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REAL-TIME MULTIVARIATE DENSITY  
FORECAST EVALUATION AND CALIBRATION:  
MONITORING THE RISK OF HIGH-FREQUENCY  
RETURNS ON FOREIGN EXCHANGE

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Real-Time Multivariate Density Forecast Evaluation  
and Calibration: Monitoring the Risk of High-  
Frequency Returns on Foreign Exchange  
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**ABSTRACT**

We provide a framework for evaluating and improving multivariate density forecasts. Among other things, the multivariate framework lets us evaluate the adequacy of density forecasts involving cross-variable interactions, such as time-varying conditional correlations. We also provide conditions under which a technique of density forecast “calibration” can be used to improve deficient density forecasts. Finally, motivated by recent advances in financial risk management, we provide a detailed application to multivariate high-frequency exchange rate density forecasts.

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## 1. Introduction

The forecasting literature has traditionally focused primarily on point forecasts. Recently, however, attention has been given to interval forecasts (Chatfield, 1993) and density forecasts (Diebold, Gunther and Tay, 1998). The reasons for the recent interest in interval and density forecasts are both methodological and substantive. On the methodological side, recent years have seen the development of powerful models of time-varying conditional variances and densities (Bollerslev, Engle and Nelson, 1994; Ghysels, Harvey and Renault, 1996; Hansen, 1994; Morvai, Yakowitz and Algoet, 1997). On the substantive side, density forecasts and summary statistics derived from density forecasts have emerged as a key part of the explosively-growing field of financial risk management (Duffie and Pan, 1997; Jorion, 1997), as well in as more traditional areas such as macroeconomic inflation forecasting (Wallis, 1998). Moreover, explicit use of predictive densities has long been a prominent feature of the Bayesian forecasting literature (Harrison and Stevens, 1976; West and Harrison, 1997), and recent advances in Markov Chain Monte Carlo (Gelman, Carlin, Stern and Rubin, 1995) have increased the pace of progress. The closely related “prequential” Bayesian literature (Dawid, 1984) also features density forecasts prominently.

Interest in forecasts of various sorts creates a derived demand for methods of *evaluating* forecasts. In parallel with the historical emphasis on point forecasts, most literature has focused on the evaluation of point forecasts (Diebold and Lopez, 1996), but recent interest in interval and density forecasts has spurred development of methods for their evaluation (Christoffersen, 1998; Diebold, Gunther and Tay, 1998). Diebold, Gunther and Tay, in particular, motivate and approach the problem of density forecast evaluation from a

risk management perspective, drawing upon an integral transform dating at least to Rosenblatt (1952), extended by Seillier-Moiseiwitsch (1993) and used creatively by Shephard (1994).

Here we extend the density forecast evaluation literature in three ways. First, we treat the multivariate case, which lets us evaluate the adequacy of density forecasts involving cross-variable interactions, such as time-varying conditional correlations, which are crucial in financial settings. Second, we provide conditions under which a technique of density forecast “calibration” can be used to improve deficient density forecasts. Finally, we provide a detailed application to the evaluation of density forecasts of multivariate high-frequency exchange rates, which is of direct substantive interest in addition to illustrating the ease with which the methods can be implemented.

## **2. Density Forecast Evaluation and Calibration**

We begin with a brief summary of certain key univariate evaluation results from Diebold, Gunther and Tay (1998), in order to establish ideas and fix notation. We then describe our methods of univariate calibration, after which we extend both the evaluation and calibration methods to the multivariate case. Throughout, we intentionally ignore parameter estimation uncertainty. We take density forecasts as primitives, and in particular, we do not require that they be based on an estimated model.<sup>1</sup> We take this approach because many density forecasts of interest do not come from models with estimated parameters, as for example when density forecasts are extracted from surveys (Diebold, Tay and Wallis, 1997) or from prices of options written at different strike prices (Aït-Sahalia and Lo, 1998;

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<sup>1</sup> Some progress, however, is also being made at evaluating density forecasts produced from estimated models; see Bai (1997) and Inoue (1997).

Soderlind and Svensson, 1997; Campa, Chang and Reider, 1998). This is the case for some of the exchange rate density forecasts that we evaluate in section 4.

### Univariate Evaluation

Consider one arbitrary period,  $t$ , and let  $y_t$  be the variable of interest with conditional density  $f(y_t | \Omega_{t-1})$ , where  $\Omega_{t-1}$  represents the collection of past information available at time  $t-1$ , and let  $p_{t-1}(y_t)$  be a density forecast of  $y_t$ . Let  $z_t$  be the probability integral transform of  $y_t$  with respect to  $p_{t-1}(y_t)$ ; that is,

$$z_t = \int_{-\infty}^{y_t} p_{t-1}(u) du \equiv P_{t-1}(y_t).$$

Then assuming that  $\frac{\partial P_{t-1}^{-1}(z_t)}{\partial z_t}$  is continuous and non-zero over the support of  $z_t$ ,  $z_t$  has

support on the unit interval with density function

$$q_t(z_t) = \left| \frac{\partial P_{t-1}^{-1}(z_t)}{\partial z_t} \right| f(P_{t-1}^{-1}(z_t) | \Omega_{t-1}) = \frac{f(P_{t-1}^{-1}(z_t) | \Omega_{t-1})}{p_{t-1}(P_{t-1}^{-1}(z_t))}.$$

In particular, if  $p_{t-1}(y_t) = f(y_t | \Omega_{t-1})$ , then

$$q_t(z_t) = \begin{cases} 1 & \text{if } 0 \leq z_t \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

so that  $z_t \sim U(0,1)$ .

Now consider a series of  $m$  density forecasts and realizations, rather than just one.

Diebold, Gunther and Tay (1998) show that if  $\{y_t\}_{t=1}^m$  is a series of realizations generated

from the series of conditional densities  $\{f(y_t|\Omega_{t-1})\}_{t=1}^m$ , and if a series of 1-step-ahead density forecasts  $\{p_{t-1}(y_t)\}_{t=1}^m$  coincides with  $\{f(y_t|\Omega_{t-1})\}_{t=1}^m$ , then assuming a non-zero Jacobian with continuous partial derivatives, the series of probability integral transforms of  $\{y_t\}_{t=1}^m$  with respect to  $\{p_{t-1}(y_t)\}_{t=1}^m$  is iid  $U(0,1)$ . That is,  $\{z_t\}_{t=1}^m \stackrel{\text{iid}}{\sim} U(0,1)$ . Thus, to assess whether a series of density forecasts coincides with the corresponding series of true conditional densities, we need only assess whether an observed series is iid  $U(0,1)$ .

### Univariate Calibration

An important practical question is how to improve suboptimal forecasts. In the point forecast case, for example, we can regress the  $y$ 's on the  $\hat{y}$ 's (the predicted values) and potentially use the estimated relationship to construct an improved point forecast. Such a regression is sometimes called a Mincer-Zarnowitz regression, after Mincer and Zarnowitz (1969). In this paper we will use an analogous procedure, which we call density forecast “calibration,” for improving density forecasts that produce an iid but non-uniform  $z$  series.<sup>2</sup>

Suppose that we are in period  $m$  and possess a density forecast of  $y_{m+1}$ . From our earlier discussion,

$$f(y_{m+1}|\Omega_m) = p_m(y_{m+1})q_{m+1}(z_{m+1}).$$

But if  $z$  is iid, then we can drop the subscript on  $q$  and write

$$f(y_{m+1}|\Omega_m) = p_m(y_{m+1})q(z_{m+1}).$$

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<sup>2</sup> We thank Charlie Thomas for pointing out that the basic idea of univariate density forecast calibration traces at least to Fackler and King (1990). Shortly, however, we will extend the calibration idea to the multivariate case. Moreover, we will emphasize that density forecast calibration is not universally applicable and develop a sufficient condition for its application in both the univariate and multivariate cases.

Thus if we knew  $q(z_{m+1})$ , we would know the actual conditional distribution  $f(y_{m+1} | \Omega_m)$ .<sup>3</sup> Because  $q(z_{m+1})$  is unknown, we use an estimate  $\hat{q}(z_{m+1})$  formed using  $\{z_t\}_{t=1}^m$  to construct an estimate  $\hat{f}(y_{m+1} | \Omega_m)$ . In finite samples, of course, there is no guarantee that the “improved” forecast will actually be superior to the original, because it is based on an estimate of  $q$  rather than the true  $q$ , and the estimate could be very poor. The practical efficacy of our improvement methods is an empirical matter, which will be assessed shortly, but with the large sample sizes typically available in financial applications we expect that precise estimation of  $q$  will often be straightforward.

### Multivariate Evaluation

The principles that govern the univariate techniques discussed thus far extend readily to the multivariate case. Suppose that  $y_t$  is now an  $N \times 1$  vector, and that we have a series of  $m$  multivariate forecasts and their corresponding multivariate realizations. Further suppose that we are able to factor each period’s joint forecast into the product of the conditionals,

$$p_{t-1}(y_{1t}, y_{2t}, \dots, y_{Nt}) = p_{t-1}(y_{Nt} | y_{N-1,t}, \dots, y_{1t}) \cdots p_{t-1}(y_{2t} | y_{1t}) p_{t-1}(y_{1t}).$$

Then for each period we can transform each element of the multivariate observation  $(y_{1t}, y_{2t}, \dots, y_{Nt})$  by its corresponding conditional distribution. This procedure produces a set of  $N$   $z$  series that will be iid  $U(0,1)$  individually, and also when taken as a whole, if the multivariate density forecasts are correct. The proof of this assertion is obtained by simply arranging the multivariate series as a series of univariate observations comprising

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<sup>3</sup> Note that  $z$  is uniquely determined by  $y$ , so that for any given value of  $y$ , we can always compute  $f(y)$  as  $p(y)q(z)$ .

