Why are Buyouts Leveraged? The Financial Structure of Private Equity Funds

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Abstract

This paper presents a model of the financial structure of private equity firms. In the model, the general partner of the firm encounters a sequence of deals over time where the exact quality of each deal cannot be credibly communicated to investors. We show that the optimal financing arrangement is consistent with a number of characteristics of the private equity industry. First, the firm should be financed by a combination of fund capital raised before deals are encountered, and deal-specific capital raised after a deal is encountered. Second, the fund investors' claim on fund cash flow is a combination of debt and levered equity, while the general partner receives a claim similar to the carry contracts observed in reality. Third, the model suggests that investments by private equity firms will exhibit cyclicality, since there is overinvestment in good states of the world and underinvestment in bad states. Fourth, investments made in recessions will on average outperform investments made in booms.

Practitioner: "Things are really tough because the banks are only lending 4 times cash flow, when they used to lend 6 times cash flow. We can't make our deals profitable anymore."

Academic: "Why do you care if banks will not lend you as much as they used to? If you are unable to lever up as much as before, your limited partners will receive lower expected returns on any given deal, but the risk to them will have gone down proportionately."

Practitioner: "Ah yes, the Modigliani-Miller theorem. I learned about that in business school. We don't think that way at our firm. Our philosophy is to lever our deals as much as we can, to give the highest returns to our limited partners."

1. Introduction

The market for buying out businesses by private equity firms is enormous, totaling $xx billion over the 1990-2002 period. These purchases range from the now legendary Beatrice and RJR Nabisco acquisitions by KKR in the 1980s, to the current market in which private equity partnerships buy both large firms like Burger King to small businesses such as funeral homes. While buyouts originally were focused in the United States, they have become increasingly common in Europe; the Wall Street Journal recently estimated that 40% of M&A activity in Germany in 2004 is from private equity firms.(WSJ, Sept. 28, 2004, p. C1) These buyouts are generally highly leveraged; indeed, when most people refer to buyouts, they invariably include the adjective 'leveraged' in their description.

Buyouts, as well as other private equity investments, are generally made by funds that share a common organizational structure. Typically, these funds raise equity at the time they are formed, make investments that are levered whenever possible using the assets of the portfolio firm but not the fund as collateral, and have a finite life (see Sahlman (1990), or Fenn, Liang and Prowse (1997) for more discussion). The funds are usually organized as limited partnerships, with the limited partners (LPs) providing the capital and the general partners (GPs) making investment decisions and receiving a substantial share of the profits (most often 20%). While the literature has spent much effort understanding some aspects of the private equity market, it is very surprising that there is no clear answers to the basic questions of how funds are structured financially, and what the impact of this structure is. Why are most private equity investments made by funds that are financed by equity and have a finite life? Why are the equity investments of these funds complemented by debt financing from third parties, such as banks? Why is this debt financing backed by the assets of the investment and not the fund? What should we expect to observe about the relation between bank lending practices, and the prices and returns of private equity investments? Why are booms and busts in the private equity industry so extreme?

According to the Modigliani-Miller theorem, capital structure decisions, including both fund structure and the financing of individual deals, is relevant only to the extent to which taxes, transactions costs, or real investment decisions are affected. Certainly the deductibility of interest payments is part of the reason why leverage is valuable at the portfolio firm level but not at the fund level since portfolio firms pay corporate taxes and funds can pass through profits to their partners tax free (see Kaplan (1989)). Yet, it seems unlikely that taxes are a complete answer: there is no evidence that buyouts are less levered when firms have tax shields limiting their corporate income taxes, and the same tax advantages are present in the targeted firms prior to the buyout, when firms typically have relatively modest leverage. Another commonly-cited explanation for leverage at the portfolio-firm level are the implicit incentives associated with leverage, in particular the fact that the commitment to pay interest limits management's discretion to waste the firm's excess cash flows (Jensen (1988)). Yet, managers of firms that are bought by private equity partnerships are monitored heavily and often replaced (Lerner 1995). It seems likely that direct monitoring by a knowledgeable practitioner personally receiving 20% of the profits would likely lead to better controls on managers than the more ad hoc constraints imposed by leverage. Furthermore, neither tax nor incentive benefits explain why the equity capital invested in portfolio firms is raised through a fund rather than deal by deal.

In this paper, we propose a new explanation for the financial structure of private equity firms. We present a model that explains a number of features of private equity markets, including the fact that private equity investments are generally done through funds that pool investments across the fund, the typical financial structure of raising equity at the fund level and supplementing it whenever possible with third-party debt at the deal level, the payoffs to GPs of a "carry-like" structure in which they receive a fraction of the profits but one that is junior to that received by the LPs, the extreme "boom and bust" nature of investments by private equity firms, and the observed empirical regularity of investments made during busts outperforming investments made during booms on average.

The model is relatively straightforward, relying mainly on one market friction that serves as the underlying source of deviations from the Modigliani-Miller benchmark. The friction we model is the notion that GPs making investment decisions have better information about the quality of their potential investments than their LPs or any potential lenders. This assumption seems plausible, given that GPs are specialists in evaluating companies who have substantial incentives to discover any relevant information.

The model is very much a dynamic extension of the famous Myers and Majluf (1984) model, in which informed firms raising capital from uninformed investors will always have an incentive to overstate the quality of potential investments so they cannot credibly communicate their information. The equilibrium of the Myers and Majluf model has debt as an optimal security because the asymmetric information leads to underpricing, and debt is less information-sensitive than equity, so it is associated with the lowest level of underpricing. This equilibrium sometimes is characterized by overinvestment, which occurs when the average project is positive NPV because both bad and good projects get financed. Alternatively, when the average project is negative NPV, since neither bad nor good projects can get financed there is underinvestment.

In our model, the GP faces several potential investment objects over time which require financing. This introduces a new financing decision for the GP relative to the static case. He can now decide whether to raise capital on a deal by deal basis (ex post financing), or raise a fund of capital to be used for several future projects (ex ante financing), or a combination of the two.

With ex post financing, the solution is the same as in the static Myers and Majluf model. Debt will be the optimal security, and GPs will choose to undertake all investments they can, even if they are value-decreasing. Whether deals will be financed at all depends on the state of the economy – in good times, where the average project is positive NPV, there is overinvestment, and in bad times there is underinvestment.

We show that ex ante financing can help to alleviate these problems. By tying the compensation of the GP to the collective performance of a fund, he has less of an incentive to invest in bad deals, since bad deals will contaminate his stake in the good deals. Thus, a fund structure often dominates deal-by-deal capital raising. Furthermore, debt is typically not the optimal security for a fund. Since the capital is raised before the GP has learned the quality of deals, there is no such thing as a "good" GP who tries to minimize underpricing by issuing debt. Indeed, issuing debt will maximize the risk shifting tendencies of a GP since it leaves him with a call option on the fund. We show that instead it is optimal to issue a security giving investors a debt contract plus a levered equity stake, leaving the GP with a "carry" at the fund level that resembles contracts observed in practice.

The downside with pure ex ante capital raising is that it leaves the GP with substantial freedom, since once the fund is raised he does not have to go back to the capital markets, and so can fund deals even in bad times. If the GP has not encountered enough good projects and is approaching the end of the investment horizon, or if economic conditions shift so that not many good deals are expected to arrive in the future, a GP with untapped funds has the incentive to "go for broke" and take bad deals. We show that it is therefore typically optimal to use a mix of ex ante and ex post capital. Giving the GP funds ex ante preserves his incentives to avoid bad deals in good times, but the ex post component has the effect of making the GP unable to finance bad deals in bad times. In addition the structure of the securities mirrors common practice; ex post deal funding is senior to the firm's claim, the LP's claim is senior to the GP's, and the GP's claim is a fraction of the profits. This financing structure turns out to be optimal in the sense that it is the one that maximizes the value of investments by minimizing the expected value of negative NPV investments undertaken and good investments ignored.

Even with this optimal financing structure however, investment nonetheless deviates from its first-best level. In particular, during good states of the world, firms are prone to overinvestment, meaning that some negative net present value investments will be undertaken. In addition, during bad states of the world there will be underinvestment, i.e., valuable projects that cannot be financed. This investment pattern is an explanation for the common observation that the private equity investment process is extremely procyclical (see Gompers and Lerner (1999)). During recessions, there not only will not be as many valuable investment opportunities, but those that do exist will have difficulty being financed. Similarly, during boom times, not only will there be more good projects than in bad times, but bad projects will be financed in addition to the good ones. The implication of this pattern is that the informational imperfections we model are likely to exacerbate normal business cycle patterns of investment. It also suggests that there is some validity to the common complaint from GPs that during tough times it is difficult to get financing for even very good projects but during good times many poor projects get financed.

An important empirical implication of this result is that returns to investments made during booms will be lower on average than the returns to investments made during poor times. This finding is consistent with anecdotal evidence about poor investments made during the internet and biotech bubbles, as well as some of the most successful deals being initiated during busts. More formally, Kaplan and Schoar (2004) find evidence documenting that this pattern of returns is more general, with investments made during bad times underperforming investments made in good times.

The next section presents the model and its implications. There is a discussion and conclusion following the model.

2. Model

There is a general partner (GP) who has the unique ability to identify candidate firms in which to invest. The GP has no funds and needs to raise the necessary funds from external investors. All agents are risk-neutral.

The timing of the model is summarized in Figure 2.1. There are two periods. Each period a candidate firm arrives. We assume it costs I to invest in a firm. Firms are of two kinds: good (G) and bad (B). The quality of the firm is only observed by the GP. A good firm has cash flow $Z > 0$ for sure and a bad firm has cash flow 0 with probability $1 - p$ and cash flow Z with probability p where

$$Z > I > pZ$$

so that good firms are positive net present value investments and bad firms are negative NPV. All cash flows are realized at the end of the second period.

Each period a good firm arrives with probability α and a bad firm with probability $1 - \alpha$.[1] To capture the concept of good and bad states of the economy, we assume the arrival probability α

[1] Equivalently, we can assume that there are always bad firms available, and a good firm arrives with probability α.

take values α_H with probability q and α_L with probability $1-q$ each period, where $\alpha_H > \alpha_L$. We assume α is observable but not verifiable, so it cannot be contracted on directly.

There is also a risk-free investment opportunity yielding the risk-free rate, which we assume to be zero.

2.1. Securities

The GP finances his investments by issuing a security $w_I(x)$ backed by the cash flow x from the investments, and keeps the residual security $w_{GP}(x) = x - w_I(x)$. The securities have to satisfy the following monotonicity condition:

Monotonicity $w_I(x), w_{GP}(x)$ are non-decreasing.

This assumption is standard in the security design literature and can be formally justified on grounds of moral hazard.[2] An equivalent way of expressing the monotonicity condition is

$$x - x' \geq w_{GP}(x) - w_{GP}(x') \geq 0 \qquad \forall x, x' \text{ s.t. } x > x'$$

Furthermore, we assume that there is an infinite supply of unserious fly-by-night operators that investors cannot distinguish from a serious GP. Fly-by-night operators only find useless firms with a pay-off of zero. However, if the security issued pays off less than the total cash flow whenever the cash flow is below the invested capital K, the fly-by-night operators can store the money and earn rents. To screen them out of the market, we therefore require that:

Fly-by-night For invested capital K, $w_{GP}(x) = 0$ whenever $x \leq K$.

2.2. Forms of Capital Raising

In a first best world, the GP will invest in all good firms and no bad firms. Because the GP has private information about the firm type, this will not be achievable - there will typically be overinvestment in bad projects and underinvestment in good projects. Our objective is to find a method of capital raising that minimizes these inefficiencies. We will look at three forms of capital raising:

- *Pure ex post* capital raising is done in each period after the GP encounters a firm. The securities investors get are backed by each individual investment's cash flow.

[2] See, for example, Innes (1990) or Nachman and Noe (1994). Suppose an investor claim $w(x)$ is decreasing on a region $a < x < b$, and that the underlying cash flow turns out to be a. The GP then has an incentive to secretly borrow money from a third party and add it on to the aggregate cash flow to push it into the decreasing region, thereby reducing the payment to the security holder while still being able to pay back the third party. Similarly, if the GP's retained claim is decreasing over some region $a < x \leq b$ and the realized cash flow is b, the GP has an incentive to decrease the observed cash flow by burning money.

