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PRECAUTIONARY SAVING AND CONSUMPTION SMOOTHING ACROSS TIME AND POSSIBILITIES

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ABSTRACT

This paper examines how aversion to risk and intertemporal substitution determine the strength of the precautionary saving motive in a two-period model with Kreps-Porteus preferences. For small risks, we derive a measure of the strength of the precautionary saving motive which generalizes to these more general preferences the concept of "prudence" introduced by Kimball [1990b] to these more general preferences. For large risks, we show that decreasing absolute risk aversion guarantees that the precautionary saving motive is stronger than risk aversion, regardless of the elasticity of intertemporal substitution. Holding risk preferences fixed, the extent to which the precautionary saving motive is stronger than risk aversion increases with the elasticity of intertemporal substitution. We derive sufficient conditions for the strength of the precautionary saving motive to decline with wealth and for a change in risk preferences alone to increase the strength of the precautionary saving motive.

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1 Introduction

Because it is built upon an intertemporal expected utility foundation which does not distinguish between aversion to risk and resistance to intertemporal substitution, the traditional theory of precautionary saving as a response to income risk does not provide a framework within which one can even ask elementary questions which are fundamental to the understanding of the determinants of consumption under uncertainty. For instance, how does the strength of the precautionary saving motive vary as risk aversion changes, holding the elasticity of intertemporal substitution constant? Or, how does the strength of the precautionary saving motive vary as the elasticity of intertemporal substitution changes, holding risk aversion constant?

Moreover, one might wonder whether some of the questions that can be posed within the intertemporal expected utility framework have different answers once one distinguishes between aversion to risk and resistance to intertemporal substitution. For instance, does decreasing absolute risk aversion imply that the precautionary saving motive is stronger than risk aversion regardless of the elasticity of intertemporal substitution? And under what condition does the precautionary saving motive decline with wealth?

To answer these questions, and to gain a better grasp on the channels through which precautionary saving may affect the economy,[1] we adopt a representation of preferences based on the Kreps-Porteus axiomatization which provides, as recent work has demonstrated,[2] a simple yet powerful separation between attitudes towards risk and attitudes towards intertemporal substitution.

The existing literature on the theory of precautionary saving under Kreps-Porteus preferences is not very extensive. Barsky [1999] implicitly addresses some of the aspects of this theory in a two-period setup, Weil [1991] analyzes a parametric infinite-horizon model with mixed isoelastic/constant absolute risk aversion preferences, while some of the other papers listed above implicitly touch on it. But there has not been as yet any systematic treatment of precautionary saving under Kreps-Porteus preferences.

1. There has recently been a considerable resurgence of interest in precautionary saving. See, for instance, Barsky et al. [1989], Caballero [1990] , Kimball [1990b], Kimball and Mankiw [1989], Skinner [1988], Weil [1991], and Zeldes [1989].

2. Selden [1978], Selden [1979], Kreps and Porteus [1978], Hall [1988], Farmer [1990], Epstein and Zin [1989], Weil [1990], etc.

Under intertemporal expected utility maximization, the strength of the precautionary saving motive is not an independent quantity, but is linked to other aspects of risk preferences. In that case, the absolute prudence $-v'''/v''$ of a von-Neumann Morgenstern second-period utility function v measures the strength of the precautionary saving motive Kimball [1990b], and there is an identity linking prudence to risk aversion under additively time- and state-separable utility:

$$-\frac{v'''(x)}{v''(x)} = a(x) - \frac{a'(x)}{a(x)} \tag{1.1}$$

where

$$a(x) = -\frac{v''(x)}{v'(x)} \tag{1.2}$$

is the Arrow-Pratt measure of absolute risk aversion. Similarly, the coefficient of relative prudence $-xv'''(x)/v''(x)$ satisfies

$$\frac{-xv'''(x)}{v''(x)} = \gamma(x) + \varepsilon(x), \tag{1.3}$$

where

$$\gamma(x) = \frac{-xv''(x)}{v'(x)} = xa(x) \tag{1.4}$$

is relative risk aversion and

$$\varepsilon(x) = \frac{-xa'(x)}{a(x)} \tag{1.5}$$

is the elasticity of risk tolerance, which approximates the wealth elasticity of risky investment.

The primary purpose of this paper is to set out the relationship that exists more generally between the strength of the precautionary saving motive, the level and rate of decline of risk aversion, and intertemporal substitution, for Kreps-Porteus preferences which allow risk preferences and intertemporal substitution to be varied independently.

In section 2, we set up the model and derive a local measure of the precautionary saving motive valid for small risks. Section 3 deals with large risks: it performs various comparative statics experiments which provide the answers to the questions we asked at the outset of the introduction and discusses the role of decreasing absolute risk aversion in guaranteeing that the precautionary saving motive is stronger than risk aversion. Consideration of two technical issues arising in the course of the paper is postponed until 4. The Conclusion takes stock of our main results and outlines directions for further research.

2 Small risks

2.1 The model

Except for departing from the assumption of intertemporal expected utility maximization, we use essentially the same two-period model of the consumption and saving decisions as in Kimball [1990b].[3] We assume that the agent can freely borrow and lend at a fixed risk-free rate and that the constraint that an agent cannot borrow against more than the minimum value of his human wealth is not binding at the end of the first period. Since the interest rate is exogenously given, all magnitudes can be represented in present-value terms, so that, without loss of generality, the real risk-free rate can be assumed to be zero. We also assume that labor supply is inelastic, so that labor income can be treated like manna from heaven. Finally, for clarity, we assume that preferences are additively time-separable until section 4, where we discuss how to extend results to the case of nonseparable utility.

The preferences of our agent can be represented in two equivalent ways. *First*, there is the Selden OCE representation,

$$u(c_1) + U(v^{-1}(\mathrm{E}\, v(\tilde{c}_2))) = u(c_1) + U(M(\tilde{c}_2)),$$

where c_1 and c_2 are first- and second-period consumption, u is the first-period utility function, U is the second-period utility function for the certainty equivalent of random second-period consumption (computed according to the atemporal von Neumann-Morgenstern utility function v), E is an expectation conditional on all information available during the first period, and M is the certainty equivalent operator associated with v:

$$M(\tilde{c}_2) = v^{-1}(\mathrm{E}\, v(\tilde{c}_2)).$$

3. Because there is enough to be said about the two-period case, we defer to another paper the discussion of the multiperiod case.

According to this representation, the utility our consumers derives from the consumption lottery $(c_1, \tilde{c}_2)$ is the sum of the felicity provided by c_1 and the felicity provided by the certainty equivalent $M(\tilde{c}_2)$ of $\tilde{c}_2$.

Second, there is the Kreps-Porteus representation,

$$u(c_1) + \phi(\mathbf{E}\, v(\tilde{c}_2)),$$

where nonlinearity of the function ϕ indicates departure from intertemporal expected utility maximization. This formulation expresses total utility as the nonlinear aggregate of current felicity and an expected future felicity.

These two representations are equivalent as long as v is a continuous, monotonically increasing function, so that $M(\tilde{c}_2)$ is well defined whenever $\mathbf{E}\, v(\tilde{c}_2)$ is well defined.[4] The link between the two representations is that

$$\phi(\nu) = U(v^{-1}(\nu)). \tag{2.1}$$

We will mainly use the Selden representation—which is more intuitive—but, when more convenient mathematically, we use the Kreps-Porteus representation.

2.2 Optimal consumption and saving

Our consumer solves the following problem:

$$\max_x \; u(w - x) + U(M(x + \tilde{y})), \tag{2.2}$$

where w is the sum of initial wealth, first-period income, and the mean of second-period income, x is "saving" out of this sum of non-human wealth and mean human wealth w, and $\tilde{y}$ is the difference of second-period income from its mean.

The first-order condition for the optimal level of saving x is

$$u'(w - x) = U'(M(x + \tilde{y}))M'(x + \tilde{y}), \tag{2.3}$$

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4. One substantive difference between the preferences Selden [1978] discusses and those of Kreps and Porteus [1978] that does not concern us here is that Selden allows v to depend on c_1, while Kreps-Porteus' assumption of recursive utility excludes this possibility. Since we assume that risk preferences are independent of c_1, we are, in effect, looking at the subset of Selden preferences constituting two-period Kreps-Porteus preferences.

where M' is defined by

$$M'(x+\tilde{y}) = \frac{\partial}{\partial x} M(x+\tilde{y}). \tag{2.4}$$

To guarantee that the solution to (2.3) is uniquely determined, we would like the marginal utility of saving,

$$U'(M(x+\tilde{y}))M'(x+\tilde{y}),$$

to be a decreasing function of x. For now, we will simply assume that it is. This is not unreasonable, as the assumption that the marginal utility of saving is downward-sloping is equivalent to the assumption that first-period consumption is a normal good. Section 4 establishes, plausible conditions on preferences sufficient to guarantee a decreasing marginal utility of saving.

From (2.3) and the assumption of a decreasing marginal utility of saving, it follows that the uncertainty represented by $\tilde{y}$ will cause additional saving if

$$U'(M(x+\tilde{y}))M'(x+\tilde{y}) > U'(x), \tag{2.5}$$

that is, if the risk $\tilde{y}$ raises the marginal utility of saving.

As in Kimball [1990b], we can study the strength of precautionary saving effects by looking at the size of the precautionary premium θ^* needed to compensate for the effect of the risk $\tilde{y}$ on the marginal utility of saving. The precautionary premium θ^* is the solution to the equation

$$U'(M(x+\theta^*+\tilde{y}))M'(x+\theta^*+\tilde{y}) = U'(x). \tag{2.6}$$

We call θ^* the *compensating Kreps-Porteus precautionary premium*, to distinguish it from the compensating von Neumann-Morgenstern precautionary premium ψ^* which is the solution to[5]

$$\mathbf{E}\, v'(x+\psi^*+\tilde{y}) = v'(x). \tag{2.7}$$

The precautionary premium θ^* is equal to the rightward shift at each point of the marginal utility of saving curve due to the risk $\tilde{y}$, as illustrated in Figure 1.

