NBER WORKING PAPER SERIES

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Working Paper 8311
http://www.nber.org/papers/w8311

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
May 2001

We thank John Beaton, Leonid Kogan, Mark Lowenstein, Dimitri Vayanos, Greg Willard and seminar participants at Chicago, Columbia, Cornell, MIT, UCLA, Stanford, Yale, the NBER 1999 Summer Institute, and the 8th World Congress of the Econometric Society for helpful comments and discussion, and Jorge Rodriguez for excellent research assistance. Research support from the MIT Laboratory for Financial Engineering and the National Science Foundation (Grant No. SBR-9709976) is gratefully acknowledged. The views expressed herein are those of the authors and not necessarily those of the National Bureau of Economic Research.

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NBER Working Paper No. 8311
May 2001
JEL No. G1

ABSTRACT

We propose a dynamic equilibrium model of asset prices and trading volume with heterogeneous agents facing fixed transactions costs. We show that even small fixed costs can give rise to large "no-trade" regions for each agent's optimal trading policy and a significant illiquidity discount in asset prices. We perform a calibration exercise to illustrate the empirical relevance of our model for aggregate data. Our model also has implications for the dynamics of order flow, bid/ask spreads, market depth, the allocation of trading costs between buyers and sellers, and other aspects of market microstructure, including a square-root power law between trading volume and fixed costs which we confirm using historical US stock market data from 1993 to 1997.

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1 Introduction

It is now well established that transactions costs in asset markets are an important factor in determining the trading behavior of market participants.[1] Consequently, transactions costs should also affect market liquidity and asset prices in equilibrium.[2] However, the direction and magnitude of their effects on asset prices, trading volume, and other market variables are still subject to considerable controversy and debate.

Early studies of transactions costs in asset markets were based primarily on partial equilibrium analysis. For example, by comparing exogenously specified returns of two assets—one with transactions costs and another without—that yield the same utility, Constantinides (1986) argued that proportional transactions costs can only have a small impact on asset prices. However, using the present value of transactions costs under a set of candidate trading policies as a measure of the liquidity discount in asset prices, Amihud and Mendelson (1986b) concluded that the liquidity discount can be substantial despite relatively small transactions costs.

More recently, several authors have developed equilibrium models to address this issue. For example, Heaton and Lucas (1996) numerically solve a model in which agents trade to share their labor-income risk, and conclude that symmetric transactions costs do not affect asset prices significantly.[3] Vayanos (1998) develops a model in which agents trade to smooth life-time consumption, and shows that the price impact of proportional transactions costs is

[1]The literature on optimal trading policies in the presence of transactions costs is vast. See, for example, Atkinson and Wilmott (1995), Constantinides (1976, 1986), Constantinides and Magill (1976), Davis and Norman (1990), Duffie and Sun (1990), Dumas and Luciano (1991), Eastham and Hastings (1988), Fleming, Grossman, Vila, and Zariphopoulou (1992), Harrison, Sellke, and Taylor (1983), Korn (1998), Morton and Pliska (1995), Schroeder (1998), and Shreve and Soner (1994). The impact of transactions costs on agents' economic behavior has been studied in many other contexts as well. See, for example, Abel and Eberly (1994), Arrow (1968), Barro (1972), Baumol (1952), Bernanke (1985), Bertola and Caballero (1990), Caplin (1985), Dixit (1989), Frenkel and Javanovic (1980), Grossman and Weiss (1983), Mankiw (1985), Miller and Orr (1966), Pindyck (1988), Romer (1986), Rothchild (1971), Scarf (1960), Sheshinski and Weiss (1977), and Tobin (1956).

[2]See, for example, Aiyagari and Gertler (1991), Allen and Gale (1994), Amihud and Mendelson (1986a, 1986b), Bensaid et al. (1992), Constantinides (1986), Demsetz (1968), Dumas (1992), Easley and O'Hara (1987), Epps (1976), Foster and Viswanathan (1990), Garman and Ohlson (1981), Grossman and Laroque (1990), Heaton and Lucas (1996), Huang (1998), Jarrow (1992), Kyle (1985, 1989), Tiniç (1972), Tuckman and Vila (1992), Uppal (1993), Vayanos (1998), and Vayanos and Vila (1999).

[3]In Heaton and Lucas (1996), agents trade two assets, a risky stock and a riskless bond. Transaction costs on the stock alone only have negligible effect on asset prices; agents can use the bond to achieve most of their risk-sharing needs. However, if transactions costs are also imposed on the bond, their effect on the prices become important. In this paper, we assume that the bond market is frictionless.

linear in the costs and for empirically plausible magnitudes, their impact is small. Huang (1998) considers agents that are exposed to surprise liquidity shocks and who are able to trade in a liquid and an illiquid financial asset. He also finds that in the absence of additional constraints, the liquidity premium is small.

A common feature of these equilibrium models is the infrequent trading needs that they imply for agents, and calibrating such models may understate the effect of transactions costs on asset prices given the much higher levels of trading that we observe empirically.[4] After all, it is the high-frequency trading needs that are affected most significantly by transactions costs. Moreover, there is a substantial empirical literature that documents the importance of trading frictions for asset prices and investment management.

[5] This suggests the need for a more plausible model of agents' trading behavior to fully capture the economic implications of transactions costs in financial markets.

In this paper, we provide such a model by investigating the impact of fixed transactions costs on asset prices and trading behavior in a continuous-time equilibrium model with heterogeneous agents. Investors are endowed with a non-tradeable risky asset, e.g., labor income, and in a frictionless economy, they wish to trade continuously in the asset market, and in amounts that are cumulatively unbounded, to hedge their non-traded risk exposure. But in the presence of a fixed transactions cost, they seek to trade only infrequently. Indeed, we find that even small fixed costs can give rise to large "no-trade" regions for each agent's optimal trading policy, and the uncertainty regarding the optimality of the agents' asset positions between trades reduces their asset demand, leading to a decrease in the equilibrium price. We show that this price decrease—a discount due to illiquidity—satisfies a power law with respect to the fixed cost, i.e., it is approximately proportional to the square root of the fixed cost, implying that small fixed costs can have a significant impact on asset prices. Moreover, the size of the illiquidity discount increases with the agents' trading needs at high frequencies and is very sensitive to their risk aversion.

Our model also allows us to examine how transactions costs can influence the level of

[4]It may be possible to calibrate the model of Heaton and Lucas (1996) to allow for high-frequency trading needs, and we hope to explore this possibility in a future study.

[5]See, for example, Brennan (1975), Hasbrouck and Schwartz (1988), Keim and Madhavan (1995a-c), Kraus and Stoll (1972), Loeb (1983), Pérold (1988), Schwartz and Whitcomb (1988), Sherrerd (1993), and Stoll (1989, 1993).

trading volume, and serves as a bridge between the market microstructure literature and the broader equilibrium asset-pricing literature. In particular, despite the many market microstructure studies that relate trading behavior to market-making activities and the price-discovery mechanism,[6] the seemingly high level of volume in financial markets has often been considered puzzling from a rational asset-pricing perspective (see, for example, Ross, 1989). Some have even argued that additional trading frictions or "sand in the gears" ought to be introduced in the form of a transactions tax to discourage high-frequency trading.[7] Yet in absence of transactions costs, most dynamic equilibrium models will show that it is quite rational and efficient for trading volume to be *infinite* when the information flow to the market is continuous, i.e., a diffusion. An equilibrium model with fixed transactions costs can reconcile these two disparate views of trading volume. In particular, our analysis shows that while fixed costs do imply less-than-continuous trading and finite trading volume, an increase in such costs has only a slight effect on volume at the margin.

Moreover, our model has significant implications for the dynamics of order flow, the evolution of bid/ask spreads and depths, and other aspects of market microstructure dynamics. In particular, we endogenize not only the price at which trades are consummated, but also the *times* at which trades occur. The standard market-clearing condition—that agents trade a market-clearing quantity in each transaction—is obviously inadequate in a dynamic context where agents can choose *when* to transact. We extend the notion of market-clearing in the following natural way: agents must wish to trade the same quantities with each other at a certain price, and they must want to do so at the same time. This feature distinguishes our model from other existing models of trading behavior in the market microstructure literature, models in which order flow is almost always specified exogenously, e.g., Glosten and Milgrom (1985) and Kyle (1985). We find that the expected time-between-trades satisfies a power law with respect to the fixed transactions cost—it is proportional to the fourth root of the fixed cost. This implies a square-root power law between trading volume and inter-arrival times, an unexpectedly sharp empirical implication that we investigate and confirm using US equity transactions data.

[6] See, for example, Admati and Pfleiderer (1988), Bagehot (1971), Easley and O'Hara (1987), Foster and Viswanathan (1990), Kyle (1985), and Wang (1994).

[7] See, for example, Stiglitz (1989), Summers and Summers (1990a,b), and Tobin (1984).

We develop the basic structure of our model in Section 2, and discuss the nature of market equilibrium in the presence of fixed transactions costs in Section 3. We derive an explicit solution for the dynamic equilibrium in Section 4, and analyze the solution in Section 5. Section 6 reports the results of a calibration exercise using empirically plausible values of the parameters drawn from the existing literature. Section 7 presents an empirical test of some of the model's implications using historical stock market transactions data from 1993 to 1997, and we conclude in Section 8.

2 The Model

Our model consists of a continuous-time dynamic equilibrium in which two agents trade with each other over time to hedge their exposure to non-traded risk. Our interest in the trading process requires that we consider more than one agent, and because we seek to capture both the time of trade as well as the quantity of trade in a dynamic equilibrium context, we develop our model in continuous time. Although some of the technical aspects of continuous-time stochastic processes are somewhat daunting, nevertheless the mathematical overhead is well worth the effort given the nature of the issues we are addressing and the remarkable tractability that continuous-time methods seem to offer us in this case.

In Section 2.1 we describe the basic framework of our model and the precise nature of the fixed transactions cost that agents must bear when they trade in the risky asset. We define the notion of equilibrium in Section 2.2, which consists of three components: an equilibrium price process for the risky asset, an allocation of the fixed transactions cost between buyer and seller, and trading policies for both agents that consist of optimal quantities and optimal trade times for each agent that coincide with the other's, i.e., market clearing. In Section 2.3, we propose several simplifying assumptions that will allow us to streamline the analysis of our model in the remaining sections without significant loss of generality.

2.1 The Economy

Our economy is defined over a continuous-time horizon $[0, \infty)$ and contains a single commodity which is also used as the numeraire. The underlying uncertainty of the economy is characterized by an n-dimensional standard Brownian motion $B = \{B_t : t \geq 0\}$ defined on its filtered probability space $(\Omega, \mathcal{F}, F, \mathcal{P})$. The filtration $F = \{\mathcal{F}_t : t \geq 0\}$ represents the information revealed by B over time.

There are two traded securities: a risk-free bond and a risky stock. The bond pays a positive, constant interest rate r. Each share of the stock pays a cumulative dividend D_t where

$$D_t = \bar{a}_D t + \int_0^t b_D dB_s = \bar{a}_D t + b_D B_t \tag{1}$$

$\bar{a}_D$ is a positive constant, and b_D is a $(1 \times n)$ constant matrix. The securities are traded competitively in a securities market. Let $P = \{P_t : t \geq 0\}$ denote the stock price process, which is progressively measurable with respect to F.

Transactions in the bond market are costless, but transactions in the stock market are costly. For each stock transaction, the buyer and seller have to pay a combined fixed cost of κ that is exogenously specified and independent of the amount transacted. However, the *allocation* of this fixed cost between buyer and seller, denoted by κ^+ and κ^-, respectively, is determined endogenously in equilibrium. More formally, the transactions cost for a trade δ is given by

$$\kappa(\delta) = \begin{cases} \kappa^+ & \text{for } \delta > 0 \\ 0 & \text{for } \delta = 0 \\ \kappa^- & \text{for } \delta < 0 \end{cases} \tag{2}$$

where δ is the signed volume (positive for purchases and negative for sales), κ^+ is the cost to the buyer, κ^- is the cost to the seller, and the sum $\kappa^+ + \kappa^- = \kappa$.

There are two agents in the economy, indexed by $i = 1, 2$, and each agent is initially endowed with no bonds and $\bar{\theta}$ shares of the stock. In addition, agent i is endowed with a stream of non-traded risky income with cumulative cash flow N_t^i, where

$$N_t^i = \int_0^t \left[(-1)^i X_s + Y_s/2\right] b_N dB_s \tag{3a}$$

$$X_t = \int_0^t (-a_X X_s ds + b_X dB_s) \tag{3b}$$

$$Y_t = \int_0^t (-a_Y Y_s ds + b_Y dB_s) \tag{3c}$$

a_X, a_Y are positive constants, b_N, b_X, b_Y are $(1 \times n)$ constant matrices. For future reference, we let $X_t^i \equiv (-1)^i X_t$. Thus, b_N specifies the non-traded risk, $X_t^i + Y_t/2$ gives agent i's total exposure to the non-traded risk. Since $X_t^1 + X_t^2 = 0$ for all t, Y_t defines the aggregate level of non-traded risk and X_t^i defines the idiosyncratic component of agent i's non-traded risk.

Each agent chooses his consumption and trading policy to maximize the expected utility of his lifetime consumption. Let C denote the agents' consumption space, which consists of F-adapted, integrable consumption processes $c = \{c_t : t \geq 0\}$. The agents' stock trading policy space consists of only "impulse" trading policies defined as:

Definition 1 *Let* $\mathbb{N}_+ \equiv \{1, 2, \ldots\}$. *An impulse trading policy* $\{(\tau_k, \delta_k) : k \in \mathbb{N}_+\}$ *is a sequence of trading times* τ_k *and trade amounts* δ_k *such that:*

(1) $0 \leq \tau_k \leq \tau_{k+1}$ *a.s.* $\forall\ k \in \mathbb{N}_+$

(2) τ_k *is a stopping time with respect to* F

(3) δ_k *is measurable with respect to* F_{τ_k}

(4) $\delta_k \leq \bar{\delta} < \infty$

(5) $\mathrm{E}_0\left[e^{\gamma\kappa n(s)}\right] < \infty$

where

$$n(s) \equiv \sum_{\{k:\, 0 \leq \tau_k \leq s\}} 1$$

gives the number of trades in $[0,\ s]$.

Conditions (1)–(3) are standard for impulse policies. Condition (4) and (5) are imposed here for technical reasons. Condition (4) requires that trade sizes are finite.[8] Condition (5) requires that trading is not frequent enough to generate infinite trading costs. These are fairly weak conditions which we expect the optimal policy to satisfy.

Agent i's stock holding at time t is θ_t^i, given by

$$\theta_t^i = \theta_{0^-}^i + \sum_{\{k:\, \tau_k^i \leq t\}} \delta_k^i \tag{4}$$

[8]Limiting trade sizes to be finite rules out potential doubling strategies. Effectively, we require the trading policy to be in the L^∞ space, which is a standard condition in continuous-time settings.

where $\theta^i_{0^-}$ is his initial endowment of stock shares, which is assumed to be $\bar{\theta}$.

Let M^i_t denote agent i's bond position at t (in value). M^i_t represents agent i's liquid financial wealth. Then,

$$M^i_t = \int_0^t \left(rM^i_s - c_s\right) ds + \int_0^t \left(\theta^i_s dD_s + dN^i_s\right) - \sum_{\{k:\, 0\le\tau^i_k\le t\}} \left(P_{\tau^i_k}\delta^i_k + \kappa^i_k\right) \tag{5}$$

where $\kappa^i_k = \kappa\left(\delta^i_k\right)$ and $\kappa(\cdot)$ is given in (2). Equation (5) defines agent i's budget constraint. Agent i's consumption/trading policy (c, δ) is budget feasible if the associated M_t process satisfies (5).[9]

Both agents are assumed to maximize expected utility of the form:

$$u(c) = \mathrm{E}_0\left[-\int_0^\infty e^{-\rho t-\gamma c_t} dt\right] \tag{6}$$

where ρ and γ (both positive) are the time-discount coefficient and the risk-aversion coefficient, respectively. To prevent agents from implementing a "Ponzi scheme", we impose the following constraint on their policies for all $\gamma > 0$:

$$\mathrm{E}_0\left[-e^{-\rho t - r\gamma(M^i_t+\theta^i_t P_t) - r\gamma p^*(X^i_t+Y^i_t)}\right] < \infty \quad \forall\, t\ge 0 \tag{7}$$

where p^* is an arbitrary positive number, representing the shadow price for future non-traded income.[10] The set of budget feasible policies that also satisfies the constraint (7) gives the set of admissible policies, which is denoted by Θ.[11]

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[9] We can also define agent i's total financial wealth W^i_t, including both his bond position and the market value of his stock holdings: $W^i_t = M^i_t + \theta^i_t P_t$. From (4) and (5), we have

$$W^i_t = \theta^i_0 P_0 + \int_0^t \left(rW^i_s + dN^i_s - c_s\right) ds + \int_0^t \theta^i_s (dD_s + dP_s - rP_s ds) - \sum_{\{k:\, \tau^i_k\le t\}} \kappa^i_k$$

which also defines agent i's budget constraint.

[10] A more conventional constraint is imposed only on the terminal date. In the absence of non-traded income, the usual terminal condition is $\lim_{t\to\infty} \mathrm{E}_0\left[-e^{-\rho t - r\gamma W_t}\right] = 0$, where $W_t = M_t + \theta P_t$ is the terminal financial wealth. This condition rules out policies that finance current consumption by running unlimited deficit. Another way to avoid such policies is to add to the objective function a bequest function of the form $\lim_{t\to\infty} -e^{-\rho t - r\gamma W_t}$ to directly penalize them. In the presence of non-traded income, the constraint also includes the non-traded income. In this paper, for convenience, we impose a stronger condition (7), which limits agents from running unbounded financial deficit at any point in time, not just in the limit.

[11] It is easy to show that Θ is not empty. An example of such a policy is to consume nothing and invest only in bonds, which is feasible and satisfies (7).

For the economy to be properly defined, we need the following condition on its parameters:

$$4\gamma^2(b_X b_X{}' + b_Y b_Y{}')(b_N b_N{}') \leq 1. \tag{8}$$

This condition limits the volatility in the amount of non-traded risk to which each agent is exposed.[12]

2.2 Definition of Equilibrium

Definition 2 *An equilibrium in the stock market is defined by:*

(a) a price process $P = \{P_t : t \geq 0\}$ progressively measurable with respect to F

(b) an allocation of the transactions cost (κ^+, κ^-), where κ^+ is the cost for purchases and κ^- is the cost for sales as defined in (2)

(c) agents' trading policies $\{(\tau_k^i, \delta_k^i) : k \in \mathbb{N}_+\}$, $i = 1, 2$, given the price process and the allocation of transactions cost

such that:

(i) each agent's trading policy solves his optimization problem:

$$J^i(M_0^i, \theta_0, \cdot) \equiv \sup_{(c,\delta) \in \Theta} \mathrm{E}\left[-\int_0^\infty e^{-\rho t - \gamma c_t} dt\right] \tag{9}$$

where "·" denotes the relevant state variables

(ii) the stock market clears:

$$\forall\, k \in \mathbb{N}_+ : \qquad \tau_k^1 = \tau_k^2 \tag{10a}$$

$$\delta_k^1 = -\delta_k^2. \tag{10b}$$

In the presence of transactions costs, the market-clearing condition consists of two parts: agents' desired trading times match, which is (10a), and their desired trade amount match, which is (10b). Thus, a double "double coincidence of wants" must always be guaranteed in equilibrium, which is a very stringent condition.

[12]See footnote 21 for a discussion of the motivation behind parameter constraint (8).

It should be pointed out that by assuming a constant interest rate, we are assuming that the bond market is exogenous. This assumption simplifies our analysis, but deserves clarification. Three comments are in order. First, our focus is on how transactions costs affect the trading and pricing of a security when agents want to trade it at high frequencies. Assuming constant interest rate allows us to focus on the stock, which is costly to trade, and to restrict our attention to simple risk-sharing motives for trading. Endogenizing the bond market would yield stochastic interest rates and introduce additional trading motives (such as intertemporal hedging). Such a complication is unnecessary for our purposes.

Second, allowing the interest rate to adjust endogenously in our model would not fundamentally change the high-frequency trading needs from simple risk-sharing motives. The bond is locally risk-free and is not used as an instrument for risk sharing at high frequencies (this is no longer the case at lower frequencies as shown in Heaton and Lucas, 1996).

Third, we can avoid the issue of an endogenous bond market altogether by considering a finite-horizon version of our model without intermediate dividend, endowment, and consumption. Dividend, endowment and consumption only occur on the terminal date. In that case, the bond becomes a numeraire and the only market-clearing condition is for the stock. The qualitative features of the model remain the same. This entails the minor inconvenience of time-dependent equilibria, but we can then take the limit as the time horizon increases without bound. For expositional clarity and parsimony, we assume that the interest rate is fixed.

2.3 Simplifying Assumptions

For parsimony, we make several simplifying assumptions about the agents' non-traded risks, which is given in (3). First, we assume that there is no aggregate non-traded risk, which requires that $Y_t = 0 \; \forall \; t \geq 0$. In the current model, in absence of differences at the individual level, non-traded risk at the aggregate level does not generate any trading needs. It is the difference between agents in their non-traded income that generates trading. Since we are mainly interested in the impact of transactions costs, it is natural to focus on the difference in non-traded risk across agents. After all, transactions costs matter only when agents want to transact.[13]

The difference in the agents' non-traded risk is fully characterized by X_t. We further assume that $a_X = 0$. From (3), X_t now follows a Brownian motion: $X_t = b_X B_t$. Thus, changes in the difference between the agents' non-traded risk are persistent. In addition, we assume that the risk in the non-traded income is instantaneously perfectly correlated with risk in stock payoffs. In particular, we set $b_N = -hb_D$, where h is a positive (scalar) constant.[14] This implies that the non-traded risk is actually marketed. (Despite this, we continue to use the term non-traded risk throughout the paper to reflect the fact that it need not be marketed in general.) These two assumptions ($a_X = 0$ and $b_N = -hb_D$) can potentially increase the agents' needs to trade. However, we do not expect them to affect our results qualitatively—they are made to simplify the model.

3 Characterization of Equilibrium

Our derivation of the equilibrium is as follows. We first conjecture a set of candidate stock price processes and a set of candidate trading policies. We then solve for each agent's optimization problem within the candidate policy set under each candidate price process. This optimal policy is further verified to be the true optimal policy among all feasible policies. Finally, we show that the stock market clears for a particular candidate price process.[15]

3.1 Candidate Price Processes and Trading Policies

In the absence of transactions costs, our model reduces to a special version of the model considered by Huang and Wang (1997). Agents trade continuously in the stock market to hedge their non-traded risk. Since their non-traded risks perfectly cancel with each other, the agents can eliminate their non-traded risk through trading. Thus, the equilibrium price remains constant over time, independent of the idiosyncratic non-traded risk as characterized

[13]The coexistence of aggregate and idiosyncratic risks can lead to interesting interactions between the two. For example, Caballero and Engle (1993a, b) consider this interaction in the context of firms' investment decisions using a partial equilibrium framework. We hope to analyze this interaction in our equilibrium setting in future research.

[14]This assumption can be partially relaxed by allowing an additional component of the non-traded income that is independent from the risk of the stock.