$\{\dots, y_{1t}, y_{2t}, \dots, y_{Nt}, \dots\}$ .<sup>4</sup> The result for the univariate case can then be applied to show that the resulting  $Nm$  vector  $z$  is iid  $U(0,1)$ , and hence the  $z$  series corresponding to any particular series of conditional forecasts is also iid  $U(0,1)$ .

Note that  $N!$   $z$  series can be produced, depending on how the joint density forecasts are factored, giving us a wealth of information with which to evaluate the forecasts. To take the bivariate case as an example, we can decompose the forecasts in two ways:

$$(i) \quad p_{t-1}(y_{1t}, y_{2t}) = p_{t-1}(y_{1t}) p_{t-1}(y_{2t} | y_{1t}) \text{ and}$$

$$(ii) \quad p_{t-1}(y_{1t}, y_{2t}) = p_{t-1}(y_{2t}) p_{t-1}(y_{1t} | y_{2t}),$$

$t = 1, \dots, m$ , and label the respective  $z$  series  $z_1, z_{2|1}, z_2$  and  $z_{1|2}$ , where we obtain  $z_i$  by taking the probability integral transform of  $y_i$  with respect to  $p_{t-1}(y_i)$ , and we obtain  $z_{ij}$  by transforming  $y_i$  with respect to  $p_{t-1}(y_i | y_j)$ . Good multivariate forecasts will produce  $z_1, z_{2|1}, z_2$  and  $z_{1|2}$  that are each iid  $U(0, 1)$ , as well as combined series  $\{\dots, z_{1,t}, z_{2|1,t}, \dots\}$  and  $\{\dots, z_{2,t}, z_{1|2,t}, \dots\}$  that are also iid  $U(0,1)$ .

In many cases of economic and financial interest, such as the application to density forecasting of DM/\$ and Yen/\$ returns in section 4 of this paper, the dimension  $N$  of the set of variables being forecast is low. A classic example is density forecasting of returns across different aggregate asset classes, such as equity, foreign exchange and fixed income. In such cases, it is straightforward to examine and learn from each of the  $N!$  sets of  $z$  series. Sometimes, however,  $N$  may be large, as for example when a density forecast is generated for each of the stocks in a broad-based market index, such as the S&P 500, in which case

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<sup>4</sup> That is, we begin with had  $m$  observations on an  $N$ -variate variable, and we convert them to a univariate series with  $Nm$  observations.



methods such as those of Clements and Smith (1998) may be used to aggregate the information in the individual  $z$  series.

### Multivariate Calibration

In parallel with the key formula underlying our discussion of univariate calibration, in the multivariate case we have that

$$f(y_{1,m+1}, y_{2,m+1}, \dots, y_{N,m+1} | \Omega_m) = p_m(y_{1,m+1}, y_{2,m+1}, \dots, y_{N,m+1}) q(z_{1,m+1}, z_{2,m+1}, \dots, z_{N,m+1}).$$

Moreover, factoring both of the right-hand-side densities, we can write the formula in a way that precisely parallels our multivariate evaluation framework,

$$f(y_{1,m+1}, y_{2,m+1}, \dots, y_{N,m+1} | \Omega_m) = \prod_{i=1}^N (p_m(y_{i,m+1} | y_{i-1,m+1}, \dots, y_{1,m+1}) q(z_{i,m+1} | z_{i-1,m+1}, \dots, z_{1,m+1})).$$

As before, an estimate of  $q$  may be used to implement the calibration empirically, yielding

$$\hat{f}(y_{1,m+1}, y_{2,m+1}, \dots, y_{N,m+1} | \Omega_m) = p_m(y_{1,m+1}, y_{2,m+1}, \dots, y_{N,m+1}) \hat{q}(z_{1,m+1}, z_{2,m+1}, \dots, z_{N,m+1}),$$

or

$$\hat{f}(y_{1,m+1}, y_{2,m+1}, \dots, y_{N,m+1} | \Omega_m) = \prod_{i=1}^N (p_m(y_{i,m+1} | y_{i-1,m+1}, \dots, y_{1,m+1}) \hat{q}(z_{i,m+1} | z_{i-1,m+1}, \dots, z_{1,m+1})).$$

Note that the multivariate calibrating density is the same (in population), regardless of which of the possible  $N!$  factorizations is used.

### **3. More on Density Forecast Calibration**

Here we elaborate on various aspects of density forecast calibration. First we give a sufficient condition for an iid integral transform series, which is required for application of the calibration method. Next we discuss subtleties associated with estimation of the density of an integral transform, which by construction has compact support on the unit interval. Finally,

we show how the calibration method can be used to generate good density forecasts even when the conditional density is unknown. In this section we use the generic notation “ $z$ ” to denote an integral transform; it is a scalar integral transform in the univariate case and a vector of integral transforms in the multivariate case.

### Conditions Producing an iid $z$ Series

In empirical work, the legitimacy of the assumption that  $z$  is iid can be assessed in a number of ways, ranging from examination of correlograms of various powers of  $z$  to formal tests such as those discussed in Brock, Hsieh and LeBaron (1991). It is nevertheless desirable to characterize theoretically the conditions under which  $z$  will be iid, in order to deepen our understanding of whether and when we can reasonably hope for an iid  $z$  series. Here we establish a sufficient condition: if the 1-step-ahead density  $f(y_t|\Omega_{t-1})$  belongs to a location-scale family, and if the forecast  $p_{t-1}(y_t)$  adequately captures dynamics (in a sense to be made precise shortly) but is mistakenly assumed to be in another location-scale family, then  $z$  will be iid.

More precisely, suppose that  $f(y_t|\Omega_{t-1})$  belongs to a location-scale family; that is, the random variable  $\frac{y_t - \mu(\Omega_{t-1})}{\sigma(\Omega_{t-1})}$  is iid with unknown density  $f(\bullet)$ . It then follows that

$$f(y_t|\Omega_{t-1}) = \frac{1}{\sigma(\Omega_{t-1})} f\left(\frac{y_t - \mu(\Omega_{t-1})}{\sigma(\Omega_{t-1})}\right).$$

Now suppose that we misspecify the model by another location-scale family,  $p_{t-1}(y_t)$ , in which we specify  $\mu(\Omega_{t-1})$  and  $\sigma(\Omega_{t-1})$  correctly, in spite of the fact that we use an incorrect

conditional density. Thus we issue the density forecast

$$p_{t-1}(y_t) = \frac{1}{\sigma(\Omega_{t-1})} p\left(\frac{y_t - \mu(\Omega_{t-1})}{\sigma(\Omega_{t-1})}\right).$$

The probability integral transform of the realization with respect to the forecast is therefore

$$z_t = \int_{-\infty}^{y_t} p_{t-1}(y_t) dy_t = \int_{-\infty}^{y_t} \frac{1}{\sigma(\Omega_{t-1})} p\left(\frac{y_t - \mu(\Omega_{t-1})}{\sigma(\Omega_{t-1})}\right) dy_t.$$

Now make the change of variable

$$u_t = \frac{y_t - \mu(\Omega_{t-1})}{\sigma(\Omega_{t-1})},$$

which yields

$$z_t = \int_{-\infty}^{u_t} \frac{1}{\sigma(\Omega_{t-1})} p(u_t) \sigma(\Omega_{t-1}) du_t = P(u_t).$$

But this means that  $z$  is iid, as claimed, because  $z_t = P(u_t)$  and  $u_t$  is iid.

### Estimation of $q(z)$

Estimation of  $q(z)$  requires care, because  $z$  has support only on the unit interval. One could use a global smoother on the unit interval, such as a simple two-parameter beta distribution, but the great workhorse beta family is unfortunately not flexible enough to accommodate the multimodal shapes of  $q(z)$  that arise routinely, as documented in Diebold, Gunther and Tay (1998). One could perhaps stay in the global smoothing framework by using some richer parametric family or a series estimator, as discussed for example by Härdle (1991).

In keeping with much of the recent literature, however, we prefer to take a local

smoothing approach, as for example with a kernel density estimator. But it has long been recognized that standard kernel estimation of densities with bounded support suffers from a “boundary problem” in that the density estimates at the boundaries are biased. One cause of this problem is that near the boundary approximately half of the kernel mass falls outside the range of the data (assuming a symmetric kernel). Therefore, kernel density estimates near boundaries have expectations approximately equal to half the true underlying density.

Let us elaborate. For i.i.d. data  $z_1, \dots, z_T$  with sufficiently smooth density  $q(z)$  on  $[0, 1]$ , the kernel estimator  $\hat{q}(z)$  of  $q(z)$  is

$$\hat{q}(z) = \frac{1}{Tb} \sum_{t=1}^T K\left(\frac{z-z_t}{b}\right),$$

where  $K(\cdot)$  is a symmetric and sufficiently smooth kernel of choice and  $b > 0$  is bandwidth.

Begin by noting that

$$E[\hat{q}(z)] = \frac{1}{b} \int_0^1 K\left(\frac{u-z}{b}\right) q(u) du$$

$$= q(z) \int_{-\frac{z}{b}}^{\frac{1-z}{b}} K(v) dv + bq'(z) \int_{-\frac{z}{b}}^{\frac{1-z}{b}} K(v) v dv + \frac{1}{2} b^2 q''(z) \int_{-\frac{z}{b}}^{\frac{1-z}{b}} K(v) v^2 dv + o(b^2).$$

Now suppose as usual that the bandwidth is chosen such that  $b \rightarrow 0$  and  $Tb \rightarrow \infty$  as  $T \rightarrow \infty$ . For  $z$

$$= \int_{-\frac{x}{b}}^{\frac{1-x}{b}} K(v) f(x+bv) dv$$

well in the interior of  $[0, 1]$ , we then have

$$E[\hat{q}(z)] = q(z) + O(b^2).$$

On the other hand, the kernel density estimates have expectations approximately equal to only half of the true underlying density at boundaries:<sup>5</sup>

$$E[\hat{q}(0)] = \frac{q(0)}{2} + O(b)$$

$$E[\hat{q}(1)] = \frac{q(1)}{2} + O(b).$$

It can be shown that similar bias occurs at points *near* the boundaries.