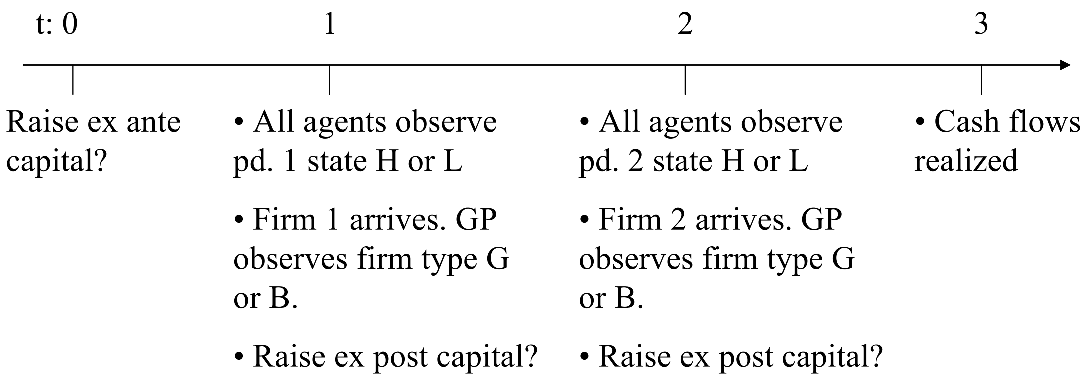

Figure 2.1: Timeline

- *Pure ex ante* capital raising is done in period zero before the GP encounters any firm. The security investors get is backed by the sum of the cash flows from the investments in both periods.

- *Ex ante and ex post* capital raising combines the forms above. Investors supplying ex post capital in a period get a security backed by the cash flow from the investment in that period only. Investors supplying ex ante capital get a security backed by the cash flows from both investments combined.

We now analyze and compare each of the financing arrangements above.

3. Pure ex post capital raising

When capital is raised ex post and firm by firm, there is no link between the first and the second period and we can look at the one period problem. This is a standard signalling problem as in Nachman and Noe (1994).

After observing the quality of the firm, the GP decides whether to seek financing. After raising capital, he decides whether to invest in the firm or in the riskless asset.

The GP will have an incentive to seek financing regardless of the observed quality, since he gets nothing otherwise. The GP needs to raise I by issuing a security $w_I(x)$ to invest in a firm, where $x \in \{0, I, Z\}$. From the fly-by-night condition, the GP cannot get anything if the cash flow from his investment is below I, so we have to have $w_I(I) = I$. The GP therefore always invests in a firm whenever he can raise capital, since his pay-off is zero if he invests in the riskless asset. There is no way for a GP with a good firm to separate himself from a GP with a bad firm, so the only equilibrium is a pooling one where all GP's issue the same security.

The security pays off only if $x = Z$, so the break even condition for investors after learning the expected fraction of good firms α in the period is

$$(\alpha + (1 - \alpha) p) w_I (Z) \geq I$$

Thus, financing is feasible as long as

$$(\alpha + (1 - \alpha) p) Z \geq I$$

and in that case, the GP will invest in all firms. The pay-off $w_I (Z)$ will be set so that investors just break even, and the security can be thought of as debt with face value $w_I (Z)$.[3] When it is impossible to satisfy the break even condition, the GP cannot invest in any firms.

We assume that the unconditional probability of success is too low for investors to break even:

Condition 3.1.

$$(E (\alpha) + (1 - E (\alpha)) p) Z < I$$

Condition 3.1 implies that ex post financing is not possible in the low state. Whether pure ex post financing is possible in the high state depends on whether $(\alpha_H + (1 - \alpha_H) p) Z \geq I$ holds.

Proposition 1. *Pure ex post financing is never feasible in the low state. If*

$$(\alpha_H + (1 - \alpha_H) p) Z \geq I$$

it is feasible in the high state, where the GP issues debt with face value F given by

$$F = \frac{I}{\alpha_H + (1 - \alpha_H) p}$$

3.1. Efficiency

The investment behavior in the pure ex post case is illustrated in Figure 3.1. Investment is inefficient in both high and low states. There is always underinvestment in the low state since good deals cannot get financed. In the high state, there is underinvestment if the break even condition of investors cannot be met, and overinvestment if it can, since then bad deals get financed.

[3] Although the exact interpretation of this contract is somewhat ambiguous given the simple two-state cash flow assumption, in a more general setting with continuous cash flows the optimal contract will correspond to risky debt, as in Nachman and Noe (1994).

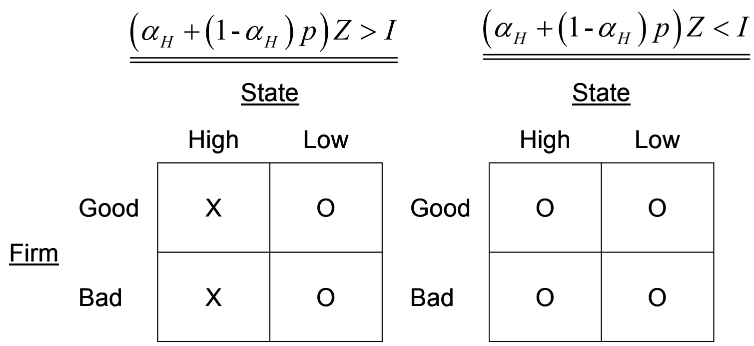

Figure 3.1: Investment behavior in the pure ex post financing case. X denotes that an investment is made, O that no investment is made.

4. Pure Ex Ante Financing

We now study the polar case where the GP raises all the capital to be used over the two periods for investment ex ante, before any investment has been undertaken. Suppose the GP raises $2I$ of ex ante capital in period zero, which implies that the GP is not capital constrained and can potentially invest in both periods.[4]

We solve for the GP's security $w_{GP}(x) = x - w_I(x)$ that maximizes investment efficiency. For all monotonic stakes, the GP will invest in all good firms he encounters over the two periods. Also, if no investment was made in period 1, he will invest in a bad firm in period 2 rather than putting the money in the riskless asset. This follows from the fly-by-night condition, since the GP's payoff has to be zero when fund cash flows are less than or equal to the capital invested.

We show that it is possible to design $w_{GP}(x)$ so that the GP avoids all other inefficiencies. Under this second best contract, he avoids bad firms in period 1, and avoids bad firms in period 2 as long as an investment took place in period 1.

To solve for the optimal security, we need to maximize GP pay-off subject to the monotonicity, fly-by-night, and investor break even conditions, and make sure that the second best investment behavior is incentive compatible. The security pay-offs $w_{GP}(x)$ must be defined over the following potential fund cash flows: $x \in \{0, I, 2I, Z, Z+I, 2I\}$. Note that under a second best contract, $x \in \{0, 2I, Z\}$ will never occur. They would result from the cases of two failed investments, no investment, and one failed and one successful investment respectively, neither of which can result from the GP's optimal investment strategy. We still need to define security pay-offs for these cash flow outcomes to ensure that the contract is incentive compatible.

The fly-by-night condition immediately implies that $w_{GP}(x) = 0$ for $x \leq 2I$. The following

[4]Below we show that in the pure ex ante case, it is never optimal to make the GP capital constrained by giving him less than $2I$.

lemma shows that incentive compatibility reduces to satisfying only one inequality.

Lemma 1. *A necessary and sufficient condition for a contract $w_{GP}(x)$ to be incentive compatible in the pure ex ante case is*

$$\begin{aligned} & (E(\alpha)+(1-E(\alpha))p)\, w_{GP}(Z+I) \\ \geq\ & ((1-p)E(\alpha)+2p(1-p)(1-E(\alpha)))\, w_{GP}(Z) \\ & +p(E(\alpha)+(1-E(\alpha))p)\, w_{GP}(2Z) \end{aligned} \tag{4.1}$$

Proof. In Appendix. ■

The left hand side is the expected pay-off for a GP who encounters a bad firm in period 1 but passes it up, and then invests in any firm that appears in period 2. The right hand side is the expected pay-off if he invests in the bad firm in period 1, and then invests in any firm in period 2. Therefore, when Condition 4.1 holds, the GP will never invest in a bad firm in period 1.[5] For incentive compatibility, we also need to ensure that the GP does not invest in a bad firm in period 2 after investing in a good firm in period 1. This turns out to be the case whenever Condition 4.1 is satisfied.

The full maximization problem can now be expressed as:

$$\begin{aligned} & \max_{w_{GP}(x)} E(w_{GP}(x)) \\ = \ & E(\alpha)^2 w_{GP}(2Z) + \left(2E(\alpha)(1-E(\alpha)) + (1-E(\alpha))^2 p\right) w_{GP}(Z+I) \end{aligned}$$

such that

$$E(x - w_{GP}(x)) \geq 2I \qquad (BE)$$

$$\begin{aligned} (E(\alpha)+(1-E(\alpha))p)\, w_{GP}(Z+I) \geq \\ ((1-p)E(\alpha)+2p(1-p)(1-E(\alpha)))\, w_{GP}(Z) \\ +p(E(\alpha)+(1-E(\alpha))p)\, w_{GP}(2Z) \end{aligned} \qquad (IC)$$

$$x - x' \geq w_{GP}(x) - w_{GP}(x') \geq 0 \quad \forall x, x' \text{ s.t. } x > x' \qquad (M)$$

$$w_{GP}(x) = 0 \quad \forall x \text{ s.t. } x \leq 2I \qquad (FBN)$$

There are two possible pay-offs to the GP in the maximand. The first pay-off, $w_{GP}(2Z)$, occurs

[5] It could be that if the GP invests in a bad firm in period 1, he would prefer to pass up a bad firm encountered in period 2. For incentive compatibility, it is necessary to ensure that the GP gets a higher pay off when avoiding a bad period 1 firm also in this case. We show in the proof, however, that 4.1 implies that this is the case.

only when good firms are encountered in both periods. The second pay-off, $w_{GP}(Z+I)$, will occur either (1) when one good firm is encountered in the first or the second period, or (2) when no good firm is encountered in any of the two periods, and the GP invests in a bad firm in period 2 which turns out to be successful. Condition BE is the investor's break-even condition. Finally, the maximization has to satisfy the monotonicity (M) and the fly-by-night condition (FBN). The feasible set and the optimal security design which solves this program is characterized in the following proposition:

Proposition 2. *Pure ex ante financing with $K = 2I$ is feasible if and only if it creates social surplus. An optimal contract (which is not always unique) is given by*

$$w_I(x) = \begin{cases} \min(x, F) & x \leq Z+I \\ F + k(x-(Z+I)) & x > Z+I \end{cases}$$

$$w_{GP}(x) = \begin{cases} \max(0, x-F) & x \leq Z+I \\ (1-k)(x-(Z+I)) + (Z+I) - F & x > Z+I \end{cases}$$

where $F \geq 2I$ and $k \in (0,1)$.

Proof: See appendix.

Figure 4.1 shows the form of the optimal securities for different levels of social surplus created, where a lower surplus will imply that a higher fraction of fund cash flow have to be pledged to investors. The security structure resembles the structure in private equity funds, where investors get all cash flows below their invested amount and a proportion of the cash flows above that. Moreover, as shown in the proof, the contracts tend to have an intermediate region, where all the additional cash flows are given to the GP. This could be interpreted as the type of "carry catch-up" which is often seen in private equity contracts.

The intuition for the pure ex ante contract is as follows. Ideally, we would like to give the GP a straight equity claim, as this would assure that he only makes positive net present value investments (i.e., invests in good firms) and otherwise invests in the risk-free asset. The problem with straight equity is that the GP receives a positive pay-off even when no capital is invested, which in turn implies that unserious fly-by-night operators can make money. To avoid this, GP's can only be paid if the fund cash flows are sufficiently high, which introduces a risk-shifting incentive. The risk-shifting problem is most severe if investors hold debt and the GP holds a levered equity claim on the fund cash flow. To mitigate this, we need to reduce the levered equity claim of the GP by giving a fraction of the high cash flows to investors.[6]

Another way to see this is by examining the IC constraint of the GP. When $Z \leq 2I$, the IC

[6] This is similar to the classic intuition of Jensen and Meckling (1976).

