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ABSTRACT

This paper addresses the question of how an institution might optimally manage the market risk of a given exposure. We provide an analytical approach to optimal risk management under the assumption that the institution wishes to minimize its Value-at-Risk (VaR) using options, and that the underlying exposure follows a geometric Brownian. The optimal solution specifies the VaR-minimizing level of moneyness of the option as a function of the asset's distribution, the risk-free rate, and the VaR hedging period. We find that the optimal strike of the put is independent of the level of expense the institution is willing to incur for its hedging program. The costs associated with a suboptimal choice of exercise price, in terms of either the increased VaR for a fixed hedging cost or the increased cost to achieve a given VaR, are economically significant. Comparative static results show that the optimal strike price of these options is increasing in the asset's drift, decreasing in its volatility for most reasonable parameterizations, decreasing in the risk-free interest rate, nonmonotonic in the horizon of the hedge, and increasing in the level of protection desired by the institution (i.e., the percentage of the distribution relevant for the VaR). We show that the most important determinant is the conditional distribution of the underlying asset exposure; therefore, the optimal exercise price is very sensitive to the relative magnitude of the drift and diffusion of this exposure.

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1 Introduction

Only recently have academics begun to study the risk management practices of financial institutions and other corporations.[1] This is surprising given that the majority of firms, according to surveys by the Wharton School in conjunction with Chase Manhattan Bank (1995) and by Ernst and Young (1995), have been applying modern financial techniques to the managing of some of their exposure to interest rates, equities, or exchange rates for some time now. One of the difficulties in analyzing these institutions' risk management programs is that their concept of risk is quite different from the standard measures implied by multifactor pricing models. Ceteris paribus, according to modern finance theory, it is cheaper for shareholders to diversify project risks on their own. Thus, a company's need to hedge either the systematic or unsystematic risk of their cash flows is limited.

However, there are several reasons why this standard argument may not hold true.[2] First, with costly external financing, firms may need risk management programs to maintain their access to cheap capital, that is, internal funds (Froot, Scharfstein and Stein (1993) and Stulz (1990)). Second, in order to reduce the value of the government's implicit call option on the firm's assets via taxes, risk management programs which lead to lower earnings volatility may be optimal (Smith and Stulz (1985)). Third, without some type of risk management at the institutional level, it may not be possible to disentangle business-related profits/losses from profits/losses associated with market exposures (DeMarzo and Duffie (1995)). Finally, risk management programs can reduce the costs of financial distress (Smith and Stulz (1985)).

Of course, the above motivations for risk management are not driven by the magnitude of the firm's *market risk*, but instead by the magnitude of its *total risk*. More specifically, it is the probability and magnitude of potential losses that determine the desire to hedge,

[1] See Allayannis and Ofek (1996), DeMarzo and Duffie (1995), Froot, Scharfstein and Stein (1993), May (1995), Mian (1996), Smith and Stulz (1985), Stulz (1990) and Tufano (1996), among others, for a discussion of the underlying theory and empirics for why firms may have incentives to hedge, and, given these incentives, how firms implement the hedges.

[2] See Stulz (1997) for a general overview of institutions' risk management practices and incentives.

especially in the case of hedging motivated by the costs of external financing and financial distress. As a result of this different criteria for risk, the Value-at-Risk (VaR) concept has become the standard tool in the exploding area of risk measurement and management. In brief, VaR is defined as an estimate of the probability and size of the potential loss to be expected over a given period. While a growing number of approaches exist to answer the question of how to measure this VaR, academics and practitioners alike have been silent on the question of how to go about *managing* this risk.

We provide an analytical approach to optimal risk management in a stripped-down framework in which an institution (whether it be a financial institution, corporation, or investment fund) wishes to minimize its VaR using options. We make two key assumptions. First, the institution's risk management criteria is VaR. While VaR is clearly not the result of some optimization over all possible risk management criteria, it may be a close first approximation. As mentioned above, VaR, and similar measures, can be motivated via capital requirements in the case of financial institutions, or through some minimum level of funds necessary to perform business as usual in the case of other corporations. In any event, VaR is becoming an industry standard.

Second, the institution's hedging strategy involves options, rather than say forwards, futures, or swaps. Of course, basis and credit risk aside, once the VaR is measured, using forwards or futures to minimize the VaR of an institution's assets is straightforward. While transacting forward is a common hedge methodology, recent surveys suggest that the use of options is also commonplace.[3] There are many motivating factors for using options as a hedging vehicle. For example, except in extreme cases, the institution may be willing, or even have the desire, to take the underlying asset exposure, leading to only a partial hedge of its cash flows. This would be true, for example, if the motivations for risk management were external financing costs, financial distress possibilities, managerial incentives or tax optimization. In addition, institutional constraints, such as GAAP hedge accounting guidelines,

[3]See, for example, recent surveys by the Wharton School and Chase Manhattan Bank (1995) and by Ernst and Young (1985).

might lead to forwards not being a viable alternative for some corporations.

Taking as given the fact that an institution hedges some of its VaR using put options, the tradeoff is between the put options' ability to reduce the VaR level and the initial cost of those put options. On one hand, at high strike prices, the puts provide substantial protection but at a high cost per option. On the other hand, at low strike prices, the less costly puts provide weaker protection, but allow the institution the opportunity to purchase more of them. For a given cost, there exists a menu of implementable strike prices and hedge ratios.

This paper provides the optimal solution to the problem of finding the put option that minimizes the VaR given a fixed cost allocated for hedging. The solution is in the form of the put option's strike price as a function of the underlying asset value, the mean and volatility of this asset, the risk-free rate, and the VaR hedging period. The analysis is performed in a Black-Scholes setting in which the stochastic differential equation defining the asset follows a geometric Brownian, and the instantaneous interest rate is a constant. As such, the analysis is better suited to hedging exposures to exchange rates, equities, or similarly distributed assets.

The main results can be summarized as follows. First, independent of the level of expense the institution is willing to incur for its hedge program, there is an optimal level of moneyness of the put option. That is, given the fundamental parameters (i.e., the asset exposure's distribution, the length of the hedging horizon, and the risk-free rate), the optimal choice of options always has the same strike price.

Second, closed-form solutions for comparative static results imply that the optimal strike price of these options is increasing in the asset's drift, decreasing in its volatility for most reasonable parameterizations, decreasing in the risk-free rate, nonmonotonic in the maturity of the hedge, and increasing in the level of protection desired by the institution (i.e., the % of the distribution relevant for VaR).

Third, we are able to characterize the functional relation between the choice of put options and the underlying parameters. Most important is the distribution of the underlying asset exposure, conditional on its current value. As one might expect, the optimal choice is very sensitive to the relative magnitude of the drift and diffusion of this exposure.

Fourth, we show that the benefits of choosing the options optimally are economically significant. For example, using parameters which are typical for equity indexes, the hedged-VaR using at-the-money options can exceed the optimally hedged VaR by more than 15%. Alternatively, using at-the-money options it would require 65% more in hedging expenditures to achieve the same VaR.

The paper is organized as follows. Section 2 describes the setting and the underlying mathematical framework for optimal VaR control using options. In addition, a graphical interpretation of the problem of minimizing VaR using options is given, showing how the distribution of the underlying assets change with the use of options. Section 3 presents the main theoretical analysis, including the solution to the VaR control problem, comparative static results, and the underlying economic intuition. Section 4 illustrates these results in the context of a numerical example and quantifies the benefits from the optimal choice of options. Section 5 concludes and discusses some possible extensions and directions for future research.

2 Optimal VaR Control

The starting point of our analysis is the classical hedging example, where an institution has an exposure to the price risk of an underlying asset. This asset may be an exchange rate, or a basket of exchange rates, in the case of a multinational corporation considering the exposure associated with a given cash flow, oil prices in the case of an energy provider, gold prices in the case of a mining company, etc. We assume that the corporation is willing to devote financial resources in order to limit the loss it may incur on its endowed position in the underlying due to adverse market conditions. We assume further that the measure of market risk with which the corporation is concerned is the position's VaR. The VaR of a position will translate to a statistical statement such as "with 95% confidence the percentage loss on the dollar value of the cash flow in one year will not exceed 10%". Clearly the position's VaR is a function of a given confidence level, which, as we show in the paper, is not an innocuous choice.

Faced with the unhedged VaR of the position, we assume that the institution chooses to use options (e.g., put options to hedge a long position in the underlying), and that the institution has access to options with various exercise prices. For simplicity we shall assume that all options and VaR estimates are evaluated in a world where the Black-Scholes option pricing model applies. In some cases exchange traded options on the underlying may not exist. For example, an energy producer may want to hedge the spread between crude oil prices and electricity prices, hence the underlying is a spread. Financial markets nowadays would gladly provide over the counter options on such an exposure. The only qualification is that the Black-Scholes assumptions still apply, and that option will still be priced according to Black-Scholes, and that

The corporation's goal is to choose a level of expenditure and an exercise price on put options from a menu of VaR/cost alternatives. It is important to note, however, that the menu of alternatives is quite large. To see this, consider a point on this VaR/cost frontier. For a given level of expenditure (cost), there is a continuum of positions which the corporation can implement. For example, the corporation may hedge the full value of the underlying with a deep out-of-the-money put, a fraction of the exposure using an out-of-the-money put with a larger exercise price, or a smaller fraction of the underlying with an at-the-money put.

This menu of options will imply vastly different distribution for the terminal value of the hedged position (i.e., the value of the underlying plus the option hedge). For example, a full hedge with a deep out of the money option will have a lower bound for the value of the portfolio, and this lower bound will be attained for any value of the underlying which is below the exercise price of the put option. The distribution of the hedged position will be a truncated log-normal distribution, with a probability mass at the exercise price. A partial hedge using a put with a higher exercise price will still provide a floor value to the hedged position (at the exercise price times the hedge ratio), but the terminal distribution will now look quite different. In fact, its distribution will be a combination of a lognormal (above the exercise price) and a shifted lognormal (below the exercise price). It is important to recognize that such different distributions have different percentiles, and thus have different VaR levels.

Hence, for a given cost, there are infinitely many pairs of exercise prices and hedge ratios which will generate different levels of VaR. We present analytical results for the optimal choice of exercise price, and hence enable the investor to generate the VaR/cost frontier, where each point on this frontier corresponds to a certain optimal exercise price.

2.1 The Distribution of the Hedged Payoff

Suppose that the institution has an exposure to an asset, S_t, whose process is governed by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dz_t,$$

where μ and σ are the drift and the diffusion of the asset value, and z_t is a standard Brownian motion. One can regard this asset either as a single asset, or as a portfolio of assets like, for example, the S&P index, or a portfolio of the institution's currency exposures. The only requirement is that this portfolio's return follows a geometric Brownian motion. As such, the analysis is better suited to an institution concerned with their exposure to commodity prices, equities, or exchange rates.

The institution is concerned about its exposure to the asset over the next τ periods, and has decided to hedge the asset's value using put options. Define the market price of the put today (i.e., time t) as $P_t = P(S_t, X, r, \tau, \sigma)$, where the strike price of the option equals X and the interest rate is r. Then the hedged future value of this asset in τ periods is given by

$$\begin{aligned} V_{t+\tau} &= S_{t+\tau} + h \text{ Max}[X - S_{t+\tau}, 0] \\ &= \text{Max}[S_{t+\tau} + h(X - S_{t+\tau}), S_{t+\tau}] \end{aligned}$$

$$= \text{Max}[hX + (1-h)S_{t+\tau}, S_{t+\tau}],$$

where h represents the hedge ratio, that is, the number of options P_t used in the hedge.