There are a number of ways to address the boundary problem. One is to use a modified kernel density estimator such as the one suggested by Müller (1993), which is designed to have less bias near boundaries. Another local smoothing approach is to estimate  $q(z)$  with a simple histogram. A histogram is a special type of kernel density estimator where the kernel never exceeds the boundaries of the distribution. Therefore we expect that it would not suffer from the problem associated with kernels that exceed the boundaries, and

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<sup>5</sup> See Härdle (1990, 130-131), for example, for related discussion in the context of kernel regression, as opposed to density estimation.

experimentation revealed it to be superior in the present context; we make extensive use of histograms as visual estimates of  $q(z)$  in the empirical application in section 5. Finally, we can bypass the density estimation problem altogether by casting our evaluation and calibration methods in terms of c.d.f.'s rather than densities, and using empirical c.d.f.'s instead of estimated densities.

Alternatively, we may write the calibration formula in terms of joint c.d.f.'s rather than joint densities, as

$$F(y_{1,m+1}, \dots, y_{N,m+1} | \Omega_m) = Q(P_m(y_{1,m+1}, \dots, y_{N,m+1})),$$

which we make operational by replacing population c.d.f.'s with estimated (empirical) c.d.f.'s,

$$\hat{F}(y_{1,m+1}, \dots, y_{N,m+1} | \Omega_m) = \hat{Q}(P_m(y_{1,m+1}, \dots, y_{N,m+1})).$$

Although empirical c.d.f.'s are not as visually digestible as estimated densities, they have a number of attractive features: they are guaranteed to be 0 for  $z=0$  and 1 for  $z=1$ , and there is no need for bandwidth selection in the estimation of  $Q$ . We make extensive use of empirical c.d.f.'s for doing the calibration transformations in the empirical application in section 5.

### Generating Density Forecasts When the Conditional Density is Unknown

It is interesting to note that our calibration methods can be used to generate density forecasts even when the conditional density is unknown. Consider, for example, the problem of 1-step-ahead density forecasting in models with unknown 1-step-ahead conditional density and time-varying volatility. The problem is of intrinsic interest, and moreover, it is quite general. In particular, it also covers  $h$ -step-ahead density forecasting, which in financial

contexts is equivalent to 1-step-ahead forecasting of h-period returns.<sup>6</sup>

As is well known, even under heroic assumptions such as conditionally Gaussian 1-step-ahead returns, the h-step-ahead conditional density, or equivalently the h-period return, will not be Gaussian and has no known closed form. Several earlier papers bear on the problem, but all are lacking in certain respects. Baillie and Bollerslev (1992), for example, study h-step-ahead prediction but must assume a Gaussian 1-step-ahead conditional density and even in that case provide only Cornish-Fisher approximations to the multi-step densities.

Engle and González-Rivera (1991) use a different but closely related strategy; let us contrast the two. Engle and González-Rivera exploit the Bollerslev-Wooldridge (1992) result that GARCH volatility parameters are consistently estimated by maximum likelihood even when the conditional density is misspecified. Hence (assuming for convenience a zero conditional mean) one can assume a Gaussian conditional distribution (incorrectly, in general), maximize the Gaussian likelihood, and standardize the data by the estimated series of conditional standard deviations. In large samples the resulting series will be iid but not, in general, Gaussian; but its density can be estimated using nonparametric techniques.

In tackling this problem, our calibration procedure also exploits the Bollerslev-Wooldridge (1992) quasi-MLE result. We proceed by maximizing the likelihood, which we assume to be Gaussian (incorrectly, in general). We then take the estimated volatility parameters and use them to make conditionally Gaussian density forecasts (again incorrectly), and we compute the probability integral transforms of the realizations with respect to those forecasts. In large samples, under the conditions given earlier, the series of integral

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<sup>6</sup> See, for example, Duan (1995) and Duan, Gauthier and Simonato (1997).

transforms will be iid but not  $U(0,1)$ ; we therefore calibrate the forecasts using either a histogram or a c.d.f. fit to the series of integral transforms.

The key difference between the Engle-González-Rivera approach and our calibration approach is that the first involves nonparametric estimation of a density with infinite support, whereas the second maps the problem into one of nonparametric estimation of a density with bounded support. Given that in risk management contexts our interest centers on tail events, the Engle and González-Rivera approach may be problematic, due to the well-known “bumpy-tail” problem.<sup>7</sup> The bumpy-tail problem may be overcome by applying larger bandwidths at the tails, but instead of applying some ad hoc method of choosing variable bandwidth, the integral transformation implicitly provides an empirically reasonable variable bandwidth, so long as the density forecasts are close to the true density. In addition, we note that our procedure is not limited to the GARCH paradigm, and is applicable to any situation where the forecasts generate  $z$  series that are iid. As discussed previously, extensions to multivariate density forecasting contexts are also straightforward.

The empirical application to which we now turn illustrates all of the ideas developed thus far: we begin by evaluating a series of multivariate density forecasts that are revealed to be poor by virtue of associated iid but non-uniform integral transform series, we estimate the appropriate calibrating density, and we use it to transform the poor density forecasts into good ones.

#### **4. Evaluating and Calibrating Multivariate Density Forecasts of High-Frequency Returns on Foreign Exchange**

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<sup>7</sup> See, for example, Silverman (1986).



Here we present a detailed application of our methods of multivariate density forecast evaluation and calibration to a bivariate system of asset returns. In particular, we study high-frequency DM/\$ and YEN/\$ exchange rate returns, and we generate density forecasts from a forecasting model in the tradition of JP Morgan's RiskMetrics (JP Morgan, 1996), which is a popular benchmark in the risk management industry.

We use our multivariate density forecast evaluation tools to assess the adequacy of the RiskMetrics approximation the dynamics in high frequency exchange rate returns, as well as the adequacy of the conditional normality assumption, after which we attempt to improve the forecasts using calibration methods. We begin with a description of the data and a discussion of the statistical properties of the returns series. We then describe the forecasting model, after which we evaluate and calibrate the density forecasts that it produces.

### Data

The data, kindly provided by Olsen and Associates, are indicative bid and ask quotes posted by banks, spanning the period from January 1, 1996, to December 31, 1996. The data are organized around a grid of half-hour intervals; Olsen provides the quotes nearest the half-hour time stamps. Foreign exchange trading occurs around the clock during weekdays, but trading is very thin during weekends, so we remove them, as is customary.<sup>8</sup> We consider the weekend to be the period from Friday 21:30 GMT to Sunday 21:00 GMT, as this period appears to correspond most closely to the weekly blocks of zero returns. Thus a trading week spans Sunday 21:30 GMT to Friday 21:00 GMT, and each of the five trading days therein spans 21:30 GMT on one day to 21:00 GMT the next day. We label the days "Monday"

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<sup>8</sup> See, for example, Anderson and Bollerslev (1997).

through “Friday.” This layout implies that we have one partial week of data, followed by 51 full weeks, followed by another partial week of data. Each full week of data has  $5 \times 48 = 240$  data points. The first partial week has 233 observations (just less than five days of data), and the sample ends with a partial week of 102 observations (about 2 days), for a total of 12,575 observations.

### Computing Returns

We compute returns by computing bid and ask rates at each grid point as the linear interpolation of the nearest previous and subsequent quotes, as in Andersen and Bollerslev (1997). The rate of return on the exchange rate is then the difference between the mid-point of the log bid and ask rate at consecutive grid points. That is, letting  $x_{t, \text{ask/bid}}$  denote the log ask / bid quote at the grid point  $t$ ,  $x_{t, \text{ask/bid, previous/next}}$  represent the nearest previous/next-occurring bid or ask quote at time  $t$ , and  $\text{time}_{t, \text{previous/next}}$  be the time between the grid point and the previous/next quote, we obtain the exchange rate at time  $t$ ,  $x_t$ , as

$$x_t = \frac{1}{2} (x_{t, \text{ask}} + x_{t, \text{bid}})$$

where

$$x_{t, \text{ask/bid}} = \ln x_{t, \text{ask/bid, previous}} + \frac{\text{time}_{t, \text{previous}}}{\text{time}_{t, \text{previous}} + \text{time}_{t, \text{next}}} (\ln x_{t, \text{ask/bid, next}} - \ln x_{t, \text{ask/bid, previous}})$$

The exchange rate return at time  $t$ ,  $r_t$ , is then  $r_t = x_t - x_{t-1}$ . It will sometimes be necessary to refer to the return at a particular time of a particular day. In such cases, we will use  $r_{\tau, i}$  to represent the return at time  $\tau$  of day  $i$ .

The returns exhibit the MA(1) conditional mean dynamics commonly found in asset returns. Because our focus in this paper is on volatility dynamics and their relation to density

forecasting, we follow standard practice and remove the MA(1) dynamics from each return. We do so by fitting MA(1) models and treating the residuals as our returns series. Hereafter,  $r_t$  will refer to returns series with the MA(1) component removed.

#### Properties of $r_t$ , $r_t^2$ and $|r_t|$

There are strong intra-day calendar effects in both Yen/\$ and DM/\$ returns; Figure 1 displays the first 200 autocorrelations of  $r_t$ ,  $r_t^2$  and  $|r_t|$ . The calendar effects are particularly pronounced in the absolute returns of both series; they occur because trading is more active at certain times of day than at others. For instance, trading is much less active during the Japanese lunch hour, and much more active when the U.S. markets open for trading. Figure 2, which plots volatility at the 48 half-hour intervals across an average day as measured by mean absolute returns, clearly reveals these phenomena. The intra-day volatility pattern corresponds closely to the pattern obtained by Andersen and Bollerslev (1997) for 5-minute DM/\$ returns.