[15]Needless to say, following this procedure we does not address the uniqueness of equilibrium, which is left for future research.

by X_t. In particular, the equilibrium price has the following form:

$$P_t = p_D - p_0 \quad \forall\ t \geq 0 \tag{11}$$

where $p_D \equiv \bar{a}_D/r$ gives the present value of expected future dividends, discounted at the risk-free rate, and p_0 gives the discount in the stock price to adjust for risk. The agents' optimal stock holding is linear in his exposure to non-traded risk:

$$\theta_t^i = \theta_0 + hX_t^i \tag{12}$$

where $\theta_0 = p_0/\left(\gamma\sigma_D^2\right)$ is a constant. It is worth noting that here each agent's stock holding is independent of his wealth.[16] When $b_N = -hb_D$, the non-traded risk is perfectly negatively correlated with the risk of the stock, and each agent's net risk exposure is given by

$$z_t^i \equiv \theta_t^i - hX_t^i. \tag{13}$$

Thus, an agent's optimal trading policy as given in (12) is to maintain his net risk exposure z_t^i to be at an optimal level θ_0, which is a constant.

In the presence of transactions costs, the agents trade only infrequently. However, whenever they trade, we expect them to reach optimal risk-sharing. This implies, as in the case of no transactions cost, that the equilibrium price at all trades should be the same, independent of the idiosyncratic non-traded risk X_t. Thus, we consider the candidate stock price processes of the form (11) even in the presence of transactions costs.[17] The discount p_0 now reflects the price adjustment of the stock for both its risk and illiquidity.

Contrary to the case of no transactions costs, agents can no longer follow trading policies that always maintain their net risk exposure at the desired level (which requires continuous trading). Instead, we consider candidate trading policies that maintain each agent's net risk exposure z_t^i within a certain band. In particular, these policies are defined by three constants (z_l, z_m, z_u), where $z_l \leq z_m \leq z_u$, as follows. When $z_t^i \in (z_l, z_u)$, no trade occurs. When z_t^i hits the lower bound z_l, agent i buys $\delta^+ \equiv z_m - z_l$ shares of the stock and moves z_t^i to z_m.

[16]The agents' optimal trading policy and the equilibrium stock price under zero transactions costs are given in Section 4.1 as a special case of the model.

[17]Given the perfect symmetry between the two agents, the economy is invariant under the following transformation: $X_t \to -X_t$. This implies that the price must be an even function of X_t. A constant is the simplest even function.

When z_t^i hits the upper bound z_u, agent i sells $\delta^- \equiv z_u - z_m$ shares of the stock and moves z_t^i to z_m. Here, since $X_0 = 0$, we assume that $\theta_{0^-} \in (z_l, z_m)$, where θ_{0^-} is the agent's initial stock position. There is no loss of generality by this latter assumption in the model defined in Section 2.

We define the stopping time τ_k to be the first time the net risk exposure z_t hits the boundary of (z_l, z_m) given the agent's net risk exposure at the previous trade $z_{\tau_{k-1}} = z_m$:

$$\tau_k = \inf\{t \geq \tau_{k-1} : z_t \notin (z_l, z_u)\} \quad \forall\ k = 1, 2, \ldots \tag{14}$$

where $\tau_0 = 0$. $\{\tau_k : k \in \mathbb{N}_+\}$ then gives the sequence of trading times. The amount of trading at τ_k is given by

$$\delta_{\tau_k} = \delta^+ 1_{\{z_{\tau_k -} = z_l\}} - \delta^- 1_{\{z_{\tau_k -} = z_u\}} \tag{15}$$

where $1_{\{\cdot\}}$ is the indicator function.

3.2 The Optimal Policy

Given the candidate stock price process and trading policies, we now examine an agent's optimization problem. We start by conjecturing that each agent's value function is of the form:

$$J(M, \theta, X, t) = -e^{-\rho t - r\gamma(M + \theta p_D) - V(\theta, X)} \tag{16}$$

where $V(\theta, X)$ is twice-differentiable. For simplicity in exposition, the index i is omitted here. Since the agent only trades at discrete times $\{\tau_k : k \in \mathbb{N}_+\}$, his stock position is constant between trades. Thus, for $t \in (\tau_{k-1},\ \tau_k)$, the Bellman equation takes the form:

$$0 = \sup_c \ \left\{-e^{-\rho t - \gamma c} + \mathrm{D}[J]\right\} \tag{17}$$

where $\mathrm{D}[\cdot]$ is the standard Itô operator.[18] The optimal consumption is given by

$$c = -\frac{1}{\gamma}\left[\ln r - r\gamma(M + \theta p_D) - V(\theta, X)\right]. \tag{18}$$

[18]Suppose that $dx_i = a_i dt + b_i dB$, where $i = 1, 2, \ldots, m$, and $f = f(x_1, \ldots, x_m)$ is twice differentiable. Let $f_i = (\partial f)/(\partial x_i)$, $f_{ij} = (\partial^2 f)/(\partial x_i \partial x_j)$, and $()'$ denotes the transpose. Then, $\mathrm{D}[f] \equiv \sum_{i=1}^m a_i f_i + \frac{1}{2}\sum_{i,j=1}^m b_i (b_j)' f_{ij}$. Note that in our case $dx_1 = dt$.

The Bellman equation then yields the following PDE for V (see the Appendix for its derivation):

$$0 = r\left(V - \bar{v}\right) + \tfrac{1}{2}\sigma_X^2\left(V_X^2 - V_{XX}\right) + \tfrac{1}{2}r^2\gamma^2\sigma_D^2\left(\theta - hX\right)^2 \tag{19}$$

where $\bar{v} \equiv (\rho - r + r\ln r)/r$, $\sigma_D^2 \equiv b_D b_D{}'$, $\sigma_X^2 \equiv b_X b_X{}'$, and $\sigma_N^2 \equiv b_N b_N{}' = h^2\sigma_D^2$.

Let $z \equiv \theta - hX$ and $V(X,\theta) \equiv v(z) + \bar{v}$. Then equation (19) reduces to a second order non-linear ordinary differential equation (ODE):

$$\sigma_z^2 v'' = \sigma_z^2 v'^2 + 2rv + (r\gamma)^2\sigma_D^2 z^2 \tag{20}$$

where $\sigma_z^2 = h^2\sigma_X^2$. Furthermore, we can rewrite the value function as follows:

$$J(M,\theta,X,t) = -e^{-\rho t - r\gamma(M+\theta p_D) - v(z) - \bar{v}}. \tag{21}$$

In general, each agent is concerned with two state variables (in addition to his bond position M_t), his exposure to non-traded risk X_t and his current stock position θ_t. Under the assumptions that the non-traded risk is permanent ($a_X = 0$) and marketed ($b_N = -hb_D$), the dimensionality of the state space is reduced. In particular, agent i only needs to be concerned with z_t^i as the state variable of interest, which characterizes his net risk exposure.

We now examine the optimal trading policy within the given candidate set, which is defined by (z_l, z_m, z_u). Solving for the optimal trading policy is equivalent to finding the optimal (z_l, z_m, z_u), given the transactions costs κ^+, κ^- ($\kappa^+ + \kappa^- = \kappa$) and price coefficient p_0.

If the trading policy (z_l, z_m, z_u) is optimal, at the trading boundaries (z_l and z_u) and with the optimal trade amounts (δ^+ and δ^-, respectively), the agent must be indifferent between trading and not trading. This leads to the "value-matching" condition:

$$v(z_l) = v(z_m) - r\gamma\left[\kappa^+ - p_0(z_m - z_l)\right] \tag{22a}$$

$$v(z_u) = v(z_m) - r\gamma\left[\kappa^- + p_0(z_u - z_m)\right]. \tag{22b}$$

In addition, the optimality of the trading boundaries requires the "smooth-pasting" condition:

$$v'(z_l) = v'(z_m) = v'(z_u) = -r\gamma p_0. \tag{23}$$

The value-matching condition (22) and the smooth-pasting condition (23) provide the boundary conditions to solve for the value function and the optimal trading policy within the candidate set.

The following theorem states that the optimal trading policy within the candidate set actually gives the optimum among all admissible policies:

Theorem 1 *Let z_l, z_m, z_u be the solution to (22) and (23) satisfying (A.10), where $v(z)$ satisfies (20) for $z \in [z_l, z_u]$. Then, v together with (21) gives the value function for the agents' optimization problem as defined in (9) and the optimal trading policies are given by (14) and (15).*

Thus, solving for the agents' optimal policies reduces to solving v under the appropriate boundary conditions.

3.3 Equilibrium Prices

An equilibrium price process is given by (11) with a particular choice of transactions cost allocation, κ^+ and κ^- ($\kappa^+ + \kappa^- = \kappa$), and price coefficients, p_0, such that the stock market clears. Given the agents' trading policies, the market-clearing condition (10) becomes

$$\delta^+ = \delta^- \tag{24a}$$

$$z_m = \bar{\theta}. \tag{24b}$$

Equation (24a) implies $z_u - z_m = z_m - z_l$. The symmetry between the two agents in their exposure to non-traded risk gives $z_t^1 - z_m = z_m - z_t^2$. Thus, their optimal trading times perfectly match when (24a) is satisfied. Furthermore, at the time of trade, the buyer wants to buy exactly the amount which the seller wants to sell. This trade amount is $\delta = \delta^+ = \delta^-$. Equation (24b) requires that both agents trade to the point where their total holdings of the stock equals the supply. Recall that $\bar{\theta}$ is the per capita endowment of shares of the risky asset.

4 Solutions to Equilibrium

Solution to the equilibrium of the conjectured form consists of two steps. The first step is to solve for each agent's value function and optimal trading policy, given κ^+ and p_0, which is to solve (20) with boundary condition (22)–(23). This is a free-boundary problem of a non-linear ODE. The second step is to solve for κ^+ and p_0 that the market-clearing condition (24) is satisfied. A general solution to the problem in closed form is not readily available. We approach the problem in two ways. We first solve the special case when transactions costs are small, and where we are able to derive approximate analytical results. We then solve the general case numerically.

4.1 Zero Transactions Costs

When $\kappa = 0$, $\delta^+ = \delta^- = 0$ and the agents trade continuously.[19] We then have the following result:

Theorem 2 *For $\kappa = 0$, agent i's optimal trading policy under a constant stock price $P_t = \bar{a}_D/r - p_0$ is*

$$\theta^i_t = \bar{z}_m + hX^i_t$$

where $\bar{z}_m = p_0/(\gamma\sigma^2_D)$, and his value function is

$$J(M^i_t, X^i_t, t) = -e^{-\rho t - r\gamma\left[M^i_t + \theta^i_t P_t + p_0 h X^i_t\right] - \frac{1}{2}r\gamma^2\sigma^2_D\bar{z}^2_m\left(1-\gamma^2\sigma^2_N\sigma^2_X\right) - \bar{v}}. \tag{26}$$

Moreover, in equilibrium, $p_0 = \bar{p}_0 \equiv \gamma\sigma^2_D\bar{\theta}$ and $\bar{z}_m = \bar{\theta}$.

Agent i's stock holding has two components. The first component $\bar{z}$, which is constant, gives his unconditional stock position. For $P_t = (\bar{a}_D/r) - p_0$, the expected excess return on one share of stock is rp_0 and the return variance is σ^2_D. Hence, rp_0/σ^2_D gives the price of per unit risk of the stock. Moreover, agent i's risk-aversion (toward uncertainty in his wealth) is $r\gamma$. Thus, his unconditional stock position, $\bar{z}_m = (1/r\gamma)(rp_0/\sigma^2_D) = p_0/(\gamma\sigma^2_D)$, is proportional to his risk tolerance and the price of risk. The second component of agent i's stock position is proportional to X^i_t, his exposure to the non-traded risk. This component reflects his hedging position against non-traded risk and the proportionality coefficient, $h = \sigma_N/\sigma_D$, gives the hedge ratio.

[19]We can think of continuous trading in this case as the limit of the progressively measurable trading policies given in Definition 1.

In equilibrium, market clearing requires that $\bar{z}_m = \bar{\theta}$. Thus, $p_0 = \bar{p}_0 \equiv \gamma\sigma_D^2\bar{\theta}$. As mentioned earlier, p_0 gives the discount in the price of the stock for its risk and illiquidity. In absence of transactions costs, the stock is liquid and $p_0 = \bar{p}_0$. Thus, $\bar{p}_0$ can be interpreted as the risk discount of the stock. In the presence of transactions costs, we define the difference between p_0 and $\bar{p}_0$, denoted by π,

$$\pi \equiv p_0 - \bar{p}_0 \tag{27}$$

to be the illiquidity discount of the stock.

4.2 Infinite Transactions Costs

To develop an intuition about the illiquidity discount and to put a bound on its magnitude, we now consider the case when the transactions costs are prohibitively high except at $t = 0$. That is, $\kappa = \widehat{\kappa}\, 1_{\{t>0\}}$ where $\widehat{\kappa} \to \infty$. Agents can trade at zero cost at $t = 0$ but cannot trade afterwards.[20] We have the following result:

Theorem 3 *For $\kappa = \widehat{\kappa}\, 1_{\{t>0\}}$ where $\widehat{\kappa} \to \infty$, agent i's stock demand is*

$$\theta_0^i = \tfrac{1}{2}\left(1 + \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}\right)\frac{p_0}{\gamma\sigma_D^2} + hX_0^i.$$

In equilibrium, $\theta_t^i = \bar{\theta}$ and the stock price at $t = 0$ is $P_0 = p_D - p_0$ where

$$p_0 = \bar{p}_0\left[1 + \frac{4\gamma^2\sigma_N^2\sigma_X^2}{\left(1 + \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}\right)^2}\right]$$

and $\bar{p}_0$ is given in Theorem 2.[21]

In this case, the stock becomes completely illiquid after the initial trade. At the same price, the demand for stock is lower compared with the case when $\kappa = 0$. In equilibrium, an

[20]This situation has been considered by Hong and Wang (2000) when they analyze the effect of market closures on asset prices. Closure of the market is equivalent to imposing prohibitive transactions costs.

[21]When agents cannot trade (after the initial point), the parameter condition (8), which becomes $4\gamma^2\sigma_N^2\sigma_X^2 \leq 1$ here, is needed. It limits the volatility of an agent's endowment risk. Unable to unload the risk to the market, the agent's consumption is forced to absorb the risk of his endowment. Conditions of the above type is needed to guarantee that his expected utility (over an infinite horizon) is well defined given his endowment. This condition is not needed when agents can trade (even infrequently) to control the risk of his consumption.

illiquidity discount is required in its price:

$$\widehat{\pi} \equiv \bar{p}_0 \frac{4\gamma^2\sigma_N^2\sigma_X^2}{\left(1+\sqrt{1-4\gamma^2\sigma_N^2\sigma_X^2}\right)^2}.$$

For σ_X^2 small, we have $\widehat{\pi} \approx \gamma^2\sigma_N^2\sigma_X^2\bar{p}_0$.

This extreme case illustrates three points. First, the agents' inability to trade in the future reduces their current demand of the stock. As a result, its price carries an additional discount in equilibrium to compensate for the illiquidity (also see Hong and Wang (2000)). Second, this illiquidity discount is proportional to agents' high frequency trading needs, which is characterized by the (instantaneous) volatility of their non-traded risk, σ_X^2. Third, the liquidity discount also increases with the risk of the stock, which is measured by σ_D^2 (or $\bar{p}_0$).

When the transactions costs are finite, agents can trade after the initial date (at a cost) and the stock becomes more liquid. We expect the magnitude of the illiquidity discount to be smaller than the extreme case above. However, the qualitative nature of the results remains the same as we show later.

4.3 Small Transactions Costs: An Approximate Solution

We now turn to the case when the transactions costs are small. We seek the solution to each agent's value function, optimal trading policy, the equilibrium cost allocation and stock price that can be approximated by powers of $\varepsilon \equiv \kappa^\nu$ where ν is a positive constant. In particular, v takes the form $v(z,\varepsilon)$ and $\kappa^\pm$ takes the form:

$$\kappa^\pm = \kappa\left(\tfrac{1}{2} \pm \sum_{n=1}^{\infty} k^{(n)}\varepsilon^n\right). \tag{29}$$

We also use $o(\kappa^\nu)$ to denote terms of higher order than κ^ν and $O(\kappa^\nu)$ to denote terms of the same order as κ^ν. The following theorem summarizes our results on optimal trading policies:

Theorem 4 *Let* $\varepsilon \equiv \kappa^{\frac{1}{4}}$. *For (a)* κ *small and* $\kappa^\pm$ *in the form of (29), and (b)* $v(z,\varepsilon)$ *analytic* *for small z and ε, an agent's optimal trading policy is given by*

$$\delta^{\pm} = \phi\kappa^{\frac{1}{4}} \pm \tfrac{6}{11}\left(k^{(1)} - \tfrac{2}{15} r\gamma p_0 \phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}}) \quad (30a)$$

$$z_m = \frac{p_0}{\gamma\sigma_D^2} + \tfrac{4}{11}\left(k^{(1)} - \tfrac{71}{120} r\gamma p_0 \phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}}) \quad (30b)$$

where $\phi = \left(\frac{6\sigma_z^2}{r\gamma\sigma_D^2}\right)^{\frac{1}{4}}$ and $\sigma_z^2 = h^2\sigma_X^2$.

Here, δ^+ and δ^- are the same to the first order of $\varepsilon = \kappa^{\frac{1}{4}}$, but differ in higher orders of ε.

The stock market equilibrium is obtained by choosing $\kappa^{\pm}$ and p_0 such that the market-clearing condition (24) is satisfied. We have the following theorem:

Theorem 5 *For (a) κ small and $\kappa^{\pm}$ in the form of (29), (b) $v(z, \varepsilon)$ analytic for small z and ε, and (c) $p(\varepsilon)$ analytic for small ε, the equilibrium stock price and transactions cost allocation are given by*

$$p_0 = \gamma\sigma_D^2\bar{\theta}\left(1 + \tfrac{1}{6} r\gamma^2\sigma_D^2\phi^2\kappa^{\frac{1}{2}}\right) + o(\kappa^{\frac{1}{2}}) \quad (31a)$$

$$\kappa^{\pm} = \kappa\left[\tfrac{1}{2} \pm \tfrac{2}{15} r\gamma p_0 \phi\kappa^{\frac{1}{4}} + o(\kappa^{\frac{1}{4}})\right] \quad (31b)$$

and the equilibrium trading policies are given by (30) with the equilibrium value of p_0 and $\kappa^{\pm}$.

4.4 General Transactions Costs: A Numerical Solution

In the general case when κ can take arbitrary values, we have to solve both the optimal trading policy and the equilibrium stock price numerically. Given p_0 and $\kappa^{\pm}$, we can solve (20) and (22-23) for each agent's optimal trading policy. We can then solve for p_0 and $\kappa^{\pm}$ that leads to the market-clearing condition (10).

In the examples shown throughout the paper, we use parameter values obtained from a calibration exercise, which is discussed in Section 6. In particular, we have: $\rho = 0.10$, $\gamma = 1.347$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = \sigma_N = 1$, and $\bar{\theta} = 5.1769$.

Figure 1 shows the numerical solution for the trade amount for various values of transactions costs. Here, we have chosen $\kappa^{\pm}$ such that $\delta^+ = \delta^- \equiv \delta$. Each circle represents the value of δ for a particular value of κ. In the left panel, δ is plotted against the value of κ. In the right panel, δ is plotted against the value of $\kappa^{\frac{1}{4}}$. This transformation is suggested by the

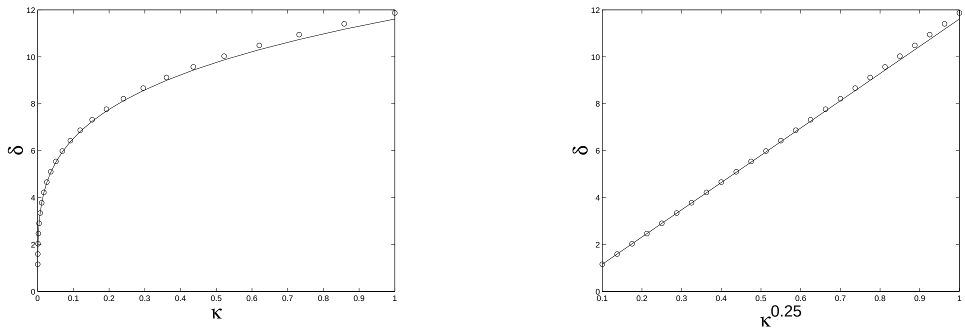


Figure 1: Trade amount δ plotted against transactions costs κ and $\kappa^{\frac{1}{4}}$. The circles represent the numerical solution. The solid line plots the analytical approximation. The parameter values are $\rho = 0.10$, $\gamma = 1.347$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = \sigma_N = 1$, and $\bar{\theta} = 5.1769$.

approximate solution when κ is small. For comparison, we have also plotted the analytical approximation obtained for small κ as the solid lines.

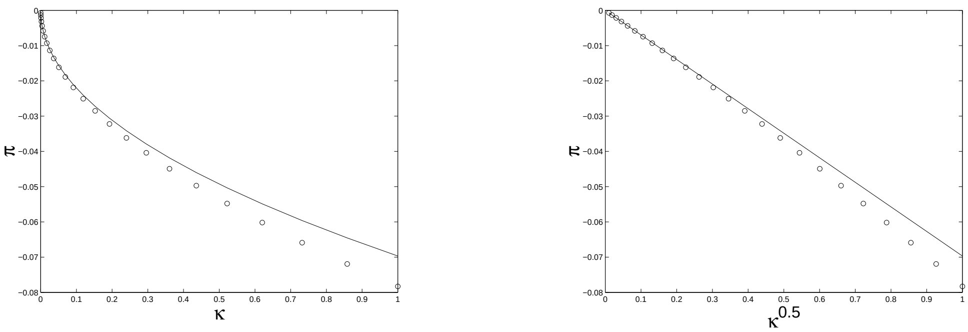


Figure 2: Illiquidity discount π plotted against κ and $\kappa^{\frac{1}{2}}$. The circles represent the numerical solution. The solid line plots the analytical approximation. The parameter values are $\rho = 0.10$, $\gamma = 1.347$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = \sigma_N = 1$, and $\bar{\theta} = 5.1769$.

Given the solution to the agents' optimal trading policies, we can further search for the p_0 and $\kappa^{\pm}$ such that the market-clearing condition (24) is satisfied. Figure 2 plots the numerical solution (circles) and the analytical approximation (solid line) for the illiquidity discount π in the stock price ($\pi = p_0 - \bar{p}_0$) for various values of the transactions cost. In the left panel, π is plotted against the value of κ. In the right panel, π is plotted against the value of $\kappa^{\frac{1}{2}}$. It is interesting to note that the analytic approximation obtained for small values of transactions costs still fits quite well for fairly large values of κ.

5 Analysis of Equilibrium

We now discuss in more detail the impact of transactions costs on agents' trading policies, the equilibrium stock price and trading volume. We focus on the case when κ is small. For convenience, we only maintain the terms up to the lowest appropriate order of κ in our discussion.

5.1 Trading Policy

When the transactions costs are zero ($\kappa = 0$), agent i trades continuously in the stock in response to changes in his exposure to non-traded risk, which is characterized by X_t^i ($i = 1, 2$). As stated in Theorem 2, the stock position is constantly adjusted such that $\theta_t^i = p_0/(\gamma\sigma_D^2) + hX_t^i$ and $z_t^i = \theta_t^i - hX_t^i = \bar{z}_m = p_0/(\gamma\sigma_D^2)$.

When the transactions costs are positive, it becomes costly to maintain $z_t^i = \bar{z}_m$ at all times. In response, agent i adopts the following policy: He does not trade when z_t^i is within a no-trade region, given by $(z_l, z_u) = (z_m - \delta^+,\ z_m + \delta^-)$. When z_t^i hits the boundary of the no-trade region, agent i trades the necessary amount (δ^+ or δ^-) to bring z_t^i back to the optimal level z_m. Two sets of parameters characterize the agent's optimal trading policy: the widths of the no-trade region, δ^+ and δ^-, and the base level he trades to, z_m, when he does trade. In general, z_m is different from $\bar{z}_m$, the position he would trade to in absence of transactions costs. We now discuss these two sets of parameters separately.

