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STRATEGIC COMPLEMENTARITY AND PERSISTENCE IN LINEAR-QUADRATIC MEAN FIELD GAMES

Fernando E. Alvarez
David Argente

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ABSTRACT

We study the relation between strategic complementarities and aggregate persistence in linear-quadratic mean field games with interactions in states and actions. The framework provides local approximations to common macroeconomic problems, including capital accumulation and multiproduct pricing. We show that the equilibrium aggregate dynamics coincide with those of a single-agent linear-quadratic control problem in which state and action curvatures are adjusted for strategic interactions. This result gives conditions for equilibrium existence, establishes uniqueness, and links strategic interactions to aggregate adjustment. We establish conditions under which stronger state complementarities increase persistence, while stronger action complementarities reduce it, even in multidimensional problems whose underlying matrices are not simultaneously diagonalizable. Under these conditions, internalizing state complementarities makes the utilitarian planner's allocation more persistent, while internalizing action complementarities makes it less persistent. With common shocks, we give conditions under which individual and aggregate dynamics can be combined to identify strategic interactions and model parameters. Finally, we extend the analysis to interactions through higher moments of the state distribution.

Fernando E. Alvarez
The University of Chicago
Department of Economics
and NBER
f-alvarez1@uchicago.edu

David Argente
Yale University
School of Management
Economics
and NBER
david.argente@yale.edu

1 Introduction

Strategic complementarities are a common source of amplification and persistence in macroeconomic models. In dynamic environments with many interacting agents, however, it is not immediate how the strength of these complementarities maps into aggregate adjustment, how they affect the planner's allocation, or what can be learned about them from individual and aggregate data. Equilibrium dynamics combine the adjustment inherent in each agent's problem with the feedback that operates through aggregate states and actions. We study these forces in a tractable class of linear-quadratic mean field games (LQ MFGs), which provide a local approximation to many standard problems in macroeconomics.

We study a continuum of infinitesimal agents who choose actions that affect the drift of an individual state. Agents face idiosyncratic shocks, and we later allow for aggregate shocks. Payoffs depend on the agent's own state and action and on the corresponding cross-sectional averages. Given paths for the aggregate state and aggregate action, each agent solves a linear-quadratic control problem in which these aggregates enter through cross terms with her own state and action. As in familiar games, such as monopolistic competition, each agent takes aggregate objects as given when making her decision. Equilibrium requires individual optimality and consistency between the conjectured aggregate paths and those implied by the cross-sectional distribution. The cross derivatives between individual and aggregate states and actions determine whether each interaction exhibits strategic complementarity or substitutability.

Our first result reduces the equilibrium problem to a familiar single-agent control problem. Given paths for the aggregate state and aggregate action, the individual value function is quadratic in the individual state, and aggregating optimal policies gives a finite-dimensional state–costate system. Once equilibrium consistency is imposed, the aggregate dynamics are characterized by the algebraic Riccati equation of a single-agent linear-quadratic regulator, with the state and action curvature matrices Q and Γ replaced by $Q + \Theta_X$ and $\Gamma + \Theta_{\mathcal{A}}$. Strategic interactions therefore enter the equilibrium by changing the effective curvature of states and actions, without changing the structure of the control problem that determines aggregate dynamics.

This result allows us to bring standard tools from linear-quadratic control directly to the equilibrium problem. We use them to establish uniqueness, derive conditions for existence, and characterize comparative statics of equilibrium dynamics. There is at most one equilibrium whose closed-loop matrix, which governs aggregate dynamics, has eigenvalues with real parts below $\rho/2$, where ρ is the discount rate. Using the associated Hamiltonian matrix, we derive a necessary condition for existence and a convenient sufficient condition.

In one dimension, the strict version of the necessary condition is also sufficient and gives an explicit threshold involving both state and action interactions. Existence is distinct from convergence: the eigenvalue restriction allows an equilibrium path to grow, provided that it does so slowly enough to satisfy the model's discounted integrability conditions.

We next study how strategic interactions affect aggregate persistence. Under our sufficient conditions for existence, stronger complementarity in either states or actions makes the stabilizing Riccati solution less negative definite, holding the rest of the model fixed. Their effects on aggregate dynamics nevertheless differ: stronger state complementarity raises the trace of the closed-loop matrix, while stronger action complementarity lowers it. In stable equilibria, state complementarity therefore slows the contraction of the volume of a set of initial aggregate states, while action complementarity speeds it up. These comparisons do not require the underlying matrices to be simultaneously diagonalizable, but they do not order every eigenvalue. A two-dimensional counterexample shows that individual eigenvalues can move in opposite directions. We then give additional conditions under which the comparison holds for every eigenvalue: a symmetry restriction on the transformed drift is sufficient for state interactions, while an exogenous drift matrix proportional to the identity gives closed-form eigenvalues and comparative statics for both interactions. Under the latter restriction, stronger state complementarities make every mode of a stable equilibrium more persistent, as agents have less incentive to return to steady state while others remain away from it. Stronger action complementarities have the opposite effect, encouraging agents to adjust when others do.

We use two examples to show how the interaction terms arise in familiar economic environments and how the general results help interpret their dynamics. The first is an industry equilibrium with capital accumulation. Decreasing returns and CES demand generate strategic complementarity in states, while an upward-sloping supply of investment goods generates strategic substitutability in actions. The second is a multiproduct price-setting model with CES demand within stores, Kimball demand across stores, and Rotemberg adjustment costs, a structure commonly used in macroeconomics. In this case, the matrix of static best responses is generally not symmetric when products have different expenditure shares, but the interaction matrix that enters the equilibrium problem is symmetric. The example also shows that the superelasticity of demand affects both the curvature of the firm's own problem and the strength of strategic complementarity, but cancels from the effective curvature governing equilibrium dynamics. Thus, stronger static complementarity need not imply greater aggregate persistence. More generally, the effect of a change in demand or adjustment costs on aggregate persistence depends on how it changes the adjusted curvature matrices $Q + \Theta_X$ and $\Gamma + \Theta_{\mathcal{A}}$, rather than on its effect on the interaction terms or static best responses alone.

We then turn to the utilitarian planner. The planner internalizes the effect of each agent's state and action on the corresponding aggregates. With symmetric interaction matrices, the planner's problem has the same structure as the decentralized equilibrium, except that both interaction terms are doubled. The comparative statics for aggregate adjustment therefore carry over directly, with the same qualifications as in the decentralized economy. Internalizing state complementarities makes the planner's allocation more persistent, while internalizing action complementarities makes it less persistent. When both interactions are present, these effects work in opposite directions, so the overall comparison between planner and equilibrium dynamics is generally ambiguous.

We next introduce aggregate shocks and use the recursive formulation to study identification. With common noise, the value function depends on both the individual and aggregate states, but the Riccati equation governing aggregate dynamics is unchanged. Writing the individual state as the sum of the aggregate state and a deviation from the cross-sectional mean separates aggregate from idiosyncratic dynamics. The law of motion for the deviation, and the corresponding action net of its cross-sectional average, do not depend on either interaction matrix. Thus, removing aggregate variation from individual states and actions also removes the information needed to identify strategic interactions. In contrast, aggregate dynamics do contain information about these interactions. We show how aggregate and individual dynamics can be combined to identify the state interaction and other parameters of the model. The state and action interaction matrices, however, cannot be separately identified without an additional normalization or outside information.

Finally, we extend the analysis to interactions through an arbitrary finite number of moments of the state distribution. For simplicity, we consider a one-dimensional state and allow payoffs to depend on the moments of its cross-sectional distribution. Around a symmetric zero-mean invariant distribution, the local dynamics retain the structure of the scalar benchmark. The coefficients on odd moments affect local persistence through a single effective interaction term, while the coefficients on even moments can affect transitional dynamics without changing the local eigenvalues. Because the weights on the odd-moment interactions depend on the invariant distribution, idiosyncratic risk can also affect aggregate persistence. The framework can also accommodate exogenous forcing processes, which we develop in the appendix.

Literature Review The mean field games framework developed by Lasry and Lions (2007) has advanced the study of dynamic games with a continuum of agents. Their formulation consists of two coupled partial differential equations: a Hamilton–Jacobi–Bellman equation for individual optimization and a Kolmogorov forward equation for the evolution of the

state distribution. This approach makes it possible to analyze individual decisions and the evolution of the entire cross section within the same system. Achdou, Han, Lasry, Lions and Moll (2017) use this formulation to study income and wealth distribution in heterogeneous-agent macroeconomic models. Several contributions in mathematics study linear-quadratic mean field games. Caines (2009) provides an early overview of this literature. Bardi (2012) studies linear-quadratic N-person games and their mean field limit with ergodic costs in a one-dimensional setting. The paper characterizes stationary equilibria through an algebraic Riccati equation, allowing for both strategic substitutability and complementarity. Bardi and Priuli (2014) extend several of these results to the n-dimensional case. Tchuendom (2018) studies how common noise can restore uniqueness in a class of linear-quadratic mean field games. Bensoussan, Sung, Yam and Yung (2016) analyze equilibrium existence and uniqueness and compare mean field games with mean field control problems, while Huang and Yang (2021) study linear-quadratic mean field social optimization. Part of our equilibrium characterization builds on this literature, particularly the reduction of equilibrium to an algebraic Riccati equation. Our use of linear-quadratic methods to study dynamic economic models also builds on Hansen and Sargent (2013). We adapt classical results from linear-quadratic control in the undiscounted setting, notably those of Kucera (1973) and Wimmer (1985), to derive conditions for equilibrium existence and uniqueness and comparative statics for the stabilizing solution.

We make four contributions to this literature. First, we distinguish interactions in states from interactions in actions and establish conditions under which they have opposite effects on aggregate persistence. In multiple dimensions, we separate results for volume contraction from stronger comparisons of individual eigenvalues. These results do not require simultaneous diagonalization, and the equilibrium characterization does not rely on the Lasry–Lions monotonicity condition. Second, the same characterization allows us to compare decentralized and planner adjustment. Third, we show why demeaned states and actions contain no information about either interaction matrix and give conditions for identification using individual and aggregate data, given a normalization of the action interaction. Fourth, we extend the local analysis to higher-moment coupling and show how it allows idiosyncratic risk to affect aggregate persistence.

Our economic applications relate to work on strategic complementarities arising from pecuniary externalities (Acemoglu, 1996) and technological externalities (Baxter and King, 1991; Romer, 1986; Cooper and John, 1988). Related work studies how strategic complementarities can generate aggregate fluctuations (Kaplan and Menzio, 2016). The capital-accumulation example distinguishes interactions through the return to holding capital from interactions through the cost of investment. The latter have an empirical counterpart in

Goolsbee (1998), who documents that investment incentives raise capital-goods prices. In our example, an upward-sloping supply of investment goods generates strategic substitutability in actions. We show how this interaction affects industry dynamics, separately from the adjustment costs faced by individual firms.

The pricing example combines two commonly used ingredients in macroeconomic models: Kimball demand and Rotemberg adjustment costs (Kimball, 1995; Rotemberg, 1982); see, for example, Santoro and Viviano (2022) for a model using both. Closely related to our analysis, Alvarez, Lippi and Souganidis (2023) study price setting with strategic complementarities as a mean field game. We study smooth rather than lumpy adjustment and characterize equilibrium dynamics through adjusted state and action curvatures. Klenow and Willis (2016) and Burstein and Hellwig (2007) study pricing complementarities and monetary propagation, while Baqaee, Farhi and Sangani (2024) emphasize heterogeneous markups, pass-through, and reallocation. Afrouzi and Yang (2021) use a dynamic linear-quadratic-Gaussian framework to study pricing and monetary propagation under rational inattention. The multiproduct setting also relates to models of joint price adjustment (Midrigan, 2009; Alvarez and Lippi, 2014; Blanco, Boar, Jones and Midrigan, 2025, 2026). In the multiproduct application, we show that even when static best responses are nonsymmetric, the payoff interaction matrix is symmetric, so our characterization still applies. We also show that stronger static pricing complementarities need not imply greater persistence: the superelasticity changes both individual curvature and the interaction term but cancels from $Q + \Theta_X$.

Our identification results relate to the missing-intercept problem in using cross-sectional evidence to study aggregate effects. Chodorow-Reich (2020) discusses why cross-regional estimates need not recover aggregate effects when shocks generate spillovers across regions or induce responses in national variables. Beraja, Hurst and Ospina (2019) show how regional and aggregate data can be combined to discipline a macroeconomic model, while Wolf (2023) gives conditions under which aggregate evidence recovers general-equilibrium effects absent from cross-sectional estimates. In our model, the distinction between individual adjustment and aggregate feedback makes clear what is lost when common variation is removed: neither interaction matrix enters the dynamics of demeaned states and actions. We then give conditions under which individual and aggregate dynamics can be combined to identify the state interaction and other parameters of the model.

The rest of the paper is organized as follows. Section 2 presents the model without aggregate shocks, and Section 3 develops the capital-accumulation and multiproduct price-setting examples. Section 4 characterizes equilibrium and studies existence, uniqueness, and aggregate persistence. Section 5 studies the utilitarian planner's problem. Section 6 introduces aggregate shocks, and Section 7 uses the recursive equilibrium to study identification. Sec-

tion 8 extends the analysis to interactions through higher moments of the state distribution. Section 9 concludes. The appendix contains proofs, additional derivations, and an extension with exogenous forcing processes.

2 Mean Field Game Setup without Aggregate Shocks

We formulate the model in multiple dimensions for two reasons. First, it maps naturally into many problems in macroeconomics and international economics, in which firms set several prices, accumulate different types of capital, or allocate activity across locations. These decisions are interrelated through returns, adjustment costs, and strategic interactions, and the matrices governing these links need not be simultaneously diagonalizable. As a result, adjustment involves not only how fast aggregate activity changes, but also how it is reallocated across dimensions and whether different combinations of states adjust at different speeds. Second, the multidimensional case highlights the tractability of our framework. The linear-quadratic structure allows us to retain these cross-dimensional links while characterizing equilibrium with finite-dimensional matrix equations rather than solving over a large state space.

Let $x \in \mathbb{R}^n$ denote the individual state and $\alpha \in \mathbb{R}^k$ the individual action, with cross-sectional averages $X \in \mathbb{R}^n$ and $\mathcal{A} \in \mathbb{R}^k$. We assume that the period return is quadratic in the individual and aggregate states and actions:

$$F(x,X) + R(\alpha,\mathcal{A}) = -\tfrac{1}{2}x'Qx - X'\Theta_X x - \tfrac{1}{2}\alpha'\Gamma\alpha - \mathcal{A}'\Theta_{\mathcal{A}}\alpha, \tag{1}$$
$$X = \int x\, m(x)\, dx, \qquad \mathcal{A} = \int \alpha_*(x)\, m(x)\, dx.$$

The density m describes the cross-sectional distribution of states, and α_* denotes the optimal policy. The matrices Q and Θ_X are $n \times n$, while Γ and $\Theta_{\mathcal{A}}$ are $k \times k$; Q and Γ govern the curvature of individual returns, and Θ_X and $\Theta_{\mathcal{A}}$ the interactions with the corresponding cross-sectional averages. The state follows

$$dx = \mu(x,\alpha)dt + \Sigma^{1/2}dW \equiv (B\alpha - Ax)dt + \Sigma^{1/2}dW,$$

so the drift is linear in (x,α), where A is $n \times n$, B is $n \times k$, and W is an n-dimensional standard Brownian motion with instantaneous variance matrix Σ.[1]

The agent discounts at rate $\rho \geq 0$. Taking paths $\{X(s), \mathcal{A}(s)\}_{s \geq t}$ as given, her value

[1] At several points below, we use the scalar case to make the intuition behind the main results easier to follow. In the scalar case, we specialize to $n = k = 1$ and write $Q = q$, $A = a$, $B = b$, $\Gamma = \gamma$, $\Theta_X = \theta_X$, and $\Theta_{\mathcal{A}} = \theta_{\mathcal{A}}$. We assume $\gamma + \theta_{\mathcal{A}} > 0$.

function $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$u(x,t) = \max_{\{\alpha(s):s\geq t\}} \int_t^\infty e^{-\rho(s-t)} \mathbb{E}\left[F(x(s),X(s)) + R(\alpha(s),\mathcal{A}(s)) \mid x(t)=x\right] ds$$
$$\text{subject to} \quad dx(s) = (B\alpha(s) - Ax(s))ds + \Sigma^{1/2} dW(s).$$

We maintain the following assumptions.

Assumptions AM.

1. $\rho > 0$;
2. Q and Γ are positive definite and Σ is positive semi-definite;
3. B has full row rank, so $B\Gamma^{-1}B'$ is invertible; and
4. Θ_X and $\Theta_{\mathcal{A}}$ are symmetric.

These conditions imply detectability and stabilizability for the uncoupled problem.

PROPOSITION 1. Under Assumptions AM, the pair $(Q, -A - \frac{\rho}{2} I_n)$ is detectable and the pair $(B\Gamma^{-1/2}, -A - \frac{\rho}{2} I_n)$ is stabilizable. Hence, when $\Theta_X = \Theta_{\mathcal{A}} = 0$, the single-agent problem has a unique finite time-invariant quadratic value function.

Strategic Complementarity and Substitutability. From equation (1), the cross derivative between own and aggregate states is $-\Theta_X$, and the cross derivative between own and aggregate actions is $-\Theta_{\mathcal{A}}$. We say that states are strategic complements when $-\Theta_X$ is positive definite and strategic substitutes when it is negative definite. Likewise, we say that actions are strategic complements when $-\Theta_{\mathcal{A}}$ is positive definite and strategic substitutes when it is negative definite. Thus, in the Loewner order, a decrease in either interaction matrix represents greater strategic complementarity, while an increase represents greater strategic substitutability.[2]

We note that the Lasry–Lions monotonicity condition for the state coupling used in Mean Field Games is equivalent to $-\Theta_X$ being negative semi-definite. Our uniqueness result below does not require this condition. In the linear-quadratic environment considered here, there is at most one admissible equilibrium even when states are strategic complements.

[2]The Loewner partial order compares matrices by saying that Θ_1 is larger than Θ_2 if $x'\Theta_1 x > x'\Theta_2 x$ for all nonzero vectors x.

MFG as Coupled P.D.E.'s. The HJB equation implied by the optimality of the value function is given by

$$\rho u(x,t) = -\tfrac{1}{2}x'Qx - X(t)'\Theta_X x + H(u_x(x,t), x, \mathcal{A}(t)) + \tfrac{1}{2}\operatorname{tr}(\Sigma u_{xx}(x,t)) + u_t(x,t). \tag{2}$$

where the Hamiltonian $H : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$ is given by:

$$H(p,x,\mathcal{A}) = \max_{\alpha\in\mathbb{R}^k} \{R(\alpha,\mathcal{A}) + p'(B\alpha - Ax)\} \equiv \max_{\alpha\in\mathbb{R}^k} \left\{-\tfrac{1}{2}\alpha'\Gamma\alpha - \mathcal{A}'\Theta_{\mathcal{A}}\alpha + p'(B\alpha - Ax)\right\} \tag{3}$$

The variable p is the marginal value of the state, and $\alpha_*(p,\mathcal{A})$ denotes the maximizing action. Note that, by the envelope theorem, the drift is given by the derivative of the Hamiltonian evaluated at the optimal action:

$$\mu\left(\alpha^*(p,\mathcal{A}), x\right) = H_p(p,x,\mathcal{A})\,.$$

The same optimal drift determines how the cross-sectional distribution evolves. Denoting the density of x at time t by $m(x,t)$, the Kolmogorov forward equation is

$$m_t(x,t) = -\operatorname{div}(H_p(u_x(x,t),x,\mathcal{A}(t))m(x,t)) + \tfrac{1}{2}\operatorname{tr}(\Sigma\, m_{xx}(x,t))\,,$$

with $m(x,0) = m_0(x)$. The compact notation for divergence *div* means:

$$\operatorname{div}(H_p(u_x(x,t),x,\mathcal{A}(t))m(x,t)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} H_{p_i}(u_x(x,t),x,\mathcal{A}(t))m(x,t)$$

where each $H_{p_i}(\cdot)$ is the derivative of the Hamiltonian with respect to p_i and where

$$\operatorname{tr}(\Sigma\, m_{xx}(x,t)) = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} m_{x_i x_j}(x,t)$$

The aggregate consistency conditions are

$$X(t) = \int x\, m(x,t)dx, \qquad \mathcal{A}(t) = \int \alpha_*(u_x(x,t),\mathcal{A}(t))m(x,t)dx \text{ for all } t \geq 0. \tag{4}$$

The aggregate paths taken as given in the individual problem must therefore coincide with those generated by optimal behavior and the resulting cross-sectional distribution. We state the standard definition of an equilibrium as a forward-backward system of value function, density, and their consistency conditions.

DEFINITION 1. An *equilibrium for the Mean Field Game* given m_0 is a (finite) value function u and a density m for all t, x, and time paths for X and $\mathcal{A}$ satisfying equation (2)–equation (4), the initial condition, and the boundary conditions stated in Appendix A.1.

3 Examples

In this section, we show how the linear-quadratic structure arises in familiar economic problems. The first example is one-dimensional and features strategic complementarity in states together with strategic substitutability in actions. The second is a multidimensional model of multiproduct pricing that can feature either strategic complementarity or substitutability in states. This example also shows that, even though the problem is not symmetric, the relevant matrix Θ_X is symmetric.

3.1 Capital Accumulation in an Industry Equilibrium

Consider a continuum of monopolistically competitive firms. A firm with capital k produces k^ν, where $0 < \nu < 1$. A constant-returns sector aggregates differentiated goods with elasticity of substitution $\eta > 1$. If all other firms have capital K, the focal firm's operating profit, in units of the numeraire, is proportional to

$$\Pi(k, K) = k^{\nu(1-1/\eta)} K^{\nu/\eta}. \tag{5}$$

Capital evolves according to

$$dk = (i - \delta k)dt + k\sigma dW.$$

The firm chooses investment i, capital depreciates at rate δ, and σ is the volatility of the shocks. The firm purchases investment at price $\mathcal{P}(I)$, where I is aggregate investment, and pays the convex adjustment cost $\psi(i)$. Aggregate investment is

$$I(t) = \int i_*(k, t) m(k, t) dk.$$

The firm's HJB is

$$\begin{aligned} \rho V(k,t) = \max_i \Pi(k, K(t)) - i\mathcal{P}(I(t)) - \psi(i) + V_k(k,t)(i - \delta k) \\ + \tfrac{1}{2} k^2 \sigma^2 V_{kk}(k,t) + V_t(k,t). \end{aligned}$$

Let $\bar{i} = \delta\bar{k}$ denote the deterministic steady-state level of investment, normalize $\mathscr{P}(\bar{i}) = 1$ and $\psi'(\bar{i}) = 0$, and define percentage deviations

$$x = \frac{k - \bar{k}}{\bar{k}}, \qquad X = \frac{K - \bar{k}}{\bar{k}}, \qquad \alpha = \frac{i - \delta\bar{k}}{\delta\bar{k}}, \qquad \mathcal{A} = \frac{I - \delta\bar{k}}{\delta\bar{k}}.$$

A second-order approximation of returns around $(k, K, i, I) = (\bar{k}, \bar{k}, \bar{i}, \bar{i})$ has the form in equation (1). Terms that depend only on aggregate variables do not affect the firm's choices, while the linear terms are absorbed by the steady-state conditions and the linear part of the value function (see Appendix A.3). Normalizing the objective by $\Pi(\bar{k}, \bar{k})$, the scalar coefficients are

$$q = -\frac{\bar{k}^2\Pi_{kk}(\bar{k}, \bar{k})}{\Pi(\bar{k}, \bar{k})}, \qquad \theta_X = -\frac{\bar{k}^2\Pi_{kK}(\bar{k}, \bar{k})}{\Pi(\bar{k}, \bar{k})},$$
$$\gamma = \frac{\delta^2\bar{k}^2\psi''(\delta\bar{k})}{\Pi(\bar{k}, \bar{k})}, \qquad \theta_{\mathcal{A}} = \frac{\delta^2\bar{k}^2\mathscr{P}'(\delta\bar{k})}{\Pi(\bar{k}, \bar{k})}.$$

The normalized state equation is $dx = \delta(\alpha - x)dt + \sigma dW$ to first order, so $a = b = \delta$ in the scalar notation used below. Direct differentiation of equation (5) gives

$$q = \nu\frac{\eta - 1}{\eta^2}\left(\eta(1 - \nu) + \nu\right) > 0,$$
$$\theta_X = -\frac{\eta - 1}{\eta}\frac{\nu^2}{\eta} < 0,$$
$$q + \theta_X = \frac{\eta - 1}{\eta}\nu(1 - \nu) > 0.$$

Higher aggregate capital raises the marginal return to a firm's own capital, generating state complementarity. This interaction partly offsets the curvature of individual profits, but decreasing returns keep $q + \theta_X$ positive. If the supply of investment goods is upward sloping, greater investment by other firms raises the price the firm pays to invest. This generates substitutability in action through $\theta_{\mathcal{A}} > 0$. This example shows that in a simple well-studied economic problem strategic complementarity in states and substitutability in actions arise naturally.