The conditional distribution of the asset τ periods from today, $S_{t+\tau}$, is well known:

$$\begin{aligned} \ln S_{t+\tau} &\sim \text{N}\left[\ln S_t + \left(\mu - \frac{1}{2}\sigma^2\right)\tau, \sigma^2\tau\right] \\ &\overset{\text{let}}{=} \text{N}\left[m, s^2\right], \end{aligned}$$

where

$$\begin{aligned} m &= \ln S_t + \left(\mu - \frac{1}{2}\sigma^2\right)\tau \\ s &= \sigma\sqrt{\tau}. \end{aligned}$$

The distribution of the hedged future value of this asset, $V_{t+\tau}$, is less straightforward. One can think of this distribution as a mixture of two separate distributions, one if the put option finishes out-of-the money, the other if finishes is in-the-money. Mathematically, define this distribution as

$$f(V_{t+\tau}) = \begin{cases} f(V_{t+\tau}|S_{t+\tau} \geq X) & \text{if } S_{t+\tau} \geq X \\ f(V_{t+\tau}|S_{t+\tau} < X) & \text{if } S_{t+\tau} < X \end{cases}$$

If $S_{t+\tau} \geq X$, then $f(V_{t+\tau})$ is a lognormal distribution, i.e.,

$$\begin{aligned} V_{t+\tau}|S_{t+\tau} \geq X &= S_{t+\tau} \\ f(V_{t+\tau}|S_{t+\tau} \geq X) &= \frac{1}{\sqrt{2\pi}sV_{t+\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln V_{t+\tau} - m}{s}\right)^2\right] \end{aligned}$$

Note that $S_{t+\tau} \geq X$ implies $V_{t+\tau} \geq X$. In contrast, when $S_{t+\tau} < X$, $f(V_{t+\tau})$ is still lognormal, but with different characteristics:

$$\begin{aligned} V_{t+\tau}|S_{t+\tau} < X &= hX + (1-h)S_{t+\tau} \\ f(V_{t+\tau}|S_{t+\tau} < X) &= \frac{1}{\sqrt{2\pi}s(V_{t+\tau} - hX)} \exp\left[-\frac{1}{2}\left(\frac{\ln(V_{t+\tau} - hX) - (\ln(1-h) + m)}{s}\right)^2\right] \end{aligned}$$

$S_{t+\tau} < X$ implies $V_{t+\tau} < X$, and the option provides a lower bound on the value of the hedged payoff of hX. Combining these results:

$$f(V_{t+\tau}) = \begin{cases} \frac{1}{\sqrt{2\pi s}V_{t+\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln V_{t+\tau} - m}{s}\right)^2\right] & \text{if } V_{t+\tau} \geq X \\ \frac{1}{\sqrt{2\pi s}(V_{t+\tau} - hX)} \exp\left[-\frac{1}{2}\left(\frac{\ln(V_{t+\tau} - hX) - (\ln(1-h) + m)}{s}\right)^2\right] & \text{if } hX < V_{t+\tau} < X \\ 0 & \text{if } V_{t+\tau} \leq hX \end{cases}$$

This result assumes the hedge ratio is less than one, i.e., $h < 1$. From a practical perspective, this is the most interesting case, and the one on which we focus below, since the expenditure on hedging tends to be small relative to the exposure, yielding low hedge ratios. However, if the exposure is overhedged (i.e., $h > 1$), then $S_{t+\tau} < X$ implies $X \leq V_{t+\tau} < hX$. The exercise price is a lower bound on the value of the hedged position, and the distribution is

$$f(V_{t+\tau}) = \begin{cases} \frac{1}{\sqrt{2\pi s}V_{t+\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln V_{t+\tau} - m}{s}\right)^2\right] & \text{if } V_{t+\tau} \geq hX \\ \frac{1}{\sqrt{2\pi s}V_{t+\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln V_{t+\tau} - m}{s}\right)^2\right] & \text{if } X \leq V_{t+\tau} < hX \\ +\frac{1}{\sqrt{2\pi s}(hX - V_{t+\tau})} \exp\left[-\frac{1}{2}\left(\frac{\ln(hX - V_{t+\tau}) - (\ln(h-1) + m)}{s}\right)^2\right] & \\ 0 & \text{if } V_{t+\tau} < X \end{cases}$$

See Appendix A.1 for the details.

To build some intuition, Figures 1 and 2 graph the probability density function of this hedged value for different choices of the hedge ratio (h) and different choices of the exercise price (X), respectively. Figures 1-4 are all based on the parameter values $\mu = 0.10$, $\sigma = 0.15$, $r = 0.05$ and $\tau = 1$. The figures illustrate the basic effects of option hedging on VaR for any set of parameters. In Figure 1 the option is purchased at-the-money, and in Figure 2 the options are at-the-money and 5%, 10%, and 15% out-of-the-money.

Figure 1 provides the distribution of the hedged position for five different hedge ratios, $h = 0.0$, $h = 0.25$, $h = 0.5$, $h = 0.75$, and $h = 1.25$. The mixture of distributions is obvious from the picture. For hedge ratios less than one, if the asset value exceeds the exercise price, and hence the option finishes out-of-the-money, the distribution of the hedged position is the same for all hedge ratios. When the option finishes in-the-money, the value of the hedged position depends materially on the hedge ratio, h. The higher the h, the greater the protection, and the more truncated the distribution of the hedged position. For hedge ratios greater than one, asset values less than the exercise price generate payoffs greater than X. Consequently, the distribution is truncated at the exercise price. Clearly, the higher the hedge ratio, the lower the VaR.

Figure 2 provides the distribution of the value of the hedged position for a hedge ratio of $h = 0.5$, and for four different choices of the exercise price. Again, the mixture of distributions is obvious from the picture. If the asset value exceeds the strike price, and hence the option finishes out-of-the-money, the distribution of the hedged position looks like the lognormal distribution of the underlying asset. For asset values below the strike price, the asset distribution is compressed due to the payoff from the option. Clearly, the higher the X, the greater the protection, the more truncated the distribution of the value of the hedged position, and the lower the VaR.

As shown in Figures 1 and 2, we can decrease $\text{VaR}_{t+\tau}$ by increasing either the strike price (X) or the number of options in our hedge (h). Unfortunately, this leads to an accompanying increase in the hedging costs. The primary question raised in this paper is

Is there a VaR-minimizing combination of strike price X and hedge ratio h for a given cost?

2.2 Minimizing VaR

There is a tradeoff between the strike price and the hedge ratio. As the hedge ratio increases, the strike price must decrease in order to maintain a fixed hedging cost. Given a cost, C, the exercise price (X) and the hedge ratio (h) must satisfy

$$C = hP(S_t, X, r, \tau, \sigma).$$

Buying options with a higher exercise price affects a larger range of the distribution, but it also results in a lower hedge ratio. Consequently, more of the extreme tail of the distribution is left unhedged.

Figure 3 shows for a given cost three combinations of exercise prices and hedge ratios out of the continuum of possible choices. Specifically, the hedging cost is fixed at 0.35% of the value of the underlying asset. For hedge ratios of 0.25, 0.5, and 0.75, the corresponding options are approximately 8%, 13%, and 15% out-of-the-money. The problem is to choose the option position to minimize the VaR at a given percentage level. The optimal exercise price (and hedge ratio) will depend on the particular percentage level chosen. Of the three choices here, as the Figure illustrates, the optimal hedge ratio is 0.5, which obtains the lowest VaR. This is generalized later to the optimal choice out of a continuum of options.

The dependence of the optimal exercise price on the VaR percentage level, and the tradeoff between the exercise price and the hedge ratio are illustrated in Figure 4. This graph presents the value of the hedged position at maturity (i.e., $V_{t+\tau}$) versus the value of the underlying asset, for the different hedge ratios and option strike prices from Figure 3. The 45° line (solid line) is the payoff assuming no hedging. The hedged payoffs for all the hedge ratios lie on this line above their respective exercise prices because the option finishes out-of-the-money. Below the exercise price, the slope of the hedged payoff depends on the hedge ratio – the higher the hedge ratio, the flatter the line. For a fully hedged position, the payoff would be completely flat below the exercise price.

From this graph it is relatively simple to calculate the $\alpha\%$ VaR for a given hedge ratio and exercise price pair, and thus to choose the best option. First, find the unhedged payoff that corresponds to the $\alpha\%$ level. The corresponding hedged payoff is the $\alpha\%$ payoff for the hedged distribution. Consequently, the hedge ratio (and exercise price) that provides the lowest VaR for a given percentage level corresponds to the highest payoff line for that corresponding underlying asset value. For small percentage levels (i.e., when the institution is concerned about larger potential losses that occur with a smaller probability), the optimal exercise price is lower and the hedge ratio is higher. For large percentage levels, it is optimal to use options with a higher exercise price but at lower hedge ratios. At intermediate percentage levels, an intermediate exercise price is optimal.

In the next section, we present the formal solution to this problem, allowing the institution to purchase an option with any strike price.

3 Hedging VaR with Options

3.1 The Solution to the Minimization Problem

The institution faces the following problem: what is the optimal choice of put options that will minimize the Value-at-Risk of the institution's exposure for a given cost? Define $\text{VaR}_{t+\tau}$ as the loss at the $\alpha\%$ level of the distribution of the institution's exposure $S_{t+\tau}$. Given the lognormality of S_t, the Value-at-Risk of the institution for our problem is given by

$$\text{VaR}_{t+\tau} = S_t - \left[(1-h)S_t \exp(\theta(\alpha)) + hX\right], \tag{1}$$

where $\theta(\alpha) \equiv (\mu - \frac{1}{2}\sigma^2)\tau + c(\alpha)\sigma\sqrt{\tau}$ under the assumption that $X \geq S_{t+\tau}\exp(\theta)$, and $c(\cdot)$ is the cut-off point of the cumulative distribution of a standard normal. Intuitively, the exercise price is above the $\alpha\%$ percent level of the unhedged payoff, so the VaR depends on the compressed lognormal distribution for the region where the option finishes in-the-money[4].

As in Section 2.1, this result assumes $h < 1$. When the exposure is overhedged ($h > 1$), the VaR is much more complex because two levels of unhedged payoffs generate the same hedged payoff. Consequently, the VaR depends on the distributions both when the option finishes in-the-money and when it finishes out-of-the-money. There is no simple closed form solution, but the VaR is characterized in Proposition 1 of Appendix A.2.