5. See Kimball [1990b].

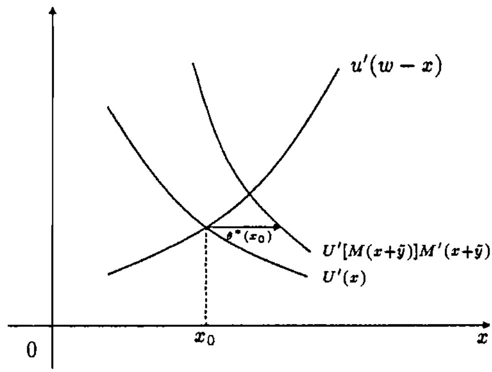

Figure 1: Precautionary saving

2.3 A local measure of prudence

In appendix A, we prove that, for a small risk $\tilde{y}$ with mean zero and variance σ^2,

$$\theta^*(x) = a(x)[1 + s(x)\varepsilon(x)]\frac{\sigma^2}{2} + o(\sigma^2), \tag{2.8}$$

where

$$s(x) = \frac{-U'(x)}{xU''(x)} \tag{2.9}$$

denotes the elasticity of intertemporal substitution for the second period utility function $U(x)$,[6] $o(\sigma^2)$ collects terms going to zero faster than σ^2, $a(x)$ is the absolute risk aversion of v defined in (1.2), and $\varepsilon(x)$ denotes the elasticity of risk tolerance given in (1.5).

Therefore, the local counterpart for Kreps-Porteus preferences to the concept of absolute prudence defined by Kimball [1990b] for time-additive expected utility preferences is

$$a(x)[1 + s(x)\varepsilon(x)].$$

6. Whenever $s(x) = \frac{-u'(w-x)}{(w-x)u''(w-x)}$ as well, s is the elasticity of intertemporal substitution defined in the usual way. More generally, $s(x)$ is the appropriate notion of intertemporal substitution for interest rate changes that are compensated so as to hold first-period consumption constant.

Similarly, the local counterpart for Kreps-Porteus preferences to relative prudence is

$$\mathcal{P}(x) = \gamma(x)[1 + s(x)\varepsilon(x)], \tag{2.10}$$

where $\gamma(x) = xa(x)$ is relative risk aversion as above. Therefore, *in the more general framework of Kreps-Porteus preferences, the strength of the precautionary saving motive is determined by attitudes towards risk and attitudes towards intertemporal substitution.*

Three important special cases should be noted:

- for intertemporal expected utility maximization, $s(x) = 1/\gamma(x)$, so that $\mathcal{P}(x) = \gamma(x) + \varepsilon(x)$—which is the expression given in (1.3);
- for constant relative risk aversion, $\gamma(x) = \gamma$ and $\varepsilon(x) = 1$, so that $\mathcal{P}(x) = \gamma[1 + s(x)]$;
- for constant relative risk aversion $\gamma(x) = \gamma$ *and* constant (but in general distinct) elasticity of intertemporal substitution[7] $s(x) = s$, $\mathcal{P}(x) = \gamma[1 + s]$.

From (2.10), the local condition for *positive* precautionary saving is simply

$$\varepsilon(x) \geq -\frac{1}{s(x)}, \tag{2.11}$$

or, by calculating $\varepsilon(x)$ and moving one piece to the right-hand side of the equation

$$\frac{-xv'''(x)}{v''(x)} \geq \gamma(x) - \rho(x), \tag{2.12}$$

where

$$\rho(x) = 1/s(x)$$

denotes the *resistance to intertemporal substitution*. The right-hand side of (2.12) is zero under intertemporal expected utility maximization, in which case (2.12) reduces to the familiar condition $v'''(x) \geq 0$.

Intriguingly, equation (2.12) shows that when one departs from intertemporal expected utility maximization, quadratic risk preferences do not in general lead to the absence of precautionary saving effects that goes under the name of "certainty equivalence."

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7. This special case characterizes the framework used by Selden [1979].

Though $\mathcal{P}(x)$ cannot be used directly to establish global results, (2.8) suggests several principles which—as the next section will show—are valid globally. *First*, a change in risk preferences v (holding the outer intertemporal utility function U fixed) which increases both risk aversion *and* the rate at which risk aversion declines increases the strength of the precautionary saving motive. *Second*, decreasing absolute risk aversion (implying $\gamma(x)\varepsilon(x) \geq 0$) guarantees that the precautionary saving motive is stronger than risk aversion. *Third*, increasing intertemporal substitution while holding risk preferences constant widens the gap between the strength of the precautionary saving motive and risk aversion. In particular, when absolute risk aversion is decreasing, raising intertemporal substitution from the intertemporal utility maximizing level so that the *resistance* to intertemporal substitution $\rho(x) = 1/s(x)$ is less than relative risk aversion[8] $\gamma(x)$ makes the precautionary saving motive stronger than it is under intertemporal expected utility maximization, while a resistance to intertemporal substitution greater than relative risk aversion makes the precautionary saving motive weaker than under intertemporal expected utility maximization.[9]

3 Large risks

3.1 Precautionary Saving and Decreasing Absolute Risk Aversion

In the case of intertemporal expected utility maximization, Drèze and Modigliani [1972] prove that decreasing absolute risk aversion leads to a precautionary saving motive stronger than risk aversion. Kimball [1991b] shows that there is a fundamental economic logic behind this result: decreasing absolute risk aversion means that greater saving makes it more desirable to take on a compensated risk. But the other side of such a complementarity between saving and a compensated risk is that a compensated risk makes saving more attractive.

8. In the multiperiod case, resistance to intertemporal substitution *less* than risk aversion ($\rho(x) < \gamma(x)$) is associated with a preference for *early* resolution of uncertainty; resistance to intertemporal substitution *greater* than risk aversion ($\rho(x) > \gamma(x)$) is associated with a preference for *late* resolution of uncertainty.

9. There are several further minor but interesting propositions suggested by (2.8) which are valid globally. First, *constant absolute risk aversion* implies that the precautionary saving motive and risk aversion are of exactly equal strength, regardless of intertemporal substitution. Second, two easy sets of sufficient conditions for a positive precautionary saving motive that are valid globally are (a) decreasing absolute risk aversion and (b) $v'''(\cdot) \geq 0$ and $\rho(x) \geq \gamma(x)$.

This argument follows not from any model specific features but rather from the logic of complementarity itself. To understand this claim, consider an arbitrary, concave indirect utility function of saving, and the presence or absence of a compensated risk: $J(x, \lambda)$, where x is saving and λ is 1 in the presence of the compensated risk $\tilde{y} + \pi^*$ and zero in its absence.[10] The definition of a compensated risk insures that

$$J(x,1) - J(x,0) = 0,$$

i.e., that the consumer is indifferent to the presence or absence of the risk $\tilde{y} + \pi^*$.

Decreasing absolute risk aversion in its most fundamental sense, independent of any particular model, is the statement that a risk that is indifferent at one level of saved wealth will become desirable at a slightly higher level of saved wealth. Thus,

$$\frac{\partial}{\partial x}[J(x,1) - J(x,0)] = J_x(x,1) - J_x(x,0) \geq 0$$

if absolute risk aversion is decreasing. But this implies that if the first-order condition for optimal saving is satisfied in the absence of the compensated risk—i.e., if $J_x(x,0) = 0$—, then

$$J_x(x,1) \geq 0,$$

implying that a compensated risk makes the agent want to save more. Thus the risk premium which makes the agent indifferent to the risk still leaves a positive effect of the risk, compensated, on saving. This is what we mean when we say that the precautionary saving motive is stronger than risk aversion.

We now examine more formally how this general logic manifests itself in the specific framework we are studying.

3.1.1 *Intertemporal expected utility maximization: a reminder*

Under intertemporal expected utility maximization, decreasing absolute risk aversion implies that if π^* is the compensating *risk* premium for $\tilde{y}$—*i.e.*, if

$$\mathbf{E}\, v(x + \pi^* + \tilde{y}) = v(x) \tag{3.1}$$

then

$$\mathbf{E}\, v(x + \delta + \pi^* + \tilde{y}) \geq v(x + \delta) \tag{3.2}$$

10. In the Kreps-Porteus model, for instance, $J(x, \lambda) = u(w - x) + U[M(x + \lambda(\tilde{y} + \pi^*))]$.

for any $\delta \geq 0$. Equation (3.1) and inequality (3.2) together imply that the derivative of the left-hand side of (3.2) at $\delta = 0$ must be greater than the derivative of the right-hand side of (3.2) at $\delta = 0$:

$$\mathrm{E}\, v'(x + \pi^* + \tilde{y}) \geq v'(x). \tag{3.3}$$

Equation (2.7) defining the ordinary compensating von Neumann-Morgenstern precautionary premium ψ^* differs from (3.3) only by having ψ^* in place of π^* and by holding with equality. Therefore, since v' is a decreasing function, $\psi^* \geq \pi^*$. In words, (3.3) says that a risk compensated by π^*, so that the agent is indifferent to the combination $\pi^* + \tilde{y}$ still raises expected marginal utility; therefore, the von Neumann-Morgenstern precautionary premium ψ^*— which brings expected marginal utility back down to what it was—must be greater than π^*.

