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THE EMPIRICAL FREQUENCY OF A PIVOTAL VOTE

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ABSTRACT

Empirical distributions of election margins are computing using data on U.S. Congressional and state legislator election returns. We present some of the first *empirical* calculations of the frequency of close elections, showing that one of every 100,000 votes cast in U.S. elections, and one of every 15,000 votes cast in state elections, "mattered" in the sense that they were cast for a candidate that officially tied or won by one vote. Very close elections are more rare than the independent binomial model predicts. The evidence also suggests that recounts, and other margin-specific election procedures, are quite relevant determinants of the frequency of a pivotal vote.

Although moderately close elections (winning margin of less than ten percentage points) are more common than landslides, the distribution of moderately close U.S. election margins is approximately uniform. The distribution of state legislature election margins is clearly monotonic, with closer margins more likely, except for very close and very lopsided elections. We find an inverse relationship between election size and the frequency of one vote margins in both data sets over a wide range of election sizes, with the exception of the smallest U.S. elections for which the frequency increases with election size.

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How often are civic elections decided by one vote? This question has been asked numerous times in the economics literature and results derived from economic and statistical models of voting have been offered as answers (eg., Beck 1975, Goode and Mayer 1975, Margolis 1977, Chamberlain and Rothschild 1981, Fischer 1999). The purpose of this paper is to carefully compute *empirical* frequencies of close elections for the first time, and compare the empirical frequencies predicted by some economic and statistical models.

Perhaps it is common knowledge that civic elections are not often decided by one vote.[1] But the costs of voting are often small, so a precise calculation of the frequency of a pivotal vote can contribute to our understanding of how many, if any, votes might be rationally and instrumentally cast. Furthermore, the distribution of electoral outcomes offers some information about the validity of various models of voting. For example, two economic theories of voting suggest that close elections should be more common than predicted by statistical models. One suggests that competing candidates try to position themselves near or at the preferences of the median voter, which makes close elections especially likely. The swing voter theory suggests that voters themselves attempt to make elections close, so that the "right" voters are pivotal.

Perhaps our data are consistent with the view that elections are never decided by a single vote, because election procedures are margin-specific in such a way that the final election returns determining the winning candidate might never exhibit a margin of zero or one votes. More work needs to be done to understand exactly how election returns are calculated, but our study suggests that such procedures are of potential theoretical importance.

Our paper begins by introducing the set of elections studied, and providing an overview

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[1]Although the Federal Election Commission appears to claim otherwise: "'Just' one vote can and often does make a difference in the outcome of an election." (http://www.fec.gov/pages/faqs.htm)

of the returns in those elections. Section II then calculates frequencies of pivotal votes, for the sample as a whole and as a function of election size. Section III compares the empirical frequencies to probabilities calculated in the theoretical literature. Section IV looks at some of the very close elections in some detail and *suggest* that elections *may* never be decided by a single vote. Section V concludes.

I. Data

Two data sets are used. The first is vote counts for elections to the United States House of Representatives for the election years 1898-1992 (hereafter "US returns"). The second is state legislative election returns in the United States for the years 1968-89 (hereafter "STATE returns").

US returns are reported electronically in ICPSR study #6311, and are divided into two samples. One sample is the "exceptions file," containing those 839 U.S. House elections where a minority candidate won, or a Democrat was not opposed by a Republican, or a Republican was not opposed by a Democrat. These 839 elections are omitted from our tabulations. Another 3228 elections were omitted for which vote counts for Democrat or Republican were omitted (7), zero (3081), or otherwise uncontested (140), leaving 16577 US returns for analysis.

STATE returns are reported electronically in ICPSR study #8907. We analyze only contested general elections in both single- and multi-member districts, except for those in multi-member free-for-all districts of which there are 5785. Of the remaining elections, 2,478 are reported as having missing votes (at either the candidate or election level) are which are excluded. Of the remaining 51,262 elections, 11,047 are coded as being uncontested, and another 179 are de facto uncontested once vote totals for candidates with similar names have been combined. This leaves 40,036 elections included in the analysis.[2]

[2]As a further check for missing elections, we examined the chronological intervals between elections for each district. If the intervals between elections were spaced irregularly then these districts were flagged for further inspection. Of the 59,525 general elections in the database, we found that there were 1495 chronological "gaps." Upon inspection of these gaps we found that 586 were cases in which a senate election was held at a two year interval rather than the usual four year interval; 280 were due to specially ordered state-wide elections; 129 were due to redistricting (e.g., changing from a single-member district to a multi-member district with positions); leaving 24 for which we found no explanation in either the raw data

As reported by the ICPSR, New York STATE candidates running under multiple party labels are listed multiple times, with their votes under each label listed separately. Since the election outcome depends each candidate's votes under all labels, we aggregated vote totals, in each district and for each post, by name. In 173 cases (out of 93,142 candidate records for the national sample of 40,036 elections satisfying the above selection criteria), we assume that names spelled very similarly were the same candidate. We compute the election margin as the number of votes cast for the winner with the least votes (usually, but not always, there was one winner) minus the number of votes cast for the loser with the most votes.

Summary statistics for both samples are displayed in Table 1. The US returns are for many more years, as compared to STATE returns, but many fewer districts each year. The median election in both samples is not close by any definition; the margin is 22% or 25% of total votes cast. STATE elections are substantially smaller, both in terms of total votes cast and absolute margins. The STATE distribution of total election votes seems to be more skewed than the US distribution, with the STATE average total votes 1.7 times median total votes (compare with 1.1 times for US elections).

or in the codebook. Among those 24 gaps, we have so far checked three in the newspaper and all three were verified to be special cases when elections were not held.

Table 1: Sample Characteristics

	US	STATE
elections	16,577	40,036
years	1898-1992 (even)	1968-1989
elections in 1988	354	3,485
districts in 1988	354†	3,485
median vote margin	18,021	3,256.5
median percentage margin	22	25
total votes:		
min	2,723	25
median	105,987	13,665.5
average	111,370	23,658
max	1,663,687	388,143
addendum: uncontested elections (dropped from sample)	3,221	11,226

†19 districts were omitted in 1988 because votes were not reported for the Democratic or Republican candidate. 62 more districts were omitted because either Democratic or Republican received zero votes.

Since the samples include several years, two rows in our table are provided to give the reader some information about the cross-section dimension of our data. Those two rows are the number of elections and districts sample in 1988, the most recent election year included in both datasets. The STATE sample has about ten times more districts in its typical cross section than does the US sample. We also see that each district has exactly one election in 1988.

II. The Empirical Distribution of Election Returns

Before comparing the returns data with particular voting theories, we first present the overall relationship between election margin and frequency. We then offer some calculations of the empirical frequency of a pivotal vote, and how that frequency varies with election size.

II.A. The Overall Relationship between Margin and Frequency

Figure 1a shows how the modal percentage US election margin (ie, the absolute election margin divided by the sum of votes cast for Democratic and Republican candidates) is four or five percent.[3] Beyond four or five percent, the density of percentage election margin appears to decline monotonically. Interestingly, as we look closer at the density in the range 0-4 percent, as in Figure 1b, the density seems to be pretty near uniform except exactly at zero. Looking closer still at elections decided by less than 100 votes (Figure 1c), margins of less than 25 seem to be particularly rare.

Figure 2a shows how the STATE distribution of percentage election margins is bimodal, with one mode near zero and another near 100%.[4] The distribution appears broadly uniform for elections decide by less than 10%. Looking closer still at elections decided by less than 100 votes, perhaps margins of less than 25 are more rare, although this is not as noticeable as with the US elections. Figure 2d shows margins for STATE elections decided by less than 20 votes, and suggests that margins of less than 5 are the most rare.

II.B. What is a Pivotal Vote?

To begin, we refer to a "pivot election" as one in which the winner's official and final vote total exceeded the loser's official and final vote total by no more than one vote. A "pivotal vote" is a vote cast for the winner in a pivot election, or cast for either candidate in a tied election. In other words, a "pivotal vote" is one for which it appears an election outcome would have been different if it had been cast differently.

Our defining "pivotal vote" may seem to be unnecessary, and verbose. But one contribution of our paper is to present some facts suggesting that, even when the official and

[3]If uncontested elections had been included, 100% would have been the modal percentage margin.

[4]Uncontested elections are omitted from the sample, but the histogram shows how there quite a few barely contested elections.