It is clear from equation (1) that $\text{VaR}_{t+\tau}$ is a decreasing function of X and h. However, increasing X and h will also increase the hedging cost. The researcher's task is to find the optimal mix of X and h, given a particular hedging cost. Using Black and Scholes (1972), the hedging cost is easy to characterize. In particular,

$$C = hP(S_t, X, r, \tau, \sigma)$$

[4]It can be easily shown that the choice of smaller X will be inefficient since $\text{VaR}_{t+\tau}$ is unaffected.

where the price of the put option equals

$$\begin{aligned} P_t &= Xe^{-r\tau}\Phi(d_1) - S_t\Phi(d_2) \\ d_1 &= \frac{\ln(X/S) - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \\ d_2 &= \frac{\ln(X/S) - \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \end{aligned}$$

and $\Phi(\cdot)$ is a cumulative normal distribution. Given these characterizations of the VaR and costs of hedging, the optimization problem faced by the institution can be written as

$$\begin{aligned} \text{Min}_{h,X} \quad & \text{VaR}_{t+\tau} = S_t - \left[(1-h)S_t e^{\theta(\alpha)} + hX\right] \\ \text{subject to} \quad & C = hP_t. \end{aligned} \tag{2}$$

Substituting in the hedging cost constraint, equation (2) can be rewritten as

$$\begin{aligned} X^* &= \text{argmin}_X \; S_t - \left[\left(1 - \frac{C}{P_t}\right) S_t e^{\theta(\alpha)} + \frac{C}{P_t} X\right] \\ &= \text{argmax}_X \; C\left[\frac{X - S_t e^{\theta(\alpha)}}{P_t}\right] && (3) \\ &= \text{argmax}_X \; \left[\frac{X - S_t e^{\theta(\alpha)}}{P_t}\right]. && (4) \end{aligned}$$

Some observations are in order. First, perhaps the most striking observation is that equation (3) indicates that the $\text{VaR}_{t+\tau}$ is an affine function of hedging cost, C, and so it will not affect the choice of X. Thus, regardless of the given hedging cost, the choice of X remains the same. Once the cash flow of the asset is given, the optimal X is determined by that, and the hedge ratio will adjust depending on the hedging costs. Recall that this result holds only if the resulting optimal hedge ratio is less than one. Below we discuss the case where hedging expenditures are sufficiently large so as to violate this assumption.

Second, equation (4) shows that the minimization of $\text{VaR}_{t+\tau}$ is equivalent to the maximization of the ratio of the distance between the exercise price and the $\alpha\%$ level of the unhedged payoff, and the price of the put option. Loosely speaking, the objective function can be interpreted as the ratio of the benefit of hedging and the cost of hedging. Increasing the strike price of the option hedges a greater fraction of the distribution, but the option becomes more expensive.

The first order condition for the maximization problem in equation (4) is

$$\frac{P_t - \left(X - S_t e^{\theta(\alpha)}\right) \frac{\partial P_t}{\partial X}}{P_t^2} = 0.$$

Hence, the solution X^* satisfies the following nonlinear equation,

$$\begin{aligned} X^* &= S_t e^{\theta(\alpha)} + \frac{P_t}{\frac{\partial P_t}{\partial X}} \\ &= S_t e^{\theta(\alpha)} + \frac{e^{-r\tau} X^* \Phi(d_1) - S_t \Phi(d_2)}{e^{-r\tau}\Phi(d_1)} \\ &= S_t e^{\theta(\alpha)} + X^* - S_t e^{r\tau} \frac{\Phi(d_2)}{\Phi(d_1)}. \end{aligned}$$

Therefore, X^* is the solution that satisfies

$$\begin{aligned} 0 &= S_t e^{\theta(\alpha)} - S_t e^{r\tau} \frac{\Phi(d_2)}{\Phi(d_1)} \\ &\overset{or}{\equiv} S_t e^{\theta(\alpha)} - E^Q[S_{t+\tau} | S_{t+\tau} \leq X^*], \end{aligned} \tag{5}$$

where Q denotes the risk-neutral probability measure.

Several comments are in order. First, it is clear that equation (5) can be rewritten as

$$e^{\theta(\alpha) - r\tau} = \frac{\Phi(d_2(X^*/S_t))}{\Phi(d_1(X^*/S_t))}$$

. As one might expect, the optimal choice of put options is equivalent to choosing a level of moneyness of the option. Second, one can interpret equation (5) in the following way. The strike price is chosen such that the α% payoff of the unhedged position is equal to the *risk-neutral* expectation of the truncated distribution of the exposure when the option is exercised.[5] This is not surprising given the nature of the optimization problem. Specifically,

[5]Note that in equation (5), the necessary condition for the existence of solution X^* is that $\theta \leq r\tau$ because $\Phi(d_2)/\Phi(d_1) < 1$. For most reasonable parameter values, this restriction will be satisfied. It essentially requires that the asset's drift not be too large relative to its diffusion. We thank Bruce Grundy for this observation.

"preferences" are specified for the expected payoff of a given percentile of the return distribution. There is no aversion to any other moments of the distribution in the cost function. As we have seem above, in section 2.2 and Figure 4 the payoff maximizing scheme is achieved at X^*, the optimal exercise price for the chosen percentile.

In arriving at the above solution, we impose the budget constraint $C = hP(X)$. Consequently, $h^* = C/P(X^*)$, i.e., the hedge ratio at the optimal exercise price is simply the cost divided by the value of the put option at that strike price. If $h^* < 1$, then this solution is correct. However, if $h^* > 1$, then the expression for the VaR in equation (1) is incorrect, and the VaR is given by Proposition 1 in Appendix A.2. Under these circumstances, what is the optimal strike price? The solution is to increase X until $h = 1$, a corner solution (see Appendix A.3 for the details). Hence, the global solution to the VaR minimization problem can be summarized as follows:

- If C is low enough that at X^*, $h^* \leq 1$, then X^* is the optimal strike price and h^* is the optimal hedge ratio.
- If C is large enough so that at X^*, $h^* > 1$, then the optimal hedge ratio is $h = 1$ and the optimal strike price is simply the solution to $C = P(X)$.

While there is no closed-form solution for X^* in the more interesting and relevant case when hedging expenditures are limited, closed form expressions are available for comparative statics using the implicit function theorem. These results are provided below.

3.2 Comparative Statics

Rewriting equation (5), the optimal choice of X satisfies the following equation:

$$\Phi(d_1)\exp(\theta(\alpha) - r\tau) = \Phi(d_2). \tag{6}$$

Define $\beta = (\mu, \sigma, r, \tau)$. Since $X = X(S_t, \beta)$, using the implicit function theorem and equation (6) yields,

$$\frac{\partial X}{\partial \beta} = \frac{N(d_2)\frac{\partial d_2}{\partial \beta} - N(d_1)\frac{\Phi(d_2)}{\Phi(d_1)}\frac{\partial d_1}{\partial \beta} - \Phi(d_2)\frac{\partial(\theta - r\tau)}{\partial \beta}}{N(d_1)\frac{\Phi(d_2)}{\Phi(d_1)}\frac{\partial d_1}{\partial X} - N(d_2)\frac{\partial d_2}{\partial X}}$$

where

$$\begin{aligned} \frac{\partial d_1}{\partial X} &= \frac{1}{\sigma\sqrt{\tau}X} \\ \frac{\partial d_2}{\partial X} &= \frac{1}{\sigma\sqrt{\tau}X} \\ N(d) &= \text{a standard normal pdf} = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}d^2\right) \end{aligned}$$

Taking the derivative of d_1 and d_2 with respect to each element of the parameter vector, β, yields the desired comparative statics results. The proofs of all these results are provided in Appendix B.

3.2.1 The Drift

The derivative of the optimal exercise price with respect to the drift of the underlying asset is

$$\frac{\partial X}{\partial \mu} = \frac{-\Phi(d_1)\Phi(d_2)\sigma X \tau^{3/2}}{N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)} \geq 0.$$

The effect of increasing the mean of the distribution is to increase the optimal strike price. The reason is that, for higher drift parameters, the future distribution of the asset is shifted to the right relative to its current value. Thus, the optimal exercise price is also increased to preserve its relation relative to the $\alpha\%$ level of the unhedged payoff.

3.2.2 The Volatility

The derivative with respect to the underlying asset's volatility is

$$\frac{\partial X}{\partial \sigma} = \frac{X\sqrt{\tau}\left[N(d_1)\Phi(d_2)d_2 - N(d_2)\Phi(d_1)d_1 + \Phi(d_1)\Phi(d_2)\left(\sigma^2\tau - c(\alpha)\sigma\sqrt{\tau}\right)\right]}{N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)} \gtrless 0,$$

which is of an indeterminate sign. The effect of σ on X^* is more complicated than that of the drift. As σ increases, the price of the put increases. Higher volatility also increases the dispersion of the distribution of the underlying asset. Consequently, the exercise price must decrease to preserve its relation relative to the $\alpha\%$ level of the unhedged distribution, for reasonable values of α. Since both these effects work in the same direction, we might expect that as σ rises, the optimal strike price falls. For most parameterizations this is true. However, if $\alpha > 50\%$, then the $\alpha\%$ level of the unhedged distribution is increasing in volatility and the unhedged $\text{VaR}_{t+\tau}$ is decreasing in volatility. For a sufficiently high α this effect can offset the cost effect, and the optimal exercise price will be increasing in volatility.

3.2.3 The Interest Rate

The derivative with respect to the risk-free rate is

$$\frac{\partial X}{\partial r} = \frac{X\left[N(d_1)\Phi(d_2)\tau - N(d_2)\Phi(d_1)\tau + \Phi(d_1)\Phi(d_2)\sigma\tau^{3/2}\right]}{N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)} \leq 0, \tag{7}$$

As the interest rate increases the optimal strike price decreases. Two observations are in order. First, the optimal strike price falls as interest rates rise because of the corresponding fall in the cost of the put. Second, because the effect on the cost is small and there is no effect on the distribution of the underlying asset, the overall effect of interest rate changes is small.

3.2.4 The Horizon

The derivative of the optimal exercise price with respect to the hedging horizon is

$$\frac{\partial X}{\partial \tau} = \frac{X\left[N(d_1)\Phi(d_2)\gamma_1 - N(d_2)\Phi(d_1)\gamma_2 - 2\sigma\tau\Phi(d_1)\Phi(d_2)\left(\mu - \frac{1}{2}\sigma^2 - r + \frac{c(\alpha)\sigma}{2\sqrt{\tau}}\right)\right]}{2\sqrt{\tau}\left[N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)\right]} \gtrless 0.$$

where

$$\gamma_1 = \sigma d_2 + 2r\sqrt{\tau}$$
$$\gamma_2 = \sigma d_1 + 2r\sqrt{\tau}.$$

The horizon over which the partial option hedge takes place can have a dramatic, yet non-monotonic, effect on the optimal level of moneyness of the option. On the one hand, as the horizon increases, the positive drift in the asset's return dominates, and the strike price rises to reflect the shift in the distribution of the asset's value away from its current value. On the other hand, the volatility of the asset increases with the horizon, and the distribution gets more disperse, leading to lower optimal exercise prices. As the horizon gets very long, the former effect dominates, and strike prices increase. For shorter horizons, the volatility effect dominates, and strike prices decrease. In general, this reversal will always occur (as long as the drift is positive); however, its point of inflection depends on the underlying parameter values themselves.

3.2.5 The Level of Protection

A further interesting question is to consider how the optimal strike price changes as a function of the institution's desired VaR level, i.e., the $\alpha\%$ of the distribution the institution wishes to protect itself against. In particular, we want to investigate the sensitivity of X to the percentile, α, where we recall that $c(\alpha)$ is a cut-off point which satisfies

$$\int_{-\infty}^{c(\alpha)} N(x)dx = \Phi(c(\alpha)) = \alpha.$$

We define the inverse function of the cumulative normal density $\Phi^{-1}(\alpha)$ (i.e., if $\alpha = 2.5\%$, then $c(\alpha) = -1.96$) such that

$$c = \Phi^{-1}(\alpha).$$

This inverse function is well-defined because $\Phi(c)$ is a monotonic function of c. Then, it is straightforward to show

$$\frac{\partial c}{\partial \alpha} = \Phi^{-1'}(\alpha) > 0.$$

An application of the implicit function theorem to equation (6) yields

$$\frac{\partial X}{\partial \alpha} = \frac{-\Phi(d_1)\Phi(d_2)X\sigma^2\tau\Phi^{-1'}(\alpha)}{N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)}. \tag{8}$$

The denominator as well as the numerator is negative, so we can conclude

$$\frac{\partial X}{\partial \alpha} \geq 0.$$

While the sign of this derivative is not surprising, equation (8) does provide an exact solution for how the level of optimal moneyness changes with the institution's desired level of protection. Of particular interest, since this level is a choice variable of the institution, one could imagine using these results to help the institution tradeoff the choice of options against the amount they are willing to pay and the desired level of protection.