3.1.2 *Kreps-Porteus preferences*

The same economic logic applies to the Kreps-Porteus case. Any property that ensures that extra saving makes compensated risk-taking more attractive guarantees that extra compensated risk makes saving more desirable. Formally, decreasing absolute risk aversion guarantees that $M'(\cdot) \geq 1$, since differentiating the identity

$$M(x + \pi^*(x) + \tilde{y}) = x \tag{3.4}$$

yields

$$(1 + \pi^{*\prime}(x))M'(x + \pi^*(x) + \tilde{y}) = 1 \tag{3.5}$$

and decreasing absolute prudence implies that $\pi^{*\prime}(x) \leq 0$. (Monotonicity of v implies that $1 + \pi^{*\prime}(x) \geq 0$.) Thus,

$$U'(M(x + \pi^* + \tilde{y}))\, M'(x + \pi^* + \tilde{y}) = U'(x)\, M'(x + \pi^* + \tilde{y}) \geq U'(x) \tag{3.6}$$

where the equality on the left follows from the definition of π^*,[11] and the inequality on the right follows from $M'(\cdot) \geq 1$. Combined with the assumption of a decreasing marginal utility of saving, (3.6) implies that $\theta^* \geq \pi^*$. We have thus proved:

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11. Note that the risk premium π^* is the same as it would be under intertemporal expected utility maximization.

Proposition 1 *Assuming a decreasing marginal utility of saving, if the risk utility function v exhibits decreasing absolute risk aversion and the marginal utility of saving is decreasing, then the Kreps-Porteus precautionary premium is always greater than the risk premium ($\theta^* \geq \pi^*$).*

Remark. By reversing the direction of the appropriate inequalities above, one can see that globally *increasing* absolute risk aversion guarantees that $\theta^* \leq \pi^*$, just as globally *decreasing* absolute risk aversion guarantees that $\theta^* \geq \pi^*$. *If absolute risk aversion is constant, both inequalities must hold, implying that* $\theta^*(x) \equiv \pi^*(x) \equiv$ constant, *regardless of the form of the outer intertemporal utility function*[12] U.

3.1.3 *Patent increases in risk*

Proposition 1 can be extended to patent increases in risk—as defined in Kimball [1991c], increases in risk for which the risk premium increases with risk aversion, at least when a utility function with decreasing absolute risk aversion is involved. The set of patent increases in risk includes increases in the scale of a risk coupled with any change of location, and the addition of any independent risk, but does not include all mean-preserving spreads.[13]

The compensating risk premium $\Pi^*(x)$ for the *difference* between two risks $\tilde{Y}$ and $\tilde{y}$ is defined by

$$\mathrm{E}\, v(x + \Pi^*(x) + \tilde{Y}) = \mathrm{E}\, v(x + \tilde{y}), \tag{3.7}$$

or equivalently, by

$$M(x + \Pi^*(x) + \tilde{Y}) = M(x + \tilde{y}). \tag{3.8}$$

If $\tilde{Y}$ is a patently greater risk than $\tilde{y}$, then decreasing absolute risk aversion guarantees that $\Pi^*(x)$ is a decreasing function of x. (Given decreasing absolute risk

12. Weil [1991] exploits the fact that $\theta^*(x) \equiv \pi^*(x)$ under constant absolute risk aversion as an aid to solving a multiperiod saving problem with Kreps-Porteus utility.

13. As noted in Kimball [1991c], "... the results of Pratt [1988] make it clear that $\tilde{X}$ is patently more risky than $\tilde{x}$ if $\tilde{X}$ can be obtained from $\tilde{x}$ by adding to $\tilde{x}$ a random variable ν that is positively related to $\tilde{x}$ in the sense of having a distribution conditional on $\tilde{x}$ which improves according to third-order stochastic dominance for higher realizations of $\tilde{x}$. This sufficient condition for patently greater risk includes as polar special cases ν perfectly correlated with $\tilde{x}$—which makes the movement from $\tilde{x}$ to $\tilde{X}$ a simple change of location and increase in scale—and ν statistically independent of $\tilde{x}$."

aversion, reducing x increases risk aversion and therefore increases Π^* by the definition of a patently greater risk.) Differentiating (3.8) with respect to x, one finds that

$$\begin{aligned} M'(x+\tilde{y}) &= [1+\Pi^{*\prime}(x)]\, M'(x+\Pi^*(x)+\tilde{Y}) \\ &\leq M'(x+\Pi^*(x)+\tilde{Y}), \end{aligned} \tag{3.9}$$

where the inequality on the second line follows from $\Pi^{*\prime}(x) \leq 0$. In combination, (3.8) and (3.9) imply that

$$U'(M(x+\Pi^*(x)+\tilde{Y}))\, M'(x+\Pi^*(x)+\tilde{Y}) \geq U'(M(x+\tilde{y}))\, M'(x+\tilde{y}) \tag{3.10}$$

Defining the compensating Kreps-Porteus precautionary premium $\Theta^*(x)$ for the difference between the two risks $\tilde{Y}$ and $\tilde{y}$ by

$$U'(M(x+\Theta^*(x)+\tilde{Y}))\, M'(x+\Theta^*(x)+\tilde{Y}) = U'(M(x+\tilde{y}))\, M'(x+\tilde{y}) \tag{3.11}$$

a decreasing marginal utility of saving implies that $\Theta^*(x) \geq \Pi^*(x)$.

Proposition 2 (extension of Proposition 1)*Assuming a decreasing marginal utility of saving, if the inner interpossibility utility function v exhibits decreasing absolute risk aversion and $\tilde{Y}$ is patently riskier than $\tilde{y}$, then the Kreps-Porteus precautionary premium for the difference between $\tilde{Y}$ and $\tilde{y}$ is always greater than the risk premium for the difference between $\tilde{Y}$ and $\tilde{y}$. (That is, $\Theta^* \geq \Pi^*$.)*

Proof: See above

3.2 Comparative statics

We now examine how the strength of the precautionary saving motive is affected by changes in risk aversion, intertemporal substitution and wealth.

3.2.1 *Risk aversion*

Given two inner interpossibility utility functions v_1 and v_2, define

$$M_i(x+\tilde{y}) = v_i^{-1}(\mathbf{E}\, v(x+\tilde{y})) \tag{3.12}$$

for $i = 1, 2$, and define θ_1^* and θ_2^* by appropriately subscripted versions of (2.6). Then one can state the following proposition:

Proposition 3 *Assuming a decreasing marginal utility of saving and a concave outer intertemporal utility function U that is held fixed when risk preferences are altered, if $M_2(x+\tilde{y}) \leq M_1(x+\tilde{y})$ and $M_2'(x+\tilde{y}) \geq M_1'(x+\tilde{y})$ for all x, then $\theta_2^*(x) \geq \theta_1^*(x)$ for all x.*

Proof: By the conditions of Proposition 3

$$\begin{aligned} U'(M_2(x+\theta_1^*(x)+\tilde{y}))\,M_2'(x+\theta_1^*(x)+\tilde{y}) &\geq U'(M_1(x+\theta_1^*(x)+\tilde{y}))\,M_1'(x+\theta_1^*(x)+\tilde{y}) \\ &= U'(x) \end{aligned} \tag{3.13}$$

Thus, the Kreps-Porteus precautionary premium θ_1^* for the first set of risk preferences is insufficient under the second set of risk preferences to bring the marginal utility of saving back down to $U'(x)$. By the assumption of a decreasing marginal utility of saving, this means that θ_2^* must be greater than θ_1^* to bring the marginal utility of saving all the way back down to $U'(x)$.

Although useful in aiding one's intuition, Proposition 3 is not as operational as we would like. If v_2 is globally more risk averse than v_1, the risk premium π_2 for v_2 is always greater than the risk premium π_1 for v_1, implying that

$$M_2(x+\tilde{y}) = x - \pi_2(x) \leq x - \pi_1(x) = M_1(x+\tilde{y}),$$

but there is no simple condition we are aware of to guarantee that $\pi_2'(x) \leq \pi_1'(x)$ so that

$$M_2'(x+\tilde{y}) = 1 - \pi_2'(x) \geq 1 - \pi_1'(x) = M_1'(x+\tilde{y}).$$

Still, if v_2 is more risk averse than v_1, so that $\pi_2(x) \geq \pi_1(x)$, and the absolute risk aversion of both $v_1(x)$ and $v_2(x)$ goes to zero as $x \to \infty$, then $\pi_2'(x) \leq \pi_1'(x)$ must hold on average since both $\pi_2(x)$ and $\pi_1(x)$ must fall to zero as $x \to \infty$. Therefore, the conditions of Proposition 3 will often be satisfied when v_2 is more risk averse than v_1, even though it is hard to find more elementary conditions to guarantee they will be satisfied.

3.2.2 *Intertemporal Substitution*

Before analyzing the effect of intertemporal substitution on the size of the precautionary premium in general, it is instructive to look at the following proposition, which indicates how departures from intertemporal expected utility maximization affect the precautionary premium:

Proposition 4 *Assuming a decreasing marginal utility of saving, if the von Neumann-Morgenstern precautionary premium ψ^* is greater than the risk premium π^* (as it will be if absolute risk aversion is decreasing), then the Kreps-Porteus precautionary premium $\theta^* \leq \psi^*$ when the function $\phi(\cdot)$ is concave, but $\theta^* \geq \psi^*$ when $\phi(\cdot)$ is convex. If $\psi^* \leq \pi^*$ (as it will be if absolute risk aversion is increasing), then $\theta^* \geq \psi^*$ when $\phi(\cdot)$ is concave but $\theta^* \leq \psi^*$ when $\phi(\cdot)$ is convex.*

Proof: Assume first that $\psi^* \geq \pi^*$ and $\phi(\cdot)$ is concave. Then

$$\begin{aligned}\phi'(\mathrm{E}\,v(x+\psi^*+\tilde{y}))\mathrm{E}\,v'(x+\psi^*+\tilde{y}) &= \phi'(\mathrm{E}\,v(x+\psi^*+\tilde{y}))\mathrm{E}\,v'(x) \\ &\leq \phi'(v(x))\mathrm{E}\,v'(x). \qquad (3.14)\end{aligned}$$

The equality on the first line follows from the definition of ψ^* (2.7), and the inequality on the second line follows from $\psi^* \geq \pi^*$, the monotonicity of v and the concavity of $\phi(\cdot)$, which makes $\phi'(\cdot)$ a decreasing function. Together with a decreasing marginal utility of saving, (3.14) implies that

$$\theta^* \leq \psi^*. \qquad (3.15)$$

Convexity of $\phi(\cdot)$ *or* $\psi^* \leq \pi^*$ alone would reverse the direction of the inequalities in both (3.14) and (3.15), while convexity of $\phi(\cdot)$ and $\psi^* \leq \pi^*$ together leave the direction of the inequalities unchanged.