As with the conditional mean of the returns process, intra-day calendar effects in return volatility are not of primary concern to us; hence we remove them. Taking  $r_t = s_{\tau,i} Z_t$  where  $Z_t$  represents the “nonseasonal” portion of the process, we remove volatility calendar effects by fitting time-of-day dummies to  $2 \log|r_t| = 2 \log s_{\tau,i} + 2 \log Z_t$ . The F tests of no time-of-day effects confirm that calendar effects are present in both series; the F(47, 12527) statistics are 17.34 and 8.927 for the DM/\$ and Yen/\$ respectively and have zero p-values. Suitably normalized so that  $s_{\tau,i}$  summed over the entire sample would be equal to one, we use these time-of-day dummies to standardize the returns.

Figure 3 displays the autocorrelations of the levels, absolute and squared standardized

returns of both currencies. The standardization removes the intraday calendar effects in volatility, and as in Andersen and Bollerslev (1997), their removal highlights a feature of the data that, at least in the case of the DM/\$, was not obvious: the autocorrelations of absolute returns decay very slowly.

### Construction of Multivariate Density Forecasts

We construct density forecasts of the two exchange rates using the exponential smoothing approach of RiskMetrics, which assumes that the returns are generated from a multivariate normal distribution, yielding the bivariate density forecasts

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{11,t|t-1} & \sigma_{12,t|t-1} \\ \sigma_{21,t|t-1} & \sigma_{22,t|t-1} \end{pmatrix} \right),$$

where

$$\begin{aligned} \sigma_{11,t|t-1} &= \lambda \sigma_{11,t-1|t-2} + (1 - \lambda) y_{1,t-1}^2 \\ \sigma_{22,t|t-1} &= \lambda \sigma_{22,t-1|t-2} + (1 - \lambda) y_{2,t-1}^2 \\ \sigma_{12,t|t-1} &= \lambda \sigma_{12,t-1|t-2} + (1 - \lambda) y_{1,t-1} y_{2,t-1}, \end{aligned}$$

and  $\lambda$  is the decay factor. RiskMetrics applies the same decay factor to each of the variances and covariances, which ensures that the variance-covariance matrix is positive definite.

### Density Forecast Evaluation

As discussed earlier, we evaluate the bivariate density forecasts generated by the RiskMetrics approach decomposing the forecasts in two ways:

- (i)  $p(y_{1t}, y_{2t}) = p(y_{1t}) p(y_{2t} | y_{1t})$  and
- (ii)  $p(y_{1t}, y_{2t}) = p(y_{2t}) p(y_{1t} | y_{2t})$ ,

$t = 1, \dots, T$ . We label the respective  $z$  series  $z_1, z_{2|1}, z_2$  and  $z_{1|2}$ , where we obtain  $z_i$  by

taking the probability integral transform of  $y_i$  with respect to  $p(y_i)$ , and we obtain  $z_{ij}$  by transforming  $y_i$  with respect to  $p(y_i | y_j)$ . Good multivariate forecasts will produce  $z_1, z_{2|1}, z_2$  and  $z_{1|2}$  that are each iid  $U(0, 1)$ .

To illustrate the multivariate density forecast evaluation procedures we split the sample in two, the first running from January 1, 00:30 GMT to June 30, 21:00 GMT (6234 observations) and the second running from June 30, 21:30 GMT to December 31, 23:30 GMT (6431 observations); we call them the “estimation sample” and the “forecast sample,” respectively. We first evaluate forecasts generated using a decay factor of  $\lambda=0.95$ , which is typical of RiskMetrics implementations. Figures 4a and 4b display the histograms and correlograms of  $z_1, z_{2|1}, z_2$  and  $z_{1|2}$ ; the histograms show clearly that the normality assumption is inappropriate, and the correlograms show clearly that the decay factor  $\lambda=0.95$  does not produce forecasts that capture the dynamics in return volatilities and correlations. Although the correlograms of levels of  $z$ 's look fine, the correlograms of squares of  $z$ 's indicate strong serial dependence. The positive serial correlation suggests that the decay factor of 0.95 produces volatilities that adapt too slowly. Thus it appears that a smaller decay factor would be appropriate.

We next turn to forecasts generated with an “optimal” decay factor, which we obtain from the estimation sample as the decay factor that produced integral transforms that visually appeared closest to iid, as assessed in our usual way, via correlograms of  $z$  and its powers. As expected, a smaller decay factor ( $\lambda=0.83$ ) turns out to be optimal. Figures 5a and 5b display the histograms and correlograms of  $z_1, z_{2|1}, z_2$  and  $z_{1|2}$  and their squares, based on the estimation sample, each with its mean removed before the correlograms are computed. The

correlograms indicate that the forecasts capture adequately the dynamics in return volatilities and correlations (that is,  $z_1$ ,  $z_{2|1}$ ,  $z_2$  and  $z_{1|2}$  and their squares appear serially uncorrelated). The histograms, in contrast, show clearly that the normality assumption is a poor one.

The smaller decay factor of 0.83 also produces much better out-of-sample joint forecasts of the two exchange rate returns. Figures 6a and 6b display the histograms and correlograms of  $z_1$ ,  $z_{2|1}$ ,  $z_2$  and  $z_{1|2}$  and their squares, based on the forecast sample, corresponding to density forecasts produced using  $\lambda=0.83$ . The histograms again show clearly that the normality assumption is inadequate. The correlograms of squares, on the other hand, all show clear improvement and indicate little, if any, serial correlation.

#### Density Forecast Calibration

We now attempt to improve the  $\lambda=.83$  RiskMetrics forecasts, which seemed to capture dynamics adequately but which were plagued by an inappropriate normality assumption, by calibrating. Recall that we use the transformation

$$\hat{f}(y_{1,m+1}, y_{2,m+1}, \dots, y_{N,m+1} | \Omega_m) = P_m(y_{1,m+1}, y_{2,m+1}, \dots, y_{N,m+1}) \hat{q}(z_{1,m+1}, z_{2,m+1}, \dots, z_{N,m+1}),$$

or equivalently,

$$\hat{F}(y_{1,m+1}, \dots, y_{N,m+1} | \Omega_m) = \hat{Q}(P_m(y_{1,m+1}, \dots, y_{N,m+1})).$$

We obtain  $\hat{Q}(\cdot)$  from the  $z$  series based on the estimation sample. Although we favor the presentation of histograms for density forecast evaluation, calibration is facilitated by using the empirical c.d.f. form. We present the histograms and correlograms of the four  $z$  series corresponding to the calibrated forecasts in Figures 7a and 7b; clearly the histograms of the calibrated forecasts are substantially improved relative to their non-calibrated counterparts,

and the correlograms remain good (that is, they are not affected by the calibration.)

## **5. Concluding Remarks and Directions for Future Research**

We have proposed a framework for multivariate density forecast evaluation and calibration. The multivariate forecast evaluation procedure is a generalization of the univariate procedure proposed in Diebold, Gunther and Tay (1998) and shares its constructive nature and ease of implementation. We illustrated the power of the procedure to detect and remove defects in bivariate exchange rate density forecasts generated by a popular method.

An interesting direction for future research involves using recursive techniques for real-time monitoring for breakdown of density forecast adequacy. Real-time monitoring using CUSUM techniques is a simple matter in the univariate case, because under the adequacy hypothesis the  $z$  series is iid  $U(0,1)$ , which is free of nuisance parameters. Appropriate boundary crossing probabilities for the CUSUM of the  $z$  series can be computed, as in Chu, Stinchcombe and White (1996), using results on boundary crossing probabilities of sample sums such as those of Robbins and Siegmund (1970). Multivariate CUSUM schemes are also possible, as with the multivariate profile charts of Fuchs and Benjamini (1994).

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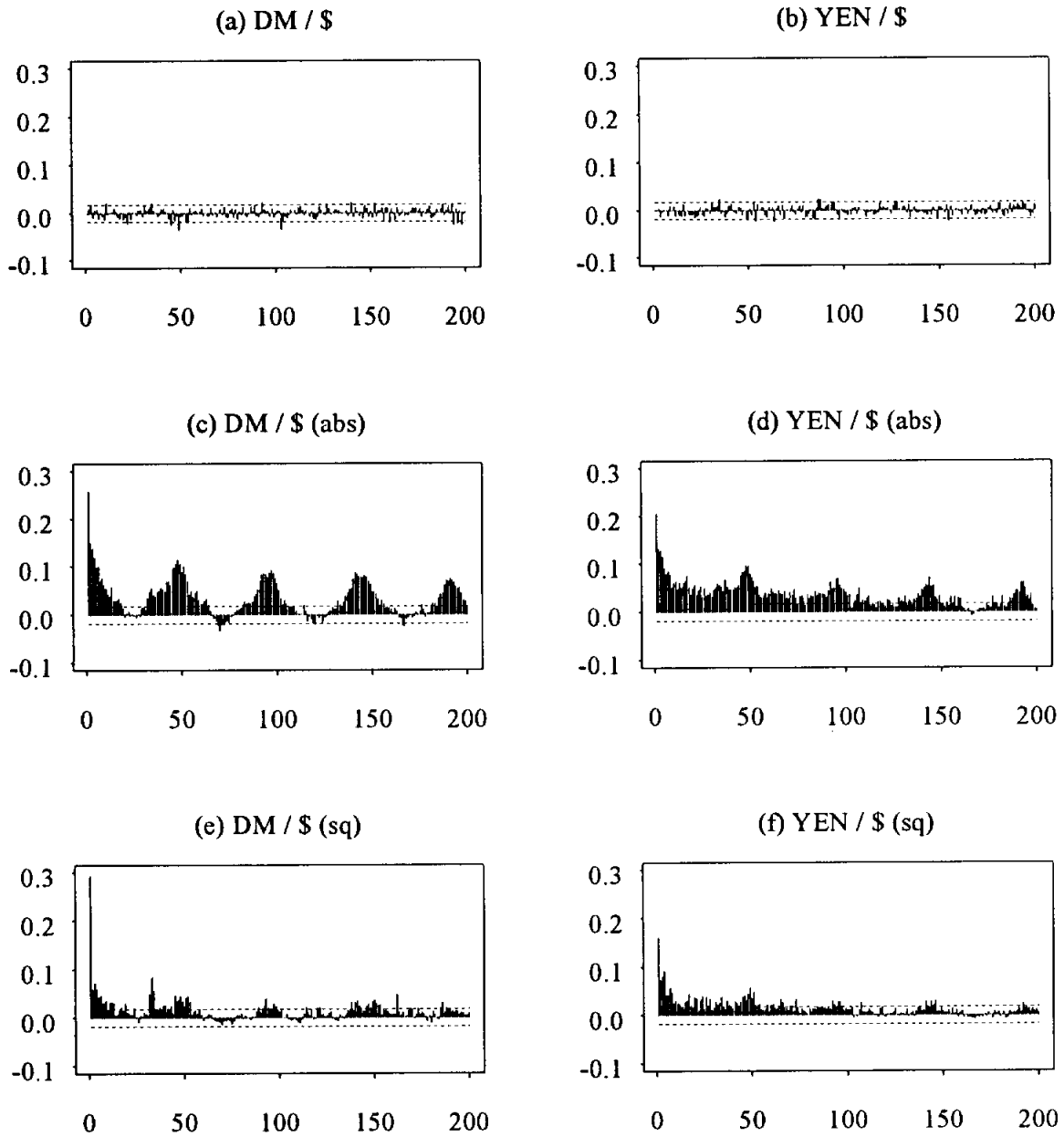