4 An Illustration of Optimal Hedging

In order to illustrate some of the above results, and to quantify the benefits associated with optimal hedging, we turn to a numerical example. Throughout this example we use the parameter values $S_t = 100$, $\mu = 0.10$, $\sigma = 0.15$, $r = 0.05$, $\tau = 1$, and $\alpha = 2.5\%$.

4.1 Hedging Costs and VaR

For the above parameter values, the optimal X^* is \$87.59, or in other words the institution should purchase options 12.41% out-of-the-money. Figure 5 shows how the optimal VaR changes as the institution increases its willingness to pay for options. For example, if no hedging takes place, the VaR is \$18.56; however, by purchasing \$0.35 worth of put options, the VaR is reduced to \$15.65. By adding another \$0.35 to these options, the VaR drops to \$12.75. The institution can then tradeoff its VaR reduction versus the cost of this reduction.

One key point, however, is that equation (5) shows that the optimal level of the moneyness of the option is invariant to these costs. In other words, as the institution increases its willingness to pay, this decision will not affect what the optimal strike prices of these options should be.

4.2 The Benefits of Optimal Hedging

It is worthwhile at this point to quantify the benefit of a judicious (i.e., optimal) choice of an exercise price relative to a suboptimal choice. We compare the VaR and cost of a hedged position using various exercise price options. We address two related questions:

1. Given a certain cost allocation for hedging, how does the VaR using the optimal exercise price options compare to the VaR using other exercise prices?

2. Given a targeted VaR level, how does the cost of implementation differ across different choices of exercise prices?

Figure 6 plots the VaR as a function of the exercise price. Each line represents a certain level of expenditure on the options hedge. Note that the absolute VaR level clearly declines as the cost allocated for the hedge increases. As expected given our parameter values, the VaR of the position is minimized for out-of-the money options with an exercise price of \$87.59. Since the optimal exercise price is independent of the total cost of the options hedge, the minimal VaR is obtained at this exercise price for any expenditure level as long as the cost is not so high that overhedging will occur at the optimal exercise price. When it does, as is the case for $C = 1$ in Figure 6, Appendix A.3 shows that the optimal hedge ratio is one. The figure vividly presents this case.

For example, at a cost level of \$0.70, the hedge ratio using at-the-money options is 18.85%, and the VaR is \$15.06. Reducing the exercise price to the optimal level affords an increase in the hedge ratio, to 94.57%, and generates a much lower VaR of \$12.75. This is an economically meaningful reduction in the VaR of the position of over 15%.

Figure 7 addresses the same issue from a slightly different perspective. We examine the cost of hedging across various exercise prices holding fixed a targeted VaR level. The figure plots the cost as a function of different exercise prices, where the lines now go though fixed VaR pairs of exercise price and cost. The similarity between this graph and the previous one (Figure 6) is not surprising, and is due to the linear relationship between cost and VaR. This figure demonstrates, for example, that if a VaR of \$12.5 is desired, implementing it using at-the-money options would cost \$1.21, while at the optimal exercise price the cost would be \$0.73.

4.3 Determinants of the Optimal Strike Price

Holding the above parameter values fixed, Figures 8-11 describe how X^* varies with μ and σ, r, τ and α. Figures 8a and 8b present the optimal exercise price in terms of the parameters of the underlying distribution of the institution's exposure, S_t. Figure 8a provides a 3-dimensional graph of this relation. As the drift μ varies from 0.05 to 0.15, the optimal strike price for the option can increase dramatically, from being 10-15% out-of-the money to 10-15% in-the-money. For high drift parameters, the distribution of the assets payoff is shifted dramatically relative to its current value; thus, the optimal exercise price also varies substantially. An effect of similar magnitude, but in the opposite direction, can be observed for the volatility. As σ increases from 5% to 15%, there is a large decrease in the optimal exercise price.

Figure 8a shows that the effect of the drift (μ) and diffusion (σ) parameters on the optimal level of moneyness of the option is similar in that they both relate to the option's ability to protect against losses at the tail of the distribution. To see their combined effect more clearly, Figure 8b provides a contour plot of the optimal exercise price. For these parameter values, and a given optimal X^*, the underlying μ and σ are proportionally related. For example, for the pairs ($\mu = .04, \sigma = .08$), ($\mu = .065, \sigma = .10$), and ($\mu = .09, \sigma = .115$), the optimal exercise price is 92, or 8% out-of-the money.

While Figures 8a and 8b illustrate the importance of the underlying distribution of the asset in determining the optimal strike price for hedging the institution's VaR, Figure 9 shows that, while this strike price is decreasing in the risk-free rate r, the effect is of second order. For example, increasing r from 5% to 20% causes the optimal level of moneyness to fall from 12.5% to only 14.4% out-of-the money.

Figure 10 shows that the horizon over which the partial option hedge takes place has a large and nonmonotonic effect on the optimal level of moneyness of the option. As the horizon increases to 1 year, the optimal strike price decreases from 6% to 12% out-of-the-money. Between 1 and 2 years the relation between horizon and strike price reverses. It is at this point that the mean effect begins to dominate the volatility effect. For a horizon of 7 years, the optimal exercise price is 10% in-the-money.

The final determinant of the exercise price is the level of desired protection. For example, the institution may wish to protect itself against losses at either the 2.5% tail of the distribution or the 10% tail. What level of moneyness provides the minimum value-at-risk at these α% levels? Figure 11 graphs the optimal strike price against the desired level of protection. Obviously, as additional protection is desired, more and more of the distribution of the asset needs to be hedged against, and the strike price rises. Figure 11, however, shows that this relation between the strike price and level is highly nonlinear. This suggests that an institution should take these results into account when deciding how much loss they should protect themselves against. For example, going from a desired $\alpha = 2.5\%$ to $\alpha = 10\%$ increases the exercise price of the option from \$87.59 to \$100.00.

5 Conclusion

This paper provides a formal analysis of optimal risk control using options in a simplified framework in which an institution wishes to minimize its VaR. The complication arises when considering a menu of possible pairs of exercise prices and hedge ratios given a level of expenditure, since such different choices imply different levels of hedged VaR. We find that the optimal strike price is independent of the level of cost. Therefore, the cost/VaR frontier is linear. That is, given the parameters governing the distribution of asset returns, and the desired confidence level, an institution faces the choice of increasing the position in an optimal exercise price option, thereby reducing its VaR. Interestingly, the choice of optimal exercise price is sensitive to the desired confidence level.

There are several natural extensions to our analysis to non-normal distributions, mean reverting processes, fixed income securities etc. The most natural extension, however, is to multiple asset exposure. Examples are the case of an exporter/importer to various exchange rates, the case of a pension fund manager to equity and bond markets, or the case of an energy company to the cost of various energy sources. The optimization can then be extended to the question of optimal choice of a *menu* of options on the different underlying exposures, taking into consideration a richer set of parameters, namely the correlations among assets (which may provide a natural hedge). Addressing such a problem may be necessary in the absence of options on baskets of securities. Since a portfolio of options is generally more expensive than an option on a portfolio, though, the risk management problem is best addressed by approaching the over-the-counter option market, and constructing an option on the compound position. In doing so, the analysis falls back within the realm of our model, so long as the distributional assumptions hold. One might argue in this context that the recent explosion in the use of over-the-counter basket-options may be related to this argument.

Appendix

A Results for $h > 1$

A.1 The Distribution

Assume that $h > 1$. Similar to the case of $h < 1$, we will have two conditional distributions. If $S_{t+\tau} \geq X$, then $f(V_{t+\tau})$ is a lognormal distribution, i.e.,

$$\begin{aligned} V_{t+\tau}|S_{t+\tau} &= S_{t+\tau} \\ f(V_{t+\tau}|S_{t+\tau} \geq X) &= \frac{1}{\sqrt{2\pi} s V_{t+\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln V_{t+\tau} - m}{s}\right)^2\right]. \end{aligned} \tag{9}$$

In contrast, when $S_{t+\tau} < X$, the distribution is still lognormal but with different characteristics. To see this,

$$V_{t+\tau}|S_{t+\tau} < X = hX + (1-h)S_{t+\tau}.$$

Hence,

$$hX - V_{t+\tau} = (h-1)S_{t+\tau} > 0.$$

Therefore,

$$\ln(hX - V_{t+\tau}) = \ln(h-1) + \ln S_{t+\tau} \sim N\left[\ln(h-1) + m, s^2\right].$$

Therefore, the conditional distribution is

$$f(V_{t+\tau}|S_{t+\tau} < X) = \frac{1}{\sqrt{2\pi} s (hX - V_{t+\tau})} \exp\left[-\frac{1}{2}\left(\frac{\ln(hX - V_{t+\tau}) - (\ln(h-1) + m)}{s}\right)^2\right]. \tag{10}$$

In fact, this distribution is the *mirror image* of that of $V_{t+\tau}|S_{t+\tau} < X$ when $h < 1$, reflected through the exercise price X. The maximum value is hX when $S_{t+\tau} = 0$.

Further, we can show that

$$\begin{aligned} S_{t+\tau} \geq X &\rightarrow V_{t+\tau} \geq X \\ S_{t+\tau} < X &\rightarrow V_{t+\tau} > X \end{aligned}. \tag{11}$$

That is, the minimum value of $V_{t+\tau}$ is X. Since $V_{t+\tau} \geq X$ for either of $(S_{t+\tau} \geq X)$ or $(S_{t+\tau} < X)$, we can combine (9) and (10) such that

$$f(V_{t+\tau}) = f(V_{t+\tau}|S_{t+\tau} \geq X) + f(V_{t+\tau}|S_{t+\tau} < X)$$

$$= \frac{1}{\sqrt{2\pi} s V_{t+\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln V_{t+\tau} - m}{s}\right)^2\right]$$
$$+ \frac{1}{\sqrt{2\pi} s (hX - V_{t+\tau})} \exp\left[-\frac{1}{2}\left(\frac{\ln(hX - V_{t+\tau}) - (\ln(h-1) + m)}{s}\right)^2\right]. \quad (12)$$

A.2 The VaR

Denote $V^* = S_t - \text{VaR}_{t+\tau}$, the level of the hedged payoff at the VaR. From equation (12) and the definition of VaR, $\text{VaR}_{t+\tau}$ at the α percentile should satisfy the following equation:

$$\int_X^{V^*} f(V_{t+\tau})dV_{t+\tau} = \int_X^{V^*} f(V_{t+\tau}|S_{t+\tau} \geq X)dV_{t+\tau} + \int_X^{(V^* \wedge hX)} f(V_{t+\tau}|S_{t+\tau} < X)dV_{t+\tau}$$
$$= \alpha, \quad (13)$$

where $a \wedge b = \min(a, b)$. Since the maximum value of $V_{t+\tau}|S_{t+\tau} < X$ is hX, we have to check V^* is greater than hX, so that in the second term of the RHS in equation (13), the upper limit of integral is $V^* \wedge hX$.