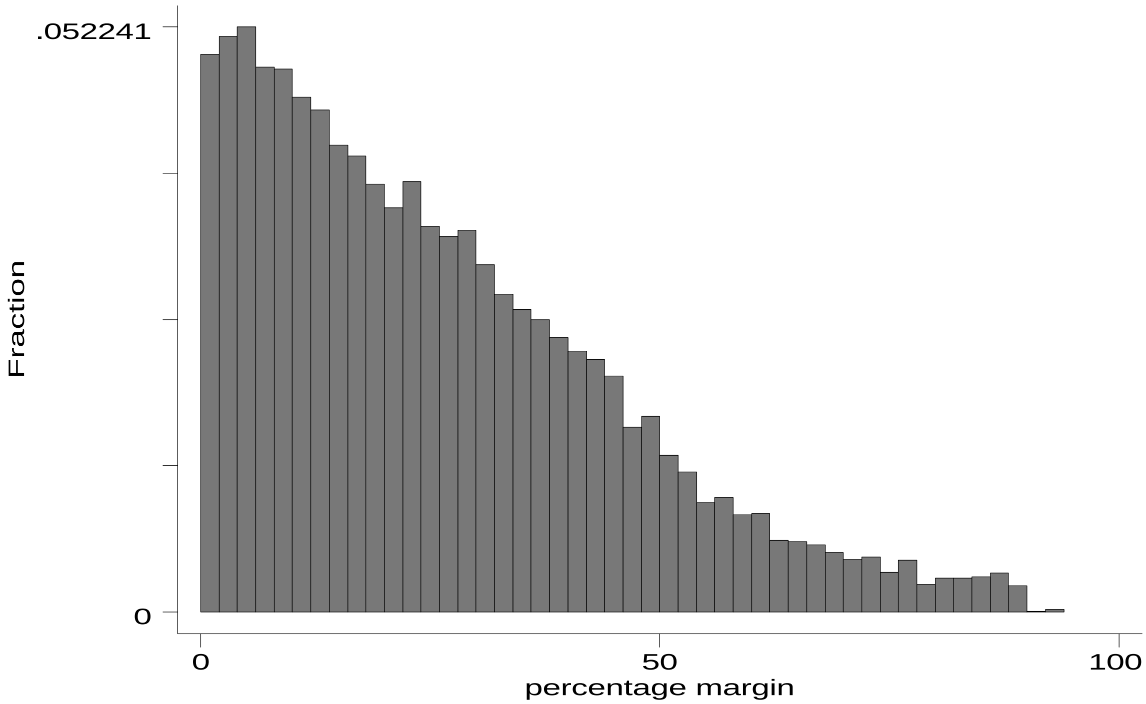

Figure 1a: all contested US elections

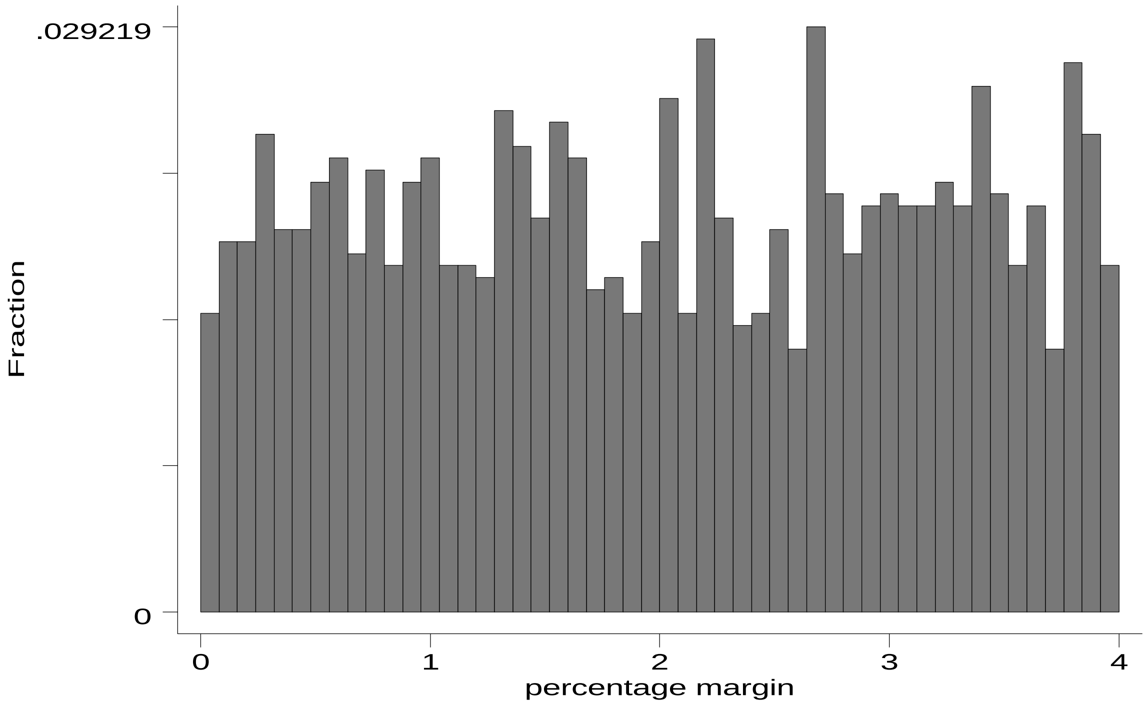

Figure 1b: US elections within 4%

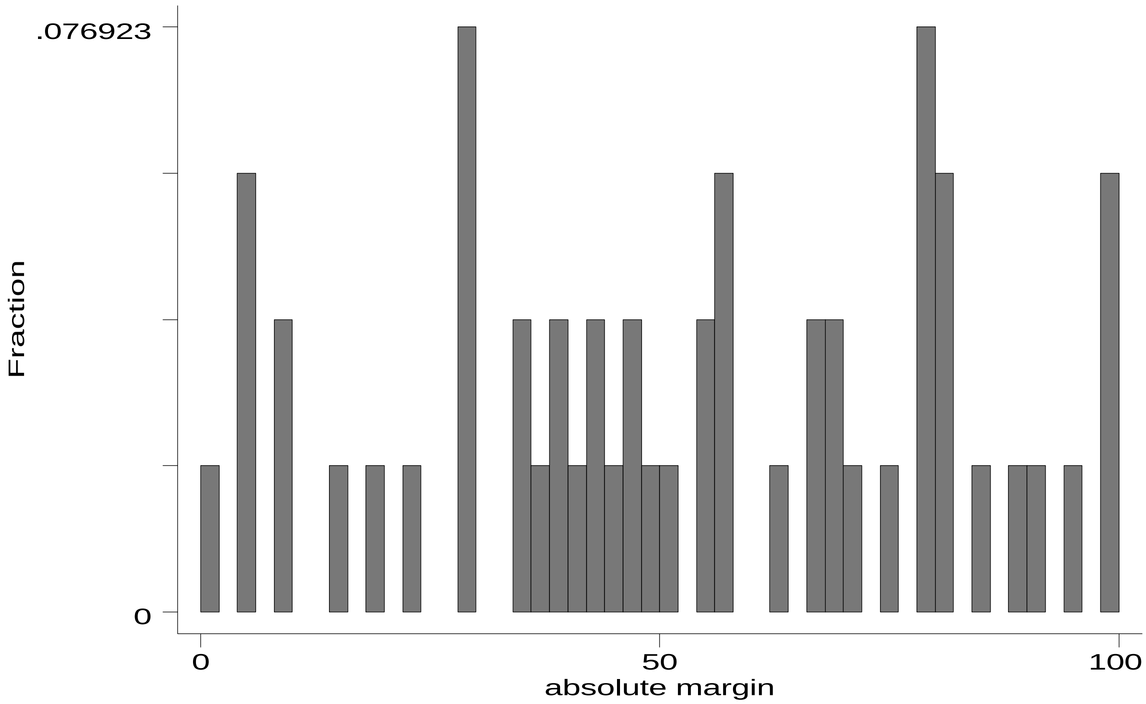

Figure 1c: US elections within 100 votes

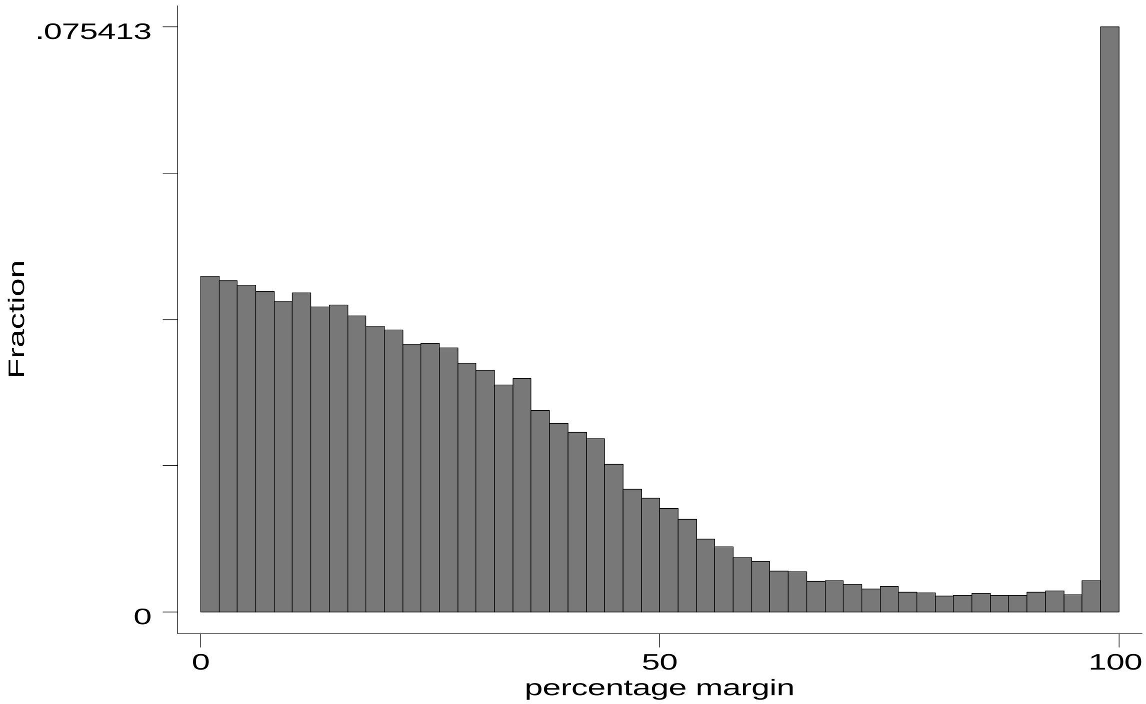

Figure 2a: all contested STATE elections

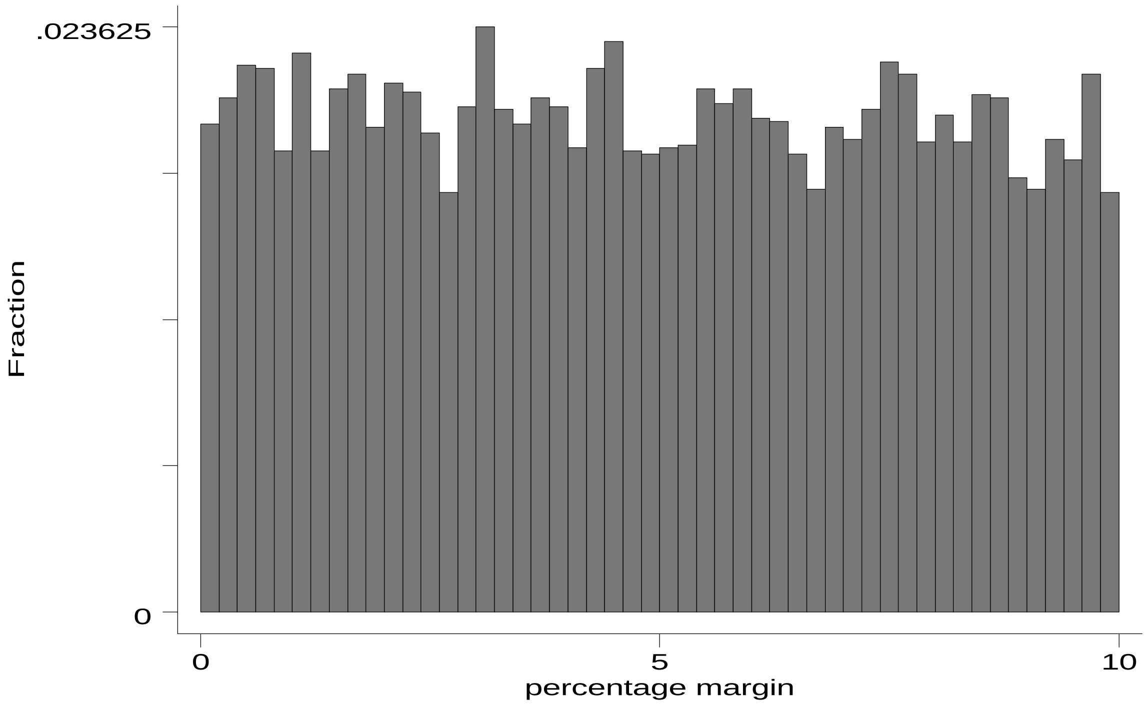

Figure 2b: STATE elections within 10%

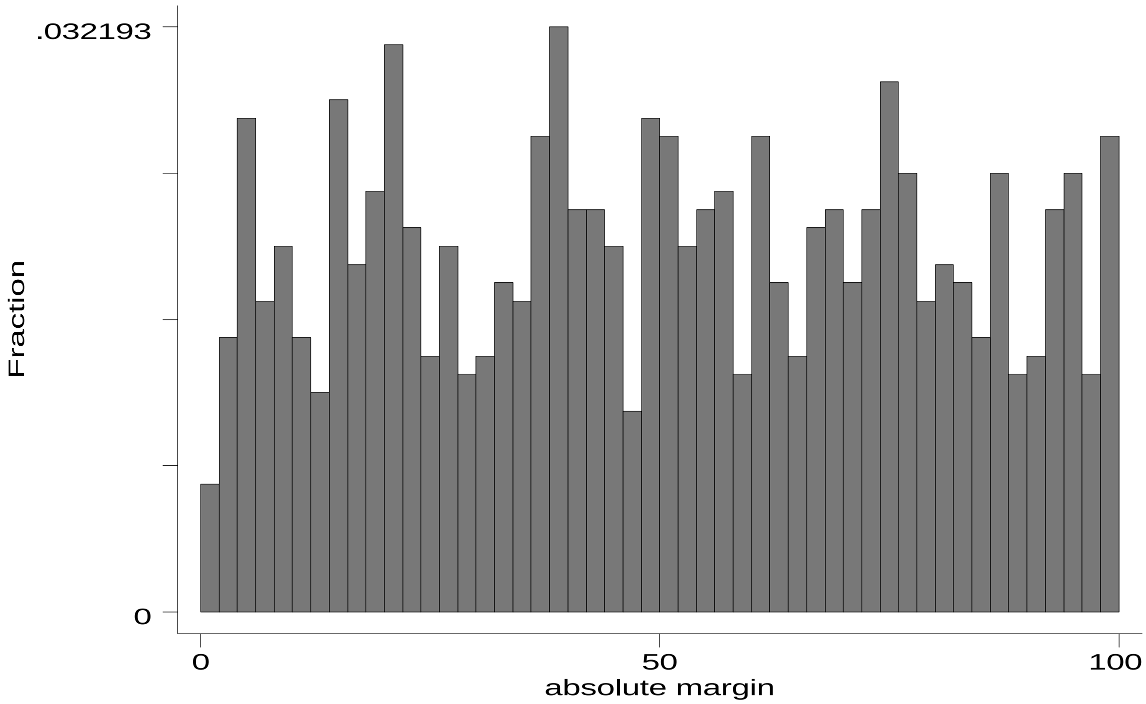

Figure 2c: STATE elections within 100 votes

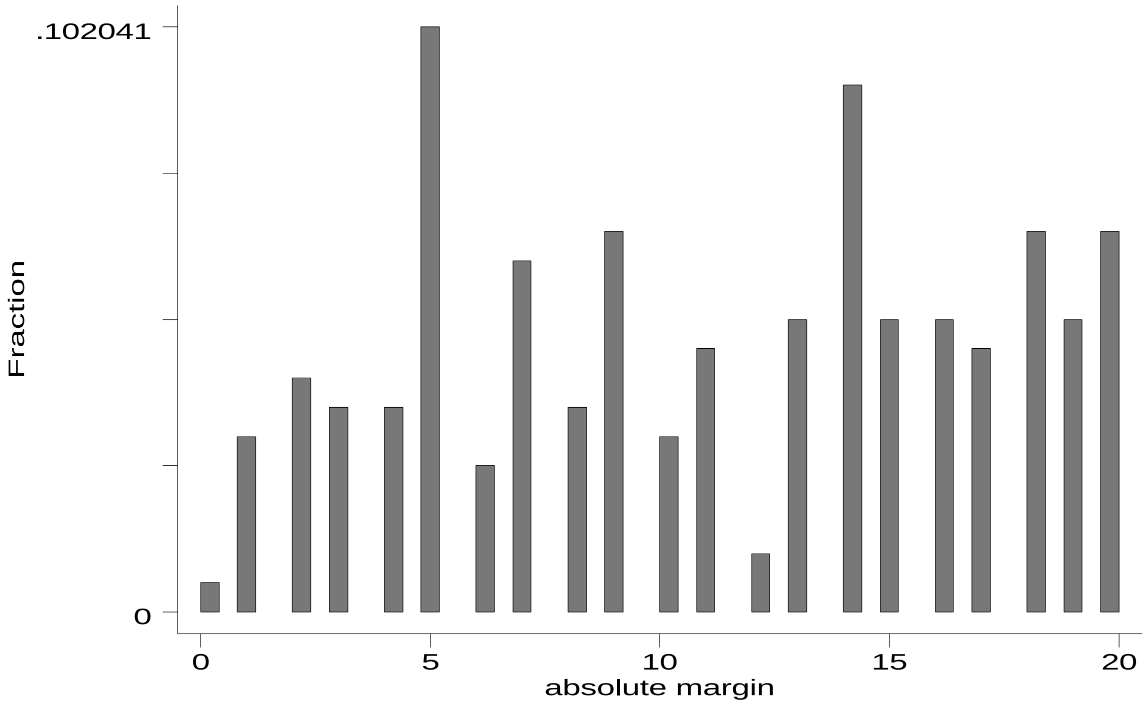

Figure 2d: STATE elections within 20 votes

final returns are known, whether or not the election was a pivot election is unclear because the procedures for determining the official and final returns are both unclear and margin-specific. We discuss this in some detail in Section IV, and for now stick with our definition above. For now, we also assume that the vote totals reported in the ICPSR studies are official and final.

Most economic models of voting assume that voters treat their vote as if it had instrumental value, and therefore cast for the candidate whose policies are expected to yield the voter higher income or utility. Pivot elections are interesting because those are the elections in which, at least with perfect foresight, a vote indeed had instrumental value. Of course, a voter typically does not know the precise election returns before he casts his ballot, so a pivot election is only a subjective probability from his point of view, and the expected instrumental value of his vote the product of that probability and the instrumental value of a vote cast in a pivot election. Presumably this probability varies across votes, because those votes are cast in elections of different sizes and different expected closeness and, within elections, cast by different voters with perhaps different assessments of the probability.

It would be nice to know how the subjective probability of a pivotal vote varies according to election and voter characteristics. We attempt to empirically describe some of those variations, but we begin by calculating the empirical frequency of a pivotal vote in our STATE and US elections samples. If subjective probabilities are unbiased, this empirical frequency might be interpreted as an estimate of the average subjective probability in the population of elections with similar election and voter characteristics as those in our samples.