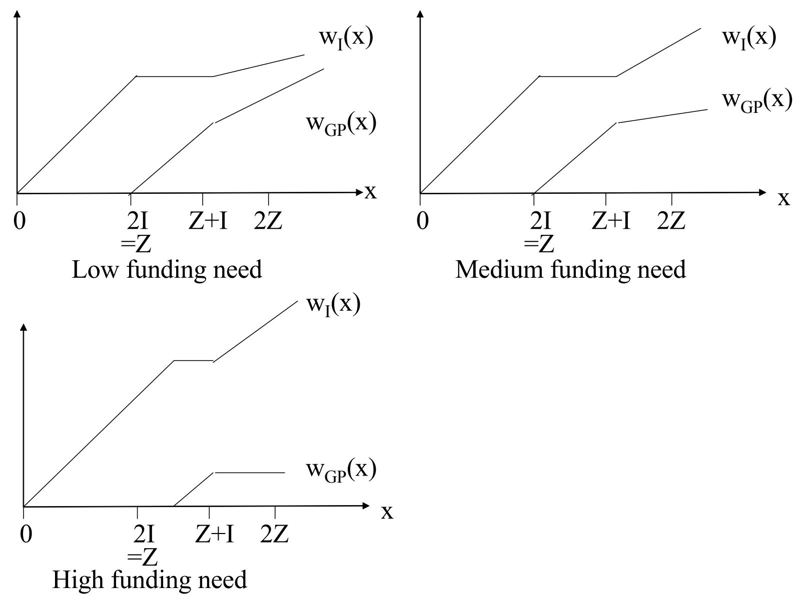

Figure 4.1: GP securities ($w_{GP}(x)$) and investor securities ($w_I(x)$) as a function of fund cash flow x in the pure ex ante case. The three graphs depict contracts under high (top left graph), medium (top right graph), and low (bottom graph) levels of $E(\alpha)$. A high level of $E(\alpha)$ corresponds to high social surplus created, which in turn means that a lower fraction of fund cash flows have to be pledged to investors.

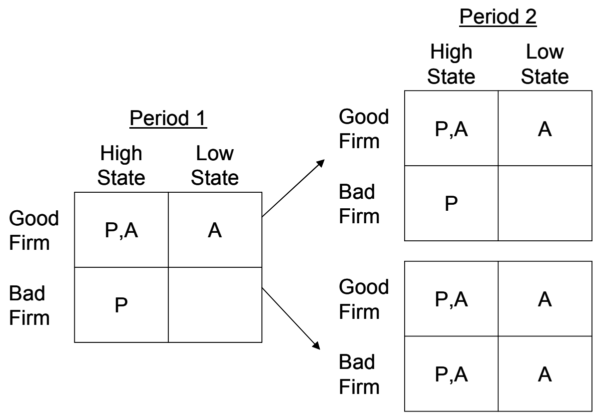

Figure 4.2: Investment behavior in the pure ex ante (A) compared to the pure ex post (P) case when ex post financing is possible in the high state.

constraint simplifies to $w_{GP}(Z+I) \geq pw_{GP}(2Z)$, implying an upper bound on the fraction that the GP can receive of the highest fund cash flows.

As the investors have to be given more rents (in order to satisfy their break-even constraint), it is optimal to increase the pay-off to investors for the highest cash flow states ($2Z$) first, while keeping the pay-offs to GPs for the intermediate cash flow states ($Z+I$) as high as possible, in order to reduce risk-shifting incentives. While our model set-up delivers this intermediate region where investor pay-offs are flat, we believe that this is not a generic feature of more general models. In particular, if we allow good projects to also have some risk, this flat region will likely disappear, in favor of a more smooth equity piece given to investors.

Finally, note that the GPs incentive compatibility condition is satisfied strictly for $w_{GP}(x) = \varepsilon$ for all $x \geq Z+I$ and $w_{GP}(x) = 0$ for $x < Z+I$ for any $\varepsilon > 0$.This implies that ex ante funding is always feasible whenever the equilibrium generates social surplus, since an arbitrarily high fraction of cash flows can be pledged to investors without distorting investment incentives.

4.1. Efficiency

The investment behavior in the pure ex ante relative to the pure ex post case is illustrated in Figure 4.2. In the ex ante case, the GP invests efficiently in period 1. If he invested in a good firm period one, the investment will be efficient in period 2 as well. The only remaining inefficiency is that the GP will invest in the bad firm in period 2 if he has not encountered any good firm in either period.

The ex ante fund structure can improve incentives relative to the ex post deal-by-deal structure by tying the pay-off of several investments together and structuring the GP security appropriately. In the ex post case, the investment inefficiency is caused by the inability to reward the GP when choosing the risk-free asset rather than investing in a bad firm, since this would violate the fly-by-night condition. In the ex ante case, the GP can now be compensated for investing in the riskless asset as long as there is a possibility of finding a good firm. By giving the GP a stake that resembles straight equity for cash flows above the invested amount, he will make efficient investment decisions as long as he anticipates being "in the money". Tying pay-offs of past and future investments together is in a sense a way to endogenously create inside wealth and circumvent the problems created by limited liability.

So far we have restricted the analysis of the ex ante case to a situation where the GP raises enough funds to invest in all firms. It is not hard to see that this dominates an ex ante structure where the GP is capital constrained. Suppose the GP only raises enough funds to invest in one firm over the two periods. He will then pass up bad firms in the first period in the hope of finding a good firm in the second period. There is no way of preventing him from investing in a bad firm in the second period, though, so that the second period overinvestment inefficiency is the same as in the unconstrained case. There is an additional inefficiency, however, in that good firms have to be passed up in period 2 whenever an investment was made in period 1. This is in contrast with the winner picking models in Stein (1997) and Inderst and Muennich (2004). Our result is in line with the empirical finding of Ljungquist and Richardson (2003), who show that private equity funds seldom use up all their capital.

Although the ex ante fund structure can improve efficiency over the pure ex post case, it is clear from the picture that it need not always be the case. Clearly, pure ex ante financing always dominates when pure ex post financing is not even feasible in the high state, i.e. when $(\alpha_H + (1-\alpha_H)\,p)\,Z < I$. Ex ante financing will still work as long as it creates any positive surplus, i.e. as long as investors break even for the contract $w_{GP}(x) = 0$ for all x. When ex post financing is feasible in the high state, ex ante financing will still be more efficient whenever

$$\begin{aligned} & (1-E(\alpha))^2 (I-pZ) \\ \leq\ & 2(q(1-\alpha_H)(I-pZ) + (1-q)\,\alpha_L (Z-I)) \end{aligned}$$

The left hand side is the NPV loss from investing in a bad project the second period, $I - pZ$, times the likelihood of this happening (probability of two bad firms in a row), $(1-E(\alpha))^2$. The right hand side is the efficiency loss for ex post raising, which is that some bad firms are financed in the high state (which happens with probability $q(1-\alpha_H)$ in each period) and some good firms are not financed in the low state (which happens with probability $(1-q)\,\alpha_L$ in each period). Ex post financing has the disadvantage that the GP will always invest in any firm he encounters in high states. However, there is also a benefit - since the contract is set up ex post, it is automatically contingent on the realized value of α and you avoid financing completely in low states. If low states are very unlikely to have good projects (α_L close to zero) and high states have almost only good projects (α_H close to one) the inefficiency with ex post fund raising is small. When the correlation between states and project quality is not so strong, pure ex ante financing will dominate.

Finally, even in the case when pure ex ante financing is more efficient, we need to check whether it is privately optimal for the GP to choose this structure. If the GP has to leave rents to the investor for the pure ex ante financing equilibrium to work, the GP may still prefer pure ex post financing. The following proposition states that the GP can indeed hold the investors to their break-even constraint in the pure ex ante structure, as long as it creates social surplus. As a result, whenever pure ex ante financing is socially optimal, it will also be privately optimal for the GP to choose this financial structure.

Proposition 3. *Whenever the pure ex ante equilibrium is feasible, the GP will capture all the rents.*

Proof. *See appendix. [incomplete]* ■

5. Mixed ex ante and ex post financing

We now examine the situation with both ex post and ex ante capital raising. The problem with unrestricted ex ante fund raising is the overinvestment tendencies in period 2 if there was no investment in a firm in period 1. This is extra costly in the low state, so efficiency is improved if we can curb investment in this contingency without destroying period 1 incentives. We show that this is possible by making the GP *somewhat* capital constrained, so that he has to go back to the market and raise extra capital when he wants to do an investment.

We now assume the GP raises $2K < 2I$ of ex ante fund capital in period 0, and is only allowed to use K for investments each period.[7] The remaining $I - K$ has to be raised ex post. Ex post investors in period i get security $w_{P,i}(x_i)$ backed by the cash flow x_i from the investment in period i. Ex ante investors and the GP get securities $w_A(x)$ and $w_{GP}(x) = x - w_A(x)$ respectively, backed by the fund cash flow $x = x_1 - w_{P,1}(x_1) + x_2 - w_{P,1}(x_2)$ (where $w_{P,i}$ is zero if no ex post financing is raised). The fly-by-night condition is now that $w_{GP}(x) = 0$ for all $x \leq 2K$.

We also assume that it is observable but not contractible whether the GP invests in the risk-free asset or a firm.

We look for the most efficient equilibrium and show that it can dominate all other forms of capital raising. The most efficient implementable equilibrium is one where the GP invests only in good firms in period 1, only in good firms in period 2 if the GP invested in a firm in period 1, and only in the high state if there was no investment in period 1.[8] As is seen in Figure 5.1, this is more efficient than the pure ex ante case since we avoid investment in the low state in period 2 after no

[7] It is common in private equity contracts to restrict the amount the GP is allowed to invest in any one deal.

[8] Note that it is impossible to implement an equilibrium where the GP only invests in good firms over both periods, since if there is no investment in period 1, he will always have an incentive to invest in period 2 whether he finds a good or a bad firm.

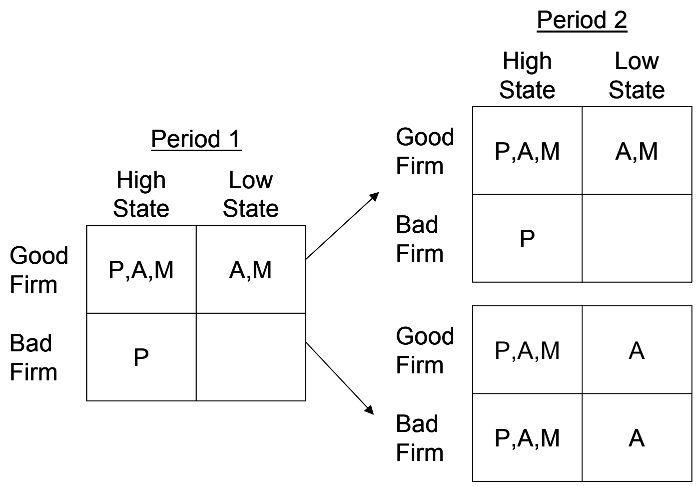

Figure 5.1: Investment behavior in the pure ex ante (A), pure ex post (P), and the postulated mixed (M) case when ex post financing is possible in the high state.

investment has been done in period 1. It is also more efficient than the pure ex post case, since pure ex post capital raising has the added inefficiencies that no good firms are taken in low states and, if the unconditional firm in high states breaks even, all bad firms are taken in high states.

We proceed by first characterizing the ex post securities under the postulated equilibrium. Then we show what the ex ante contract must look like for the investment behavior of the GP to be incentive compatible, and under what circumstances the equilibrium is feasible.

5.1. Ex Post Securities

If the GP raises ex post capital in period i, the cash flow x_i can potentially take on values in $\{0, I, Z\}$, corresponding to a failed investment, a risk-free investment, and a successful investment. If the GP does not raise any ex post capital, he cannot invest in a firm, and saves the ex ante capital K for that period so that $x_i = K$.

We assume that GP's have to issue debt to raise the needed $I - K$ in ex post capital each period, so that $w_{P,i}(x_i) = \min(x_i, F)$. Note that $F \geq I - K$ for ex post investors to break even.

This implies that there is no gain for a GP to raise ex post capital just to invest in a risk-free asset. The fund's stake in such a risk-free investment will be $I - \min(I, F) \leq K$, while if the GP instead saves the ex ante capital K in a period without raising ex post financing, the fund's stake in the saved capital will be K. Since the GP's stake is increasing in fund cash flow, the result follows. Without loss of generality, we therefore assume that the GP only raises ex post capital to invest in a firm.