The results of propositions 4 are summarized in Table 1. Concavity of ϕ brings

	ϕ concave	ϕ convex
v DARA	$\pi^* \leq \theta^* \leq \psi^*$	$\pi^* \leq \psi^* \leq \theta^*$
v IARA	$\psi^* \leq \theta^* \leq \pi^*$	$\theta^* \leq \psi^* \leq \pi^*$

Table 1: Summary of propositions 3 and 1

the Kreps-Porteus precautionary premium θ^* closer than the von Neumann-Morgenstern precautionary premium ψ^* to the risk premium π^*. Convexity of ϕ pushes θ^* further than ψ^* from π^*. In particular, for quadratic utility—which exhibits IARA and for which $\psi^* = 0$— the precautionary premium θ^* is positive if ϕ is concave and negative if ϕ is convex.

Since concavity of ϕ is equivalent to intertemporal substitution below the reciprocal of risk aversion and convexity of ϕ is equivalent to intertemporal substitution above the reciprocal of risk aversion, one can see that the pattern is one of greater intertemporal substitution increasing the distance between the risk premium π^* and the Kreps-Porteus precautionary premium θ^*. Proposition 5 indicates that, holding risk preferences fixed, higher intertemporal substitution leads quite generally to a Kreps-Porteus precautionary premium that is further from the risk premium.

Proposition 5 *If, for two Kreps-Porteus utility functions $u_i(c_1)+U_i(M(\tilde{c}_2))$ with the same risk preferences, (a) the optimal amount of saving under certainty is the same, (b) the marginal utility of saving is decreasing for both utility functions, and (c) U_2 has greater resistance to intertemporal substitution than U_1—that is, $\frac{-U_2''(x)}{U_2'(x)} \geq \frac{-U_1''(x)}{U_1'(x)}$ for all x—then $|\theta_2^* - \pi^*| \leq |\theta_1^* - \pi^*|$, where θ_i is the Kreps-Porteus precautionary premium for utility function i and π^* is the risk premium for both utility functions. Also, $\theta_2^* - \pi^*$ and $\theta_1^* - \pi^*$ have the same sign.*

Proof: The optimal amount of saving for the two utility functions is the same under certainty only if there is an x_0 for which

$$u_i'(w - x_0) = U_i'(x_0) \quad i = 1, 2. \tag{3.16}$$

Define the normalized marginal utility of saving functions f_1 and f_2 by

$$f_i(x) = \frac{U_i'(M(x+\tilde{y}))M'(x+\tilde{y})}{U_i'(x_0)}. \tag{3.17}$$

For either utility function, the normalized marginal utility of saving f_i equals one when $x = x_0 + \theta_i^*(x_0)$, as can be seen be careful inspection of (3.17). Therefore, f_1 and f_2 tell us what we need to know in order to establish comparative statics about the Kreps-Porteus precautionary premium θ^*.

Equation (3.16) says that f_1 and f_2 meet at $x = x_0 + \pi^*(x_0)$, since

$$\begin{aligned} \frac{U_i'(M(x_0+\pi^*(x_0)+\tilde{y}))\,M'(x_0+\pi^*(x_0)+\tilde{y})}{U_i'(x_0)} &= \frac{U_i'(x_0)\,M'(x_0+\pi^*(x_0)+\tilde{y})}{U_i'(x_0)} \\ &= M'(x_0+\pi^*(x_0)+\tilde{y}) \end{aligned} \tag{3.18}$$

for $i = 1, 2$. Since the equation $M(x + \pi^*(x) + \tilde{y}) = x$ can be differentiated to obtain

$$[1 + \pi^{*\prime}(x_0)]M'(x_0 + \pi^*(x_0) + \tilde{y}) = 1, \tag{3.19}$$

both $f_1(x_0 + \pi^*(x_0))$ and $f_2(x_0 + \pi^*(x_0))$ are greater than 1 if $\pi'(x_0) \leq 0$ and both $f_1(x_0 + \pi^*(x_0))$ and $f_2(x_0 + \pi^*(x_0))$ are less than 1 if $\pi^{*\prime}(x_0) \geq 0$.

The ratio between f_2 and f_1 simplifies as follows:

$$\frac{f_2(x)}{f_1(x)} = \frac{U_2'(M(x+\tilde{y}))}{U_1'(M(x+\tilde{y}))}\frac{U_1'(x)}{U_2'(x_0)}. \tag{3.20}$$

The condition $\frac{-U_2''(x_0)}{U_2'(x_0)} \geq \frac{-U_1''(x_0)}{U_1'(x_0)}$ implies that the ratio $\frac{U_2'(x_0)}{U_1'(x_0)}$ is a decreasing function[14] of x_0 since

$$\frac{d}{dx}\ln\left(\frac{U_2'(x)}{U_1'(x)}\right) = \frac{U_2''(x)}{U_2'(x)} - \frac{U_1''(x)}{U_1'(x)} \leq 0. \tag{3.21}$$

Since $M(x+\tilde{y})$ is an increasing function of x, this means that $\frac{f_2(x)}{f_1(x)}$ is a decreasing function of x. In other words, $f_2(x) \geq f_1(x)$ for $x \leq x_0 + \pi^*(x_0)$ and $f_2(x) \leq f_1(x)$ for $x \geq x_0 + \pi^*(x_0)$.

We are now in a position to draw an instructive graph of f_1 and f_2. Figures 2 and 3 depict the two main cases. Decreasing marginal utility of saving means that f_i is decreasing for both utility functions. If $\pi^{*\prime}(x_0) \leq 0$, then f_2 and f_1 are both above one at $x = x_0 + \pi^*(x_0)$ and f_2 must hit one first as x moves to the right from $x_0 + \pi^*(x_0)$ since f_2 is below f_1 to the right of $x_0 + \pi^*(x_0)$. Therefore $x_0 + \theta_2^*(x_0) \leq x_0 + \theta_1^*(x_0)$ and $\theta_2^*(x_0) \leq \theta_1^*(x_0)$.

If $\pi^{*\prime}(x_0) \geq 0$, then f_2 and f_1 are both below one at $x = x_0 + \pi^*(x_0)$ and f_2 must hit one first as x moves to the left from $x_0 + \pi^*(x_0)$ since f_2 is above f_1 to the left of $x_0 + \pi^*(x_0)$. Therefore $x_0 + \theta_2^*(x_0) \geq x_0 + \theta_1^*(x_0)$ and $\theta_2^*(x_0) \geq \theta_1^*(x_0)$.

If one or the other of $f_i(x)$ never hits one, these inequalities remain valid if one writes $\theta_i^*(x_0) = +\infty$ when $f_i(x) \geq 1$ for all x, and $\theta_i^*(x_0) = -\infty$ when $f_i(x) \leq 1$ for all x. In any of these cases, $\theta_1(x_0)$ and $\theta_2(x_0)$ are on the same side of $\pi^*(x_0)$ and $|\theta_2^*(x_0) - \pi^*(x_0)| \leq |\theta_1^*(x_0) - \pi^*(x_0)|$.

Figures 2 and 3 can be used to illustrate not only the effects of changing the elasticity of intertemporal substitution but also the effects of changing the level and rate of decline of risk aversion. Focusing on the point $(x + \pi^*, M'(x + \pi^* + \tilde{y}))$ at which the curves f_1 and f_2 intersect, increases in risk aversion move this point to the right, more quickly declining risk aversion tends to move this point upward, while increases in the elasticity of intertemporal substitution swivel the curve describing f counterclockwise around this point (increases in the *resistance* to intertemporal substitution swivelling the curve clockwise around this point.) The effects of each change on the strength of the precautionary saving motive can be seen in the horizontal movement of the intersection with the dashed line $f(x) = 1$. The message of (2.10) about the determinants of the strength of the precautionary saving motive under Kreps-Porteus preferences is amply confirmed by such an exercise.

14. Pratt [1964] proves a result for comparative risk aversion that is mathematically identical to what we prove here for comparative resistance to intertemporal substitution.

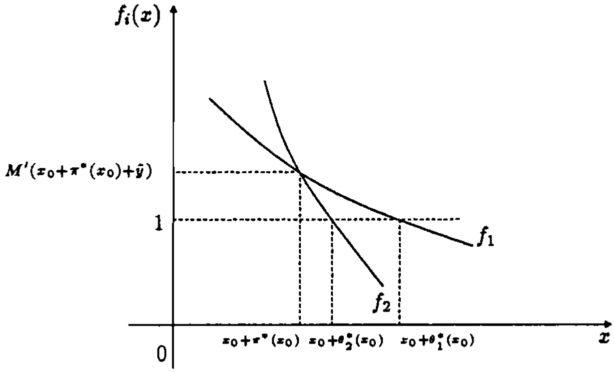

Figure 2: $\pi^{*\prime}(x_0) < 0$

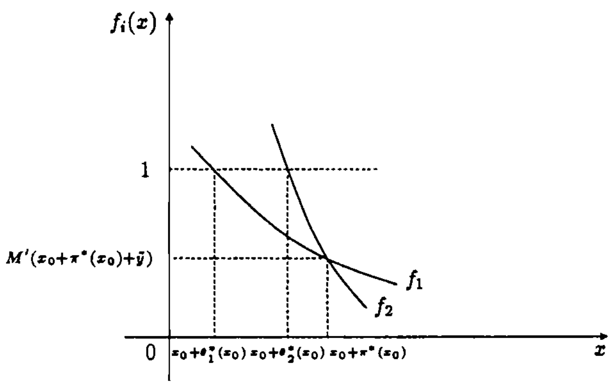

Figure 3: $\pi^{*\prime}(x_0) > 0$

3.2.3 *Wealth*

Kimball [1990b] shows that the precautionary premium is equal to the rightward shift in the graph of consumption as a function of wealth that results from a risk $\tilde{y}$. Thus, if the precautionary saving motive decreases in strength with wealth, a risk $\tilde{y}$ raises the marginal propensity to consume out of wealth at a given level of consumption.[15] We now discuss conditions sufficient to guarantee that the precautionary saving motive becomes weaker as wealth w—and therefore "saving" x—increases.