II.C. Calculations of the Frequency of a Pivotal Vote

Of the 16577 US returns analyzed, only one (1/16577=0.00006) was decided by a single vote.[5] 41369 votes were cast in that election, or a 0.00002 share of all of the votes in the US elections analyzed.[6] Two others were decided by 4 votes, one by 5, and two by 9 votes. All 16577

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[5]That election was in 1910 for the Representative for New York's 36th congressional district, with the Democratic candidate winning 20685-20684. See Section IV for more details on this and other apparently pivot elections.

[6]Of course, 20684 of those votes "did not matter" in the sense that they were cast for the losing candidate, so we are left with a 0.00001 share of all of the votes in U.S. elections

(16571/16577=0.9996) others were decided by at least 14 votes. Of the 40036 STATE elections (with almost one billion votes) analyzed, two were tied and seven were decided by a single vote (9/40036=0.0002). 61,328 votes were cast in those nine elections, or a 0.00006 share of all of the votes in the STATE elections analyzed.

These calculations do not make use of two dimensions of the data that are potentially informative about the subjective probability of a pivotal vote: the frequency of "close" (but not as close as 10 votes) elections and relationships between margins and election size. With a simple model of the probability of a pivotal vote, we can use results from other close – but not pivot – elections to increase the precision of our estimates. For example, suppose that the subjective probability of margin m is independent of m for $m \in [1,M]$, for M very small relative to total votes cast,[7] and that the probability of a tied election is half of that.[8] Then two estimates of the frequency of a pivotal vote (ie, the probability of m=0 or m=1) are of interest, q and p:

$$q(M) = \frac{3}{2(M+1)} \frac{\sum_i [x_i(M) + x_i(0)]}{\sum_i 1} \quad , \quad p(M) = \frac{3}{2(M+1)} \frac{\sum_i v_i [x_i(M) + x_i(0)]}{\sum_i v_i}$$

$$\text{where} \quad x_i(M) = \begin{cases} 1 \text{ if } m_i \in [0,M] \\ 0 \text{ otherwise} \end{cases}$$

and where m_i and v_i denote the margin and total votes, respectively, in the ith election. p and q

that mattered.