Now we establish the following proposition about the VaR for $h > 1$.

Proposition 1: *Given α, the VaR of the hedged payoff $V_{t+\tau}$ when $h \geq 1$, equals*

$$\text{VaR}_{t+\tau} = S_t - V^*,$$

where V^* satisfies the following equation

$$\alpha = \begin{cases} \Phi\left(\frac{\ln V^* - m}{s}\right) - \Phi\left(\frac{\ln\left(\frac{hX - V^*}{h-1}\right) - m}{s}\right) & \text{if } \alpha < \Phi\left(\frac{\ln(hX) - m}{s}\right) \\ \Phi\left(\frac{\ln V^* - m}{s}\right) & \text{if } \alpha \geq \Phi\left(\frac{\ln(hX) - m}{s}\right) \end{cases} \quad (14)$$

where m and s are

$$m = \ln S_t + \left(\mu - \frac{1}{2}\sigma^2\right)\tau$$
$$s = \sigma\sqrt{\tau}.$$

Proof: The first term of the right hand side in equation (13) is

$$\int_X^{V^*} f(V_{t+\tau}|S_{t+\tau} \geq X)dV_{t+\tau} = \int_X^{V^*} \frac{1}{\sqrt{2\pi} s V_{t+\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln V_{t+\tau} - m}{s}\right)^2\right] dV_{t+\tau}$$

$$
\begin{aligned}
&= \int_X^{V^*} \frac{1}{\sqrt{2\pi} s S_{t+\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln S_{t+\tau} - m}{s}\right)^2\right] dS_{t+\tau} \\
&= \Phi\left(\frac{\ln V^* - m}{s}\right) - \Phi\left(\frac{\ln X - m}{s}\right),
\end{aligned} \tag{15}
$$

since $V_{t+\tau} | S_{t+\tau} \geq X = S_{t+\tau}$. The second term is more complicated.

1. $V^* < hX$.

   In this case, the second term will be

$$
\begin{aligned}
& \int_X^{V^*} f(V_{t+\tau} | S_{t+\tau} < X) dV_{t+\tau} \\
= & \int_X^{V^*} \frac{1}{\sqrt{2\pi} s (hX - V_{t+\tau})} \exp\left[-\frac{1}{2}\left(\frac{\ln(hX - V_{t+\tau}) - (\ln(h-1) + m)}{s}\right)^2\right] dV_{t+\tau} \\
= & \int_X^{S_2^*} \frac{1}{\sqrt{2\pi} s (h-1) S_{t+\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln((h-1)S_{t+\tau}) - (\ln(h-1) + m)}{s}\right)^2\right] (1-h) dS_{t+\tau} \\
= & \int_{S^*}^{X} \frac{1}{\sqrt{2\pi} s S_{t+\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln S_{t+\tau} - m}{s}\right)^2\right] dS_{t+\tau} \\
= & \Phi\left(\frac{\ln X - m}{s}\right) - \Phi\left(\frac{\ln S_2^* - m}{s}\right),
\end{aligned} \tag{16}
$$

   where

$$
S^* = \frac{hX - V^*}{h-1}.
$$

   Therefore, from equations (15) and (16),

$$
\begin{aligned}
\alpha &= \Phi\left(\frac{\ln V^* - m}{s}\right) - \Phi\left(\frac{\ln X - m}{s}\right) + \Phi\left(\frac{\ln X - m}{s}\right) - \Phi\left(\frac{\ln S^* - m}{s}\right) \\
&= \Phi\left(\frac{\ln V^* - m}{s}\right) - \Phi\left(\frac{\ln S^* - m}{s}\right).
\end{aligned}
$$

   One important thing is that given X, there is the upper bound for V^*, or equivalently, the lower bound for $\mathrm{VaR}_{t+\tau}$. Using L'Hospital's rule, it can be easily shown that

$$
\sup V^*(h) = \lim_{h \uparrow \infty} V^*(h),
$$

   is the solution to

$$
\alpha = \Phi\left(\frac{\ln V^* - m}{s}\right) - \Phi\left(\frac{\ln X - m}{s}\right).
$$

   Therefore, some deep-out-of-the-money options cannot obtain certain levels of VaRs if those values are lower than the lower limit.

2. $V^* \geq hX$.

In this case, the second term on the right hand side in (13) will be

$$\begin{aligned}
&\int_X^{hX} f(V_{t+\tau}|S_{t+\tau} < X)dV_{t+\tau} \\
= &\int_X^{hX} \frac{1}{\sqrt{2\pi}s(hX - V_{t+\tau})} \exp\left[-\frac{1}{2}\left(\frac{\ln(hX - V_{t+\tau}) - (\ln(h-1)+m)}{s}\right)^2\right] dV_{t+\tau} \\
= &\int_0^{X} \frac{1}{\sqrt{2\pi}sS_{t+\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln S_{t+\tau} - m}{s}\right)^2\right] dS_{t+\tau} \\
= &\Phi\left(\frac{\ln X - m}{s}\right). \qquad (17)
\end{aligned}$$

Then, the V^* should satisfy

$$\begin{aligned}
\alpha &= \Phi\left(\frac{\ln V^* - m}{s}\right) - \Phi\left(\frac{\ln X - m}{s}\right) + \Phi\left(\frac{\ln X - m}{s}\right) \\
&= \Phi\left(\frac{\ln V^* - m}{s}\right).
\end{aligned}$$

Finally, the condition $V^* \geq hX$ implies that $\alpha \geq \text{prob}(V_{t+\tau} \leq hX)$. Further

$$\begin{aligned}
\alpha \geq \text{prob}(V_{t+\tau} \leq hX) &= \Phi\left(\frac{\ln hX - m}{s}\right) - \Phi\left(\frac{\ln X - m}{s}\right) + \Phi\left(\frac{\ln X - m}{s}\right) \\
&= \Phi\left(\frac{\ln(hX) - m}{s}\right),
\end{aligned}$$

which yields the desired results. In this case, however, the hedging is inefficient since $\text{VaR}_{t+\tau}$ is unaffected. *Q.E.D.*

Even though we do not have a closed-form expression for the VaR in this case, we can easily find the numerical solution using a numerical search such as the Newton-Raphson method.

A.3 The Optimal Exercise Price

In Section 3 we solve for the optimal exercise price X^* under the constraint that $h < 1$. Recall that we impose the budget constraint $C = hP(X)$. Consequently, $h^* = C/P(X^*)$, i.e., the hedge ratio at the optimal exercise price is simply the cost divided by the value of the put option at that strike price. If $h^* < 1$, then this solution is correct. However, if $h^* > 1$, then we must use the VaR derived above to find the optimal exercise price. In this case, the solution is to increase X until

$h = 1$, a corner solution. Hence, the solution to the VaR minimization problem can be summarized as follows:

- If C is low enough that at X^*, $h^* < 1$, then X^* is the optimal strike price and $h^* < 1$ is the optimal hedge ratio.
- If C is large enough so that at X^*, $h^* > 1$, then the optimal hedge ratio is $h = 1$ and the optimal strike price is simply the solution to $C = P(X)$.

B Comparative Statics

We will prove the set of equations (7). $N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)$ is a common term in the denominators of all the equations. Define $\Lambda = N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)$. It is possible to rewrite Λ as

$$\begin{aligned}
\Lambda &= N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1) \\
&= N(d_2)[\Phi(d_2) - \Phi(d_1)] + \Phi(d_2)[N(d_1) - N(d_2)] \\
&= -N(d_2)\int_{d_2}^{d_1} N(z)dz + \Phi(d_2)\int_{d_2}^{d_1}(-zN(z))dz \\
&= -\int_{d_2}^{d_1}[N(d_2) + z\Phi(d_2)]N(z)dz,
\end{aligned}$$

using the fact that $\partial N(z)/\partial z = -zN(z)$ for any $z \geq d_2$. Now, we need to prove that $N(d_2) + z\Phi(d_2) \geq 0$ for any $z \geq d_2$. To do that, we establish the following lemmas:

Lemma 1: *If z is a standard normal variate, then*

$$N(z) + z\Phi(z) \geq 0.$$

Proof: The lemma is true if $z \geq 0$. If $z < 0$, we need $-\frac{N(z)}{z} \geq \Phi(z)$. For notation convenience we define a positive normal variate, $\omega = -z$. Then, the above inequality is equivalent to $\frac{N(\omega)}{\omega} \geq 1 - \Phi(\omega)$, using the fact $N(z) = N(-z)$ and $\Phi(z) = 1 - \Phi(-z)$. Differentiating each side yields

$$-N(\omega) - \frac{N(\omega)}{\omega^2} = -\left(1 + \omega^{-2}\right)N(\omega)$$

and $-N(\omega)$, respectively. Thus, the LHS of the equation is

$$\frac{N(\omega)}{\omega} = \int_{\omega}^{\infty}\left(1 + y^{-2}\right)N(y)dy, \tag{18}$$

and the RHS is

$$1 - \Phi(\omega) = \int_{\omega}^{\infty} N(y)dy. \tag{19}$$

For $\omega > 0$, we have $(1 + y^{-2})N(y) \geq N(y)$ for all $y \geq \omega$, so that equation (18) exceeds (19); and the lemma holds for $z < 0$. *Q.E.D.*

Lemma 2: *For any $z \geq d_2$,*

$$-\frac{N(z)}{\Phi(z)} \geq -\frac{N(d_2)}{\Phi(d_2)}.$$

Proof: For any z,

$$\frac{\partial(N(z)/\Phi(z))}{\partial z} = \frac{-zN(z)\Phi(z) - N(z)^2}{\Phi(z)^2}.$$

Consider the following two cases:

1. If $z \geq 0$, the numerator is negative, so $\frac{\partial(N(z)/\Phi(z))}{\partial z} \leq 0$.

2. If $z < 0$, the numerator is $-N(z)(N(z) + z\Phi(z)) < 0$ from Lemma 1, which results in $\frac{\partial(N(z)/\Phi(z))}{\partial z} < 0$.

Therefore, $\frac{\partial(N(z)/\Phi(z))}{\partial z} \leq 0$ for any z, which is equivalent to $\frac{\partial(-N(z)/\Phi(z))}{\partial z} \geq 0$. Then, $-\frac{N(z)}{\Phi(z)} \geq -\frac{N(d_2)}{\Phi(d_2)}$, for any $z \geq d_2$. *Q.E.D.*

From Lemma 1 and Lemma 2, for any $z \geq d_2$,

$$z \geq -\frac{N(z)}{\Phi(z)} \geq -\frac{N(d_2)}{\Phi(d_2)}.$$

Therefore, $N(d_2) + z\Phi(d_2) \geq 0$ for any $z \geq d_2$, which yields $\Lambda < 0$; thus, the sign$(\partial X/\partial \beta)$ =-sign(the numerator of $(\partial X/\partial \beta)$).

B.1 The Drift (μ)

With respect to μ, using

$$\frac{\partial d_1}{\partial \mu} = 0, \quad \frac{\partial d_2}{\partial \mu} = 0, \quad \frac{\partial(\theta - r\tau)}{\partial \mu} = \tau,$$

the numerator of $(\partial X/\partial \mu)$ will be simplified to

$$-\Phi(d_1)\Phi(d_2)\sigma X \tau^{3/2} < 0,$$

which results in $(\partial X/\partial \mu) > 0$.