3.2 Multiproduct Price Setting with Kimball Demand

In this example there is a continuum of firms, denoted as "stores" which compete by setting prices of n products. Each store sells n products with constant marginal costs z_j. Products

within a store are aggregated with a CES price index

$$a(p) = \left(\sum_{j=1}^{n} \omega_j p_j^{1-\bar{\eta}_d} \right)^{1/(1-\bar{\eta}_d)},$$

so conditional product demand is proportional to $\omega_j (p_j / a(p))^{-\bar{\eta}_d}$. Across stores, demand is generated by a symmetric Kimball aggregator $\mathcal{C}$ satisfying

$$1 = \int_0^1 \Upsilon\left(\frac{C^s}{\mathcal{C}} \right) ds, \qquad \Upsilon' > 0, \quad \Upsilon'' < 0.$$

The index s denotes a store, C^s is consumption of its composite good, and $\mathcal{C}$ is aggregate consumption. Let p be the store's price vector and P the common price vector of other stores. Profits are proportional to

$$\Pi(p, P) = \sum_{j=1}^{n} (p_j - z_j) \omega_j d\left(\frac{p_j}{a(p)} \right) D\left(\frac{a(p)}{a(P)} \right), \tag{6}$$

where $d(y) = y^{-\bar{\eta}_d}$ and

$$D(y) = (\Upsilon')^{-1}(y \Upsilon'(\bar{q})).$$

Define the store-demand elasticity $\eta_D(y) = -y D'(y)/D(y)$, and let $\bar{\eta}_D$ and $\bar{\eta}'_D$ denote its level and derivative at $y = 1$. The flexible-price equilibrium satisfies

$$\bar{p}_i = \bar{z}_i \frac{\bar{\eta}_D}{\bar{\eta}_D - 1}, \qquad i = 1, \ldots, n,$$

with $\bar{\eta}_D > 1$. Let x and X denote log deviations of p and P from $\bar{p}$, and let $\tilde{\Pi}(x, X)$ denote equation (6) expressed in these deviations. The state curvature and interaction matrices are

$$Q = -\frac{1}{\tilde{\Pi}(0,0)} \frac{\partial^2 \tilde{\Pi}(0,0)}{\partial x \partial x}, \qquad \Theta_X = -\frac{1}{\tilde{\Pi}(0,0)} \frac{\partial^2 \tilde{\Pi}(0,0)}{\partial x \partial X}.$$

Writing $\bar{s}_i$ for the steady-state expenditure share and $\bar{s}$ for the associated column vector gives

$$\begin{aligned} Q &= (\bar{\eta}_D - 1) \left[\left(\bar{\eta}_D - \bar{\eta}_d + \frac{\bar{\eta}'_D}{\bar{\eta}_D - 1} \right) \bar{s}\bar{s}' + \bar{\eta}_d \operatorname{diag}(\bar{s}_i) \right], \\ \Theta_X &= -\bar{\eta}'_D \bar{s}\bar{s}', \\ Q + \Theta_X &= (\bar{\eta}_D - 1) \left[(\bar{\eta}_D - \bar{\eta}_d) \bar{s}\bar{s}' + \bar{\eta}_d \operatorname{diag}(\bar{s}_i) \right]. \end{aligned}$$

The matrix Q is positive definite if and only if $\bar{\eta}'_D/\bar{\eta}_D > 1 - \bar{\eta}_D$. The matrix Θ_X is negative semi-definite when $\bar{\eta}'_D > 0$ and positive semi-definite when $\bar{\eta}'_D < 0$. By contrast, $Q + \Theta_X$ is positive definite and independent of $\bar{\eta}'_D$. The static best response has derivative

$$\frac{\partial x_i^*(X)}{\partial X_j} = \bar{s}_j \frac{\dfrac{\bar{\eta}'_D}{(\bar{\eta}_D - 1)\bar{\eta}_D}}{1 + \dfrac{\bar{\eta}'_D}{(\bar{\eta}_D - 1)\bar{\eta}_D}}, \qquad i, j = 1, \ldots, n.$$

The matrix above need not be symmetric and, in general, is not symmetric when products have different expenditure shares. Importantly, both its sign and the sign of Θ_X depend on the superelasticity, while the effective curvature $Q+\Theta_X$, which will determine the equilibrium Riccati equation, does not. At the same time, although $\frac{\partial x_i^*(X)}{\partial X_j}$ is generally nonsymmetric, the interaction matrix Θ_X is always symmetric. This distinction will be important for the equilibrium analysis, since the symmetry required by our general results applies to Θ_X, not to $\frac{\partial x_i^*(X)}{\partial X_j}$. We return to the role of $Q + \Theta_X$ after deriving the equilibrium characterization. Details of the multiproduct derivation are in Appendix D.

Finally, in this example stores have to pay quadratic "Rotemberg" adjustment costs to change prices. Thus, α_i is simply the percentage (log) change in the price of product i. The matrix Γ allows the Rotemberg price adjustment cost to depend on individual product prices as well as on bundles of products. Adjustment costs that depend on bundles are important in Midrigan (2009), Alvarez and Lippi (2014), and Blanco, Boar, Jones and Midrigan (2024). We can incorporate non-zero cost inflation by letting the matrix A be diagonal, with inflation on its diagonal. In this example, there is no aggregate-action interaction, so $\Theta_{\mathcal{A}} = 0$. The use of Kimball demand in price-setting models is quite common because strategic complementarity can generate additional propagation, as in Dotsey and King (2005), Smets and Wouters (2007), Klenow and Willis (2016), and Alvarez, Lippi and Souganidis (2023), with some models featuring state-dependent price adjustment and others time-dependent price adjustment. We return to this property after deriving one of our main results.

4 Analysis of the MFG without Aggregate Shocks

We first solve the agent's problem for given paths of the aggregate state and aggregate action. We then aggregate the optimal decision rules and characterize an equilibrium as a saddle path of a finite-dimensional dynamical system. The resulting Riccati equation makes clear how the two forms of interaction affect equilibrium dynamics.

4.1 The Individual Problem and Aggregation

For later use, the Hamiltonian in equation (3) can be written explicitly.

LEMMA 1. **Hamiltonian.** For $p \in \mathbb{R}^n$,

$$\begin{aligned}
\alpha_*(p, \mathcal{A}) &= \Gamma^{-1} \left(B'p - \Theta_{\mathcal{A}} \mathcal{A}\right), \\
H(p, x, \mathcal{A}) &= \tfrac{1}{2} \left(B'p - \Theta_{\mathcal{A}} \mathcal{A}\right)' \Gamma^{-1} \left(B'p - \Theta_{\mathcal{A}} \mathcal{A}\right) - \tfrac{1}{2} \left(p'Ax + x'A'p\right), \\
H_p(p, x, \mathcal{A}) &= B\Gamma^{-1} \left(B'p - \Theta_{\mathcal{A}} \mathcal{A}\right) - Ax.
\end{aligned}$$

Given paths $\{X(t), \mathcal{A}(t)\}_{t \geq 0}$, the individual problem is a time-varying linear-quadratic regulator. The value function is therefore quadratic in x:

$$u(x, t) = \beta_0(t) + \beta_1(t)'x + \tfrac{1}{2} x'\beta_2(t)x, \tag{7}$$

where $\beta_2(t)$ is symmetric. Substituting equation (7) into the HJB and matching the constant, linear, and quadratic terms gives

$$\dot{\beta}_0 = \rho\beta_0 - \tfrac{1}{2} \left(B'\beta_1 - \Theta_{\mathcal{A}} \mathcal{A}\right)' \Gamma^{-1} \left(B'\beta_1 - \Theta_{\mathcal{A}} \mathcal{A}\right) - \tfrac{1}{2} \operatorname{tr}(\Sigma\beta_2), \tag{8}$$

$$\dot{\beta}_1 = (\rho I_n + A')\beta_1 + \Theta_X X - \beta_2 B\Gamma^{-1}B'\beta_1 + \beta_2 B\Gamma^{-1}\Theta_{\mathcal{A}} \mathcal{A}, \tag{9}$$

$$\dot{\beta}_2 = Q - \beta_2 B\Gamma^{-1}B'\beta_2 + \beta_2 A + A'\beta_2 + \rho\beta_2. \tag{10}$$

Equation (10) is autonomous and does not depend on either interaction matrix. Thus, this is also the solution of the stationary LQ agent problem, where the only concave solution is the stationary minimal solution to this Riccati equation, which we denote by $\bar{\beta}_2$. Since the agent problem is also concave, this must then be the same solution, i.e. $\beta_2(t) = \bar{\beta}_2$.

PROPOSITION 2. In any equilibrium, $\beta_2(t) = \bar{\beta}_2$ for all $t \geq 0$, where $\bar{\beta}_2$ is the unique negative definite solution of

$$\bar{\beta}_2 B\Gamma^{-1}B'\bar{\beta}_2 = Q + \rho\bar{\beta}_2 + \bar{\beta}_2 A + A'\bar{\beta}_2. \tag{11}$$

Notice that equation (11) does not depend on either interaction matrix. Taking the aggregate paths as given, strategic interactions affect $\beta_1(t)$ but not the curvature of the value function, $\bar{\beta}_2$. Thus, $\bar{\beta}_2$ governs individual adjustment relative to the aggregate, while the interaction terms enter when we impose the equilibrium consistency conditions.

The optimal action is

$$\alpha_*(x,t) = \Gamma^{-1}\left[B'\left(\beta_1(t) + \bar{\beta}_2 x\right) - \Theta_{\mathcal{A}}\mathcal{A}(t)\right].$$

Integrating this expression with respect to $m(x,t)$ and using the consistency condition for aggregate actions yields

$$\mathcal{A}(t) = (\Gamma + \Theta_{\mathcal{A}})^{-1} B'\left(\beta_1(t) + \bar{\beta}_2 X(t)\right). \tag{12}$$

where we assume that $\Gamma + \Theta_{\mathcal{A}}$ is invertible. Equation (12) shows how the action interaction enters the aggregate problem. At the individual level, Γ governs the curvature of the action cost and $\Theta_{\mathcal{A}}$ governs the response to the aggregate action. Imposing consistency on aggregate actions combines the two into the effective curvature $\Gamma + \Theta_{\mathcal{A}}$.

PROPOSITION 3. Assume $\Gamma + \Theta_{\mathcal{A}}$ is invertible. Then $\beta_1(t)$ must satisfy the following linear o.d.e. given a path $\{X(t)\}$:

$$\begin{aligned}\dot{\beta}_1(t) = &\left(\rho I_n + A' - \bar{\beta}_2\left[B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'\right]\right)\beta_1(t) \\ &+ \left[\Theta_X + \bar{\beta}_2\left(B\Gamma^{-1}B' - \left[B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'\right]\right)\bar{\beta}_2\right]X(t).\end{aligned}$$

Next we obtain a constant coefficient o.d.e. for $X(t)$ given the path of $\{\beta_1(t)\}$.

PROPOSITION 4. Assume $\Gamma + \Theta_{\mathcal{A}}$ is invertible. Then $X(t)$ must satisfy the following linear o.d.e. given a path $\{\beta_1(t)\}$:

$$\dot{X}(t) = \left[B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'\right]\beta_1(t) + \left(-A + \left[B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'\right]\bar{\beta}_2\right)X(t).$$

An informal derivation of this equation is as follows. Start with the individual law of motion $dx(t) = [B\alpha(x,t) - Ax(t)]\,dt + \Sigma^{1/2}dW$ and integrate across agents using the density $m(x,t)$. By a law of large numbers, the idiosyncratic dW terms average to zero. Using the definitions of $X(t)$ as the average state and $\mathcal{A}(t)$ as the average optimal action gives $\dot{X}(t) = B\mathcal{A}(t) - AX(t)$. Substituting the expression $\mathcal{A}(t) = (\Gamma + \Theta_{\mathcal{A}})^{-1}B'\left(\beta_1(t) + \bar{\beta}_2 X(t)\right)$ obtained above yields the desired equation. We give a rigorous proof using the Kolmogorov forward equation in the appendix.

We can collect the two o.d.e.'s obtained in Proposition 3 and Proposition 4 into a $2n$

system of constant coefficient o.d.e's which has the interpretation of a state–costate system:

$$\begin{bmatrix}\dot{\beta}_1\\ \dot{X}\end{bmatrix} = \underbrace{\begin{bmatrix}\rho I_n + A' - \bar{\beta}_2\left[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'\right] & \Theta_X + \bar{\beta}_2(B\Gamma^{-1}B' - [B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'])\bar{\beta}_2\\ \left[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'\right] & -A + \left[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'\right]\bar{\beta}_2\end{bmatrix}}_{\mathcal{H}}\begin{bmatrix}\beta_1\\ X\end{bmatrix}. \tag{13}$$

This o.d.e. system has the initial condition $X(0) = \int x\, m(x,0)\, dx$, but no initial condition for $\beta_1(0)$. Indeed, β_1 must be solved for as a function of X. In particular, once (β_1, X) and $\mathcal{A}$ have been found, equation (8) determines β_0. The eigenvalues associated with the relevant solution of equation (13) must have real parts below $\rho/2$ so that the discounted value $\beta_0(t)$ is finite. We turn to this next, but first note a result about the eigenvalues of $\mathcal{H}$.

Because Θ_X, $B\Gamma^{-1}B'$, and $[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B']$ are symmetric, $\mathcal{H} - \frac{\rho}{2}I_{2n}$ is a Hamiltonian matrix. The eigenvalues of $\mathcal{H}$ are therefore symmetric around $\rho/2$.

PROPOSITION 5. If λ is an eigenvalue of $\mathcal{H}$, then $\rho - \lambda$, λ^*, and $\rho - \lambda^*$ are also eigenvalues, where * denotes complex conjugation.

This proposition shows that there is at most one equilibrium, as we will see below. This is a well-known result; see Lancaster and Rodman (1995).

4.2 The Equilibrium Riccati Equation

We now describe the equilibrium. Given the state–costate nature of the o.d.e. in equation (13), we are looking for a "saddle path" that makes β_1 a linear function of X. Thus, we look for a matrix S such that $\beta_1(t) = SX(t)$ and hence $\dot{\beta}_1(t) = S\dot{X}(t)$. Substituting these two expressions into equation (13), we obtain that S satisfies

$$\begin{aligned} S\left[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'\right]S = {} & SA + A'S + \rho S + \Theta_X \\ & - S\left[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'\right]\bar{\beta}_2 - \bar{\beta}_2\left[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'\right]S \\ & + \bar{\beta}_2\left(B\Gamma^{-1}B' - \left[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'\right]\right)\bar{\beta}_2. \end{aligned} \tag{14}$$

This is a complicated quadratic matrix equation in S, some of whose coefficients depend on $\bar{\beta}_2$, which is itself the solution of an algebraic Riccati equation. Nevertheless, define

$$P \equiv S + \bar{\beta}_2.$$

Combining equation (14) with the single-agent Riccati equation (11) gives the following proposition, which has a much simpler expression.

PROPOSITION 6. An equilibrium for the MFG is given by a matrix P solving

$$P\left[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'\right]P = Q+\Theta_X+\rho P+PA+A'P. \tag{15}$$

with the associated law of motion of the aggregate state given by

$$\begin{aligned}&\dot{X} = (J+G)X = (\left[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'\right]P - A)X, \text{ where} \\ &G \equiv B\Gamma^{-1}B'\bar{\beta}_2 - A, \quad J \equiv \left[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'\right]S + \left(\left[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B'\right] - B\Gamma^{-1}B'\right)\bar{\beta}_2.\end{aligned} \tag{16}$$

The equilibrium requires that all the eigenvalues of $J+G$ have real part smaller than $\rho/2$.[3]

Since the aggregate dynamics are linear and homogeneous, $X = 0$ is always a steady state. A nonzero steady state requires the closed-loop matrix $J+G$ to be singular. Thus, generically, the origin is the unique steady state.

The Riccati structure allows us to use standard results on existence and comparative statics. If $\Theta_X = \Theta_{\mathcal{A}} = 0$, then $S = 0$ and $P = \bar{\beta}_2$. The Riccati equation for $\bar{\beta}_2$, with associated law of motion G, describes individual adjustment in the absence of strategic interactions, while the equation for P, with associated law of motion $J+G$, describes aggregate equilibrium adjustment. Thus, J captures the additional feedback generated by strategic interactions, separating the adjustment inherent in the individual problem from equilibrium feedback. Comparing equation (11) and equation (15) shows that the two Riccati equations have the same form, with Q and Γ replaced in equilibrium by the adjusted curvatures $Q+\Theta_X$ and $\Gamma+\Theta_{\mathcal{A}}$. This gives the following result.

PROPOSITION 7. The equilibrium of a Mean Field Game with interaction terms Θ_X and $\Theta_{\mathcal{A}}$ and its law of motion correspond to a single agent decision problem where the curvature on the period return function is given by $Q+\Theta_X$ instead of Q, and where the curvature on the actions is given by $\Gamma+\Theta_{\mathcal{A}}$ instead of Γ.

The multiproduct price-setting example provides a useful illustration. The superelasticity affects both the curvature of the store's own problem, Q, and the strength of the strategic interaction, Θ_X. These effects enter with opposite signs, so the superelasticity drops out of their sum, $Q+\Theta_X$, which is the curvature relevant for equilibrium dynamics. This cancellation holds for the class of Kimball demand considered here, rather than only for a particular

[3] In the scalar case, the Riccati equation is quadratic in p: $\frac{b^2}{\gamma+\theta_{\mathcal{A}}}p^2 = q+\theta_X+(\rho+2a)p$. When an equilibrium exists, exactly one of its two roots gives an eigenvalue below $\rho/2$. For this root, the aggregate eigenvalue is $\lambda = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}+a\right)^2 + \frac{b^2(q+\theta_X)}{\gamma+\theta_{\mathcal{A}}}}$.

parameterization. Changes in the superelasticity therefore differ from changes in Θ_X holding Q fixed: with the remaining parameters fixed, static pricing complementarity can change without changing aggregate persistence.

PROPOSITION 8. There is at most one equilibrium for the Linear Quadratic Mean Field Game.

This is a consequence of Proposition 5. As is well known from the theory of algebraic Riccati equations, the eigenvalues of the closed-loop matrix $J + G$ correspond to eigenvalues of the matrix $\mathcal{H}$, as in the classical result of Theorem 7.1.2 in Lancaster and Rodman (1995). Thus, regardless of whether there is strategic complementarity, a linear-quadratic mean field game has at most one equilibrium. There may nevertheless be no equilibrium, a topic to which we turn next.

The associated existence conditions can be stated directly in terms of equation (15). Define

$$E \equiv Q + \Theta_X + \left(A' + \tfrac{\rho}{2} I_n\right) \left[B(\Gamma + \Theta_{\mathcal{A}})^{-1} B'\right]^{-1} \left(A + \tfrac{\rho}{2} I_n\right). \tag{17}$$

PROPOSITION 9. Suppose $B(\Gamma + \Theta_{\mathcal{A}})^{-1} B'$ is positive definite.

1. A necessary condition for an equilibrium is that E in equation (17) be positive semi-definite.

2. If $Q + \Theta_X$ is positive definite, an equilibrium exists and is unique.

In the scalar case, an equilibrium exists if and only if the strict scalar version of the necessary condition holds.[4]

Some remarks are in order. The condition that E in equation (17) be positive semi-definite puts a limit on the degree of strategic complementarity in states and actions, i.e. on "how" negative definite Θ_X and $\Theta_{\mathcal{A}}$ can be. Moreover, strategic substitutability in one interaction can compensate for strategic complementarity in the other, since E contains one term involving Θ_X and another involving $\Theta_{\mathcal{A}}$.

The first condition, i.e. the condition on E, is not sufficient in general dimensions: one can construct a two-dimensional example in which E is positive definite but the Hamiltonian matrix has no admissible n-dimensional invariant subspace. The sufficient condition is particularly convenient: it requires positive effective curvature in states and actions, $Q + \Theta_X \succ 0$ and $\Gamma + \Theta_{\mathcal{A}} \succ 0$. This condition is stronger than necessary, but it is simple to verify directly from the effective curvature matrices.

[4] In the scalar case, this condition becomes $q + \theta_X + \left(a + \frac{\rho}{2}\right)^2 \frac{\gamma + \theta_{\mathcal{A}}}{b^2} > 0$.

The scalar case also makes the distinction between existence and stability particularly transparent. Define

$$-\theta^* \equiv q + a(a+\rho)\frac{\gamma+\theta_{\mathcal{A}}}{b^2} \quad \text{and} \quad -\theta^{**} \equiv q + \left(\frac{\rho}{2}+a\right)^2 \frac{\gamma+\theta_{\mathcal{A}}}{b^2}.$$

Then $-\theta^{**} = -\theta^* + (\rho/2)^2(\gamma+\theta_{\mathcal{A}})/b^2$. As state complementarity increases, there are four regions:

1. if $-\theta_X < -\theta^*$, the equilibrium is stable and $X(t)$ converges to zero;
2. if $-\theta_X = -\theta^*$, then $\lambda = 0$ and $X(t) = X(0)$;
3. if $-\theta^* < -\theta_X < -\theta^{**}$, an equilibrium exists but $0 < \lambda < \rho/2$, so $X(t)$ moves away from zero; and
4. if $-\theta_X \geq -\theta^{**}$, there is no equilibrium.

Thus, existence and stability are distinct: for an intermediate range of complementarity, an equilibrium exists and satisfies the eigenvalue condition $\lambda < \rho/2$, even though the aggregate state moves away from the steady state. As complementarity increases, stability is lost before existence.

4.3 Comparative Statics and Persistence

We first characterize how the stabilizing solution P changes with the two interaction matrices. Throughout this subsection, we assume that $Q + \Theta_X$ and $\Gamma + \Theta_{\mathcal{A}}$ are positive definite.

PROPOSITION 10. Consider two equilibria that differ only in one interaction matrix.

1. If $\Theta_X^1 \preceq \Theta_X^0$, then $P^1 \succeq P^0$. Thus, more strategic complementarity in states makes P less negative definite.
2. If $\Theta_{\mathcal{A}}^1 \preceq \Theta_{\mathcal{A}}^0$, then $P^1 \succeq P^0$. Thus, more strategic complementarity in actions also makes P less negative definite.

Both forms of complementarity raise P, but this does not mean that they have the same effect on aggregate dynamics. State interactions affect $J + G$ through P, while action interactions also change $B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'$. Moreover, when $n > 1$, an ordering of P is not sufficient to order every eigenvalue of $J + G$. We first establish a comparison of their sum, given by the trace.

LEMMA 2. Assume that $\Theta^i_X + Q$ and $\Theta^i_{\mathcal{A}} + \Gamma$ are both positive definite for $i = 0, 1$. Denote by $[J+G]^i$ the matrix corresponding to the equilibrium law of motion of the aggregate state.

1. If $\Theta^1_X \preceq \Theta^0_X$, then $\mathrm{tr}\left([J+G]^1\right) \geq \mathrm{tr}\left([J+G]^0\right)$. Thus, more strategic complementarity in states increases the sum of the eigenvalues of the aggregate law of motion.

2. If $\Theta^1_{\mathcal{A}} \preceq \Theta^0_{\mathcal{A}}$, then $\mathrm{tr}\left([J+G]^1\right) \leq \mathrm{tr}\left([J+G]^0\right)$ Thus, more strategic complementarity in actions decreases the sum of the eigenvalues of the aggregate law of motion.

In words, greater state complementarity or greater action substitutability raises the trace of $J+G$, while greater action complementarity lowers it. In a stable equilibrium, a higher trace means that the sum of the eigenvalues is closer to zero. In one dimension, this immediately implies that greater state complementarity makes the aggregate state more persistent, while greater action complementarity makes it less persistent. In the capital-accumulation example, state complementarity and action substitutability therefore work in the same direction: both make aggregate capital more persistent than it would be if the corresponding interaction were absent. In multiple dimensions, however, the trace does not determine how each individual eigenvalue changes. We can instead give the following comparative-static result for the evolution of a set of initial conditions:

PROPOSITION 11. Let $[J+G]^i$ be the equilibrium law of motion corresponding to Θ^i_X and $\Theta^i_{\mathcal{A}}$ for $i = 0, 1$. Assume that $\Theta^i_X + Q$ and $\Theta^i_{\mathcal{A}} + \Gamma$ are both positive definite for $i = 0, 1$. Let $\Omega_0 \subset \mathbb{R}^n$ be an arbitrary set of initial conditions and let $\Omega^i_t = \{X(t) \in \mathbb{R}^n : \dot{X}(s) = [J+G]^i X(s), 0 \leq s \leq t, \text{ and } X(0) \in \Omega_0\}$ be the points reached after $t > 0$ periods. Let $vol(\cdot)$ denote the volume, or Lebesgue measure, of a set in $\mathbb{R}^n$.

1. If $\Theta^1_X \preceq \Theta^0_X$, then $vol(\Omega^1_t) \geq vol(\Omega^0_t)$. Thus, more strategic complementarity in states keeps the volume of states larger.

2. If $\Theta^1_{\mathcal{A}} \preceq \Theta^0_{\mathcal{A}}$, then $vol(\Omega^1_t) \leq vol(\Omega^0_t)$. Thus, more strategic complementarity in actions shrinks the volume faster.

The proof of Proposition 11 is immediate from the continuity equation, which shows that $vol(\Omega_t)$ solves a simple scalar constant-coefficient o.d.e., giving $vol(\Omega^i_t) = e^{\mathrm{tr}([J+G]^i)t} vol(\Omega_0)$, together with Lemma 2. State complementarity therefore reduces the rate of volume contraction, while action complementarity increases it. A change in the trace implies that the sum of the real parts of the eigenvalues changes, and hence that at least one of them moves. It does not imply that every eigenvalue moves in the same direction.

The following numerical example makes this distinction explicit. Set $\rho = 0$ and hold the action interaction fixed. For $i = 1, 2$, let P_i solve

$$P_i \left[B(\Gamma + \Theta_{\mathcal{A}})^{-1} B'\right] P_i = Q_i + P_i A + A' P_i, \qquad (J + G)_i = \left[B(\Gamma + \Theta_{\mathcal{A}})^{-1} B'\right] P_i - A,$$

with

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \qquad \left[B(\Gamma + \Theta_{\mathcal{A}})^{-1} B'\right] = \begin{pmatrix} 1 & -3 \\ -3 & 10 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 3 & 5 \\ 5 & 51 \end{pmatrix}.$$

Here A, $[B(\Gamma + \Theta_{\mathcal{A}})^{-1} B']$, Q_1, and $Q_2 - Q_1$ are positive definite, but A and $[B(\Gamma + \Theta_{\mathcal{A}})^{-1} B']$ do not commute. The stabilizing solutions satisfy $P_2 \prec P_1 \prec 0$, and the traces move in the order implied by Lemma 2. Nevertheless,

$$\text{eig}((J + G)_1) \simeq \{-3.02, -6.55\}, \qquad \text{eig}((J + G)_2) \simeq \{-1.56, -22.68\}.$$

One eigenvalue rises and the other falls. Thus, even a strict Loewner ordering of the Riccati solutions does not imply an eigenvalue-by-eigenvalue ordering of the closed-loop matrices. Stronger assumptions are therefore required to obtain an unambiguous result on persistence for each eigenvalue. We give two sets of results, with increasingly strong assumptions and correspondingly stronger conclusions.

The first result imposes a symmetry condition after the natural change of coordinates.

PROPOSITION 12. Hold $B(\Gamma + \Theta_{\mathcal{A}})^{-1} B'$ fixed. Suppose $\left[B(\Gamma + \Theta_{\mathcal{A}})^{-1} B'\right]^{-1} A$ is symmetric. Order the eigenvalues of $J + G$ from smallest to largest. If $\Theta_X^1 \preceq \Theta_X^0$, then

$$\text{eig}_j(J^1 + G) \geq \text{eig}_j(J^0 + G), \qquad j = 1, \ldots, n.$$

Assuming both systems are stable, under this condition, more strategic complementarity in states makes every eigenvalue of the equilibrium law of motion closer to zero, i.e. the equilibrium is more persistent.