To the lowest order of κ, $\delta^+ = \delta^- = \phi\kappa^{\frac{1}{4}}$ as shown in Theorem 4. In other words, the width of the no-trade region exhibits a quartic-root "law" for small transactions costs. We argue that this quartic-root law arises from the boundary conditions, reflecting mainly the nature of the transaction costs. To see this, consider the simple case when $p_0 = 0$ and $\kappa^+ = \kappa^- = \kappa/2$. Then, $z_m = 0$. We can re-express the boundary conditions in (22) and (23) as follows:

$$v(-\delta^-) - v(0) = -r\gamma\kappa = v(\delta^+) - v(0) \tag{32a}$$

$$v'(-\delta^-) = v'(0) = v'(\delta^+) = 0. \tag{32b}$$

The symmetry between the boundary conditions for the upper and lower no-trade band implies that the band should be symmetric around z_m to the lowest order of κ. That is,

$\delta^+ \approx \delta^- \equiv \delta$. Hence, to the zero-th order of κ, $v_1(0) \approx 0 \approx v_3(0)$, where $v_k(0)$ denotes the k-th derivative of v at 0, and furthermore, $v_2(0) \approx 0$ by (32b). It follows that for small z, $v(z) \approx \frac{1}{4!}v_4(0)z^4$. The value matching condition (32a) then implies that $\delta \propto \kappa^{\frac{1}{4}}$ (if $v_4(0) \neq 0$).[22] The above argument suggests that the quartic-root relation between δ and κ for small κ is determined by the boundary conditions, especially (32a), which in turn reflects the form of the transactions cost. For this reason, the quartic-root relation between the width of no-trade region and the fixed transactions cost may be a more general result for optimal trading policies.

In the above argument, the quartic-root relation between δ and κ is closely related to the fact that $v(z)$ is quartic in z for small z. The economic intuition behind this property of the value function is as follows. The optimality of the point $z = 0$ requires that $v_1(0) = 0$. If the agent was not allowed to trade, we would have $v(z) \propto z^2$ for z small. However, since the agent always trades back to the optimal position when the trading boundaries are hit, the quadratic term vanishes (to the zero-th order of κ). The symmetry of the boundary conditions further requires that the cubic term vanishes. Thus, $v(z)$ is quartic in z. Intuitively, under fixed transaction costs, the agent always trades back to the optimal stock position. Thus, he can minimize his utility loss without trading too frequently.[23]

Having established that the width of no-trade region should be proportional to the quartic root of κ (i.e., $\delta = \phi\kappa^{\frac{1}{4}}$), we now examine the proportionality coefficient ϕ. From Theorem 4, we have $\phi = [6\sigma_z^2/(r\gamma\sigma_D^2)]^{\frac{1}{4}}$. Note that $r\gamma\sigma_D^2$ corresponds to the certainty equivalence of the (per unit time) expected utility loss for bearing the risk of one stock share. It is then not surprising that ϕ (and δ) is negatively related to $r\gamma\sigma_D^2$. Moreover, σ_z^2 gives the variability of

[22] More precisely, (32b) leads to $v_2(0) + \frac{1}{6}v_4(0)\delta^2 \approx 0$, or $v_2(0) = -\frac{1}{6}v_4(0)\delta^2$. From (32a), we have $\frac{1}{12}v_4(0)\delta^4 \approx r\gamma\kappa$, or $\delta \propto \kappa^{\frac{1}{4}}$ if $v_4(0) \neq 0$. In fact, (30) gives that $v_4(0) = 2(r\gamma)^2\sigma_D^2/\sigma_z^2$. See the appendix for more details.

[23] The above result on optimal trading policies under fixed transactions costs are closely related to the results of Morton and Pliska (1995) and Atkinson and Wilmott (1995) (see also Schroeder (1998)). Morton and Pliska solve for the optimal trading policy when the agent maximizes his asymptotic growth rate of wealth and pays a cost as a fixed fraction of his total wealth for each transaction. The optimization problem reduces to a free-boundary ODE, which has a closed form solution up to a set of coefficients to be determined by the boundary conditions. They numerically solve for these coefficients. Atkinson and Wilmott (1995), using perturbation techniques, derive an analytic approximations for the solution to the Morton and Pliska model when the transactions cost is small. Interestingly, they also find that the no-trade region is proportional in size to the fourth root of the transactions cost. Note that in their model, the transactions cost is a fixed fraction of the total wealth.

the agent's non-traded risk. For larger σ_z^2, the agent's hedging need is changing more quickly. Given the cost of changing his hedging position, the agent is more cautious in trading on immediate changes in his hedging need. Thus, ϕ (and δ) is positively related to σ_z^2.

Under the optimal trading policy, agents trade only infrequently. Define $\Delta\tau \equiv \mathrm{E}\left[\tau_{k+1} - \tau_k\right]$ to be the average time between two neighboring trades. It is easy to show that

$$\Delta\tau = \delta^2/\sigma_z^2 \approx \left(\phi^2/\sigma_z^2\right)\kappa^{\frac{1}{2}} \tag{33}$$

(see, e.g., Harrison, 1990). Not surprisingly, the average waiting time between trades is inversely related to σ_z, the volatility in the agent's hedging need, and $r\gamma\sigma_D^2$, the cost of bearing the risk of one stock share. Moreover, it is proportional to the square root of the transactions cost. Figure 3 plots the average trading interval $\Delta\tau$ versus different values of transactions cost κ as well as the appropriate power law for small κ's.

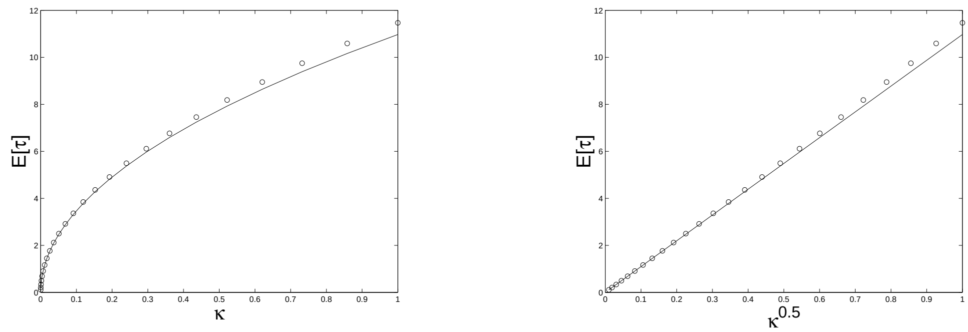


Figure 3: Trading Interval. The two panels show the expected inter-arrival times plotted against κ and its square root, respectively. The circles represent the numerical solution. The solid line plots the analytical approximation. The parameter values are $\rho = 0.10$, $\gamma = 1.347$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = \sigma_N = 1$, and $\bar{\theta} = 5.1769$.

When each agent chooses to trade, he trades to a base position z_m. In absence of transactions costs, each agent trades to a position ($\bar{z}_m$) that is most desirable given his current non-traded risk. As his non-traded risk changes, he maintains this desirable position by constantly trading. In the presence of transactions costs, however, an agent only trades infrequently. A position desirable now becomes less desirable later. But he has to stay in this position until the next trade when the gain from trading exceeds the transactions cost. As a result, the agent chooses a position that takes into account the deterioration of its desirability over time and the inability to revise it immediately.

From Theorems 4 and 5, the shift in the base position is given by $\Delta z_m \equiv \bar{z}_m - z_m =$

$\frac{1}{6}r\gamma p_0(\sigma_z^2\Delta\tau)$. It is not surprising that Δz_m is proportional to the total volatility of an agent's non-traded risk over the no-trade period, which is $\sigma_z^2\Delta\tau$. Moreover, Δz_m is proportional to p_0, the risk discount on the stock. In order to further understand this result, let us consider the following heuristic argument. Suppose that the current level of the agent's non-traded asset is zero. The uncertainty in its level over the next no-trade period, denoted by $\tilde{z}$, gives rise to an additional uncertainty in his wealth: $-\tilde{z}(-p_0 + \tilde{d})$, where $\tilde{d}$ denotes the stock dividend over the period. (Here, we set $h = 1$ for simplicity.) Although $\tilde{z}$ has a zero mean, its impact on the overall uncertainty in wealth is not zero. Averaging over $\tilde{z}$ (assumed to be normally distributed with variance σ_z^2), the agent's utility over his future wealth is proportional to $\mathrm{E}_{\tilde{z}}\left[-e^{-r\gamma(\theta-\tilde{z})(-p_0+\tilde{d})}\right] = -e^{-r\gamma[\theta-\frac{1}{2}r\gamma(-p_0+\tilde{d})\sigma_z^2\Delta\tau](-p_0+\tilde{d})}$, where $\mathrm{E}_{\tilde{z}}$ denotes the average over $\tilde{z}$, θ is the agent's stock position and $\Delta\tau$ is the length of no-trade period. In other words, the uncertainty in $\tilde{z}$ leads to an effective risk in the agent's wealth that is equivalent to an average stock position of size $\frac{1}{2}r\gamma p_0(\sigma_z^2\Delta\tau)$. The size is proportional to p_0 because the uncertainty in wealth generated by uncertainty in $\tilde{z}$ is proportional to p_0. Consequently, the agent reduces his base stock position by the same amount. This shift in the agent's base position reflects the decrease in his demand of the stock in response to its illiquidity.

5.2 Stock Prices and the Illiquidity Discount

In equilibrium, the stock price has to adjust in response to the negative effect of illiquidity on agents' stock demand, giving rise to an illiquidity discount π. For small transactions costs, the illiquidity discount is proportional to the square root of κ. Figure 2 further shows that this square-root relation provides a reasonable approximation even for fairly large transaction costs. From Theorem 5, we have

$$\pi \approx \gamma\sigma_D^2\Delta z_m \approx \tfrac{1}{6}r\gamma^3\sigma_D^4\phi^2\bar{\theta}\kappa^{\frac{1}{2}} = \tfrac{1}{6}\gamma\left(r\gamma\sigma_D^2\right)\bar{p}_0(\sigma_z^2\Delta\tau). \tag{34}$$

As we have shown, fluctuations in his non-traded risk and the cost of adjusting stock positions to hedge this risk reduce an agent's stock demand by Δz_m. Given the linear relation between the agents' stock demand and the stock price, the price has to decrease proportionally to the decrease in demand to clear the market, which gives the illiquidity discount in the first expression of (34). Moreover, the decrease in agents' stock demand is proportional to the total risk discount of the stock (p_0) and the volatility of their non-traded risk between trades ($\sigma_z^2 \Delta\tau$), which leads to the second expression.

We thus conclude that the illiquidity discount of the stock is proportional to the product of the cost of bearing the risk of one stock share ($r\gamma\sigma_D^2$), the total risk discount of the stock ($\bar{p}_0$), and the variability of agents' desired positions between trades ($\sigma_z^2 \Delta\tau$). The proportionality constant is the risk-aversion coefficient γ.

Note that the illiquidity discount is proportional to the cubic power of γ. Comparing with the risk discount which is proportional to γ, we infer that the illiquidity discount is highly sensitive to the agents' risk aversion.

Using a model similar to ours but with proportional transactions costs and deterministic trading needs, Vayanos (1998) finds that the illiquidity discount on the stock is linear in the transactions costs (when they are small). Our result shows that small fixed transaction costs can give rise to a non-trivial illiquidity discount when agents have high frequency trading needs. Given the difference in the nature of transactions costs between our model and Vayanos's, our result is not directly comparable to his. However, our result does suggest that the presence of high frequency trading needs is important in analyzing the effect of transactions costs on asset prices.

To confirm this, we consider a special variation of our model, in which $X_t = \bar{a}_X t$. In this case, the agents' non-traded risk evolves deterministically. This gives rise to deterministic needs to trade among agents since they differ in their non-traded risk. We have the following result:

Theorem 6 *Let* $\varepsilon = \kappa^{\frac{1}{3}}$. *For (a)* $X_t = \bar{a}_X t$ $(\bar{a}_X \geq 0)$, *(b)* $\kappa^{\pm} = \kappa/2$, *(c)* $v(z,\varepsilon)$ *analytic for small* z *and* ε, *and (d)* $p(\varepsilon)$ *analytic for small* ε, *agents' optimal trading policies are given by*

$$\delta^{1+} = \lambda\kappa^{\frac{1}{3}} + o(\kappa^{\frac{2}{3}}), \quad \delta^{1-} = 0, \quad \delta^{2+} = 0, \quad \delta^{2-} = \delta^{1+} \tag{35a}$$

$$z_m^1 = \frac{p_0}{\gamma\sigma_D^2} + \tfrac{1}{2}\lambda\kappa^{\frac{1}{3}} - \tfrac{1}{2}\left(\lambda\gamma\sigma_D^2\right)^{-1}\kappa^{\frac{2}{3}} + o(\kappa^{\frac{2}{3}}), \quad z_m^2 = \bar{\theta} - (z_m^1 - \bar{\theta}) \tag{35b}$$

where $\lambda = \left(\frac{6\sigma_N}{r\gamma\sigma_D^3}\right)^{\frac{1}{3}} (\bar{a}_X)^{\frac{1}{3}}$; *the equilibrium stock price is given by* $p_0 = \bar{p}_0 + o(\kappa)$.

It is indeed the case that in the absence of high frequency trading needs, the transactions cost does not lead to significant liquidity discount on the stock. Also the power law for the trade amount has now become $\frac{1}{3}$, rather than $\frac{1}{4}$. This is a result of the fact that each agent has only 1 trade boundary (each agent either always sells or always buys), because of the deterministic nature of the endowment process.

5.3 Trading Volume

Economic intuition suggests that an increase in transactions costs must reduce the volume of trade. Our model suggests a specific form for this relation. In particular, the equilibrium trade size is a constant. From our solution to equilibrium, the volume of trade between time interval t and $t+1$ is given by:

$$\nu_{t+1} = \sum_{\{k:\, t<\tau_k\leq t+1\}} \left|\delta_k^i\right| \tag{36}$$

where $i = 1$ or 2. The average trading volume per unit of time is

$$\mathrm{E}\left[\nu_{t+1}\right] = \mathrm{E}\left[\sum_k 1_{\{\tau_k\in(t,\,t+1]\}}\right]\delta \equiv \omega\delta$$

where ω is the frequency of trade (i.e., the number of trades per unit of time). For convenience, we define another measure of average trading volume as the number of shares traded per average trading time, or

$$\nu = \frac{\delta}{\Delta\tau} = \sigma_z^2/\delta \tag{37}$$

where $\Delta\tau \equiv \mathrm{E}\left[\tau_{k+1}-\tau_k\right] \approx \delta^2/\sigma_z^2$ is the average time between trades.[24] From (30), we have

$$\nu = \sigma_z^2\phi^{-1}\kappa^{-\frac{1}{4}}\left[1+O\left(\kappa^{\frac{1}{4}}\right)\right].$$

Clearly, as κ goes to zero, trading volume goes to infinity. However, we also have

$$\frac{\Delta\nu}{\nu} \approx -\tfrac{1}{4}\frac{\Delta\kappa}{k}.$$

In other words, (for positive transactions costs) one percentage increase in the transactions cost only decreases trading volume by a quarter of a percent. In this sense, within the range of positive transactions costs, an increase in the cost only reduce the volume mildly at the

[24]Of course, ν is different from $\mathrm{E}[\nu_{t+1}]$ by Jensen's inequality.

margin.

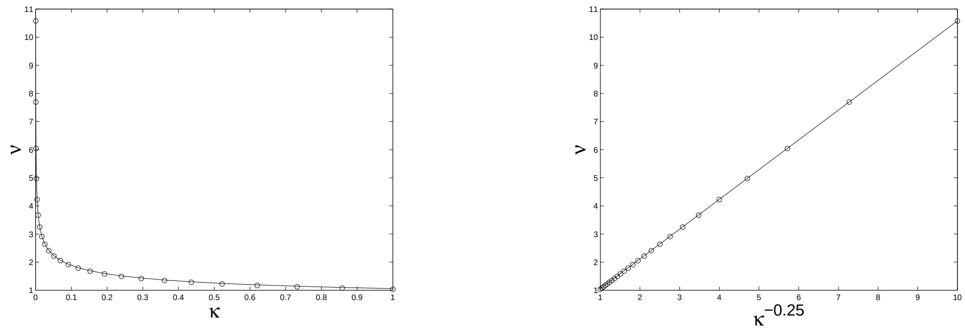


Figure 4: Trading volume. The two panels show the volume measure ν plotted against κ (left) and $\kappa^{-\frac{1}{4}}$ (right). The circles represent the numerical solution. The solid line plots the analytical approximation. The parameter values are $\rho = 0.10$, $\gamma = 1.347$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = \sigma_N = 1$, and $\bar{\theta} = 5.1769$.

Figure 4 plots the average volume measure ν versus different values of transactions cost κ as well as the appropriate power laws.

5.4 Bid/Ask Prices and Quote Sizes

Even in the presence of a transactions cost, each agent is willing to transact at the right prices: he is willing to buy at a low enough price, P_t^B, and sell at a high enough price, P_t^A. For prices in between these two extremes, the agent prefers not to transact. We define these two critical prices as the agent's bid and ask prices. As it turns out under fixed transactions costs, an agent is willing to buy/sell a finite amount at his bid/ask prices. We define the amount that an agent is willing to transact at his bid and ask prices as the bid and ask quote size, denoted by δ_t^B and δ_t^A, respectively. Clearly, an agent's bid and ask prices and their quote sizes depend on his current stock position, the current state of his non-traded risk, as well as on the transactions costs.

In Theorem 4, we have shown the explicit dependence of the agent's trading policy on the stock prices and the allocation of transactions costs. In particular, we have expressed his transaction boundaries (z_l and z_u) as a function of his current risk state (z_t) and the stock price. For a particular allocation of transaction costs (κ^+, κ^-), an agent's bid and ask prices are those prices that put him right at the lower and upper transaction boundaries, respectively. That is

$$z_l(P_t^B) = z_t \tag{38a}$$

$$z_u(P_t^A) = z_t. \tag{38b}$$

The agent is willing to buy only δ_t^B shares at the bid price P_t^B and to sell only δ_t^A shares at the ask price P_t^A. From Theorem 4, the quote sizes at the bid and ask are given by

$$\delta_t^B = z_m(P_t^B) - z_l(P_t^B) \tag{39a}$$

$$\delta_t^A = z_u(P_t^A) - z_m(P_t^A). \tag{39b}$$

The following lemma characterizes the bid and ask prices, as well as their associated quote sizes:

Theorem 7 *Let $\varepsilon \equiv \kappa^{\frac{1}{4}}$. For (a) κ small and $\kappa^{\pm}$ has the form in (29), and (b) $v(z,\varepsilon)$ analytic for small z and ε, each agent's bid and ask prices are*

$$P_t^B = \frac{\bar{a}_D}{r} - \gamma\sigma_D^2\left[\bar{\theta} + \phi\kappa^{\frac{1}{4}} + (z_t - \bar{\theta}) + \left(\tfrac{2}{11}k^{(1)} + \tfrac{47}{330}r\gamma\bar{p}_0\phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}})\right] \tag{40a}$$

$$P_t^A = \frac{\bar{a}_D}{r} - \gamma\sigma_D^2\left[\bar{\theta} - \phi\kappa^{\frac{1}{4}} + (z_t - \bar{\theta}) + \left(\tfrac{2}{11}k^{(1)} + \tfrac{47}{330}r\gamma\bar{p}_0\phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}})\right]. \tag{40b}$$

The corresponding quote sizes δ^B and δ^A are given by

$$\delta_t^B = \phi\kappa^{\frac{1}{4}} + \left(\tfrac{6}{11}k^{(1)}\phi - \tfrac{4}{55}r\gamma\bar{p}_0\phi^2\right)\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}}) \tag{41a}$$

$$\delta_t^A = \phi\kappa^{\frac{1}{4}} - \left(\tfrac{6}{11}k^{(1)}\phi - \tfrac{4}{55}r\gamma\bar{p}_0\phi^2\right)\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}}) \tag{41b}$$

where ϕ is given in Theorem 4.

To the order of $\kappa^{\frac{1}{2}}$, the quote sizes of the bid and ask are constant, independent of the agent's risk state. This contrasts sharply with the behavior of bid/ask prices themselves, which tend to vary linearly with the agent's risk state z_t. Moreover, with an arbitrary allocation of transactions costs between buyer and seller, the depth at the bid and ask differ.

Given the bid/ask prices of individual agents, we define the bid/ask prices of the market as the best bid/ask prices currently available across all agents in the market. They are denoted by P_t^{MB} and P_t^{MA}, respectively. Thus, $P_t^{MB} = \max\left[P_t^{1B},\, P_t^{2B}\right]$ and $P_t^{MB} = \min\left[P_t^{1A},\, P_t^{2A}\right]$.

For convenience, we define $\tilde{z}_t^i \equiv z_t^i - \bar{\theta}$. Obviously, $\tilde{z}_t^1 = -\tilde{z}_t^2$. Let $\tilde{z}_t \equiv |\tilde{z}_t^1| = |\tilde{z}_t^2|$. Then, $\max[\tilde{z}_t^1, \tilde{z}_t^2] = \tilde{z}_t$ and $\min[\tilde{z}_t^1, \tilde{z}_t^2] = -\tilde{z}_t$. We have the following expression for the bid/ask prices of the market:

$$P_t^{MB} = \frac{\bar{a}_D}{r} - \gamma\sigma_D^2\left[\bar{\theta} + \phi\kappa^{\frac{1}{4}} + \tilde{z}_t + \left(\tfrac{2}{11}k^{(1)} + \tfrac{47}{330}r\gamma\bar{p}_0\phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}})\right] \tag{42a}$$

$$P_t^{MA} = \frac{\bar{a}_D}{r} - \gamma\sigma_D^2\left[\bar{\theta} - \phi\kappa^{\frac{1}{4}} + \tilde{z}_t + \left(\tfrac{2}{11}k^{(1)} + \tfrac{47}{330}r\gamma\bar{p}_0\phi\right)\phi\kappa^{\frac{1}{2}} + o(\kappa^{\frac{1}{2}})\right]. \tag{42b}$$

The depth of the market bid and ask prices can be determined from the quote sizes of individual bid and ask prices in Theorem 7.

5.5 Allocation of Transactions Costs

Given each agent's bid/ask prices, we can now examine the trading process. In the presence of transactions costs, agents do not trade most of the time because one agent's bid price sits below the other agent's ask price. Trading occurs when two things happen at the same time: the bid price of one agent coincides with the ask price of another agent, *and* at this price the two agents want to buy/sell the same amount. In other words, trading occurs when the market bid/ask spread shrinks to zero *and* the depth at the bid equals the depth at the ask.

The market bid and ask prices and their depth given above indicate that it can be difficult to meet both of these conditions simultaneously for an arbitrary allocation of the transactions cost (i.e., $\kappa^{\pm}$). In particular, when $\tilde{z}_t = \phi\kappa^{\frac{1}{4}}$, $P_t^{MB} = P_t^{MA}$ and the agents agree on a transaction price. However, they do not agree on the amount to transact at that price because in general, $\delta^B \neq \delta^A$. This situation should not be surprising. Under fixed transactions costs, agents always transact a finite amount when they trade. In general, there is no reason to expect any symmetry between the amount they choose to buy and the amount they choose to sell when they decide to trade. This is different from the situation when they face zero transactions costs, in which case only infinitesimal amount is transacted (hence, the symmetry is guaranteed). The lack of symmetry between the depth at the bid and ask prices would prevent the existence of an equilibrium.