[7]In elections with thousands of votes cast, it seems that voters cannot, before the votes are tallied, have a much different subjective probability for the outcome of m=1 or m=2, or m=3, etc. Can a voter in an election with 1000 votes really claim to know that the outcome 504-496 any more (or less) likely than 502-408? Hence, as long as M is small relative to total votes cast, we assume the subjective probability of margin m is independent of m for $m \in [1,M]$.

[8]With an even number v of voters, there only one way Democrat and Republican can tie (namely, both get $v/2$), but two ways they can differ by two votes (namely, D gets $(v/2)$-1, or $(v/2)$+1), two ways they can differ by four votes (namely, D gets $(v/2)$-2, or $(v/2)$+2), etc. When v is odd, there are two ways to differ by one vote, two ways to differ by three votes, etc.

are 1.5/(M+1) times a ratio of sums. q's ratio of sums is the sample frequency of elections with margin less than or equal to M, counting tied elections twice, and dividing this ratio by (M+1) gives us an estimate of the probability that an election is decided by exactly, say, 1 vote. Multiplying by 3/2 yields an estimate that an election is either decided by 1 or decided by 0 votes. For example, if $M = 9$, we have $q(M) = (3/20)(90/40036)=0.0003$ in the STATE returns data because there are 86 elections decided by 1, 2, 3, 4, 5, 6, 7, 8, or 9 votes and two tied elections, which means, in expectation, 9 (=90/10) elections would be decided by exactly 1 vote, 9 by exactly 2 votes, 9 by exactly 9 votes, and 9 by exactly 4 votes – or 13.5 (=9*3/2) elections decided by exactly 0 or one votes.

p differs from q in only that it weights by votes cast in the election. Hence, while q is an estimate of the probability that a randomly chosen election will be decided by zero or one votes, p is an estimate of the probability that a randomly chosen *ballot* is cast in an election decided by zero or one votes. Hereafter, we refer to q as the "unweighted frequency" and p as the "vote-weighted frequency."

If all sample elections had the same number of votes cast, it would be straightforward to compute standard errors for p and q, since both are averages of a binomial variable x. But sample elections do vary in size, so we need to say something about how x's success probability – ie, the probability of margin m for some $m \in [1,M]$ – varies with election size v. Consider two possible assumptions: (a) the probability of margin m is independent of v and (b) the probability of margin m varies inversely with v. In either case, it is straightforward to compute the standard error, or t-ratio, of q or p. With assumption (a), the expression for q's t-ratio $t_q(M)$ is quite intuitive.

$$t_q(M) = \sqrt{\frac{N}{\frac{3}{2q(M)}\frac{M+2}{(M+1)^2} - 1}} \approx \sqrt{\frac{Nq}{3}(2M+1)}\,\frac{M+1}{\sqrt{(M+2)(M+1/2)}}$$

$$t_p(M) = \sqrt{\frac{N}{\frac{3}{2p(M)}\frac{M+2}{(M+1)^2} - 1}} \approx \sqrt{\frac{Np}{3}(2M+1)}\,\frac{M+1}{\sqrt{(M+2)(M+1/2)}}$$

The same is true for p with assumption (b), in which case we denote the t-ratio as $t_p(M)$. In both cases, because p and q are so much less than one, the squared t-statistic can be very closely approximated by a product of two terms. The first term is the expected number of elections (unweighted for q, and vote weighted for p, and counting tied elections twice) where the margin is less than or equal to M. The second term is a function of M only, and is approximately one. Hence, the confidence intervals for p and q can, relative to those for $p(1)$ and $q(1)$, be shrunk substantially by assuming the probability of a margin m does not vary on $[1,M]$, even for M quite small, and thereby including more "close" elections in the calculation. For example, for $M = 1$ we have only 7 close STATE elections, but we have 92 for $M = 10$ and 196 for $M = 20$.

Figures 3a and 3b graph $p(M)$ and $q(M)$ vs M for the STATE data, and also display 95% p and q confidence intervals as $p(M)[1 \pm 2/t_p(M)]$ and $q(M)[1 \pm 2/t_q(M)]$, respectively. We see that p and q rise with M for M between 0 and 50, which is another version of what we see from the histograms (Figures 1 and 2), namely that margins of 0 - 4 are substantially less like than, say, margins of 20-24. We also see the confidence intervals shrink with M but, given the sensitive of p and q to M for M less than 10, perhaps the increased confidence is not enough to justify using M larger than 4.

There are at least 2 STATE elections with margin m, for any $m \in [1,50]$, but there are much fewer very close US elections. For example, there were no US elections that tied, or were decided by 2, 3, 6, 7, or 8 votes. Hence a much larger M is needed to estimate the frequency of a pivotal vote from the US sample with some confidence. Figures 4a and 4b graph $p(M)$ and $q(M)$ vs M for the US data on $M \in [0,100]$, and also display 95% p and q confidence intervals. M does not affect the point estimates as noticeably as in the STATE sample.