The restriction is needed because $[B(\Gamma + \Theta_{\mathcal{A}})^{-1} B'] P - A$ is otherwise nonsymmetric, and a change in P need not translate into an ordering of the eigenvalues of the closed-loop matrix. A stronger restriction gives a closed-form characterization and permits comparative statics with respect to both interactions.

PROPOSITION 13. Suppose $A = aI_n$. Let λ_j denote the eigenvalues of $J + G$. Then they are

real and can be indexed so that

$$\lambda_j = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2} + a\right)^2 + \mu_j}, \tag{18}$$
$$\mu_j = \text{eig}_j\left((Q + \Theta_X)^{1/2} B(\Gamma + \Theta_{\mathcal{A}})^{-1} B'(Q + \Theta_X)^{1/2}\right),$$

for $j = 1, \ldots, n$. Consequently:

1. more strategic complementarity in states raises every λ_j;
2. more strategic complementarity in actions lowers every λ_j; and
3. the jth mode is stable if and only if $\mu_j + a(a + \rho) > 0$.

It is useful to keep the sequence of results in mind. In the general multidimensional model, stronger complementarity orders both the Riccati solution and the trace of the closed-loop matrix, and therefore gives a prediction for the rate of volume contraction. This allows us to characterize how strategic interactions affect aggregate persistence without assuming that the different dimensions of the problem are separable. What cannot be signed in general is the response of each individual eigenvalue. For state interactions, eigenvalue-by-eigenvalue comparative statics are obtained when $[B(\Gamma+\Theta_{\mathcal{A}})^{-1}B']^{-1}A$ is symmetric. Under the stronger restriction $A = aI_n$, the same conclusion holds for both state and action interactions, and the eigenvalues are available in closed form.

Importantly, the last result remains multidimensional. The matrices $Q + \Theta_X$ and $B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'$ need not commute or be simultaneously diagonalizable, so the model does not reduce to a collection of independent one-dimensional decisions. The restriction $A = aI_n$ is also natural in the multiproduct price-setting example. There, A captures trend cost inflation, so common inflation across products gives $A = aI_n$, while demand and adjustment costs can continue to link prices across products.

5 Planner's Problem

The utilitarian planner chooses $\alpha(x, t)$ for every state and date, taking into account how these choices affect the aggregate state and aggregate action. The planner's period objective is

$$\mathcal{U}(m, \alpha) = -\tfrac{1}{2}\int x'Qx\, m(x)dx - X'\Theta_X X - \tfrac{1}{2}\int \alpha(x)'\Gamma\alpha(x)m(x)dx - \mathcal{A}'\Theta_{\mathcal{A}}\mathcal{A},$$

where $X = \int x\, m(x)dx$ and $\mathcal{A} = \int \alpha(x)m(x)dx$. The planner maximizes the discounted integral of this objective subject to the Kolmogorov forward equation. The planner internalizes both sides of each interaction: changing an individual state or action affects the agent's own payoff and, through the aggregate, the payoffs of other agents. With symmetric interaction matrices, this doubles the interaction terms in the first-order conditions.

Let $\lambda(x,t)$ be the multiplier on the Kolmogorov forward equation. The functional derivative of the state interaction is

$$-X'(\Theta_X' + \Theta_X)x,$$

while the derivative of the aggregate-action interaction replaces $\Theta_{\mathcal{A}}$ in the individual Hamiltonian with $\Theta_{\mathcal{A}}' + \Theta_{\mathcal{A}}$. Define the planner's Hamiltonian by

$$H^*(p,x,\mathcal{A}) = \max_{\alpha\in\mathbb{R}^k} \left\{ -\tfrac{1}{2}\alpha'\Gamma\alpha - \mathcal{A}'(\Theta_{\mathcal{A}}' + \Theta_{\mathcal{A}})\alpha + p'(B\alpha - Ax) \right\}.$$

Its maximizer is

$$\alpha^*(p,\mathcal{A}) = \Gamma^{-1}\left[B'p - (\Theta_{\mathcal{A}}' + \Theta_{\mathcal{A}})\mathcal{A}\right].$$

Proposition 14. The solution of the utilitarian planner problem is characterized by the following coupled partial differential equations for all $t \geq 0$ and $x \in \mathbb{R}^n$:

$$\begin{aligned} \rho\lambda(x,t) = &-\tfrac{1}{2}x'Qx - X(t)'(\Theta_X' + \Theta_X)x + H^*(\lambda_x(x,t),x,\mathcal{A}(t)) \\ &+ \tfrac{1}{2}\operatorname{tr}(\Sigma\lambda_{xx}(x,t)) + \lambda_t(x,t), \\ m_t(x,t) = &-\operatorname{div}\big(H_p^*(\lambda_x(x,t),x,\mathcal{A}(t))m(x,t)\big) + \tfrac{1}{2}\operatorname{tr}(\Sigma m_{xx}(x,t)). \end{aligned}$$

These equations are coupled through

$$X(t) = \int x\, m(x,t)dx, \qquad \mathcal{A}(t) = \int \alpha^*(\lambda_x(x,t),\mathcal{A}(t))m(x,t)dx,$$

for all $t \geq 0$, with $m(x,0) = m_0(x)$ for all $x \in \mathbb{R}^n$, and with the corresponding boundary and transversality conditions in Appendix A.1.

Proposition 15. Suppose Θ_X and $\Theta_{\mathcal{A}}$ are symmetric. The planner's HJB–KFE system for interaction matrices $(\Theta_X, \Theta_{\mathcal{A}})$ is identical to the decentralized equilibrium system for interaction matrices $(2\Theta_X, 2\Theta_{\mathcal{A}})$. Hence, whenever the corresponding allocation exists, the planner's allocation coincides with that decentralized equilibrium. If $\Gamma + 2\Theta_{\mathcal{A}}$ is invertible,

the planner's Riccati equation is

$$P^*B(\Gamma + 2\Theta_{\mathcal{A}})^{-1}B'P^* = Q + 2\Theta_X + \rho P^* + P^*A + A'P^*. \tag{19}$$

Under the conditions of Lemma 2, doubling the state interaction raises the trace of the closed-loop matrix when states are strategic complements, while doubling the action interaction lowers it when actions are strategic complements. Since the planner doubles both interaction matrices, these two effects work in opposite directions, and the comparison between the planner and decentralized rates of volume contraction is generally ambiguous when both interactions are present. If only one interaction is present, stronger conclusions are possible under the additional conditions discussed above. For state interactions, Proposition 12 gives an eigenvalue-by-eigenvalue comparison. For action interactions, the corresponding result follows under the stronger restriction in Proposition 13.

The Kimball example gives an additional implication. The superelasticity cancels from $Q + \Theta_X$ and therefore does not affect decentralized aggregate price dynamics, but it does not cancel from the planner's effective curvature $Q + 2\Theta_X$. Thus, a parameter that has no effect on decentralized aggregate persistence can still affect the planner's aggregate dynamics and the gap between the planner and decentralized allocation.

6 MFG with Aggregate Shocks and the Master Equation

We now add a common shock. Let $N^+(t)$ and $N^-(t)$ be independent Poisson processes with intensity $\kappa/2$, and define $\mathcal{J}(t) = N^+(t) - N^-(t)$. The individual state follows

$$dx(t) = (B\alpha(t) - Ax(t))dt + \Sigma^{1/2}dW(t) + \Upsilon d\mathcal{J}(t),$$

where $\Upsilon \in \mathbb{R}^n$. Because the common shock shifts every individual's state by the same amount, it also shifts the aggregate state by Υ. Thus, we conjecture, and verify below, that aggregate states and actions satisfy

$$dX(t) = MX(t)dt + \Upsilon d\mathcal{J}(t), \qquad \mathcal{A}(t) = CX(t), \tag{20}$$

for constant matrices M and C. In this section, C denotes the coefficient that maps the aggregate state into the aggregate action. The aggregate state must now enter the recursive

problem because it follows its own stochastic law of motion and affects the agent's current payoff. Given (M, C), the recursive value function has arguments (x, X) and solves

$$\begin{aligned}\rho v(x,X) = &-\tfrac{1}{2}x'Qx - X'\Theta_X x + H(v_x(x,X), x, CX) + \tfrac{1}{2}\operatorname{tr}(\Sigma v_{xx}(x,X)) \\ &+ v_X(x,X)'MX + \kappa\left[\tfrac{1}{2}v(x+\Upsilon, X+\Upsilon) + \tfrac{1}{2}v(x-\Upsilon, X-\Upsilon) - v(x,X)\right]. \quad (21)\end{aligned}$$

The corresponding stochastic KFE shifts the density by Υ at each common jump. Integrating that equation against x gives the aggregate law of motion in equation (20).

We use the quadratic conjecture

$$v(x,X) = S_0 + x'S_1X + \tfrac{1}{2}x'S_2x + \tfrac{1}{2}X'S_3X, \quad (22)$$

where S_2 and S_3 are symmetric. For given (M, C), matching coefficients in equation (21) gives

$$\rho S_0 = \tfrac{1}{2}\operatorname{tr}(\Sigma S_2) + \kappa\Upsilon'\left(S_1 + \tfrac{1}{2}S_2 + \tfrac{1}{2}S_3\right)\Upsilon, \quad (23)$$

$$\rho S_1 = -\Theta_X + S_2B\Gamma^{-1}B'S_1 - S_2B\Gamma^{-1}\Theta_{\mathcal{A}}C - A'S_1 + S_1M, \quad (24)$$

$$\rho S_2 = -Q + S_2B\Gamma^{-1}B'S_2 - (S_2A + A'S_2), \quad (25)$$

$$\begin{aligned}\tfrac{\rho}{2}S_3 = &\tfrac{1}{2}S_1'B\Gamma^{-1}B'S_1 - \tfrac{1}{2}\left[S_1'B\Gamma^{-1}\Theta_{\mathcal{A}}C + C'\Theta_{\mathcal{A}}\Gamma^{-1}B'S_1\right] \\ &+ \tfrac{1}{2}C'\Theta_{\mathcal{A}}\Gamma^{-1}\Theta_{\mathcal{A}}C + \tfrac{1}{2}(S_3M + M'S_3). \quad (26)\end{aligned}$$

The system is recursive for given (M, C): first solve for S_2, then S_1, S_3, and S_0. In particular, the equation for S_2, which determines the curvature of the value function in the individual state, does not depend on the common shock.

Proposition 16. Suppose $\Gamma + \Theta_{\mathcal{A}}$ is invertible. In an equilibrium with common noise, let $P \equiv S_1 + S_2$. The equilibrium condition for aggregate actions and aggregation of the drift imply

$$(\Gamma + \Theta_{\mathcal{A}})C = B'P, \qquad M = BC - A. \quad (27)$$

Thus,

$$C = (\Gamma + \Theta_{\mathcal{A}})^{-1}B'P, \qquad M = B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'P - A. \quad (28)$$

Moreover, $S_2 = \bar{\beta}_2$, and P solves equation (15), the same Riccati equation as without common noise.

This is the relevant certainty-equivalence result. Unlike the formulation without aggregate shocks, the recursive value function must include both x and X as states. The common shock changes the realized path of the aggregate state, but not the matrices governing optimal

adjustment around that path. Nevertheless, the matrix that governs aggregate persistence is unchanged.

7 Recursive Equilibrium and Identification

The common-noise formulation gives a convenient recursive representation. Throughout this section, we assume that $\Gamma + \Theta_{\mathcal{A}}$ is invertible and that Σ is nonsingular. Define the deviation from the cross-sectional mean by

$$z(t) = x(t) - X(t).$$

Using equation (28), the equilibrium can be written as

$$dX(t) = \left(\left[B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'\right]P - A\right)X(t)dt + \Upsilon d\mathcal{G}(t), \tag{29}$$

$$dz(t) = (B\Gamma^{-1}B'\bar{\beta}_2 - A)z(t)dt + \Sigma^{1/2}dW(t), \tag{30}$$

$$\alpha(t) = (\Gamma + \Theta_{\mathcal{A}})^{-1}B'PX(t) + \Gamma^{-1}B'\bar{\beta}_2 z(t), \tag{31}$$

$$\mathcal{A}(t) = (\Gamma + \Theta_{\mathcal{A}})^{-1}B'PX(t). \tag{32}$$

These equations separate individual from aggregate adjustment. Deviations from the cross-sectional mean are governed by the single-agent solution $\bar{\beta}_2$, while aggregate dynamics are governed by the equilibrium solution P. The common jump cancels from the law of motion of z, and, more importantly, neither interaction matrix enters the dynamics of z or of actions relative to their cross-sectional average. The reason is that strategic interactions operate through the aggregate state and aggregate action, which disappear when individual variables are expressed relative to their cross-sectional averages. Thus, the difference between individual and aggregate adjustment reflects equilibrium feedback, and estimates of how quickly individuals close gaps relative to others need not reveal how quickly the aggregate adjusts.

To make the link with observed data explicit, let

$$U_{\dot{X},X} \equiv \left[B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'\right]P - A, \qquad U_{\dot{z},z} \equiv B\Gamma^{-1}B'\bar{\beta}_2 - A,$$
$$U_{\alpha,X} \equiv (\Gamma + \Theta_{\mathcal{A}})^{-1}B'P, \qquad U_{\alpha,z} \equiv \Gamma^{-1}B'\bar{\beta}_2.$$

Thus, $U_{\dot{X},X}$ and $U_{\dot{z},z}$ are the conditional-drift coefficients in equation (29) and equation (30), while $U_{\alpha,X}$ and $U_{\alpha,z}$ are the policy coefficients in equation (32) and equation (31), respectively.

Proposition 17. A data set containing paths of $x(t) - X(t)$ and $\alpha(t) - \mathcal{A}(t)$ has no

information about Θ_X or $\Theta_{\mathcal{A}}$.

Such data identify $U_{\dot{z},z}$ and $U_{\alpha,z}$, neither of which depends on the interaction matrices. Thus, removing time effects from both states and actions removes information about strategic interactions while retaining information about individual adjustment. This is the analogue of the missing-intercept problem in this setting, applying to both state and action coupling.

The next results show what can be recovered once one of the interaction matrices is normalized. We begin by taking the parameters of the individual problem as known and then progressively relax this requirement.

PROPOSITION 18. Suppose $\Theta_{\mathcal{A}}$, Γ, ρ, Q, A, and B are known, and $B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'$ is invertible. A time series for $X(t)$ identifies Θ_X.

The time series identifies $U_{\dot{X},X}$. Since the remaining objects are known, this coefficient identifies P, and equation (15) then identifies Θ_X. The proof is constructive and shows how Θ_X can be recovered from the reduced-form drift coefficient.

We can relax the assumption that Q and A are known if aggregate actions and individual deviations are also observed.

PROPOSITION 19. Suppose $\Theta_{\mathcal{A}}$, Γ, ρ, and B are known. Time series for $X(t)$, $z(t)$, and $\mathcal{A}(t)$ identify Q, Θ_X, and A.

The three time series identify $U_{\dot{X},X}$, $U_{\dot{z},z}$, and $U_{\alpha,X}$. The aggregate state and action equations identify A and P, while the demeaned state equation identifies $\bar{\beta}_2$. The single-agent and equilibrium Riccati equations then identify Q and Θ_X.

Finally, we can also allow B to be unknown. This requires observing individual actions and a rank condition that allows B to be recovered from the difference between aggregate and individual dynamics.

PROPOSITION 20. Suppose $\Theta_{\mathcal{A}}$, Γ, and ρ are known. Suppose also that $n = k$ and $U_{\alpha,z} - U_{\alpha,X}$ is nonsingular. Time series for $X(t)$, $z(t)$, $\alpha(t)$, and $\mathcal{A}(t)$ identify Q, Θ_X, A, and B.

The four time series identify the four reduced-form matrices above. Their differences identify B; the aggregate equation then identifies A and P, while the demeaned equation identifies $\bar{\beta}_2$. Equations equation (11) and equation (15) finally identify Q and Θ_X.

The normalization of $\Theta_{\mathcal{A}}$ in these propositions is essential. Even with aggregate states and actions, the data reveal $U_{\alpha,X}$ and $U_{\dot{X},X}$, not $\Theta_{\mathcal{A}}$ and P separately. For any admissible value of $\Theta_{\mathcal{A}}$, one can recover a corresponding P from the observed policy coefficient and then choose Θ_X to satisfy equation (15). Hence Θ_X and $\Theta_{\mathcal{A}}$ are not separately identified without an additional normalization or outside information.

8 Coupling with Higher Moments

In this section, we extend the MFG so that the coupling in the agent's period return can depend on the first $\bar{r}$ moments of the cross-sectional distribution, rather than only on its mean. Distributions with the same mean can then generate different incentives to adjust, and idiosyncratic risk can affect aggregate persistence. For simplicity, we assume that the state is one-dimensional, so $x \in \mathbb{R}$ and $n = 1$, and abstract from interactions through aggregate actions. We denote the rth moment of the distribution $m(\cdot, t)$ by $M_r(t)$:

$$M_r(t) \equiv \int x^r \, m(x,t) dx \text{ for all } t \geq 0 \text{ and } r = 1, 2, \ldots, \bar{r}$$

In this case the period return function is given by

$$F(x,m) + R(\alpha) \equiv -\tfrac{1}{2} x \, q \, x - \left(\theta_0 + \sum_{r=1}^{\bar{r}} M_r(t) \theta_r \right) x - \tfrac{1}{2} \alpha' \, \Gamma \, \alpha$$
$$\text{where } M_r \equiv \int x^r \, m(x) \, dx$$

Three comments are in order. First, we have also added a linear term to the period return, namely the product of x with the scalar θ_0. We do so in order to choose θ_0 so that the cross-sectional mean of x is zero, as in the benchmark case. Second, since $X(t) = M_1(t)$, the benchmark case is obtained by setting $\bar{r} = 1$ and $\theta_0 = 0$. Third, because the state is one-dimensional, the corresponding coefficients q, a, and γ_B are scalars.

$$\frac{\partial^2 F(x,m)}{\partial x \partial M_r(t)} = -\theta_r \text{ for } r = 1, 2, \ldots, \bar{r}$$

A possible definition of strategic complementarity (substitutability) is that, if the measure associated with the density m_a is stochastically higher than the one associated with the density m_b, then $F(x, m_a) - F(x, m_b)$ is weakly increasing (decreasing) in x. Under this definition and our specification of F, strategic complementarity (substitutability) implies $\theta_r \leq 0$ (≥ 0) for $r = 1, 3, 5, \ldots$. This definition places no restriction on θ_r for $r = 2, 4, 6, \ldots$.

An equilibrium for the mean field game is now defined as:

DEFINITION 2. An equilibrium for the MFG given m_0 is a value function $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$,

a density $m : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\bar{r}$ moments $M_r : \mathbb{R}_+ \to \mathbb{R}$ for $r = 1, \ldots, \bar{r}$ satisfying:

$$\rho u(x,t) = -\tfrac{1}{2} x\, q\, x - \left(\sum_{r=1}^{\bar{r}} M_r(t)\theta_r + \theta_0 \right) x + H(u_x(x,t), x) + \tfrac{1}{2}\sigma^2 u_{xx}(x,t) + u_t(x,t)$$
$$m_t(x,t) = -\left(H_p(u_x(x,t), x)\, m(x,t)\right)_x + \frac{1}{2}\sigma^2\, m_{xx}(x,t)$$

for all $x \in \mathbb{R}$ and $t \geq 0$ where

$$H(p,x) = \max_{\alpha \in \mathbb{R}^k} -\tfrac{1}{2}\,\alpha'\,\Gamma\alpha + p\,(B\alpha - ax) \text{ for all } (p,x) \in \mathbb{R}^2$$
$$M_r(t) = \int x^r\, m(x,t)\, dx \text{ for all } t \geq 0 \text{ and } r = 1, \ldots, \bar{r}$$

with $m(x,0) = m_0(x)$ and appropriate boundary conditions for m and u.

We can specialize the definition of an equilibrium for a steady state as follows:

Definition 3. A steady state is given by moments $\bar{M}_r \in \mathbb{R}$ for $r = 1, \ldots, \bar{r}$ and two functions $\bar{u} : \mathbb{R} \to \mathbb{R}$ and $\bar{m} : \mathbb{R} \to \mathbb{R}_+$ satisfying:

$$\rho \bar{u}(x) = -\tfrac{1}{2} x\, q\, x - \left(\sum_{r=1}^{\bar{r}} \bar{M}_r \theta_r + \theta_0 \right) x + H(\bar{u}_x, x) + \tfrac{1}{2}\sigma^2 \bar{u}_{xx}(x)$$
$$0 = -\left(H_p(\bar{u}_x(x), x)\bar{m}(x)\right)_x + \tfrac{1}{2}\sigma^2\, \bar{m}_{xx}(x)$$

for all x and the vector of moments $\bar{M}$ satisfies

$$\bar{M}_r = \int x^r\, \bar{m}(x)\, dx \text{ for } r = 1, \ldots, \bar{r}$$

We now turn to the analysis of the equilibrium. The value function must still be of the same form:

$$u(x,t) = \beta_0(t) + \beta_1(t)x + \tfrac{1}{2}\beta_2(t)x^2$$

We follow the same steps as before and emphasize the parts of the analysis that differ from the benchmark case with $\bar{r} = 1$.

Hamiltonian. In this case, the first-order condition for the Hamiltonian gives $\alpha^* = \Gamma^{-1}B'p$. Substituting α^* back into the Hamiltonian and differentiating with respect to p gives

$$H(p,x) = \frac{\gamma_B^{-1}}{2}p^2 - pax \text{ and } H_p(p,x) = \gamma_B^{-1}p - ax \text{ where } \gamma_B \equiv 1/(B\Gamma^{-1}B')$$

Following the same steps as in the benchmark case we get that the Hamiltonian and its derivative evaluated at the derivative of the value function are:

$$\begin{aligned} H(u_x(x,t),x) &= \frac{\gamma_B^{-1}}{2}(\beta_1(t) + \beta_2(t)x)^2 - ax(\beta_1(t) + \beta_2(t)x) \\ H_p(u_x(x,t),x) &= \gamma_B^{-1}(\beta_1(t) + \beta_2(t)x) - ax \end{aligned}$$

Dynamical System for the Coefficients of the Value Function. We can now proceed as before. Matching coefficients on the different powers of x in the HJB equation gives the following system of o.d.e.'s:

$$\begin{aligned} \dot{\beta}_0 &= \rho\beta_0 - \tfrac{1}{2}\beta_1'\gamma_B^{-1}\beta_1 - \tfrac{1}{2}\sigma^2\beta_2 \\ \dot{\beta}_1 &= (\rho + a)\,\beta_1 + \theta_0 + \sum_{r=1}^{\bar{r}}\theta_r M_r - \beta_2\gamma_B^{-1}\beta_1 \\ \dot{\beta}_2 &= q - \beta_2\gamma_B^{-1}\beta_2 + \beta_2 a + a\beta_2 + \rho\beta_2 \end{aligned}$$

As before, these o.d.e.'s take as given the path for $M_1, \ldots, M_{\bar{r}}$. Again we must have $\beta_2(t) = \bar{\beta}_2 < 0$, i.e. the same negative constant for all $t \geq 0$, and the same negative constant as in the benchmark case. In this scalar case the solution of the Riccati equation $\bar{\beta}_2$ is simply the negative root of a scalar quadratic equation:

$$\gamma_B^{-1}\bar{\beta}_2 - a = \tfrac{\rho}{2} - \sqrt{(a + \rho/2)^2 + \gamma_B^{-1}q}$$

Dynamics of Moments. We can extend the previous analysis to obtain an o.d.e. for each of the $\bar{r}$ moments. The useful feature of this representation is that the moments satisfy a relatively simple finite-dimensional system, rather than requiring us to solve directly for the entire density m.

PROPOSITION 21. Given the expression for the Hamiltonian, and the definition of the moments $M_r(t)$, the solution of the Kolmogorov forward equation implies the following $\bar{r}$

dimensional system of linear o.d.e.s

$$\dot{M}_r(t) = r\left(\gamma_B^{-1}\bar{\beta}_2 - a\right) M_r(t) + r\gamma_B^{-1}\beta_1(t)M_{r-1}(t) + \frac{r(r-1)\sigma^2}{2}M_{r-2}(t) \tag{33}$$

for $r = 1, \ldots, \bar{r}$ and $t \geq 0$, for a given path β_1 and given initial conditions $M_r(0)$ for $r = 1, \ldots, \bar{r}$.

Next, we combine the o.d.e. for β_1 with those for the $\bar{r}$ moments. Once we replace $\beta_2(t)$ by the Riccati solution $\bar{\beta}_2$, we obtain a system of o.d.e.'s for the scalar β_1 and the vector $(M_1, \ldots, M_{\bar{r}})$.

State and Costate System. We can combine the costate-type variable β_1 with the $\bar{r}$ aggregate states $M_1, \ldots, M_{\bar{r}}$ to obtain a system of $\bar{r}+1$ nonlinear ordinary differential equations:

$$\begin{aligned}
\dot{\beta}_1(t) &= \left(\rho + a - \bar{\beta}_2\gamma_B^{-1}\right)\beta_1(t) + \theta_0 + \sum_{r=1}^{\bar{r}}\theta_r M_r \\
\dot{M}_r(t) &= r\left(\gamma_B^{-1}\bar{\beta}_2 - a\right) M_r(t) + r\gamma_B^{-1}\beta_1(t)M_{r-1}(t) + \frac{r(r-1)\sigma^2}{2}M_{r-2}(t)
\end{aligned} \tag{34}$$

The equations for $\dot{\beta}_1$ and $\dot{M}_1$ are linear in $(\beta_1, M_1, \ldots, M_{\bar{r}})$. For $r = 2, \ldots, \bar{r}$, however, the equation for $\dot{M}_r$ contains the product $\beta_1 M_{r-1}$ and is therefore nonlinear.