To allow trading to occur effectively, we need to choose a particular allocation of the fixed cost such that the depth at the bid and ask prices always match when the two prices coincide. From the expressions for the bid/ask depth, this is achievable by setting $k^{(1)} = \frac{2}{15}r\gamma\bar{p}_0\phi$ (to the order of $\kappa^{\frac{1}{2}}$). In this case, we have $\delta_t^{MB} = \delta_t^{MA} = \phi\kappa^{\frac{1}{4}}$. Trading occurs whenever the market bid-ask spread shrinks to zero and the amount $\delta = \phi\kappa^{\frac{1}{4}}$ is transacted. Thus, an equilibrium exists.

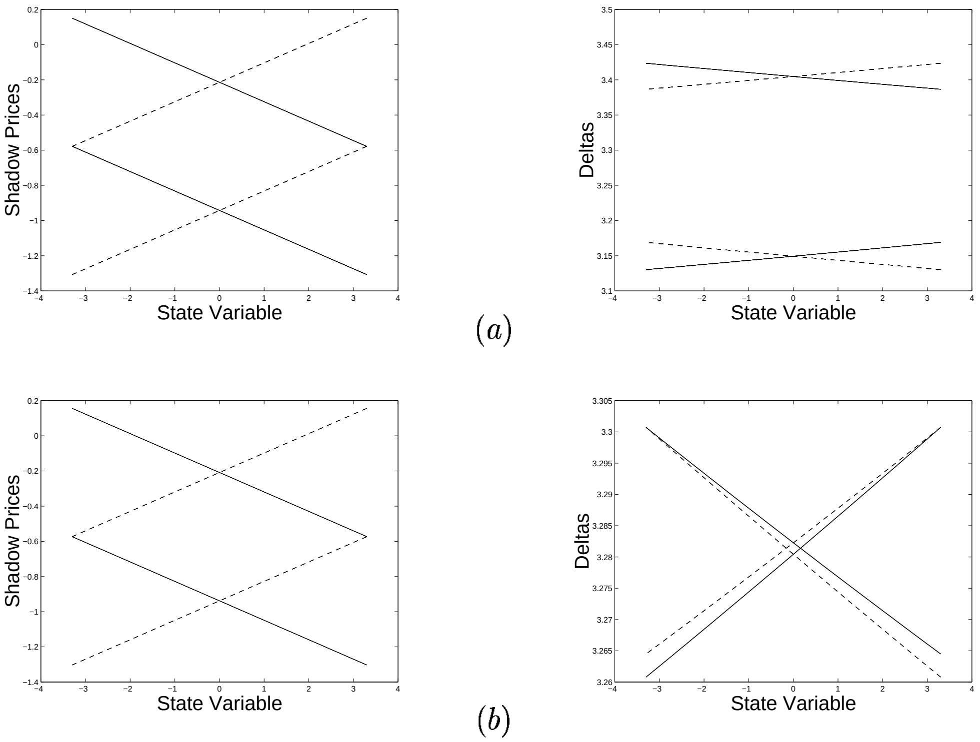


Figure 5: Agents' bid-ask prices and depth. The x-axis represents the value of each agent's state variable $z_t^i = \theta_t^i - hX_t^i$ with $X_t^1 = -X_t^2$. Each agent's prices and demands are mirror images around the point $z_t^i = 0$. The dashed and solid lines represent the shadow prices and demands of agents' 1 and 2, respectively. The parameter values are $\rho = 0.10$, $\gamma = 1.347$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = \sigma_N = 1$, $\bar{\theta} = 5.1769$, and $\kappa = 0.828\% \times \bar{P}$. In (a), $\kappa^+ = 1.1\kappa_e^+$, where κ_e^+ is the equilibrium allocation for the buy side transactions cost. In (b), κ^+ and κ^- are assigned their equilibrium values.

The discussion above can be illustrated by looking at agents' bid/ask prices/depth graphically. Figure 5 shows the bid and ask prices and their depth of both agents for various values of $z_t^i = \theta_{t\text{-}}^i - hX_t^i$, within the no-trade region. Since the agents' endowment of non-traded income is opposite to each other, the agents' prices and demands are mirror images of each other around the point $z_t^i = 0$.

Figure 5(a) describes the case when $\kappa^+ = \kappa^- = \kappa/2 = 0.025$. The left panel plots the bid/ask prices of the two agents and the right panel plots the depth of the bid and ask prices, respectively. Notice that as deviations in the risk exposure, which has the opposite sign for the two agents, approaches the boundary of no-trade region, the bid price of one agent approaches the ask price of the other agent. At the boundary, the two prices coincide and the two agents would agree on the price to transact. However, their desirable trade amount is different. As shown in the right panel of Figure 5(a), at the boundary of no-trade region, the depth of the selling price is lower than the depth of the buying price. This implies that trade would not occur, even though both agents can agree on a price.

The above situation can be avoided if we adjust the allocation of transactions cost. In particular, if we choose κ^+ and κ^- such that the depth of bid and ask prices also coincide at the boundary of no-trade region, trade would occur at the boundary because the agents agree on both the price and the amount of the transaction. Figure 5(b) illustrates this case. In this case, an equilibrium exists.

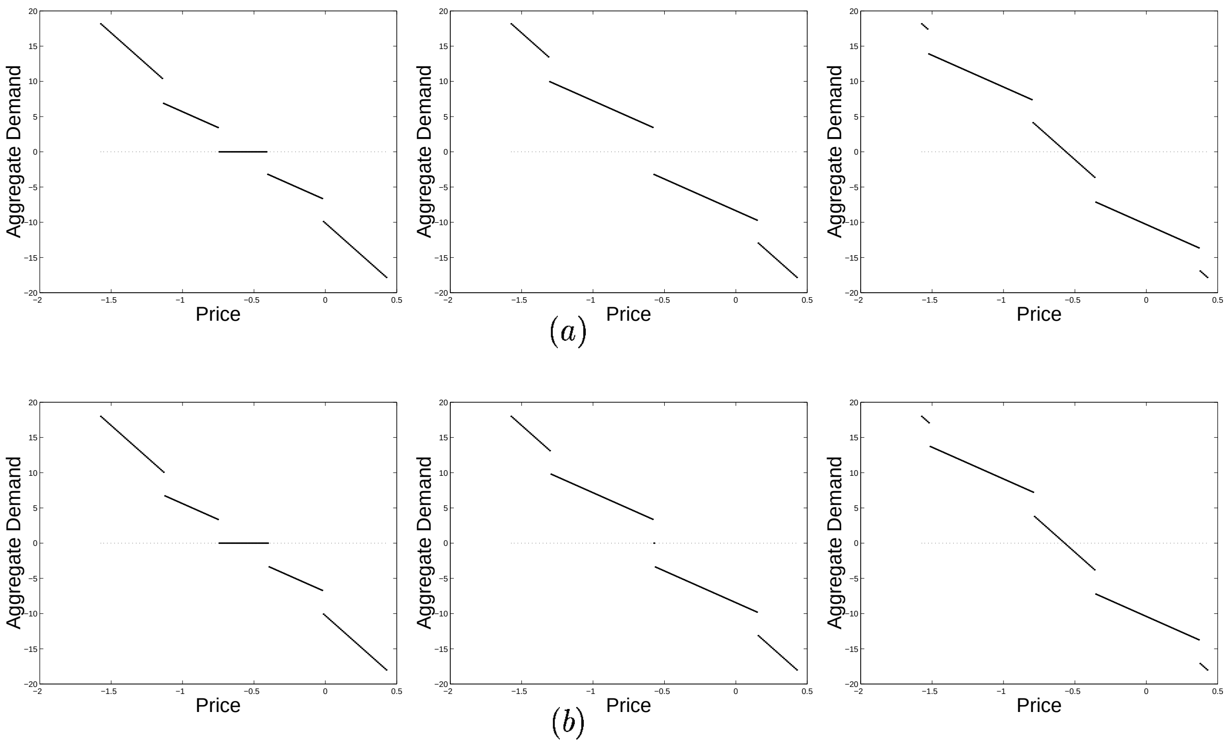


Figure 6: Aggregate demand curves for different values of X_t. In both (a) and (b), the three panels correspond to $X_t = 10, 15, 16.90, 24$, respectively. The parameter values are $\rho = 0.10$, $\gamma = 1.347$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = \sigma_N = 1$, $\bar{\theta} = 5.1769$, and $\kappa = 0.828\% \times \bar{P}$. In (a), $\kappa^+ = 1.1\kappa_e^+$, and in (b), $\kappa^+ = \kappa_e^+$, where k_e^+ is the equilibrium allocation for the buy side transactions cost.

Another way to see that an equilibrium may not exist for arbitrary cost allocations is to examine the corresponding aggregate demand curve. It can be seen from Figure 6(a) that the aggregate demand curve exhibits a discontinuity through 0 for some values of X_t. For small values of X_t, both agents have demands of 0 and the market could clear for a range of prices, as can be seen in the first panel of Figure 6(a). For values of X_t which bring both agents outside of their optimal control region (z_l, z_m) (if for example this was the initial endowment in the economy), both agents would like to trade immediately to their optimal allocation, and the trade amount would be $\delta^{\pm} = |z^i - z_m|$. In this case it follows from (30) that the market clearing price is

$$p_0 = \gamma\sigma_D^2 \left[\bar{\theta} - \frac{4}{11}\left(k^{(1)} - \frac{71}{120}\bar{\theta}r\gamma^2\sigma_D^2\phi\right)\phi\kappa^{\frac{1}{2}}\right] + o(\kappa^{\frac{1}{2}}). \tag{43}$$

The last panel of Figure 6(a) illustrates this situation. Only for values of X_t such that one agent's state variable is in the vicinity of z_l, the market does not clear for an arbitrary transactions cost allocation as shown in the middle panel of Figure 6(a). But because X_t evolves continuously, an equilibrium in the economy does not exist almost surely.

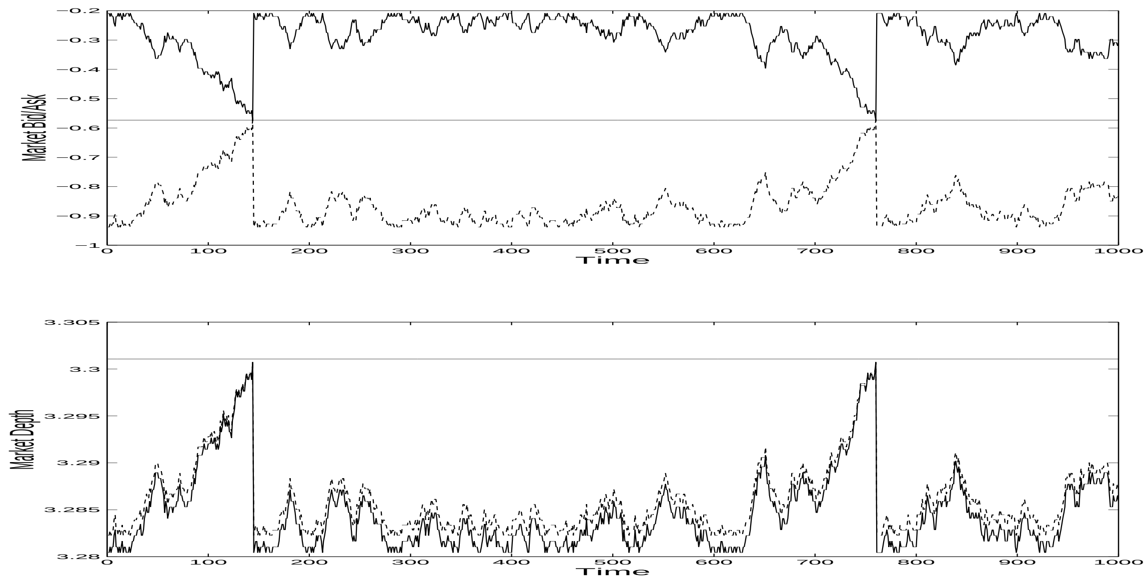
For the equilibrium allocation of transactions costs, κ_e^+, the aggregate demand curve remains discontinuous. However, it always passes through 0 for all values of X_t. Figure 6(b) shows the aggregate demand for $\kappa^+ = \kappa_e^+$ at various values of X_t.

It is well known that the existence of an equilibrium in the presence of fixed transactions costs is not always guaranteed. In our case, a particular allocation of the cost between the two trading parties is needed to reach an equilibrium. From a practical point of view, one may ask if such an allocation can be implemented through an actual trading process. The answer is affirmative. Let us imagine an electronic trading system through which agents can post their limit orders. Whenever a transaction occurs, the buyer pays κ^+ in addition to the dollar amount of his purchase and the seller receives κ^- less than the dollar amount of his sale. The sum of the charges, $\kappa^+ + \kappa^- = \kappa$ is used to cover the total fixed cost. Such a mechanism can then support the trading process as we discussed.

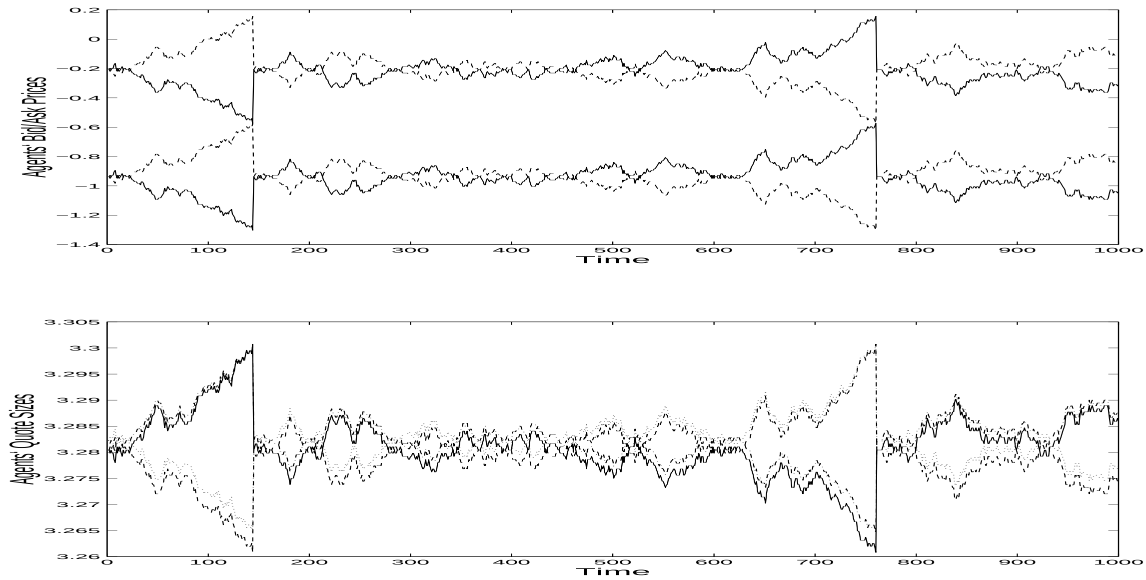
5.6 Trading Process

Let us now examine the actual trading process. As the risk exposure of each agent changes over time, their bid/ask prices and the respective depth also change. A transaction occurs when the market's bid and ask prices as well as their depth coincide. Figure 7(a) and 7(b) show the time path of a single realization of the economy.

Figure 7(a) shows the time evolution of the market bid/ask prices and of the number of shares offered and sought at the ask and bid, respectively. Note that the depth of the bid/ask prices is not constant over time, but its variation is much smaller than that in the



(a)



(b)

Figure 7: A single realization of the economy. In the top panel of (a), the dashed line represents the market bid price, and the solid line represents the market ask price. In the bottom panel of (a) , the dashed line represents the depth at the market bid price, and the solid line represents the depth at the market ask price. In the top panel of (b), the dashed lines represents one agent's bid and ask prices, and the solid lines represent the other's. In the bottom panel of (b), the solid and dotted lines represent one agent's ask and bid amounts respectively. The dashed and dot-dashed lines represent the other agent's ask and bid amounts, respectively. The parameter values are $\rho = 0.10$, $\gamma = 1.347$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = \sigma_N = 1$, $\bar{\theta} = 5.1769$, and $\kappa = 0.828\% \times \bar{P}$. Here, $T = 1/2$, $N = 1000$, where T is the number of years in the simulation and N is the number of points in the simulated Brownian motion.

bid/ask prices. We observe that the bid-ask spread approaches zero as a trade occurs and widens discontinuously right after the trade. This is intuitive because right after a trade, the desire for another trade is minimal. We also observe that the difference in depth between the bid and ask prices exhibits the same pattern, diminishing to zero as a trade occurs and widening discontinuously after the trade.

Figure 7(b) plots each agent's bid/ask spread and desired buy/sell amounts. Immediately after a trade, all of these variables revert discontinuously back to their level when each agent's endowment z_t is equal to $\bar{\theta}$. Interestingly the ultimate trade price is always the half-way point between the market bid and ask (the solid line in the top panel of Figure 7(a) is the mean of the current bid-ask prices).

6 A Calibration Exercise

Our model shows that even small fixed transactions costs imply a significant reduction in trading volume and an illiquidity discount in asset prices. To further examine the impact of fixed costs, we calibrate our model using historical data and derive numerical implications for the illiquidity discount, trading frequency, and trading volume. From (34), for small fixed costs κ we can re-express the illiquidity premium π as:

$$\pi = \frac{1}{\sqrt{6}} r^{-\frac{1}{2}} \gamma^{\frac{3}{2}} \sigma_N \sigma_X \bar{p}_0 \kappa^{\frac{1}{2}}. \tag{44}$$

Without loss of generality, we set $\sigma_N = 1$. The remaining parameters to be calibrated are: the interest rate r, the risk discount $\bar{p}_0$, the volatility of the idiosyncratic non-traded risk σ_X, the agents' coefficient of absolute risk aversion γ, and the fixed transactions cost κ. To do so, we review the empirical analysis of aggregate consumption and stock-market data in Campbell and Kyle (1993) and Heaton and Lucas (1996) in Sections 6.1 and Section 6.2, respectively, and draw on these results to perform our calibration in Section 6.3.

6.1 Campbell and Kyle (1993)

The starting point for our calibration exercise is a study by Campbell and Kyle (1993), in which they propose and estimate a detrended stock-price process of the following form:[25]

$$P_t^{\text{CK}} = V_t^{\text{CK}} - \frac{\lambda}{r} - Y_t^{\text{CK}} \tag{45}$$

where V_t^{CK} (the present value of future dividends discounted at the risk-free rate) is assumed to follow a Gaussian process, Y_t^{CK} (fluctuations in stock demand) is assumed to follow an AR(1) Gaussian process, r is the risk-free rate, and λ/r is the risk discount. In the Appendix, we show that in the absence of transactions costs, the general non-traded income process (3) of our model yields the following price process:

$$P_t = \bar{a}_D/r - \bar{p}_0 - p_Y Y_t \tag{46}$$

which is formally the same as (45) with $\bar{a}_D/r$ in our model corresponding to V_t^{CK} in Campbell and Kyle, $\bar{p}_0$ corresponding to λ/r, and $p_Y Y_t$ corresponding to Y_t^{CK}. Here, Y_t is the aggregate exposure of non-traded risk, which generates changes in stock demand and follows an AR(1) process, and p_Y is a constant, depending on the parameters of the model, which is given in the Appendix. Thus, we can rely on the estimates of the price process (45) by Campbell and Kyle, especially the estimates for r, λ, γ and σ_Y^{CK} (the instantaneous volatility of Y_t^{CK}) to calibrate the values of r, $\bar{p}_0$, γ and σ_Y in our model.

Campbell and Kyle based their estimates on annual time series of the U.S. real stock prices and dividends from 1871 to 1986. The real stock price of each year is defined by the Standard & Poors Composite Stock Price Index in January, normalized by the Producer Price Index (PPI) in the same month. The real dividend each year is taken to be the annual dividend per-share normalized by the PPI (over this sample period, the average annual dividend growth rate is 0.013). The price and dividend series are then detrended by an exponential detrending factor $e^{-0.013\,t}$ and the detrended series are used to estimate (45) via maximum likelihood. In particular, they obtain the following estimates for the price process:

$$r = 0.0370, \quad \lambda = 0.0210, \quad \bar{V} = 1.3514, \quad a_Y^{\text{CK}} = 0.0890$$
$$\sigma_Y^{\text{CK}} = 0.1371, \quad \sigma_P^{\text{CK}} = 0.3311, \quad \rho_{PY}^{\text{CK}} = -0.5176$$

[25] See Campbell and Kyle (1993, equation (2.3), p. 3).

where $\bar{V}$ denotes the unconditional mean of $V_t^{\rm CK}$, $a_Y^{\rm CK}$ and $\sigma_Y^{\rm CK}$ denote the mean-reversion coefficient and the instantaneous volatility of $Y_t^{\rm CK}$, $\sigma_P^{\rm CK}$ denotes the instantaneous volatility of $P_t^{\rm CK}$, and $\rho_{PY}^{\rm CK}$ denotes the instantaneous correlation between $P_t^{\rm CK}$ and $Y_t^{\rm CK}$.[26] From these estimates, we are able to compute values for the following parameters in our model (in addition to the value of r):

$$\bar{a}_D = 0.0500, \quad \bar{p}_0 = 0.5676, \quad \gamma\sigma_Y = 1.3470, \quad \sigma_D = 0.2853, \quad \bar{P} = 0.7838$$

(see the Appendix for the computation of these parameter values in our model from the estimates of Campbell and Kyle). These estimates do not allow us to fully specify the values of γ and σ_Y. However, they do allow us to fix the product of the two. Thus, a choice of γ uniquely specifies the value of σ_Y.

6.2 Heaton and Lucas (1996)

Another parameter to be calibrated is σ_X, the volatility of idiosyncratic non-traded risk. Because it is the *aggregate* non-traded risk that affects prices, Campbell and Kyle (1993) only provide an estimate for the volatility σ_Y of *aggregate* non-traded risk as a function of the coefficient of absolute risk aversion γ.[27] Obtaining an estimate for the magnitude of σ_X requires data at a more disaggregated level about individual agents' heterogeneous risk exposures. Heaton and Lucas (1996) have performed such an analysis using PSID data. They show that the residual variability in the growth rate of individual income—the variability of the component that is uncorrelated with aggregate income—is 8 to 13 times larger than the variability in the growth rate of aggregate income. Based on this result, we use values for σ_X that are 1, 4, 8, and 16 times the value of σ_Y in our exercise.[28]

[26]See Campbell and Kyle's (1993, p. 20) estimates for "Model B".

[27]This is simply due to the fact that in absence of transactions costs, the pure idiosyncratic component of the non-traded risk does not affect prices. Thus, estimation of the model based only prices provides little information about the idiosyncratic component without additional assumptions.

[28]The results of Heaton and Lucas are on the growth rates of individual and aggregate income. Our model does not exactly map into their setting. The income in our model is expressed in levels. At very short horizon, which is what we focus on, the difference between levels and growth rates are less significant. But their calibration is based on data over relatively long horizons. Our use of their results is merely suggestive.

6.3 Calibration Results

The two remaining parameters to be calibrated are the coefficient of absolute risk aversion γ and the fixed cost κ. Since there is little agreement as to what the natural choices are for these two parameters, we calibrate our model for a range of values for both.

Tables 1.1–1.4 report the results of our calibrations. Each of the four tables corresponds to a different ratio between the variability of the idiosyncratic component of agents' non-traded risk σ_X and the variability of the aggregate component σ_Y: Table 1.1 sets $\sigma_X = \sigma_Y$, Table 1.2 sets $\sigma_X = 4\sigma_Y$, Table 1.3 sets $\sigma_X = 8\sigma_Y$, and Table 1.4 sets $\sigma_X = 16\sigma_Y$.