B.2 The Volatility (σ)

Using the expressions for d_1, d_2 and the definition of $\theta(\alpha)$, we have

$$\frac{\partial d_1}{\partial \sigma} = -\frac{d_2}{\sigma}, \quad \frac{\partial d_2}{\partial \sigma} = -\frac{d_1}{\sigma}, \quad \frac{\partial(\theta - r\tau)}{\partial \sigma} = -\sigma\tau + c(\alpha)\sqrt{\tau}$$

Then, the numerator of $\partial X/\partial\sigma$ will reduce to

$$\frac{1}{\sigma\Phi(d_1)}\left[N(d_1)\Phi(d_2)d_2 - N(d_2)\Phi(d_1)d_1 + \Phi(d_1)\Phi(d_2)\left(\sigma^2\tau - c(\alpha)\sigma\sqrt{\tau}\right)\right]. \tag{20}$$

$N(d_1)\Phi(d_2)d_2 - N(d_2)\Phi(d_1)d_1$ can be shown to be negative using Lemma 1, while the sign of $\Phi(d_1)\Phi(d_2)\left(\sigma^2\tau - c(\alpha)\sigma\sqrt{\tau}\right)$ is indeterminate. As long as we are concerned about the payoff below mean $\alpha < .5$, the term will be positive. Hence, the sign of $(\partial X/\partial\sigma)$ will be determined by which term will dominate.

B.3 The Interest Rate (r)

We can show that

$$\frac{\partial d_1}{\partial r} = -\frac{\sqrt{\tau}}{\sigma}, \quad \frac{\partial d_2}{\partial r} = -\frac{\sqrt{\tau}}{\sigma}, \quad \frac{\partial(\theta - r\tau)}{\partial r} = -\tau.$$

Then, the numerator of $(\partial X/\partial r)$ can be written

$$\frac{1}{\sigma\Phi(d_1)}\left[N(d_1)\Phi(d_2)\sqrt{\tau} - N(d_2)\Phi(d_1)\sqrt{\tau} + \Phi(d_1)\Phi(d_2)\sigma\tau\right], \tag{21}$$

which will give us the desired result.

It is possible to rewrite equation (21) as

$$\begin{aligned} & \frac{\sqrt{\tau}}{\sigma\Phi(d_1)}\left[N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1) + \Phi(d_1)\Phi(d_2)\sigma\sqrt{\tau}\right] \\ = & \frac{\sqrt{\tau}}{\sigma\Phi(d_1)}\left[N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1) + \Phi(d_1)\Phi(d_2)(d_1 - d_2)\right]. \end{aligned}$$

Now we need to prove

$$\Gamma(d_2) \equiv N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1) + \Phi(d_1)\Phi(d_2)(d_1 - d_2) \geq 0.$$

Using $d_1 - d_2 = \sigma\sqrt{\tau}$, we can show

$$\frac{d\,\Gamma}{d\,d_2} = -\left[N(d_1)\Phi(d_2)d_2 - N(d_2)\Phi(d_1)d_1\right].$$

From Lemma 1 and Lemma 2,

$$N(d_1)\Phi(d_2) \;\leq\; N(d_2)\Phi(d_1).$$

Further, since $N(d_1)\Phi(d_2) \geq 0$, $N(d_2)\Phi(d_1) \geq 0$, and $d_2 < d_1$,

$$N(d_1)\Phi(d_2)d_2 - N(d_2)\Phi(d_1)d_1 \leq 0,$$

which yields $\frac{d\Gamma}{d d_2} \geq 0$. In addition,

$$\inf \Gamma(d_2) = \lim_{d_2 \downarrow -\infty} \Gamma(d_2) = 0,$$

so $\Gamma(d_2) \geq 0$ for any d_2. This results in $(\partial X/\partial r) \leq 0$.

B.4 The Time to Maturity (τ)

We can show that

$$\begin{aligned}
\frac{\partial d_1}{\partial \tau} &= -\frac{d_2}{2\tau} - \frac{r}{\sigma\sqrt{\tau}} \\
\frac{\partial d_2}{\partial \tau} &= -\frac{d_1}{2\tau} - \frac{r}{\sigma\sqrt{\tau}} \\
\frac{\partial(\theta - r\tau)}{\partial \tau} &= \left(\mu - \frac{1}{2}\sigma^2 - r\right) + \frac{c(\alpha)\sigma}{2\sqrt{\tau}}.
\end{aligned}$$

Substituting these into $(\partial X/\partial \beta)$ will give us the desired result. The sign is indeterminate as shown in Figure 10.

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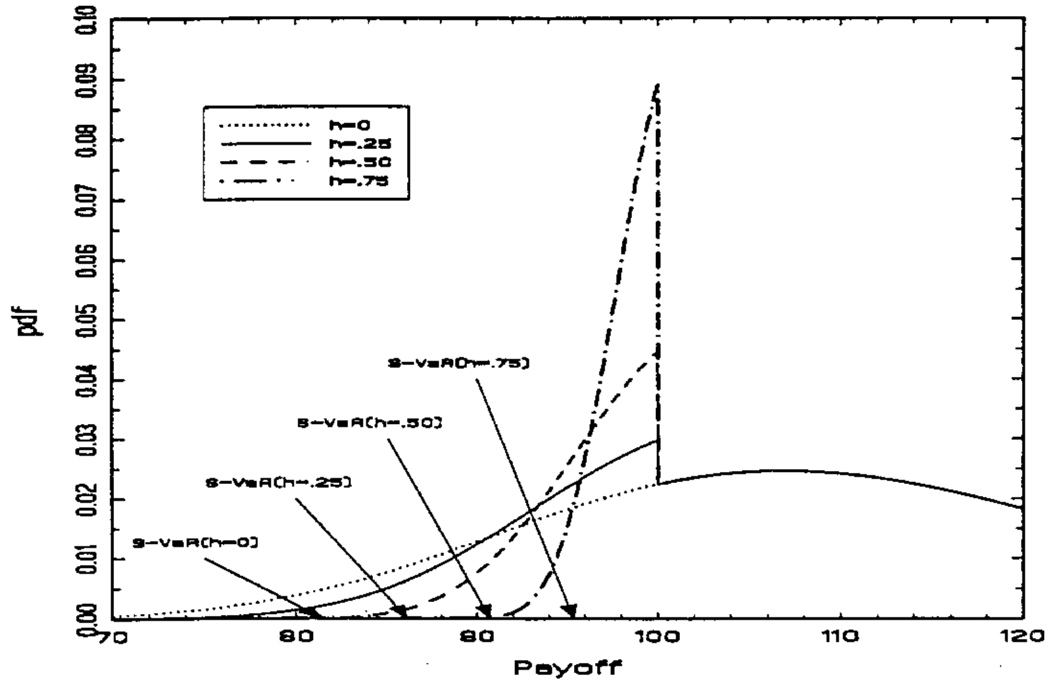

Figure 1: The probability density function of the hedged value for different choices of the hedge ratio (h). Parameter values are $\mu = 0.10$, $\sigma = 0.15$, $r = 0.05$ and $\tau = 1$.

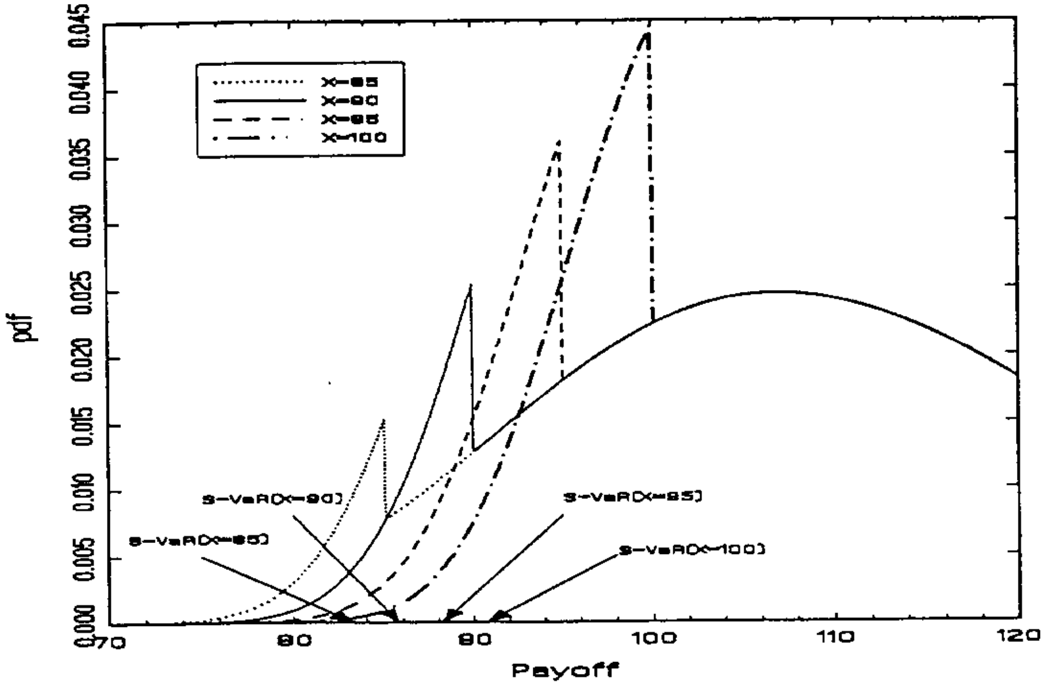

Figure 2: The probability density function of the hedged value for different choices of the exercise price (X) using a hedge ratio of 0.5 ($h = 0.50$). Parameter values are $\mu = 0.10$, $\sigma = 0.15$, $r = 0.05$ and $\tau = 1$.

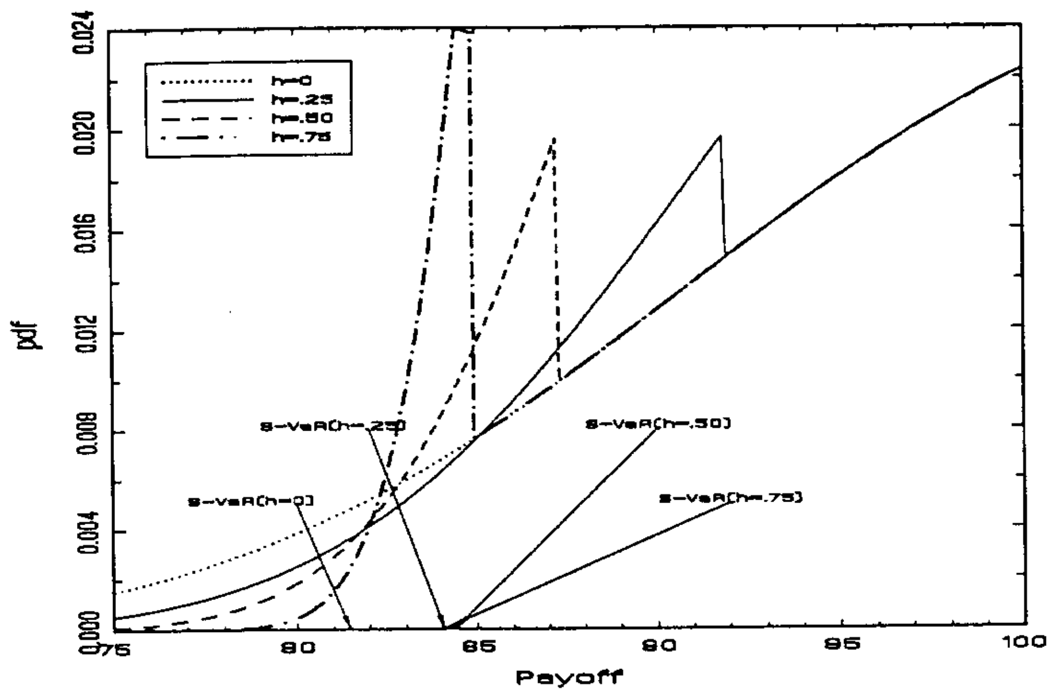

Figure 3: The probability density function of the hedged value for different choices of h and X, given a fixed hedging cost. Parameter values are $\mu = 0.10$, $\sigma = 0.15$, $r = 0.05$ and $\tau = 1$.