Steady States. The steady state of the system for $(\beta_1, M_1, \ldots, M_{\bar{r}})$ is, in general, different from zero. There are two reasons. First, the even moments M_r are nonzero. Second, the constant θ_0 enters F. Given $\bar{\beta}_2$, the steady state $(\bar{\beta}_1, \bar{M}_1, \ldots, \bar{M}_{\bar{r}})$ solves

$$\begin{aligned}
0 &= \left(\rho + a - \bar{\beta}_2\gamma_B^{-1}\right)\bar{\beta}_1 + \theta_0 + \sum_{r=1}^{\bar{r}}\theta_r \bar{M}_r \\
0 &= r\left(\gamma_B^{-1}\bar{\beta}_2 - a\right)\bar{M}_r + r\gamma_B^{-1}\bar{\beta}_1\bar{M}_{r-1} + \frac{r(r-1)\sigma^2}{2}\bar{M}_{r-2}
\end{aligned}$$

This system is non-linear, so the existence and uniqueness of a steady state should be analyzed. Also, not every solution to this set of equations gives a steady state, since the $\bar{M}_r$ are moments, and thus they have to satisfy the Hamburger inequality restrictions. To ensure they are satisfied we can write this system in two steps. In the first step we solve for $\bar{M} = (\bar{M}_1, \ldots, \bar{M}_{\bar{r}})$ as a function of $\bar{\beta}_1$. Given $\bar{\beta}_1$ this is a linear system with recursive structure given by:

$$\bar{M}_r = \frac{1}{a - \gamma_B^{-1}\bar{\beta}_2}\left[\gamma_B^{-1}\bar{\beta}_1\bar{M}_{r-1} + \frac{(r-1)\sigma^2}{2}\bar{M}_{r-2}\right] \text{ for } r = 1, 2\ldots, \bar{r}$$

with $\bar{M}_0 = 1$. We let $\{\mathcal{m}_r(\bar{\beta}_1)\}_{r=1}^{\bar{r}}$ denote the unique solution of these linear equations. In the second step we replace this solution into the equation for $\bar{\beta}_1$, defining

$$\mathcal{B}(\bar{\beta}_1) \equiv \frac{1}{\bar{\beta}_2\gamma_B^{-1} - a - \rho}\left[\theta_0 + \sum_{r=1}^{\bar{r}}\theta_r \mathcal{m}_r(\bar{\beta}_1)\right]$$

Thus, a steady state is a fixed point $\bar{\beta}_1 = \mathcal{B}(\bar{\beta}_1)$.

This operator can, in general, have several fixed points. To see this, consider the example of $\bar{r} = 2$. In this case the function $\mathcal{B}$ is quadratic in $\bar{\beta}_1$. First, solve:

$$\bar{M}_1 = \frac{\gamma_B^{-1}\bar{\beta}_1}{a - \gamma_B^{-1}\bar{\beta}_2} \text{ and } \bar{M}_2 = \frac{\left(\gamma_B^{-1}\bar{\beta}_1\right)^2}{(a - \gamma_B^{-1}\bar{\beta}_2)^2} + \frac{\sigma^2/2}{a - \gamma_B^{-1}\bar{\beta}_2}$$

Replacing them back we get:

$$\mathcal{B}(\bar{\beta}_1) = \frac{1}{\bar{\beta}_2\gamma_B^{-1} - a - \rho}\left[\theta_0 + \theta_1\frac{\gamma_B^{-1}\bar{\beta}_1}{a - \gamma_B^{-1}\bar{\beta}_2} + \theta_2\frac{\left(\gamma_B^{-1}\bar{\beta}_1\right)^2}{(a - \gamma_B^{-1}\bar{\beta}_2)^2} + \theta_2\frac{\sigma^2/2}{a - \gamma_B^{-1}\bar{\beta}_2}\right]$$

The fixed-point condition is quadratic in $\bar{\beta}_1$, so it can have up to two real solutions.

We are interested in steady states with $\bar{M}_1 = 0$, since we are interpreting x to be a deviation from steady state, it should have mean zero. The steady state expression for $\bar{M}_1$ implies that $\bar{M}_1 = 0$ if and only if $\bar{\beta}_1 = 0$. This requirement has several implications, summarized in the next proposition.

PROPOSITION 22. Consider the model with coupling up to $\bar{r}$ moments, where $\bar{r}$ is even. Suppose $\sigma^2 > 0$, and $a - \gamma_B^{-1}\bar{\beta}_2 > 0$. At any steady state, $\bar{M}_1 = 0$ if and only if $\bar{\beta}_1 = 0$. Conditional on this restriction, the steady state is unique: $\bar{M}_r = 0$ for $r = 1, 3, \ldots, \bar{r} - 1$, while the even moments satisfy

$$\bar{M}_r = \frac{(r-1)\sigma^2}{2\left(a - \gamma_B^{-1}\bar{\beta}_2\right)}\bar{M}_{r-2} = (r-1)!!\left(\frac{\sigma^2}{2\left(a - \gamma_B^{-1}\bar{\beta}_2\right)}\right)^{r/2}, \qquad r = 2, 4, 6, \ldots, \bar{r}. \tag{35}$$

Such a steady state exists if and only if θ_0 is set to

$$\theta_0 = -\sum_{r=2,4,6,\ldots}^{\bar{r}} \theta_r \bar{M}_r, \tag{36}$$

where !! denotes the double factorial.

Four comments are in order. First, the invariant distribution $\bar{m}$ of x is normal and independent of the vector θ. Second, as an example, if $\bar{r} = 2$, then

$$\theta_0 = -\theta_2 \frac{\sigma^2/2}{a - \gamma_B^{-1}\bar{\beta}_2}$$

and there is a steady state with $\bar{\beta}_1 = \bar{M}_1 = 0$. Third, even with these parameter values, there may be other steady states. Fourth, the MFG we consider here, with coupling through higher moments, does not satisfy certainty equivalence, and hence the invariant distribution $\bar{m}$ depends on the variance of the idiosyncratic shock σ^2. To maximize comparability, we choose θ_0 so that the zero-mean invariant distribution is the same as in the MFG with coupling only through the first moment.

Local Analysis around the "Zero Mean" Invariant Distribution We now use Proposition 22 and focus on the invariant distribution with $\bar{\beta}_1 = \bar{M}_r = 0$ for odd r. The dynamical system equation (34)–equation (33) for $(\beta_1, M_1, \ldots, M_{\bar{r}})$ is nonlinear. To study its behavior around the steady state of interest, we define the deviations

$$\hat{\beta}_1(t) = \beta_1(t) - \bar{\beta}_1 \text{ and } \hat{M}_r(t) = M_r(t) - \bar{M}_r, \text{ for all } t \geq 0 \text{ and } r = 1, \ldots, \bar{r}$$

The linearized system around the zero-mean steady state is

$$\frac{d}{dt}\hat{\beta}_1(t) = \left(\rho + a - \bar{\beta}_2\gamma_B^{-1}\right)\hat{\beta}_1(t) + \sum_{r=1}^{\bar{r}} \theta_r \hat{M}_r(t) \tag{37}$$

$$\frac{d}{dt}\hat{M}_r(t) = r\gamma_B^{-1}\bar{M}_{r-1}\hat{\beta}_1(t) + r\left(\gamma_B^{-1}\bar{\beta}_2 - a\right)\hat{M}_r(t) + \frac{r(r-1)\sigma^2}{2}\hat{M}_{r-2}(t) \tag{38}$$

For the main result of this section, we rewrite the state-costate system in equation (37) and equation (38) as

$$\dot{Y}(t) = \widetilde{\mathscr{H}}\, Y(t) \text{ where } Y \equiv \left(\hat{\beta}_1, \hat{M}_1, \hat{M}_2, \ldots, \hat{M}_{\bar{r}}\right)' \tag{39}$$

so that Y is an $(\bar{r}+1)$-dimensional column vector and $\widetilde{\mathscr{H}}$ is an $(\bar{r}+1)\times(\bar{r}+1)$ real matrix, whose

coefficients are described by equation (37) and equation (38). Without loss of generality, we take $\bar{r}$ to be even.

The existence and uniqueness of an equilibrium around the zero-mean invariant distribution depend on the eigenvalues of $\widetilde{\mathcal{H}}$. We denote these eigenvalues by $\widetilde{\lambda}_j$. If this matrix has one unstable (positive) eigenvalue and $\bar{r}$ stable (negative) eigenvalues, we say that the zero-mean invariant distribution is locally determinate. In this case, the speed of convergence along the equilibrium path is determined by the $\bar{r}$ stable eigenvalues. The next proposition shows that these eigenvalues have a particularly simple structure.

PROPOSITION 23. Fix $\bar{r}$ as an arbitrary even integer. Let $\{\widetilde{\lambda}_r\}_{r=1}^{\bar{r}+1}$ be the eigenvalues of the matrix $\widetilde{\mathcal{H}}$ associated with the linearization in equation 39 around the zero-mean invariant distribution. Define $\bar{\theta} \equiv \sum_{\substack{1 \leq r \leq \bar{r}-1 \\ r \text{ odd}}} r \bar{M}_{r-1} \theta_r$. Then

$$\widetilde{\lambda}_1 = \frac{\rho}{2} - \sqrt{\left(a + \frac{\rho}{2}\right)^2 + \gamma_B^{-1}(q + \bar{\theta})}\,, \widetilde{\lambda}_{\bar{r}+1} = \frac{\rho}{2} + \sqrt{\left(a + \frac{\rho}{2}\right)^2 + \gamma_B^{-1}(q + \bar{\theta})}\,, \text{ and}$$
$$\widetilde{\lambda}_r = r\left[\frac{\rho}{2} - \sqrt{\left(a + \frac{\rho}{2}\right)^2 + \gamma_B^{-1} q}\right] \text{ for } r = 2, \ldots, \bar{r}.$$

Moreover, the path of the even moment deviations $\widehat{M}_r(t)$, $r = 2, 4, \ldots, \bar{r}$, is independent of $\{\theta_r\}_{r=1}^{\bar{r}}$ and of the initial conditions for the odd moment deviations. If $q + \bar{\theta} + \gamma_B a(a + \rho) > 0$ and $b \equiv a - \gamma_B^{-1}\bar{\beta}_2 > 0$, then the zero-mean invariant distribution is locally determinate:

$$\widetilde{\lambda}_1 < 0 < \widetilde{\lambda}_{\bar{r}+1}, \qquad \widetilde{\lambda}_r < 0, \quad r = 2, \ldots, \bar{r}.$$

Several comments are in order.

1. The coefficients $\{\theta_r\}_{r=1}^{\bar{r}}$ enter only two eigenvalues: $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_{\bar{r}+1}$.
2. Moreover, only the coefficients $\{\theta_1, \theta_3, \ldots, \theta_{\bar{r}-1}\}$ on the odd moments enter $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_{\bar{r}+1}$.
3. The formula for $\widetilde{\lambda}_1$ is identical to that for λ_1 in the scalar case with $n = 1$, with $\bar{\theta}$ taking the place of the scalar θ.
4. The interpretation of $\bar{\theta}$ is the same as in the scalar case: an increase in any θ_r for odd r makes convergence to the steady state faster.
5. There is a critical value $-\bar{\theta}^* \equiv q + a(a + \rho)\gamma_B$ such that if $-\bar{\theta} > -\bar{\theta}^*$, then there is no locally determinate equilibrium; that is, for those values of $\bar{\theta}$, $\widetilde{\lambda}_1 > 0$.

6. If $\bar{\theta} = 0$, then all the stable eigenvalues satisfy $\widetilde{\lambda}_r = r\widetilde{\lambda}_1 < 0$ for $r = 1, 2, \ldots, \bar{r}$. If there is strategic complementarity, so that $\bar{\theta} < 0$, then $\widetilde{\lambda}_1 > \widetilde{\lambda}_r$ for all $r = 2, 3, \ldots, \bar{r}$. Thus, with strategic complementarity, $\widetilde{\lambda}_1$ is the dominant stable eigenvalue.

7. If $\sigma^2 = 0$, then $\bar{\theta} = \theta_1$. If all $\theta_r < 0$ (> 0) for odd r, then $\bar{\theta}$ is decreasing (increasing) in σ^2. Thus, with strategic complementarity (substitutability), convergence to equilibrium is slower (faster) when the idiosyncratic variance σ^2 is higher.

8. Although only the coefficients on the odd moments $\{\theta_1, \theta_3, \ldots, \theta_{\bar{r}-1}\}$ enter the eigenvalues, the coefficients on the even moments affect the equilibrium, as the case $\bar{r} = 2$ below illustrates.

The case $\bar{r} = 2$ illustrates how an interaction through an even moment can affect transition dynamics even though it does not affect the local eigenvalues. We turn to the complete solution for the case $\bar{r} = 2$. Assume $\hat{\beta}_1(t) = s_1\hat{M}_1(t) + s_2\hat{M}_2(t)$. Substituting this expression for $\hat{\beta}_1$ into equation (37) and equation (38) for $r = 1$ and $r = 2$, and matching coefficients on $\hat{M}_1(t)$ and $\hat{M}_2(t)$, gives

$$\begin{aligned}
\gamma_B^{-1}\theta_1 &= \gamma_B^{-1}s_1\left(-\rho + 2\left(\gamma_B^{-1}\bar{\beta}_2 - a\right) + \gamma_B^{-1}s_1\right) \text{ and} \\
\gamma_B^{-1}\theta_2 &= \gamma_B^{-1}s_2\left(-\rho + 3\left(\gamma_B^{-1}\bar{\beta}_2 - a\right) + \gamma_B^{-1}s_1\right)
\end{aligned}$$

Substituting these expressions, together with the expression for $\bar{\beta}_2$, back into the system gives the equilibrium laws of motion for the first and second moments:

$$\begin{aligned}
&\frac{d}{dt}\hat{M}_2(t) = \tilde{\lambda}_2\hat{M}_2(t) \text{ and} \\
&\frac{d}{dt}\hat{M}_1(t) = \tilde{\lambda}_1\hat{M}_1(t) + \frac{\gamma_B^{-1}\theta_2}{\frac{\rho}{2} - 2\sqrt{(a+\rho/2)^2 + \gamma_B^{-1}q} - \sqrt{(a+\rho/2)^2 + \gamma_B^{-1}(\theta_1 + q)}}\hat{M}_2(t)
\end{aligned}$$

Thus, around the zero-mean steady state, the second moment converges at the rate given by $\widetilde{\lambda}_2$, which is independent of $\{\theta_1, \theta_2\}$. Nevertheless, while the second moment is away from its steady-state value, it affects the control α and, through it, the first moment, as shown by the second term in the equilibrium o.d.e. for $\hat{M}_1$.

9 Conclusion

We study how strategic interactions shape aggregate dynamics in linear-quadratic mean field games with interactions through both states and actions. We show that equilibrium can be characterized using the same Riccati structure as the corresponding single-agent problem,

with strategic interactions changing the effective curvature of states and actions. This representation gives a tractable characterization of existence and uniqueness and separates the dynamics inherited from individual optimization from the additional feedback generated by strategic interactions.

Our main results characterize how these interactions affect aggregate persistence. State and action complementarities have opposite effects: stronger state complementarity increases persistence, while stronger action complementarity reduces it. Importantly, these results extend to multidimensional environments without requiring the different dimensions of the problem to be separable or the underlying matrices to be simultaneously diagonalizable. The framework can therefore be used in a broad range of macroeconomic settings in which individual decisions interact through aggregate states or actions, as we illustrate with applications to capital accumulation and multiproduct price setting.

We also study the planner's allocation. Because the planner internalizes the effects of each agent's choices on others, the effects of strategic interactions differ from those in the decentralized equilibrium. Internalizing state complementarity increases persistence, while internalizing action complementarity reduces it, so the two forces can work in opposite directions when both interactions are present.

Finally, we study what can be learned about strategic interactions from individual and aggregate data. Demeaned states and actions contain no information about strategic interactions, giving rise to a missing-intercept problem. Combining individual and aggregate dynamics, we give conditions under which strategic interactions and other model parameters can be identified. We also extend the framework to interactions through an arbitrary finite number of moments of the cross-sectional distribution, allowing idiosyncratic risk to affect aggregate persistence.

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ONLINE APPENDIX

A Model Details

A.1 Boundary Conditions

We assume that m_0 has a finite moment of order $r > 2$ and seek solutions for which the same is true of $m(\cdot, t)$ at every date. In particular,

$$\begin{aligned}
1 &= \int m(x,t)dx, \\
0 &= \lim_{x_i \to +\infty} m(x,t) = \lim_{x_i \to -\infty} m(x,t) = \lim_{x_i \to +\infty} m_{x_j}(x,t) = \lim_{x_i \to -\infty} m_{x_j}(x,t), \\
0 &= \lim_{x_i \to +\infty} x_i m(x,t) = \lim_{x_i \to -\infty} x_i m(x,t),
\end{aligned}$$

for every i, j and $t \geq 0$. These conditions justify the integrations by parts used to aggregate the Kolmogorov forward equation. For the value function we impose the transversality condition

$$\lim_{t \to \infty} e^{-\rho t} \mathbb{E}\left[u(x(t), t) \mid x(0) = x\right] = 0.$$

We also restrict attention to aggregate paths satisfying

$$\int_0^\infty e^{-\rho t} \|X(t)\|^2 dt < \infty, \qquad \int_0^\infty e^{-\rho t} \|\mathcal{A}(t)\|^2 dt < \infty.$$

The same conditions, with the natural stochastic interpretation, are imposed in the common-noise model.

A.2 Riccati Equations: MFG versus the Linear Regulator

The algebraic Riccati equation for a standard infinite-horizon regulator can be written as

$$0 = \mathbf{Q} + \mathbf{A}'\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}\mathbf{B}'\mathbf{X},$$

where the stabilizing solution makes $\mathbf{A} - \mathbf{B}\mathbf{B}'\mathbf{X}$ stable. The MFG equation (15) is obtained from this equation under the correspondence

$$\mathbf{Q} = Q + \Theta_X, \qquad \mathbf{A} = -A - \frac{\rho}{2} I_n, \qquad \mathbf{B}\mathbf{B}' = \left[B(\Gamma + \Theta_{\mathcal{A}})^{-1} B'\right], \qquad \mathbf{X} = -P.$$

Indeed, substitution gives

$$0 = Q + \Theta_X + \rho P + PA + A'P - P\left[B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'\right]P.$$

The stabilizing regulator solution $\mathbf{X} \succ 0$ therefore corresponds to the negative definite solution $P \prec 0$ for which the eigenvalues of $[B(\Gamma + \Theta_{\mathcal{A}})^{-1}B']\,P - A$ have real parts below $\rho/2$. The single-agent equation (11) has the same correspondence with $[B(\Gamma + \Theta_{\mathcal{A}})^{-1}B']$ replaced by $B\Gamma^{-1}B'$ and $Q + \Theta_X$ replaced by Q.

This mapping also makes transparent the role of the interaction matrices. The state interaction changes the regulator's state curvature from Q to $Q+\Theta_X$. The action interaction changes its control matrix from $B\Gamma^{-1}B'$ to $B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'$. Thus, both interactions enter the MFG Riccati equation through objects that have direct analogues in the single-agent regulator.

A.3 Second-Order Approximation of the Capital-Accumulation Example

Let $\bar{i} = \delta\bar{k}$ and write $\bar{\Pi} = \Pi(\bar{k}, \bar{k})$. Terms in the return that depend only on the aggregate variables can be omitted from the firm's problem. The second-order expansion of profits is therefore

$$\begin{aligned}\Pi(k, K) = \Pi_k(\bar{k}, \bar{k})(k - \bar{k}) + \frac{1}{2}\Pi_{kk}(\bar{k}, \bar{k})(k - \bar{k})^2 \\ + \Pi_{kK}(\bar{k}, \bar{k})(k - \bar{k})(K - \bar{k}) + \omega_\Pi(K) + o\big(\|(k - \bar{k}, K - \bar{k})\|^2\big)\,,\end{aligned}$$

where ω_Π does not affect the firm's choice. Likewise,

$$\begin{aligned}i\mathscr{P}(I) &= \mathscr{P}(\bar{i})(i - \bar{i}) + \mathscr{P}'(\bar{i})(i - \bar{i})(I - \bar{i}) + \omega_i(I) + o\big(\|(i - \bar{i}, I - \bar{i})\|^2\big)\,, \\ \psi(i) &= \psi'(\bar{i})(i - \bar{i}) + \frac{1}{2}\psi''(\bar{i})(i - \bar{i})^2 + \omega_\psi + o\big((i - \bar{i})^2\big)\,.\end{aligned}$$

The terms that are linear in the firm's state and action determine the steady-state level and the linear term in the value function. As in a standard linear-quadratic approximation, translating the problem into deviations from the deterministic steady state eliminates these terms from the problem expressed in deviations.

Using

$$x = \frac{k - \bar{k}}{\bar{k}}, \quad X = \frac{K - \bar{k}}{\bar{k}}, \quad \alpha = \frac{i - \bar{i}}{\bar{i}}, \quad \mathcal{A} = \frac{I - \bar{i}}{\bar{i}},$$

and dividing flow returns by $\bar{\Pi}$, the quadratic part of the return is

$$\frac{1}{2}\frac{\bar{k}^2\Pi_{kk}(\bar{k},\bar{k})}{\bar{\Pi}}x^2 + \frac{\bar{k}^2\Pi_{kK}(\bar{k},\bar{k})}{\bar{\Pi}}xX \\ -\frac{1}{2}\frac{\delta^2\bar{k}^2\psi''(\delta\bar{k})}{\bar{\Pi}}\alpha^2 - \frac{\delta^2\bar{k}^2\mathscr{P}'(\delta\bar{k})}{\bar{\Pi}}\alpha\mathscr{A}.$$

This gives the coefficients stated in Section 3.1. The state equation becomes, to first order,

$$dx = \delta(\alpha - x)dt + \sigma dW.$$

For $\Pi(k,K) = k^{\nu(1-1/\eta)}K^{\nu/\eta}$, direct differentiation gives

$$\begin{aligned} -\frac{\bar{k}^2\Pi_{kk}}{\bar{\Pi}} &= \nu\frac{\eta-1}{\eta^2}\left[\eta(1-\nu)+\nu\right], \\ -\frac{\bar{k}^2\Pi_{kK}}{\bar{\Pi}} &= -\frac{\eta-1}{\eta^2}\nu^2, \\ -\frac{\bar{k}^2(\Pi_{kk}+\Pi_{kK})}{\bar{\Pi}} &= \frac{\eta-1}{\eta}\nu(1-\nu). \end{aligned}$$

Hence $q > 0$, $\theta_X < 0$, and $q + \theta_X > 0$. If $\mathscr{P}'(\delta\bar{k}) > 0$, then $\theta_{\mathscr{A}} > 0$.

B Proofs

For compactness within the proofs in this appendix, write

$$D \equiv B(\Gamma + \Theta_{\mathscr{A}})^{-1}B'.$$

Proof. (of Proposition 1) Write $Q = C'C$. Since $Q \succ 0$, C has full column rank. Hence no nonzero eigenvector of $-A-\rho I_n/2$ can belong to the null space of C, which gives detectability.

Because B has full row rank, so does $B\Gamma^{-1/2}$. Let R be a right inverse, $B\Gamma^{-1/2}R = I_n$. For any $c > 0$, choose

$$L = R\left(A + \frac{\rho}{2}I_n - cI_n\right).$$

Then

$$B\Gamma^{-1/2}L - A - \frac{\rho}{2}I_n = -cI_n.$$

Thus $(B\Gamma^{-1/2}, -A - \rho I_n/2)$ is stabilizable. The standard infinite-horizon regulator result then gives the unique stabilizing quadratic value function for the uncoupled problem.

Proof. (of Proposition 2) For given paths of X and $\mathscr{A}$, the quadratic coefficient of the

HJB is independent of both paths. Matching the terms that are quadratic in x gives equation (10). This is the Riccati differential equation of the uncoupled single-agent regulator. Under Assumptions AM, detectability and stabilizability imply that its infinite-horizon solution is the unique negative definite stabilizing solution of equation (11). The infinite-horizon transversality condition therefore requires $\beta_2(t) = \bar{\beta}_2$ at every date. □.

Proof. of Proposition 3. We replace $\mathcal{A}(t)$ from equation (12) into equation (9). This gives us

$$\begin{aligned}\dot{\beta}_1(t) = &\left(\rho I_n + A' - \bar{\beta}_2 B \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\right) \beta_1(t) \\ &+ \left[\bar{\beta}_2 \left(B\Gamma^{-1}B' - B \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\right) \bar{\beta}_2 + \Theta_X\right] X(t)\end{aligned}$$

$$\begin{aligned}\dot{\beta}_1(t) &= \left(\rho I_n + A' - \bar{\beta}_2 B\Gamma^{-1}B'\right) \beta_1(t) + \Theta_X X(t) \\ &+ \bar{\beta}_2 B\Gamma^{-1}\Theta_{\mathcal{A}} \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B' \left(\beta_1(t) + \bar{\beta}_2 X(t)\right) \\ &= \left(\rho I_n + A' - \bar{\beta}_2 B\Gamma^{-1}B' + \bar{\beta}_2 B\Gamma^{-1}\Theta_{\mathcal{A}} \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\right) \beta_1(t) \\ &+ \left[\bar{\beta}_2 B\Gamma^{-1}\Theta_{\mathcal{A}} \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\bar{\beta}_2 + \Theta_X\right] X(t) \\ &= \left(\rho I_n + A' - \bar{\beta}_2 B \left(\Gamma^{-1} - \Gamma^{-1}\Theta_{\mathcal{A}} \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1}\right) B'\right) \beta_1(t) \\ &+ \left[\bar{\beta}_2 B\Gamma^{-1}\Theta_{\mathcal{A}} \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\bar{\beta}_2 + \Theta_X\right] X(t) \\ &= \left(\rho I_n + A' - \bar{\beta}_2 B \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\right) \beta_1(t) \\ &+ \left[\bar{\beta}_2 \left(B\Gamma^{-1}B' - B \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B'\right) \bar{\beta}_2 + \Theta_X\right] X(t)\end{aligned}$$

The last equality uses that

$$\Gamma^{-1} - \Gamma^{-1}\Theta_{\mathcal{A}} \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} = \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1}$$

□

Proof. (of Proposition 4) The proof proceeds in several steps.