Each table has five sub-panels, reporting different variables of interest for different values of the risk-aversion coefficient γ (and its corresponding variability σ_X of idiosyncratic non-traded risk as implied by the estimates of Campbell and Kyle) and different values of the transactions cost κ. Columns from the left to the right of each table are for values of γ being 0.001, 0.010, 0.100, 1.000, 1.500, 2.000 and 5.000, respectively. Rows in each sub-panel are for different values of κ. Here, we express the transactions cost as a fraction of the share price of the stock $\bar{P}$.[29] Thus, rows from the top to the bottom of each sub-panel are for the values of $\kappa/\bar{P}$ being 0.010%, 0.050%, 0.100%, 0.300%, 0.500%, 1.000%, and 5.000%, respectively.

The first sub-panel reports the expected time between trades in the stock. The second sub-panel reports the illiquidity discount in the stock price (as a percentage of the price $\bar{P} \equiv \bar{a}_D/r - \bar{p}_0$. The third sub-panel reports the illiquidity premium in the rate of returns on the stock. Here, the illiquidity return premium is defined as the increase in the expected rate of return on the stock when the transactions cost is positive. The fourth sub-panel reports the annual turnover ratio of the stock. The last panel reports the fixed transactions cost as a fraction of the average trade size, which is given by $\delta \cdot \bar{P}$.

From Tables 1.1–1.4, we observe that for a given level of risk aversion (and the variability of idiosyncratic non-traded risk), the time between trades, the illiquidity price discount and the illiquidity return premium all increase with the transactions cost, and the average turnover decreases with the transactions cost. For example, in Table 1.1, for a risk aversion parameter of 1.000, the average time between trades increases from 0.084 years (12 trades

[29]The purpose of normalizing the transactions cost by $\bar{P}$ is merely to provide relative measure of their magnitude. Since κ is a fixed cost, its value is, by definition, scale-dependent and must therefore be considered in the complete context of the calibration exercise.

per year) to 1.886 years (1 trade per two years) when the transactions cost increases from 0.010% of the share price to 5.000%. For the same increase in the transactions cost, the illiquidity discount in the share price increases from 0.068% to 1.547% and the illiquidity premium in the rate of return increases from 0.004% to 0.100%. The turnover, however, decreases from 117.01% to 24.65% per year.

For a given transactions cost (as a fraction of share price), the time between trades, the illiquidity price discount and the illiquidity return premium all increase with the risk aversion, and the average turnover decreases with the risk aversion. For example, in Table 1.1, for a transactions cost of 1.000% of the share price, the average time between trades ranges from 0.026 years (38 trades per year) to 1.886 years (1 trade per two years) when the value of risk aversion coefficient γ increases from 0.001 to 5.000. For the same range of γ, the illiquidity discount in price increases from 0.021% to 1.547% and the illiquidity premium in return increases from 0.001% to 0.100%. The turnover, on the other hand, decreases from 208.09% to 24.65%.

The inverse relation between trading and risk aversion might seem counter-intuitive. Holding everything else constant, when agents become more risk averse, they would like to trade more to unload his non-traded risk. However, in our calibration, what is held constant is the product of γ and σ_Y (or σ_X). Thus, when we increase the risk aversion, we also decrease the value of σ_X. Lower σ_X implies reduced needs for high frequency trading, thus longer times between trades and lower turnover.

Although qualitatively, we see the same dependence of average trade time, illiquidity price discount, illiquidity return premium and turnover on the risk aversion and transactions cost in Tables 1.1–1.4, the magnitudes are different. For example, for the same risk aversion of 1.000 and transactions cost of 1.000% of share price, the average time between trades is 0.840 years (1 trade per year) when $\sigma_X/\sigma_Y = 1$ versus 0.053 years (19 trades per year) when $\sigma_X/\sigma_Y = 16$, and the turnover is 37.94% versus 2,361.10% per year. More interestingly, for the same two cases, the illiquidity price discount is 0.684% versus 12.669% an the illiquidity return premium is 0.044% versus 0.925%.

Tables 1.1–1.4 show that our model is capable of yielding empirically plausible values for trading frequency, trading volume, and the illiquidity discount. In contrast to much of the existing literature, e.g., Schroeder (1998) and Vayanos (1998), we find that transactions costs can have very significant impact on both the trading frequency as well as the illiquidity discount in the stock price. For example, Schroeder (1998) finds that when faced with a fixed transactions cost of 0.1% of the total trade amount, an agent with a coefficient of relative risk aversion of 5.0 trades once every 10 years. In Table 1.1, we see that for a fixed cost of approximately 0.1% of the total trade amount, agents in our model trade anywhere between once every 0.015 years (or 67 times a year) and once every 0.026 years (or 38 times a year).

Clearly, for larger values of σ_X, agents have stronger needs for high-frequency trading and the transactions cost has a larger impact on equilibrium prices and expected returns.

Our model can easily generate the level of trading volume observed in the market. Even with relatively low levels of high-frequency trading needs, i.e., when $\sigma_X/\sigma_Y = 1$, the turnover can range from 16.41% to 658.08% for different values of risk aversion and transactions cost (see Table 1.1). With relatively high levels of high-frequency trading needs, i.e., when $\sigma_X/\sigma_Y = 16$, the turnover can range from 982.77% to 42,117.08%. The range of turnover covered by the different scenarios is compatible with the average turnover in the U.S. stock market, which is 92.56% per year for NYSE and AMEX from 1962 to 1998 (see Lo and Wang, 2000).

For the impact of transactions cost on prices, the calibration shows that small transactions cost can have significant contributions. For example, in Table 1.3 where σ_X/σ_Y is set at 8, which is in the range that Heaton an Lucas (1996) reports from the PSID data, a transactions cost of one percent of the share price can give rise to a 5.847% discount in the stock price and an increase of 0.396% in expected returns when the risk aversion is 1. If the transactions cost becomes five percent of the share price and the risk aversion is 5.000, the price discount due to illiquidity becomes 41.509% and the return premium becomes 4.527%, which are very significant. (As discussed below, the magnitude of the transactions cost in this case is merely 0.902% of the average transaction amount.) The significant impact of small transactions cost in our model is in clear contrast to the results in Constantinides (1986), Heaton and Lucas (1996) and Vayanos (1998).

The striking difference between our results and those of the existing literature stems from the fact that agents in our model have a strong need to trade frequently and not trading can be very costly. Furthermore, not trading means that holding the market-clearing levels of the stock is riskier. Most of the other transactions cost models fail to account for a high-frequency component in trading needs.[30] Our model strongly suggests that it is important to appropriately capture the agents' high-frequency trading needs in understanding the impact of transactions cost on the trading and pricing of financial securities.

In choosing the values of the transactions cost in our calibration exercise, we have used the transactions cost as a fraction of the stock price. However, the level of the stock price we use is derived from the estimates of Campbell and Kyle for the detrended prices. Thus, the interpretation of its magnitude is somewhat ambiguous. To better gauge the magnitude of the transactions cost as implied by our choice of fixed transactions cost as a fraction of the share price, we report in the bottom panel of Tables 1.1–1.4 the cost κ as a percentage of the total transaction amount $\delta \cdot \bar{P}$, that is, $100 \times \kappa/(\delta \cdot \bar{P})$. This normalized measure of the transactions cost also depends on the choice of fixed cost and the risk-aversion parameter. From Table 1.1, for example, we see that it ranges from 0 to 2.6% of the total transaction amount, which seems to be a plausible range empirically.

In our calibration exercise, we considered the range of γ from 0.001 to 5.000. Little empirical guidance is available on the reasonable range of γ. There has, however, been a lot of discussion on the reasonable values of relative risk aversion, which we denote by α (see, for example, Blume and Friend (1974); and Hansen and Jagannathan (1991)). Let W denote the wealth of a representative agent. We have $\alpha = \gamma W$. From the model (in the $\kappa = 0$ case), $W = \theta \bar{P}$, $\theta = \lambda/(r\gamma\sigma_D^2)$ and $\bar{P} = \bar{a}_D/r - \lambda/r$. Thus, $\alpha = \lambda(\bar{a}_D - \lambda)/(r^2\sigma_D^2)$. Interestingly, the values of r, λ, $\bar{a}_D$, and σ_D, uniquely determines the value of α, which is independent of the value of γ. From the calibration, $r = 0.0370$, $\lambda = 0.0210$ and $\sigma_D = 0.2853$, we have the relative risk aversion $\alpha = 5.4653$, which is consistent with the values suggested in the literature. From the value of λ, we have $\gamma = \lambda/(r\sigma_D^2\theta) = 6.9729/\theta$. If we let $\theta = 1$ (per capita stock holding is one) , then $\gamma = 6.9729$, which is comparable to the range of γ used in the tables.

[30]While partial equilibrium models such as Constantinides (1986) and Amihud and Mendelson (1986b), do contain a high-frequency component in the uncertainty faced by the agents, they do not take into account the unwillingness of agents to even hold the market-clearing level of the risky asset in the presence of transactions costs.

7 An Empirical Test

Fixed costs have a number of empirical implications for asset prices, trading volume, trading frequency, trade size, and bid/ask spreads and depths, as Sections 4–6 demonstrate. Perhaps the most direct implications are the power laws for trade sizes δ and inter-arrival times $\Delta\tau$ implied by Theorem 4:

$$\delta \approx \phi\kappa^{\frac{1}{4}}, \qquad \Delta\tau \approx \left(\phi^2/\sigma_z^2\right)\kappa^{\frac{1}{2}}. \tag{47}$$

A direct test of (47) can be readily performed by regressing $\log\delta$ and $\log\Delta\tau$, respectively, on $\log\kappa$ and testing the null hypotheses that the slope coefficients are $\frac{1}{4}$ and $\frac{1}{2}$, respectively. However, κ is generally not observable, hence the direct approach is difficult to implement.[31]

An indirect test of (47) can be performed by combining the two equations to yield the rather unexpected relation:

$$\delta \approx \sigma_z(\Delta\tau)^{\frac{1}{2}} \tag{48}$$

which can be tested by regressing the logarithm of trade size on the logarithm of inter-arrival times and testing the null hypothesis that the slope coefficient is $\frac{1}{2}$. This is a less-than-satisfying test of the impact of fixed costs on δ and $\Delta\tau$ because κ does not appear in (48).

A more compelling test of our model of fixed costs can be developed by applying (47) and (48) to the case of stock splits. The typical motivation for stock splits is to enhance liquidity in the face of indivisibilities associated with round-lot trading conventions, high share prices, or exchange-mandated minimum price variation rules.[32] For example, if round lots are cheaper to trade than odd lots, then a 2:1 stock split will reduce the cost of trading 50 pre-split shares. Such arguments for increased post-split liquidity are based on a decrease

[31]While certain components of fixed costs for stock trading are observable, e.g., ticket charges, there are other unobservable components that may be considerably larger, such as the opportunity cost of the time and effort spent on information acquisition and processing as well as the decision making and implementation involved in the trading process.

[32]See, for example, Angel (1997). Another motivation for stock splits is a signaling mechanism for revealing private information to agents; see Brennan and Copeland (1988), McNichols and Dravid (1990), and Pilotte and Timothy (1996). Muscarella and Vetsuypens (1996) attempt to differentiate between these two motives empirically using ADR "solo splits" and conclude that the liquidity effect dominates.

in fixed costs.[33] If we denote by κ_b and κ_a the fixed cost of trading before and after an $s\!:\!1$ split, respectively, and denote by δ_b and δ_a the optimal number of shares traded before and after an $s\!:\!1$ split, respectively, then we have:

$$\delta_b \approx \phi \kappa_b^{\frac{1}{4}}, \qquad \frac{\delta_a}{s} \approx \phi \kappa_a^{\frac{1}{4}} \tag{49}$$

where δ_a is renormalized by the split factor s because the split should have no impact on the optimal trade size (other than through its impact on κ). This yields the relation

$$\xi_\delta \equiv \log\left(\frac{\delta_a/s}{\delta_b}\right) = \tfrac{1}{4}\,\log\left(\frac{\kappa_a}{\kappa_b}\right). \tag{50}$$

A similar relation for $\Delta\tau$ follows from (47):

$$\xi_{\Delta\tau} \equiv \log\left(\frac{\Delta\tau_a}{\Delta\tau_b}\right) = \tfrac{1}{2}\,\log\left(\frac{\kappa_a}{\kappa_b}\right) \tag{51}$$

and combining (50) with (51) yields

$$\zeta \equiv \frac{\xi_\delta}{\xi_{\Delta\tau}} \approx \frac{1}{2} \tag{52}$$

which is an empirically testable implication that has the advantage of involving a clear and significant change in fixed costs (otherwise companies would not go to the expense of a split) without the need to observe the magnitudes of those costs.[34] Moreover, (50) and (51) provide two theoretically independent estimates of the change in fixed costs after a split. We examine these implications in Sections 7.1–7.3.

7.1 Data

To empirically test (52), we begin by identifying all stock splits that occurred during the period from January 1, 1993 to December 31, 1997 using the University of Chicago's Center

[33]Proportional costs are also affected by a split, but the evidence seems to suggest that these costs *increase*. For example, Conroy, Harris, and Benet (1990) conclude that the percentage bid/ask spreads of NYSE-listed companies typically increase after splits. Therefore, if liquidity enhancement is indeed an outcome of a stock split, it must be accomplished through a reduction in fixed costs. An indirect indication that splits reduce fixed costs is the fact that the number of shareholders tends to increase after a split, documented by Barker (1956) and Lamoureux and Poon (1987). Curiously, Copeland (1979) finds that during the era of fixed commissions, liquidity—as measured by trading volume, brokerage revenues, and bid/ask spreads—declines after splits.

[34]Here, we implicitly assume that the split is unexpected by the agents. Thus, the equilibrium before the split is not affected the forthcoming split. Of course, in practice the split is announced before it occurs. In this case, the agents' trading behavior may well reflect the impact of the expected split.

for Research in Security Prices (CRSP) event file. To ensure that our sample consists only of splits, we select only those stocks whose "share factor" changes match their "price factor" changes, and we eliminate all split events in which the split factor "`facshr(i)`" is not an integer when multiplied by 1, 2, 3, 4, or 5. This yields 2,842 split events during the five-year sample period.

For each of these split events, we use the New York Stock Exchange's Trades and Quotes (TAQ) database to obtain the trades and time stamps for these stocks over a 14-day windowed centered symmetrically around the split date. Some of the stocks identified in the CRSP database were not present in the TAQ database, hence we dropped the events of such stocks from our sample. For the remaining events, we collect all trades from the TAQ database for the 14-day window surrounding each event, dropping TAQ observations with correction codes 7–12 (see the TAQ User's Guide for more information), or observations missing a time stamp, trade size, or price. This leaves a total of 2,169 split events and 6,495,403 trades. Table 2 summarizes the number of split events in our sample according to split factor and year. Note that the more extreme split events, 2 : 1 and 3 : 2, dominate the sample in all years, accounting for at least 80% of all the split events in each year.

For each stock and each event, we eliminate the lowest and highest 5% of the trade sizes and inter-arrival times during the 14-day window to reduce the impact of outliers, and use the remaining trade sizes and inter-arrival times to perform our empirical analysis. If a stock had no data for trade size or inter-arrival times either before or after the split, we eliminate that event from our sample.

Table 3 reports means and standard deviations for trade size δ and inter-arrival times τ over 1-day, 2-day, 3-day, and 7-day intervals before and after splits. For a 1-day window and the entire sample of split events, the pre-split average trade size and inter-arrival time are 1,139 shares and 728 seconds, respectively; the post-split average trade size and inter-arrival time are 740 shares and 503 seconds, respectively. Using a longer window yields similar results as the rest of Table 3 shows—splits do enhance liquidity in the sense that average trade sizes and inter-arrival times always decline after splits, i.e., more frequent trading of smaller lots. Therefore, it is likely that fixed costs have declined after the split date.

7.2 Empirical Results

To compute the ratio ζ in (52), we first construct the quantities

$$\log\left(\overline{\delta}^{a}/(s\overline{\delta}^{b})\right) \qquad \text{and} \qquad \log\left(\overline{\Delta\tau}^{a}/\overline{\Delta\tau}^{b}\right)$$

for each split event using the pre- and post-split average trade sizes and inter-arrival times for 1-day to 7-day windows. Here, s is the split factor (e.g., 2 for a 2:1 split). We eliminate the lowest and highest 5% of these log-ratios from our sample to reduce the impact of outliers, and with the remaining sample we compute the ratio ζ for each event and summarize the sampling distributions of these ratios in Table 4.

The entries in the '1-Day' sub-panel show that the average ζ using the entire sample of split events is 0.482, which is remarkably consistent with the theoretical value of $\frac{1}{2}$ given in (52). Similar averages are obtained for 2:1 and 3:2 splits. However, 4:3 and 5:4 splits yield an average ζ of -0.583 and -0.150, respectively, for the 1-day window. The same patterns emerge from 2-day and 3-day windows: the average ζ is approximately $\frac{1}{2}$ when the entire sample of split events is used, but deviates significantly from $\frac{1}{2}$ for 4:3 and 5:4 splits. Not surprisingly, the 7-day window results are the farthest from (52)—over longer periods, factors other than fixed costs will influence trade size and inter-arrival times, adding noise to the power laws on which (52) is based. But overall, the relation (52) seems to be well supported by the majority of splits in our sample, especially those that involve more extreme split factors, which are precisely the cases in which the reduction in fixed costs are expected to be the greatest.

7.3 A Control

A natural control to our empirical analysis in Section 7.2 is to consider the implications of (47) for non-split dates. In particular, let $(\delta_b, \Delta\tau_b)$ and $(\delta_a, \Delta\tau_a)$ denote the optimal trade size and inter-arrival time before and after an arbitrary non-split date, respectively. Then $\delta_b = \delta_a$, $\Delta\tau_b = \Delta\tau_a$, the split factor $s = 1$, and that transactions costs have not changed, which implies

$$\xi_\delta = \xi_{\Delta\tau} = 0. \tag{53}$$

Table 5 reports estimates of ξ_δ and $\xi_{\Delta\tau}$ for the same data set used in Table 4, but where the "before" and "after" windows are centered either before the split date or after the split date, and where a 1-day window is used to compute average trade sizes and inter-arrival times. For example, the first sub-panel labeled '*Dates* -5 *and* -4' contains estimates for ξ_δ and $\xi_{\Delta\tau}$ where the "before" period is the fifth day before the split and the "after period is the fourth day before the split. In contrast to the entries in Table 4, the estimates of ξ_δ and $\xi_{\Delta\tau}$ are considerably smaller in magnitude and fluctuate around 0.000 without any discernible pattern. These results, and those of Table 4, suggest that our model of fixed costs may be a reasonable approximation for US equity markets.

8 Conclusion

We have developed a continuous-time equilibrium model of asset prices and trading volume with heterogeneous agents and fixed transactions costs. With prices, trading volume, and inter-arrival times determined endogenously, we show that even a small fixed cost of trading can have a substantial impact on the frequency of trade. Investors follow an optimal policy of not trading until their risk level reaches either a lower or upper boundary, at which point they incur the fixed cost and trade back to an optimal level of risk exposure. As the agents' endowment uncertainty increases, their "no-trade" region increases as well, despite the fact that the expected time between trades declines. Investors optimally balance their desire to hedge their endowment risk exposure against the fixed cost of transacting.

We also show that small fixed costs can induce a relatively large premium in asset prices. The magnitude of this illiquidity premium is more sensitive to the risk aversion of agents than is the risk premium. Because agents must incur a transactions cost with every trade, they do not rebalance very often. In between trades, they face some uncertainty as to the level of their holdings of the risky asset. This increases the effective risk faced by the agent for holding the risky asset, which reduces his demand for the risky asset at any given price, and to clear the market, the equilibrium price must compensate agents for the illiquidity of the shares that they hold. The price effect, then, relies heavily on the market-clearing motive, hence partial equilibrium models are likely to underestimate the effect of transactions costs on asset returns because they ignore this mechanism.

Because our model is dynamic, the market-clearing condition we propose has an auxiliary requirement: agents must want to trade at the same time. Imposing this "double coincidence of wants" endogenizes the market's order flow, and inter-arrival times between trades are determined in equilibrium as well as the quantities traded. Despite the fact that every buyer must have a seller and vice versa, we allow the fixed cost to be divided endogenously between the buyer and seller so that one agent can bear a larger share of the cost to induce the other agent to trade earlier than he otherwise would. This division of the fixed cost between buyer and seller is a means of representing the compensation for the provision of "immediacy" that typically accrues to market makers, and provides a natural bridge between the asset-pricing literature (in which risk sharing is the prime motive for trading) and the market microstructure literature (in which the facilitation of trade through market making activities is the main focus).

Although our model has many interesting theoretical and empirical implications, it is admittedly a rather simple parameterization of a considerably more complex set of phenomena. In particular, our assumption of perfect correlation between the dividend and endowment flows is likely to exaggerate the hedging motive in our economy. If a perfect hedging vehicle were not available, then agents may trade less often. The persistence of the endowment shocks in our economy may increase both the illiquidity discount and the desire to trade. Moreover, we do not allow for an aggregate endowment component (indeed our aggregate endowment is exactly zero), which certainly does exist in reality. All of these are interesting and important extensions of our model.

Another set of questions has to do with the effects of investor and security heterogeneity. For example, Vayanos and Vila (1999) and Huang (1998) consider the implications of transactions costs that are asymmetric across different securities. Also, fixed costs may differ across individuals. Who, then, is the marginal, or price-setting agent? It is unclear what effect transactions costs may have in the presence of many small heterogeneous agents. The behavior of risk and illiquidity discounts in the presence of heterogeneous securities and transactions costs remains an unanswered question. A more complete understanding of transactions costs will involve a resolution of some of these outstanding issues.

9 Appendix

Proof of Theorem 1

The proof of Theorem 1 consists of three steps. We first define the quasi-variational inequalities (QVI) for the optimization problem (9). Next, we show that under the conditions in Theorem 1, a solution to the QVI exists, which gives a candidate value function and a candidate optimal policy. Finally, we verify that the solution to the QVI is indeed the solution to (9). Our proof is similar to those given in Eastham and Hastings (1988), Korn (1998), Cadenillas and Zapataro (2000).

Quasi-Variational Inequalities (QVI)

Consider a candidate value function $e^{-\rho t}I(M,\theta,X)$. Define the optimal trade operator $\mathrm{T}[\cdot]$ as follows:

$$\mathrm{T}[I] \equiv \sup_{\delta} I(M - \delta P - \kappa(\delta), \theta + \delta, X)$$

where $\kappa(\delta)$ is the transactions cost function given in (2).

Definition A.1 *A twice-differentiable function $I(M,\theta,X)$ satisfies the quasi-variational-inequalities for optimization problem (9) if for all θ and c,*

$$\mathrm{D}[I] - \rho I - e^{-\gamma c} \leq 0 \tag{A.1a}$$

$$\mathrm{T}[I] \leq I \tag{A.1b}$$

$$(I - \mathrm{T}[I])\left(\sup_{c} \left\{\mathrm{D}[I] - \rho I - e^{-\gamma c}\right\}\right) = 0 \tag{A.1c}$$

where $\mathrm{D}[\cdot]$ is the Itô operator (defined in Footnote 18).

A solution I of the QVI (A.1) separates the state space $S \equiv (M,\theta,X)$ into two disjoint regions: an intervention (action) region A and a continuation (no action) region Ã:

$$\mathrm{A} \equiv \left\{S : I = \mathrm{T}[I], \ \sup_{c} \mathrm{D}[I] - \rho I - e^{-\gamma c} \leq 0\right\}$$

$$\tilde{\mathrm{A}} \equiv \left\{S : I < \mathrm{T}[I], \ \sup_{c} \mathrm{D}[I] - \rho I - e^{-\gamma c} = 0\right\}.$$

Given a solution to the QVI, we can construct a corresponding policy.