For the special case of small risks, one can differentiate[16] (2.8) with respect to x to get a simple condition for the Kreps-Porteus precautionary premium to be decreasing:

$$\frac{\partial}{\partial x}\{a(x)[1+s(x)\varepsilon(x)]\} \leq 0. \tag{3.22}$$

For large risks, however, matters are more difficult.

First, we can show that the combination of $M(x+\tilde{y})$ concave (true for any v belonging to the hyperbolic absolute risk aversion class) and $1/U'(x)$ convex (true for a constant elasticity of intertemporal substitution less than one) is enough to guarantee that the Kreps-Porteus precautionary premium θ^* is decreasing in wealth:

Proposition 6 *If v exhibits decreasing absolute risk aversion, $M(x+\tilde{y})$ is increasing and concave, and $1/U'(x)$ is increasing and convex, then $\theta^*(x)$ is decreasing in x.*

Proof: See Appendix B.

In the important case of constant relative risk aversion *and* constant elasticity of intertemporal substitution, Proposition 6 guarantees a decreasing Kreps-Porteus precautionary premium whenever the intertemporal substitution is less than one. (See below.) Even when intertemporal substitution is greater than one, Proposition 7 guarantees that the precautionary premium will be decreasing as long as a constant intertemporal substitution is less than or equal to the reciprocal of a constant relative risk aversion.

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15. See Figures 1–3 in Kimball [1990a].
16. As above, one must make the assumption that the next derivative beyond the one for which the limit is to be calculated is bounded as $\sigma^2 \to 0$.

Proposition 7 *If intertemporal substitution is constant and less than the reciprocal of relative risk aversion, also constant, then the Kreps-Porteus precautionary premium $\theta^*(x)$ is decreasing.*

Proof: Proposition 6 takes care of the case in which intertemporal substitution is less than or equal to one, as can be verified from the calculation $\frac{1}{U'(x)} = x^{\frac{1}{s}}$ when

$$U(x) = \frac{x^{1-1/s}}{1-\frac{1}{s}}.$$

Appendix C proves the result for the case $1 < s \leq \frac{1}{\gamma}$.

Remark. Figure 4 graphs the combinations of relative risk aversion γ and the resistance to intertemporal substitution $\rho = 1/s$ that we have shown by Propositions 6 and 7 to imply a decreasing Kreps-Porteus precautionary premium when both relative risk aversion and intertemporal substitution are constant ($\rho \geq \min(1, \gamma)$). We do not know what happens for other possible values of the parameters ρ and γ for large risks, although for *small risks* (3.22) indicates that the Kreps-Porteus precautionary premium is always decreasing for constant intertemporal substitution and constant relative risk aversion.

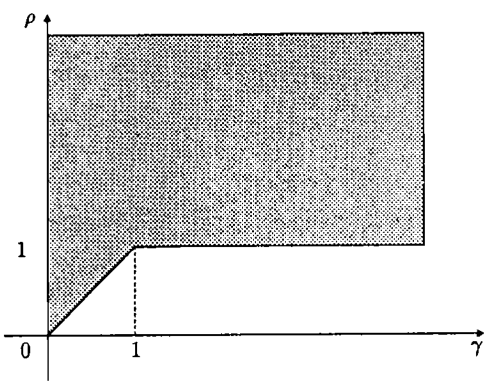

Figure 4: Parameter values known to imply a decreasing θ^*

Both Propositions 6 and 7 are difficult to apply when the inner interpossibility utility function v is not in the hyperbolic absolute risk aversion class.[17] For a restricted range of functions $\phi(\cdot)$, the following proposition allows one to establish

17. Proposition 6 can easily be extended to constant relative risk aversion utility functions

a decreasing precautionary saving motive for any interpossibility utility function v with decreasing absolute risk aversion that yields a decreasing precautionary saving motive under intertemporal expected utility maximization.

Proposition 8 *If $1/\phi'$ is increasing and convex, and v increasing, concave, and has both decreasing absolute risk aversion and decreasing absolute prudence ($\frac{d}{dx}\frac{-v'''(x)}{v''(x)} \leq 0$), then the Kreps-Porteus precautionary premium $\theta^*(x)$ is decreasing.*

Proof: See appendix D.

4 Technical issues

4.1 Conditions Guaranteeing a Decreasing Marginal Utility of Saving

Almost every proof in the preceding sections relies in some way on the assumption of a decreasing marginal utility of saving. If that assumption fails, the marginal utility of saving $U'(M(x+\tilde{y}))M'(x+\tilde{y})$ can intersect first-period marginal utility $u'(w-x)$ more than once when the first-period utility function u is close enough to linear. Regardless of the shape of the first-period utility function, a failure of the assumption of a decreasing marginal utility of saving results in a negative first-period marginal propensity to consume out of wealth, as illustrated in Figure 5. In Figure 5, as wealth rises from w_1 to w_2, the first-period marginal utility of consumption u' rises, indicating that first-period consumption *falls* as wealth rises from w_1 to w_2. In order to assess how restrictive this assumption is, we now derive conditions sufficient to guarantee that the marginal utility of saving will, indeed, be decreasing.

The simplest sufficient condition for a decreasing marginal utility of saving is concavity of ϕ—the function showing the direction and extent of the departure from intertemporal expected utility maximization—together with concavity of the inner interpossibility utility function v.

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with a shifted origin—that is, to all of the hyperbolic absolute risk aversion functions that have decreasing absolute risk aversion—as long as U has the same shifted origin.

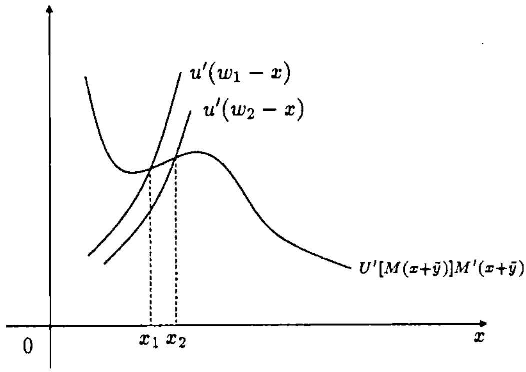

Figure 5: A marginal utility of saving increasing on one interval

Proposition 9 *If both v and ϕ are (monotonically increasing and) concave—that is, if the resistance to intertemporal substitution is greater than risk aversion, which in turn is positive—then the marginal utility of saving is decreasing.*

Proof: Concavity of both v and ϕ guarantees that both factors of the Kreps-Porteus representation of the marginal utility of saving, $\phi'(\mathrm{E}\, v(x+\tilde{y}))\mathrm{E}\, v'(x+\tilde{y})$, decrease as x increases.

Alternatively, concavity of both the outer intertemporal utility function U and of the certainty equivalent function M, is enough to guarantee a decreasing marginal utility of saving. This result becomes very useful in conjunction with the result of Kimball [1991a] that the certainty equivalent function M is always concave for any atemporal-von Neumann Morgenstern utility function v in the hyperbolic absolute risk aversion class, including quadratic, exponential, linear, and constant relative risk aversion utility functions.[18]

Proposition 10 *If both the outer intertemporal utility function U and the certainty equivalent function $M(x+\tilde{y})$ are increasing and concave in x (as M will be if v belongs to the hyperbolic absolute risk aversion class), then the marginal utility of saving is decreasing.*

18. Utility functions in the hyperbolic absolute risk aversion class are those that can be expressed in the form $v(x) = \frac{\gamma}{1-\gamma}\left\{\left(\frac{x-b}{\gamma}\right)^{1-\gamma} - 1\right\}$ together with the logarithmic limit as $\gamma \longrightarrow 1$ and the exponential limit as $\gamma \longrightarrow \infty$ with $\frac{b}{\gamma} \longrightarrow -a$.

Proof: Concavity of both U and M guarantees that both factors of the Selden representation of the marginal utility of saving, $U'(M(x+\tilde{y}))M'(x+\tilde{y})$, decrease as x increases.

Remark. Looking at the Selden representation of the marginal utility of saving makes it clear that a necessary condition for the inner interpossibility utility function v to guarantee a decreasing marginal utility of saving for *any* concave U (that is, regardless of intertemporal substitution) is for M to be concave. In turn, a necessary condition for M to be concave is for absolute risk aversion to be a convex function ($a''(x) \geq 0$), since for small risks one can differentiate (A.3) to find that (as long as $M'''(x+\tilde{y})$ is bounded in the neighborhood of x), $M''(x+\tilde{y}) = -a''(x)\frac{\sigma^2}{2} + o(\sigma^2)$.

It stands to reason that strict concavity of U would allow one to expand the set of inner interpossibility utility functions v which would lead to a decreasing marginal utility of saving. The following proposition confirms that notion.

Proposition 11 *If there exist constants A, B, and $k \geq 0$ such that $A+Bv(x) \geq 0$ for all x and*

$$\frac{[A+Bv(x)]}{v'(x)}\left(\frac{-v''(x)}{v'(x)}\right) \geq k \geq \frac{[A+Bv(x)]}{v'(x)}\left\{-\frac{v''(x)}{v'(x)}+\frac{U''(x)}{U'(x)}\right\} \quad (4.1)$$

for all x, then the marginal utility of saving is decreasing.

Proof: See Appendix E.

Remark. Proposition 11 includes the combination of U concave and v in the hyperbolic absolute risk aversion class as a special case, since the left-hand inequality in (4.1) can be made to hold with equality for any hyperbolic absolute risk aversion utility function, garanteeing that the right-hand inequality holds.[19] Proposition 11 ensures a decreasing marginal utility of saving for inner interpossibility utility functions v in a wider domain around hyperbolic absolute risk aversion the more concave U is.