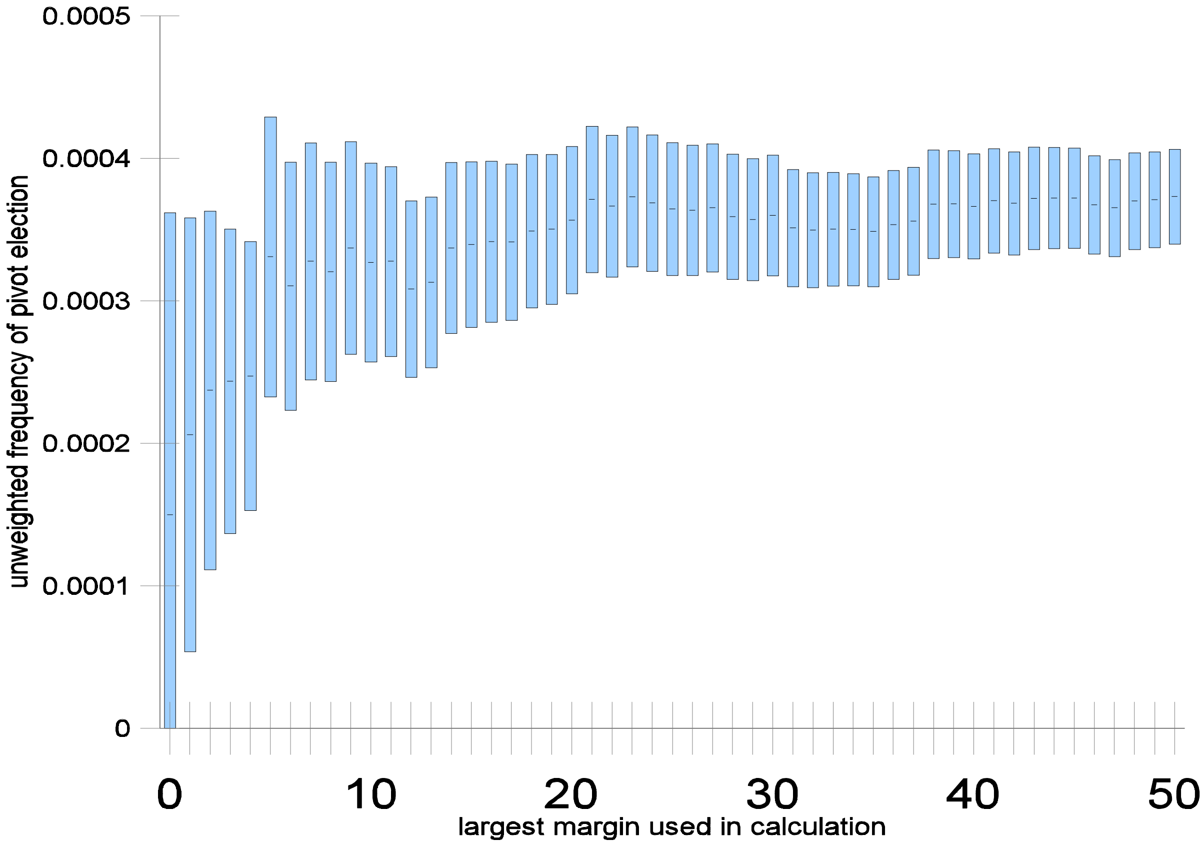

Figure 3a STATE Calculations of the Unweighted Frequency of a Pivot election

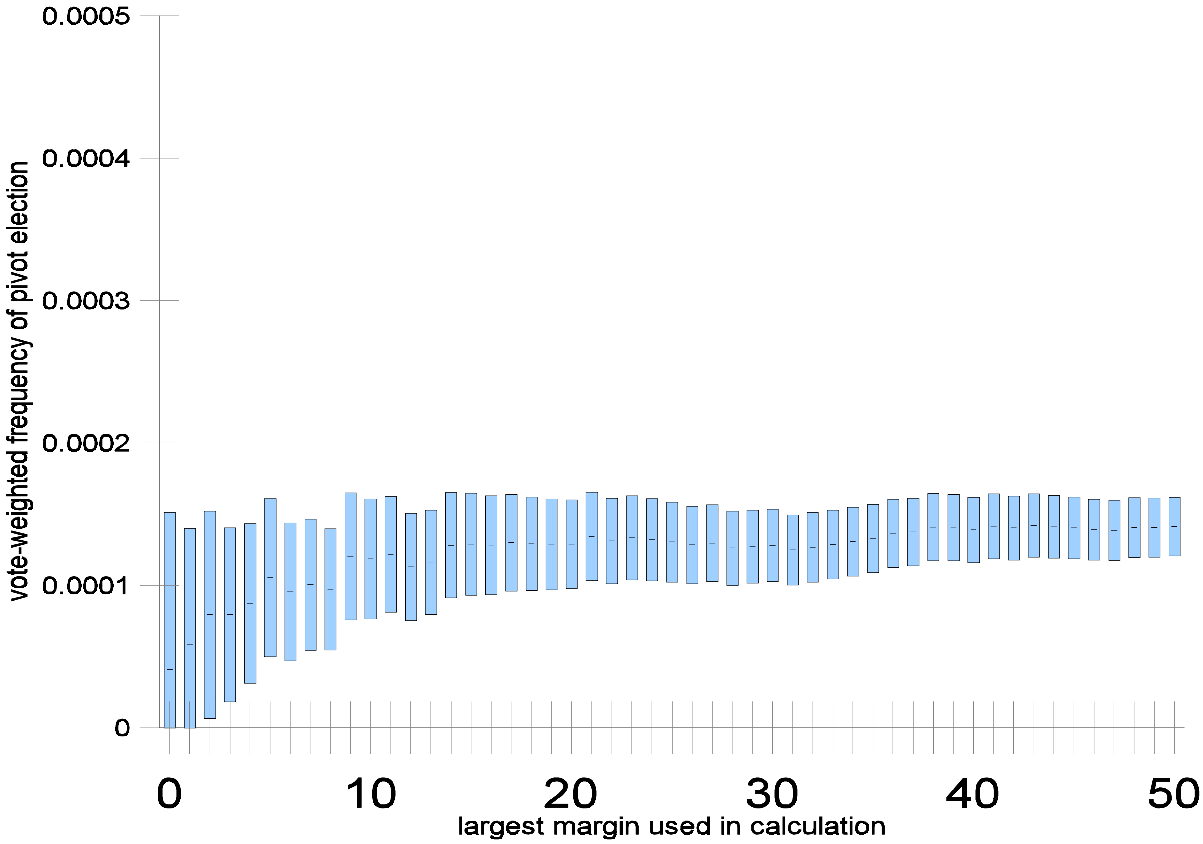

Figure 3b STATE Calculations of the Vote-Weighted Frequency of a Pivot election

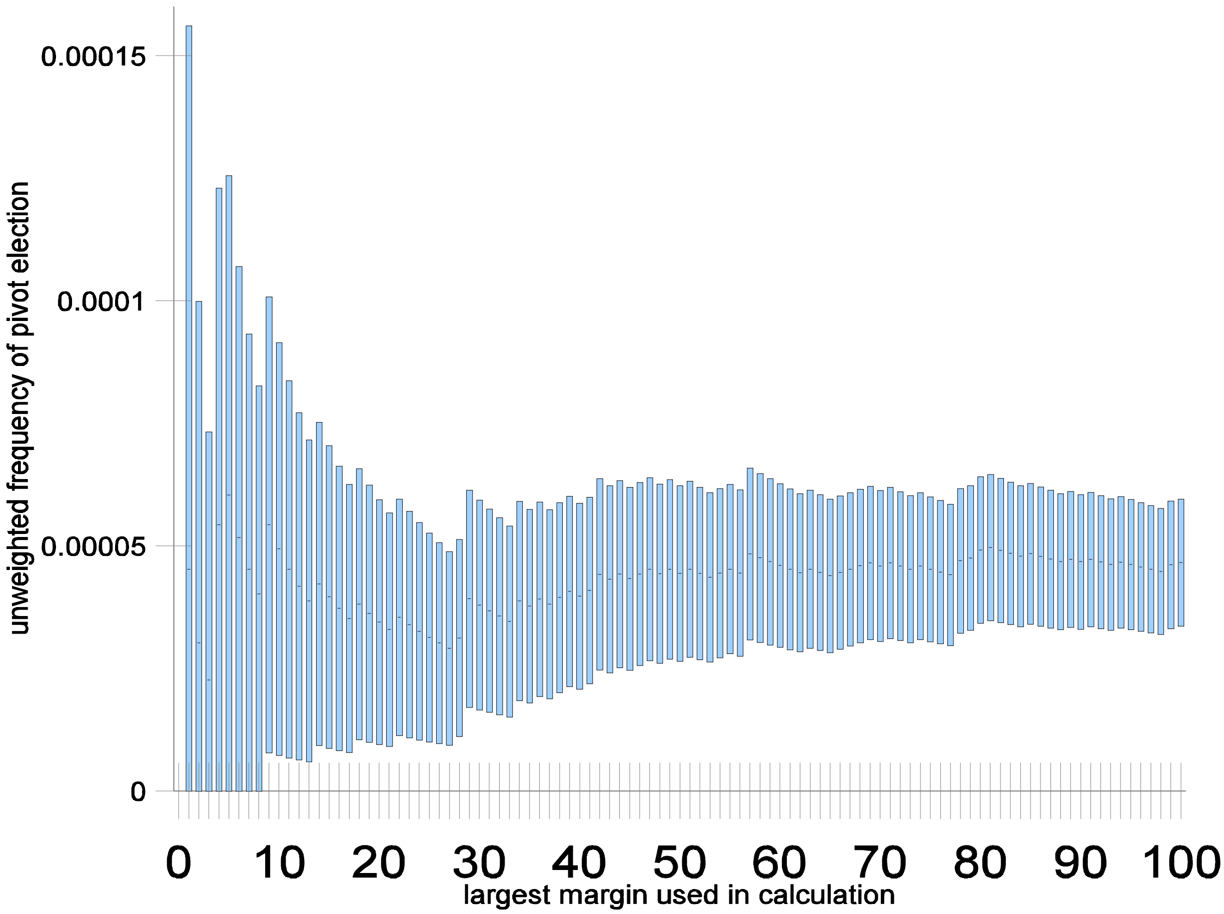

Figure 4a US Calculations of the Unweighted Frequency of a Pivot Election

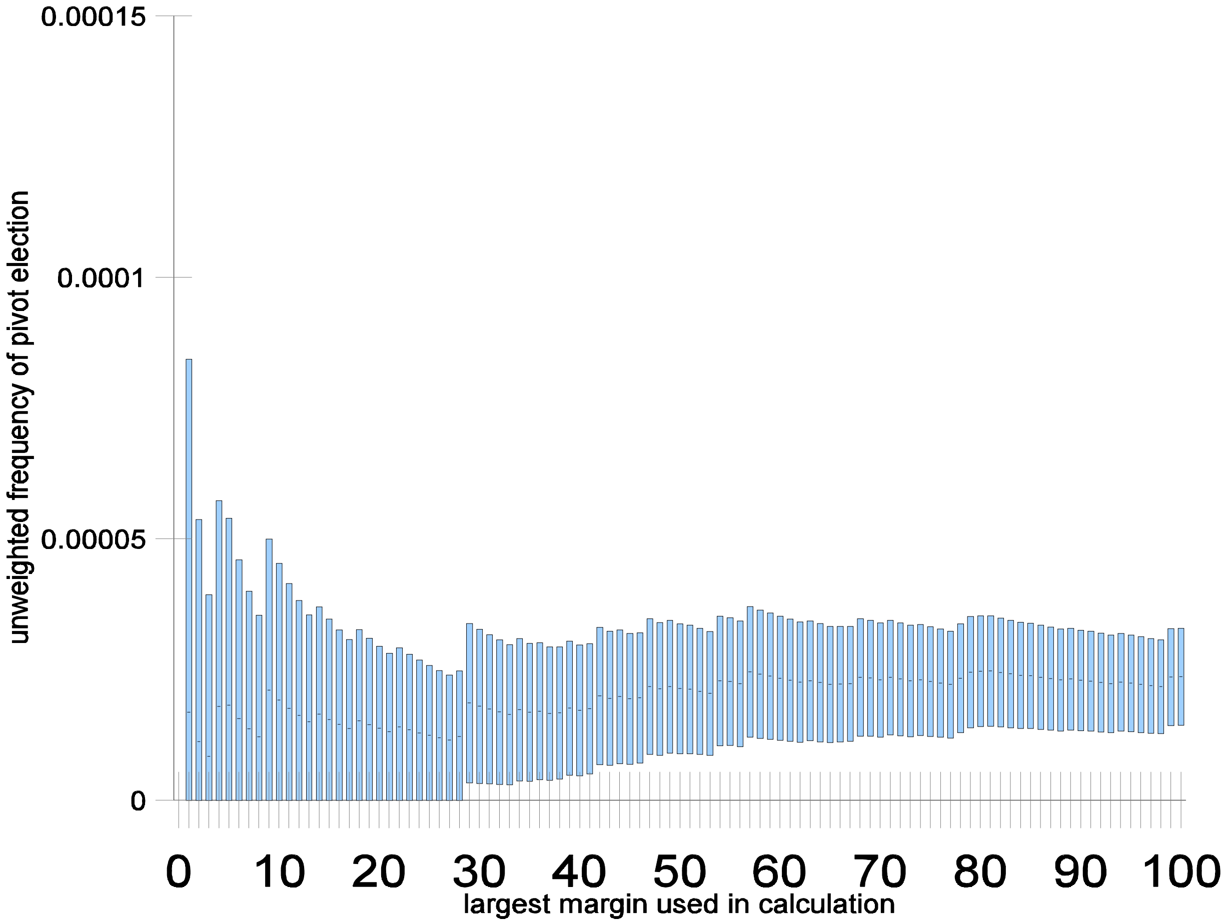

Figure 4b US Calculations of the Vote-Weighted Frequency of a Pivot Election

In calculating the confidence intervals for our estimate of the average subjective probability in the population of elections with similar election and voter characteristics as those in our samples, we assume a form of "independence" across elections. More specifically, we assume that, in a set of elections with the same election and voter characteristics, the factors determining whether or not one of those elections has $m \leq M$ are independently determined across elections. Perhaps the more interesting inference regards the average subjective probability in a subpopulation of elections, such as the group of elections with 10,000 votes cast, to which we turn below.

II.D. The Frequency of a Pivotal Vote as a Function of Election Size

Figures 3a and 3b suggest that using M larger than 4 may be misleading, but there were only 29 STATE and 3 US elections decided by a margin of 4 or less, so only a little can be said about the relationship between election size and the frequency of pivotal vote without considering $M > 4$. For example, we can divide the STATE sample into 3 subsamples, with equal numbers of elections, by total votes in the election and compute q separately for each subsample. Table 2 reports the results:

Table 2: Pivot Election Frequency by Election Size (STATE)

subsample (each with 13345 or 6 elections)	total votes in election				$\times 10^{-4}$		
	min	avg	max	elections w/ $m \leq 4$	$q(4)$	$\sigma_q(4)$	$p(4)$
small	25	5,570	9,181	18	4.5	1.1	3.8
medium	9,183	14,236	21,398	12	2.7	0.9	2.4
large	21,399	51,169	388,143	1	0.2	0.2	0.1

Not surprisingly, we see that the frequency of a pivotal vote falls with election size, and the differences across STATE subsamples are statistically significant (standard errors for $q(4)$, $\sigma_q(4)$, are reported in the end-to-last column). Interestingly, the relationship does not appear to be $1/v$, where v is total votes cast in the election. For example, the average election in the medium subsample is almost 3 times as large as the average small election, but the unweighted frequency of a pivotal vote is only 2/3 as large, and the weighted frequency only 3/4 as large. $1/v$ also misses the differences between the medium and large subsamples, although in the other direction, where votes differ by a factor of 3.6 and pivotal frequencies by a factor of 10 or more. However, it should be noted that q or p of 0.5 is not outside the confidence interval for the large subsample.