Definition A.2 *Let I be a solution of the QVI (A.1). The policy $(c; \{\tau, \delta\})$ defined by*

$$c = \arg\sup\left\{\mathrm{D}[I] - \rho I - e^{-\gamma c}\right\} \quad a.s. \quad \forall\ S \in \tilde{\mathrm{A}} \tag{A.2a}$$

$$\tau_k = \inf\{t > \tau_{k-1} : I = \mathrm{T}[I]\} \tag{A.2b}$$

$$\delta_k = \arg \mathrm{T}[I] \tag{A.2c}$$

(when it exists) is called the QVI-policy associated with I, where $\tau_0 = 0$.

We now seek the solution of the QVI (A.1). In the continuation region $\tilde{\mathrm{A}}$, we have

$$\begin{aligned} 0 &= \sup_c \left\{\mathrm{D}[I] - \rho I - e^{-\gamma c}\right\} \\ &= \sup_c \left\{I_M(rM - c + \theta\bar{a}_D) + \tfrac{1}{2}I_{MM}(\theta - hX)^2\sigma_D^2 + \tfrac{1}{2}I_{XX}\sigma_X^2 - \rho I - e^{-\gamma c}\right\} \end{aligned}$$

where I_M and I_X denote I's first order partial derivative with respect to M and X, respectively. The optimal c is

$$c = \frac{1}{\gamma}(\ln\gamma - \ln I_M). \tag{A.3}$$

I then satisfies

$$0 = I_M\left(rM + \frac{1}{\gamma}\ln r - \frac{1}{\gamma}\ln I_M + \bar{a}_D\theta\right) + \tfrac{1}{2}I_{MM}(\theta - hX)^2\sigma_D^2 + \tfrac{1}{2}I_{XX}\sigma_X^2 - \rho I - \frac{1}{\gamma}I_M. \tag{A.4}$$

Without loss of generality, we assume that I takes the form

$$I(M, \theta, X) = -e^{-r\gamma(M + p_D\theta) - V(\theta, X)}. \tag{A.5}$$

Equation (A.4) for I then reduces to the following equation for V:

$$0 = r\left(V - \bar{v}\right) + \tfrac{1}{2}\sigma_X^2\left(V_X^2 - V_{XX}\right) + \tfrac{1}{2}r^2\gamma^2\sigma_D^2\left(\theta - hX\right)^2 \tag{A.6}$$

where $\bar{v} \equiv (\rho - r + r\ln r)/r$, $\sigma_X^2 \equiv b_X b_X'$, and $\sigma_D^2 \equiv b_D b_D'$. This is Equation (19). Let $z \equiv \theta - hX$ and $V(X, \theta) \equiv v(z) + \bar{v}$. (A.6) reduces to

$$0 = \sigma_z^2 v'' - \sigma_z^2 v'^2 - 2rv - (r\gamma)^2\sigma_D^2 z^2 \equiv \mathrm{G}[v(z)] \tag{A.7}$$

with $\sigma_z^2 \equiv h^2\sigma_X^2$, which is Equation (20). We can now rewrite the value function as

$$I(M, \theta, X) = -e^{-r\gamma(M + p_D\theta) - v(z) - \bar{v}} \tag{A.8}$$

which is Equation (21).

The QVI can be rewritten in terms of $v(z)$ as follows:

$$\mathrm{G}[v(z)] \equiv \sigma_z^2 v''(z) - \sigma_Z^2 v'(z)^2 - 2rv(z) - (r\gamma)^2 \sigma_D^2 z^2 \leq 0 \tag{A.9a}$$

$$\mathrm{T}[v(z+\delta) - \delta p_0 - \kappa(\delta)] \leq v(z) \tag{A.9b}$$

$$[v(z) - v(z+\delta) - \delta p_0 - \kappa(\delta)]\, \mathrm{G}[v(z)] = 0. \tag{A.9c}$$

Now we need to solve (A.9). The result is summarized in the following lemma:

Lemma A.1 *Let $\widehat{v}(\cdot)$ be a solution of (A.7) for $z \in [z_l, z_u]$ such that*

$$z_l \leq \frac{rp_0 - \sqrt{(rp_0)^2 + \sigma_D^2 q_l}}{r\gamma\sigma_D^2}, \quad z_u \geq \frac{rp_0 + \sqrt{(rp_0)^2 + \sigma_D^2 q_u}}{r\gamma\sigma_D^2} \tag{A.10}$$

and

$$\widehat{v}\,'(z) > -r\gamma p_0, \quad z \in (z_l, z_m) \tag{A.11a}$$

$$\widehat{v}\,'(z) < -r\gamma p_0, \quad z \in (z_m, z_u) \tag{A.11b}$$

where

$$\begin{aligned} q_l &\equiv -(r\gamma p_0)^2 \sigma_z^2 - 2r\left[\widehat{v}(z_m) - r\gamma(\kappa^+ - p_0 z_m)\right] \\ q_u &\equiv -(r\gamma p_0)^2 \sigma_z^2 - 2r\left[\widehat{v}(z_m) - r\gamma(\kappa^- - p_0 z_m)\right] \end{aligned}$$

and $z_l \leq z_m \leq z_u$. Then $v(z)$ defined by

$$v(z) \equiv \begin{cases} \widehat{v}(z_m) - r\gamma[\kappa^+ - p_0(z_m - z)], & z \leq z_l \\ \widehat{v}(z), & z_l \leq z \leq z_u \\ \widehat{v}(z_m) - r\gamma[\kappa^- + p_0(z - z_m)], & z_u \leq z \end{cases} \tag{A.12}$$

is a solution to the QVI (A.9).

Proof. Given the form of v in (A.12), we have

$$\mathrm{G}[v] = \begin{cases} q_l + 2(r\gamma)rp_0 z - (r\gamma)^2\sigma_D^2 z^2, & z \leq z_l \\ 0, & z_l \leq z \leq z_u \\ q_u + (r\gamma)rp_0 z - (r\gamma)^2 \sigma_D^2 z^2, & z_u \leq z \end{cases}. \tag{A.13}$$

It is easy to verify that as long as (A.10) holds, (A.9a) is satisfied. We next turn to (A.9b). The first order condition for $\mathrm{T}[v(z+\delta) - r\gamma p_0 \delta - \kappa(\delta)]$ is

$$-v'(z+\delta) - r\gamma p_0 = 0.$$

From (A.11), $v''(z_m) < 0$. Thus, whenever $z \notin (z_l, z_u)$, it is optimal to trade an amount $\delta = z_m - z$. As a result, $v(z) = \mathrm{T}[v(z+\delta) - r\gamma p_0 \delta - \kappa(\delta)]$ for $z \notin (z_l, z_u)$. Moreover,

$$-r\gamma p_0 - v'(z) < 0, \quad z \in (z_l, z_m) \quad \text{and} \quad -r\gamma p_0 - v'(z) > 0, \quad z \in (z_m, z_u).$$

Thus, when $z \in (z_l, z_u)$, any trade with $\delta \neq 0$ is suboptimal. Since z_m is the optimal point to trade to, $v > \mathrm{T}[v(z+\delta) - r\gamma p_0 \delta - \kappa(\delta)]$ for $z \in (z_l, z_u)$. Thus, we have (A.9b). Combine the above with (A.13), we have (A.9c). Q.E.D.

The following lemma on the bounds of v is useful later.

Lemma A.2 *Let v be a solution to the QVI given in Lemma A.1. Then, v has a lower bound linear in z*

$$v(z) \geq v^{\min} - r\gamma p_0 z$$

where $v^{\min}$ is a constant, and $|v'|$ has a constant upper bound $\bar{v}_1$.

Proof. Let $\kappa^{\max} = \max\{\kappa^+, \kappa^-\}$ and $\delta^{\max} = \max\{\delta^+, \delta^-\}$. As a solution to the QVI, $v(z)$ is smooth for $z \in \tilde{\mathrm{A}}$ and $v(z) \geq [\widehat{v}(z_m) + r\gamma p_0 z_m - r\gamma\kappa^{\max}] - r\gamma p_0 z$ for $z \in \mathrm{A}$, where $z = \theta - hX$. Since $\tilde{\mathrm{A}}$ is bounded, $v(z)$ is bounded below and above on $\tilde{\mathrm{A}}$. Let $\tilde{v}$ denote v's lower bound on $\tilde{\mathrm{A}}$. We then have the lower bound for v for all z:

$$v(z) \geq \min\left\{\tilde{v} - r\gamma p_0 \delta^{\max}, \widehat{v}(z_m) - r\gamma\kappa^{\max}\right\} + r\gamma p_0 z_m - r\gamma p_0 z.$$

Letting $v^{\min} \equiv \min\left\{\tilde{v} + r\gamma p_0 \delta^{\max}, \widehat{v}(z_m) - r\gamma\kappa^{\max}\right\} + r\gamma p_0 z_m$, we have the lower bound for v in Lemma A.2. Also, the smoothness of v on $\tilde{\mathrm{A}}$ implies that v' is bounded on $\tilde{\mathrm{A}}$. Let $\tilde{v}_1$ be the upper bound of $|v'|$ on $\tilde{\mathrm{A}}$. On A, $v' = -r\gamma p_0$. Thus, $|v'|$ has a upper bound $\bar{v}_1 = \max\{\tilde{v}_1, r\gamma p_0\}$ for all z. Q.E.D.

Admissibility of the QVI-Policy

Next, we confirm that the QVI-policy given by the solution to the QVI is an admissible policy satisfying both Definition 1 and the financial condition (7).

Given a solution to the QVI, it is obvious that the QVI-policy exists. The consumption policy is given by (A.3). The trading policy is fully characterized by (z_l, z_m, z_u), from which the trading times and the trade amounts, as defined in (14-15), can be explicitly computed.

What remains to be shown is that the QVI-policy is admissible, i.e., it satisfies the conditions in Definition 1 and financial condition (7).

Lemma A.3 *The QVI-policy, given by the solution to v, (A.3) and (14)-(15), is admissible.*

Proof. We first verify conditions (1)-(5) in Definition 1. It is obvious that the QVI-policy satisfies Conditions (1)-(4). We only need to verify Condition (5). In particular, we need to show that for the QVI-trading policy,

$$\mathrm{E}_0\left[e^{\gamma\kappa n(s)}\right] < \infty \tag{A.14}$$

for all γ and κ positive, where $n(s)$ is the number of trades in $[0,\ s]$. For the QVI-policy, the time between trades is given by the first-passage time of a (one-dimensional standard) Brownian motion, $B(t)$, to hit $z = \delta$ (thus a stock sale occurs) or $z = -\delta$ (thus a stock purchase occurs).[35] The density function to hit $x = \delta$ (i.e., a sale) is given by

$$f(s,z) = \frac{z}{\sqrt{2\pi}s^{3/2}}e^{-\frac{z^2}{2s}} \qquad (s > 0)$$

(see, e.g., Øksendal, p. 130). Define $u \equiv s/z^2$. It is easy to verify the following upper bound on $f(s,z)$:

$$f(s,z) = (3/e)^{3/2}\frac{1}{\sqrt{2\pi}z^2} \qquad (s > 0).$$

Let $F(t,z) \equiv \int_0^t f(s,z)ds$ denote the cumulative density function for a sale to occur in $[0,\ s]$. Similarly, we can define $F_\cup(t,z) \equiv \int_0^t f_\cup(s,z)ds$ to the cumulative density function for a trade, either sale or a purchase, to occur in $[0,\ s]$ and $f_\cup(s,z)$ the corresponding density function. Note that

$$\begin{aligned}
F_\cup(t,z) &\equiv \mathrm{Prob}\big(\inf B(u) \leq -z \text{ or } \sup B(u) \geq z, u \leq t\big) \\
&= \mathrm{Prob}\big(\inf B(u) \leq -z, u \leq t\big) + \mathrm{Prob}\big(\sup B(u)u \leq t \geq z, u \leq t\big) \\
&\quad - \mathrm{Prob}\big(\inf B(u) \leq -z \text{ and } \sup B(u) \geq z, u \leq t\big) \\
&= 2F(t,z) - F_\cap(t,z) = 2\int_0^t f(s,z)ds - \int_0^t f_\cap(s,z)ds
\end{aligned}$$

[35]For simplicity, here we consider the case that $\delta^+ = \delta^- = \delta$. Extending to the case of $\delta^+ \neq \delta^-$ is straightforward.

where $F_{\cap}(t,z)$ is the cumulative density function for both a sale and a purchase to occur in $[0,\ s]$ and $f_{\cap}(t,z)$ is the corresponding density function. Since $f_{\cap}(t,z) \geq 0$,

$$f_{\cup}(t,z) \leq 2f(t,z) \leq (3/e)^{3/2}\sqrt{2/(\pi z)}.$$

The probability to have n trades in $[0,\ s]$, denoted by $p_n(s)$, is given by (see Karlin and Taylor, 1975, p. 169)

$$\begin{aligned} p_n(s) &= \int_0^s \cdots \int_0^{s-s_1-\cdots-s_n} dF_{\cup}(s_n)\cdots dF_{\cup}(s_1) = \int_{\sum_{k=1}^n s_k \leq s} \prod_{k=1}^n f_{\cup}(s_k) \prod_{k=1}^n ds_k \\ &\leq \left[(3/e)^{3/2}(2/\pi)^{1/2}\right]^n \int_{\sum_{k=1}^n u_k \leq u} \prod_{k=1}^n du_k = \left[(3/e)^{2/3}(2/\pi)^{1/2}\right]^n \frac{1}{n!}u^n \end{aligned}$$

where $u \equiv t/z^2$. Now, we can write

$$\mathrm{E}_0\left[e^{\gamma\kappa n(s)}\right] = \sum_{n=0}^{\infty} e^{n\gamma\kappa}p_n(s) \leq \sum_{n=0}^{\infty} \frac{1}{n!}\left[(3/e)^{3/2}(2/\pi)^{1/2}\, e^{\gamma\kappa}\right]^n.$$

Obviously, the sum on the right-hand-side is finite. Thus, (A.14) is satisfied by the QVI trade policy.

We now examine the financial condition (7). Substitute the QVI consumption policy into the equation for M_t, we have

$$M_t = \int_0^t \left[\frac{1}{\gamma}\ln r - \theta\bar{a}_D - \frac{1}{\gamma}(v+\bar{v})\right] ds + \int_0^t (\theta - hX)b_D dB_s - \sum_{\tau_k \leq t} \left[(p_D - p_0)\delta_k + \kappa_k\right].$$

Furthermore, under the QVI trade policy, $|v|$ and $|\theta - hX|$ are both bounded. We then have

$$\begin{aligned} 0 &\leq E_0\left[e^{-r\gamma(M_t + p_D\theta_t - hp_0X_t)}\right] \\ &\leq \mathrm{E}_0\left[e^{-r\gamma[K_M - \bar{\delta}n(t)]t + r\gamma\kappa^{\max}n(t)}e^{-r\gamma\int_0^t(\theta - hX)b_D dB_s + r\gamma p_0 X_t}\right] \end{aligned} \quad \text{(A.15)}$$

where $K_M = (1/\gamma)\ln r - \bar{v}/\gamma - v^{\min}/\gamma$. (For simplicity, we have assumed that $\theta_0 = 0$.) Given the bounds for $p_n(t)$, the bound on $|\theta - hX|$ (for the QVI-policy), and the normality of B_t and X_t, it is obvious that

$$0 \leq \mathrm{E}_0\left[e^{-kr\gamma[K_M - \bar{\delta}n(t)]t + r\gamma\kappa^{\max}n(t)}\right] < \infty \quad \text{and} \quad 0 \leq \mathrm{E}_0\left[e^{-kr\gamma\int_0^t(\theta - hX)b_D dB_s + r\gamma p_0 X_t}\right] < \infty$$

for $k \geq 0$. By Hölder's inequality, the right-hand-side of (A.15) is finite. Thus, the QVI-policy does satisfy the financial condition (7). This completes our of proof that the QVI-policy is admissible. Q.E.D.

Optimality of the QVI-Policy

Our last step is to show that the QVI-policy is optimal. To proceed further, we need two technical conditions on the solution to the QVI, $e^{-\rho t}I$: For (M, θ, X) generated by any admissible policy,

$$\lim_{t\to\infty} \mathrm{E}_0\left[e^{-\rho t}I(M_t,\theta_t,X_t)\right] = 0 \tag{A.16a}$$

$$\mathrm{E}_0\left[\int_0^t e^{-\rho s}\left(I_M b_M + I_X b_X\right)\left(I_M b_M + I_X b_X\right)' e^{-\rho s} ds\right] < \infty. \tag{A.16b}$$

The following two lemmas establish these two conditions for admissible policies.

Lemma A.4 *For any admissible policy in Θ, (A.16a) is satisfied.*

Proof. From Lemma A.2, $v \geq v^{\min} - r\gamma p_0 z$. Consequently,

$$0 \leq -e^{-\rho t}I(M_t,\theta_t,X_t) = e^{-\rho t}e^{-r\gamma(M_t+p_D\theta_t)-v(\theta_t-hX_t)-\bar{v}} \leq e^{-\rho t}e^{-r\gamma(M_t+\theta_t P_t+p_0 hX_t)-(\bar{v}+v^{\min})}.$$

Taking the expectation and the limit of $t \to \infty$, we have

$$0 \leq \lim_{t\to\infty} \mathrm{E}_0\left[-e^{-\rho t}I(M_t,\theta_t,X_t)\right] \leq \lim_{t\to\infty} \mathrm{E}_0\left[e^{-\rho t}e^{-r\gamma(M_t+\theta_t P_t+p_0 hX_t)}\right] e^{-(\bar{v}+v^{\min})}$$

where $P_t = p_D - p_0$ and the last equality comes from financial condition (7). By the dominated convergence, (A.16a) follows. Q.E.D.

Lemma A.5 *For any admissible policy in Θ, (A.16b) is satisfied.*

Proof. Let $F \equiv (I_M b_M + I_X b_X)(I_M b_M + I_X b_X)'$. It is sufficient to show that $\mathrm{E}_0[F] < \infty$. Since $I_M = -r\gamma I$ and $I_X = v'(\theta - hX)hI$, we have

$$F = I^2\left[(r\gamma)^2(\theta - hX)^2\sigma_D^2 + h^2 v'(\theta - hX)^2\sigma_X^2\right].$$

By Lemma A.2, $|v'|$ has a constant upper bound, $(v')^2 \leq \bar{v}_1^2$. Thus,

$$0 \leq F \leq I^2\left[(r\gamma)^2\sigma_D^2(\theta - hX)^2 + h^2\sigma_X^2\bar{v}_1^2\right].$$

By Hölder's inequality,

$$0 \leq \mathrm{E}_0[F] \leq \left(\mathrm{E}_0\left[I^4\right] E_0\left[(r\gamma)^2\sigma_D^2(\theta - hX)^2 + h^2\sigma_X^2\bar{v}_1^2\right]^2\right)^{1/2}.$$

First, we want to show that $\mathrm{E}_0[I^4] < \infty$. This follows directly from the financial condition (7) for the admissible policies. Next, we what to show that

$$E_0\left[(r\gamma)^2\sigma_D^2(\theta - hX)^2 + h^2\sigma_X^2\bar{v}_1^2\right]^2 < \infty.$$

It suffices to show that $\mathrm{E}_0\left[|\theta|^n\right] < \infty$ and $\mathrm{E}_0\left[|X|^n\right] < \infty$ for $n \leq 4$. The latter condition is obviously satisfied since X_s is normal conditional on X_0. For the former condition, it is sufficient to show that $\forall\ 0 \leq s < \infty$, $\mathrm{E}_0[e^{\lambda\theta_s}] < \infty$ for $\lambda \in (-\bar{\lambda}, \bar{\lambda})$ with $\bar{\lambda} > 0$ (see, e.g., Billingsley, p. 285). Note that $\theta_s = \theta_0 + \sum_{\tau_k \leq s}\delta_k$. $\forall\ \lambda$, $e^{\lambda\theta_s} \leq e^{|\lambda|(|\theta_0| + \bar{\delta}n(s))}$. By Condition (5) in Definition 1 for the admissible policies, $\mathrm{E}_0[e^{\lambda\bar{\delta}n(s)}] < \infty$. Thus, $\mathrm{E}_0[|\theta_s|^n] < \infty$. This completes our proof. Q.E.D.

Given (A.16a) and (A.16b), we now proceed to show that the solution to the QVI, which is generated by the QVI-policy, gives an upper bound on the value function of the optimization problem (9).

$$\begin{aligned}
&\int_0^{\tau_k\wedge t} -e^{\rho s - \gamma c_s}ds + e^{-\rho(\tau_k\wedge t)}I(S_{\tau_k\wedge t}) \\
= &\int_0^{\tau_k\wedge t} -e^{\rho s-\gamma c_s}ds + I(S_0) + \int_0^{\tau_k\wedge t} e^{-\rho s}(\mathrm{D}[I] - \rho I) + \int_0^{\tau_k\wedge t} e^{-\rho s}\left(I_M b_M + I_X b_X\right)dB_s \\
&+ \sum_{m:\,0\leq\tau_m\leq t} e^{-\rho\tau_m}\left[I(S_{\tau_m}) - I(S_{\tau_m^-})\right] \\
= &\ I(S_0) + \int_0^{\tau_k\wedge t} e^{-\rho s}\left(\mathrm{D}[I] - \rho I - e^{-\gamma c_s}\right) + \int_0^{\tau_k\wedge t} e^{-\rho s}\left(I_M b_M + I_X b_X\right)dB_s \\
&+ \sum_{m:\,0\leq\tau_m\leq t} e^{-\rho\tau_m}\left\{\mathrm{T}[I(S_{\tau_m})] - I(S_{\tau_m^-})\right\} \\
\leq &\ I(S_0) + \int_0^{\tau_k\wedge t} e^{-\rho s}\left(I_M b_M + I_X b_X\right)dB_s
\end{aligned}$$

where the first equality involves simple application of Itô's formula and the last inequality comes from the fact that I is a solution to the QVI, i.e., inequalities (A.1a) and (A.1b). Taking the expectation of both sides of the inequality, we have

$$\mathrm{E}_0\left[\int_0^{\tau_k\wedge t} -e^{-\rho s-\gamma c_s}ds\right] + \mathrm{E}_0\left[e^{-\rho(\tau_k\wedge t)}I(S_{\tau_k\wedge t})\right] \leq I(S_0) + \mathrm{E}_0\left[\int_0^{\tau_k\wedge t} e^{-\rho s}\left(I_M b_M + I_X b_X\right)dB_s\right].$$

If we let k goes to infinity, $\tau_k > t$ almost surely. (A.16b) insures that

$$\lim_{k\to\infty}\mathrm{E}_0\left[\int_0^{\tau_k\wedge t} e^{-\rho s}\left(I_M b_M + I_X b_X\right)dB_s\right] = 0.$$

Thus, in the limit of $k \to \infty$, we have

$$\mathrm{E}_0 \left[\int_0^t -e^{-\rho s - \gamma c_s} ds \right] + \mathrm{E}_0 \left[e^{-\rho t} I(S_t) \right] \le I(S_0).$$

If we further take the limit of $t \to \infty$, by the dominated convergence theorem and (A.16a), we have

$$\mathrm{E}_0 \left[\int_0^\infty -e^{-\rho s - \gamma c_s} ds \right] \le I(S_0)$$

for all admissible policies. Thus,

$$J(M, \theta, X, t) \equiv \sup_{(c,\,\delta) \in \Theta} \mathrm{E}_0 \left[\int_0^\infty -e^{-\rho s - \gamma c_s} ds \right] \le e^{-\rho t} I(S_0)$$

and the equality holds for the QVI-policy.