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19. In terms of the parametrization of the previous footnote for hyperbolic absolute risk aversion utility functions, let $B = \frac{1-\gamma}{\gamma}$, $A = 1$ and $k = 1$.

4.2 Intertemporally Nonseparable Utility

If the utility function must be expressed in the non-time-separable form

$$\Upsilon(c_1, M(\tilde{c}_2)) = \Phi(c_1, \mathrm{E}\, v(\tilde{c}_2)),$$

the first order condition for optimal saving can be expressed as

$$\frac{\Upsilon_2(w - x, M(x + \tilde{y}))}{\Upsilon_1(w - x, M(x + \tilde{y}))} M'(x + \tilde{y}) = 1 \tag{4.2}$$

or

$$\frac{\Phi_2(w - x, \mathrm{E}\, v(x + \tilde{y}))}{\Phi_1(w - x, \mathrm{E}\, v(x + \tilde{y}))} \mathrm{E}\, v'(x + \tilde{y}) = 1, \tag{4.3}$$

where subscripts on Υ and Φ indicate partial derivatives.

In sections 2, 3.2.1, 3.1 and 3.2.2, all of the results involve finding θ^* to compensate for $\tilde{y}$ so that first period consumption $c_1 = w - x$ need not change. Therefore, the way to reinterpret all of the results in this section is to replace $U'(x)$ by the marginal rate of substitution function under certainty,

$$\frac{\Upsilon_2(c_1, x)}{\Upsilon_1(c_1, x)},$$

treating c_1 as a constant, and to replace $\phi'(\nu)$ by

$$\frac{\Phi_2(c_1, \nu)}{\Phi_1(c_1, \nu)},$$

again treating c_1 as a constant. (Note that none of the propositions depends on U itself, but only on its derivatives.) The marginal rate of substitution Υ_2/Υ_1, as well as Φ_2/Φ_1, are unaffected by monotonic transformations of Υ or Φ, since they are the slopes of indifference curves.

In sections 4.1 and 3.2.3, the effect of changes in x on $c_1 = w - x$ must also be taken into account. Fortunately, this does not disturb properties such as a decreasing marginal rate of substitution, and as long as second-period consumption is a normal good, the marginal rate of substitution Υ_2/Υ_1 or the slope Φ_2/Φ_1 must change more when the effect of x on c_1 is taken into account than when c_1 is treated as a constant. Therefore, the rate of change in the logarithm of the marginal rate of substitution represented by U''/U' in (4.1) will always be greater when measured inclusive of the effect of x on c_1 than when c_1 is treated as a constant. In Section 3.2.3, Proposition 7 needs no extension, since any intertemporal utility function with a constant elasticity of intertemporal substitution can be represented in an additively separable form. Proposition 6 remains true if on top of the conditions that M is concave and

$$\frac{\Upsilon_1(c_1, x)}{\Upsilon_2(c_1, x)}$$

is increasing and convex treating c_1 as a constant, the additional condition

$$\frac{\partial^2}{\partial c_1 \partial c_2} \{\ln \Upsilon_2(c_1, c_2) - \ln \Upsilon_1(c_1, c_2)\} \geq 0 \tag{4.4}$$

is satisfied. Proposition 8 remains true if on top of v having positive and decreasing absolute risk aversion and decreasing absolute prudence, and

$$\frac{\Phi_1(c_1, \nu)}{\Phi_2(c_1, \nu)}$$

being increasing and convex, the additional condition

$$\frac{\partial^2}{\partial c_1 \partial \nu} \{\ln \Phi_2(c_1, \nu) - \ln \Phi_1(c_1, \nu)\} \geq 0 \tag{4.5}$$

is satisfied. In both of these cases the additional inequality is satisfied with equality when utility is intertemporally separable.[20]

5 Conclusion

We have made considerable progress in understanding the determinants of the strength of the precautionary saving motive under Kreps-Porteus preferences in the two-period case. Perhaps one of the more surprising results is that greater risk

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20. The seeming inconsistency between the results of this section and those of Kimball [1990b] Appendix C, arises because under intertemporal expected utility maximization, a utility function that is nonseparable intertemporally must also have nonseparable risk preferences, so that c_1 affects one's preferences for gambles over $\tilde{c}_2$. Here we assume that risk preferences are separable between c_1 and c_2 even if the outer intertemporal utility function is nonseparable. Kimball [1990b] Appendix C gives a good indication of some of the extra issues that arise if risk preferences are nonseparable.

aversion tends to increase the strength of the precautionary saving motive, while, in the typical case of decreasing absolute risk aversion, greater resistance to intertemporal substitution *reduces* the strength of the precautionary saving motive. Thus, distinguishing between risk aversion and the resistance to intertemporal substitution is very important in discussing the determinants of the strength of the precautionary saving motive since these two parameters, —which are forced to be equal under intertemporal expected utility maximization—have opposite effects when allowed to vary separately.

Our results also shed new, and important, light on the likely empirical magnitude of the precautionary saving motive. Remember that we showed in section 2.3 that, for preferences exhibiting a constant elasticity of intertemporal substitution and constant relative risk aversion, the local measure (in relative terms) of the strength of the precautionary saving motive is

$$\mathcal{P} = \gamma + \frac{\gamma}{\rho},$$

where γ is the coefficient of relative risk aversion and ρ the inverse of the elasticity of intertemporal substitution. If one believes in the intertemporal expected utility model and thinks that a plausible value of the coefficient of relative risk aversion is 4, then one will set $\gamma = \rho = 4$ and conclude that $\mathcal{P} = 4 + 4/4 = 5$. If, on the other hand, one judges—using some recent empirical evidence[21] that the intertemporal expected utility restriction $\gamma = \rho$ is unreasonable, that the coefficient of relative risk aversion is smaller than 4 and the elasticity of intertemporal substitution is lower than $1/4$,[22] then one might set $\gamma = 2$ and $\rho = 10$, in which case $\mathcal{P} = 2 + 2/10 = 2.2$. Thus a moderate coefficient of relative risk aversion coupled with a strong insensitivity of consumption profiles to interest rates—a not implausible description of consumer tastes—yields a precautionary saving motive that is weaker than the (counterfactually too strong) motive predicted by the standard intertemporal expected utility model.

Finally, the most important direction in which to extend the results here is to the multi-period case.[23] The propositions of this paper point to natural conjectures

21. See, for instance, Epstein and Zin [1991], Hall [1988], and Giovannini and Weil [1989].
22. Obviously, the intertemporal expected utility framework can't even entertain the possibility that *both* risk aversion and intertemporal substitution are smaller.
23. Weil [1991] solves a infinite horizon model with isoelatic intertemporal preferences and exponential risk preferences, and van der Ploeg [1991] one with quadratic intertemporal pref-

about the determinants of the strength of the precautionary saving motive in the multi-period case.

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erences and exponential risk preferences. But the general multi-period case has not yet beem studied.

Appendix A. Local approximation

Define the *equivalent Kreps-Porteus precautionary premium* θ, expressed as a function of x, as the solution to the equation

$$U'(M(x))M'(x) = U'(x - \theta(x)). \tag{A.1}$$

It is easiest to find the small-risk approximation for the equivalent Kreps-Porteus precautionary premium θ first, and then the small risk approximation for the compensating Kreps-Porteus precautionary premium θ^*. For a small risk $\tilde{y}$ with mean zero and variance σ^2, Pratt [1964] shows that

$$M(x + \tilde{y}) = x - a(x)\frac{\sigma^2}{2} + o(\sigma^2). \tag{A.2}$$

As long as $M''(\cdot)$ is bounded in the neighborhood of x, (A.2) can be differentiated to obtain

$$M'(x + \tilde{y}) = 1 - a'(x)\frac{\sigma^2}{2} + o(\sigma^2). \tag{A.3}$$

Finally, Substituting from (A.2) and (A.3) into (A.1), and doing a Taylor expansion of $U(x - \theta)$ around x,

$$\begin{aligned} U'(M(x))M'(x) &= [U'(x) - U''(x)a(x)\frac{\sigma^2}{2} + o(\sigma^2)]\,[1 - a'(x)\frac{\sigma^2}{2} + o(\sigma^2)] \\ &= U'(x) - [U''(x)a(x) + U'(x)a'(x)]\frac{\sigma^2}{2} + o(\sigma^2) \\ &= U'(x) - U''(x)\theta(x) + o(\theta(x)). \end{aligned} \tag{A.4}$$

Therefore,

$$\theta(x) = \left[a(x) + \frac{U'(x)}{U''(x)}a'(x)\right]\frac{\sigma^2}{2} + o(\sigma^2). \tag{A.5}$$

Inspecting (2.6) and (A.1) and then using (A.5) makes it clear that

$$\begin{aligned} \theta^*(x) &= \theta(x + \theta^*(x)) \\ &= \left[a(x + \theta^*(x)) + \frac{U'(x + \theta^*(x))}{U''(x + \theta^*(x))}a'(x + \theta^*(x))\right]\frac{\sigma^2}{2} + o(\sigma^2) \\ &= \left[a(x) + \frac{U'(x)}{U''(x)}a'(x)\right]\frac{\sigma^2}{2} + o(\sigma^2), \end{aligned} \tag{A.6}$$

as long as a' and $\frac{U'}{U''}$ are continuous at x. Using (2.9), (A.6) can be rewritten as

$$\theta^*(x) = a(x)[1 + s(x)\varepsilon(x)]\frac{\sigma^2}{2} + o(\sigma^2), \tag{A.7}$$

which establishes (2.8) in the text.