To analyze more subtle relationships between election size and the frequency of a pivotal vote, we need to expand M beyond 4. We set $M = 100$ for the STATE sample, so that there are

996 elections with $m < M$, and make 50 equally sized subsamples by total votes. Estimates of q,[9] and a confidence interval for q are displayed in Figure 5, as functions of election size, for each of the 50 subsamples. Both axes are on a logarithmic scale, and the relationship is apparently linear with slope equal to -1, so the probability of pivotal vote appears to vary with $1/v$ rather than e^{-v}. Figure uses $M = 100$, and suggests a different functional form for the relationship between q and M than suggested by Table 2, which uses $M = 4$.

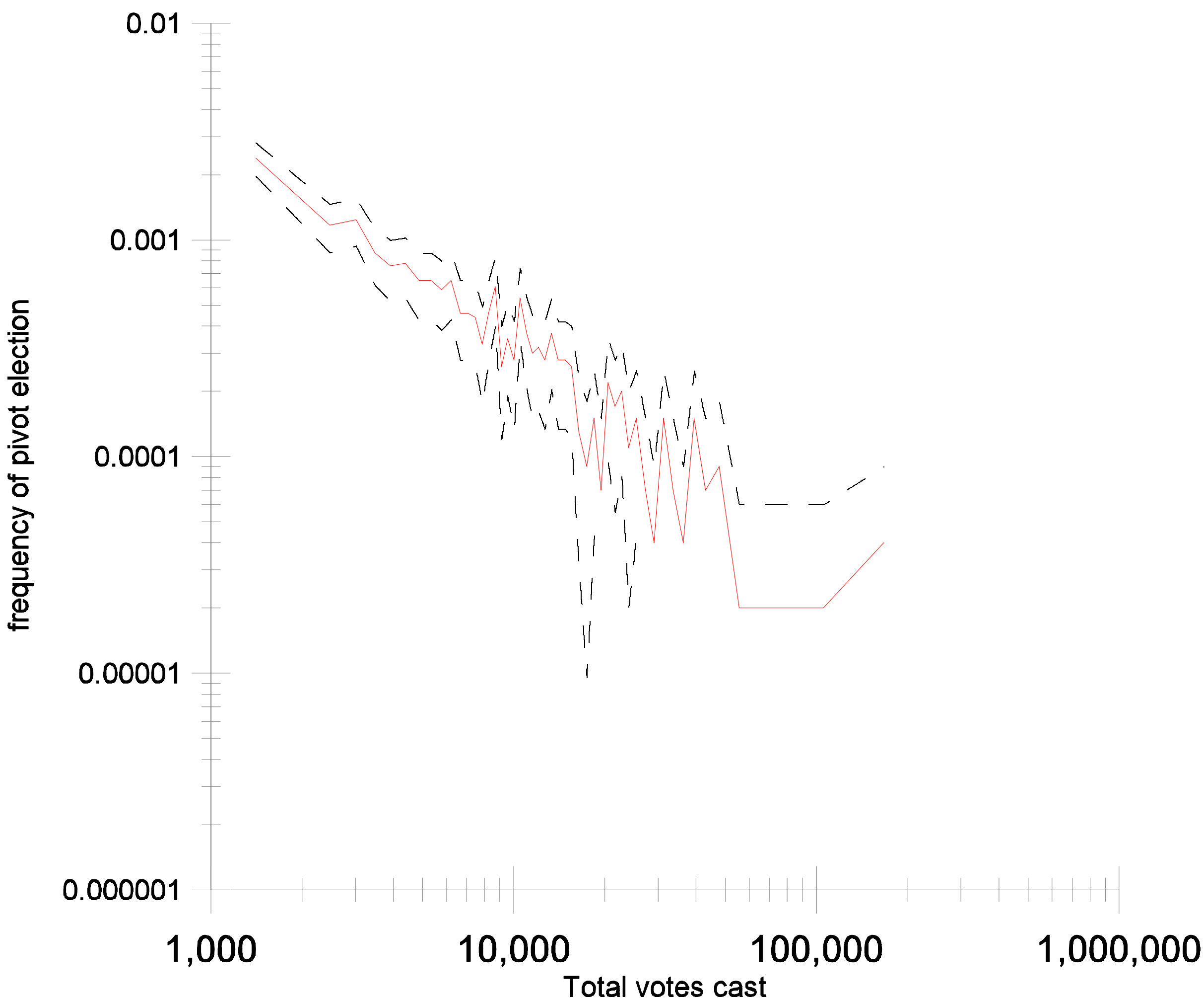

Figure 5 Frequency of a Pivot STATE Election, as a function of Votes Cast

We set $M = 500$ for the US sample, so that there are 304 elections with $m < M$, and make

[9]Since elections are grouped by total votes, any subsample's weighted (p) and unweighted (q) frequencies are very similar.

34 subsamples by total votes.[10] Estimates of q, and a confidence interval for q are displayed in Figure 6, as functions of election size, for each of the subsamples. Both axes are on a logarithmic scale. Notice that the US sample has larger elections than the STATE sample, with little overlap in terms of total votes cast between Figures 5 and 6. For elections with more than 30,000 or 40,000 votes, the U.S. relationship between pivot election frequency and total votes cast is apparently linear with slope equal to -1, as with the STATE elections. However, the frequency of a pivot US election *increases* with total votes cast for elections smaller than 30,000 or 40,000 votes.[11] We speculate that the increase is due to a large response of voter turnout to closeness, but leave verifying our conjecture to future research.

[10]The subsamples are not equally sized, but rather are larger with more total votes.

[11]The increase over that range can also be seen if we set M=100, or smaller.

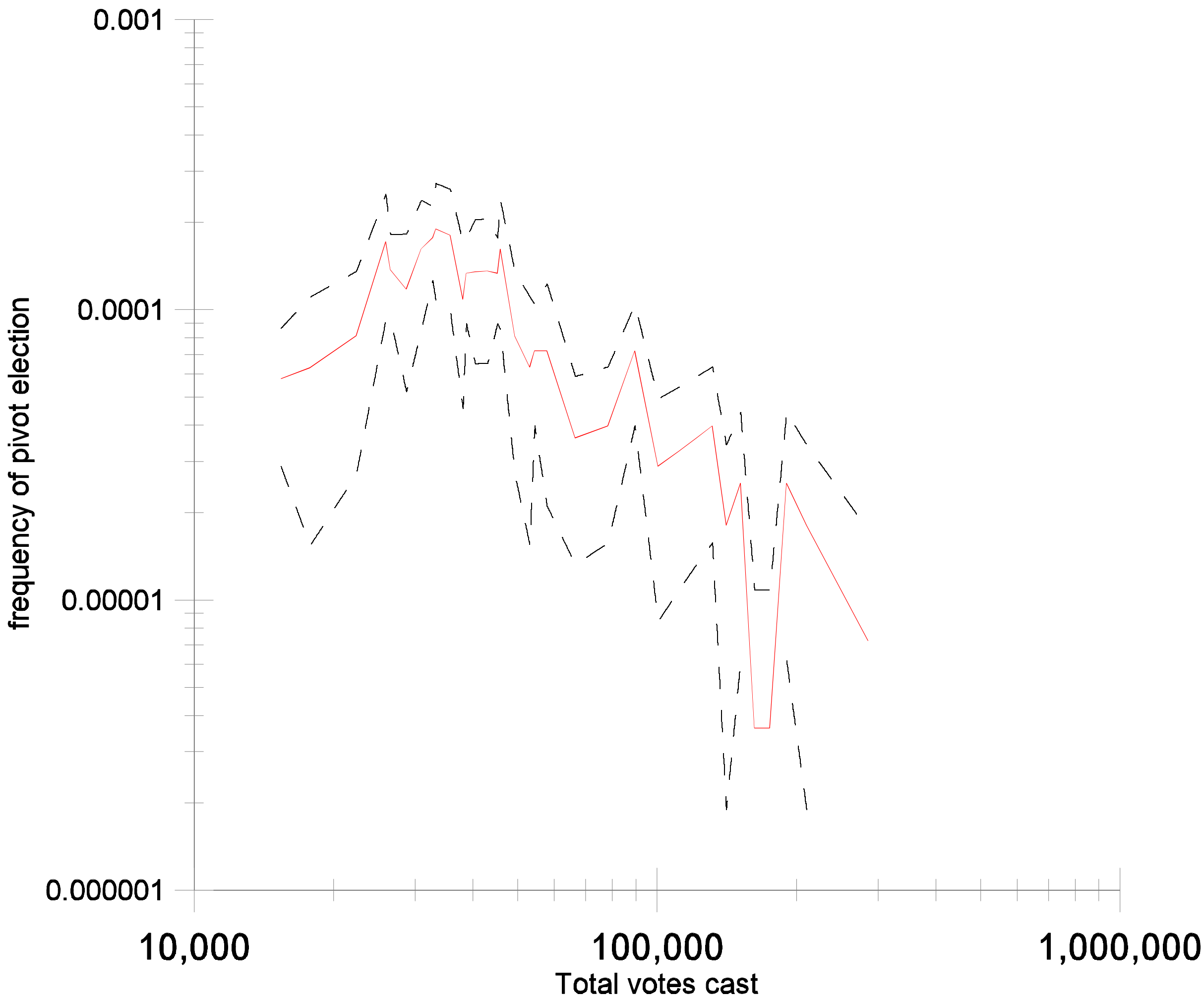

Figure 6 Frequency of a Pivot US Election, as a function of Votes Cast

One practical use of the frequencies calculated in Figures 5 and 6 is to estimate the average subjective probability for an election of a particular size. It should be noted that, for this purpose, the standard errors calculated in Table 2, Figure 5, and Figure 6 require a form of "independence" across elections. More specifically, we assume that, in our sample of elections with the same turnout, the factors determining whether or not one of those elections has $m \leq M$ were independently determined across elections. Might it be that there are, say, district effects on closeness conditional on turnout, so that our independence assumption is violated because we have multiple elections from the same district? This is hard to gauge with our data because there are not so many close elections in our data, but the cross-district incidence of uncontested elections suggests yes. Namely, conditional on voter turnout, some districts are more like to have an uncontested election.[12] Hence we suggest that readers interpret our confidence intervals with some caution.