Thus, the QVI-policy is the optimal policy within the admissible set and the solution to the QVI gives the value function. This completes our proof of Theorem 1.

Proof of Theorem 2

When $\kappa = 0$, the conjectured price process is a constant. The agents' conjectured value function has the form: $J(W, X, t) = -e^{-\rho t - r\gamma W - v(X) - \bar{v}}$. The Bellman equation for each agent's optimization problem has the same form as in (17), except that the agent trades continuously to choose the optimal θ. His budget constraint can be expressed in terms of his wealth, given in Footnote 9 without the terms associated with transactions costs. In particular, we have for $t \ge 0$:

$$\begin{aligned} 0 &= \sup_{c,\,\theta} \left\{ -e^{-\rho t - \gamma c} + \mathrm{D}[J] \right\} \\ &= \sup_{c,\,\theta} \left\{ -e^{-\rho t - \gamma c} + J \left[-\rho - r\gamma(rW + \theta r p_0 - c) + \tfrac{1}{2}(r\gamma)^2 \sigma_D^2 (\theta - hX)^2 + \tfrac{1}{2}\sigma_X^2 \left(v'^2 - v'' \right) \right] \right\}. \end{aligned}$$

The optimal policies are given by

$$c = -\frac{1}{\gamma}[\ln r - r\gamma W - v(X) - \bar{v}] \quad \text{and} \quad \theta_t = \frac{p_0}{\gamma \sigma_D^2} + hX_t.$$

Substituting the optimal policies into (A.17), we find the solution for the value function: $v(X) = v_0 + r\gamma p_0 h X$, where $v_0 = \frac{1}{2} r\gamma^2 \sigma_D^2 \bar{z}_m^2 (1 - \gamma^2 \sigma_N^2 \sigma_X^2)$ and $\bar{z}_m = p_0/(\gamma \sigma_D^2)$. The same argument as in the $\kappa > 0$ case shows that the transversality condition is satisfied.

Since for agents $i = 1, 2$, $X_t^1 = -X_t^2$. Market clearing only requires that $p_0 = \gamma\sigma_D^2\bar{\theta}$ and the equilibrium price is indeed constant.

Proof of Theorem 3

When $k = \widehat{\kappa}\,1_{\{t>0\}}$ with $\widehat{\kappa} \to \infty$, agents do not trade for $t > 0$. We conjecture that for $t > 0$, $J(M, \theta, X, t) = -e^{-\rho t - r\gamma(M - p_D\theta) - V(\theta, X)}$. The optimal choice of c is given by (A.3). The corresponding Bellman equation for V is identical to (A.6). V has the following solution: $V(\theta, X) = v_0 - v_2(\theta - hX)^2$ where

$$v_0 = \bar{v} - 1 - \tfrac{1}{4}r\left(1 - \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}\right) \quad \text{and} \quad v_2 = \frac{r\gamma^2\sigma_D^2}{1 + \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}}.$$

Apparently, we must require that $4\gamma^2\sigma_N^2\sigma_X^2 < 1$.

At $t = 0$, agents are free to trade at no cost. They choose the optimal θ to maximize their expected utility:

$$\theta = \arg\sup\ J(M - \theta P, \theta, X, 0) = \frac{r\gamma p_0}{2v_2} + hX.$$

where $P = p_D - p_0$. Since for agents $i = 1, 2$ we have $X_t^1 = -X_t^2$, the market clearing condition, $\theta_1 + \theta_2 = \bar{\theta}$, requires that $\bar{\theta} = 2r\gamma p_0/v_2$. Thus, the risk discount is

$$p_0 = \frac{2\bar{\theta}v_2}{r\gamma} = \bar{p}_0 \frac{2}{1 + \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}} = \bar{p}_0\left(1 + \frac{1 - \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}}{1 + \sqrt{1 - 4\gamma^2\sigma_N^2\sigma_X^2}}\right)$$

where $\bar{p}_0 = \gamma\sigma_D^2\bar{\theta}$.

Proof of Theorem 4

When the value function is analytic in the interval (z_l, z_u), we can express it in the form of a Taylor series:

$$v(z) = \sum_{k=0}^{\infty} \tfrac{1}{k!} v_k (z - z_m)^k. \tag{A.17}$$

Substituting this into (20), we obtain

$$\sigma_Z^2 \sum_{n=0}^{\infty} \tfrac{1}{n!} v_{n+2}(z-z_m)^n \;=\; \sigma_Z^2 \sum_{n=0}^{\infty}\sum_{m=0}^{n} \tfrac{1}{(n-m)!m!} v_{n-m+1} v_{m+1}(z-z_m)^n + 2r \sum_{n=0}^{\infty} \tfrac{1}{n!} v_n (z-z_m)^n + (r\gamma)^2 \sigma_D^2 z^2. \tag{A.18}$$

Since the expansion is around z_m, we can write z^2 as $z^2 = (z-z_m)^2 + 2z_m(z-z_m) + z_m^2$. Matching powers on both sides of equation (A.18), we have the following conditions for the coefficients in the Taylor series:

$$0 = \sigma_z^2 \left(v_1^2 - v_2\right) + 2rv_0 + (r\gamma)^2 \sigma_D^2 z_m^2 \tag{A.19a}$$

$$0 = \sigma_z^2 \left(2v_1 v_2 - v_3\right) + 2rv_1 + 2(r\gamma)^2 \sigma_D^2 z_m \tag{A.19b}$$

$$0 = \sigma_z^2 \left(v_2^2 + v_1 v_3 - \tfrac{1}{2} v_4\right) + rv_2 + (r\gamma)^2 \sigma_D^2 \tag{A.19c}$$

$$0 = \sigma_z^2 \left[\sum_{l=0}^{k} \tfrac{1}{l!(k-l)!} v_{l+1} v_{k-l+1} - \tfrac{1}{k!} v_{k+2} \right] + \tfrac{2}{k!} r v_k \quad \forall\, k > 2. \tag{A.19d}$$

It is obvious that the other coefficients in the Taylor series can be expressed as polynomials of only two coefficients, v_1 and v_2. Note that v_0 does not enter into any of the higher order coefficients in (A.19d). Solving the value function (and the optimal trading policy) now reduces to solving v_1 and v_2. It is immediate that the smooth-pasting condition gives

$$v_1 = -r\gamma p_0. \tag{A.20}$$

The remaining conditions determine v_2 and the policy parameters, z_l, z_m, and z_u, which depend on κ^+, κ^-, p_0 and the other parameters of the model.

When κ^+ and κ^- are small, we consider the solution to v_k, z_l, z_m, and z_u of the form:

$$v_k = \sum_{n=0}^{\infty} v_k^{(n)} \varepsilon^n \tag{A.21a}$$

$$z_m - z_l \equiv \delta^+ = \sum_{n=0}^{\infty} b^{(n)} \varepsilon^n \tag{A.21b}$$

$$z_u - z_m \equiv \delta^- = \sum_{n=0}^{\infty} s^{(n)} \varepsilon^n \tag{A.21c}$$

where $\varepsilon \equiv \kappa^{\frac{1}{4}}$.

The two value-matching conditions are

$$\tfrac{1}{2}v_2\delta^{-2} + \tfrac{1}{6}v_3\delta^{-3} + \tfrac{1}{24}v_4\delta^{-4} + \tfrac{1}{120}v_5\delta^{-5} + \cdots = -\, r\gamma\kappa^- \tag{A.22a}$$

$$\tfrac{1}{2}v_2\delta^{+2} - \tfrac{1}{6}v_3\delta^{+3} + \tfrac{1}{24}v_4\delta^{+4} - \tfrac{1}{120}v_5\delta^{+5} + \cdots = -\, r\gamma\kappa^+ \tag{A.22b}$$

One of the smooth pasting conditions is satisfied by $v_1 = -r\gamma p_0$. We write the remaining two as follows

$$v_2\delta^- + \tfrac{1}{2}v_3\delta^{-2} + \tfrac{1}{6}v_4\delta^{-3} + \tfrac{1}{24}v_5\delta^{-4} + \cdots = 0 \tag{A.23a}$$

$$v_2\delta^+ - \tfrac{1}{2}v_3\delta^{+2} + \tfrac{1}{6}v_4\delta^{+3} - \tfrac{1}{24}v_5\delta^{+4} + \cdots = 0 \tag{A.23b}$$

We have four equations and four unknowns: $v_2, z_m, \delta^-, \delta^+$ (the dependence on z_m enters through v_3 by (A.19b)). Using the equations for the coefficients given in (A.19), and the expansion of $v_k, z_m, \delta^-, \delta^+$ given in (A.21), we match powers of ε in (A.23) and (A.22). That is, for every $n = \{0, 1, 2, 3, \dots\}$, we write the system of equations involving ε^n. Each system is linear in the i-th order coefficients. Proceeding in this way we obtain

$$v_2 = -2r\gamma\phi^{-2}\varepsilon^2 + o(\varepsilon^2) \tag{A.24a}$$

$$v_3 = \frac{r\gamma}{\phi^3}\left(\tfrac{240}{55}k^{(1)} + \tfrac{78}{55}r\gamma p_0\phi\right)\varepsilon^2 + o(\varepsilon^2). \tag{A.24b}$$

With more work, we can compute higher order approximations for all the coefficients.

Proof of Theorem 5

First we set $\delta^+ = \delta^-$ in equation (30). This gives us the value of κ^+ in equation (31a). Then we set $z_m = \bar{\theta}$ in equation(30). This gives us p_0 in (31a).

The Numerical Solution

To solve the boundary value ODE problem, we use a first-order expansion finite difference scheme set out in Press, et. al. (1992). The general idea is to convert our equation into a set of two coupled first-order finite difference equations of the form

$$\begin{aligned} y_2' &= y_1 \\ \sigma_z^2 y_1' &= \sigma_z^2 y_1^2 + 2ry_2 + (r\gamma)^2\sigma_D^2 z^2 \end{aligned}$$

where $y_2(\cdot) = v(\cdot)$ and the derivatives are understood to mean $y'(z) = (y(z+\Delta) - y(z))/\Delta$ for a grid spacing Δ. The system is then iterated using a first-order Taylor approximation until convergence. The free boundaries are found by using a numerical root finder in Matlab to find values of (z_l, z_m, z_u) such that the value matching and smooth pasting conditions are satisfied.

We can solve for an equilibrium by finding values of (p_0, κ^+) such that the optimal policy (z_l, z_m, z_u) satisfies the market clearing conditions. However, this requires a nested iteration: to solve for the equilibrium price and transactions cost allocation we need to solve the free-boundary problem for each candidate (p_0, κ^+). A faster approach is to find values of (p_0, κ^+, δ) such that the solution to a free boundary problem with boundaries $(\bar{\theta} - \delta, \bar{\theta} + \delta)$ satisfies the optimality conditions for the policy $(\bar{\theta} - \delta, \bar{\theta}, \bar{\theta} + \delta)$. This avoids the nested iterations required by the first approach.

Proof of Theorem 6

Consider the case where the endowment level is deterministic $X_t = \bar{a}_X t$ ($\bar{a}_X \geq 0$) and $\kappa^\pm = \kappa/2$. We conjecture that $J(M, \theta, X, t) = -e^{-\rho t - r\gamma(M - \theta\bar{a}_D/r) - V(\theta, X)}$. The Bellman equation reduces to a differential equation for V:

$$0 = \tfrac{1}{2}(r\gamma)^2\sigma_D^2(\theta - hX)^2 + r(V - \bar{v}) - \bar{a}_X V_X. \tag{A.25}$$

This is essentially the same equation as (19) (with a similar derivation), except that here $\sigma_X = 0$ and there is an additional term $-\bar{a}_X V_X$ due to the deterministic drift in X. Because the σ_X term drops out in this case, the differential equation for V is linear. Letting $z = \theta - hX$ and $V(\theta, X) = v(z) + \bar{v}$, we can reduce the PDE above to the following first-order linear free boundary problem

$$0 = \tfrac{1}{2}(r\gamma)^2\sigma_D^2 z + rv + \bar{a}_X h v'. \tag{A.26}$$

The boundary conditions are the same smooth-pasting and value matching conditions we had before, with the exception that the optimal policy consists of only two points: for the agent with endowment X_t, the optimal policy is (z_l, z_m) and for the agent with endowment $-X_t$, the optimal policy if (z_m, z_u) (z_m in both policies is the same). The reason for this type of policies is that the risk state z_t for one agent only decreases between trades, and for the other agent the risk state only increase. Thus for the agent with endowment X_t, z_t can only decrease and when it deviates sufficiently from the optimal point z_m, the agent rebalance back to the optimal point.

The solution to (A.26) is given by

$$v(z) = v_0 + 2v_1 z + v_2 z^2 + \beta e^{\alpha t}$$

where β is determined using the boundary conditions, and the other constants are given by

$$v_0 = -\bar{a}_X^2\gamma^2\sigma_N^2/r, \quad v_1 = \tfrac{1}{2}\bar{a}_X\gamma^2\sigma_N\sigma_D, \quad v_2 = -\tfrac{1}{2}r\gamma^2\sigma_D^2, \quad \alpha = -r\sigma_D/(\bar{a}_X\sigma_N).$$

For $P_t = \bar{a}_D/r - p_0$ and $\kappa^+ = \kappa^- = \kappa/2$, the boundary conditions for the X_t agent are

$$\begin{aligned} v(z_l) &= v(z_m) - r\gamma(\kappa/2 - p_0\delta^+) \\ v'(z_l) &= v'(z_m) = -r\gamma p_0 \end{aligned}$$

and the boundary conditions for the $-X_t$ agent are

$$\begin{aligned} v(z_u) &= v(z_m) - r\gamma(\kappa/2 + p_0\delta^-) \\ v'(z_u) &= v'(z_m) = -r\gamma p_0. \end{aligned}$$

Given the solution for v, the boundary conditions for the $X_t = \bar{a}_X t$ agent are

$$-r\gamma p_0 = 2v_2 z_m + 2v_1 + \beta\alpha e^{\alpha z_m} \tag{A.27a}$$

$$0 = -2v_2\delta + \beta\alpha e^{\alpha z_m}(e^{-\alpha\delta} - 1) \tag{A.27b}$$

$$0 = v_2\delta^2 - (2v_2 z_m + r\gamma p_0)\delta + r\gamma\kappa/2 \tag{A.27c}$$

where $\delta^+ = \delta$. Note that for the $-X_t$ agent, if we replace $\bar{a}_X$ in (A.27) with $-\bar{a}_X$ and let $\delta^- = -\delta$, then the algebraic form of the boundary conditions remains exactly the same. Hence solving (A.27) for $\bar{a}_X$ and for $-\bar{a}_X$ gives us solutions for both agents' control problems. The unknown variables are (β, z_m, δ). We are unable to solve these non-linear algebraic equations in closed form. We expand the unknowns as follows

$$z_m = \sum_{i=0}^{\infty} z_m^{(i)}\varepsilon^i, \quad \delta = \sum_{i=1}^{\infty} \delta^{(i)}\varepsilon^i, \quad \text{and} \quad \beta = \sum_{i=0}^{\infty} \beta^{(i)}\varepsilon^i$$

where the appropriate power law is given by $\varepsilon = \kappa^{\frac{1}{3}}$. Substituting the expansions into (A.27), and collecting terms for successive powers of ε, we are left with a series of linear equations for the coefficients in the above expansions. Hence, we are able to solve for (β, z_m, δ) in the approximate form.

Proof of Lemma 7

We first show how to compute the agent's bid price and bid amount. The ask price and ask amount are handled in the analogous way. For an agent with risk level z_t, we find a price $P_t^B = p_D - p_0^B$, such that $z_t + \delta^+(P_t^B) = z_m(P_t^B)$. Using the values of $\delta^\pm$ and z_m in Theorem 4 and doing a little algebra, we can solve for P_t^B, the agent's bid price, and for $\delta^+(P_t^B)$, the agent's bid amount.

Calibration

Here, we establish the equivalence between the model estimated by Campbell and Kyle (1993), with a price process given in (45), and our model in absence of transactions costs. Our model without transactions costs is analyzed in detail in Huang and Wang (1997). Given the dividend process (1), the agents' non-traded income (3), and their preferences, we have the following result:

Theorem A.8 *In the economy defined in Section 2 with $\kappa = 0$, the equilibrium stock price is*

$$P_t = p_D - p_0 - p_Y Y_t \tag{A.28}$$

where

$$\begin{aligned} p_0 &= \gamma\bar{\theta}\left(\sigma_D^2 + 2p_Y\sigma_{DY} + p_Y^2\sigma_Y^2\right) \\ p_Y &= \left[r\gamma\sigma_{DN} + (\sigma_{DY}/\sigma_Y^2)(r/2 + a_Y + r\gamma\sigma_{NY} - u)\right]/(r+u) \\ u &= \sqrt{-r^2\gamma^2\sigma_Y^2 + (r/2 + a_Y + r\gamma\sigma_{NY})^2}. \end{aligned}$$

Equation(A.28) has exactly the same form as (45), with $V_t = \bar{a}_D/r$ and an additional scaling constant p_Y for Y_t. To match the correlation structure in Campbell and Kyle (1993), we require $\sigma_{NY}/(\sigma_N\sigma_Y) = \sigma_{DY}/(\sigma_D\sigma_Y) \equiv \rho_Y$ and $\sigma_{DN} = -\sigma_D\sigma_N$ with $\sigma_N = 1$.

Let $Q_t \equiv \int_0^t (dP_t + dD_t - rP_t dt)$ denote the excess dollar return on one share of the stock and $R_t \equiv (r + a_Y)p_Y Y_t$. Then we have

$$dQ_t = (rp_0 + R_t)\, dt + b_Q dB_t \tag{A.29a}$$

$$dR_t = -a_R R_t dt + b_R dB_t \tag{A.29b}$$

where $a_R = a_Y$, $b_Q = b_D - p_Y b_Y$ and $b_R = (r + a_Y)p_Y b_Y$. Equation (A.29) is identical to the equations (3.3) and (3.4) in Campbell and Kyle (1993, p.10) to be estimated, except that they use M_t for the excess share return on the stock and N_t for the varying mean-return variable while we use Q_t and R_t, respectively. Following our notation, we have $\sigma_Q^2 = b_Q{}'b_Q = \sigma_D^2 + 2p_Y\sigma_{DY} + p_Y^2\sigma_Y^2$, $\sigma_R^2 = b_R{}'b_R = (r+a_Y)^2p_Y^2\sigma_Y^2$, and $\sigma_{QR}^2 = b_Q{}'b_R = (r+a_Y)^2p_Y^2(\sigma_{DY} + p_Y\sigma_Y^2)^2$.

Campbell and Kyle gave the estimates $\sigma_Q = 0.3311$, $\sigma_R = 0.0173$ and $\sigma_{QR}/(\sigma_Q\sigma_R) = -0.5176$. Together with their estimates for $r = 0.0370$, $\bar{a}_D = 0.050$ (the unconditional mean of V_t in equation (45) times r), $\lambda = 0.0210 = rp_0$ and $a_Y = 0.0890$, we have $\sigma_D = 0.2853$, $\rho_Y = -0.1194$, $\gamma\sigma_Y = 1.347$, and $\bar{P} = p_D - p_0 = 0.7838$.

10 References

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Table 1.1: Calibration results using parameter estimates from Campbell and Kyle's (1993) Model B, with the ratio of idiosyncratic to aggregate volatility set to 1 (i.e. $\sigma_X = 1 \times \sigma_Y$). The first sub-panel reports expected trade inter-arrival times $\Delta\tau$ (in years), the second sub-panel reports the illiquidity discount in the stock price (as a percentage of the price $\bar{P} = \bar{a}_D/r - \bar{p}_0$ in the frictionless economy), the third sub-panel reports the return premium (defined as $\bar{a}_D/P - \bar{a}_D/\bar{P}$ where P is the price under the transactions cost), the fourth sub-panel reports the annual turnover in percent ($100 \times \frac{\delta}{2\bar{\theta}\tau}$), and the fifth sub-panel reports the transactions cost as a percent of the transaction amount ($100 \times \frac{\kappa}{\delta P}$). These quantities are reported as functions of the transactions cost $\kappa_P \equiv \kappa/\bar{P}$ (in percentages), and the absolute risk aversion coefficient γ. Given γ, a unique value of σ_X^2 is implied by Campbell and Kyle's Model B, and $\bar{p}_0$ is determined from their estimates of λ and r.

γ	0.001	0.010	0.100	0.500	1.000	1.500	2.000	5.000
σ_X	1347.026	134.703	13.470	2.694	1.347	0.898	0.674	0.269
$\kappa/\bar{P}$ (%)				$\Delta\tau$ (Years)				
0.010	0.003	0.008	0.026	0.059	0.084	0.103	0.118	0.187
0.050	0.006	0.019	0.059	0.132	0.187	0.229	0.265	0.419
0.100	0.008	0.026	0.084	0.187	0.265	0.325	0.375	0.593
0.300	0.015	0.046	0.145	0.325	0.459	0.563	0.650	1.029
0.500	0.019	0.059	0.187	0.419	0.593	0.727	0.840	1.331
1.000	0.026	0.084	0.265	0.593	0.840	1.029	1.190	1.886
5.000	0.059	0.187	0.593	1.331	1.886	2.314	2.676	4.257
$\kappa/\bar{P}$ (%)				Illiquidity Discount (% of $\bar{P}$)				
0.010	0.002	0.007	0.021	0.048	0.068	0.083	0.096	0.152
0.050	0.005	0.015	0.048	0.107	0.152	0.186	0.215	0.341
0.100	0.007	0.021	0.068	0.152	0.215	0.264	0.304	0.483
0.300	0.012	0.037	0.118	0.264	0.373	0.458	0.529	0.840
0.500	0.015	0.048	0.152	0.341	0.483	0.592	0.684	1.088
1.000	0.021	0.068	0.215	0.483	0.684	0.840	0.971	1.547
5.000	0.048	0.152	0.483	1.088	1.547	1.903	2.206	3.546
$\kappa/\bar{P}$ (%)				Return Premium (%)				
0.010	0.000	0.000	0.001	0.003	0.004	0.005	0.006	0.010
0.050	0.000	0.001	0.003	0.007	0.010	0.012	0.014	0.022
0.100	0.000	0.001	0.004	0.010	0.014	0.017	0.019	0.031
0.300	0.001	0.002	0.008	0.017	0.024	0.029	0.034	0.054
0.500	0.001	0.003	0.010	0.022	0.031	0.038	0.044	0.070
1.000	0.001	0.004	0.014	0.031	0.044	0.054	0.063	0.100
5.000	0.003	0.010	0.031	0.070	0.100	0.124	0.144	0.235
$\kappa/\bar{P}$ (%)				Annual Turnover (%)				
0.010	658.08	370.06	208.09	139.15	117.01	105.72	98.38	78.23
0.050	440.13	247.47	139.15	93.04	78.23	70.68	65.77	52.29
0.100	370.06	208.09	117.01	78.23	65.77	59.43	55.30	43.96
0.300	281.18	158.11	88.89	59.43	49.96	45.13	41.99	33.37
0.500	247.47	139.15	78.23	52.29	43.96	39.71	36.94	29.35
1.000	208.09	117.01	65.77	43.96	36.94	33.37	31.04	24.65
5.000	139.15	78.23	43.96	29.35	24.65	22.26	20.70	16.41
$\kappa/\bar{P}$ (%)				Cost as % of Transaction Amount				
0.010	0.000	0.000	0.001	0.004	0.007	0.010	0.012	0.024
0.050	0.000	0.001	0.004	0.015	0.024	0.033	0.041	0.082
0.100	0.000	0.001	0.007	0.024	0.041	0.056	0.069	0.137
0.300	0.001	0.003	0.017	0.056	0.094	0.127	0.158	0.313
0.500	0.001	0.004	0.024	0.082	0.137	0.186	0.231	0.459
1.000	0.001	0.007	0.041	0.137	0.231	0.313	0.388	0.771
5.000	0.004	0.024	0.137	0.459	0.771	1.044	1.295	2.567

Table 1.2: Calibration results using parameter estimates from Campbell and Kyle's (1993) Model B, with the ratio of idiosyncratic to aggregate volatility set to 4 (i.e. $\sigma_X = 4 \times \sigma_Y$). The first sub-panel reports expected trade inter-arrival times $\Delta\tau$ (in years), the second sub-panel reports the illiquidity discount in the stock price (as a percentage of the price $\bar{P} = \bar{a}_D/r - \bar{p}_0$ in the frictionless economy), the third sub-panel reports the return premium (defined as $\bar{a}_D/P - \bar{a}_D/\bar{P}$ where P is the price under the transactions cost), the fourth sub-panel reports the annual turnover in percent ($100 \times \frac{\delta}{2\,\bar{\theta}\,\tau}$), and the fifth sub-panel reports the transactions cost as a percent of the transaction amount ($100 \times \frac{\kappa}{\delta P}$). These quantities are reported as functions of the transactions cost $\kappa_P \equiv \kappa/\bar{P}$ (in percentages), and the absolute risk aversion coefficient γ. Given γ, a unique value of σ_X^2 is implied by Campbell and Kyle's Model B, and $\bar{p}_0$ is determined from their estimates of λ and r.