Appendix B. Proof of proposition 6

Taking a total derivative of both sides of (2.6) with respect to x, one finds that

$$[1+\theta^{*\prime}(x)]\frac{\partial}{\partial x}[U'(M(x+\theta^*(x)+\tilde{y}))M'(x+\theta^*(x)+\tilde{y})] \leq U''(x), \qquad \text{(B.1)}$$

where the *partial* derivative of the marginal utility of saving with respect to x excludes the effect of x through $\theta^*(x)$. A decreasing marginal utility of saving is guaranteed by the assumption that M is concave and the $\frac{1}{U'}$ is increasing, and means that the partial derivative is negative, and therefore that $\theta^*(x)$ will be decreasing in x if and only if

$$\frac{\partial}{\partial x}[U'(M(x+\theta^*+\tilde{y}))M'(x+\theta^*+\tilde{y})] \leq U''(x), \qquad \text{(B.2)}$$

that is, if an increase in x without an adjustment in θ^* tends to push the marginal utility of saving in the presence of the risk $\tilde{y}$ compensated by θ^* down faster than the marginal utility of saving in the absence of the risk.[24] Rewriting (B.2), $\theta^*(x)$ is decreasing if

$$\begin{aligned} &U''(M(x+\theta^*(x)+\tilde{y}))[M'(x+\theta^*(x)+\tilde{y})]^2 \\ &\quad + U'(M(x+\theta^*(x)+\tilde{y}))M''(x+\theta^*(x)+\tilde{y}) \leq U''(x). \end{aligned} \qquad \text{(B.3)}$$

The condition that M is concave guarantees that the second term on the left hand side is negative. Therefore it is sufficient to prove that

$$U''(M(x+\theta^*(x)+\tilde{y}))[M'(x+\theta^*(x)+\tilde{y})]^2 \leq U''(x). \qquad \text{(B.4)}$$

Dividing both sides of (B.4) by

$$[U'(M(x+\theta^*(x)+\tilde{y}))M'(x+\theta^*(x)+\tilde{y})]^2 = [U'(x)]^2 \qquad \text{(B.5)}$$

one finds that (B.4) is equivalent to

$$\frac{U''(M(x+\theta^*(x)+\tilde{y}))}{[U'(M(x+\theta^*(x)+\tilde{y}))]^2} \leq \frac{U''(x)}{[U'(x)]^2}. \qquad \text{(B.6)}$$

By Proposition 1, the assumption of decreasing absolute risk aversion implies that $\theta^*(x) \geq \pi^*(x)$ and

$$M(x+\theta^*+\tilde{y}) \geq x, \qquad \text{(B.7)}$$

24. Appendix B uses the Kreps-Porteus representation of (B.2).

while $\frac{1}{U'(x)}$ convex implies that

$$\frac{d^2}{dx^2}\frac{-1}{U'(x)} = \frac{d}{dx}\frac{U''(x)}{[U'(x)]^2} \leq 0 \tag{B.8}$$

so that $\frac{U''}{[U'(x)]^2}$ is a decreasing function. Inequalities (B.7) and (B.8) together imply (B.6), as desired.

Remark. If $M''(\cdot) \leq 0$ but v exhibits *increasing* absolute risk aversion and $1/U'(x)$ is *concave*, the inequalities in both (B.7) and (B.8) are reversed, but (B.6) remains true, allowing an extension of Proposition 6 to this other case.

Appendix C. Proof of Proposition 7

If

$$U(x) = \frac{x^{1-1/s}}{1-\frac{1}{s}}$$

and

$$v(x) = \frac{x^{1-\gamma}}{1-\gamma}$$

then

$$\phi(\nu) = U(v^{-1}(\nu)) = \frac{[(1-\gamma)\nu]^{\frac{1-1/s}{1-\gamma}}}{1-\frac{1}{s}}. \tag{C.1}$$

The assumption that $1 < s \leq \frac{1}{\gamma}$ implies that the exponent of ϕ—which we will label ζ—is between zero and 1:

$$1 \geq \frac{1-1/s}{1-\gamma} = \zeta > 0. \tag{C.2}$$

Of course,

$$1 \geq 1-\gamma > 0 \tag{C.3}$$

as well. The composite function $\phi(\mathrm{E}\,v(\cdot))$ is therefore homogeneous, of degree $\zeta(1-\gamma)$:

$$\begin{aligned} \phi(\mathrm{E}\,v(\lambda(x+\tilde{y}))) &= \phi(\mathrm{E}\,\lambda^{1-\gamma}v(x+\tilde{y}) \\ &= \phi(\lambda^{1-\gamma}\mathrm{E}\,v(x+\tilde{y}) \\ &= \lambda^{\zeta(1-\gamma)}\phi(\mathrm{E}\,v(x+\tilde{y})). \end{aligned} \tag{C.4}$$

Taking the derivative of both sides of (C.4) with respect to x and dividing by $\lambda\zeta(1-\gamma)$ yields

$$\phi'(\mathrm{E}\,v(\lambda(x+\tilde{y}))\mathrm{E}\,v'(\lambda(x+\tilde{y})) = \lambda^{\zeta(1-\gamma)-1}\phi'(\mathrm{E}\,v(x+\tilde{y}))\mathrm{E}\,v'(x+\tilde{y}). \tag{C.5}$$

As a consequence, if θ^* is the Kreps-Porteus precautionary premium for $\tilde{y}$ at x, then it is also the Kreps-Porteus precautionary premium for $\lambda\tilde{y}$ at x:

$$\begin{aligned}
&\phi'(\mathrm{E}\,v(\lambda(x+\theta^*+\tilde{y}))\mathrm{E}\,v'(\lambda(x+\theta^*+\tilde{y})) \\
&= \lambda^{\zeta(1-\gamma)-1}\phi'(\mathrm{E}\,v(x+\theta^*+\tilde{y}))\mathrm{E}\,v'(x+\tilde{y}) \\
&= \lambda^{\zeta(1-\gamma)-1}\phi'(v(x))v'(x) \\
&= \phi'(v(\lambda x))v'(\lambda x). \qquad \text{(C.6)}
\end{aligned}$$

Since (C.6) holds for all positive λ, the derivatives of the two sides with respect to λ at $\lambda = 1$ must be equal. That equality of derivatives becomes useful for our purposes when the derivative of the left-hand side of (C.6) is broken into two pieces:

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\{\phi'(\mathrm{E}\,v(\lambda x+\beta(\theta^*+\tilde{y}))\mathrm{E}\,v'(\lambda x+\beta(\theta^*+\tilde{y}))\} \\
&+ \frac{\partial}{\partial\beta}\{\phi'(\mathrm{E}\,v(\lambda x+\beta(\theta^*+\tilde{y}))\mathrm{E}\,v'(\lambda x+\beta(\theta^*+\tilde{y}))\} \\
&= \frac{\partial}{\partial\lambda}\{\phi'(v(\lambda x))v'(\lambda x)\} \qquad \text{(C.7)}
\end{aligned}$$

at $\lambda = \beta = 1$. Equation (C.7) implies that

$$\begin{aligned}
&\frac{\partial}{\partial x}\{\phi'(\mathrm{E}\,v(x+\theta^*+\tilde{y})\mathrm{E}\,v'(x+\theta^*+\tilde{y}))\} \\
&= \frac{\lambda}{x}\frac{\partial}{\partial\lambda}\{\phi'(\mathrm{E}\,v(\lambda x+\beta(\theta^*+\tilde{y}))\mathrm{E}\,v'(\lambda x+\beta(\theta^*+\tilde{y}))\}\Big|_{\lambda=\beta=1} \\
&\leq \frac{\lambda}{x}\frac{\partial}{\partial\lambda}\{\phi'(v(\lambda x))(\lambda x)\}\Big|_{\lambda=1} \\
&= \frac{\partial}{\partial x}\{\phi'(v(x)v'(x)\}, \qquad \text{(C.8)}
\end{aligned}$$

if and only if

$$\frac{\partial}{\partial\beta}\{\phi'(\mathrm{E}\,v(x+\beta(\theta^*+\tilde{y}))\mathrm{E}\,v'(x+\beta(\theta^*+\tilde{y}))\}\Big|_{\beta=1} \geq 0. \qquad \text{(C.9)}$$

Since (C.8) is in turn equivalent to (B.2), the Kreps-Porteus precautionary premium $\theta^*(x)$ is decreasing if and only if (C.9) is true. (Note that positive intertemporal substitution and constant relative risk aversion guarantee a decreasing marginal utility of saving.)

Figure 6 graphs

$$g(\beta) = \phi'(\mathrm{E}\,v(x+\beta(\theta^*+\tilde{y}))\mathrm{E}\,v'(x+\beta(\theta^*+\tilde{y})) \qquad \text{(C.10)}$$

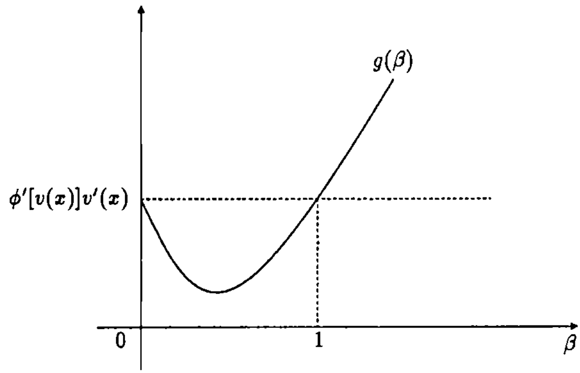

Figure 6: The function $g(\beta) = \phi'(\mathrm{E}\, v(x + \beta(\theta^* + \tilde{y}))\mathrm{E}\, v'(x + \beta(\theta^* + \tilde{y}))$

against β. By the definition of θ^*, $g(\beta) = \phi'(v(x))v'(x)$ at both $\beta = 0$ and $\beta = 1$. We need to prove that $g(\beta)$ is upward sloping at $\beta = 1$ as shown in Figure 6. To show that $g(\beta)$ is upward sloping at $\beta = 1$, we will show that $\ln g(\beta)$ is convex, so that once $g(\beta)$ passes its minimum and starts upward, it must continue going up. This means that $g(\beta)$ can intersect $\phi'(v(x))v'(x)$ only twice, and must be upward sloping at the intersection of $g(\beta)$ with $\phi'(v(x))v'(x)$ at $\beta = 1$.