III. Modeling Votes as Independent Binomial Variables

III.A. Theoretical Calculations of the Probability of a Pivot election

The most common voting model in the literature used to study the probability of a pivotal vote might be called the "independent binomial" model. The model election is between two candidates, say, Democrat and Republican. There is a fixed number of voters v. A vote for the Democrat is denoted z=1 and for z=0 the Republican. Denoting as z_j the vote of the jth voter ($j \in \{1,\ldots,v\}$), the election margin is (as a share of total votes cast):

$$\left| 2\frac{\sum_j z_j}{v} - 1 \right|$$

Modeling each vote as independently drawn from a binomial distribution with "success probability" Π, the sum (ie, aggregate Democrat votes) is the sum of i.i.d. binomial variables with expectation $v\Pi$ and variance $\Pi(1\text{-}\Pi)v$. The expected Democratic percentage margin is $\Pi\text{-}(1\text{-}\Pi) = 2\Pi\text{-}1$.

Computing the probability that Democrat votes total exactly $v/2$ is straightforward, although extremely tedious and difficult to relate precise to one of the key parameters, v:

$$Pr\left\{ \sum_j z_j = v/2 \,\middle|\, \Pi, v \text{ even} \right\} = \binom{v}{v/2} \Pi^{v/2}(1-\Pi)^{v/2} \tag{1}$$

[12]About half of our 40,036 STATE elections have an election margin of less than 25% of the total votes cast. Among these, the 5040 district-office fixed effects predict 30% of the variance of the percentage margin, with an adjusted R-squared of 0.06. The F-statistic is 1.27 for the null hypothesis that district-office fixed effects have no joint predictive power, with critical value 1.00. These calculation ignore the bias involved with selecting elections based on their percentage margin, but suggests that district-office fixed effects are of some limited help in predicting closeness.

This is the probability of a pivotal vote conditional on the success probability Π. Whether or not voters know the success probability has been discussed in the literature (eg., Good and Mayer 1975, Chamberlain and Rothschild 1981, Fischer 1999), and in the later case the probability (from the perspective of a voter) is computed by integrating (1) over a prior distribution F for Π:

$$Pr\left\{ \sum_j z_j = v/2 \,\middle|\, v \text{ even} \right\} = \int_0^1 \binom{v}{v/2} \Pi^{v/2}(1-\Pi)^{v/2} dF(\Pi) \tag{2}$$

Of course, we do not know Π for any one of our 56,613 elections, let alone know what voters in each election knew about Π. But, in order to generate predictions for the occurrence of pivot elections in our data, the relevant calculation is functionally equivalent to a voter's: integrate over a prior distribution for Π. The only difference is in the interpretation, with the voter's having expectations about Π for his election, and our having priors for the distribution of Π across elections.

Interestingly, not much has to be known about F to compute the unconditional probability of a pivotal vote, because (2)'s integrand can be neglected everywhere except in the neighborhood of Π = ½. So (2) simplifies to (3):

$$Pr\left\{ \sum_j z_j = v/2 \,\middle|\, v \text{ even} \right\} \approx f(1/2) \int_0^1 \binom{v}{v/2} \Pi^{v/2}(1-\Pi)^{v/2} d\Pi = \frac{f(1/2)}{v+1} \tag{3}$$

where f is the density function corresponding to F.

(3) calculates the probability of a tied election conditional on an even number of votes. Of course, this probability is literally zero when the number of votes is odd. But, with v odd, (3) does calculate half of the probability that the election margin is exactly one (either in favor of Democrat or of Republican) so, heuristically, we can compute the probability of a pivot election

as 1.5 times (3):[13]

$$Pr\left\{ \sum_j z_j = v/2 \text{ or } \sum_j z_j = (v\pm1)/2 \right\} \approx 1.5 \frac{f(1/2)}{v+1} \qquad \textbf{(4)}$$

The probability of a pivotal vote is just (3) because only half of the votes in a pivot election with odd total votes are pivotal.

The estimates reported in Figure 5 suggest that $f(½) \approx 2$. For example, Figure 5 shows the frequency of a pivot election, for elections of size $v = 20{,}000$, to be 1/6000, so $f(½) = (1/6000)(2/3)(20001) \approx 2$. The model seems sensible in this regard, since $f(½) = 2$ if, say, Π were uniformly distributed on [0.25,0.75].[14]

However, there are two senses in which the binomial model overpredicts the frequency of a pivotal vote. First, the independence assumption is presumably invoked merely for convenience, and it seems natural that voters are subject to correlated influences. For example, few if any voters are completely unique in terms of their economic situation, social network, perceptions of the candidates, and other latent factors expressed in his vote. We leave it to future research to work this out more carefully,[15] but we suspect that the effective number of voters is less than the actual number of voters v, so that (4) overpredicts the frequency of a

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[13]With an even number of total votes, the probability of a pivot election is approximately twice (3). If even and odd total votes are equally likely, the probability of a pivot election is approximately the average of (3) and twice (3).

[14]As emphasized by Fisher (1999), the formula (3) yields a *very* different estimate then, say, (1) evaluated at some average percentage election margin. For example, setting v = 100,000 and $\Pi = .61$ (roughly the averages for the US sample), (1) yields an estimated probability of a pivot election of 10^{-1108} (here I follow Margolis 1977 and others in the liteture using the normal density to compute (1)). Compare 10^{-1108} to my US sample frequency of 0.00005.

[15]Glaeser, Sacerdote, and Scheinkman (1996) study a binomial model of crime, and emphasize the relationship between sample size and crime rates as a test for independent behavior – presumably there are analogous tests for independent voting behavior.

pivotal vote.[16]

Second, we find $f(\mu)$ to be significantly larger than $f(½)$ for μ quite close to ½. This is seen in Figures 2c and 2d where the frequency of STATE elections decided by 0-4 votes is significantly less than, say, the frequency of elections decided by 20-24 votes, and in Figures 3a and 3b where estimates of q and p increase significantly when the largest margin M included in the calculation is expanded beyond 4 or 5. We discuss these extensions of the binomial model below.

III.B. Recounts and other Margin-Specific Election Procedures

Election procedures are simple, and independent of margin, in the binomial model above. In particular, votes are counted for both candidates, and the winner is the one with the highest count. If election procedures are not independent of margin, this can affect our calculations of pivot frequencies, and interpretations of those calculations. To see this, consider a illustrative model of such procedures.

In any particular election, votes $z \in \{0,1\}$ are indexed by $j = 1,\ldots,v$ – as in the binomial model. However, these votes are initially measured by election officials with error, so that the combined quantity $(z_j + \varepsilon_j)$ is observed, where $\varepsilon_j \in \{-1,0,1\}$ is a measurement error.[17] Denoting the true and initially observed absolute margins in the ith election as m_i and $\hat{m}_i$, respectively, we have:

$$m_i = \left| 2\sum_j z_j - v \right|$$

$$\hat{m}_i = \left| 2\sum_j (z_j + \varepsilon_j) - v \right|$$

[16]For example, if husbands and wives always voted alike, but each household's vote were independent drawn from a binomial distribution, then the probability of a pivotal vote is $(1.5)f(½)/(2v+1)$ rather than $(1.5)f(½)/(v+1)$.

[17]Reasons for errors include vote machine errors, misplacement of ballots, and judgements about the legality of particular ballot.

If the initially observed margin exceeds g_i in the ith election, then the official election margin is $\hat{m}_i$. Otherwise, a "recount" is held to eliminate[18] measurement errors, and the official margin is the true margin, m_i . g_i varies by election, because these procedures may depend on juridiction, or on the willingness of candidates to demand (and pay for) a recount.

Although this is a stylized model of margin-specific election procedures, it clearly reveals how such procedures could reduced the frequency of close elections as measured by the official margin – some close elections are misclassified because $\hat{m}_i > g_i$, but no "far" elections are misclassified as close. Furthermore, a vote can have (instrumental) value even the election in which it was not cast in a pivot election (as we have defined it) because a single vote can affect whether there is a recount, which in turn has some probability of changing the outcome of the election.

III.C. Other Extensions of the Binomial Model

There two types of extensions of the binomial model that may be relevant for interpreting our findings. First, dependence across voters could be modeled. Such dependence might be created by correlation in the latent factors affecting a person's vote, or may affect strategic interactions as in the "swing voter models" (eg., Feddersen and Pesendorfer 1996). A second extension might allow for competition among candidates, competition which tends to make an election close, and competition which may be (endogenously) more intense when the election is close.

Building and studying such models are beyond the scope of our paper, but we believe that these models would produce frequencies of pivot elections that are closer than in the independent binomial model. Hence, our calculations with the latter model may overstate a frequency of pivot elections that more faithfully represents the various strategic and competitive phenomena studied in economic theories of elections.

IV. Can a Vote Ever Be Pivotal?: Some Evidence on Margin-Specific Voting Procedures

[18]This is for simplicity only. A more realistic model would have separate (nonzero) measurement errors before and after recount, perhaps with a smaller variance after the recount.

IV.A. What Really Happened to the Ballots in the Pivot elections?

