \text{ (change of variable: } \frac{\hat{z}+2x}{\hat{\sigma}} = \frac{z+2x}{\sigma}) \\
&< [\int_0^{z_{0H}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(z+x)^2}{2\sigma^2}\right) dz]\bar{w} = A^x_{0H}.
\end{aligned}$$

Therefore, the IC conditions become strict inequalities with our construction, and we can improve the contract. ■

Proof of Proposition 5

Proof. We prove the proposition by contradiction. Assume there exists a q-Irregular Equilibrium with non-zero q_{xx}, q_{00} or q_{0x}, where the market maker's pricing function is denoted by some $p(z)$, where z is the total market order. It is easy to check that $p(z) \in (p_L, p_H)$ for any given q and z. Therefore , the direct investor will always hold his position when he receives a good signal and sell when he receives a bad signal.

If $q_{x0} = 0$, notice that the strategy of "hold with a bad signal and sell with a good signal" is always strictly dominated either by the strategy of "hold no matter what" or by "sell no matter what." Therefore, instead of paying a non-negative wage to the trader, the principal/indirect investor can just choose not to hire a professional trader and then choose to either hold or liquidate the position, whichever is better in expected payoff.[7]

The case where $q_{0x} > q_{x0} > 0$ is equivalent to the case where $q''_{0x} > q''_{x0} = 0$, according to Lemma 8.

If $q_{x0} > q_{0x} \geq 0$, according to Lemma 8, without loss of generality, we assume $q_{0x} = 0$. If $q_{xx} > 0$ and $q_{00} > 0$, then we know that, when the trader receives a good signal or a bad signal, he sometimes holds and sometimes sells, i.e., the trader is indifferent between holding and selling. Abusing the notation in (3), we have:

$$\begin{aligned}
\frac{\pi_v \theta}{\pi_s} A^+_{xH} + \frac{(1-\pi_v)(1-\theta)}{\pi_s} A^+_{xL} &= \frac{\pi_v \theta}{\pi_s} A^x_{0H} + \frac{(1-\pi_v)(1-\theta)}{\pi_s} A^x_{0L} \\
\frac{\pi_v (1-\theta)}{1-\pi_s} A^-_{0H} + \frac{(1-\pi_v)\theta}{1-\pi_s} A^-_{0L} &= \frac{\pi_v (1-\theta)}{1-\pi_s} A^x_{xH} + \frac{(1-\pi_v)\theta}{1-\pi_s} A^x_{xL}.
\end{aligned}$$

At the same time, the incentive constraints for the trader to produce information imply:

$$\begin{aligned}
& \pi_v \theta A^+_{xH} + (1-\pi_v)(1-\theta) A^+_{xL} + \pi_v (1-\theta) A^-_{0H} + (1-\pi_v)\theta A^-_{0L} \\
\geq \; & \pi_v \theta A^x_{0H} + (1-\pi_v)(1-\theta) A^x_{0L} + \pi_v (1-\theta) A^x_{xH} + (1-\pi_v)\theta A^x_{xL} + \frac{2\kappa}{q_{x0}},
\end{aligned}$$

which is a contradiction. The special cases where $q_{xx} = 0$ or $q_{00} = 0$ can be similarly proved. ■

[7] Recall that we assume that the market maker can only observe the total market order. He can not observe the contract between the principal and the trader nor the effort choice made by the trader.

Proof for Lemma 10

Proof. We prove this by contradiction. Assume (18) is not binding. Let:

$$
\begin{aligned}
R_1^+ &\equiv \frac{\pi_v \theta \phi^+(z_{xH}^*)}{(1-\pi_v)(1-\theta)\phi^+(z_{xL}^*)},\ R_1^x \equiv \frac{\pi_v(1-\theta)\phi^x(z_{xH}^*)}{(1-\pi_v)\theta\phi^x(z_{xL}^*)}, \\
R_2^+ &\equiv \frac{\pi_v(1-\theta)\phi^+(z_{xH}^*)}{(1-\pi_v)\theta\phi^+(z_{xL}^*)},\ R_2^x \equiv \frac{\pi_v \theta \phi^x(z_{xH}^*)}{(1-\pi_v)(1-\theta)\phi^x(z_{xL}^*)}, \\
R_3^+ &\equiv \frac{\pi_v(q_{x0}\theta + q_{xx})\phi^+(z_{xH}^*)}{(1-\pi_v)[q_{x0}(1-\theta)+q_{xx}]\phi^+(z_{xL}^*)}, \\
R_3^x &\equiv \frac{\pi_v[q_{x0}(1-\theta)+q_{00}]\phi^x(z_{xH}^*)}{(1-\pi_v)(q_{x0}\theta+q_{00})\phi^x(z_{xL}^*)}.
\end{aligned}
\tag{30}
$$

It is easy to show that $R_1^+ > R_3^+ > R_2^+$ and $R_1^x < R_3^x < R_2^x$. Consider two cut-paste strategies: (i) cut a piece from A_{xL}^+ and paste on A_{xH}^+; and (ii) cut a piece from A_{xH}^+ and paste on A_{xL}^+. For (i), we increase z_{xL}^* by $R_1^+\delta$ and decrease z_{xH}^* by δ, and we know the LHS of (14) remains the same while the LHS of (18) decreases since $R_1^+ > R_3^+$. The IC constraints in (15) (17) and (19) remain unaffected (no matter whether it is binding or not) only if $\max\{R_1^x, R_2^x, R_3^x\} \leq R_1^+$, or $\frac{\phi^x(z_{xH}^*)}{\phi^x(z_{xL}^*)} \leq \frac{\phi^+(z_{xH}^*)}{\phi^+(z_{xL}^*)}$. Similarly, for (ii), we increase z_{xH}^* by δ, and decrease z_{xL}^* by $R_2^+\delta$, and we know the LHS of (16) remains the same while the LHS of (18) decreases since $R_2^+ < R_3^+$. The IC constraint in (15) (17) and (19) remain unaffected (no matter whether it is binding or not) only if $\min\{R_1^x, R_2^x, R_3^x\} \geq R_2^+$, or $\frac{\phi^x(z_{xH}^*)}{\phi^x(z_{xL}^*)} \geq \frac{\phi^+(z_{xH}^*)}{\phi^+(z_{xL}^*)}$. Therefore, for any value of z_{xH}^* and z_{xL}^*, we can always find a strategy to improve the contract without affecting the incentive constraints. We conclude that (18) must be binding.

Similarly, we can also show (19) is binding. ■