We now solve for the face value of debt that makes ex post investors break even. In period 1, under the postulated equilibrium, only a GP who finds a good firm will raise ex post capital. Since good firms are risk-free, the face value can be set at $F = I - K$.

In period 2, if a GP has made an investment in period 1, he is also assumed to raise ex post capital and investonly if he finds a good firm. Since the market can observe whether an investment was made in period 1, the GP can again raise risk-free debt with face value $F = I - K$.

For the case when there has been no investment in period 1, the postulated equilibrium says that all GP's will seek ex post capital to invest. The break even condition for ex post investors in state $\alpha \in \{\alpha_L, \alpha_H\}$ is then

$$\begin{aligned} (\alpha + (1-\alpha)\,p)\,F &= I - K \Leftrightarrow \\ F &= \frac{I-K}{\alpha + (1-\alpha)\,p} \end{aligned}$$

When $F > Z$, ex post financing is infeasible. Since we want to make financing infeasible in the low state, the internal capital K per period has to be set high enough so that the GP can invest in the high state but low enough such that the GP cannot invest in the low state. The condition for this is:

$$I - (\alpha_H + (1-\alpha_H)\,p)\,Z \leq K \leq I - (\alpha_L + (1-\alpha_L)\,p)\,Z \tag{5.1}$$

5.2. Ex Ante Securities

We can now solve for the ex ante securities $w_A(x)$ and $w_{GP}(x) = x - w_A(x)$ and the amount of per period ex ante capital K that implement the proposed equilibrium investment behavior. The security pay-offs must be defined over the following potential fund cash flows, which are net of payments to ex post investors:

Fund cash flow x	Investments
0	2 failed investments.
$Z - (I - K)$	1 failed and 1 successful investment.
K	1 failed investment.
$Z - \frac{I-K}{\alpha_H + (1-\alpha_H)p} + K$	1 successful investment in period 2.
$Z - (I - K) + K$	1 successful investment in period 1.
$2\,(Z - (I - K))$	2 successful investments.

Note that the first three cash flows cannot happen in the proposed equilibrium. Also, note that the last three cash flows are in strictly increasing order. In particular, as opposed to the pure ex ante case, the fund cash flow now differs for the case where there is only one good firm depending on whether the firm is encountered in the first or second period. This is because if the good firm is encountered in the second period, the GP is pooled with GP's encountering bad firms, so that ex post investors will demand a higher face value to finance the investment.

The following lemma provides a necessary and sufficient condition on the GP pay-offs to implement the desired equilibrium investment behavior. Just as in the pure ex ante case, it is enough to ensure that the GP does not invest in bad firms in period 1.

Lemma 2. *A necessary and sufficient condition for a contract $w_{GP}(x)$ to be incentive compatible in the mixed ex ante and ex post case is*

$$\begin{aligned} & q\left(\alpha_H+\left(1-\alpha_H\right)p\right)w_{GP}\left(Z-\frac{I-K}{\alpha_H+\left(1-\alpha_H\right)p}+K\right) \\ > \; & E(\alpha)\left(pw_{GP}\left(2\left(Z-\left(I-K\right)\right)\right)+\left(1-p\right)w_{GP}\left(Z-\left(I-K\right)\right)\right)+\left(1-E(\alpha)\right)* \\ & *p\max\left[w_{GP}\left(Z-\left(I-K\right)+K\right),pw_{GP}\left(2\left(Z-\left(I-K\right)\right)\right)+2\left(1-p\right)w_{GP}\left(Z-\left(I-K\right)\right)\right] \end{aligned} \tag{5.2}$$

Proof. In appendix. ■

The left hand side is the expected pay-off of the GP if he passes up a bad firm in period 1. He will then get to invest in period 2 if the state is high (probability q), and will get rewarded if the second period firm is successful (probability $\alpha_H+(1-\alpha_H)p$). If the state in period 2 is low, he cannot invest, and will get a zero pay-off from the fly-by-night constraint. The right hand side is the expected pay-off if the GP deviates and invests in a bad firm in period 1. In this case, he will be able to raise debt at face value $F=I-K$ in both periods, since the market assumes that he is investing efficiently. The first line on the right hand side is his pay-off if he finds a good firm in period 2. The last line is his pay-off when he finds a bad firm in period 2, in which case he will either invest in it or not depending on which pay-off is expected to be higher.

Just as in the pure ex ante case, the incentive compatibility condition 5.2 shows that it is necessary to give part of the upside to investors to avoid risk-shifting by the GP. The GP stake after two successful investments ($w_{GP}\left(2\left(Z-\left(I-K\right)\right)\right)$) cannot be too high relative to his stake if he passes up a period 1 bad firm ($w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)p}+K\right)$).

To solve for the optimal contract, we maximize GP expected pay-off subject to the investor break even constraint, the incentive compatibility condition 5.2, the fly-by-night condition, the monotonicity condition, and Condition 5.1 on the required amount of per period ex ante capital K. The full maximization problem is given in the Appendix. The optimal security design is characterized in the following proposition.

Proposition 4. *The ex ante capital K per period should be set maximal at $K^*=I-\left(\alpha_L+\left(1-\alpha_L\right)p\right)Z$. An optimal contract (which is not always unique) is given by*

$$\begin{aligned} w_A(x) &= \min\left(x,F\right)+k\left(\max\left(x-S,0\right)\right) \\ w_{GP}(x) &= x-w_A(x) \end{aligned}$$

where $F \in \left[2K^*, Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*\right)$, $S. \in \left[Z - \frac{I-K^*}{\alpha_H+(1-\alpha_H)p} + K^*, Z - (I-K^*) + K^*\right]$ *and* $k \in (0,1]$.

Proof. In appendix. ■

The mixed financing contracts look similar to the pure ex ante contracts. As in the pure ex ante case, it is essential to give the ex ante investors an equity part to avoid the risk shifting tendencies of the GP so that he does not pick bad firms whenever he has invested in good firms or has the chance to do so in the future. At the same time, a debt part is necessary in order to screen out fly-by-night operators.

The intuition for why fund capital K per period should be set as high as possible is the following. The higher GP pay-offs are if he passes up bad firms in period 1, the easier it is to implement the equilibrium. The GP only gets a positive pay-off if he reaches the good state in period 2 and succeeds with the period 2 investment, so it would help to transfer some of his expected profits to this state from states where he has two successful investments. This is possible to do by changing the ex ante securities, since ex ante investors only have to break even unconditionally. However, ex post investors break even state by state, so the more ex post capital the GP has to rely on, the less room there is for this type of transfer, and the harder it is to satisfy the GP incentive compatibility condition.

5.3. Feasibility

Compared to both the pure ex ante and pure ex post equilibrium, the equilibrium which combines ex ante and ex post financing is strictly more efficient, as is seen in Figure 5.1. A shortcoming of the mixed financing equilibrium is that it is not always implementable even when it creates surplus (as opposed to the pure ex ante and pure ex post equilibria). This is because it is now harder to provide the GP with incentives to avoid investing in bad firms in the first period. If he deviates and invests, not only will he be allowed to invest also in the low state in period 2, but he will be perceived as being good in the high state, which means that he can raise ex post capital more cheaply. The following proposition gives the conditions under which the equilibrium is implementable.

Proposition 5. *: Necessary and sufficient conditions for the equilibrium to be implementable are that it creates social surplus, that*

$$q\left(\alpha_H + (1-\alpha_H)\,p\right) \geq p \tag{5.3}$$

and

$$\frac{\alpha_L + (1-\alpha_L)\,p}{\alpha_H + (1-\alpha_H)\,p} < \min\left(\frac{I}{Z}, 1 - \frac{I}{Z} + \alpha_L + (1-\alpha_L)\,p\right)$$

Proof: See appendix.

We need that the average project quality in high states (i.e. $\alpha_H + (1 - \alpha_H)\, p$) is sufficiently good, compared both to the overall quality of bad projects (p) and the average in project quality in low states ($\alpha_L + (1 - \alpha_L)\, p$). In other words, if the project quality does not improve sufficiently in high states, it will not be possible to implement this equilibrium. If project quality does not improve much in the high state, however, the efficiency gain from combining ex ante and ex post financing will be small compared to pure ex ante financing. Hence, when the efficiency gain from this equilibrium is large, it will also be feasible to implement.

5.4. Features of the equilibrium and robustness

A few things are worth noting about the nature of the equilibrium.

First, even though the solution is the most efficient that can be implemented, there are still investment distortions. As is seen in Figure 5.1, there is overinvestment in the good state since some bad investments are made, and there is underinvestment in bad states since some good investments get passed up. Thus, our model predicts that deals made in good times will tend to underperform deals made in worse times. In addition, some deals that are economically sound will not get financed in bad times. This is consistent both with anecdotal evidence as well as the private equity return study of Kaplan and Schoar (forthcoming).

Second, note also that the solution involves two types of financiers, ex ante and ex post investors. In fact, it is essential that these be different agents. One could imagine that the GP might just as well go back to the ex ante investors in period 2 and ask for the ex post capital. But after no investment has been made in period 1, the ex ante investors have an incentive to veto the investment if it does not break even so they can get their capital back. However, if this was done, the incentives for the GP in period 1 would unravel, and he would take all deals that come along. For example, the mixed financing equilibrium may still be feasible in cases when ex post financing is not. But in this case, it is clearly not ex post optimal for the ex ante investors to provide financing period 2 in the case when no investment was made period 1.

Third, another essential feature of our equilibrium is that the GP is not allowed to invest more than K of the fund's capital in any given investment. In the case when the GP did not invest in period 1, the GP would otherwise have an incentive to use the whole fund capital, $2K$, to finance a deal in the second period, and the equilibrium would break down. Similarly, if the GPs were allowed to back the ex post securities with total fund cash flows, rather than just the cash flows from an individual deal, this would be equivalent to using the first period capital to back the second period ex post debt. Hence, we need that the contract imposes a restriction on the amount of fund capital that can be used in a given deal, and that the GPs are not allowed to raise financing ex post backed by total fund cash flows. In fact, both these restrictions are commonplace in real world private equity partnership agreements, as have been shown in Gompers and Lerner (1996).

Fourth, although the mixed financing equilibrium is more efficient than pure ex ante or pure ex post, it is sometimes harder to implement. In addition, it is possible (although we have not yet proved this in the current version of the paper) that the ex ante investors cannot always be held to their break-even constraint and will sometimes be left with some rents in equilibrium. If this is the case, it could still be privately optimal for the GP to choose pure ex ante or pure ex post financing, even though the mixed financing equilibrium is socially more efficient. There is another mixed financing equilibrium, however, that may be easier to implement, which we plan to analyze in a future version of the paper. In this equilibrium, the GP cannot be forced to only invest in good projects in period 1. As a result, the first period investment will be a pooling equilibrium, where the GP takes all projects in high states, and no projects in low states. In the second period, however, the GP will only take good projects in the case where an investment occurred in period one. Although this equilibrium is less efficient than the mixed financing equilibrium we have analyzed, it may still be more efficient than pure ex ante or pure ex post, and may be easier to implement.

Fifth, the model we have analyzed is a simple two period model, and the question is to what extent our results would hold up in a multi-period set-up. We plan to analyze this extension in a future version of the paper, but we can still state some conjectures about this case. First, it is clear that if we were to let the fund life go towards an infinite number of periods, we could achieve first best investment with pure ex ante financing. As the fund life goes to infinity, the GP will be certain that he will eventually encounter enough good investments to keep him from taking any bad ones. Clearly infinitely lived funds are not very realistic, and there could be a number of reasons outside of the model for why we do not observe this in the real world, such as LP liquidity constraints. If we take a finite fund life as given, it is likely that the mixed ex ante and ex post financing will still be optimal, as long as the state of the world is sufficiently persistent. Say that the GP encounters a low state early in the fund's life, where good deals are very scarce, and that this low state is expected to last for a long time. In a pure ex ante financing set-up, the GP might then have an incentive to take a bad deal even in a multi-period set up, since the likelihood of encountering any good projects in the future is very low. This would give a role for ex post financing, which prevents the GP from taking any deals in low states.