1. **Law of motion for X_i:** We substitute the p.d.e. for m_t into the definition of $X_i(t)$ to write an equation for the time derivative of X_i, denoted by $\dot{X}_i(t)$, in terms of integrals of m and $H\left(u_x, t\right) \equiv H(u_x(x,t), x, \mathcal{A}(t))$ to simplify the notation.

$$\begin{aligned}\dot{X}_i(t) &= \int \cdots \int x_i \, m_t(x,t) \, dx_1 \ldots dx_n \\ &= -\int \cdots \int x_i \sum_{j=1}^{n} \left(H_{p_j}\left(u_x, t\right) m(x,t) \right)_{x_j} \, dx_1 \ldots dx_n \\ &+ \tfrac{1}{2} \int \cdots \int x_i \sum_k \sum_j \sigma_{kj} m_{x_j x_k}\left(x,t\right) \, dx_1 \ldots dx_n\end{aligned}$$

2. **No contribution from the Laplacian into $\dot{X}$.** We show that the term on $\dot{X}_i$ with integrals of second derivatives of m is zero, i.e. we show that $0 = \int x_i \sum_{k,j} \sigma_{kj} m_{x_k x_j}\left(x,t\right) \, dx$. For this we use the assumed boundary conditions of m. Consider

$$A^i \equiv \int \cdots \int x_i \sum_k \sum_j \sigma_{kj} m_{x_k x_j}\left(x,t\right) \, dx_1 \ldots dx_n = \sum_k \sum_j A^i_{kj}$$

where for an arbitrary combination k, j we have

$$A^i_{kj} \equiv \sigma_{kj} \underbrace{\int \cdots \int}_{n-1} \left[\int x_i \, m_{x_k x_j}\left(x,t\right) \, dx_k \right] dx_1 \ldots dx_{k-1} dx_{k+1} \ldots dx_n$$

We will argue that $A^i_{kj} = 0$. We first analyze the integral inside the braces, namely $\int x_i \, m_{x_k x_j}\left(x,t\right) \, dx_k$. For this we consider two cases. First, assume that $i \neq k$. In this case we write:

$$\begin{aligned}&\int x_i \, m_{x_k x_j}\left(x,t\right) \, dx_k = x_i \int \frac{\partial}{\partial x_k} m_{x_j}\left(x,t\right) \, dx_k \\ &= x_i \left(m_{x_j}\left(x_1, \ldots, +\infty, \ldots, x_n, t\right) - m_{x_j}\left(x_1, \ldots, -\infty, \ldots, x_n, t\right) \right) \\ &= 0\end{aligned}$$

where the equality follows from the assumed boundary conditions on m. If $k = i$ we use integration by parts:

$$\begin{aligned}&\int x_i m_{x_i x_j}\left(x,t\right) \, dx_i = \int x_i \frac{\partial}{\partial x_i} m_{x_j}\left(x,t\right) \, dx_i \\ =&x_i m_{x_j}\left(x_1, \ldots, x_i, \ldots, x_n, t\right) |_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} m_{x_j}\left(x_1, \ldots, x_i, \ldots, x_n, t\right) \, dx_i \\ =&-\int_{-\infty}^{\infty} \frac{\partial}{\partial x_j} m\left(x_1, \ldots, x_i, \ldots, x_n, t\right) \, dx_i\end{aligned}$$

where the first term is zero from the assumed boundary conditions on m. Thus we have:

$$
\begin{aligned}
A_{ij}^{i} \equiv & \sigma_{ij} \underbrace{\int \cdots \int}_{n-1} \left[\int x_i \, m_{x_i x_j}(x,t) \, dx_i \right] dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \\
= & - \sigma_{ij} \underbrace{\int \cdots \int}_{n-1} \left[\int \frac{\partial}{\partial x_j} m(x_1, \ldots, x_i, \ldots, x_n, t) \, dx_i \right] dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \\
= & - \sigma_{ij} \int \cdots \int \frac{\partial}{\partial x_j} m(x_1, \ldots, x_n, t) \, dx_1 \ldots dx_n \\
= & - \sigma_{ij} \underbrace{\int \cdots \int}_{n-1} \left[\int \frac{\partial}{\partial x_j} m(x_1, \ldots, x_j, \ldots, x_n, t) \, dx_j \right] dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n \\
= & - \sigma_{ij} \underbrace{\int \cdots \int}_{n-1} \left[m(x_1, \ldots, +\infty, \ldots, x_n, t) - m(x_1, \ldots, -\infty, \ldots, x_n, t) \right] dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n \\
= & - \sigma_{ij} \underbrace{\int \cdots \int}_{n-1} 0 \; dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n \\
= & \, 0
\end{aligned}
$$

where we use the previous result, reorder the integral twice, and use the boundary conditions for m.

3. **Simplifying the expression for $\dot{X}_i$.** Using our previous results we obtain:

$$
\begin{aligned}
\dot{X}_i(t) &= - \int \cdots \int x_i \sum_{j=1}^{n} \left(H_{p_j}(u_x(x,t),t) \, m(x,t) \right)_{x_j} dx_1 \ldots dx_n \\
&= \int \cdots \int H_{p_i}(u_x(x,t),t) \, m(x,t) \, dx_1 \ldots dx_n
\end{aligned}
$$

We can also write it in vector form as:

$$
\dot{X}'(t) = \int \cdots \int H_p(u_x(x,t),t) \, m(x,t) \, dx_1 \ldots dx_n
$$

where X' and H_p are row vectors.

To see why this has to hold we write:

$$\begin{aligned}
\dot{X}_i(t) &= -\int\cdots\int x_i \sum_{j=1}^{n} \left(H_{p_j}\left(u_x(x,t),t\right) m(x,t)\right)_{x_j} dx_1 \ldots dx_n \\
&= -\int\cdots\int x_i \left(H_{p_i}\left(u_x(x,t),t\right) m(x,t)\right)_{x_i} dx_1 \ldots dx_n \\
&- \int\cdots\int x_i \sum_{j=1, j\neq i}^{n} \left(H_{p_j}\left(u_x(x,t),t\right) m(x,t)\right)_{x_j} dx_1 \ldots dx_n \\
&= -\underbrace{\int\cdots\int}_{n-1} \left[\int x_i \left(H_{p_i}\left(u_x(x,t),t\right) m(x,t)\right)_{x_i} dx_i\right] dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \\
&- \int\cdots\int x_i \sum_{j\neq i} \left(H_{p_j}\left(u_x(x,t),t\right) m(x,t)\right)_{x_j} dx_1 \ldots dx_n \\
&= \int\cdots\int H_{p_i}\left(u_x(x,t),t\right) m(x,t)\, dx_1 \ldots dx_n \\
&- \int\cdots\int x_i \sum_{j\neq i} \left(H_{p_j}\left(u_x(x,t),t\right) m(x,t)\right)_{x_j} dx_1 \ldots dx_n
\end{aligned}$$

where we have used integration by parts, and the boundary conditions. Fix a $j \neq i$ and write

$$\begin{aligned}
&- \int\cdots\int x_i \left(H_{p_j}\left(u_x(x,t),t\right) m(x,t)\right)_{x_j} dx_1 \ldots dx_n \\
&= -\underbrace{\int\cdots\int}_{n-1} x_i \left[\int \left(H_{p_j}\left(u_x(x,t),t\right) m(x,t)\right)_{x_j} dx_j\right] dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n \\
&= -\underbrace{\int\cdots\int}_{n-1} x_i \left[H_{p_j}\left(u_x(x,t),t\right) m(x,t)|_{x_j=-\infty}^{x_j=+\infty}\right] dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n \\
&= 0
\end{aligned}$$

where the last term follows from the boundary conditions on m and from the fact that $H_p(u_x, t)$ is linear in x.

4. Derivative of Hamiltonian on $\dot{X}$. The next step is to substitute the expression we have for the derivative of the Hamiltonian in terms of β's into the law of motion for $\dot{X}$.

$$\dot{X}'(t) = \int\cdots\int H_p\left(u_x(x,t), x, \mathcal{A}(t)\right) m(x,t)\, dx_1 \ldots dx_n$$

Using:

$$H_p(u_x(x,t),x,\mathcal{A}(t)) = \left[\left(\beta_1'(t) + x'\bar{\beta}_2\right) B - \mathcal{A}'\Theta_{\mathcal{A}}\right]\Gamma^{-1}B' - x'A'$$

$$\begin{aligned}
\dot{X}'(t) &= \int\cdots\int H_p\left(u_x(x,t),t\right) m(x,t)\, dx_1 \ldots dx_n \\
&= \int\cdots\int \left(\left(\left[\beta_1'(t) + x'\bar{\beta}_2\right] B - \mathcal{A}(t)'\Theta_{\mathcal{A}}\right)\Gamma^{-1}B' - x'A'\right) m(x,t)\, dx_1 \ldots dx_n \\
&= \int\cdots\int \left[\beta_1'(t) + x'\bar{\beta}_2\right] m(x,t)\, dx_1 \ldots dx_n\, B\Gamma^{-1}B' \\
&- \int\cdots\int \mathcal{A}(t)'\Theta_{\mathcal{A}}\Gamma^{-1}B' m(x,t)\, dx_1 \ldots dx_n \\
&- \int\cdots\int x' m(x,t)\, A'\, dx_1 \ldots dx_n
\end{aligned}$$

We now use that $X = \int x m(x,t) dx$, and that $\int m(x,t) dx = 1$ so that:

$$\begin{aligned}
\dot{X}'(t) &= \int \left[\beta_1'(t) + x'\bar{\beta}_2\right] m(x,t)\, dx\, B\Gamma^{-1}B' - \mathcal{A}(t)'\Theta_{\mathcal{A}}\Gamma^{-1}B' - \int x' m(x,t) dx\, A' \\
&= \beta_1'(t)B\Gamma^{-1}B' - \mathcal{A}(t)'\Theta_{\mathcal{A}}\Gamma^{-1}B' + X(t)'\left(\bar{\beta}_2 B\Gamma^{-1}B' - A'\right)
\end{aligned}$$

Transposing this expression we get:

$$\dot{X}(t) = B\Gamma^{-1}B'\beta_1(t) + \left(B\Gamma^{-1}B'\bar{\beta}_2 - A\right) X(t) - B\Gamma^{-1}\Theta_{\mathcal{A}}\mathcal{A}(t)$$

5. Now we are ready to obtain our final expression. Using the aggregate optimal action obtained above,

$$\mathcal{A}(t) = \left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1} B' \left(\beta_1(t) + \bar{\beta}_2 X(t)\right)$$

and substitute it into the transpose of the expression we obtain in the previous step:

$$\dot{X}(t) = B\Gamma^{-1}B'\beta_1(t) + \left(B\Gamma^{-1}B'\bar{\beta}_2 - A\right) X(t) - B\Gamma^{-1}\Theta_{\mathcal{A}}\mathcal{A}(t)$$

Then we obtain:

$$\begin{aligned}\dot{X}(t) &= B\Gamma^{-1}B'\beta_1(t) + \left(B\Gamma^{-1}B'\bar{\beta}_2 - A\right)X(t) \\ &\quad - B\Gamma^{-1}\Theta_{\mathcal{A}}\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1}B'\left(\beta_1(t) + \bar{\beta}_2 X(t)\right) \\ &= B\Gamma^{-1}\left[I_n - \Theta_{\mathcal{A}}\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1}\right]B'\left(\beta_1(t) + \bar{\beta}_2 X(t)\right) - AX(t) \\ &= B\left(\Gamma + \Theta_{\mathcal{A}}\right)^{-1}B'\left(\beta_1(t) + \bar{\beta}_2 X(t)\right) - AX(t)\end{aligned}$$

which is the desired expression.

□

Proof. (of Proposition 5) Subtracting $\rho I_{2n}/2$ from $\mathcal{H}$ in equation (13), its upper-left block is minus the transpose of its lower-right block, while both off-diagonal blocks are symmetric. Hence $\mathcal{H} - \rho I_{2n}/2$ is Hamiltonian. The spectrum of a real Hamiltonian matrix is invariant under multiplication by -1 and under complex conjugation. Re-centering the spectrum at $\rho/2$ gives the four eigenvalues stated in the proposition.

□

Proof. (of Proposition 8) Suppose P^0 and P^1 are two symmetric solutions of equation (15) whose closed-loop eigenvalues have real parts below $\rho/2$. Let $\Delta = P^1 - P^0$ and define

$$\widetilde{\mathcal{G}}^i = DP^i - A - \frac{\rho}{2}I_n, \qquad i = 0, 1.$$

Subtracting the two Riccati equations and collecting terms gives

$$(\widetilde{\mathcal{G}}^1)'\Delta + \Delta\widetilde{\mathcal{G}}^0 = 0.$$

Both matrices in this Sylvester equation are stable. Hence the equation has the unique solution $\Delta = 0$. Therefore $P^1 = P^0$, and the associated saddle path and aggregate law of motion are unique.

□

Proof. (of Proposition 9) Let $\widetilde{Q} = Q + \Theta_X$ and $C = A + \rho I_n/2$. The Riccati equation can be written as

$$PDP - PC - C'P = \widetilde{Q}.$$

Completing the square gives

$$\left(P - C'D^{-1}\right)D\left(P - D^{-1}C\right) = \widetilde{Q} + C'D^{-1}C = E.$$

The left side is of the form $L'DL$ and is positive semi-definite. Thus $E \succeq 0$ is necessary.

If $\widetilde{Q} \succ 0$ and $D \succ 0$, use the regulator correspondence in Appendix A.2. The associated regulator is detectable because $\widetilde{Q}$ is positive definite and stabilizable because D is positive definite. It therefore has a unique positive definite stabilizing solution. Under the correspondence $\mathbf{X} = -P$, this gives a unique negative definite solution of equation (15) for which $DP - A - \rho I_n/2$ is stable.

In the scalar case, write $D > 0$, $\widetilde{q} = q + \theta_X$, and $A = a$. The two roots are

$$p = \frac{\rho + 2a \ \pm \sqrt{(\rho + 2a)^2 + 4D\widetilde{q}}}{2D}.$$

The root with the minus sign is admissible exactly when the discriminant is strictly positive. This is equivalent to

$$\widetilde{q} + \frac{(a + \rho/2)^2}{D} > 0,$$

which is the strict scalar version of the necessary condition. □

Proof. (of Proposition 10) Consider differentiating the expression for P with respect to a small change in Θ_X and $\Theta_{\mathcal{A}}$. We will denote the derivatives with respect to these changes with a dot. Differentiating equation (15) with respect to the changes in Θ_X and $\Theta_{\mathcal{A}}$ gives

$$\dot{P}\,\widetilde{\mathcal{G}} + \widetilde{\mathcal{G}}'\dot{P} = \dot{\Theta}_X - P\dot{D}P, \qquad \widetilde{\mathcal{G}} = DP - A - \frac{\rho}{2}I_n.$$

Since $\widetilde{\mathcal{G}}$ is stable, the Sylvester equation has the solution:

$$\dot{P} = -\int_0^\infty e^{\widetilde{\mathcal{G}}'s}\left(\dot{\Theta}_X - P\dot{D}P\right)e^{\widetilde{\mathcal{G}}s}ds. \tag{40}$$

If only the state interaction changes and strategic complementarity increases, then $\dot{\Theta}_X \preceq 0$ and $\dot{D} = 0$, so $\dot{P} \succeq 0$.

If only the action interaction changes, then

$$\dot{D} = -B(\Gamma + \Theta_{\mathcal{A}})^{-1}\dot{\Theta}_{\mathcal{A}}(\Gamma + \Theta_{\mathcal{A}})^{-1}B'.$$

Thus $\dot{\Theta}_{\mathcal{A}} \preceq 0$ implies $\dot{D} \succeq 0$. In that case $\dot{\Theta}_X = 0$ and $-P\dot{D}P \preceq 0$, again implying $\dot{P} \succeq 0$. Integrating along a path between the two interaction matrices gives the two orderings in the proposition.

□

Proof. (of Lemma 2) We prove a lemma for the differentiable version of the comparative

static, which implies the discrete one by integration. Let $\Theta_X(\epsilon) = \Theta_X(0) + \epsilon\dot{\Theta}_X$ and $\Theta_{\mathcal{A}}(\epsilon) = \Theta_{\mathcal{A}}(0) + \epsilon\dot{\Theta}_{\mathcal{A}}$, where ϵ is a small positive scalar. Assume that $\dot{\Theta}_X$ is negative definite, so that $\Theta_X(\epsilon)$ has more strategic complementarity than $\Theta_X(0)$, and $\dot{\Theta}_{\mathcal{A}}$ is positive definite, so that $\Theta_{\mathcal{A}}(\epsilon)$ has more strategic substitutability than $\Theta_{\mathcal{A}}(0)$. Then $\operatorname{tr}(J+G)(\epsilon) > \operatorname{tr}(J+G)(0)$.

We will differentiate the Riccati equation and the corresponding closed-loop matrix with respect to changes in Θ_X and $\Theta_{\mathcal{A}}$, which we denote with a dot. Differentiate the closed-loop matrix $J + G \equiv M = DP - A$ and take the trace

$$\operatorname{tr}\dot{M} = \operatorname{tr}(\dot{D}P) + \operatorname{tr}(D\dot{P}),$$

since A does not change. First we will consider the term $tr(D\dot{P})$. Using the expression for $\dot{P}$ in the proof of Proposition 10 in equation (40),

$$\dot{P} = -\int_0^\infty e^{\widetilde{\mathcal{G}}'s}\left(\dot{\Theta}_X - P\dot{D}P\right)e^{\widetilde{\mathcal{G}}s}ds.$$

Premultiplying by D gives

$$D\dot{P} = -\int_0^\infty De^{\widetilde{\mathcal{G}}'s}\left(\dot{\Theta}_X - P\dot{D}P\right)e^{\widetilde{\mathcal{G}}s}ds.$$

Taking the trace,

$$\operatorname{tr}(D\dot{P}) = -\int_0^\infty \operatorname{tr}\left[De^{\widetilde{\mathcal{G}}'s}\left(\dot{\Theta}_X - P\dot{D}P\right)e^{\widetilde{\mathcal{G}}s}\right]ds.$$

By cyclicity of the trace,

$$\operatorname{tr}\left[De^{\widetilde{\mathcal{G}}'s}\left(\dot{\Theta}_X - P\dot{D}P\right)e^{\widetilde{\mathcal{G}}s}\right] = \operatorname{tr}\left[e^{\widetilde{\mathcal{G}}s}De^{\widetilde{\mathcal{G}}'s}\left(\dot{\Theta}_X - P\dot{D}P\right)\right].$$

Therefore,

$$\begin{aligned}\operatorname{tr}(D\dot{P}) &= -\int_0^\infty \operatorname{tr}\left[\left(\dot{\Theta}_X - P\dot{D}P\right)e^{\widetilde{\mathcal{G}}s}De^{\widetilde{\mathcal{G}}'s}\right]ds = -\operatorname{tr}\left[\int_0^\infty\left(\dot{\Theta}_X - P\dot{D}P\right)e^{\widetilde{\mathcal{G}}s}De^{\widetilde{\mathcal{G}}'s}ds\right] \\ &= -\operatorname{tr}\left[\left(\dot{\Theta}_X - P\dot{D}P\right)\int_0^\infty e^{\widetilde{\mathcal{G}}s}De^{\widetilde{\mathcal{G}}'s}ds\right]\end{aligned}$$

Let the matrix W be

$$W = \int_0^\infty e^{\widetilde{\mathcal{G}}s}De^{\widetilde{\mathcal{G}}'s}ds,$$

which is the solution of the Sylvester equation

$$\widetilde{\mathcal{G}}W + W\widetilde{\mathcal{G}}' = -D.$$

Using the definition of W,

$$\operatorname{tr}(D\dot{P}) = -\operatorname{tr}\left[\left(\dot{\Theta}_X - P\dot{D}P\right)W\right] = -\operatorname{tr}\left[W\left(\dot{\Theta}_X - P\dot{D}P\right)\right],$$

and hence

$$\operatorname{tr}(D\dot{P}) = -\operatorname{tr}(W\dot{\Theta}_X) + \operatorname{tr}(WP\dot{D}P).$$

Adding the remaining term and using the cyclicality of the trace, we obtain

$$\begin{aligned}\operatorname{tr}\dot{M} &= -\operatorname{tr}(W\dot{\Theta}_X) + \operatorname{tr}(WP\dot{D}P) + \operatorname{tr}(\dot{D}P) = -\operatorname{tr}(W\dot{\Theta}_X) + \operatorname{tr}(\dot{D}PWP) + \operatorname{tr}(\dot{D}P)\\ &= -\operatorname{tr}(W\dot{\Theta}_X) + \operatorname{tr}\left(\dot{D}(P + PWP)\right)\end{aligned}$$

The first term gives the comparative static of $\operatorname{tr}\dot{M}$ with respect to $\dot{\Theta}_X$. The second term gives the comparative static of $\operatorname{tr}\dot{M}$ with respect to $\dot{\Theta}_{\mathcal{A}}$. Each of the two is the trace of the product of symmetric matrices. The first term has matrix W which is positive definite, given that D is positive definite. To establish the sign of each of the two terms, we first work with the matrix in the second one. Let $V = -P^{-1} \succ 0$. Multiplying the Riccati equation on the left and right by P^{-1} and rearranging gives

$$\widetilde{\mathcal{G}}V + V\widetilde{\mathcal{G}}' = -D - V(Q + \Theta_X)V.$$

Subtracting the Lyapunov equation for W yields

$$\widetilde{\mathcal{G}}(V - W) + (V - W)\widetilde{\mathcal{G}}' = -V(Q + \Theta_X)V.$$

Because $\widetilde{\mathcal{G}}$ is stable and $Q + \Theta_X \succ 0$, $V - W \succ 0$. Hence

$$P + PWP = -P(V - W)P \prec 0.$$

Now we are ready to show the main result. Recall that the trace of the product of two (symmetric) positive definite matrices is positive, and likewise a positive times a negative definite is negative. If state complementarity increases, then $\dot{\Theta}_X \preceq 0$ and the first term is nonnegative. If action complementarity increases, then $\dot{D} \succeq 0$ and the second term is nonpositive. Strict changes give strict inequalities.

□

Proof. (of Proposition 12) The closed-loop matrix is similar to

$$D^{-1/2}(DP - A)D^{1/2} = D^{1/2}PD^{1/2} - D^{-1/2}AD^{1/2}.$$

The condition that $D^{-1}A$ is symmetric implies that $D^{-1/2}AD^{1/2}$ is symmetric. Thus the displayed matrix is symmetric. If $\Theta_X^1 \preceq \Theta_X^0$, Proposition 10 gives $P^1 - P^0 \succeq 0$, and hence

$$D^{1/2}(P^1 - P^0)D^{1/2} \succeq 0.$$

The monotonicity of the ordered eigenvalues of symmetric matrices in the Loewner order gives the conclusion.

□

Proof. (of Proposition 13) Let $\widetilde{Q} = Q + \Theta_X$ and define

$$T = D^{1/2}PD^{1/2}.$$

When $A = aI_n$, multiplying equation (15) by $D^{1/2}$ on both sides gives

$$T^2 - (\rho + 2a)T = D^{1/2}\widetilde{Q}D^{1/2}.$$

The right side is a polynomial in the symmetric matrix T, so the two matrices commute and can be diagonalized by the same orthogonal matrix. If μ_j is an eigenvalue of $D^{1/2}\widetilde{Q}D^{1/2}$, the corresponding eigenvalue t_j of T satisfies

$$t_j^2 - (\rho + 2a)t_j = \mu_j.$$

The stabilizing root is

$$t_j = a + \frac{\rho}{2} - \sqrt{\left(a + \frac{\rho}{2}\right)^2 + \mu_j}.$$

The matrices DP and T are similar, and the eigenvalues of $D^{1/2}\widetilde{Q}D^{1/2}$ are the same as those of $\widetilde{Q}^{1/2}D\widetilde{Q}^{1/2}$. Therefore $\lambda_j = t_j - a$ gives equation (18).

For fixed D, a decrease in $\widetilde{Q}$ lowers every μ_j and therefore raises every λ_j. For fixed $\widetilde{Q}$, a decrease in $\Theta_{\mathcal{A}}$ raises D, which raises every μ_j and lowers every λ_j. Finally, $\lambda_j < 0$ if and only if

$$\left(a + \frac{\rho}{2}\right)^2 + \mu_j > \left(\frac{\rho}{2}\right)^2,$$

which is equivalent to $\mu_j + a(a+\rho) > 0$.

□

Uniqueness of the Steady State. The Riccati representation can equivalently be written as the state–costate system

$$\frac{d}{dt}\begin{bmatrix} PX \\ X \end{bmatrix} = \begin{bmatrix} \rho I_n + A' & Q + \Theta_X \\ D & -A \end{bmatrix}\begin{bmatrix} PX \\ X \end{bmatrix}.$$

At a steady state, the second row implies $DPX = AX$. When D is invertible, this gives $PX = D^{-1}AX$. Substitution into the first row gives

$$\left[Q + \Theta_X + A'D^{-1}A + \rho D^{-1}A\right] X = 0.$$

Thus, a nonzero steady state requires

$$\det\left(Q + \Theta_X + A'D^{-1}A + \rho D^{-1}A\right) = 0.$$

Since this determinant is a nonzero polynomial in Q and Θ_X, this condition fails generically. Hence, generically, $X = 0$ is the unique steady state.

Scalar Case for $P = p$. In the scalar case, the Riccati equation is

$$Dp^2 = q + \theta_X + (\rho + 2a)p, \qquad D = \frac{b^2}{\gamma + \theta_{\mathcal{A}}}.$$

The admissible root is

$$p = \frac{\rho + 2a - \sqrt{(\rho+2a)^2 + 4D(q+\theta_X)}}{2D}.$$

Since $\lambda = Dp - a$, the lower root gives the aggregate eigenvalue reported in the main text. The condition $\lambda < \rho/2$ is equivalent to

$$q + \theta_X + \left(a + \frac{\rho}{2}\right)^2 \frac{\gamma + \theta_{\mathcal{A}}}{b^2} > 0,$$

while stability, $\lambda < 0$, is equivalent to

$$q + \theta_X + a(a+\rho)\frac{\gamma + \theta_{\mathcal{A}}}{b^2} > 0.$$

Together, these two thresholds generate the four regions described in the main text, distinguishing a stable equilibrium, the boundary case $\lambda = 0$, an equilibrium with $0 < \lambda < \rho/2$, and nonexistence.

Proof. (of Proposition 14). We start noticing that the minimization of the Lagrangian L with respect to λ is equivalent to the Kolmogorov Forward equation, hence we include that as part of the solution. The second step is to rewrite the part of the Lagrangian which involves the constraints by repeatedly integrating by parts. This will produce a linear expression on m, with the exception of the terms in $\mathcal{U}$.