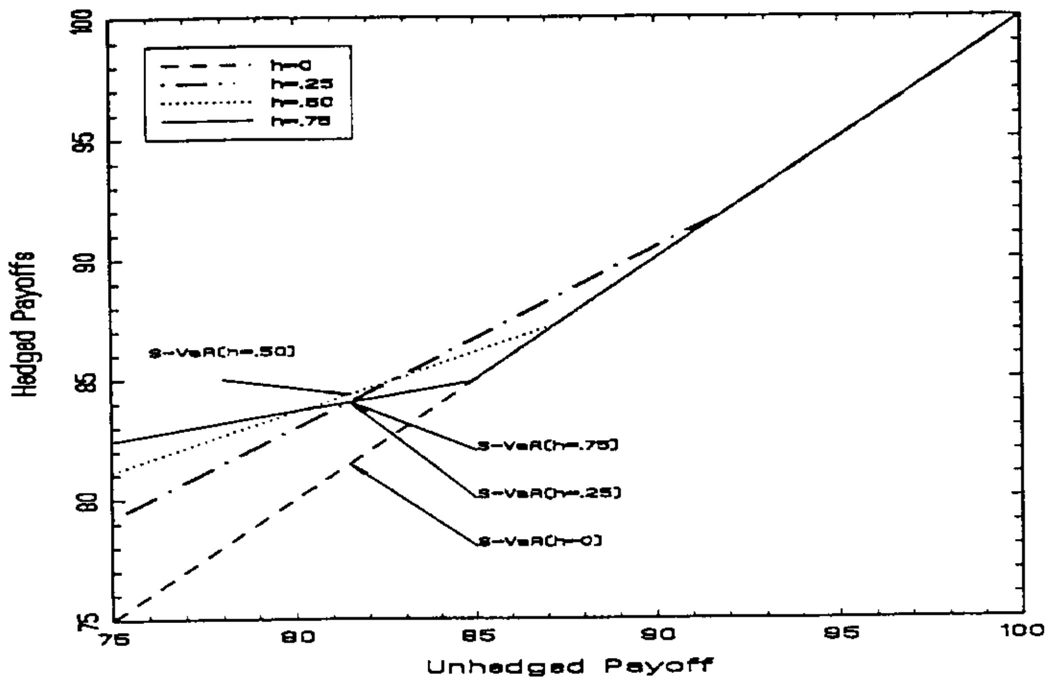

Figure 4: The payoff of the hedged position at maturity τ versus the unhedged payoff, for different choices of h and X, given a fixed hedging cost. Parameter values are $\mu = 0.10$, $\sigma = 0.15$, $r = 0.05$ and $\tau = 1$.

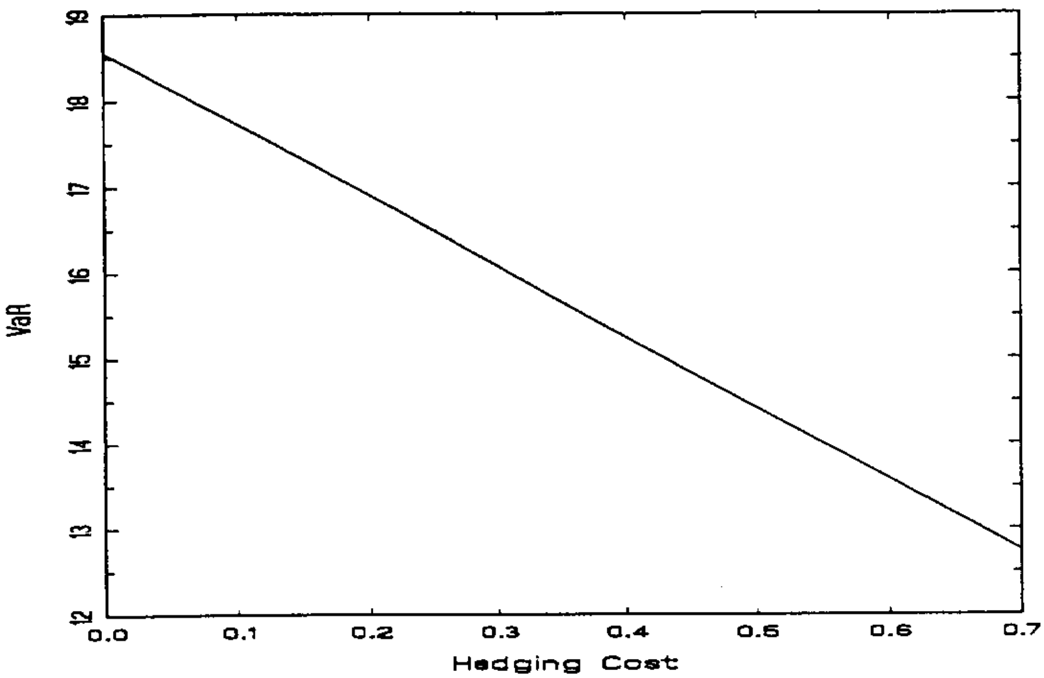

Figure 5: The changes in the optimal $\text{VaR}_{t+\tau}$ for different hedging cost C. The parameter values used are $S_t = 100$, $\mu = .10$, $\sigma = .15$, $r = .05$, $\tau = 1$, and $\alpha = 2.5\%$.

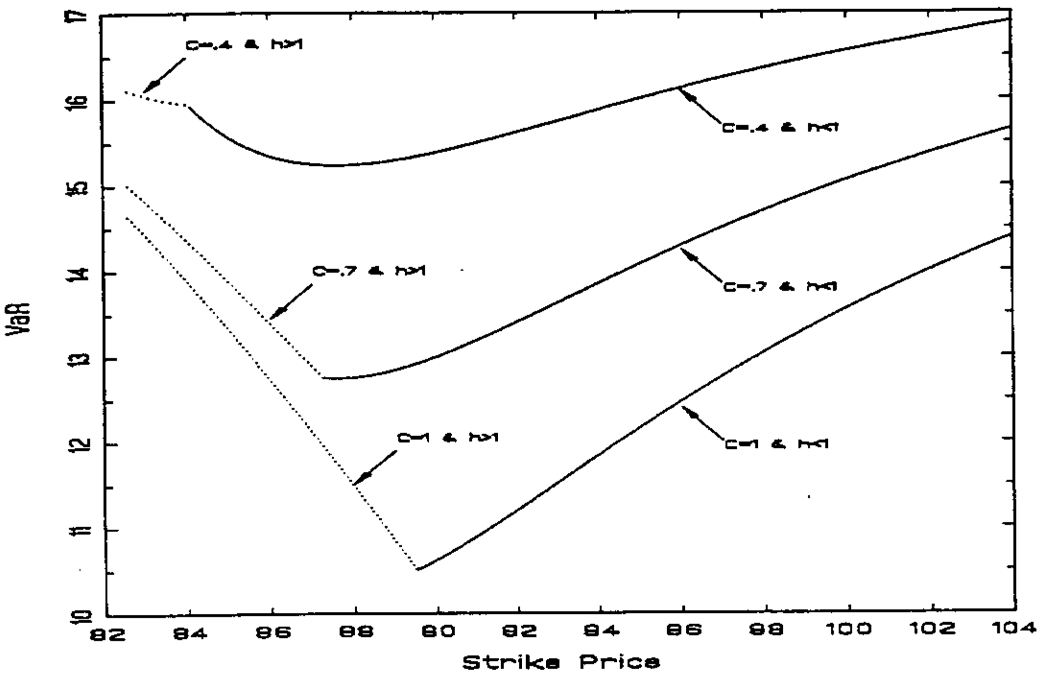

Figure 6: The $\text{VaR}_{t+\tau}$ as a function of the exercise price. Each line represents a certain level of expenditure on option hedge. Dotted lines illustrate the functional relationship when overhedging is needed. The parameter values used are $S_t = 100$, $\mu = .10$, $\sigma = .15$, $r = .05$, $\tau = 1$ and $\alpha = 2.5\%$.

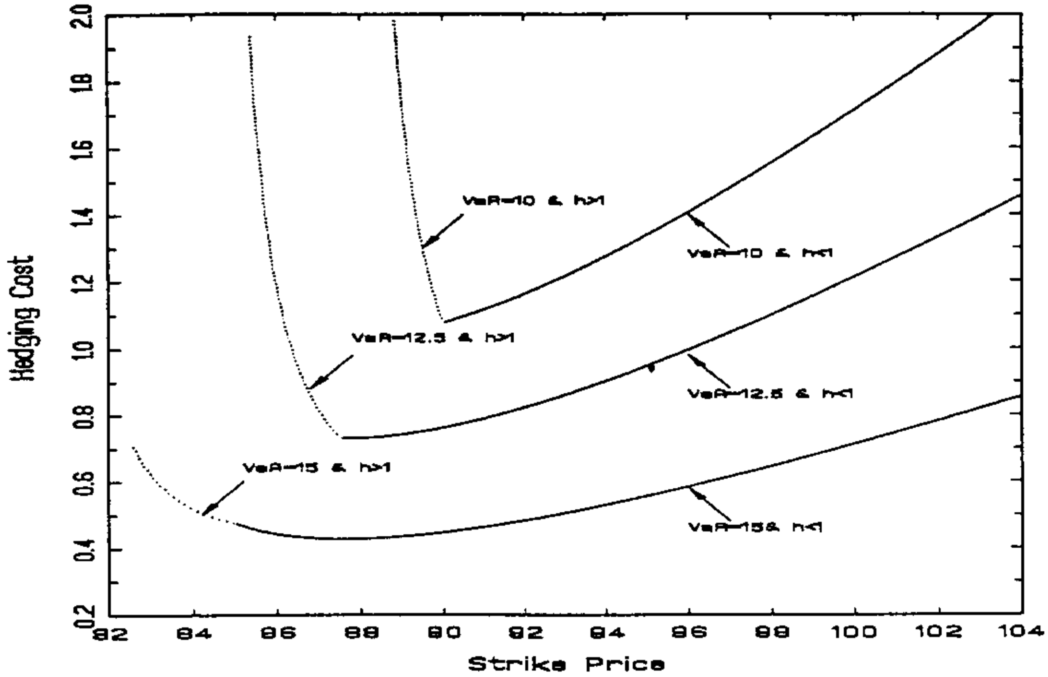

Figure 7: The cost of hedging across various exercise price required to obtain targeted VaR levels. Dotted lines illustrate the functional relationship when overhedging is needed. The parameter values used are $S_t = 100$, $\mu = .10$, $\sigma = .15$, $r = .05$, $\tau = 1$ and $\alpha = 2.5\%$.