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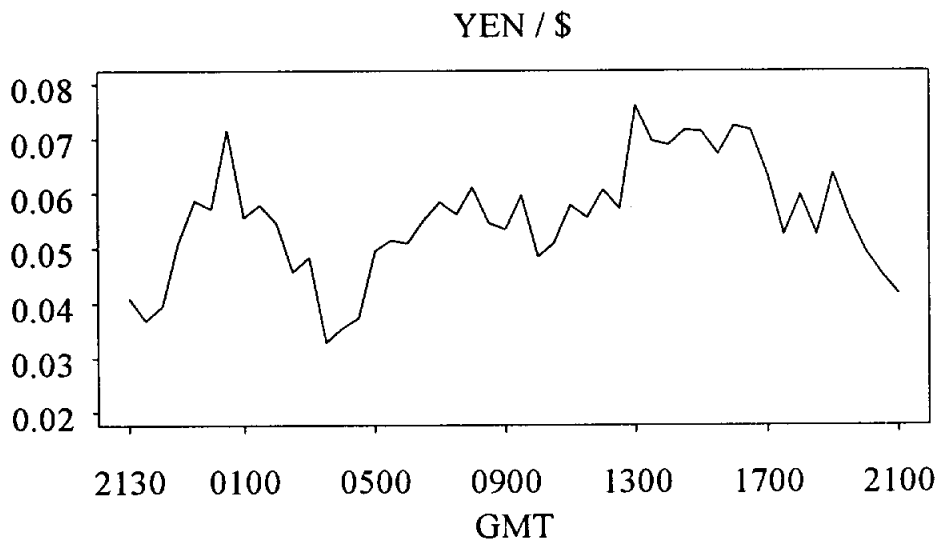
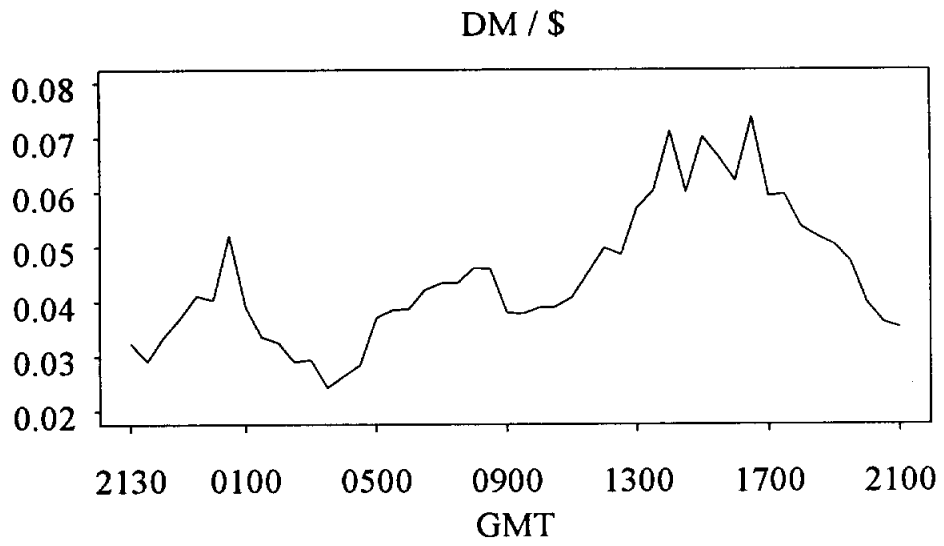
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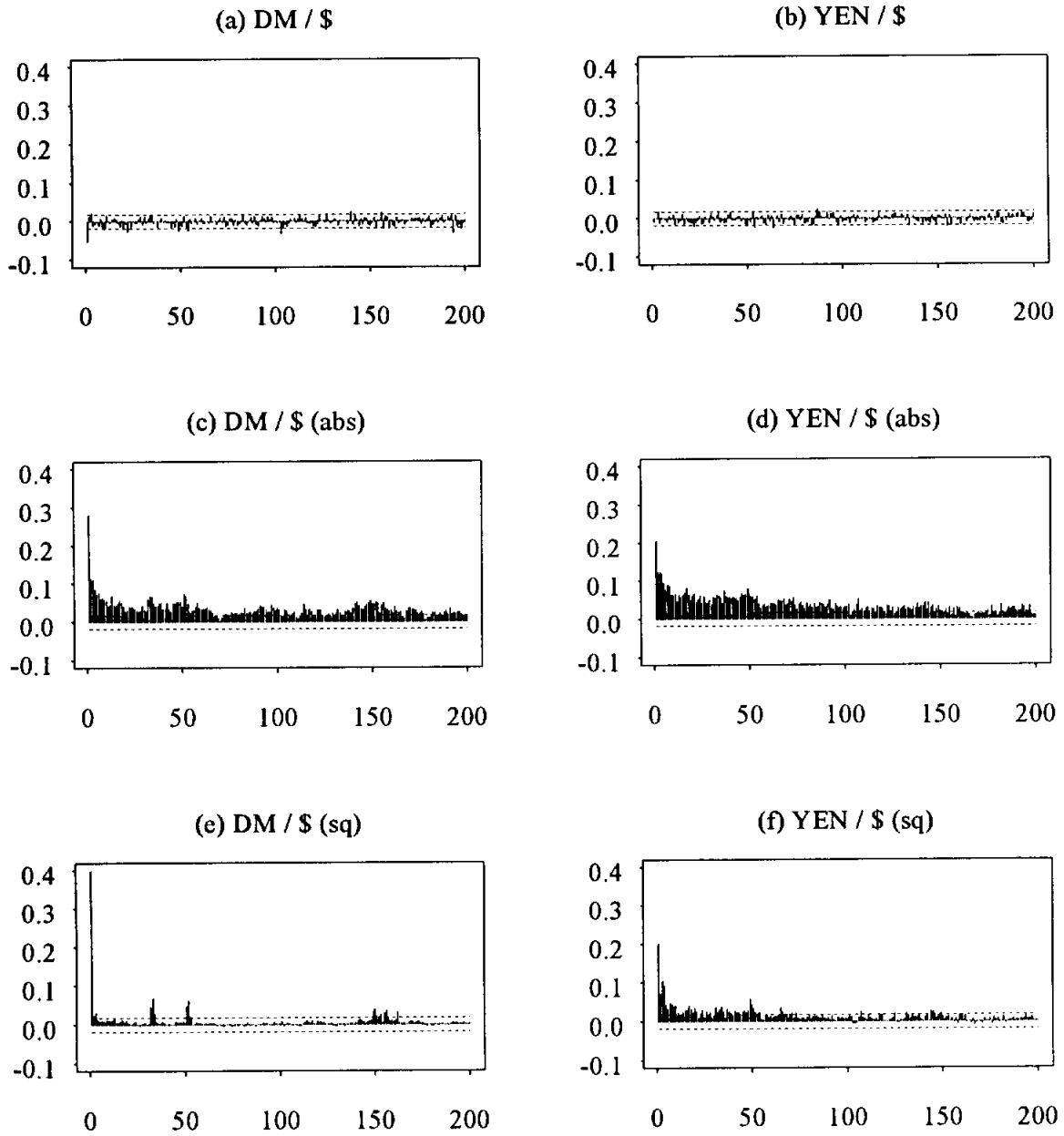
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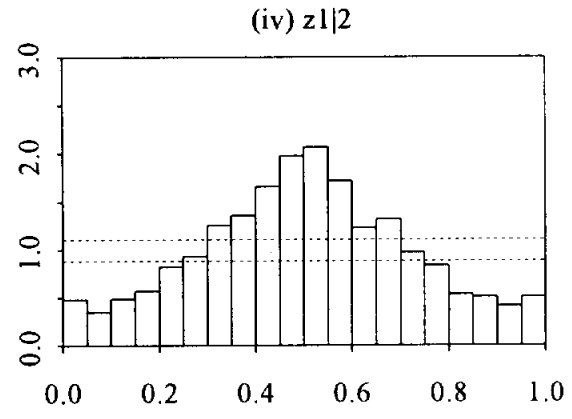
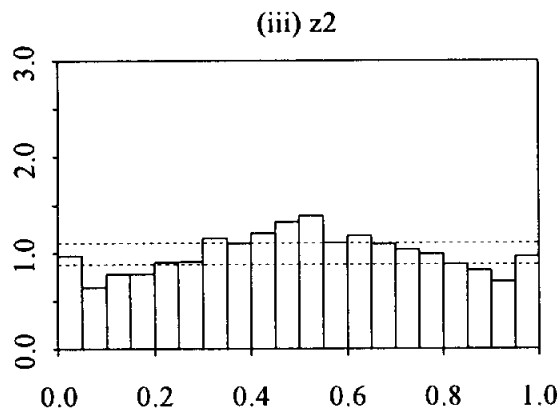
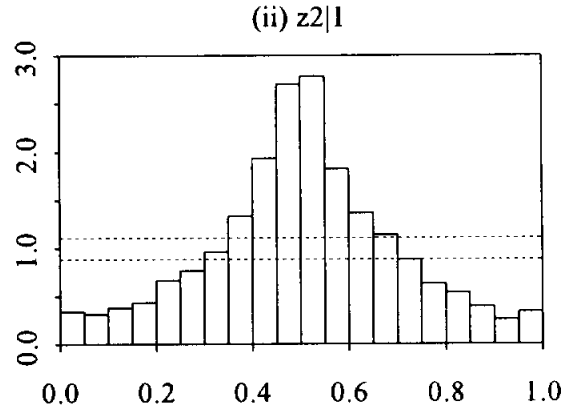
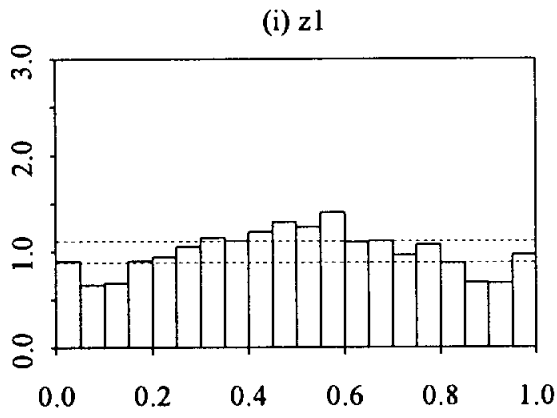
**Figure 1**  
**A.c.f.s of Returns, Absolute Returns and Squared Returns**  
**MA(1) Component Removed**