We write the Lagrangian as

$$\begin{aligned} L(a, m, \lambda) = &\lim_{T\to\infty} \int_0^T e^{-\rho t}\mathcal{U}(m(t), \alpha(t))dt \\ &- \int_0^T \int e^{-\rho t}\lambda(x,t)\,\mathrm{div}\left(\left(B\alpha(x,t) - Ax\right)m(x,t)\right)dx\,dt \\ &+ \int_0^T \int e^{-\rho t}\lambda(x,t)\left(\tfrac{1}{2}tr\left(\Sigma m_{xx}(x,t)\right) - m_t(x,t)\right)dx\,dt. \end{aligned}$$

1. We start with the first term involving the divergence:

$$\begin{aligned} &- \int e^{-\rho t}\lambda(x,t)\mathrm{div}\left(\left(B\alpha(x,t) - Ax\right)m(x,t)\right)dx \\ &= - \int e^{-\rho t}\lambda(x,t)\sum_{i=1}^{n}\left(\left(\sum_{r=1}^{k} B_{i,r}\alpha_r(x,t) - \sum_{j=1}^{n} A_{ij}x_j\right)m(x,t)\right)_{x_i} dx. \end{aligned}$$

For each term in the sum of the divergence, we reorder the integral to consider:

$$\begin{aligned} &- \int \lambda(x,t)\left(\left(\sum_{r=1}^{k} B_{i,r}\alpha_r(x,t) - \sum_{j=1}^{n} A_{ij}x_j\right)m(x,t)\right)_{x_i} dx_i \\ &= \int \lambda_{x_i}(x,t)\left(\left(\sum_{r=1}^{k} B_{i,r}\alpha_r(x,t) - \sum_{j=1}^{n} A_{ij}x_j\right)m(x,t)\right)dx_i \\ &- \lambda(x,t)\left(\left(\sum_{r=1}^{k} B_{i,r}\alpha_r(x,t) - \sum_{j=1}^{n} A_{ij}x_j\right)m(x,t)\right)\Big|_{x_i=-\infty}^{x_i=+\infty}. \end{aligned}$$

which using the assumed integrability of λ and λ_x and the maintained assumptions

about the limits of m and m_{x_j}. Thus we have:

$$
\begin{aligned}
&-\int_0^T \int e^{-\rho t}\lambda(x,t)\,\mathrm{div}\left(\left(B\alpha(x,t)-Ax\right)m(x,t)\right)dx\,dt \\
&=\int_0^T \int e^{-\rho t}m(x,t)\sum_{i=1}^{n}\lambda_{x_i}(x,t)\left(\sum_{r=1}^{k}B_{i,r}\alpha_r(x,t)-\sum_{j=1}^{n}A_{ij}x_j\right)dx\,dt.
\end{aligned}
$$

2. For the second term, we focus on the inner integral

$$
\begin{aligned}
&\int \lambda(x,t)tr\left(\Sigma m_{xx}(x,t)\right)dx \\
&=\sum_k\sum_j \sigma_{kj}\int\cdots\int \lambda(x,t)m_{x_kx_j}(x,t)dx_1\ldots dx_n.
\end{aligned}
$$

Take the case of $j=k$ and concentrate on the dx_j integral:

$$
\begin{aligned}
&\int \lambda(x,t)m_{x_jx_j}(x,t)dx_j \\
&=-\int \lambda_{x_j}(x,t)m_{x_j}(x,t)dx_j+\lambda(x,t)m_{x_j}(x,t)\big|_{x_j=-\infty}^{x_j=+\infty} \\
&=\int \lambda_{x_jx_j}(x,t)m(x,t)dx_j+\lambda(x,t)m_{x_j}(x,t)\big|_{x_j=-\infty}^{x_j=+\infty}+\lambda_{x_j}(x,t)m(x,t)\big|_{x_j=-\infty}^{x_j=+\infty} \\
&=\int \lambda_{x_jx_j}(x,t)m(x,t)dx_j.
\end{aligned}
$$

where we eliminate the boundary terms under the assumption that $\lambda(x,t)$ and $\lambda_{x_j}(x,t)$ satisfy the corresponding growth conditions and under the maintained assumptions about the limits of m and m_{x_j}.

Now we consider the case where $j\neq k$ and change the order of the integrals so it reads:

$$
\begin{aligned}
&\int\int \lambda(x,t)m_{x_jx_k}(x,t)dx_jdx_k \\
&=-\int\int \lambda_{x_j}(x,t)m_{x_k}(x,t)dx_jdx_k+\int \lambda(x,t)m_{x_k}(x,t)\big|_{x_j=-\infty}^{x_j=+\infty}dx_k \\
&=-\int\int \lambda_{x_j}(x,t)m_{x_k}(x,t)dx_jdx_k.
\end{aligned}
$$

So we have:

$$
\begin{aligned}
&-\int\int \lambda_{x_j}(x,t)m_{x_k}(x,t)dx_j dx_k = -\int\int \lambda_{x_j}(x,t)m_{x_k}(x,t)dx_k dx_j \\
&= \int \left[\int \lambda_{x_j x_k}(x,t)m(x,t)dx_k - \lambda_{x_j}(x,t)m(x,t)\Big|_{x_k=-\infty}^{x_k=+\infty}\right] dx_j \\
&= \int \lambda_{x_j x_k}(x,t)m(x,t)dx_k.
\end{aligned}
$$

Thus we obtained:

$$
\begin{aligned}
&\int_0^T \int e^{-\rho t}\lambda(x,t)\tfrac{1}{2}tr\left(\Sigma m_{xx}(x,t)\right)dx\,dt \\
&= \int_0^T \int e^{-\rho t}m(x,t)\tfrac{1}{2}tr\left(\Sigma \lambda_{xx}(x,t)\right)dx\,dt.
\end{aligned}
$$

3. Third, the second part of the last term, we change the order of integration and write for the inner integral:

$$
\begin{aligned}
-\int_0^T \lambda(x,t)e^{-\rho t}m_t(x,t)dt = &\int_0^T e^{-\rho t}\left[\lambda_t(x,t) - \rho\lambda(x,t)\right]m(x,t)dt \\
&- e^{-\rho t}\lambda(x,t)m(x,t)\Big|_{t=0}^{t=T}.
\end{aligned}
$$

Let's define $L^*(m,\lambda) = \max_\alpha L(\alpha, m, \lambda)$ as the Lagrangian maximized with respect to α. We use the terms we computed above to rewrite the Lagrangian as:

$$
\begin{aligned}
&L^*(m,\lambda) = \lim_{T\to\infty}\int_0^T e^{-\rho t}\left[-\tfrac{1}{2}\int x'Qx\, m(x,t)dx - X(t)'\Theta_X X(t)\right]dt \\
&+ \max_\alpha \lim_{T\to\infty}\int_0^T e^{-\rho t}\left[-\tfrac{1}{2}\int \alpha'(x,t)\Gamma\alpha(x,t)m(x,t)dx - \mathcal{A}(t)'\Theta_{\mathcal{A}}\mathcal{A}(t)\right]dt \\
&+ \max_\alpha \lim_{T\to\infty}\int_0^T\int e^{-\rho t}\left[\sum_{i=1}^n \lambda_{x_i}(x,t)\left(\sum_{r=1}^k B_{i,r}\alpha_r(x,t) - \sum_{j=1}^n A_{ij}x_j\right)\right]m(x,t)dxdt \\
&+ \lim_{T\to\infty}\int_0^T\int e^{-\rho t}\left[\tfrac{1}{2}tr\left(\Sigma\lambda_{xx}(x,t)\right) - \rho\lambda(x,t) + \lambda_t(x,t)\right]m(x,t)dxdt \\
&- \lim_{T\to\infty}\int e^{-\rho t}\lambda(x,t)m(x,t)\Big|_{t=0}^{t=T}dx.
\end{aligned}
$$

The maximization with respect to $\alpha(x,t)$ now also takes into account that $\mathcal{A}(t) =$

$\int \alpha(x,t)m(x,t)dx$. The first order condition is

$$0 = -\Gamma\alpha(x,t) - (\Theta'_{\mathcal{A}} + \Theta_{\mathcal{A}})\mathcal{A}(t) + B'\lambda_x(x,t).$$

Thus, the optimal action is also the maximizer of the planner's Hamiltonian H^*.

We differentiate $L^*(m,\lambda)$ with respect to $m(x,t)$ obtaining:

$$\begin{aligned}\frac{\partial L^*(m,\lambda;x,t)}{\partial m} &= e^{-\rho t}\left[-\tfrac{1}{2}x'Qx - X(t)'(\Theta'_X + \Theta_X)x\right] \\ &+ \max_{\alpha(x,t)} e^{-\rho t}\left[-\tfrac{1}{2}\alpha'(x,t)\Gamma\alpha(x,t) - \mathcal{A}(t)'(\Theta'_{\mathcal{A}} + \Theta_{\mathcal{A}})\alpha(x,t)\right. \\ &\qquad \left. + \sum_{i=1}^{n}\lambda_{x_i}(x,t)\left(\sum_{r=1}^{k}B_{i,r}\alpha_r(x,t) - \sum_{j=1}^{n}A_{ij}x_j\right)\right] \\ &+ e^{-\rho t}\left[\tfrac{1}{2}tr\left(\Sigma\lambda_{xx}(x,t)\right) + \lambda_t(x,t) - \rho\lambda(x,t)\right]\end{aligned}$$

for all $0 < t < T$ and all $x \in \mathbb{R}^n$.

Note we can write:

$$\begin{aligned}H^*(\lambda_x(x,t),x,\mathcal{A}(t)) &= \max_{\alpha} -\tfrac{1}{2}\alpha'\Gamma\alpha - \mathcal{A}(t)'(\Theta'_{\mathcal{A}} + \Theta_{\mathcal{A}})\alpha \\ &\quad + \sum_{i=1}^{n}\lambda_{x_i}(x,t)\left(\sum_{r=1}^{k}B_{i,r}\alpha_r - \sum_{j=1}^{n}A_{ij}x_j\right) \\ &= \max_{\alpha} -\tfrac{1}{2}\alpha'\Gamma\alpha - \mathcal{A}(t)'(\Theta'_{\mathcal{A}} + \Theta_{\mathcal{A}})\alpha + \lambda'_x(x,t)(B\alpha - Ax).\end{aligned}$$

where $\lambda_x(x,t)$ is the n-dimensional vector of derivatives of λ. Thus we can write

$$\begin{aligned}0 &= \frac{\partial L^*(m,\lambda;x,t)}{\partial m} \text{ as equivalent to} \\ \rho\lambda(x,t) &= -\tfrac{1}{2}x'Qx - X(t)'(\Theta'_X + \Theta_X)x + H^*(\lambda_x(x,t),x,\mathcal{A}(t)) \\ &\quad + \tfrac{1}{2}tr\left(\Sigma\lambda_{xx}(x,t)\right) + \lambda_t(x,t).\end{aligned}$$

The derivative with respect to $m(x,T)$ gives:

$$\lim_{T\to\infty} e^{-\rho T}\lambda(x,T) = 0.$$

Finally, we can use the envelope in the Hamiltonian to obtain the optimal value for $\alpha(x,t)$ and thus write the Kolmogorov forward equation as:

$$m_t(x,t) = -div\left(H^*_p(\lambda_x(x,t),x,\mathcal{A}(t))m(x,t)\right) + \tfrac{1}{2}tr\left(\Sigma m_{xx}(x,t)\right).$$

The definitions of the aggregate state and aggregate action give

$$X(t) = \int x\, m(x,t)\, dx,$$
$$\mathcal{A}(t) = \int \alpha^*(\lambda_x(x,t), \mathcal{A}(t)) m(x,t)\, dx.$$

□

Proof. (of Proposition 15) The aggregate state term in the planner's instantaneous objective is $-X'\Theta_X X$. A perturbation of the density at x changes this term by

$$-X'(\Theta_X + \Theta_X')x.$$

Similarly, the aggregate-action term is $-\mathcal{A}'\Theta_{\mathcal{A}}\mathcal{A}$. A perturbation of the action selected at a state changes this term by

$$-\mathcal{A}'(\Theta_{\mathcal{A}} + \Theta_{\mathcal{A}}')\alpha.$$

These are the two terms that distinguish the planner's first-order conditions from those of an individual agent. If the interaction matrices are symmetric, the planner's HJB and optimal action coincide with the decentralized HJB and optimal action after replacing

$$\Theta_X \quad \text{by} \quad 2\Theta_X, \qquad \Theta_{\mathcal{A}} \quad \text{by} \quad 2\Theta_{\mathcal{A}}.$$

The consistency conditions and the KFE are otherwise unchanged. Applying the equilibrium construction in Section 4 with the doubled matrices gives equation (19).

□

Proof. (of Proposition 16) We first derive the system equation (23)–equation (26). From equation (22),

$$v_x = S_1 X + S_2 x, \qquad v_X = S_1' x + S_3 X.$$

The optimal action for given (M, C) is therefore

$$\alpha_*(x, X) = \Gamma^{-1}\left[B'(S_1 X + S_2 x) - \Theta_{\mathcal{A}} C X\right].$$

Substitution in the Hamiltonian gives

$$H(v_x, x, CX) = \frac{1}{2}\left[B'(S_1 X + S_2 x) - \Theta_{\mathcal{A}} C X\right]' \Gamma^{-1}\left[B'(S_1 X + S_2 x) - \Theta_{\mathcal{A}} C X\right] - (S_1 X + S_2 x)' A x.$$

The common jump shifts x and X by the same amount. For the quadratic value function,

$$\begin{aligned}\frac{1}{2}v(x+\Upsilon,X+\Upsilon)+\frac{1}{2}v(x-\Upsilon,X-\Upsilon)-v(x,X)\\ =\Upsilon'\left(S_1+\frac{1}{2}S_2+\frac{1}{2}S_3\right)\Upsilon.\end{aligned}$$

Matching the constant, xX, x^2, and X^2 terms gives equation (23)–equation (26).

We next obtain the law of motion for the aggregate state. We fix a coordinate and evaluate the stochastic process of X_j:

$$\begin{aligned}dX_j(t)&=d\int x_j\,m(x,t)dx\\ &=\int_{\mathbb{R}^n}\left[-\operatorname{div}\left(H_p(v_x(x,X(t)),x,CX(t))m(x,t)\right)+\frac{1}{2}\operatorname{tr}\left(\Sigma m_{xx}(x,t)\right)\right]x_j\,dx\,dt\\ &\quad+\int_{\mathbb{R}^n}\left(m(x-\Upsilon,t)-m(x,t)\right)x_j\,dx\,dN^+(t)\\ &\quad+\int_{\mathbb{R}^n}\left(m(x+\Upsilon,t)-m(x,t)\right)x_j\,dx\,dN^-(t)\\ &=-\int_{\mathbb{R}^n}\sum_i\left(H_{p_i}(v_x(x,X(t)),x,CX(t))m(x,t)\right)_{x_i}x_j\,dx\,dt\\ &\quad+\Upsilon_j\,dN^+(t)-\Upsilon_j\,dN^-(t),\end{aligned}$$

where the term involving the second derivatives of m is zero using the boundary conditions in Appendix A.1.

Now we split the sum in two terms:

$$\begin{aligned}dX_j(t)=&-\int_{\mathbb{R}^n}\left(H_{p_j}(v_x(x,X(t)),x,CX(t))m(x,t)\right)_{x_j}x_j\,dx\,dt\\ &-\sum_{i\neq j}\int_{\mathbb{R}^n}\left(H_{p_i}(v_x(x,X(t)),x,CX(t))m(x,t)\right)_{x_i}x_j\,dx\,dt\\ &+\Upsilon_j\,dN^+(t)-\Upsilon_j\,dN^-(t).\end{aligned}$$

Take a typical term $i\neq j$. Then

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(H_{p_i}(v_x(x,X(t)),x,CX(t))m(x,t)\right)_{x_i}dx\\ &=\int_{x'\in\mathbb{R}^{n-1}}\int\left(H_{p_i}(v_x(x,X(t)),x,CX(t))m(x,t)\right)_{x_i}dx_i\,dx'\\ &=\int_{x'\in\mathbb{R}^{n-1}}\left[\lim_{x_i\to\infty}H_{p_i}(v_x(x,X(t)),x,CX(t))m(x,t)-\lim_{x_i\to-\infty}H_{p_i}(v_x(x,X(t)),x,CX(t))m(x,t)\right]dx',\end{aligned}$$

which is zero using the boundary conditions in Appendix A.1. Thus,

$$\begin{aligned} dX_j(t) &= -\int_{\mathbb{R}^n} \left(H_{p_j}(v_x(x, X(t)), x, CX(t))m(x,t)\right)_{x_j} x_j \, dx \, dt \\ &\quad + \Upsilon_j \, dN^+(t) - \Upsilon_j \, dN^-(t) \\ &= \int_{\mathbb{R}^n} H_{p_j}(v_x(x, X(t)), x, CX(t))m(x,t) \, dx \, dt + \Upsilon_j \, d\mathcal{G}(t). \end{aligned}$$

In vector form,

$$dX(t) = \int_{\mathbb{R}^n} H_p(v_x(x, X(t)), x, CX(t))m(x,t) \, dx \, dt + \Upsilon \, d\mathcal{G}(t).$$

Using the derivative of the Hamiltonian,

$$H_p(p, x, CX) = B\Gamma^{-1}\left(B'p - \Theta_{\mathcal{A}} CX\right) - Ax,$$

we obtain

$$\begin{aligned} dX(t) = \int_{\mathbb{R}^n} &\left[B\Gamma^{-1}\left(B' v_x(x, X(t)) - \Theta_{\mathcal{A}} CX(t)\right) - Ax\right] m(x,t) \, dx \, dt \\ &+ \Upsilon \, d\mathcal{G}(t). \end{aligned}$$

Using $v_x(x, X) = S_1 X + S_2 x$ and $X(t) = \int x \, m(x,t) dx$, this becomes

$$dX(t) = \left[B\Gamma^{-1}\left(B'(S_1 + S_2) - \Theta_{\mathcal{A}} C\right) - A\right] X(t) dt + \Upsilon \, d\mathcal{G}(t).$$

Since $P = S_1 + S_2$, comparison with equation (20) gives

$$M = B\Gamma^{-1}\left(B'P - \Theta_{\mathcal{A}} C\right) - A.$$

We next impose the equilibrium condition for aggregate actions:

$$CX(t) = \int \alpha_*(v_x(x, X(t)), CX(t))m(x,t) \, dx.$$

Using the optimal action,

$$\alpha_*(p, \mathcal{A}) = \Gamma^{-1}\left(B'p - \Theta_{\mathcal{A}} \mathcal{A}\right),$$

together with $v_x(x, X) = S_1 X + S_2 x$ and $X(t) = \int x \, m(x,t) dx$, gives

$$CX = \Gamma^{-1}\left[B'(S_1 + S_2) - \Theta_{\mathcal{A}} C\right] X.$$

Since this holds for every X,

$$(\Gamma + \Theta_{\mathcal{A}})C = B'P.$$

Using this expression in the equation for M gives

$$M = BC - A.$$

Since $\Gamma + \Theta_{\mathcal{A}}$ is invertible, we obtain

$$C = (\Gamma + \Theta_{\mathcal{A}})^{-1}B'P, \qquad M = B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'P - A,$$

which gives equation (27) and equation (28).

The equation for S_2 in equation (25) is the same single-agent Riccati equation as in the case without common noise. Hence

$$S_2 = \bar{\beta}_2.$$

Finally, we add equation (24) and equation (25) obtaining

$$\begin{aligned}\rho(S_1 + S_2) = & -(Q + \Theta_X) + S_2B\Gamma^{-1}B'(S_1 + S_2) - S_2B\Gamma^{-1}\Theta_{\mathcal{A}}C \\ & - A'(S_1 + S_2) - S_2A + S_1M.\end{aligned}$$

Using $P = S_1 + S_2$ and

$$B'P - \Theta_{\mathcal{A}}C = \Gamma C,$$

the terms involving S_2 satisfy

$$S_2B\Gamma^{-1}\left(B'P - \Theta_{\mathcal{A}}C\right) - S_2A = S_2BC - S_2A = S_2M.$$

Therefore,

$$\rho P = -(Q + \Theta_X) - A'P + PM.$$

Substituting

$$M = B(\Gamma + \Theta_{\mathcal{A}})^{-1}B'P - A$$

gives

$$\rho P = -(Q + \Theta_X) - A'P + PB(\Gamma + \Theta_{\mathcal{A}})^{-1}B'P - PA.$$

Rearranging,

$$PB(\Gamma + \Theta_{\mathcal{A}})^{-1}B'P = Q + \Theta_X + \rho P + PA + A'P,$$

which is equation (15), the same Riccati equation as in the case without common noise.

□

Proof. (of Proposition 17) Subtracting equation (29) from the individual state equation and using equation (31) gives

$$dz = (B\Gamma^{-1}B'\bar{\beta}_2 - A)zdt + \Sigma^{1/2}dW.$$

The common jump cancels because it moves the individual and aggregate states by the same amount. Likewise,

$$\alpha - \mathcal{A} = \Gamma^{-1}B'\bar{\beta}_2 z.$$

Hence the two reduced-form coefficients available from demeaned data are

$$U_{\dot{z},z} = B\Gamma^{-1}B'\bar{\beta}_2 - A, \qquad U_{\alpha,z} = \Gamma^{-1}B'\bar{\beta}_2.$$

Neither contains Θ_X or $\Theta_{\mathcal{A}}$, which proves Proposition 17.

□

Proof. (of Proposition 18) The conditional drift of the observed aggregate process identifies

$$U_{\dot{X},X} = DP - A.$$

Since D and A are known and D is invertible,

$$P = D^{-1}(U_{\dot{X},X} + A).$$

The Riccati equation then gives

$$\Theta_X = PDP - \rho P - PA - A'P - Q.$$

Every object on the right is known or identified from the aggregate time series, so Θ_X is uniquely identified.

□

Proof. (of Proposition 19) The observed processes $X(t)$, $z(t)$, and $\mathcal{A}(t)$ identify

$$U_{\dot{X},X}, \qquad U_{\dot{z},z}, \qquad U_{\alpha,X}.$$

Using equation (29) and equation (32),

$$U_{\dot{X},X} = BU_{\alpha,X} - A,$$

so, since B is known,

$$A = BU_{\alpha,X} - U_{\dot{X},X}.$$

Next, equation (30) gives

$$U_{\dot{z},z} = B\Gamma^{-1}B'\bar{\beta}_2 - A.$$

Under Assumptions AM, $B\Gamma^{-1}B'$ is invertible. Hence

$$\bar{\beta}_2 = \left(B\Gamma^{-1}B'\right)^{-1}\left(U_{\dot{z},z} + A\right).$$

From equation (32),

$$(\Gamma + \Theta_{\mathcal{A}})U_{\alpha,X} = B'P.$$

Since B has full row rank, BB' is invertible, and therefore

$$P = (BB')^{-1}B(\Gamma + \Theta_{\mathcal{A}})U_{\alpha,X}.$$

The single-agent Riccati equation then identifies

$$Q = \bar{\beta}_2 B\Gamma^{-1}B'\bar{\beta}_2 - \rho\bar{\beta}_2 - \bar{\beta}_2 A - A'\bar{\beta}_2.$$

Finally, the equilibrium Riccati equation identifies

$$\Theta_X = PB(\Gamma + \Theta_{\mathcal{A}})^{-1}B'P - \rho P - PA - A'P - Q.$$

Thus Q, Θ_X, and A are uniquely recovered.

□

Proof. (of Proposition 20) The observed processes identify

$$U_{\dot{X},X} = BU_{\alpha,X} - A, \qquad U_{\dot{z},z} = BU_{\alpha,z} - A,$$
$$U_{\alpha,X} = (\Gamma + \Theta_{\mathcal{A}})^{-1}B'P, \qquad U_{\alpha,z} = \Gamma^{-1}B'\bar{\beta}_2.$$

Subtracting the two drift coefficients gives

$$U_{\dot{z},z} - U_{\dot{X},X} = B\left(U_{\alpha,z} - U_{\alpha,X}\right).$$

Since $n = k$ and $U_{\alpha,z} - U_{\alpha,X}$ is nonsingular, this identifies

$$B = \left(U_{\dot{z},z} - U_{\dot{X},X}\right)\left(U_{\alpha,z} - U_{\alpha,X}\right)^{-1}.$$

The aggregate drift then identifies

$$A = BU_{\alpha,X} - U_{\dot{X},X}.$$

Since $n = k$, Assumptions AM imply that B is nonsingular. The two policy coefficients therefore identify

$$P = (B')^{-1}(\Gamma + \Theta_{\mathcal{A}})U_{\alpha,X}, \qquad \bar{\beta}_2 = (B')^{-1}\Gamma U_{\alpha,z}.$$

The single-agent Riccati equation identifies

$$Q = \bar{\beta}_2 B\Gamma^{-1}B'\bar{\beta}_2 - \rho\bar{\beta}_2 - \bar{\beta}_2 A - A'\bar{\beta}_2.$$

Finally, the equilibrium Riccati equation identifies

$$\Theta_X = PB(\Gamma + \Theta_{\mathcal{A}})^{-1}B'P - \rho P - PA - A'P - Q.$$

Thus Q, Θ_X, A, and B are uniquely recovered.

□

Proof. (of Proposition 21) Using the definition of M_r, differentiating with respect to time, and substituting the Kolmogorov forward equation gives

$$\dot{M}_r(t) = -\int x^r \left(H_p(u_x(x,t),x)m(x,t)\right)_x dx + \frac{\sigma^2}{2}\int x^r m_{xx}(x,t)dx.$$

The boundary conditions in Appendix A.1 and two integrations by parts imply

$$\int x^r m_{xx}(x,t)dx = r(r-1)M_{r-2}(t).$$

Here and below, $M_0(t) = 1$ and $M_{-1}(t) = 0$. A further integration by parts gives

$$-\int x^r \left(H_p(u_x(x,t),x)m(x,t)\right)_x dx = r\int x^{r-1}H_p(u_x(x,t),x)m(x,t)dx.$$

Since

$$H_p(u_x(x,t),x) = \gamma_B^{-1}\left(\beta_1(t) + \bar{\beta}_2 x\right) - ax,$$

we obtain

$$\dot{M}_r(t) = r\left(\gamma_B^{-1}\bar{\beta}_2 - a\right)M_r(t) + r\gamma_B^{-1}\beta_1(t)M_{r-1}(t) + \frac{r(r-1)\sigma^2}{2}M_{r-2}(t),$$

which is equation (33).