The calculations above assume that the vote totals reported in the ICPSR studies are official and final. Systematically verifying this assumption for each of our 56,613 elections is beyond the scope of our project, but we have done some research that reflects on this assumption. Namely, for those 10 elections coded as tied or decided by one vote, we searched for newspaper articles written after the election to verify that the election was in fact close, and to verify that the winner and vote margin were coded "correctly" in our file. Somewhat to our surprise, we learned from these ten cases that the procedure for determining the winner may be flexible and margin-specific, and report some of our findings in Table 3.

We report in the fifth column the candidate vote totals as reported in the computer file which, with one exception, we found (according to newspaper reports) to be the official vote totals recorded with the state election commission. But, except in the five cases indicated in bold, it was later and different vote totals, reported in the next Table 3 column, that determined the winner of the election. Generally speaking, it seems that the counting of votes is not an exact science and, much like the model above, exact counts are not attempted unless the official (and error-ridden) margin is close. Reasons for errors include human arithmetic errors (as in the federal election), vote machine errors, misplacement of ballots (especially mail-in),[19] and judgements about the legality of particular ballot. These later counts typically reveal a larger margin of victory than officially recorded – as in four cases shown in the Table – and the official counts are not adjusted when the winner does not change.

[19]Mail in ballots are a problem because they are usually not counted and are sometimes not filed according to the correct office, district, etc. Even when one of the races is close enough to count, most others in the jurisdiction are not, and it usually is not know whether one of the mail ins was misplaced with another district's mail ins, which are not being counted (Evening Bulletin, 18 Nov. 1970).

Table 3: General Elections Coded with Margins of 0 or 1
Were They Really Decided by One Vote?

state or fed?	state	yr	dist/ off*	returns official	returns latest	Newspaper Account (source)
ST	RI	78	29/S	4110-4110	2546-2038**	After some argument whether a disputed 8221st ballot should be included, the tie was broken with another election run two months later with returns 2546-2038 (Evening Bulletin, 11/12,14,16/1978 and 1/4,12/1979).
ST	NM	80	19/H	2327-2327	**2327-2327**	(Albuquerque Journal, 5, 6, 11-13, 20, 21, 23 and 25-27 Nov. 1980 and 7 Dec. 1980 and 24 Jan. 1981, Associated Press 6 Nov. 1980)
FED	NY	10	36/H	20685-20684	20690-20684	Another recount broke the tie by one, when an arithmetic error was discovered. The winner wrote to the *New York Times* that five more of his votes were uncovered, there were further machine errors in the loser's favor, and both parties recognize that the election was not that close. (*New York Times* 11/11,12/1910)
ST	ME	82	69/H	1387-1386	**1387-1386**	Unofficial counts said it was tied, but all subsequent counts were 1387-1386 (Bangor Daily News 4, 16 and 18 Nov. and 1 Dec. 1982)
ST	MA	82	8/S	5352-5351	5378-5347	A recount was requested, revealing (13 days after the election) a 31 vote margin (The Union Leader, Manchester 9, 11 and 16 Nov. 1982).
ST	RI	70	26/H	1760-1759	**1760-1759**	1760-1759 was the count after a recount or two (Evening Bulletin, 18 Nov. 1970).
ST	WI	68	25/H	6522-6521	6523-6521	Margin was 3 votes before a recount, and a case was filed with the circuit court, where the judge said the margin was two votes. An election committee investigated, but could make no proclamation of the true margin (The Milwaukee Journal 6, 14 and 21 Nov. 1968; 7, 16 and 23-24 Jan. 1969)
ST	MO	70	116/H	4819-4818	**4819-4818**	A recount showed the same vote totals (Kansas City Star 6 and 7 Nov. 1970)
ST	ND	78	27/S	2459-2458	2448-2454	A first recount yielded the one vote margin. Another recount 25 days after the election showed a six vote margin *for the other candidate*, who took the office after subsequent appeals were reversed. (*Fargo-Moorhead Forum*; 10, 15, 16 and 28 Nov. 1978; 1, 5, 7, 8 and 12 Dec. 1978; 7 Jan. 1979)
ST	UT	80	44/H	1931-1930	**1931-1930**	(The Salt Lake Tribune, 6 Nov. 1980; Deseret News 18-19 Nov. 1980)

*"dist" is state or federal district number, and "off" is Senate or House
**the "late count" was from another election run two months later to break the tie

In one Rhode Island case, another election was held to break the originally "tied" election, even though according to (our reading of) Rhode Island election law, rerunning the election is not one of the legal options. Exhaustive legal research on the conduct of close elections is beyond the scope of our paper, but perhaps one moral of the ten stories told in Table 3 is that election procedures are flexible when official counts are close, and cannot be accurately modeled as the simple counting of a fixed population of ballots without regards for the situations of the candidates, their parties, and other legal and political factors. In other words, even when the election is over, one cannot know for sure whether or not the outcome hinged on one vote.

Up to this point, we have used the fifth "official" column to compute the frequency of a pivot election. One alternative calculation is to substitute the sixth column for the fifth in those ten cases where we have the data. In other words, rather than two ties and eight one-vote victories in the combined data set of 56,613 elections (an unweighted pivot election frequency $q(1) = (10+2)/56613 = 0.0002$), we have one tie and four one-vote victories (a frequency $q(1) = (5+1)/56613 = 0.0001$). This is a 50% reduction in the unweighted frequency and, since 76% of the votes analyzed in Table 3 were not cast in a pivot election according to the "latest" numbers, a 76% reduction in the vote-weighted frequency. On the other hand, we might have increased these frequencies if we had verified official margins in some of the other 56,603 elections decided by a margin of two or more votes.

IV.B. Dips in the Margin Density Near Zero

Our bar graph 3b also suggests that voting procedures might be margin-specific, at least in the STATE elections. Remember that 3b's point estimates would be flat if the sample frequency of vote margins were independent of the margin m in the neighborhood of a tie. Instead, we see that the frequency is nearly 50% higher for $m > 4$ vs. $m \leq 4$, and noted above how the frequency for $m \leq 4$ is still too high because the official vote totals are not those determining the winners.

Is the change in frequency with m due to the behavior of voters, or candidates? A complete answer is beyond the scope of this paper, but it may be that the procedures similar to those revealed by our newspaper investigations (which transformed the "official" counts to counts determining the winner) also played out *before* the official count was determined.[20] In other words, think of elections as being counted in three stages: a first count, which is followed by an independent "official" count only if the first count is close, which is in turn followed by an independent "latest" count only if the official count is close. If we had data on all three counts, we would expect, in the first count, to find the frequency of any margin m to be independent of m (for m small), but the find a dip near m=0 in the official count (as we did in Figure 3b), and an even bigger dip in the "latest" count (as our Table 3 suggests).

V. Conclusions

If we take the official counts as perfectly identifying pivotal votes, our empirical findings are perhaps best summarized in reference to the binomial voting model that has been used in the literature to calculate the probability of a pivotal vote. First, the frequency of a pivotal ballot (ie, one cast for a winner in an election decided by one vote, or for either candidate in a tied election), and its relationship with total votes v cast in the election, is closely approximated by $2/v$, which is broadly consistent with a binomial model. Second, smaller U.S. Congressional elections are an interesting exception to the $2/v$ rule, because the frequency of a pivotal ballot appears to *increase* with total votes cast. Third, $2/v$ seems to be somewhat less than a reasonably calibrated binomial model would predict and, because the frequency of (official) STATE election margin m is smaller for $m < 5$ than for $m \geq 5$. For example, only two STATE elections tied and only 29 were decided by 1-4 votes while 39 were decided by 11-14 votes, 46 by 21-24 votes, 29 by 31-34 votes, 46 by 41-44 votes, 49 by 51-54 votes, etc.

It might also be argued that the official counts do not accurately identify pivotal votes, because officially close elections would be recounted. So, conditional on a recount, a pivotal vote is one cast for a candidate winning by no more than one *according to the recount*. Some preliminary investigation of recounts suggest that the frequency of a pivotal ballot would only be $1/v$, rather than $2/v$, according to this definition. $1/v$ is perhaps a further departure from the standard binomial model, although not from one what is modified to include margin-specific recounts. Such a model would not only emphasize the value of a ballot because it might be

[20]Indeed, our newspaper investigations revealed some cases of this, but we did not sytematically report them in the Table since our objective was to calculate the sixth column.

pivotal (ie, the election might be decided by one vote in the final recount), but also the value of a ballot in affecting the probability of recounts that would change the winner of the election.

We believe that the theory of voting can be enhanced by fitting it to the empirical distribution of election returns documented here, but we leave the theoretical analysis to future research (Becker and Mulligan 1999 is one attempt). Perhaps some relevant questions are "Is the Swing Voter Model consistent with the empirical frequency of swing votes?", or "Can models of candidate competition for votes explain the empirical distribution of election returns?" Future empirical research could also investigate the prevalence of votes in uncontested elections – these were omitted from our calculations but are conspicuous in our data and probably relevant to the theory of voting.

V. References

Beck, N. "A Note on the Probability of a Tied Election." *Public Choice*. 23, Fall 1975: 75-80.

Becker, Gary S. and Casey B. Mulligan. "Is Voting Rational or Instrumental?" Working paper, University of Chicago, October 1999.

Chamberlain, Gary and Michael Rothschild. "A Note on the Probability of Casting a Decisive Vote." *Journal of Economic Theory*. 25, 1981: 152-62.

Feddersen, Timothy and Wolfgang Pesendorfer. "The Swing Voter's Curse." *American Economic Review*. 86(3), June 1996: 408-24.

Fischer, A.J. "The Probability of Being Decisive." *Public Choice*. 101(3-4), December 1999: 267-83.

Glaeser, Edward L., Bruce Sacerdote, and Jose A. Scheinkman. "Crime and Social Interactions." *Quarterly Journal of Economics*. 111(2), May 1996: 507-48.

Good, I. J. and Lawrence S. Mayer. "Estimating the Efficacy of a Vote." *Behavioral Science*. 20(1), January 1975: 25-33.

Margolis, Howard. "Probability of A Tie Election." *Public Choice*. 31, Fall 1977: 135-8.

Margolis, Howard. "Pivotal voting and the Emperor's new clothes." *Social Choice and Welfare*, in press.

Mulligan, Casey B. and Charles G. Hunter. "The Empirical Frequency of a Pivotal Vote." Harris School Working Paper 00.25, October 2000.