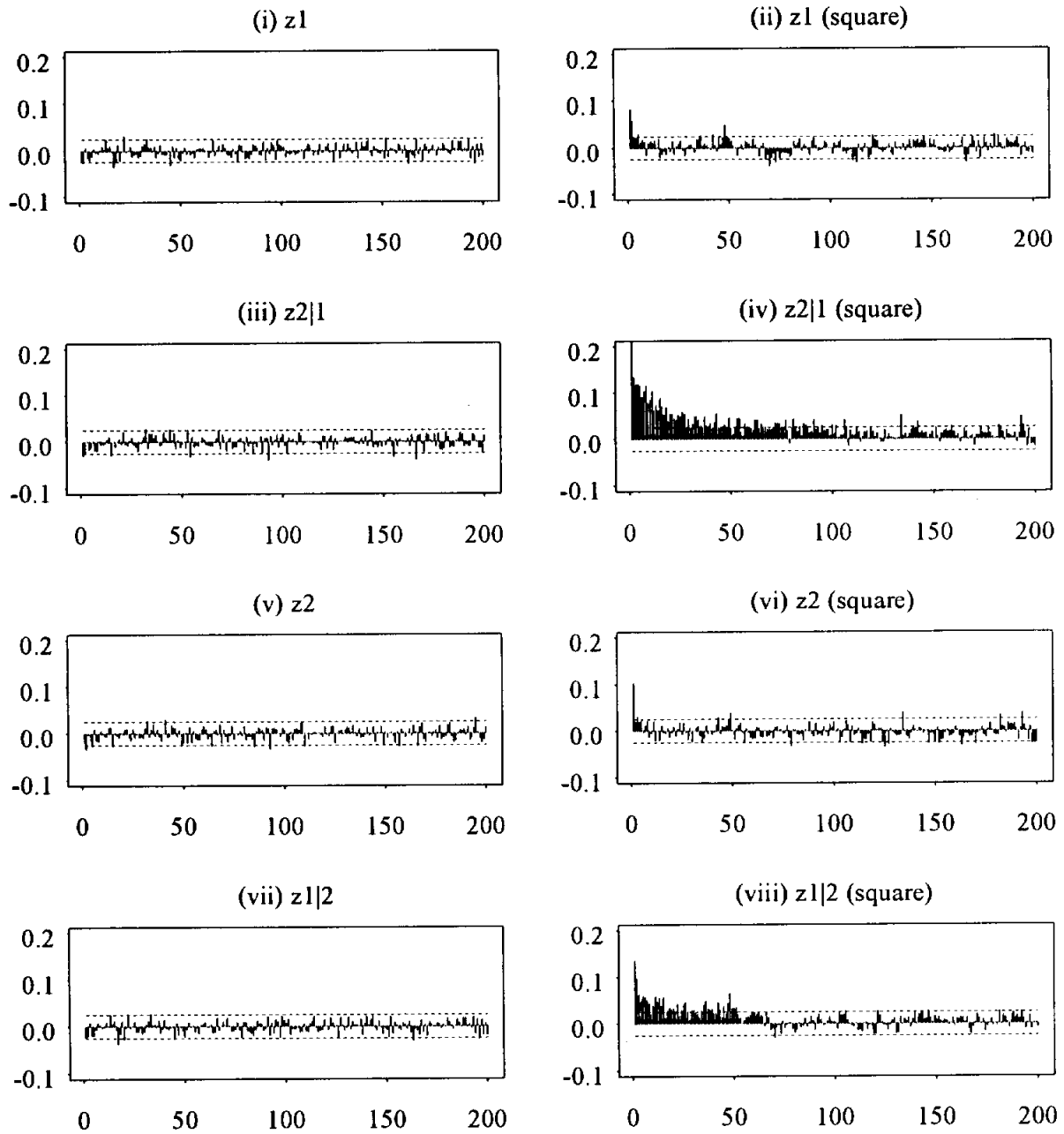
**Figure 2**  
**Mean Intraday Absolute Returns**



**Figure 3**  
**A.c.f.s of Returns, Absolute Returns and Squared Returns**  
**Conditional Mean MA(1) Component Removed**  
**Volatility Intra-Day Calendar Effects Removed**