Proof. (of Proposition 22) At a steady state, the equation for the first moment is

$$0 = \left(\gamma_B^{-1}\bar{\beta}_2 - a\right)\bar{M}_1 + \gamma_B^{-1}\bar{\beta}_1.$$

Since $a - \gamma_B^{-1}\bar{\beta}_2 > 0$, this equation implies $\bar{M}_1 = 0$ if and only if $\bar{\beta}_1 = 0$. Conditional on $\bar{\beta}_1 = 0$, the remaining steady-state moment equations reduce to

$$\bar{M}_r = \frac{(r-1)\sigma^2}{2\left(a - \gamma_B^{-1}\bar{\beta}_2\right)}\bar{M}_{r-2}, \qquad r = 2, \ldots, \bar{r}.$$

Starting from $\bar{M}_0 = 1$ and $\bar{M}_1 = 0$, this recursion gives zero for every odd moment and gives equation (35) for the even moments. These moments are uniquely determined. Finally, setting $\bar{\beta}_1 = 0$ in the steady-state equation for β_1 gives

$$0 = \theta_0 + \sum_{r=1}^{\bar{r}} \theta_r \bar{M}_r.$$

The odd moments vanish, so this condition is equivalent to equation (36).

Proof. (of Proposition 23) Let

$$b \equiv a - \gamma_B^{-1}\bar{\beta}_2 > 0.$$

Equations equation (37) and equation (38) can then be written as

$$\begin{aligned}
\frac{d}{dt}\widehat{\beta}_1(t) &= (\rho + b)\widehat{\beta}_1(t) + \sum_{r=1}^{\bar{r}} \theta_r \widehat{M}_r(t), \\
\frac{d}{dt}\widehat{M}_r(t) &= r\gamma_B^{-1}\bar{M}_{r-1}\widehat{\beta}_1(t) - rb\widehat{M}_r(t) + \frac{r(r-1)\sigma^2}{2}\widehat{M}_{r-2}(t), \qquad r = 1, \ldots, \bar{r},
\end{aligned}$$

where $\widehat{M}_0(t) = 0$. At the zero-mean invariant distribution,

$$\bar{M}_r = 0 \quad \text{for odd } r, \qquad \bar{M}_r = (r-1)\frac{\sigma^2}{2b}\bar{M}_{r-2} \quad \text{for even } r.$$

Permute the coordinates so that the even moments precede $\widehat{\beta}_1$ and the odd moments. In this ordering the coefficient matrix is block lower triangular. The block for $(\widehat{M}_2, \widehat{M}_4, \ldots, \widehat{M}_{\bar{r}})'$ is lower triangular because $\bar{M}_{r-1} = 0$ for even r. Its diagonal entries are $-2b, -4b, \ldots, -\bar{r}b$,

and therefore its characteristic polynomial is

$$\prod_{\substack{2\le r\le \bar{r}\\ r\ \text{even}}} (\lambda + rb). \tag{41}$$

This also shows directly that the path of the even moment deviations is independent of the interaction coefficients and of the initial odd moment deviations.

The remaining block corresponds to $(\widehat{\beta}_1, \widehat{M}_1, \widehat{M}_3, \dots, \widehat{M}_{\bar{r}-1})'$. Its lower-right part is lower triangular with diagonal entries $-b, -3b, \dots, -(\bar{r}-1)b$. To evaluate the Schur complement, solve the associated lower-triangular system with right-hand side $r\gamma_B^{-1}\bar{M}_{r-1}$ in the row for the odd moment r. The component in that row is

$$\frac{r\gamma_B^{-1}\bar{M}_{r-1}}{\lambda + b}, \qquad r = 1, 3, \dots, \bar{r}-1. \tag{42}$$

For $r = 1$ the expression follows immediately. If it holds for $r-2$, substitution in the row for r gives

$$\begin{aligned}(\lambda + rb)\frac{r\gamma_B^{-1}\bar{M}_{r-1}}{\lambda + b} - \frac{r(r-1)\sigma^2}{2}\frac{(r-2)\gamma_B^{-1}\bar{M}_{r-3}}{\lambda + b}\\ = \frac{r\gamma_B^{-1}\bar{M}_{r-1}}{\lambda + b}\left[\lambda + rb - (r-1)b\right] = r\gamma_B^{-1}\bar{M}_{r-1},\end{aligned}$$

where the stationary-moment recursion was used. This proves equation (42) by induction.

Using the definition

$$\bar{\theta} = \sum_{\substack{1\le r\le \bar{r}-1\\ r\ \text{odd}}} r\bar{M}_{r-1}\theta_r,$$

the Schur complement gives the characteristic polynomial of this block:

$$\prod_{\substack{3\le r\le \bar{r}-1\\ r\ \text{odd}}} (\lambda + rb)\left[\lambda^2 - \rho\lambda - b(\rho + b) - \gamma_B^{-1}\bar{\theta}\right]. \tag{43}$$

The calculation is first valid away from the finite set at which the lower-right block is singular. Since both sides are polynomials in λ, the identity holds for all λ.

Combining equation (41) and equation (43) gives

$$\det(\lambda I - \widetilde{\mathscr{H}}) = \left[\lambda^2 - \rho\lambda - b(\rho + b) - \gamma_B^{-1}\bar{\theta}\right]\prod_{r=2}^{\bar{r}}(\lambda + rb).$$

It follows that $\widetilde{\lambda}_r = -rb$ for $r = 2, \ldots, \bar{r}$, while the remaining two roots are

$$\frac{\rho}{2} \mp \sqrt{\left(\frac{\rho}{2} + b\right)^2 + \gamma_B^{-1}\bar{\theta}}.$$

The scalar Riccati equation for $\bar{\beta}_2$ implies

$$\gamma_B^{-1}\bar{\beta}_2 - a = \frac{\rho}{2} - \sqrt{\left(a + \frac{\rho}{2}\right)^2 + \gamma_B^{-1}q},$$

and therefore

$$\frac{\rho}{2} + b = \sqrt{\left(a + \frac{\rho}{2}\right)^2 + \gamma_B^{-1}q}.$$

Substitution gives the eigenvalues stated in the proposition. The condition $q+\bar{\theta}+\gamma_B a(a+\rho) > 0$ makes the lower exceptional root negative and the upper exceptional root positive. The maintained condition $b > 0$ makes all remaining roots negative. □

C Forcing Processes

This appendix considers the extension in which part of the state is outside the agent's control. We retain the state-coupling formulation and set $\Theta_{\mathcal{A}} = 0$. Write

$$x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}, \qquad n = n_1 + n_2,$$

where $x^{(1)}$ contains the endogenous states and $x^{(2)}$ contains the forcing processes. The latter are autonomous and cannot be affected by either $x^{(1)}$ or the action. We impose

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0_{n_2 \times n_1} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ 0_{n_2 \times k} \end{bmatrix}.$$

Thus $x^{(2)}$ can affect $x^{(1)}$ through A_{12}, but its own law of motion is exogenous.

Partition the state curvature and interaction matrices as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & \tilde{q} I_{n_2} \end{bmatrix}, \qquad \Theta_X = \begin{bmatrix} \Theta_{X,11} & \Theta_{X,12} \\ \Theta_{X,21} & \tilde{\theta} I_{n_2} \end{bmatrix}.$$

The scalars $\tilde{q}$ and $\tilde{\theta}$ do not affect the policy for the endogenous state except through the continuation value associated with the forcing process. They can be chosen sufficiently large to make Q and $Q + \Theta_X$ positive definite whenever Q_{11} and $Q_{11} + \Theta_{X,11}$ are positive definite.

The two Riccati equations are

$$\bar{\beta}_2 B\Gamma^{-1}B'\bar{\beta}_2 = Q + \rho\bar{\beta}_2 + \bar{\beta}_2 A + A'\bar{\beta}_2, \tag{44}$$

$$PB\Gamma^{-1}B'P = Q + \Theta_X + \rho P + PA + A'P. \tag{45}$$

Because $B\Gamma^{-1}B'$ is singular, the full equations are no longer standard full-rank Riccati equations. Their block structure is nevertheless recursive.

PROPOSITION 24. Suppose $B_1\Gamma^{-1}B_1'$ is positive definite, $Q_{11} \succ 0$, $Q_{11} + \Theta_{X,11} \succ 0$, and the eigenvalues of $-A_{22}$ have real parts below $\rho/2$. Choose $\tilde{q}$ and $\tilde{\theta}$ so that $Q \succ 0$ and $Q+\Theta_X \succ 0$. Then equation (44) and equation (45) have unique admissible symmetric solutions. Their upper-left blocks solve

$$\bar{\beta}_{2,11} B_1\Gamma^{-1}B_1'\bar{\beta}_{2,11} = Q_{11} + \rho\bar{\beta}_{2,11} + \bar{\beta}_{2,11}A_{11} + A_{11}'\bar{\beta}_{2,11},$$

$$P_{11} B_1\Gamma^{-1}B_1'P_{11} = Q_{11} + \Theta_{X,11} + \rho P_{11} + P_{11}A_{11} + A_{11}'P_{11}.$$

The eigenvalues of the aggregate closed-loop matrix are the n_1 eigenvalues of

$$B_1\Gamma^{-1}B_1'P_{11} - A_{11}$$

and the n_2 eigenvalues of $-A_{22}$.

The proposition allows the benchmark results to be applied to the endogenous block. In particular, the full-rank requirement is imposed on $B_1\Gamma^{-1}B_1'$, and the effective-curvature condition is imposed on $Q_{11} + \Theta_{X,11}$. The forcing block contributes its own autonomous eigenvalues but does not alter the Riccati equation governing the endogenous block.

Proof. (of Proposition 24) Let Z and W be symmetric matrices partitioned conformably with the state, and consider

$$ZB\Gamma^{-1}B'Z = Z\widetilde{A} + \widetilde{A}'Z + \rho Z + W,$$

where

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ 0 & A_{22} \end{bmatrix}.$$

Set $R_1 = B_1\Gamma^{-1}B_1'$. Since

$$B\Gamma^{-1}B' = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix},$$

the quadratic term is

$$ZB\Gamma^{-1}B'Z = \begin{bmatrix} Z_{11}R_1Z_{11} & Z_{11}R_1Z_{12} \\ Z_{21}R_1Z_{11} & Z_{21}R_1Z_{12} \end{bmatrix}.$$

Matching blocks gives, first,

$$Z_{11}R_1Z_{11} = Z_{11}\widetilde{A}_{11} + \widetilde{A}'_{11}Z_{11} + \rho Z_{11} + W_{11}.$$

This is an n_1-dimensional Riccati equation. Once Z_{11} has been found, the upper-right block solves the linear matrix equation

$$Z_{11}R_1Z_{12} = Z_{11}\widetilde{A}_{12} + Z_{12}A_{22} + \widetilde{A}'_{11}Z_{12} + \rho Z_{12} + W_{12}.$$

Symmetry gives $Z_{21} = Z'_{12}$. The lower-right block then solves the Lyapunov equation

$$Z_{21}R_1Z_{12} = Z_{21}\widetilde{A}_{12} + \widetilde{A}'_{12}Z_{12} + Z_{22}A_{22} + A'_{22}Z_{22} + \rho Z_{22} + W_{22}.$$

Thus the blocks are obtained recursively: a Riccati equation for Z_{11}, followed by linear equations for Z_{12} and Z_{22}.

For $Z = \bar{\beta}_2$, take $\widetilde{A} = A$ and $W = Q$. For $Z = P$, take $\widetilde{A} = A$ and $W = Q + \Theta_X$. The assumptions on R_1, Q_{11}, and $Q_{11} + \Theta_{X,11}$ give the unique stabilizing solutions of the two upper-left Riccati equations. The condition on A_{22} makes the linear equations for the remaining blocks nonsingular and ensures that the forcing modes satisfy the admissibility restriction. This proves existence and uniqueness of the full symmetric solutions.

Finally,

$$B\Gamma^{-1}B'P - A = \begin{bmatrix} R_1P_{11} - A_{11} & R_1P_{12} - A_{12} \\ 0 & -A_{22} \end{bmatrix}.$$

The matrix is block upper triangular, so its characteristic polynomial is the product of the characteristic polynomials of the two diagonal blocks. This gives the eigenvalue decomposition stated in the proposition. □

D Derivations of Multiproduct Price Setting with Kimball Demand

D.1 Set Up of Static Firm's Problem

There is a continuum of stores indexed by $s \in [0, 1]$. Each store sells n products with price vector $p_s = (p_{1s}, \ldots, p_{ns})$. The products within a store are aggregated by a CES subutility,

whose unit expenditure function is

$$a(p_s) = \left(\sum_{j=1}^{n} \omega_j (p_{js})^{1-\bar{\eta}_d} \right)^{1/(1-\bar{\eta}_d)}.$$

The conditional demand for product j per unit of the store composite is

$$\frac{\partial a(p_s)}{\partial p_{js}} = \omega_j \left(\frac{p_{js}}{a(p_s)} \right)^{-\bar{\eta}_d} \equiv \omega_j d\left(\frac{p_{js}}{a(p_s)} \right).$$

Hence, if the household spends e_s at store s, it purchases $e_s/a(p_s)$ units of that store's composite and its demand for product j is

$$c_{js} = \frac{e_s}{a(p_s)} \omega_j d\left(\frac{p_{js}}{a(p_s)} \right).$$

Across stores, the household has a symmetric Kimball aggregator $\mathcal{C}(\{C_s\})$ defined implicitly by

$$1 = \int_0^1 \Upsilon\left(\frac{C_s}{\mathcal{C}} \right) ds, \qquad \Upsilon' > 0, \quad \Upsilon'' < 0,$$

where C_s is the quantity of the store-s composite of the n products sold by store s. The household has fixed total expenditure E and satisfies

$$\int_0^1 a(p_s) C_s \, ds = E.$$

Consider one infinitesimal focal store charging the n-dimensional vector of prices p, while all other stores charge the common $n-$dimensional vector P, thus omitting the index s. Let

$$p = (p_1, p_2, \ldots, p_n), \qquad P = (P_1, P_2, \ldots, P_n), \qquad y \equiv \frac{a(p)}{a(P)}, \qquad \bar{q} \equiv \Upsilon^{-1}(1),$$

and define the Kimball relative quantity-demand function

$$D(y) \equiv \left(\Upsilon'\right)^{-1} (y \, \Upsilon'(\bar{q})).$$

The focal store's composite demand is

$$C(p, P; E) = \frac{E}{a(P)\bar{q}} D\left(\frac{a(p)}{a(P)} \right),$$

and product-j demand is therefore

$$c_j(p, P; E) = \frac{E}{a(P)\bar{q}} D\left(\frac{a(p)}{a(P)}\right) \omega_j d\left(\frac{p_j}{a(p)}\right).$$

Consequently, actual static profit is

$$\Pi^{\text{act}}(p, P; E) = \frac{E}{a(P)\bar{q}} D\left(\frac{a(p)}{a(P)}\right) \sum_{j=1}^{n} (p_j - z_j)\omega_j d\left(\frac{p_j}{a(p)}\right).$$

Because $E/(a(P)\bar{q})$ is positive and independent of the focal store's own price vector p, we work with normalized profit. The static problem for a representative multi-product firm is

$$\Pi(p, P) = \sum_{j=1}^{n} (p_j - z_j)\omega_j d\left(\frac{p_j}{a(p)}\right) D\left(\frac{a(p)}{a(P)}\right)$$

We let $\eta_D(y) \equiv \frac{-yD'(y)}{D(y)}$ be the demand elasticity, and $\eta_D'(y)$ its derivative with respect to its relative price.

D.2 Optimal Price Setting

Proposition 1 *Consider a firm facing the rest of the firms charging prices given by vector P. The optimal price $p_i^*(P)$ for each product i satisfies the Lerner condition using the demand elasticity of the store aggregator:*

$$p_i^*(P) = z_i \frac{\eta_D\left(\frac{a(p)}{a(P)}\right)}{\eta_D\left(\frac{a(p)}{a(P)}\right) - 1} \text{ for all } i = 1, \ldots, n \tag{46}$$

At this level of generality, the expression in equation 46 should be thought of as a fixed point, since the vector p is on both sides of the expression. Nevertheless, this expression makes clear that the markups are the same for all products. The proof of Proposition 1 is in Section D.5. We are interested in the behavior of the firm when it is representative in the following sense.

Proposition 2 *Let $\bar{p}$ be a vector of prices that satisfies $\bar{p} = p^*(\bar{p})$, so it is a fixed point of the optimal price. Let $\bar{\eta}_D \equiv \eta_D(1)$. It is simply given by the Lerner formula evaluated at price one*

$$\bar{p}_i = z_i \frac{\bar{\eta}_D}{\bar{\eta}_D - 1} \text{ for all } i = 1, \ldots, n$$

The proof of this proposition is an immediate result from Proposition 1. From this proposition, a necessary condition for an interior solution is that $\bar{\eta}_D > 1$. For future reference we define the shares of each product j and the related function g_j as:

$$s_j(p) \equiv \frac{p_j}{a(p)} \frac{\partial a(p)}{\partial p_j} \text{ and } g_j(p) \equiv \frac{1}{p_j} s_j(p) \text{ for all } j = 1, \ldots, n$$

Proposition 3 *Consider $p = P = \bar{p}$, and let $\bar{D} = D(1)$ and $\bar{a} = a(\bar{p})$. Profits are given by*

$$\Pi(\bar{p}, \bar{p}) = \frac{\bar{D}\,\bar{a}}{\bar{\eta}_D}\,.$$

We note that $\bar{D} = \Upsilon^{-1}(1)$. From now on, we freeze the values of z at a vector $\bar{z}$, which we use to define $\bar{p}$. We can differentiate the Lerner condition and evaluate it at $p = P = \bar{p}$ to obtain the following.

Proposition 4 *Let $\bar{p}$ be a vector of prices that satisfies $\bar{p} = p^*(\bar{p})$, so it is a fixed point of the optimal price. Let $\bar{\eta}'_D \equiv \eta'_D(1)$. Then*

$$\frac{\partial p_i^*(\bar{p})}{\partial P_j} = \bar{p}_i \, \frac{\frac{\bar{\eta}'_D}{(\bar{\eta}_D - 1)\bar{\eta}_D}}{1 + \frac{\bar{\eta}'_D}{(\bar{\eta}_D - 1)\bar{\eta}_D}} \, g_j(\bar{p}) \quad \textit{or} \quad \frac{\partial \log p_i^*(\bar{p})}{\partial \log P_j} = \frac{\frac{\bar{\eta}'_D}{(\bar{\eta}_D - 1)\bar{\eta}_D}}{1 + \frac{\bar{\eta}'_D}{(\bar{\eta}_D - 1)\bar{\eta}_D}} \, s_j(\bar{p})$$

Note that the matrix $\partial \log p^*(\bar{p}) / \partial \log P$ is only symmetric in the case in which all the shares $\bar{s}_i$ are the same.

D.3 Second Derivatives at Equilibrium Prices

Proposition 5 *The cross derivative of the profit function $\Pi(p, P)$ evaluated at $p = P = \bar{p}$ is a symmetric positive semi-definite (resp. negative semidefinite) rank 1 matrix if $\bar{\eta}'_D > 0$ (resp. $\bar{\eta}'_D < 0$) given by*

$$\frac{\partial^2 \Pi(\bar{p}, \bar{p})}{\partial p \partial P} = D(1)\, a(\bar{p})\, \frac{\bar{\eta}'_D}{\bar{\eta}_D}\, g(\bar{p})\, g(\bar{p})^\top, \quad \textit{where } g(p) = (s_1(p)/p_1, s_2(p)/p_2, \ldots, s_n(p)/p_n)^\top$$

where $s(\bar{p})$ are the shares corresponding to the aggregator a, and $\bar{\eta}'_D$ is the derivative of the elasticity η_D evaluated at price one.

The proof of Proposition 5 is in Section D.5. The next proposition characterizes the Hessian.

Proposition 6 *Let $\bar{p} = p^*(\bar{p})$. The Hessian of the profit function evaluated at this point is given by:*

$$\frac{\partial^2 \Pi(\bar{p},\bar{p})}{\partial p \partial p} = -\bar{D}\,\bar{a}\left(\frac{\bar{\eta}_D - 1}{\bar{\eta}_D}\right)\left[\left(\bar{\eta}_D + \frac{\bar{\eta}'_D}{\bar{\eta}_D - 1} - \bar{\eta}_d\right)\bar{g}\,\bar{g}^\top + \bar{\eta}_d\,\mathrm{diag}\left(\frac{\bar{g}_i}{\bar{p}_i}\right)\right]$$

where $\bar{D} = D(1)$, and $\bar{a} = a(\bar{p})$.

The proof of Proposition 6 is in Section D.5.

Proposition 7 *Let $\bar{p} = p^*(\bar{p})$ be a fixed point at which the first-order conditions hold for $\bar{p} = p = P$. Consider the own-price Hessian $\frac{\partial^2 \Pi(\bar{p},\bar{p})}{\partial p \partial p}$ and the cross-price matrix $\frac{\partial^2 \Pi(\bar{p},\bar{p})}{\partial p \partial P}$. Their sum is*

$$\frac{\partial^2 \Pi(\bar{p},\bar{p})}{\partial p \partial p} + \frac{\partial^2 \Pi(\bar{p},\bar{p})}{\partial p\,\partial P} = -\frac{\bar{D}\,\bar{a}}{\bar{\eta}_D}\left(\bar{\eta}_D - 1\right)\left[\left(\bar{\eta}_D - \bar{\eta}_d\right)\bar{g}\,\bar{g}^\top + \bar{\eta}_d\,\mathrm{diag}\left(\frac{\bar{g}_i}{\bar{p}_i}\right)\right]$$

where $\bar{D} = D(1)$, and $\bar{a} = a(\bar{p})$. In particular, this sum does not depend on $\bar{\eta}'_D$, the parameter that controls the strategic complementarity.

D.4 Log Approximation and Summary

We define the log price deviations, and profits in terms of them as:

$$x_i \equiv \log\left(\frac{p_i}{\bar{p}_i}\right),\ X_i \equiv \log\left(\frac{P_i}{\bar{p}_i}\right),\ \text{and}\ \tilde{\Pi}(x,X) \equiv \Pi\left(e^{x_1}\bar{p}_1,\ldots,e^{x_n}\bar{p}_n,e^{X_1}\bar{p}_1,\ldots,e^{X_n}\bar{p}_n\right)$$

This gives the following expression for the quadratic approximation:

Proposition 8 *Assume that $\bar{\eta}_d > 0$ and $\bar{\eta}_D > 1$. Define the second derivatives as:*

$$Q \equiv -\frac{1}{\tilde{\Pi}(0,0)}\frac{\partial^2 \tilde{\Pi}(0,0)}{\partial x \partial x}\ \text{and}\ \Theta \equiv -\frac{1}{\tilde{\Pi}(0,0)}\frac{\partial^2 \tilde{\Pi}(0,0)}{\partial x \partial X}.$$

Then

$$\begin{aligned}
Q &= \left(\bar{\eta}_D - 1\right)\left[\left(\bar{\eta}_D - \bar{\eta}_d + \frac{\bar{\eta}'_D}{\bar{\eta}_D - 1}\right)\bar{s}\,\bar{s}^\top + \bar{\eta}_d\,\mathrm{diag}\left(\bar{s}_i\right)\right], \\
\Theta &= -\bar{\eta}'_D\,\bar{s}\,\bar{s}^\top, \\
Q + \Theta &= \left(\bar{\eta}_D - 1\right)\left[\left(\bar{\eta}_D - \bar{\eta}_d\right)\bar{s}\,\bar{s}^\top + \bar{\eta}_d\,\mathrm{diag}\left(\bar{s}_i\right)\right].
\end{aligned}$$

Then $Q+\Theta$ is positive definite, and Q is positive definite if and only if $\frac{\bar{\eta}'_D}{\bar{\eta}_D} > 1-\bar{\eta}_D$. Finally

$$\frac{\partial x_i^*(X)}{\partial X_j} = \frac{\frac{\bar{\eta}'_D}{(\bar{\eta}_D-1)\bar{\eta}_D}}{1+\frac{\bar{\eta}'_D}{(\bar{\eta}_D-1)\bar{\eta}_D}}\bar{s}_j \text{ for all } i,j=1,\ldots,n \text{ where } x^*(X) = \arg\max_x \tilde{\Pi}(x,X)$$

The condition for Q to be positive definite involves only the elasticity and super-elasticity of the aggregator. Note that since Proposition 2 implies $\bar{\eta}_D > 1$, the only extra condition is that $\bar{\eta}'_D$ cannot be "too" negative. Note that while $\frac{\partial x^*(X)}{\partial X}$ is not symmetric unless $\bar{s}_i = 1/n$, the matrix Θ is always symmetric. Moreover, the denominator of the expression for $\partial x_i^*/\partial X_j$ is positive under concavity of $\tilde{\Pi}(\cdot, X)$, and hence its sign is the same as that of $\bar{\eta}'_D$.