6. Discussion and Conclusions

An enormous literature in corporate finance concerns the capital structure of firms and the manner in which firms decide to finance investments. Yet, much financing today is done through private capital markets, by private equity firms who receive funding from limited partners and use this money to finance investments in both new ventures and buyouts of existing companies. These firms follow a common financial structure: They are finite-lived limited partnerships who raise equity capital from limited partners before any investments are made (or even discovered) and then supplement this equity financing with third party outside financing at the individual deal level whenever possible. General partners have most decision rights, and receive a percentage of the profits (usually 20%), which is junior to all other securities. Yet, while this financial structure is responsible for a very large quantity of investment, we have no theory explaining why it should be so prevalent.

This paper presents a model of the financial structure of a private equity firm. The firm can finance its investments either ex ante, by pooling capital across future deals, or ex post, by financing deals once knows about them. The financial structure chosen is the one that maximizes the value of the fund, which will depend on financial structure because managers have better information about deal quality than potential investors. The value maximizing financial structure of the firm minimizes the losses both from expected bad investments that are undertaken and good investments that are ignored.

The model leads to a number of predictions that are consistent with commonly observed features of the private equity industry. First, deals are financed by a combination of ex ante and ex post financing, i.e., private equity funds raise capital both initially at the fund level and subsequently at the deal level. Second, the nature of the optimal securities derived by the model appears to mimic the contracts used by private equity firms. Ex post financing provided by third party financiers is backed by the individual deal, while payments to LPs are backed by the cash flow of the fund. Third, our model can explain some of the most frequent covenants seen in private equity partnership agreements, such as restrictions on levering up the whole fund (rather than individual investments), and specifying a maximum amount of fund capital that can be invested in any individual deal (see Gompers and Lerner, 199X). Fourth, the model predicts that observed investments in the private equity market will be procyclical, with the already procyclical nature of investment opportunities augmented by the overinvestment in good times and underinvestment in bad times. Fifth, consistent with both casual observation (the internet and biotech bubbles) as well as more formal empirical evidence, this overinvestment and underinvestment predicts that average returns to investments made during booms will be worse than returns to investments made during recessions.

In addition, the intuitions coming from our model are consistent with other common observations about the private equity industry. For example, there are circumstances where investors do provide financing for individual deals. Sellers sometimes pay for partial financing of their firms, GPs syndicate investments across funds, and approach LPs for coinvestment opportunities. Each of these types of financing can be thought of in terms of our model in that they all occur in circumstances where the degree of information asymmetry is likely to be low. For example, when a seller helps to finance a deal, it typically supplements bank financing and is likely to occur when the seller has better information about his firm than the bank. When GPs syndicate deals, they usually agree on the prospects for an investment, so that information asymmetry is potentially minimized. When funds ask LPs to coinvest, our model suggests that they tend to more sophisticated LPs, who can evaluate the deal themselves and be assured that it is a good investment. Finally, in those circumstances where specific funds are raised to finance particular deals, there should be a good reason why the initiating GP did not do the entire investment by himself. One potential reason is that the fund could be constrained in the size of its investment by its charter; An example of such a situation is Exxel's acquisitions of Argencard and Norte (see Hoye and Lerner (1995), Ballve and Lerner (2001)).

However, our model falls short in that it fails to explain a number of important features of private equity funds. First, private equity funds tend to be finitely-lived; We provide no rationale for such a finite life. Second, while one might expect much of our analysis to apply equally to hedge funds, it is not clear that it does. Hedge funds are financed predominately by levered equity and we have no explanation for this phenomenon. Third, most venture deals are staged in a number of rounds. While a number of explanations for staging are in the literature (see Gompers and Lerner (1999)), ideally staging should come as an implication of a more general model of private equity firms' capital structures such as the one presented here. Finally, while we identify potential investment distortions arising even when funds use the optimal financial structure, we do not have a clear understanding of what practitioners and policy-makers could conceivably do to minimize these distortions. Knowing about any conceivable such policies clearly is a potentially valuable contribution to the study of, as well as the practice of, private equity.

7. Appendix

7.1. Proof of Lemma 1

To implement the investment behavior in Figure 4.2, we first check that the GP always invests in good firms regardless of what other investments he has made. For any random variable $y \in \{0, I, Z\}$ resulting from investment behavior in a period, the condition for this is:

$$E\left(w_{GP}\left(Z+y\right)\right) \geq E\left(w_{GP}\left(I+y\right)\right)$$

This holds automatically from the monotonicity condition. It remains to check that the GP does not invest in bad firms in period 2 after investing in a good firm in period 1, and that the GP does not invest in bad firms in period 1. Using that $w_{GP}\left(0\right) = w_{GP}\left(I\right) = w_{GP}\left(2I\right) = 0$ from the fly-by-night constraint, the incentive compatibility conditions are:

$$w_{GP}\left(Z+I\right) \geq \left(1-p\right)\ w_{GP}\left(Z\right) + p\ w_{GP}\left(2Z\right) \tag{7.1}$$

$$\begin{aligned} & \left(E\left(\alpha\right) + \left(1-E\left(\alpha\right)\right)p\right)\ w_{GP}\left(Z+I\right) \\ \geq\ & \left(1-p\right)E\left(\alpha\right)\ w_{GP}\left(Z\right) + p\left(1-E\left(\alpha\right)\right)\ w_{GP}\left(Z+I\right) \\ & +pE\left(\alpha\right)\ w_{GP}\left(2Z\right) \end{aligned} \tag{7.2}$$

$$\begin{aligned} & \left(E\left(\alpha\right) + \left(1-E\left(\alpha\right)\right)p\right)\ w_{GP}\left(Z+I\right) \\ \geq\ & \left(\left(1-p\right)E\left(\alpha\right) + 2p\left(1-p\right)\left(1-E\left(\alpha\right)\right)\right)\ w_{GP}\left(Z\right) \\ & +\left(pE\left(\alpha\right) + p^2\left(1-E\left(\alpha\right)\right)\right)\ w_{GP}\left(2Z\right) \end{aligned} \tag{7.3}$$

The first condition assures that the GP does not invest in a bad firm in period 2 after investing in a good firm in period 1. The two last conditions assure that the GP does not invest in a bad firm in period 1. The two conditions differ only on the right hand side, corresponding to the two possible off-equilibrium investment decisions in period 2: Only investing in good firms in period 2 after making a bad investment in period 1 (Condition 7.2), or investing in all firms in period 2 (Condition 7.3).

Deducting $\left(1-E\left(\alpha\right)\right)pw_{GP}\left(Z+I\right)$ from both sides of Condition 7.2 and dividing by $E\left(\alpha\right)$, we see that it is identical to Condition 7.1.

Rearranging Condition 7.3, we get

$$w_{GP}(Z+I) \geq \frac{(E(\alpha)+(1-E(\alpha))p)+(1-E(\alpha))p}{(E(\alpha)+(1-E(\alpha))p)}(1-p)\,w_{GP}(Z)+p\,w_{GP}(2Z) \tag{7.4}$$

Note that this implies Condition 7.1, and is therefore a necessary and sufficient condition for incentive compatibility.

7.2. Proof of Proposition 2:

We need to solve for optimal values of $w_{GP}(Z), w_{GP}(Z+I)$, and $w_{GP}(2Z)$. We start by establishing the following lemma:

Lemma 3. *In the optimal contract under pure ex ante financing,* $w_{GP}(Z) = \max(0, w_{GP}(Z+I) - I)$, $w_I(Z) = \min(Z, w_I(Z+I))$

Proof: We cannot have $w_{GP}(Z) < \max(0, w_{GP}(Z+I) - I)$, since this would violate monotonicity. Suppose contrary to the claim that $w_{GP}(Z) > \max(0, w_{GP}(Z+I) - I)\,w_I(Z)$ in an optimal contract. Then, we can relax the IC constraint by decreasing $w_{GP}(Z)$ without violating M or FBN. The maximand and the break even constraint are unaffected by this, since $x = Z$ does not happen in equilibrium so that $w_{GP}(Z)$ does not enter the maximand or the break even constraint. QED.

Given this, the program now becomes

$$\max_{w_{GP}(x)} E(\alpha)^2 w_{GP}(2Z) + \left(2E(\alpha)(1-E(\alpha)) + (1-E(\alpha))^2 p\right) w_{GP}(Z+I)$$

such that

$$\begin{aligned} E(\alpha)^2(2Z - w_{GP}(2Z)) \\ +\left(2E(\alpha)(1-E(\alpha)) + (1-E(\alpha))^2 p\right)(Z+I-w_{GP}(Z+I)) \\ +(1-E(\alpha))^2(1-p)I \geq 2I \end{aligned} \tag{BE}$$

$$\begin{aligned} (E(\alpha)+(1-E(\alpha))p)\,w_{GP}(Z+I) \geq \\ ((1-p)E(\alpha) + 2p(1-p)(1-E(\alpha)))\max(0, w_{GP}(Z+I) - I) \\ +p(E(\alpha)+(1-E(\alpha))p)\,w_{GP}(2Z) \end{aligned} \tag{IC}$$

$$\begin{aligned} x - x' \geq w_{GP}(x) - w_{GP}(x') \geq 0 \quad & \forall x, x' \text{ s.t. } x > x' \quad (M) \\ w_{GP}(x) = 0 \quad & \forall x \text{ s.t. } x \leq 2I \quad (FBN) \end{aligned}$$

Lemma 4. *In the optimal contract under pure ex ante financing, we will either have (1)* $w_{GP}(Z+I) = Z - I$ *and* $w_{GP}(2Z) = (1-k)(2Z-2I)$ *(and* $w_I(Z+I) = 2I, w_I(2Z) = 2I + k(2Z-2I)$*),* *where* $0 < k < 1$ *or (2)* $w_{GP}(Z+I) < Z-I$ *and* $w_{GP}(2Z) = w_{GP}(Z+I)$ *(and* $w_I(Z+I) > 2I, w_I(2Z) = w_I(Z+I) + (Z-I)$*).*

Proof: Suppose not. Then, we will show that you can relax the IC constraint by increasing $w_{GP}(Z+I)$ and decreasing $w_{GP}(2Z)$ without violating FBN, M or BE. Note that if $w_{GP}(Z+I) = Z-I$ or $w_{GP}(2Z) = w_{GP}(Z+I)$, $w_{GP}(Z+I)$ cannot be increased without violating monotonicity.

Case 1: Suppose $w_{GP}(Z) = 0 > w_{GP}(Z+I) - I$. Then, increase $w_{GP}(Z+I)$ and decrease $w_{GP}(2Z)$ to keep the break even constraint and the maximand constant:

$$-dw_{GP}(2Z) = \frac{2E(\alpha)(1-E(\alpha)) + (1-E(\alpha))^2 p}{E(\alpha)^2} dw_{GP}(Z+I)$$

This relaxes IC.

Case 2: $w_{GP}(Z) = w_{GP}(Z+I) - I$. Doing the same perturbation, we show that IC is relaxed. Moving all terms to the LHS of IC, the change in the LHS is equal to

$$1 - \frac{E(\alpha) + (1-E(\alpha))2p}{E(\alpha) + (1-E(\alpha))p}(1-p) + \frac{2E(\alpha)(1-E(\alpha)) + (1-E(\alpha))^2 p}{E(\alpha)^2} p$$

We show that this is positive. The derivative w.r.t. to p of the expression above is equal to

$$\begin{aligned}
& 1 - (1-E(\alpha))\frac{(1-2p)E(\alpha) - p^2(1-E(\alpha))}{(E(\alpha) + (1-E(\alpha))p)^2} + \frac{2E(\alpha)(1-E(\alpha)) + 2(1-E(\alpha))(1-E(\alpha))p}{E(\alpha)^2} \\
> \; & -(1-E(\alpha))\frac{(1-2p)E(\alpha)}{(E(\alpha)+(1-E(\alpha))p)^2} + \frac{2E(\alpha)(1-E(\alpha))}{E(\alpha)^2} \\
> \; & 0
\end{aligned}$$

since $E(\alpha)^2 \le (E(\alpha) + (1-E(\alpha))p)^2$, and since $(1-2p) < 2$. Thus, if the change is non-negative for $p = 0$, IC is relaxed. Substituting for p=0, the change becomes zero. *QED*

Thus, the optimal investor security w_I is debt with face value $w_I(Z+I) \ge 2I$, and an equity piece $w_I(2Z) - w_I(Z+I)$ given by

$$\begin{aligned}
w_I(2Z) - w_I(Z+I) &= Z - I \quad \text{if } w_I(Z+I) > 2I \\
w_I(2Z) - w_I(Z+I) &\in [0, Z-I] \quad \text{if } w_I(Z+I) = 2I
\end{aligned}$$

, so that $w_I(x)$ and $w_{GP}(x) = x - w_I(x)$ are given as in the proposition.