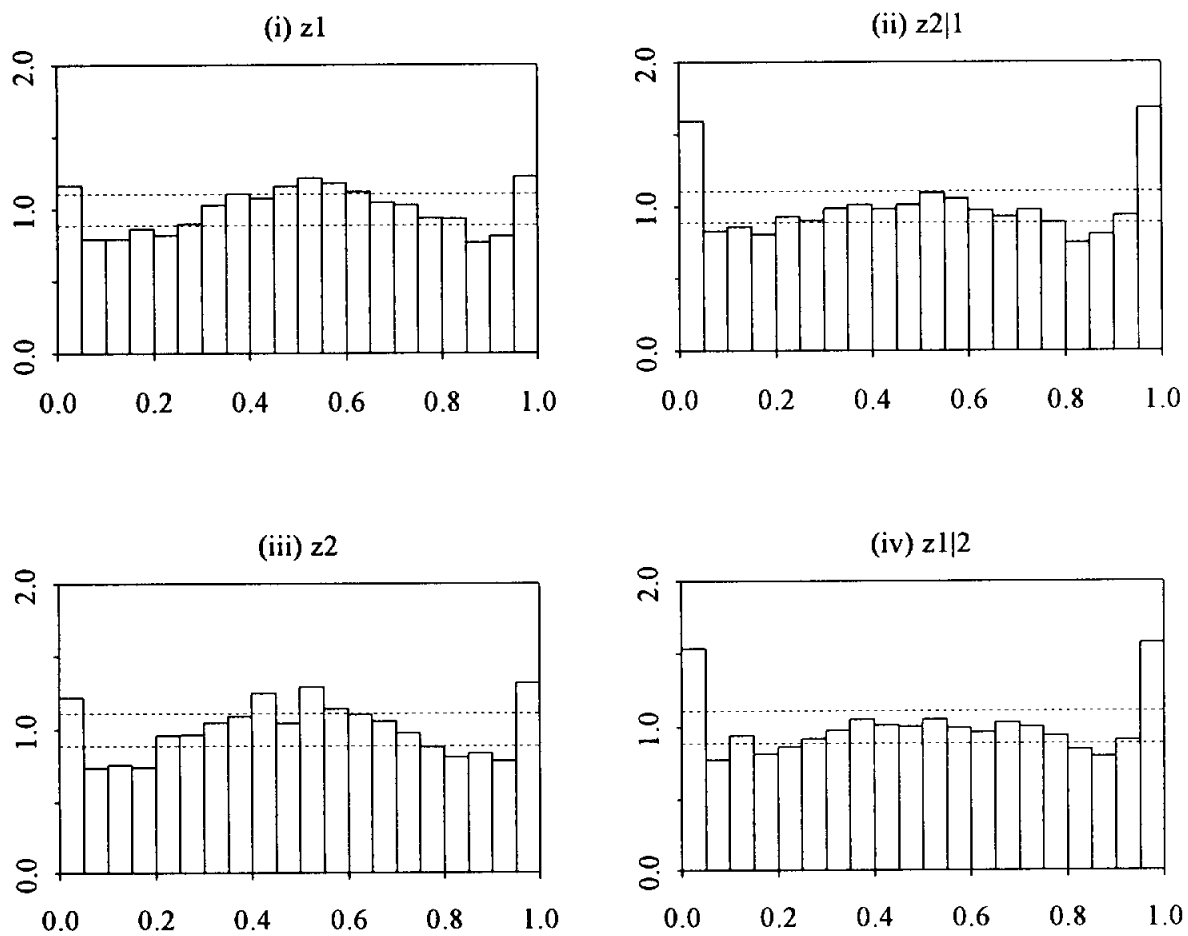


**Figure 4a**  
**Histograms of  $z$  (decay factor=0.95, forecast sample)**

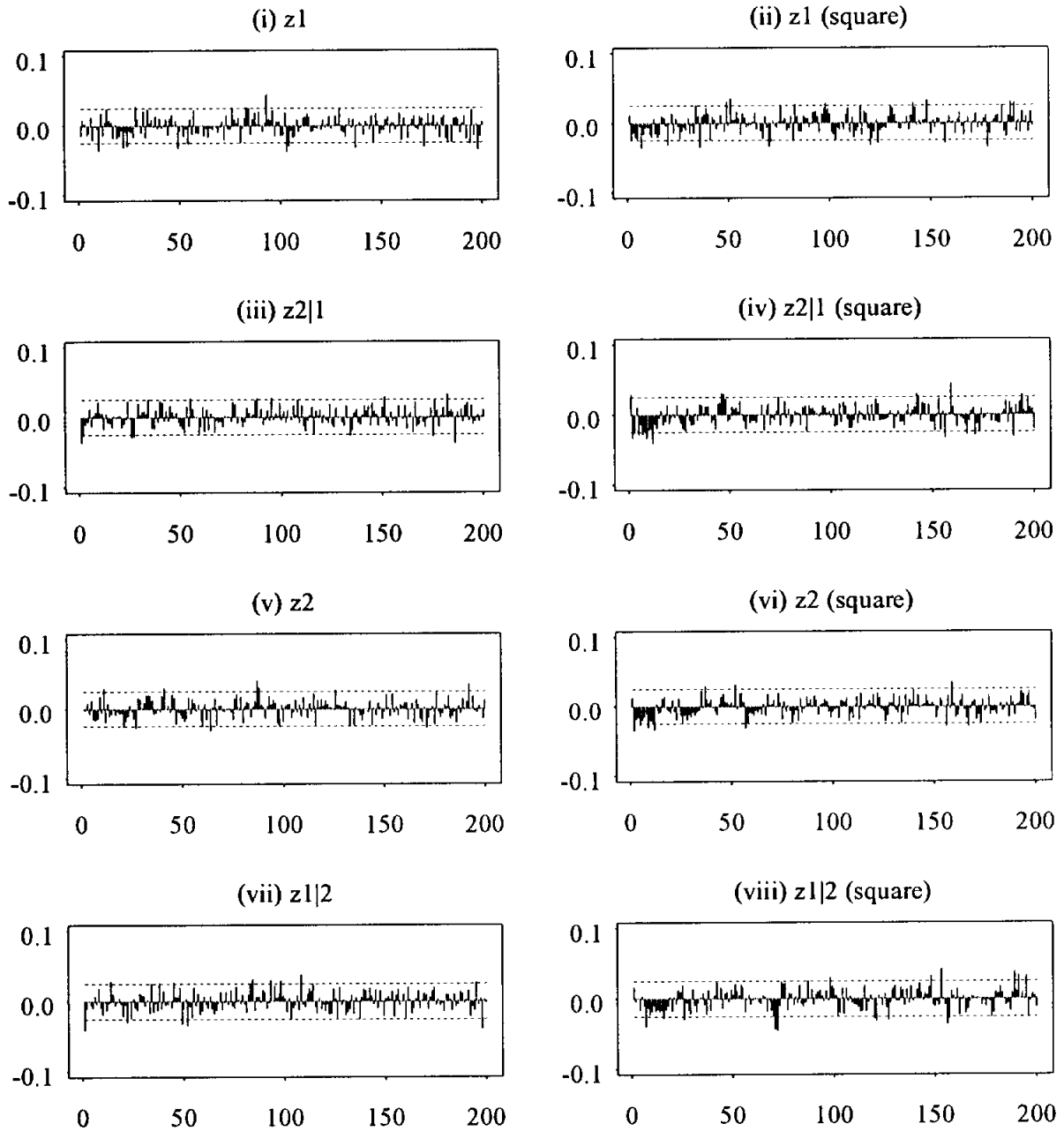


**Figure 4b**  
**Correlograms of powers of  $z$  (decay factor=0.95, forecast sample)**

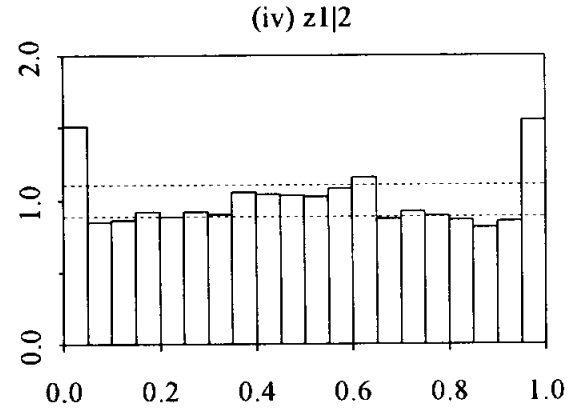
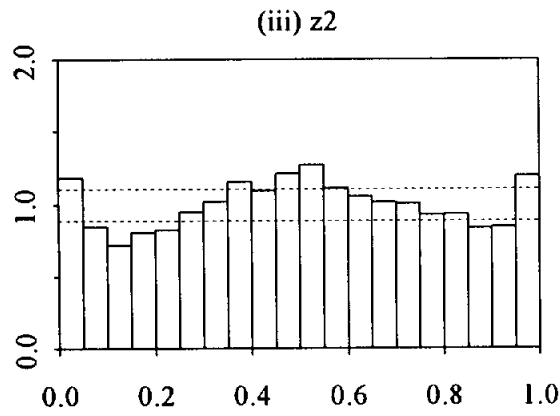
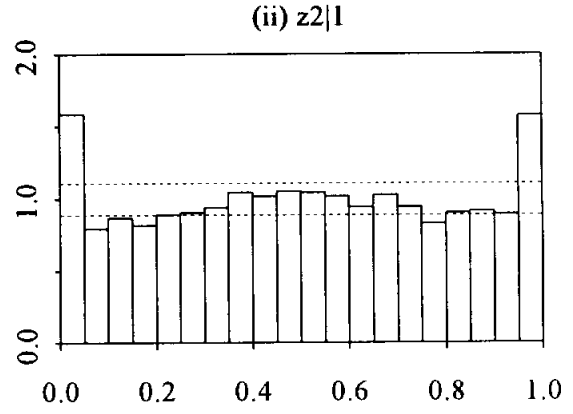
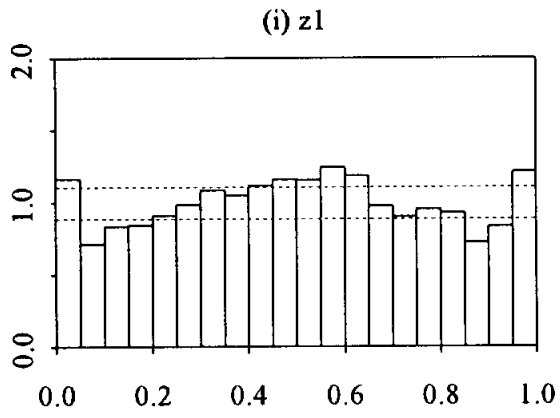




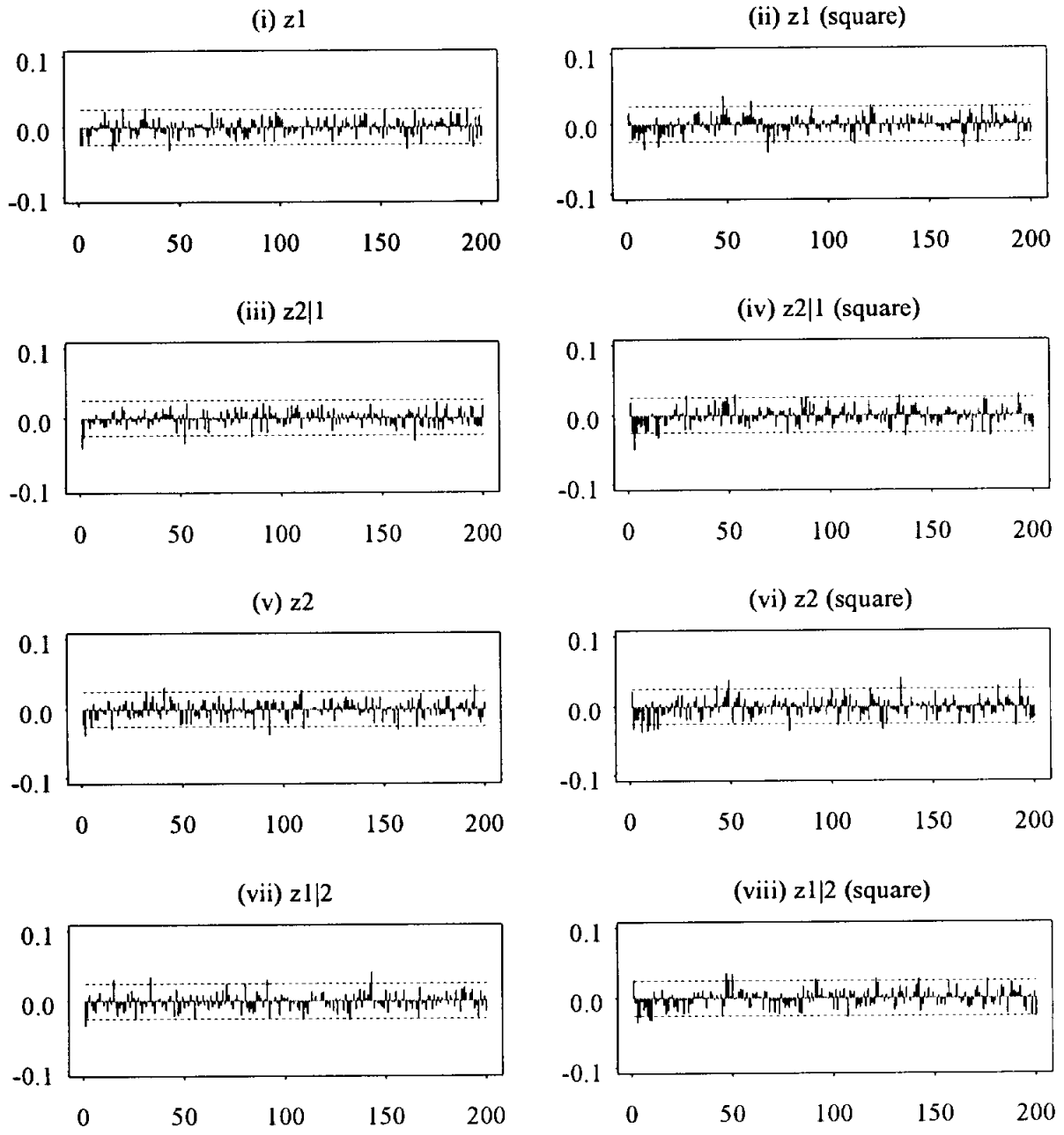
**Figure 5a**  
**Histograms of  $z$  (decay factor=0.83, estimation sample)**