γ	0.001	0.010	0.100	0.500	1.000	1.500	2.000	5.000
σ_X	5388.103	538.810	53.881	10.776	5.388	3.592	2.694	1.078
$\kappa/\bar{P}$ (%)				$\Delta\tau$ (Years)				
0.010	0.001	0.002	0.007	0.015	0.021	0.026	0.030	0.047
0.050	0.001	0.005	0.015	0.033	0.047	0.057	0.066	0.105
0.100	0.002	0.007	0.021	0.047	0.066	0.081	0.094	0.148
0.300	0.004	0.011	0.036	0.081	0.115	0.141	0.162	0.257
0.500	0.005	0.015	0.047	0.105	0.148	0.182	0.210	0.332
1.000	0.007	0.021	0.066	0.148	0.210	0.257	0.297	0.470
5.000	0.015	0.047	0.148	0.332	0.470	0.576	0.666	1.059
$\kappa/\bar{P}$ (%)				Illiquidity Discount (% of $\bar{P}$)				
0.010	0.009	0.027	0.086	0.192	0.272	0.334	0.386	0.611
0.050	0.019	0.061	0.192	0.431	0.611	0.750	0.868	1.381
0.100	0.027	0.086	0.272	0.611	0.868	1.065	1.233	1.968
0.300	0.047	0.149	0.473	1.065	1.516	1.865	2.161	3.476
0.500	0.061	0.192	0.611	1.381	1.968	2.425	2.814	4.548
1.000	0.086	0.272	0.868	1.968	2.814	3.476	4.042	6.596
5.000	0.192	0.611	1.968	4.548	6.596	8.239	9.675	16.495
$\kappa/\bar{P}$ (%)				Return Premium (%)				
0.010	0.001	0.002	0.005	0.012	0.017	0.021	0.025	0.039
0.050	0.001	0.004	0.012	0.028	0.039	0.048	0.056	0.089
0.100	0.002	0.005	0.017	0.039	0.056	0.069	0.080	0.128
0.300	0.003	0.010	0.030	0.069	0.098	0.121	0.141	0.230
0.500	0.004	0.012	0.039	0.089	0.128	0.159	0.185	0.304
1.000	0.005	0.017	0.056	0.128	0.185	0.230	0.269	0.450
5.000	0.012	0.039	0.128	0.304	0.450	0.573	0.683	1.260
$\kappa/\bar{P}$ (%)				Annual Turnover (%)				
0.010	5264.66	2960.52	1664.79	1113.27	936.12	845.86	787.15	625.95
0.050	3520.68	1979.79	1113.27	744.43	625.95	565.58	526.31	418.48
0.100	2960.52	1664.79	936.12	625.95	526.31	475.54	442.52	351.83
0.300	2249.49	1264.94	711.25	475.54	399.82	361.23	336.13	267.19
0.500	1979.79	1113.27	625.95	418.48	351.83	317.86	295.77	235.07
1.000	1664.79	936.12	526.31	351.83	295.77	267.19	248.60	197.53
5.000	1113.27	625.95	351.83	235.07	197.53	178.38	165.92	131.61
$\kappa/\bar{P}$ (%)				Cost as % of Transaction Amount				
0.010	0.000	0.000	0.001	0.002	0.004	0.005	0.006	0.012
0.050	0.000	0.000	0.002	0.007	0.012	0.017	0.021	0.041
0.100	0.000	0.001	0.004	0.012	0.021	0.028	0.035	0.069
0.300	0.000	0.001	0.008	0.028	0.047	0.064	0.079	0.157
0.500	0.000	0.002	0.012	0.041	0.069	0.093	0.116	0.230
1.000	0.001	0.004	0.021	0.069	0.116	0.157	0.194	0.386
5.000	0.002	0.012	0.069	0.230	0.386	0.523	0.649	1.287

Table 1.3: Calibration results using parameter estimates from Campbell and Kyle's (1993) Model B, with the ratio of idiosyncratic to aggregate volatility set to 8 (i.e. $\sigma_X = 8 \times \sigma_Y$). The first sub-panel reports expected trade inter-arrival times $\Delta\tau$ (in years), the second sub-panel reports the illiquidity discount in the stock price (as a percentage of the price $\bar{P} = \bar{a}_D/r - \bar{p}_0$ in the frictionless economy), the third sub-panel reports the return premium (defined as $\bar{a}_D/P - \bar{a}_D/\bar{P}$ where P is the price under the transactions cost), the fourth sub-panel reports the annual turnover in percent ($100 \times \frac{\delta}{2\,\bar{\theta}\,\tau}$), and the fifth sub-panel reports the transactions cost as a percent of the transaction amount ($100 \times \frac{\kappa}{\delta P}$). These quantities are reported as functions of the transactions cost $\kappa_P \equiv \kappa/\bar{P}$ (in percentages), and the absolute risk aversion coefficient γ. Given γ, a unique value of σ_X^2 is implied by Campbell and Kyle's Model B, and $\bar{p}_0$ is determined from their estimates of λ and r.

γ	0.001	0.010	0.100	0.500	1.000	1.500	2.000	5.000
σ_X	10776.2	1077.62	107.762	21.552	10.776	7.184	5.388	2.155
$\kappa/\bar{P}$ (%)				$\Delta\tau$ (Years)				
0.010	0.000	0.001	0.003	0.007	0.010	0.013	0.015	0.023
0.050	0.001	0.002	0.007	0.017	0.023	0.029	0.033	0.052
0.100	0.001	0.003	0.010	0.023	0.033	0.041	0.047	0.074
0.300	0.002	0.006	0.018	0.041	0.057	0.070	0.081	0.129
0.500	0.002	0.007	0.023	0.052	0.074	0.091	0.105	0.166
1.000	0.003	0.010	0.033	0.074	0.105	0.129	0.149	0.236
5.000	0.007	0.023	0.074	0.166	0.236	0.290	0.335	0.538
$\kappa/\bar{P}$ (%)				Illiquidity Discount (% of $\bar{P}$)				
0.010	0.017	0.054	0.172	0.386	0.546	0.670	0.775	1.233
0.050	0.038	0.121	0.386	0.868	1.233	1.516	1.756	2.814
0.100	0.054	0.172	0.546	1.233	1.756	2.161	2.507	4.042
0.300	0.094	0.298	0.951	2.161	3.094	3.824	4.451	7.287
0.500	0.121	0.386	1.233	2.814	4.042	5.011	5.847	9.678
1.000	0.172	0.546	1.756	4.042	5.847	7.287	8.542	14.443
5.000	0.575	1.233	4.042	9.678	14.443	18.462	22.123	41.509
$\kappa/\bar{P}$ (%)				Return Premium (%)				
0.010	0.001	0.003	0.011	0.025	0.035	0.043	0.050	0.080
0.050	0.002	0.008	0.025	0.056	0.080	0.098	0.114	0.185
0.100	0.003	0.011	0.035	0.080	0.114	0.141	0.164	0.269
0.300	0.006	0.019	0.061	0.141	0.204	0.254	0.297	0.501
0.500	0.008	0.025	0.080	0.185	0.269	0.337	0.396	0.684
1.000	0.011	0.035	0.114	0.269	0.396	0.501	0.596	1.077
5.000	0.037	0.080	0.269	0.684	1.077	1.444	1.812	4.527
$\kappa/\bar{P}$ (%)				Annual Turnover (%)				
0.010	14890.69	8374.52	4708.68	3148.72	2647.65	2392.35	2226.27	1770.29
0.050	9959.04	5599.66	3148.72	2105.43	1770.29	1599.53	1488.44	1183.40
0.100	8374.52	4708.68	2647.65	1770.29	1488.44	1344.83	1251.39	994.81
0.300	6363.26	3577.71	2011.58	1344.83	1130.60	1021.42	950.39	755.25
0.500	5599.66	3148.72	1770.29	1183.40	994.81	898.69	836.14	664.26
1.000	4708.68	2647.65	1488.44	994.81	836.14	755.25	702.59	557.79
5.000	3149.32	1770.29	994.81	664.26	557.79	503.38	467.87	369.24
$\kappa/\bar{P}$ (%)				Cost as % of Transaction Amount				
0.010	0.000	0.000	0.000	0.002	0.003	0.004	0.004	0.009
0.050	0.000	0.000	0.002	0.005	0.009	0.012	0.015	0.029
0.100	0.000	0.000	0.003	0.009	0.015	0.020	0.024	0.049
0.300	0.000	0.001	0.006	0.020	0.033	0.045	0.056	0.111
0.500	0.000	0.002	0.009	0.029	0.049	0.066	0.082	0.162
1.000	0.000	0.003	0.015	0.049	0.082	0.111	0.137	0.273
5.000	0.002	0.009	0.049	0.162	0.273	0.369	0.457	0.902

Table 1.4: Calibration results using parameter estimates from Campbell and Kyle's (1993) Model B, with the ratio of idiosyncratic to aggregate volatility set to 16 (i.e. $\sigma_X = 16 \times \sigma_Y$). The first sub-panel reports expected trade inter-arrival times $\Delta\tau$ (in years), the second sub-panel reports the illiquidity discount in the stock price (as a percentage of the price $\bar{P} = \bar{a}_D/r - \bar{p}_0$ in the frictionless economy), the third sub-panel reports the return premium (defined as $\bar{a}_D/P - \bar{a}_D/\bar{P}$ where P is the price under the transactions cost), the fourth sub-panel reports the annual turnover in percent ($100 \times \frac{\delta}{2\,\theta\,\tau}$), and the fifth sub-panel reports the transactions cost as a percent of the transaction amount ($100 \times \frac{\kappa}{\delta P}$). These quantities are reported as functions of the transactions cost $\kappa_P \equiv \kappa/\bar{P}$ (in percentages), and the absolute risk aversion coefficient γ. Given γ, a unique value of σ_X^2 is implied by Campbell and Kyle's Model B, and $\bar{p}_0$ is determined from their estimates of λ and r.

γ	0.001	0.010	0.100	0.500	1.000	1.500	2.000	5.000
σ_X	21552.410	2155.241	215.524	43.105	21.552	14.368	10.776	4.310
$\kappa/\bar{P}$ (%)				$\Delta\tau$ (Years)				
0.010	0.000	0.001	0.002	0.004	0.005	0.006	0.007	0.012
0.050	0.000	0.001	0.004	0.008	0.012	0.014	0.017	0.026
0.100	0.001	0.002	0.005	0.012	0.017	0.020	0.023	0.037
0.300	0.001	0.003	0.009	0.020	0.029	0.035	0.041	0.065
0.500	0.001	0.004	0.012	0.026	0.037	0.046	0.053	0.084
1.000	0.002	0.005	0.017	0.037	0.053	0.065	0.075	0.120
5.000	0.004	0.012	0.037	0.084	0.120	0.148	0.173	0.304
$\kappa/\bar{P}$ (%)				Illiquidity Discount (% of $\bar{P}$)				
0.010	0.034	0.109	0.345	0.775	1.101	1.353	1.566	2.507
0.050	0.077	0.243	0.775	1.756	2.507	3.094	3.595	5.847
0.100	0.108	0.345	1.101	2.507	3.595	4.451	5.187	8.542
0.300	0.188	0.599	1.927	4.451	6.453	8.057	9.459	16.113
0.500	0.233	0.775	2.507	5.847	8.542	10.733	12.669	22.132
1.000	0.345	1.101	3.595	8.542	12.669	16.113	19.222	35.322
5.000	0.775	2.507	8.542	22.132	35.322	47.713	59.958	131.419
$\kappa/\bar{P}$ (%)				Return Premium (%)				
0.010	0.002	0.007	0.022	0.050	0.071	0.087	0.102	0.164
0.050	0.005	0.016	0.050	0.114	0.164	0.204	0.238	0.396
0.100	0.007	0.022	0.071	0.164	0.238	0.297	0.349	0.596
0.300	0.012	0.038	0.125	0.297	0.440	0.559	0.666	1.225
0.500	0.015	0.050	0.164	0.396	0.596	0.767	0.925	1.813
1.000	0.022	0.071	0.238	0.596	0.925	1.225	1.518	3.484
5.000	0.050	0.164	0.596	1.813	3.484	5.821	9.552	-26.683
$\kappa/\bar{P}$ (%)				Annual Turnover (%)				
0.010	42117.08	23683.81	13317.72	8905.28	7487.89	6765.70	6295.90	5005.90
0.050	28168.42	15837.87	8905.28	5954.05	5005.90	4522.74	4208.40	3345.04
0.100	23686.72	13317.72	7487.89	5005.90	4208.40	3801.97	3537.51	2810.94
0.300	17995.50	10118.73	5688.53	3801.97	3195.52	2886.32	2685.06	2131.37
0.500	15837.71	8905.28	5005.90	3345.04	2810.94	2538.51	2361.10	1872.30
1.000	13317.72	7487.89	4208.40	2810.94	2361.10	2131.37	1981.58	1566.79
5.000	8905.28	5005.90	2810.94	1872.30	1566.79	1408.08	1302.25	982.77
$\kappa/\bar{P}$ (%)				Cost as % of Transaction Amount				
0.010	0.000	0.000	0.000	0.001	0.002	0.002	0.003	0.006
0.050	0.000	0.000	0.001	0.004	0.006	0.008	0.010	0.020
0.100	0.000	0.000	0.002	0.006	0.010	0.014	0.017	0.034
0.300	0.000	0.001	0.004	0.014	0.023	0.032	0.039	0.078
0.500	0.000	0.001	0.006	0.020	0.034	0.047	0.058	0.114
1.000	0.000	0.002	0.010	0.034	0.058	0.078	0.097	0.191
5.000	0.001	0.006	0.034	0.114	0.191	0.258	0.318	0.600

Table 2

Number of split events of NYSE/AMEX/NASDAQ stocks in our sample after filtering for errors and other irregularities, from January 1, 1993 to December 31, 1997.

Split Factor	1993	1994	1995	1996	1997	Total
2:1	174	140	189	223	276	1002
3:2	158	108	140	170	235	811
4:3	11	13	9	13	14	60
5:4	37	31	39	38	45	190
Other	12	19	13	26	37	107
Total	392	311	390	470	607	2170

Table 3

Summary statistics for trade sizes δ and inter-arrival times $\Delta\tau$ before and after stock splits, from January 1, 1993 to December 31, 1997.

Window	Split Factor	Before or After	Sample Size	δ (Shares) Mean	δ (Shares) S.D.	$\Delta\tau$ (Seconds) Mean	$\Delta\tau$ (Seconds) S.D.
1-Day	All	Before	1626	1139	930	728	807
	All	After	1621	740	657	503	535
	2:1	Before	791	1116	817	519	609
	2:1	After	791	641	490	364	428
	3:2	Before	598	1174	960	921	923
	3:2	After	599	836	777	635	586
	4:3	Before	37	961	727	650	516
	4:3	After	37	754	406	520	432
	5:4	Before	118	1272	1297	1286	1178
	5:4	After	118	944	730	1031	1093
2-Day	All	Before	1749	1149	896	871	950
	All	After	1749	739	579	604	660
	2:1	Before	827	1122	712	599	718
	2:1	After	827	652	431	400	473
	3:2	Before	654	1219	1057	1095	1072
	3:2	After	654	841	693	763	751
	4:3	Before	45	1136	1260	761	613
	4:3	After	45	813	592	695	561
	5:4	Before	143	1130	901	1617	1516
	5:4	After	143	869	602	1388	1386
3-Day	All	Before	1805	1141	800	932	1010
	All	After	1805	757	615	662	730
	2:1	Before	846	1118	670	655	768
	2:1	After	846	667	418	446	543
	3:2	Before	677	1213	899	1144	1112
	3:2	After	677	843	634	816	803
	4:3	Before	49	1077	936	762	646
	4:3	After	49	1007	1754	799	574
	5:4	Before	148	1122	831	1628	1490
	5:4	After	148	910	621	1355	1228
7-Day	All	Before	1869	1144	680	972	997
	All	After	1869	793	548	747	799
	2:1	Before	870	1122	602	724	851
	2:1	After	870	701	422	525	649
	3:2	Before	699	1194	698	1159	1049
	3:2	After	699	886	601	899	859
	4:3	Before	53	1100	806	816	563
	4:3	After	53	835	584	766	559
	5:4	Before	160	1153	870	1610	1378
	5:4	After	160	921	595	1438	1183

Table 4

Summary statistics for the log-ratios $\xi_\delta \equiv \log(\overline{\delta}^a/(s\overline{\delta}^b))$ and $\xi_{\Delta\tau} \equiv \log(\overline{\Delta\tau}^a/\overline{\Delta\tau}^b)$, the implied ratios of fixed costs $(\kappa^a/\kappa^b)_\delta$ and $(\kappa^a/\kappa^b)_{\Delta\tau}$ based on trade sizes and inter-arrival times, respectively, and ratios $\zeta \equiv \xi_\delta/\xi_{\Delta\tau}$, across stock splits from January 1, 1993 to December 31, 1997, where $\overline{\delta}^b$ and $\overline{\delta}^a$ are the average trade size (in shares) before and after a split, respectively, s is the split factor, and $\overline{\Delta\tau}^b$ and $\overline{\Delta\tau}^a$ are the average inter-arrival times before and after a split, respectively.

Window	Split Factor	Sample Size	ξ_δ		$\xi_{\Delta\tau}$		$(\kappa_i^a/\kappa_i^b)_\delta$		$(\kappa_i^a/\kappa_i^b)_\tau$		ζ	
			Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
1-Day	All	1619	−0.418	0.018	−0.396	0.018	0.984	0.061	0.739	0.018	0.482	0.053
	2:1	791	−0.524	0.023	−0.411	0.024	0.456	0.036	0.695	0.023	0.494	0.085
	3:2	598	−0.298	0.028	−0.368	0.030	1.759	0.185	0.810	0.034	0.558	0.085
	4:3	36	−0.089	0.128	−0.291	0.153	6.401	3.360	1.216	0.199	−0.583	0.251
	5:4	118	−0.264	0.080	−0.296	0.065	3.062	0.664	0.821	0.077	−0.150	0.226
2-Day	All	1746	−0.416	0.015	−0.392	0.018	0.661	0.036	0.672	0.015	0.419	0.047
	2:1	827	−0.531	0.019	−0.424	0.024	0.329	0.022	0.620	0.018	0.503	0.075
	3:2	654	−0.300	0.024	−0.378	0.027	1.324	0.153	0.714	0.027	0.361	0.072
	4:3	44	−0.146	0.130	−0.109	0.110	1.947	0.561	1.264	0.170	−0.337	0.520
	5:4	142	−0.224	0.059	−0.119	0.112	2.736	0.676	1.305	0.217	−0.036	0.170
3-Day	All	1801	−0.405	0.013	−0.387	0.017	0.582	0.026	0.664	0.013	0.484	0.046
	2:1	846	−0.502	0.018	−0.440	0.021	0.351	0.023	0.602	0.016	0.582	0.075
	3:2	677	−0.319	0.022	−0.352	0.027	0.822	0.062	0.699	0.024	0.329	0.070
	4:3	47	−0.174	0.101	0.037	0.100	2.615	1.435	1.661	0.269	−0.278	0.372
	5:4	145	−0.164	0.054	−0.142	0.086	1.690	0.274	1.131	0.148	0.280	0.166
7-Day	All	1864	−0.383	0.011	−0.323	0.014	0.454	0.017	0.703	0.012	0.597	0.045
	2:1	870	−0.458	0.015	−0.380	0.020	0.278	0.013	0.624	0.015	0.820	0.065
	3:2	698	−0.305	0.018	−0.285	0.022	0.590	0.035	0.747	0.022	0.406	0.070
	4:3	51	−0.251	0.066	−0.043	0.059	1.173	0.276	1.147	0.114	−0.252	0.483
	5:4	159	−0.198	0.046	−0.092	0.059	1.305	0.165	1.144	0.090	0.021	0.264

Table 5

Summary statistics for the log-ratios $\xi_\delta \equiv \log(\overline{\delta}^a/\overline{\delta}^b)$ and $\xi_{\Delta\tau} \equiv \log(\overline{\Delta\tau}^a/\overline{\Delta\tau}^b)$ across dates prior to and after stock splits stock splits, from January 1, 1993 to December 31, 1997, where $\overline{\delta}^b$ and $\overline{\delta}^a$ are the average trade size (in shares) in the "before" and "after" periods, respectively, and $\overline{\Delta\tau}^b$ and $\overline{\Delta\tau}^a$ are the average inter-arrival times in the "before" and "after" periods, respectively. '*Dates* -5 *and* -4' indicates that the "before" period is the fifth day prior to a split event and the "after" period is the fourth day prior to the same event.

Split Factor	Sample Size	ξ_δ Mean	ξ_δ S.D.	$\xi_{\Delta\tau}$ Mean	$\xi_{\Delta\tau}$ S.D.
		Dates -5 *and* -4			
All	1513	0.010	0.019	−0.021	0.018
2:1	741	0.013	0.024	−0.006	0.023
3:2	562	0.039	0.034	−0.030	0.034
4:3	36	0.059	0.147	−0.047	0.166
5:4	108	−0.138	0.080	−0.079	0.073
		Dates $+4$ *and* $+5$			
All	1646	0.003	0.016	0.014	0.017
2:1	803	0.019	0.022	0.020	0.020
3:2	609	−0.011	0.028	0.000	0.032
4:3	40	0.034	0.152	−0.026	0.143
5:4	117	−0.190	0.067	0.046	0.072
		Dates -7 *and* -6			
All	1526	0.008	0.018	0.024	0.017
2:1	749	−0.031	0.022	0.014	0.023
3:2	569	0.015	0.035	0.037	0.030
4:3	36	0.073	0.124	−0.041	0.134
5:4	108	0.113	0.075	0.005	0.080
		Dates $+6$ *and* $+7$			
All	1621	0.028	0.019	0.005	0.017
2:1	792	0.024	0.024	0.006	0.022
3:2	599	0.017	0.033	−0.024	0.029
4:3	37	−0.159	0.143	−0.043	0.138
5:4	119	0.179	0.088	0.099	0.066