We now show that the equilibrium is always implementable as long as it generates social surplus. Suppose you give the GP the following contract:

$$\begin{aligned} w_{GP}(Z) &= 0 \\ w_{GP}(Z+I) &= \varepsilon \\ w_{GP}(2Z) &= \varepsilon \end{aligned}$$

For $\varepsilon > 0$, the IC condition holds strictly. Making ε small, an arbitrarily large fraction of cash flows can be given to investors, and M and FBN hold. Therefore, the BE condition can always be made to hold as long as the equilibrium creates social surplus.

End proof.

7.3. Proof of Proposition 3:

[This proof is somewhat incomplete.] We show that the IC constraint is always satisfied for $w(Z+I) > 2I$ (where $w(2Z) - w(Z+I) = Z - I$), and for $w(Z+I) = 2I$ if $w(2Z) - w(Z+I) = Z - I$. The IC constraint becomes

$$w_{GP}(Z+I) \geq \frac{E(\alpha) + (1 - E(\alpha))\,2p}{E(\alpha) + (1 - E(\alpha))\,p}(1-p)\max(0, w_{GP}(Z+I) - I) + p w_{GP}(Z+I)$$

or

$$w_{GP}(Z+I) \geq \frac{E(\alpha) + (1 - E(\alpha))\,2p}{E(\alpha) + (1 - E(\alpha))\,p}\max(0, w_{GP}(Z+I) - I)$$

Note that this always holds for $\max(0, w_{GP}(Z+I) - I) = 0$, so it remains to show that it holds for $\max(0, w_{GP}(Z+I) - I) = w_{GP}(Z+I) - I$. [To be completed.]

Also, note that by decreasing $w_{GP}(Z+I)$ (and decreasing $w_{GP}(2Z)$ according to the optimal contract specified above), we can always satisfy the break even constraint as long as the social surplus is positive. Whether investors get a strictly positive surplus depends on whether the break even constraint is slack at the contract that gives them the lowest possible surplus while satisfying IC. This contract is given by

$$\begin{aligned} w_{GP}(Z+I) &= Z - I \\ Z - I &= \frac{E(\alpha) + (1 - E(\alpha))\,2p}{E(\alpha) + (1 - E(\alpha))\,p}(1-p)\max(0, Z - 2I) + p w_{GP}(2Z) \\ Z - I &= \frac{E(\alpha) + (1 - E(\alpha))\,2p}{E(\alpha) + (1 - E(\alpha))\,p}(1-p)\max(0, Z - 2I) + p w_{GP}(2Z) \end{aligned}$$

Case 1:$\max(0, Z - 2I) = 0$.

Then,

$$w_{GP}(2Z) = \frac{Z-I}{p}$$

Note that monotonicity is violated if $w_{GP}(2Z) > 2(Z-I)$, or if

$$\frac{Z-I}{p} > 2(Z-I) \Leftrightarrow \frac{1}{2} > p$$

Then, the IC constraint is slack at $w_{GP}(2Z) = 2(Z-I)$, a contract at which investors do not break even, so we can increase the pay offs to investors until they break even while still satisfying IC and the GP captures all the surplus.

Assume $\frac{1}{2} < p$. Plugging in $w_{GP}(Z+I) = Z-I$ and $w_{GP}(2Z) = \frac{Z-I}{p}$ into the break even constraint, it becomes

$$E(\alpha)^2\left(2Z - \frac{Z-I}{p}\right) + \left(2E(\alpha)(1-E(\alpha)) + (1-E(\alpha))^2 p\right)2I + (1-E(\alpha))^2(1-p)I \geq 2I \qquad (BE)$$

We can rewrite this as

$$E(\alpha)^2\left(2Z - \frac{Z-I}{p}\right) + (1-E(\alpha))^2(1-p)I \geq \left(E(\alpha)^2 + (1-E(\alpha))^2(1-p)\right)2I$$

which becomes

$$\frac{E(\alpha)^2}{(1-E(\alpha))^2}\left(2Z - \frac{Z-I}{p}\right) \geq \frac{E(\alpha)^2}{(1-E(\alpha))^2}2I + (1-p)I$$

Since the left hand side increases faster in $E(\alpha)$ than the right hand side, this is easiest to satisfy when $E(\alpha)$ is high. The maximal value is given by the assumption that the unconditional project is negative NPV:

$$\begin{aligned}
(E(\alpha) + (1-E(\alpha))p)Z &= I \\
E(\alpha) &= \frac{\frac{I}{Z} - p}{(1-p)} \\
\frac{E(\alpha)}{1-E(\alpha)} &= \frac{\frac{I}{Z} - p}{1 - \frac{I}{Z}}
\end{aligned}$$

Plugging this in gives

$$\begin{aligned}
\frac{\left(\frac{\frac{I}{Z}-p}{(1-p)}\right)^2}{\left(1-\frac{\frac{I}{Z}-p}{(1-p)}\right)^2}\left(2Z-\frac{Z-I}{p}\right) &\geq \frac{\left(\frac{\frac{I}{Z}-p}{(1-p)}\right)^2}{\left(1-\frac{\frac{I}{Z}-p}{(1-p)}\right)^2}2I+(1-p)\,I \\
\frac{\left(\frac{\frac{I}{Z}-p}{(1-p)}\right)^2}{\left(\frac{1-\frac{I}{Z}}{(1-p)}\right)^2}\left(2Z-\frac{Z-I}{p}\right) &\geq \frac{\left(\frac{\frac{I}{Z}-p}{(1-p)}\right)^2}{\left(\frac{1-\frac{I}{Z}}{(1-p)}\right)^2}2I+(1-p)\,I \\
\frac{\left(\frac{I}{Z}-p\right)^2}{\left(1-\frac{I}{Z}\right)^2}\left(2Z-\frac{Z-I}{p}\right) &\geq \frac{\left(\frac{I}{Z}-p\right)^2}{\left(1-\frac{I}{Z}\right)^2}2I+(1-p)\,I \\
\left(2Z-\frac{Z-I}{p}\right) &\geq 2I+\frac{\left(1-\frac{I}{Z}\right)^2}{\left(\frac{I}{Z}-p\right)^2}(1-p)\,I \\
\left(\frac{I}{Z}-p\right)^2\left(2-\frac{1}{p}\right) &\geq \left(1-\frac{I}{Z}\right)(1-p)\,\frac{I}{Z}
\end{aligned}$$

Moving everything to the left hand side and taking the derivative w.r.t. to p gives

$$\begin{aligned}
& -2\left(\frac{I}{Z}-p\right)\left(2-\frac{1}{p}\right)+\left(\frac{I}{Z}-p\right)^2\frac{1}{p^2}+\left(1-\frac{I}{Z}\right)\frac{I}{Z} \\
= & \left(\frac{I}{Z}-p\right)\left(\frac{\frac{I}{Z}-p}{p^2}-2\left(2-\frac{1}{p}\right)\right)+\left(1-\frac{I}{Z}\right)\frac{I}{Z}
\end{aligned}$$

Note that

$$\frac{\frac{I}{Z}-p}{p^2}-2\left(2-\frac{1}{p}\right)$$

is increasing in p. For the minimal value $p=\frac{1}{2}$, this expression is positive, so the whole derivative is always positive. Thus, the constraint is easiest to satisfy for high values of p. The maximal value of p is $\frac{I}{Z}$, and plugging this in shows that the break even constraint is not satisfied. Thus, the GP captures all the surplus.

Case 2:$\max(0, Z-2I)=Z-2I$.

The break even constraint is

$$E(\alpha)^2(2Z-w_{GP}(2Z))+\left(2E(\alpha)(1-E(\alpha))+(1-E(\alpha))^2p\right)2I+(1-E(\alpha))^2(1-p)\,I\geq 2I$$

Suppose we force this to hold with equality and solve for $w_{GP}(2Z)$:

$$\begin{aligned} E(\alpha)^2 (2Z - w_{GP}(2Z)) + (1 - E(\alpha))^2 (1-p) I &= \left(E(\alpha)^2 + (1 - E(\alpha))^2 (1-p)\right) 2I \\ 2Z - w_{GP}(2Z) &= 2I + \left(\frac{1 - E(\alpha)}{E(\alpha)}\right)^2 (1-p) I \end{aligned}$$

The IC constraint is slack if

$$Z - I \geq \frac{E(\alpha) + (1 - E(\alpha)) 2p}{E(\alpha) + (1 - E(\alpha)) p} (1-p)(Z - 2I) + p w_{GP}(2Z)$$

$$2Z - w_{GP}(2Z) > 2Z - \frac{Z-I}{p} + \frac{E(\alpha) + (1 - E(\alpha)) 2p}{E(\alpha) + (1 - E(\alpha)) p} (1-p) \frac{Z - 2I}{p}$$

or

$$\begin{aligned} 2I + \left(\frac{1 - E(\alpha)}{E(\alpha)}\right)^2 (1-p) I &> 2Z - \frac{Z-I}{p} + \frac{1 + \left(\frac{1-E(\alpha)}{E(\alpha)}\right) 2p}{1 + \left(\frac{1-E(\alpha)}{E(\alpha)}\right) p} (1-p) \frac{Z-2I}{p} \\ 2I + x^2 (1-p) I &> 2Z - \frac{Z-I}{p} + \frac{1 + x2p}{1 + xp} (1-p) \frac{Z - 2I}{p} \end{aligned}$$

Taking everything to the left hand side and taking the derivative w.r.t. to x gives

$$\begin{aligned} & 2x(1-p) I - (1-p) \frac{Z-2I}{p} \frac{2p(1+xp) - (1+x2p)p}{(1+xp)^2} \\ \sim \quad & 2xI - \frac{Z-2I}{(1+xp)^2} \end{aligned}$$

This is increasing in x. Thus, if it is positive for the lowest possible x, it is always positive. The lowest possible x is

$$x = \frac{1 - \frac{I}{Z}}{\frac{I}{Z} - p}$$

Plugging this in gives

$$\begin{aligned} & 2\frac{Z-I}{I-pZ} I - \frac{Z-2I}{\left(1 + \frac{Z-I}{I-pZ} p\right)^2} \\ = \quad & 2\frac{Z-I}{1 - \frac{pZ}{I}} - \frac{Z-2I}{\left(1 + \frac{Z-I}{I-pZ} p\right)^2} \\ > \quad & 0 \end{aligned}$$

Thus, the derivative w.r.t. to x is everywhere positive, and we should set x as low as possible to make it hard to satisfy the IC constraint.