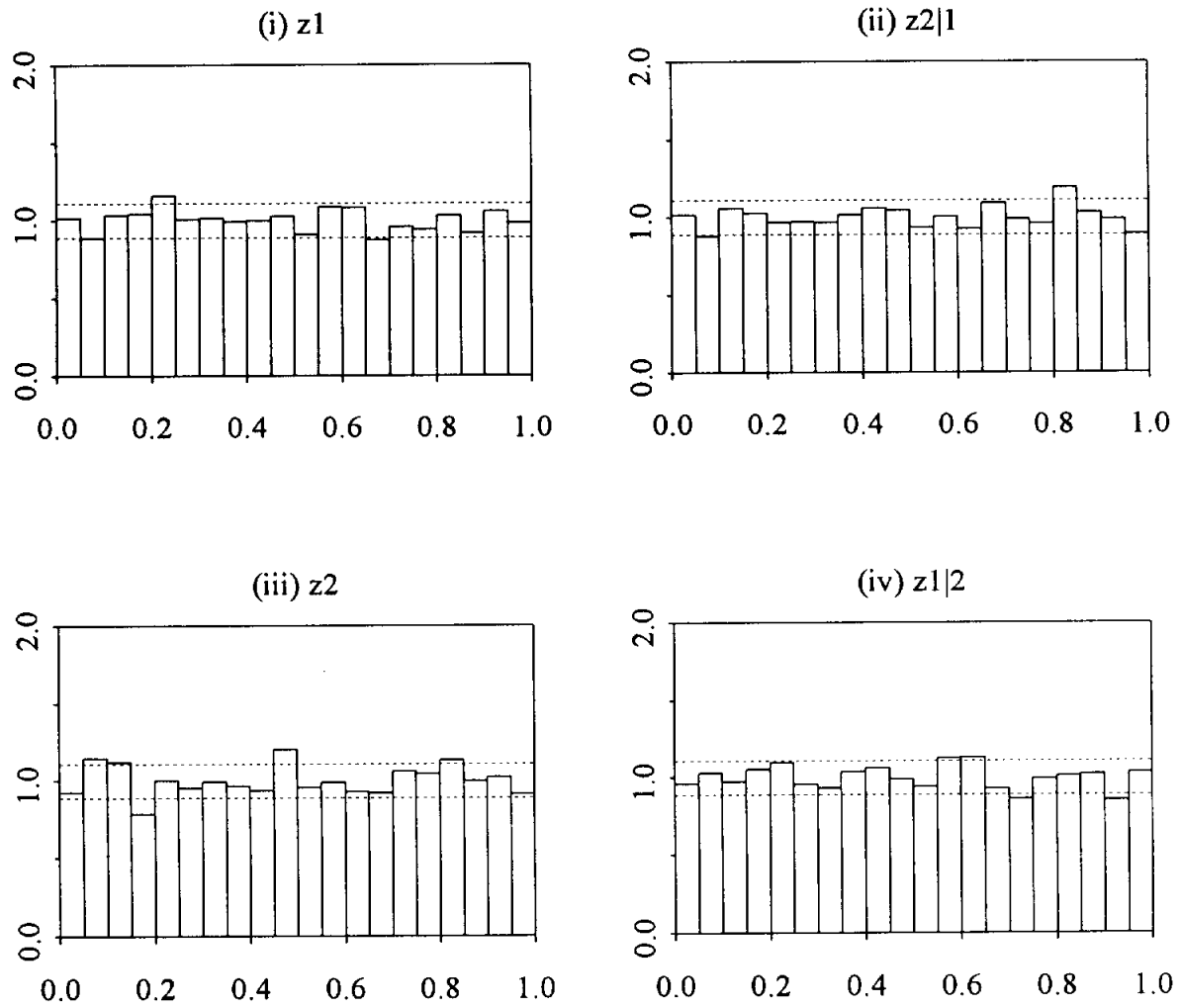
**Figure 5b**  
**Correlograms of powers of z (decay factor=0.83, estimation sample)**



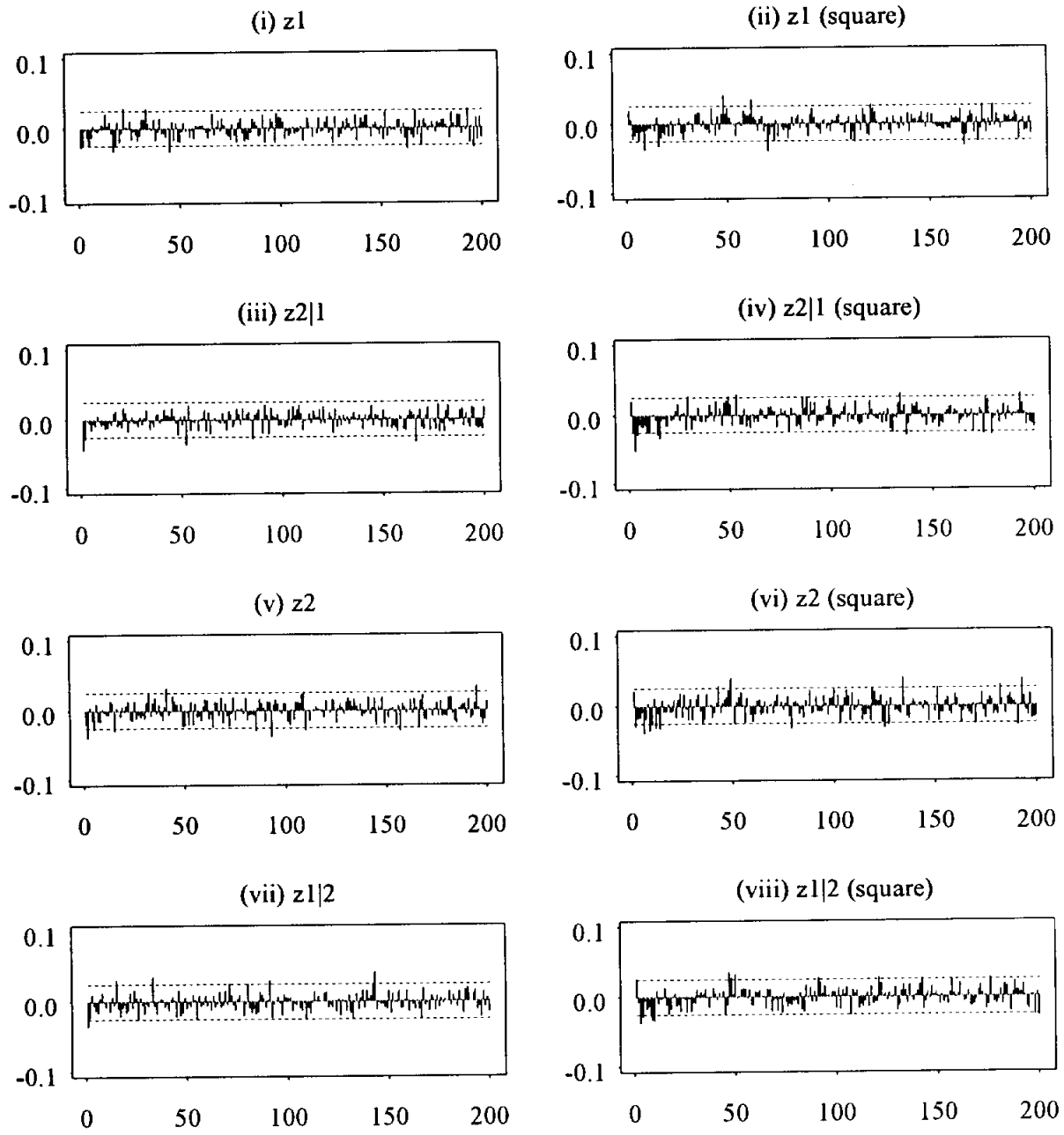
**Figure 6a**  
**Histograms of  $z$  (decay factor=0.83, forecast sample)**



**Figure 6b**  
**Correlograms of powers of  $z$  (decay factor=0.83, forecast sample)**



**Figure 7a**  
**Histograms of z (decay factor=0.83, forecast sample, calibrated)**



**Figure 7b**  
**Correlograms of powers of z (decay factor=0.83, forecast sample, calibrated)**