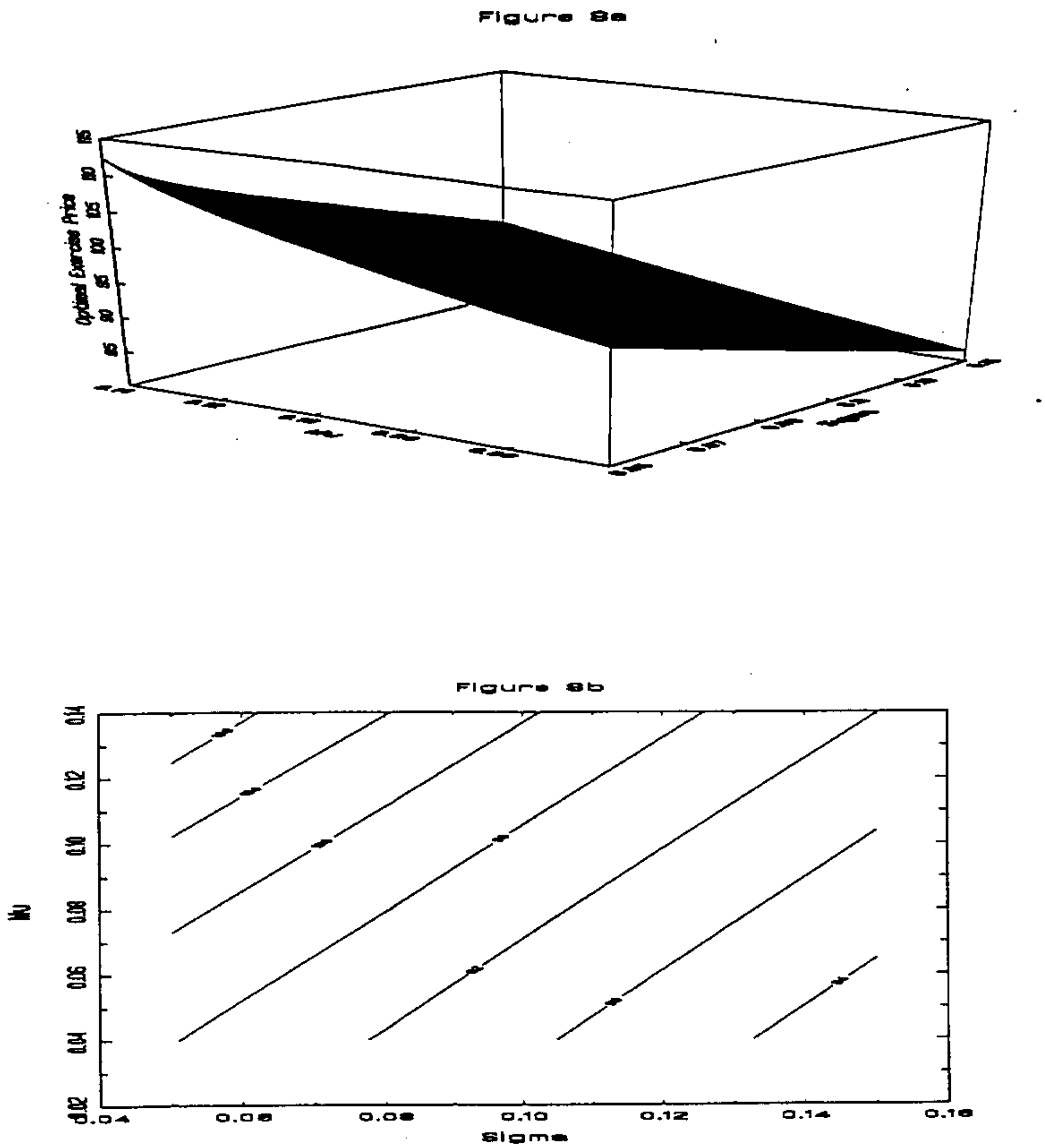

Figure 8: The effect of the underlying distribution of S_t on the optimal exercise price X. The parameter values used are $S_t = 100$, $r = .05$, $\tau = 1$, and $\alpha = 2.5\%$.

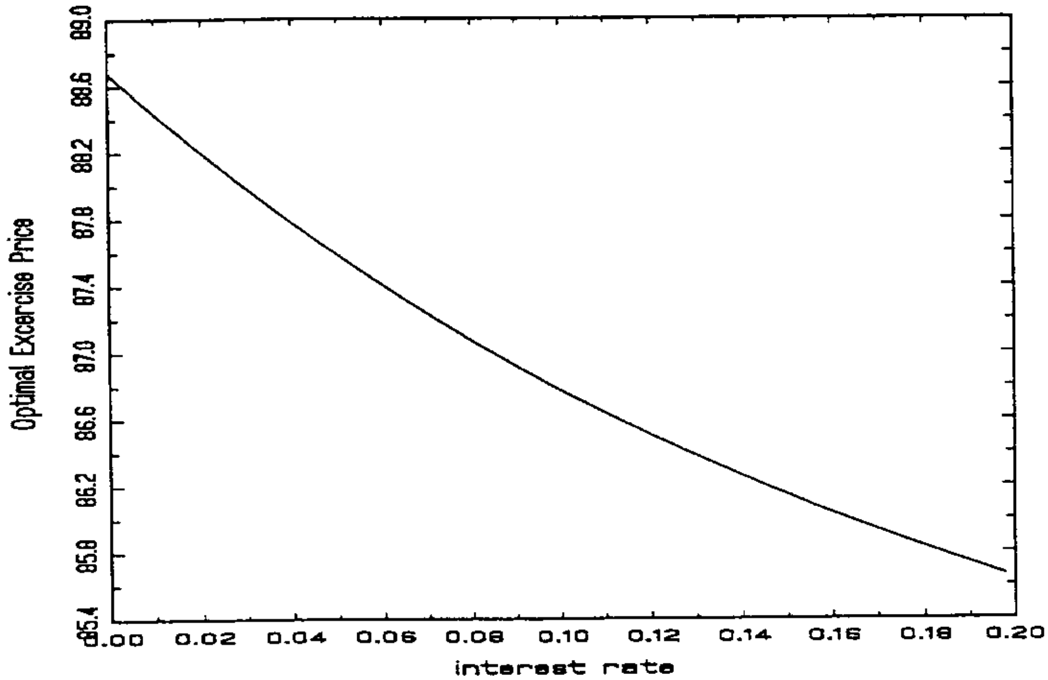

Figure 9: The effect of the interest rate r on the optimal choice of exercise price X. The parameter values used are $S_t = 100$, $\mu = .10$, $\sigma = .15$, $\tau = 1$, and $\alpha = 2.5\%$.

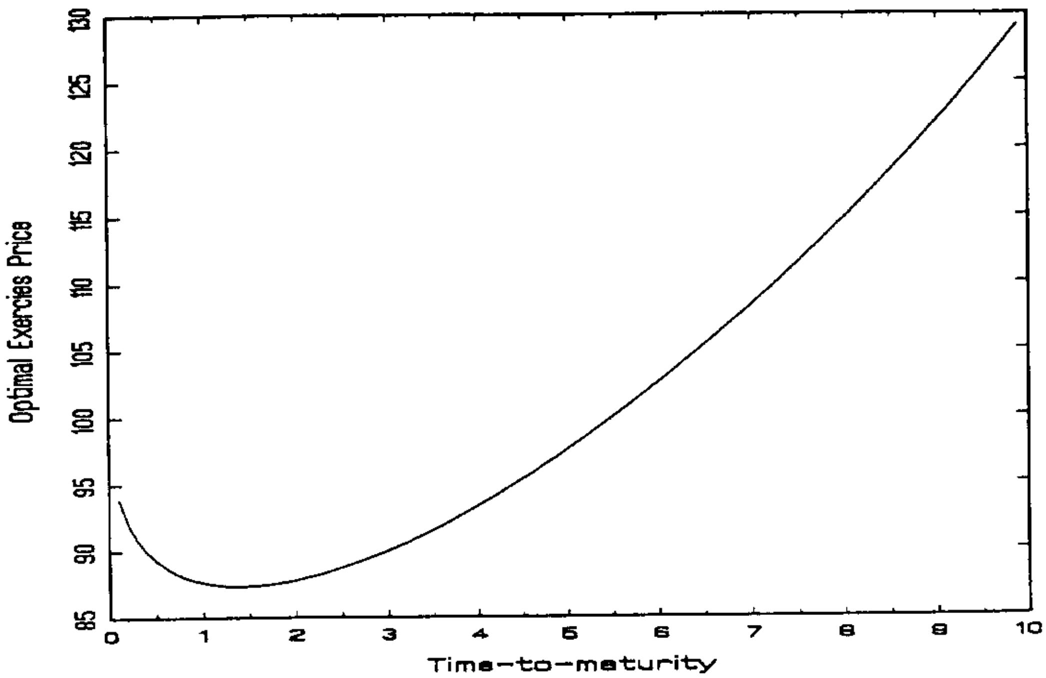

Figure 10: The effect of the maturity over which the option hedge takes place, τ on the optimal choice of exercise price X. The parameter values used are $S_t = 100$, $\mu = .10$, $\sigma = .15$, $r = .05$, and $\alpha = 2.5\%$.

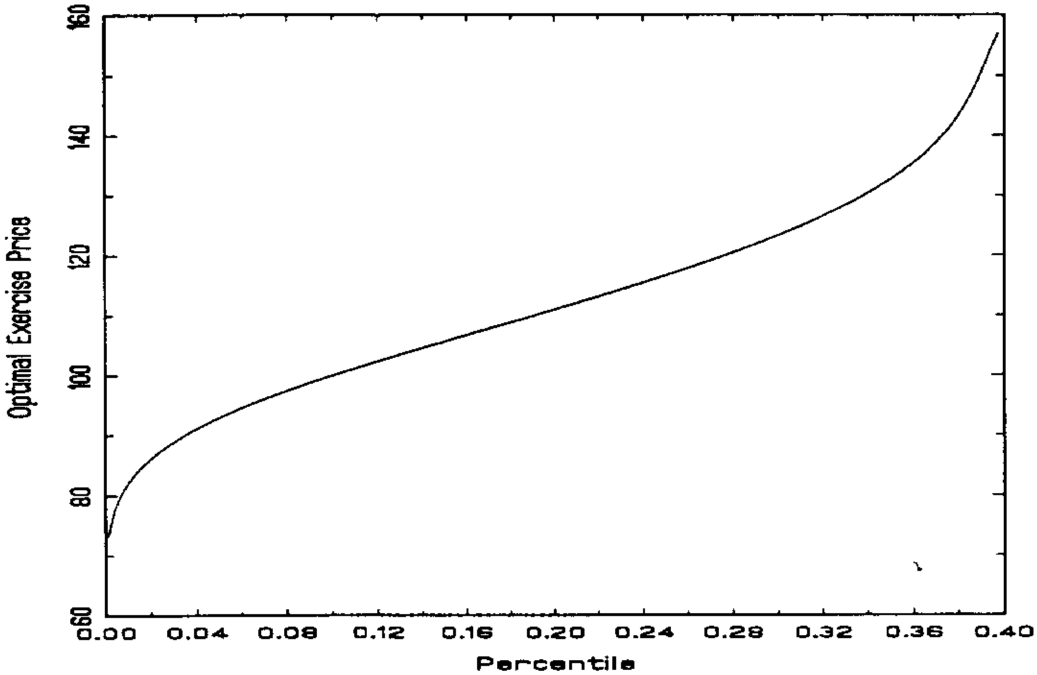

Figure 11: The effect of the desired level of protection α on the optimal exercise price X. The parameter values used are $S_t = 100$, $\mu = .10$, $\sigma = .15$, $r = .05$, and $\tau = 1$.