Plugging this in to the IC constraint gives

$$
\begin{aligned}
2I+\left(\frac{1-\frac{I}{Z}}{\frac{I}{Z}-p}\right)^{2}(1-p)\,I &> 2Z-\frac{Z-I}{p}+\frac{1+\frac{1-\frac{I}{Z}}{\frac{I}{Z}-p}2p}{1+\frac{1-\frac{I}{Z}}{\frac{I}{Z}-p}p}(1-p)\,\frac{Z-2I}{p} \\
2I+\left(\frac{1-\frac{I}{Z}}{\frac{I}{Z}-p}\right)^{2}(1-p)\,I &> 2Z-\frac{Z-I}{p}+\frac{I+p\,(Z-2I)}{(1-p)\,I}(1-p)\,\frac{Z-2I}{p} \\
2I+\left(\frac{1-\frac{I}{Z}}{\frac{I}{Z}-p}\right)^{2}(1-p)\,I &> 2Z-\frac{Z-I}{p}+\frac{Z-2I}{p}+\frac{p\,(Z-2I)}{I}\frac{Z-2I}{p}
\end{aligned}
$$

$$
\begin{aligned}
2I+\left(\frac{Z-I}{I-Zp}\right)^{2}(1-p)\,I &> 2Z-\frac{I}{p}+\frac{(Z-2I)^{2}}{I} \\
\left(\frac{Z-I}{I-Zp}\right)^{2}(1-p)\,I+\frac{I-Zp}{p} &> Z-2I+\frac{(Z-2I)^{2}}{I} \\
\left(\frac{Z-I}{I-Zp}\right)^{2}(1-p)\,I+\frac{I-Zp}{p} &> (Z-2I)\left(\frac{Z-I}{I}\right)
\end{aligned}
$$

Noting that

$$
\frac{Z-I}{I-Zp}>\frac{Z-I}{I}
$$

it is harder to satisfy the constraint if we divide the LHS by $\frac{Z-I}{I-Zp}$ and the RHS with $\frac{Z-I}{I}$, which gives

$$
\begin{aligned}
\left(\frac{Z-I}{I-Zp}\right)(1-p)\,I+\frac{I-Zp}{p} &> (Z-2I) \\
(Z-I)\,\frac{1-p}{1-\frac{Z}{I}p}+\frac{I-Zp}{p} &> Z-2I
\end{aligned}
$$

This always holds, since

$$
\frac{1-p}{1-\frac{Z}{I}p}>1
$$

. Thus, IC is always satisfied when the investor just breaks even. QED

7.4. Proof of Lemma 2:

If the GP invested in a good firm in period 1, he will pass up a bad firm if:

$$
\begin{aligned}
& w_{GP}\,(Z-(I-K)+K) \\
> \quad & p w_{GP}\,(2\,(Z-(I-K)))+(1-p)\,w_{GP}\,(Z-(I-K))
\end{aligned} \tag{7.5}
$$

The last term is the case where the bad firm does not pay off, and the fund defaults on its period 2 ex post debt.

We also have to check the off-equilibrium behavior where the GP invested in a bad firm in period 1. If the GP invested in a bad firm in period 1 he will pass up a bad firm in period 2 if:

$$\begin{aligned} & pw_{GP}\left(Z-(I-K)+K\right)+(1-p)\,w_{GP}\left(K\right) \\ > \; & p^{2}w_{GP}\left(2\left(Z-(I-K)\right)\right)+p\left(1-p\right)w_{GP}\left(Z-(I-K)\right) \\ & +\left(1-p\right)pw_{GP}\left(Z-(I-K)\right) \end{aligned}$$

The two last terms are, respectively, the case where the first bad firm pays off and the second does not, and the case where the first bad firm does not pay off and the second does. Since $w_{GP}\left(K\right)=0$ from the fly by night condition, this can be rewritten as

$$\begin{aligned} & w_{GP}\left(Z-(I-K)+K\right) \\ > \; & pw_{GP}\left(2\left(Z-(I-K)\right)\right)+2\left(1-p\right)w_{GP}\left(Z-(I-K)\right) \end{aligned} \tag{7.6}$$

Note that this is a stricter condition than condition 7.5.

Given the period two incentive compatibility constraints, we can now consider the GP's investment incentives in the first period. In period 1, it is always optimal to invest in a good project. We must check that the GP does not want to invest in a bad project to sustain the separating equilibrium.

The condition for not investing in a bad project in period 1 becomes

$$\begin{aligned} & q\left(\alpha_{H}+(1-\alpha_{H})\,p\right)w_{GP}\left(Z-\frac{I-K}{\alpha_{H}+(1-\alpha_{H})\,p}+K\right) \\ > \; & E(\alpha)\left(pw_{GP}\left(2\left(Z-(I-K)\right)\right)+(1-p)\,w_{GP}\left(Z-(I-K)\right)\right)+\left(1-E(\alpha)\right)* \\ & *p\max\left(w_{GP}\left(Z-(I-K)+K\right),pw_{GP}\left(2\left(Z-(I-K)\right)\right)+2\left(1-p\right)w_{GP}\left(Z-(I-K)\right)\right) \end{aligned}$$

The last line is the GP pay off when he has invested in a bad firm in period 1 and encounters another bad firm in period 2, in which case he will either invest in it or not, depending on whether Condition 7.6 holds or not. Note that this condition implies condition 7.5, since

$$w_{GP}\left(Z-\frac{I-K}{\alpha_{H}+(1-\alpha_{H})\,p}+K\right)\leq w_{GP}\left(Z-(I-K)+K\right)$$

and

$$\begin{aligned}
&\frac{E(\alpha)}{q\left(\alpha_H+\left(1-\alpha_H\right)p\right)}\left(pw_{GP}\left(2\left(Z-(I-K)\right)\right)+\left(1-p\right)w_{GP}\left(Z-(I-K)\right)\right)\\
&+\frac{\left(1-E(\alpha)\right)p}{q\left(\alpha_H+\left(1-\alpha_H\right)p\right)}\max\left(\begin{array}{c} w_{GP}\left(Z-(I-K)+K\right), pw_{GP}\left(2\left(Z-(I-K)\right)\right)\\ +2\left(1-p\right)w_{GP}\left(Z-(I-K)\right)\end{array}\right)
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{\left(E(\alpha)+\left(1-E(\alpha)\right)p\right)}{q\left(\alpha_H+\left(1-\alpha_H\right)p\right)}\left(pw_{GP}\left(2\left(Z-(I-K)\right)\right)+\left(1-p\right)w_{GP}\left(Z-(I-K)\right)\right)\\
&\geq pw_{GP}\left(2\left(Z-(I-K)\right)\right)+\left(1-p\right)w_{GP}\left(Z-(I-K)\right)
\end{aligned}$$

Thus, the only relevant incentive constraint is the period 1 IC constraint.

7.5. Proof of Proposition 4:

The full maximization problem can now be expressed as

$$\begin{aligned}
&\max E\left(w_{GP}\left(x\right)\right)\\
=\ &E\left(\alpha\right)^2 w_{GP}\left(2\left(Z-(I-K)\right)\right)+E\left(\alpha\right)\left(1-E\left(\alpha\right)\right)w_{GP}\left(Z-(I-K)+K\right)\\
&+\left(1-E\left(\alpha\right)\right)q\left(\alpha_H+\left(1-\alpha_H\right)p\right)w_{GP}\left(Z-\frac{I-K}{\alpha_H+\left(1-\alpha_H\right)p}+K\right)
\end{aligned}$$

such that

$$E\left(x-w_{GP}\left(x\right)\right)\geq 2K \qquad (BE)$$

$$\begin{aligned}
&q\left(\alpha_H+\left(1-\alpha_H\right)p\right)w_{GP}\left(Z-\frac{I-K}{\alpha_H+\left(1-\alpha_H\right)p}+K\right)\\
>\ &E(\alpha)\left(pw_{GP}\left(2\left(Z-(I-K)\right)\right)+\left(1-p\right)w_{GP}\left(Z-(I-K)\right)\right)+\left(1-E(\alpha)\right)*\\
&*p\max\left[w_{GP}\left(Z-(I-K)+K\right), pw_{GP}\left(2\left(Z-(I-K)\right)\right)+2\left(1-p\right)w_{GP}\left(Z-(I-K)\right)\right] \qquad (IC)
\end{aligned}$$

$$x-x'\geq w_{GP}\left(x\right)-w_{GP}\left(x'\right)\geq 0 \qquad \forall x,x' \text{ s.t. } x>x' \qquad (M)$$

$$w_{GP}\left(x\right)=0 \qquad \forall x \text{ s.t. } x\leq 2K, \qquad (FBN)$$

and

$$I-\left(\alpha_H+\left(1-\alpha_H\right)p\right)Z\leq K\leq I-\left(\alpha_L+\left(1-\alpha_L\right)p\right)Z$$

7.5.1. Proof that $K = I - (\alpha_L + (1-\alpha_L)\, p)\, Z$:

We want to show that the ex ante capital K should be set maximal at $K^* = I - (\alpha_L + (1-\alpha_L)\, p)\, Z$. First, we have to have $Z - \frac{I-K}{\alpha_H + (1-\alpha_H)p} + K > Z - (I-K)$ and $Z - \frac{I-K}{\alpha_H+(1-\alpha_H)p} + K > 2K$ for the equilibrium to be feasible. These two conditions are given respectively by

$$\frac{I-K}{\alpha_H + (1-\alpha_H)\, p} < I$$

and

$$\frac{I-K}{\alpha_H + (1-\alpha_H)\, p} < Z - K$$

Suppose we start at a contract that satisfies monotonicity and fly by night. Thus, the cash flow states are ordered by

$$\begin{aligned} & 2\,(Z - (I-K)) \\ > \; & Z - (I-K) + K \\ > \; & Z - \frac{I-K}{\alpha_H + (1-\alpha_H)\, p} + K \\ > \; & \max\,(Z - (I-K)\,, 2K) \end{aligned}$$

Note that from the fly by night condition,

$$w_A\,(Z - (I-K)) \geq \min\,(Z - (I-K)\,, 2K)$$

Now suppose we increase K by Δ arbitrarily small, increase $w_A\,(2K)$ by 2Δ, increase $w_A\,(Z - (I-K))$ by

$$\begin{aligned} \Delta \text{ if} \quad & w_A\,(Z - (I-K)) = Z - (I-K) \\ 2\Delta \text{ if} \quad & w_A\,(Z - (I-K)) < Z - (I-K) \end{aligned}$$

Suppose $w_A\,(Z - (I-K) + K) = w_A\left(Z - \frac{I-K}{\alpha_H+(1-\alpha_H)p} + K\right) + k\left(\frac{I-K}{\alpha_H+(1-\alpha_H)p} - (I-K)\right)$ where $k \in (0,1]$. Then, increase $w_A\left(Z - \frac{I-K}{\alpha_H+(1-\alpha_H)p} + K\right)$ by

$$\left(\Delta + \frac{\Delta}{\alpha_H + (1-\alpha_H)\, p}\right)$$

and increase $w_A\,(2\,(Z - (I-K)))$ and $w_A\,(Z - (I-K) + K)$ by 2Δ. For small Δ, none of these changes violate monotonicity or fly by night. Also, the maximand and the break even constraint are unchanged. The IC constraint is weakly relaxed, as all states $w_{GP}\,(x)$ are unchanged except possible $w_{GP}\,(Z - (I-K))$ which can go down, which relaxes the IC constraint.

Now suppose $w_A\,(Z - (I-K) + K) = w_A\left(Z - \frac{I-K}{\alpha_H+(1-\alpha_H)p} + K\right)$. Then, we cannot increase

$w_A\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)p}+K\right)$ faster than $w_A\left(Z-(I-K)+K\right)$ without violating monotonicity. Increasing $w_A\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)p}+K\right)$, $w_A\left(Z-(I-K)+K\right)$, and $w_A\left(2\left(Z-(I-K)\right)\right)$ by the same amount $A\geq 2\Delta$ does not violate monotonicity. There always exists such an A in $\left[2\Delta,\Delta+\frac{\Delta}{\alpha_H+(1-\alpha_H)p}\right]$ such that the maximand and the break even constraint are unchanged. But this A relaxes the IC constraint, since $w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)p}+K\right)$ goes up weakly and $w_{GP}\left(Z-(I-K)+K\right)$ and $w_{GP}\left(2\left(Z-(I-K)\right)\right)$ go down weakly. End Proof.

7.5.2. Proof of optimal contract:

The second issue is how the investor and GP securities should be designed. We will derive the securities under two different cases.