The function $g(\beta)$ is log-convex because both of its two factors are log-convex and

$$\ln g(\beta) = \ln\{\phi'(\mathrm{E}\, v(x + \beta(\theta^* + \tilde{y}))\} + \ln\{\mathrm{E}\, v'(x + \beta(\theta^* + \tilde{y}))\}. \qquad \text{(C.11)}$$

The second factor of $g(\beta)$ is log-convex because (a) decreasing absolute risk aversion guarantees that $\ln v'(\cdot)$ in its argument, which makes $\ln v'(x + \beta(\theta^* + \tilde{y}))$ convex in β for any realization of $\tilde{y}$, and (b) Artin's theorem guarantees that an expectation over functions that are all log-convex in β is log convex in β. (In economics, Artin's theorem was rediscovered as the "preservation of decreasing absolute risk aversion under expectations" Nachman [1982] and Kihlstrom et al. [1981]. The simplest proof of Artin's theorem is that log-convexity is equivalent to positive definiteness of the matrix

$$\begin{bmatrix} v'(x + \beta(\theta^* + \tilde{y})) & v'(x + (\beta + \delta)(\theta^* + \tilde{y})) \\ v'(x + (\beta + \delta)(\theta^* + \tilde{y})) & v'(x + (\beta + 2\delta)(\theta^* + \tilde{y})) \end{bmatrix},$$

which is preserved under expectations (See Marshall and Olkin [1979]). The second factor of $g(\beta)$ is log-convex because $\ln \phi'(\cdot)$ is a decreasing, convex function (as can be verified by direct calculation or by noting that $\phi(\cdot)$ has the same functional form as a utility function with decreasing absolute risk aversion) and $\mathrm{E}\, v(x + \beta(\theta^* + \tilde{y}))$ is a concave function of β—as can be verified by calculating

$$\frac{\partial^2}{\partial \beta^2} \mathrm{E}\, v(x + \beta(\theta^* + \tilde{y})) = \mathrm{E}\,(\theta^* + \tilde{y})^2 v''(x + \beta(\theta^* + \tilde{y})) \leq 0.$$

A decreasing, convex function $h(\cdot) = \ln(\phi'(\cdot))$ of a concave function $j(\beta) = \mathrm{E}\, v(x + \beta(\theta^* + \tilde{y}))$ is convex since

$$\frac{\partial^2}{\partial \beta^2} h(j(\beta)) = h''(j(\beta))[j'(\beta)]^2 + h'(j(\beta))j''(\beta) \geq 0. \tag{C.12}$$

Appendix D. Proof of Proposition 8

The Kreps-Porteus representation of (B.3) is

$$\begin{aligned} &\phi''(\mathrm{E}\, v(x + \theta^* + \tilde{y}))[\mathrm{E}\, v'(x + \theta^* + \tilde{y})]^2 \\ &+ \ \phi'(\mathrm{E}\, v(x + \theta^* + \tilde{y}))\mathrm{E}\, v''(x + \theta^* + \tilde{y}) \\ &\leq \phi''(v(x))[v'(x)]^2 + \phi'(v(x))v''(x). \end{aligned} \tag{D.1}$$

The marginal utility of saving is decreasing because $\frac{1}{\phi'(\cdot)}$ increasing means that $\phi''(\cdot) \leq 0$, so (D.1) is what is needed to guarantee that the Kreps-Porteus precautionary premium $\theta^*(x)$ is decreasing. We will prove (D.1) by showing that each of the two terms on the left-hand side of (D.1) is less than or equal to the corresponding term on the right-hand side of (D.1). Using the identity

$$\phi'(\mathrm{E}\, v(x + \theta^* + \tilde{y})\mathrm{E}\, v'(x + \theta^* + \tilde{y}) = \phi'(v(x))v'(x),$$

one finds that

$$\begin{aligned} \frac{\phi''(\mathrm{E}\, v(x + \theta^* + \tilde{y}))[\mathrm{E}\, v'(x + \theta^* + \tilde{y})]^2}{[\phi'(\mathrm{E}\, v(x + \theta^* + \tilde{y})\mathrm{E}\, v'(x + \theta^* + \tilde{y})]^2} &= \frac{\phi''(\mathrm{E}\, v(x + \theta^* + \tilde{y}))}{[\phi'(\mathrm{E}\, v(x + \theta^* + \tilde{y})]^2} \\ &\leq \frac{\phi''(v(x))}{[\phi'(v(x))]^2} \\ &= \frac{\phi''(v(x))[v'(x)]^2}{[\phi'(v(x))v'(x)]^2}. \end{aligned} \tag{D.2}$$

The inequality in the middle of (D.2) is analogous to (B.6). Decreasing absolute risk aversion guarantees that $\theta^* \geq \pi^*$ and therefore that $\mathrm{E}\, v'(x + \theta^* + \tilde{y}) \geq v'(x)$ and convexity of $\frac{1}{\phi'(\cdot)}$ means that $\frac{\phi''(\cdot)}{[\phi'(\cdot)]^2}$ is a decreasing function. Using the same identity again,

$$\begin{aligned}
\frac{\phi'(\mathrm{E}\,v(x+\theta^*+\tilde{y}))\mathrm{E}\,v''(x+\theta^*+\tilde{y})}{\phi'(\mathrm{E}\,v(x+\theta^*+\tilde{y})\mathrm{E}\,v'(x+\theta^*+\tilde{y})} &= \frac{\mathrm{E}\,v''(x+\theta^*+\tilde{y})}{\mathrm{E}\,v'(x+\theta^*+\tilde{y})} \\
&\le \frac{v''(x)}{v'(x)} \\
&= \frac{\phi'(v(x))v''(x)}{\phi'(v(x))v'(x)}. \qquad \text{(D.3)}
\end{aligned}$$

The inequality in the middle of (D.3) is a consequence of the decreasing absolute prudence of v. Kimball [1991c] shows among other things that decreasing absolute prudence means that the derived utility function $\hat{v}(x) = \mathrm{E}\,v(x+\theta^*+\tilde{y})$ is locally more risk averse than $v(x)$ at any point x at which

$$\mathrm{E}\,v'(x+\theta^*+\tilde{y}) \ge v'(x). \qquad \text{(D.4)}$$

The middle inequality of (D.3) is precisely an inequality between the absolute risk aversion $\hat{v}(x)$ and the absolute risk aversion of $v(x)$ at x. Inequality (D.4) is guaranteed by the concavity of ϕ, which implies, by Proposition 4 that $\theta^* \le \psi^*$ where ψ^* is the quantity which would make (D.4) hold with equality.

Appendix E. Proof of Proposition 11

Since $M(x+\tilde{y}) = v^{-1}(\mathrm{E}\,v(x+\tilde{y}))$,

$$M'(x+\tilde{y}) = \frac{\mathrm{E}\,v'(x+\tilde{y})}{v'(M(x+\tilde{y}))} \qquad \text{(E.1)}$$

and

$$\frac{M''(x+\tilde{y})}{M'(x+\tilde{y})} = \frac{\mathrm{E}\,v''(x+\tilde{y})}{\mathrm{E}\,v'(x+\tilde{y})} - \frac{v''(M(x+\tilde{y}))}{v'(M(x+\tilde{y}))}M'(x+\tilde{y}). \qquad \text{(E.2)}$$

Also, by (E.1),

$$\frac{\mathrm{E}\,[A+Bv(x+\tilde{y})]}{\mathrm{E}\,v'(x+\tilde{y})} = \frac{[A+Bv(M(x+\tilde{y}))]}{v'(M(x+\tilde{y}))M'(x+\tilde{y})}. \qquad \text{(E.3)}$$

Therefore,

$$\frac{\mathrm{E}\,[A+Bv(x+\tilde{y})]}{\mathrm{E}\,v'(x+\tilde{y})}\frac{\partial}{\partial x}\ln(U'(M(x+\tilde{y}))M'(x+\tilde{y}))$$

$$
\begin{aligned}
&= \frac{\mathrm{E}[A + Bv(x+\tilde{y})]}{\mathrm{E}\,v'(x+\tilde{y})} \left[\frac{U''(M(x+ty))}{U'(M(x+\tilde{y}))} M'(x+\tilde{y}) + \frac{M''(x+\tilde{y})}{M'(x+\tilde{y})} \right] \\
&= \frac{[A + Bv(M(x+\tilde{y}))]}{v'(M(x+\tilde{y}))} \left[\frac{U''(M(x+\tilde{y}))}{U'(M(x+\tilde{y}))} - \frac{v''(M(x+\tilde{y}))}{v'(M(x+\tilde{y}))} \right] \\
&\quad + \frac{\mathrm{E}[A + Bv(x+\tilde{y})]\mathrm{E}\,v''(x+\tilde{y})}{[\mathrm{E}\,v'(x+\tilde{y})]^2} \\
&\leq 0, \qquad \text{(E.4)}
\end{aligned}
$$

where the inequality follows from condition (3.22) of Proposition 11. The one difficult leap between (3.22) and (E.4) is made by noting that the left-hand inequality in (3.22) along with the condition that $A + Bv(x)$ is positive, are equivalent to positive definiteness of the matrix

$$
\begin{bmatrix} A + Bv(x) & \sqrt{k}v'(x) \\ \sqrt{k}v'(\xi) & -v''(\xi) \end{bmatrix},
$$

for any value of ξ, which implies positive definiteness of

$$
\begin{bmatrix} \mathrm{E}\,(A + Bv(x+\tilde{y})) & \sqrt{k}\mathrm{E}\,v'(x+\tilde{y}) \\ \sqrt{k}\mathrm{E}\,v'(x+\tilde{y}) & -\mathrm{E}\,v''(x+\tilde{y}) \end{bmatrix}.
$$

Thus, the marginal utility of saving is decreasing.

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