7.5.3. Case 1: $Z-(I-K^*)\leq 2K^*$

This is the case when the GP gets no pay-off if he fails with one project, so $w_{GP}\left(Z-(I-K^*)\right)=0$. For this case, the IC condition reduces to

$$\begin{aligned} & q\left(\alpha_H+(1-\alpha_H)p\right)w_{GP}\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right) \\ > \; & E(\alpha)pw_{GP}\left(2\left(Z-(I-K^*)\right)\right) \\ & +\left(1-E(\alpha)\right)p\max\left(w_{GP}\left(Z-(I-K^*)+K^*\right),pw_{GP}\left(2\left(Z-(I-K^*)\right)\right)\right) \end{aligned}$$

Given a certain expected pay-off $E\left(x-w_{GP}(x)\right)$ to investors, the optimal contract should relax the IC condition maximally without violating the fly by night condition or the monotonicity constraints. Any decrease of $w_{GP}\left(2\left(Z-(I-K^*)\right)\right)$ or $w_{GP}\left(Z-(I-K^*)+K^*\right)$ and increase of $w_{GP}\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$ that keeps the expected value of the security constant relaxes the constraint. The optimal contract is given in the following proposition:

Proposition 6. *: Suppose $Z-(I-K^*)\leq 2K^*$. The optimal investor security $w_A(x)$ is debt with face value $F=w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)\in\left[2K^*,Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$ plus a carry $k\left(\max\left(x-S,0\right)\right)$ starting at $S.\in\left[Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*,Z-(I-K^*)+K^*\right]$. For $F=2K^*$, we have $k\in(0,1)$, $S\in\left[Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*,Z-(I-K^*)+K^*\right]$ and for $F>2K^*$, we have $k=1$ (call option) and $S=Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*$. For a fixed expected value $E\left(w_A(x)\right)$ given to investors, F is set minimal.*

Proof: First, suppose $w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)>2K^*$. The optimal contract in the proposition then claims that

$$\begin{aligned} w_A(Z-(I-K^*)+K^*) &= w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)+(I-K^*)-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p} \\ w_A(2(Z-(I-K^*))) &= w_A(Z-(I-K^*)+K^*)+Z-I \end{aligned}$$

Suppose this is not true. First, suppose

$$\begin{aligned} w_A(Z-(I-K^*)+K^*) &< w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)+(I-K^*)-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p} \\ w_A(2(Z-(I-K^*))) &\leq w_A(Z-(I-K^*)+K^*)+Z-I \end{aligned}$$

Then, we can increase $w_A(Z-(I-K^*)+K^*)$ and decrease $w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$ (which means we decrease $w_{GP}(Z-(I-K^*)+K^*)$ and increase $w_{GP}\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$) to keep the break even constraint and the maximand constant without violating monotonicity. This relaxes the IC constraint and so improves the contract.

Now, suppose

$$\begin{aligned} w_A(Z-(I-K^*)+K^*) &= w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)+(I-K^*)-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p} \\ w_A(2(Z-(I-K^*))) &< w_A(Z-(I-K^*)+K^*)+Z-I \end{aligned}$$

Then, we can increase $w_A(2(Z-(I-K^*)))$ by ε and decrease $w_A(Z-(I-K^*)+K^*)$ and $w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$ by

$$\frac{\varepsilon E(\alpha)^2}{E(\alpha)(1-E(\alpha))+(1-E(\alpha))q(\alpha_H+(1-\alpha_H)p)}$$

to keep the break even constraint and the maximand constant without violating monotonicity. This relaxes the IC constraint and so improves the contract.

Next suppose $w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)=2K^*$. Then, $w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$ cannot be lowered without violating the fly by night condition.

First, note that increasing $w_A(2(Z-(I-K^*)))$ by ε and reducing $w_A(Z-(I-K^*)+K^*)$ by

$$\varepsilon\frac{E(\alpha)}{(1-E(\alpha))}$$

to keep the break even constraint constant leaves the IC constraint unchanged if

$$w_{GP}(Z-(I-K^*)+K^*)>pw_{GP}(2(Z-(I-K^*)))$$

and relaxes it if

$$w_{GP}\left(Z-(I-K^{*})+K^{*}\right)<pw_{GP}\left(2\left(Z-(I-K^{*})\right)\right)$$

. Therefore, if such a transfer does not violate monotonicity, it (weakly) relaxes the IC constraint. Thus, a contract that maximally relaxes the IC constraint keeping the expected value $E(w)$ constant should have

$$w_{A}\left(2\left(Z-(I-K^{*})\right)\right)=w_{A}\left(Z-(I-K^{*})+K^{*}\right)+Z-I$$

if $w\left(X\left(Z,K^{*}\right)\right)>2K^{*}$. However, for such a contract we have

$$\begin{aligned} pw_{GP}\left(2\left(Z-(I-K^{*})\right)\right) &= p\left[2\left(Z-(I-K^{*})\right)-\left(w_{A}\left(Z-(I-K^{*})+K^{*}\right)\right)+Z-I\right] \\ &= pw_{GP}\left(Z-(I-K^{*})+K^{*}\right) \\ &< w_{GP}\left(Z-(I-K^{*})+K^{*}\right) \end{aligned}$$

and therefore the IC constraint is unchanged if we lower $w_{A}\left(2\left(Z-(I-K^{*})\right)\right)$ and increase $w_{A}\left(Z-(I-K^{*})+K^{*}\right)$ slightly so that

$$w_{A}\left(2\left(Z-(I-K^{*})\right)\right)=w_{A}\left(Z-(I-K^{*})+K^{*}\right)+k\left(Z-I\right)$$

where $k<1$. Thus, this contract can be expressed as a carry. End proof.

7.5.4. Case 2: $Z-(I-K^{*})>2K^{*}$

This is the case when the GP can get some pay-off even if he fails with one project, so it is possible to have $w_{GP}\left(Z-(I-K^{*})\right)>0$. It is always optimal to set $w_{A}\left(Z-(I-K^{*})\right)$ as high as possible at $\min\left(Z-(I-K^{*}),w_{A}\left(Z-\frac{I-K^{*}}{\alpha_{H}+(1-\alpha_{H})p}+K^{*}\right)\right)$, so the contract will have a debt piece as before with face value $w_{A}\left(Z-\frac{I-K^{*}}{\alpha_{H}+(1-\alpha_{H})p}+K^{*}\right)$. However, it is no longer true that we want to set this face value as low as possible given a fixed $E(w_{A})$ by increasing the higher pay offs. This is because when we reduce the face value, we also increase the pay off to the GP if he fails with one and succeeds with one firm, which can worsen incentives. The following proposition characterizes the optimal contract.

Proposition 7. *Suppose $Z-(I-K^{*})>2K^{*}$. The optimal investor security $w_{A}(x)$ is debt with face value $F=w_{A}\left(Z-\frac{I-K^{*}}{\alpha_{H}+(1-\alpha_{H})p}+K^{*}\right)\in\left[2K^{*},Z-\frac{I-K^{*}}{\alpha_{H}+(1-\alpha_{H})p}+K^{*}\right)$ plus a carry $k\left(\max\left(x-S,0\right)\right)$ starting at $S.\in\left[Z-\frac{I-K^{*}}{\alpha_{H}+(1-\alpha_{H})p}+K^{*},Z-(I-K^{*})+K^{*}\right]$. For $S<Z-(I-K^{*})+K^{*}$, we have $k=1$ (call option), and for $S=Z-(I-K^{*})+K^{*}$, we have $k\in(0,1)$.*

Proof: We start with the following Lemma:

Lemma 5. $w_{A}\left(Z-(I-K^{*})\right)=\min\left(Z-(I-K^{*}),w_{A}\left(Z-\frac{I-K^{*}}{\alpha_{H}+(1-\alpha_{H})p}+K^{*}\right)\right)$.

Proof. First, note that given $w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$, the highest we can set $w_A\left(Z-(I-K^*)\right)$ is the expression in the lemma from monotonicity and the fact that $Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*>Z-(I-K^*)$ in feasible contracts. Suppose $w_A\left(Z-(I-K^*)\right)$ is lower than this upper bound. Then, we can increase it without changing the break even constraint and the maximand, since the outcome $Z-(I-K^*)$ does not happen in equilibrium. This relaxes the IC constraint and so improves the contract. ■

This proves that the first piece is debt with face value $w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$.

Next, suppose $w_A\left(Z-(I-K^*)+K^*\right)>w_A\left(Z-\frac{I-K^*}{\alpha_H+(1-\alpha_H)p}+K^*\right)$. Then, the proposition states that

$$w\left(2\left(Z-(I-K^*)\right)\right)=w_A\left(Z-(I-K^*)+K^*\right)+Z-I$$

which is the highest possible value for $w_A\left(2\left(Z-(I-K^*)\right)\right)$ given $w_A\left(Z-(I-K^*)+K^*\right)$. Suppose this is not the case. Then, we can lower $w_A\left(Z-(I-K^*)+K^*\right)$ and increase $w_A\left(2\left(Z-(I-K^*)\right)\right)$ to keep the break even constraint and the maximand constant without violating monotonicity. If

$$w_{GP}\left(Z-(I-K^*)+K^*\right)>pw_{GP}\left(2\left(Z-(I-K^*)\right)\right)+2p\left(1-p\right)w_{GP}\left(Z-(I-K^*)\right)$$

this does not change the IC constraint, but if

$$w_{GP}\left(Z-(I-K^*)+K^*\right)<pw_{GP}\left(2\left(Z-(I-K^*)\right)\right)+2p\left(1-p\right)w_{GP}\left(Z-(I-K^*)\right)$$

the IC constraint is relaxed and so this improves the contract. End Proof.

7.6. Proof of Proposition 5:

Proof: First, it is necessary that

$$Z+K-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}>2K$$

or else the left hand side of the IC condition is zero from monotonicity. Second, it is necessary that

$$Z+K-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}>Z-(I-K)$$

since otherwise

$$w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}+K\right)\leq Z+(I-K)-w\left(Z-(I-K)\right)$$

This would violate the IC condition, since in that case

$$E(\alpha)\left[pw_{GP}\left(2\left(Z-(I-K)\right)\right)+(1-p)\,w_{GP}\left(Z-(I-K)\right)\right]+(1-E(\alpha))\,p* $$
$$\max\left(w_{GP}\left(Z-(I-K)+K\right),pw_{GP}\left(2\left(Z-(I-K)\right)\right)+2\left(1-p\right)w_{GP}\left(Z-(I-K)\right)\right)$$

$$\begin{aligned}
&\geq (E(\alpha)+(1-E(\alpha))\,p)\left[pw_{GP}\left(2\left(Z-(I-K)\right)\right)+(1-p)\,w_{GP}\left(Z-(I-K)\right)\right]\\
&\geq (E(\alpha)+(1-E(\alpha))\,p)\,w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}+K\right)\\
&> q\left(\alpha_H+(1-\alpha_H)\,p\right)w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}+K\right)
\end{aligned}$$

These two necessary conditions can be rewritten as

$$\begin{aligned}
\frac{I-K}{\alpha_H+(1-\alpha_H)\,p} &< Z-K\\
Z+K-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p} &> Z-(I-K)
\end{aligned}$$

and

$$\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}<I$$

Note that both these are easier to satisfy for higher K, and for K maximal we get

$$\frac{\alpha_L+(1-\alpha_L)\,p}{\alpha_H+(1-\alpha_H)\,p}Z<Z-\left(I-\left(\alpha_L+(1-\alpha_L)\,p\right)Z\right)$$

and

$$\frac{\alpha_L+(1-\alpha_L)\,p}{\alpha_H+(1-\alpha_H)\,p}Z<I$$

Dividing by Z gives the last expression in the proposition.

The first part from the proposition is proved as follows. The right hand side of the IC constraint is given by

$$E(\alpha)\left(pw_{GP}\left(2\left(Z-(I-K)\right)\right)+(1-p)\,w_{GP}\left(Z-(I-K)\right)\right)+(1-E(\alpha))\,p* $$
$$*\max\left(w_{GP}\left(Z-(I-K)+K\right),pw_{GP}\left(2\left(Z-(I-K)\right)\right)+2\left(1-p\right)w_{GP}\left(Z-(I-K)\right)\right)$$

$$\begin{aligned}
&\geq E(\alpha)pw_{GP}\left(2\left(Z-(I-K)\right)\right)+(1-E(\alpha))\,pw_{GP}\left(Z-(I-K)+K\right)\\
&\geq pw_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}+K\right)
\end{aligned}$$

where the last step follows from monotonicity. Therefore, the IC condition can only be satisfied if

$$q\left(\alpha_H + (1-\alpha_H)\,p\right) \geq p$$

Thus, this is a necessary condition for the equilibrium to be implementable. To show that it together with the other conditions are sufficient, suppose they are satisfied. Then, for ε small enough, it is always possible to set

$$\begin{aligned}
w_{GP}\left(Z-(I-K)\right) &= 0 \\
w_{GP}\left(2\left(Z-(I-K)\right)\right) &= \varepsilon \\
w_{GP}\left(Z-\frac{I-K}{\alpha_H+(1-\alpha_H)\,p}+K\right) &= \varepsilon \\
w_{GP}\left(Z-(I-K)+K\right) &= \varepsilon
\end{aligned}$$

For this contract, the IC condition reduces to

$$q\left(\alpha_H + (1-\alpha_H)\,p\right) \geq p$$

For ε small enough, investors always break even as long as social surplus is created. End Proof.

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