D.5 Proofs

In some of the calculations that follow we will denote the ideal price index $a(P)$ simply as $A = a(P)$, and we define a profit function accordingly, i.e. Let's denote the profits as:

$$\Pi(p,P) = \pi(p,a(P)) \text{ where } \pi(p,A) = \sum_{j=1}^n (p_j - z_j)\omega_j d\left(\frac{p_j}{a(p)}\right) D\left(\frac{a(p)}{A}\right)$$

The derivative of π with respect to p_i is given by:

$$\begin{aligned}\frac{\partial \pi(p,A)}{\partial p_i} = &\sum_{j=1}^n (p_j - z_j)\left[-\omega_j d'\left(\frac{p_j}{a}\right) D\left(\frac{a}{A}\right)\frac{p_j}{a^2} + \omega_j d\left(\frac{p_j}{a}\right) D'\left(\frac{a}{A}\right)\frac{1}{A}\right]\frac{\partial a}{\partial p_i} \\ &+ (p_i - z_i)\omega_i d'\left(\frac{p_i}{a}\right) D\left(\frac{a}{A}\right)\frac{1}{a} + \omega_i d\left(\frac{p_i}{a}\right) D\left(\frac{a}{A}\right)\end{aligned}$$

which can be written as

$$\begin{aligned}\frac{\partial \pi(p,A)}{\partial p_i} = &\frac{D(a(p)/A)}{p_i}\Big(\sum_{j=1}^n (p_j - z_j)\omega_j\left[-d'\left(\frac{p_j}{a}\right)\frac{p_j}{a} - d\left(\frac{p_j}{a}\right)\eta_D\left(\frac{a}{A}\right)\right]\frac{p_i}{a}\frac{\partial a}{\partial p_i} \\ &+ (p_i - z_i)\omega_i d'\left(\frac{p_i}{a}\right)\frac{p_i}{a} + \omega_i d\left(\frac{p_i}{a}\right) p_i\Big)\end{aligned}$$

Proof. (of Proposition 1) The first order condition of this problem for price p_i is $\frac{\partial \pi(p,A)}{\partial p_i} = 0$ or

$$\begin{aligned}0 = &\sum_{j=1}^n (p_j - z_j)\omega_j\left[-d'\left(\frac{p_j}{a}\right) D\left(\frac{a}{A}\right)\frac{p_j}{a^2} + d\left(\frac{p_j}{a}\right) D'\left(\frac{a}{A}\right)\frac{1}{A}\right]\frac{\partial a}{\partial p_i} \\ &+ (p_i - z_i)\omega_i d'\left(\frac{p_i}{a}\right) D\left(\frac{a}{A}\right)\frac{1}{a} + \omega_i d\left(\frac{p_i}{a}\right) D\left(\frac{a}{A}\right)\end{aligned}$$

where a is evaluated at the vector p. We can rewrite this expression as:

$$0 = \sum_{j=1}^{n}(p_j - z_j)\omega_j d\left(\frac{p_j}{a}\right)\left[-\frac{d'\left(\frac{p_j}{a}\right)}{d\left(\frac{p_j}{a}\right)}\frac{p_j}{a} + \frac{D'\left(\frac{a}{A}\right)}{D\left(\frac{a}{A}\right)}\frac{a}{A}\right]\frac{p_i}{a}\frac{\partial a}{\partial p_i}$$
$$+ \omega_i d\left(\frac{p_i}{a}\right)\left[(p_i - z_i)\frac{d'\left(\frac{p_i}{a}\right)}{d\left(\frac{p_i}{a}\right)}\frac{p_i}{a} + p_i\right]$$

or

$$0 = \sum_{j=1}^{n}\frac{(p_j - z_j)\omega_j d\left(\frac{p_j}{a}\right)}{(p_i - z_i)\omega_i d\left(\frac{p_i}{a}\right)}\left[\bar{\eta}_d + \frac{D'\left(\frac{a}{A}\right)}{D\left(\frac{a}{A}\right)}\frac{a}{A}\right]s_i(p) + \left[-\bar{\eta}_d + \frac{p_i}{p_i - z_i}\right]$$

or

$$0 = \sum_{j=1}^{n}\frac{\frac{p_j - z_j}{p_j}s_j(p)}{\frac{p_i - z_i}{p_i}s_i(p)}\left[\bar{\eta}_d + \frac{D'\left(\frac{a}{A}\right)}{D\left(\frac{a}{A}\right)}\frac{a}{A}\right]s_i(p) + \left[-\bar{\eta}_d + \frac{p_i}{p_i - z_i}\right]$$

Defining $\mu_j \equiv \frac{p_j}{p_j - z_j}$:

$$\mu_i = \sum_{j=1}^{n}\frac{\frac{p_j - z_j}{p_j}s_j(p)}{\frac{p_i - z_i}{p_i}}\left[-\bar{\eta}_d - \frac{D'\left(\frac{a}{A}\right)}{D\left(\frac{a}{A}\right)}\frac{a}{A}\right] + \bar{\eta}_d$$

or

$$\mu_i = \sum_{j=1}^{n}\frac{\frac{s_j(p)}{\mu_j}}{\frac{1}{\mu_i}}\left[-\bar{\eta}_d - \frac{D'\left(\frac{a}{A}\right)}{D\left(\frac{a}{A}\right)}\frac{a}{A}\right] + \bar{\eta}_d$$

or

$$1 = \sum_{j=1}^{n}\frac{s_j(p)}{\mu_j}\left[-\bar{\eta}_d - \frac{D'\left(\frac{a}{A}\right)}{D\left(\frac{a}{A}\right)}\frac{a}{A}\right] + \frac{1}{\mu_i}\bar{\eta}_d$$

or

$$\frac{\bar{\eta}_d}{\mu_i} = 1 - \sum_{j=1}^{n}s_j(p)\frac{p_j - z_j}{p_j}\left[-\bar{\eta}_d - \frac{D'\left(\frac{a}{A}\right)}{D\left(\frac{a}{A}\right)}\frac{a}{A}\right]$$

so all the markups are the same, say $\bar{\mu} = p_i/(p_i - z_i)$

$$\frac{\bar{\eta}_d}{\bar{\mu}} = 1 + \frac{1}{\bar{\mu}}\left[\bar{\eta}_d + \frac{D'\left(\frac{a}{A}\right)}{D\left(\frac{a}{A}\right)}\frac{a}{A}\right]$$

or

$$0 = 1 + \frac{1}{\bar{\mu}} \frac{D'\left(\frac{a}{A}\right)}{D\left(\frac{a}{A}\right)} \frac{a}{A}$$

or

$$\mu_i = \bar{\mu} = -\frac{D'\left(\frac{a}{A}\right)}{D\left(\frac{a}{A}\right)} \frac{a}{A}$$

Using the definition of μ_i and $A = a(P)$ we obtain the desired result.

■

Proof. (of Proposition 3)

$$\begin{aligned}\Pi(\bar{p}, \bar{p}) &= D(1)\left[1 - \frac{\bar{\eta}_D - 1}{\bar{\eta}_D}\right] \sum_{j=1}^{n} \bar{p}_j \omega_j d\left(\frac{\bar{p}_j}{a(\bar{p})}\right) = \frac{D(1)}{\bar{\eta}_D} \sum_{j=1}^{n} \bar{p}_j \omega_j \left(\frac{\bar{p}_j}{a(\bar{p})}\right)^{-\bar{\eta}_d} \\ &= \frac{D(1)}{\bar{\eta}_D} a(\bar{p})^{\bar{\eta}_d} \sum_{j=1}^{n} \omega_j \bar{p}_j^{1-\bar{\eta}_d} = \frac{D(1)}{\bar{\eta}_D} a(\bar{p})^{\bar{\eta}_d} a(\bar{p})^{1-\bar{\eta}_d} = \frac{\bar{D}\,\bar{a}}{\bar{\eta}_D}\end{aligned}$$

■

Proof. (of Proposition 4)

Let us rewrite the optimal price as $p_i^* = z_i\, r(A, z)$ where A stands for $A = a(P)$ and where $a(p) = r(A, z)a(z)$

$$r(A, z) = \frac{\eta_D\left(r(A, z)\frac{a(z)}{A}\right)}{\eta_D\left(r(A, z)\frac{a(z)}{A}\right) - 1}$$

In what follows we suppress z and A as arguments of r and z as the argument of a. Differentiating this expression with respect to A

$$\begin{aligned} r_A &= -\frac{\eta_D'\left(r\frac{a}{A}\right)}{\left(\eta_D\left(r\frac{a}{A}\right) - 1\right)^2}\left[r_A \frac{a}{A} - r\frac{a}{A^2}\right] \\ &= -\frac{\eta_D\left(r\frac{a}{A}\right)}{\left(\eta_D\left(r\frac{a}{A}\right) - 1\right)} \frac{\eta_D'\left(r\frac{a}{A}\right)}{\left(\eta_D\left(r\frac{a}{A}\right) - 1\right)\eta_D\left(r\frac{a}{A}\right)}\left[r_A \frac{a}{A} - r\frac{a}{A^2}\right] \\ &= -\frac{\eta_D'\left(r\frac{a}{A}\right)}{\left(\eta_D\left(r\frac{a}{A}\right) - 1\right)\eta_D\left(r\frac{a}{A}\right)}\left[r_A \frac{ra}{A} - r\frac{ra}{A^2}\right] \end{aligned}$$

Then

$$r_A = -\frac{\eta_D'\left(r\frac{a}{A}\right)}{\left(\eta_D\left(r\frac{a}{A}\right)-1\right)\eta_D\left(r\frac{a}{A}\right)}\, r_A \frac{ra}{A} + \frac{\eta_D'\left(r\frac{a}{A}\right)}{\left(\eta_D\left(r\frac{a}{A}\right)-1\right)\eta_D\left(r\frac{a}{A}\right)}\, r\frac{ra}{A^2}$$

or

$$r_A = \frac{\frac{\eta_D'\left(r\frac{a}{A}\right)}{\left(\eta_D\left(r\frac{a}{A}\right)-1\right)\eta_D\left(r\frac{a}{A}\right)}}{1+\frac{\eta_D'\left(r\frac{a}{A}\right)}{\left(\eta_D\left(r\frac{a}{A}\right)-1\right)\eta_D\left(r\frac{a}{A}\right)}\frac{ra}{A}}\, r\frac{ra}{A^2}$$

Or completing elasticities

$$\frac{A\, r_A}{r} = \frac{\frac{\eta_D'\left(r\frac{a}{A}\right)}{\left(\eta_D\left(r\frac{a}{A}\right)-1\right)\eta_D\left(r\frac{a}{A}\right)}}{1+\frac{\eta_D'\left(r\frac{a}{A}\right)}{\left(\eta_D\left(r\frac{a}{A}\right)-1\right)\eta_D\left(r\frac{a}{A}\right)}\frac{ra}{A}}\frac{ra}{A}$$

Evaluating it at $p = P = \bar{p}$ we have $ra/A = 1$. Then

$$\frac{A\, r_A(A,z)}{r(A,z)} = \frac{\frac{\bar{\eta}_D'}{(\bar{\eta}_D-1)\bar{\eta}_D}}{1+\frac{\bar{\eta}_D'}{(\bar{\eta}_D-1)\bar{\eta}_D}}$$

Using the chain rule:

$$\begin{aligned}\frac{\partial p_i^*(\bar{p})}{\partial P_j} = z_i \frac{\partial r(A,z)}{\partial A}\frac{\partial a(\bar{p})}{\partial p_j} &= z_i\, r(A,z)\, \frac{a(\bar{p})}{r(A,z)}\frac{\partial r(A,z)}{\partial A}\frac{1}{a(\bar{p})}\frac{\partial a(\bar{p})}{\partial p_j} \\ &= \frac{\bar{p}_i}{\bar{p}_j}\, \frac{\frac{\bar{\eta}_D'}{(\bar{\eta}_D-1)\bar{\eta}_D}}{1+\frac{\bar{\eta}_D'}{(\bar{\eta}_D-1)\bar{\eta}_D}}\, \frac{\bar{p}_j}{a(\bar{p})}\frac{\partial a(\bar{p})}{\partial p_j} \\ &= \bar{p}_i\, \frac{\frac{\bar{\eta}_D'}{(\bar{\eta}_D-1)\bar{\eta}_D}}{1+\frac{\bar{\eta}_D'}{(\bar{\eta}_D-1)\bar{\eta}_D}}\, g_j(\bar{p})\end{aligned}$$

■

Consistency Check: We use the expression derived in Proposition 4 for $\partial p^*/\partial P$, and the expressions derived in Propositions 5 and 6 for $\frac{\partial^2 \bar{\Pi}}{\partial p \partial P}$ and $\frac{\partial^2 \bar{\Pi}}{\partial p \partial p}$ to verify that $\frac{\partial^2 \bar{\Pi}}{\partial p \partial p}\, \partial p^*/\partial P + C = 0$. The best response $p^*(P)$ satisfies the first-order condition $\nabla_p \Pi\big(p^*(P), P\big) = 0$ identically

in P. Differentiating in P_j and using the chain rule,

$$\sum_k \frac{\partial^2 \Pi}{\partial p_i \partial p_k} \frac{\partial p_k^*}{\partial P_j} + \frac{\partial^2 \Pi}{\partial p_i \partial P_j} = 0, \qquad \text{i.e.} \qquad \tfrac{\partial^2 \Pi(\bar{p},\bar{p})}{\partial p \partial p} \frac{\partial p^*}{\partial P} + C = 0,$$

where $C = \partial^2 \Pi / \partial p \, \partial P$. We verify this identity directly at $p = P = \bar{p}$ using the closed forms of Propositions 4, 5 and 6, without inverting $\frac{\partial^2 \Pi(\bar{p},\bar{p})}{\partial p \partial p}$.

Recall, with $g_i \equiv s_i(\bar{p})/\bar{p}_i$ and $\kappa \equiv \bar{\eta}_d \dfrac{\bar{\eta}_D - 1}{\bar{\eta}_D}$,

$$\tfrac{\partial^2 \Pi(\bar{p},\bar{p})}{\partial p_i \partial p_k} = D(1)\, a(\bar{p}) \Big[\bar{\chi}\, g_i g_k - \kappa\, \delta_{ik} \frac{s_i}{\bar{p}_i^2} \Big], \qquad C_{ij} = D(1)\, a(\bar{p})\, \frac{\bar{\eta}'_D}{\bar{\eta}_D}\, g_i g_j, \qquad \frac{\partial p_k^*}{\partial P_j} = \bar{p}_k\, c\, g_j,$$

where $\bar{\chi} = 1 + \bar{\eta}_d - \bar{\eta}_D - \dfrac{\bar{\eta}_d}{\bar{\eta}_D} - \dfrac{\bar{\eta}'_D}{\bar{\eta}_D}$ and $c = \dfrac{K}{1+K}$ with $K = \dfrac{\bar{\eta}'_D}{(\bar{\eta}_D - 1)\bar{\eta}_D}$.

Contraction. Form $\big(\frac{\partial^2 \bar{\Pi}}{\partial p \partial p}\, \partial p^* / \partial P\big)_{ij} = \sum_k \frac{\partial^2 \bar{\Pi}}{\partial p_i \partial p_k} \dfrac{\partial p_k^*}{\partial P_j}$:

$$\sum_k D(1) a \Big[\bar{\chi}\, g_i g_k - \kappa\, \delta_{ik} \tfrac{s_i}{\bar{p}_i^2} \Big] \bar{p}_k\, c\, g_j = D(1)\, a\, c\, g_j \Big[\bar{\chi}\, g_i \sum_k g_k \bar{p}_k - \kappa\, \tfrac{s_i}{\bar{p}_i^2}\, \bar{p}_i \Big].$$

Two elementary identities close it. First, $g_k \bar{p}_k = \dfrac{s_k}{\bar{p}_k} \bar{p}_k = s_k$, so $\sum_k g_k \bar{p}_k = \sum_k s_k = 1$. Second, $\dfrac{s_i}{\bar{p}_i^2}\, \bar{p}_i = \dfrac{s_i}{\bar{p}_i} = g_i$. Hence

$$\big(\frac{\partial^2 \bar{\Pi}}{\partial p \partial p}\, \partial p^* / \partial P\big)_{ij} = D(1)\, a\, c\, g_j \big[\bar{\chi}\, g_i - \kappa\, g_i \big] = D(1)\, a\, c\, (\bar{\chi} - \kappa)\, g_i g_j.$$

Therefore

$$\big(\frac{\partial^2 \bar{\Pi}}{\partial p \partial p}\, \partial p^* / \partial P + C\big)_{ij} = D(1)\, a\, g_i g_j \Big[c\, (\bar{\chi} - \kappa) + \frac{\bar{\eta}'_D}{\bar{\eta}_D} \Big].$$

The scalar bracket vanishes. Compute $\kappa - \bar{\chi}$:

$$\kappa - \bar{\chi} = \bar{\eta}_d \frac{\bar{\eta}_D - 1}{\bar{\eta}_D} - \Big(1 + \bar{\eta}_d - \bar{\eta}_D - \frac{\bar{\eta}_d}{\bar{\eta}_D} - \frac{\bar{\eta}'_D}{\bar{\eta}_D} \Big) = \frac{\bar{\eta}_D^2 - \bar{\eta}_D + \bar{\eta}'_D}{\bar{\eta}_D}.$$

On the other hand,

$$c = \frac{K}{1+K} = \frac{\bar{\eta}'_D}{\bar{\eta}_D \big(\bar{\eta}_D - 1\big) + \bar{\eta}'_D} = \frac{\bar{\eta}'_D / \bar{\eta}_D}{\kappa - \bar{\chi}},$$

using the previous display. Hence $c\,(\bar{\chi}-\kappa) = -\dfrac{\bar{\eta}'_D}{\bar{\eta}_D}$, and the bracket is

$$c\,(\bar{\chi}-\kappa) + \frac{\bar{\eta}'_D}{\bar{\eta}_D} = -\frac{\bar{\eta}'_D}{\bar{\eta}_D} + \frac{\bar{\eta}'_D}{\bar{\eta}_D} = 0.$$

Therefore $\frac{\partial^2 \bar{\Pi}}{\partial p \partial p}\, \partial p^*/\partial P + C = 0$ at $p = P = \bar{p}$, confirming that the Jacobian of Proposition 4 is consistent with the Hessian and cross-derivative of Propositions 6 and 5. □

Proof. (of Proposition 5)

$$\frac{\partial^2 \Pi(p,P)}{\partial p_i \partial P_k} = \frac{\partial^2 \pi(p, a(P))}{\partial p_i \partial A} \frac{\partial a(P)}{\partial P_k}$$

at a value for p, given P for which $\partial \pi(p, a(P))/\partial p_j = 0$ for all j. Moreover, we are interested in a particular type of P which we label $\bar{p}$ for which: $\partial \pi(\bar{p}, a(\bar{p}))/\partial p_j = 0$ for all j. Thus, evaluated at a vector p for which the first order condition holds, we have:

$$\frac{\partial^2 \pi(p,A)}{\partial p_i \partial A} = \frac{D(a(p)/A)}{p_i} \Big(\sum_{j=1}^{n} (p_j - z_j)\omega_j d\left(\frac{p_j}{a}\right) \Big) \frac{a}{A^2}\, \eta'_D\left(\frac{a}{A}\right) \left(\frac{p_i}{a}\frac{\partial a}{\partial p_i}\right)$$

Hence:

$$\frac{\partial^2 \pi(p, a(P))}{\partial p_i \partial A} \frac{\partial a(P)}{\partial P_k} = \frac{D(a(p)/A)}{p_i} \Big(\sum_{j=1}^{n} (p_j - z_j)\omega_j d\left(\frac{p_j}{a}\right) \Big) \frac{a}{A^2}\, \eta'_D\left(\frac{a}{A}\right) \left(\frac{p_i}{a}\frac{\partial a}{\partial p_i}\right) \frac{\partial a(P)}{\partial P_k}$$

Imposing that $A = a(\bar{P})$ and that $\bar{P} = \bar{p}$ gives

$$\begin{aligned}
\frac{\partial^2 \pi(\bar{p}, a(\bar{p}))}{\partial p_i \partial A} \frac{\partial a(\bar{p})}{\partial P_k} &= \frac{D(a(p)/A)}{\bar{p}_i} \left(\sum_{j=1}^{n} (\bar{p}_j - z_j)\omega_j d\left(\frac{\bar{p}_j}{a(\bar{p})}\right) \right) \bar{\eta}'_D \left(\frac{\bar{p}_i}{a}\frac{\partial a(\bar{p})}{\partial p_i}\right) \left(\frac{1}{a(\bar{p})}\frac{\partial a(\bar{p})}{\partial P_k}\right) \\
&= \frac{D(a(p)/A)}{\bar{p}_i \bar{p}_k} \left(\sum_{j=1}^{n} (\bar{p}_j - z_j)\omega_j d\left(\frac{\bar{p}_j}{a(\bar{p})}\right) \right) \bar{\eta}'_D \left(\frac{\bar{p}_i}{a}\frac{\partial a(\bar{p})}{\partial p_i}\right) \left(\frac{\bar{p}_k}{a(\bar{p})}\frac{\partial a(\bar{p})}{\partial p_k}\right)
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial^2 \Pi(\bar{p}, \bar{p})}{\partial p \partial P} &= D(1)\, \bar{\eta}'_D \left(\sum_{j=1}^{n} \omega_j (\bar{p}_j - z_j) d\left(\frac{\bar{p}_j}{a(\bar{p})}\right) \right) g(\bar{p})\, g(\bar{p})^\top \\
\text{where } g(p) &= \left(s_1(p)/p_1, s_2(p)/p_2, \ldots, s_n(p)/p_n\right)^\top
\end{aligned}$$

where $s(\bar{p})$ are the shares corresponding to the aggregator a, and $\bar{\eta}'_D$ is the derivative of the

elasticity evaluated at price one. Now we can use that

$$\begin{aligned} p_j - z_j &= z_j \left(\frac{\bar{\eta}_D}{\bar{\eta}_D - 1} - \frac{\bar{\eta}_D - 1}{\bar{\eta}_D - 1} \right) = z_j \left(\frac{1}{\bar{\eta}_D - 1} \right) = p_j \left(\frac{\bar{\eta}_D - 1}{\bar{\eta}_D} \right) \left(\frac{1}{\bar{\eta}_D - 1} \right) \\ &= p_j \frac{1}{\bar{\eta}_D} \end{aligned}$$

thus

$$\omega_j (\bar{p}_j - z_j) d \left(\frac{\bar{p}_j}{a(\bar{p})} \right) = \frac{1}{\bar{\eta}_D} p_j \omega_j d \left(\frac{\bar{p}_j}{a(\bar{p})} \right)$$

Thus

$$\begin{aligned} \frac{\partial^2 \Pi(\bar{p}, \bar{p})}{\partial p \partial P} &= D(1) \, \frac{\bar{\eta}_D'}{\bar{\eta}_D} \left(\sum_{j=1}^{n} p_j \omega_j d \left(\frac{\bar{p}_j}{a(\bar{p})} \right) \right) g(\bar{p}) \, g(\bar{p})^\top \\ &= D(1) \, \frac{\bar{\eta}_D'}{\bar{\eta}_D} a(\bar{p}) \, g(\bar{p}) \, g(\bar{p})^\top \end{aligned}$$

■

Proof. (of Proposition 6) Since A enters π only through $D(a(p)/A)$, write $\pi(p, A) = D\left(\frac{a(p)}{A}\right) S(p)$ with

$$S(p) \equiv \sum_{k=1}^{n} \omega_k \, (p_k - z_k) \, d\left(\frac{p_k}{a(p)} \right).$$

Let $u_k = p_k/a$, $a_{p_i} = \partial a / \partial p_i$, $\mu_i = p_i/(p_i - z_i)$, and write $\beta \equiv \bar{\eta}_d$, $g_i \equiv s_i/p_i$. Because $d(u) = u^{-\beta}$ and a is the CES index with the same elasticity β, we have

$$a_{p_i} = \frac{s_i a}{p_i}, \qquad \omega_i \, d(u_i) = \frac{s_i a}{p_i}, \qquad a_{p_i p_k} = \frac{\beta \, a}{p_i p_k} \, s_i \big(s_k - \delta_{ik} \big), \qquad \frac{\partial s_i}{\partial p_k} = \frac{1 - \beta}{p_k} \, s_i \big(\delta_{ik} - s_k \big).$$

Gradient. With $x = a/A$,

$$\frac{\partial \pi}{\partial p_i} = D'(x) \, \frac{a_{p_i}}{A} \, S + D(x) \, S_{p_i}, \qquad S_{p_i} = \omega_i d(u_i) \Big(1 - \frac{\beta}{\mu_i} \Big) + \beta \, \frac{a_{p_i}}{a} \, S,$$

where S_{p_i} uses $u_i d'(u_i) = -\beta \, d(u_i)$ and $\sum_k \omega_k (p_k - z_k) d'(u_k) \, p_k = -\beta \, a \, S$.

Second derivative. Differentiating in p_k with $\partial x / \partial p_k = a_{p_k}/A$,

$$\frac{\partial^2 \pi}{\partial p_i \partial p_k} = D(x) \Big[\tfrac{D''}{D} \tfrac{a_{p_i} a_{p_k}}{A^2} S + \tfrac{D'}{D} \tfrac{a_{p_i p_k}}{A} S + \tfrac{D'}{D} \tfrac{a_{p_i}}{A} S_{p_k} + \tfrac{D'}{D} \tfrac{a_{p_k}}{A} S_{p_i} + S_{p_i p_k} \Big], \qquad (*)$$

which we label terms (1)–(5).

Fixed-point simplifications. At $\bar{p} = \bar{P}$ we have $x = 1$, so $D'/D = -\bar{\eta}_D$ and $D''/D =$

$\bar\eta_D^2+\bar\eta_D-\bar\eta_D'$. By Proposition 1 the markup is common, $\mu_i=\bar\eta_D$, hence $(p_k-z_k)/p_k=1/\bar\eta_D$, and, exactly as in the proof of Proposition 5,

$$S=\sum_k \omega_k(p_k-z_k)d(u_k)=\frac{1}{\bar\eta_D}\sum_k p_k\,\omega_k d(u_k)=\frac{a}{\bar\eta_D}\sum_k s_k=\frac{a}{\bar\eta_D}.$$

Moreover the within-store gradient collapses to the aggregator gradient,

$$S_{p_i}=\frac{s_i a}{p_i}\Big(1-\frac{\beta}{\bar\eta_D}\Big)+\beta\,\frac{s_i}{p_i}\,S=\frac{s_i a}{p_i}=a_{p_i}.$$

Since also $A=a$, we have $a_{p_i}/A=g_i$ and $S_{p_i}=ag_i$.

The five terms. Substituting $S=a/\bar\eta_D$, $S_{p_i}=ag_i$, the off-diagonal ($i\neq k$) contributions to $(*)$, each divided by the common factor $D(1)\,a\,g_ig_k$, are

$$\begin{aligned}
(1):&\ \frac{D''}{D}\,\frac{S}{a}=\big(\bar\eta_D^2+\bar\eta_D-\bar\eta_D'\big)\frac{1}{\bar\eta_D}=\bar\eta_D+1-\frac{\bar\eta_D'}{\bar\eta_D},\\
(2):&\ \frac{D'}{D}\,\frac{a_{p_ip_k}}{a\,g_ig_k}\,\frac{S}{a}=(-\bar\eta_D)\,\beta\,\frac{1}{\bar\eta_D}=-\beta\qquad(\text{the }-\delta_{ik}\text{ part is diagonal, treated below}),\\
(3)+(4):&\ 2\,\frac{D'}{D}\,\frac{S_{p_k}}{ag_k}=2(-\bar\eta_D)=-2\bar\eta_D,\\
(5):&\ \beta\Big(1-\frac{\beta}{\bar\eta_D}\Big)-\beta(1-\beta)\frac{S}{a}+\beta=2\beta-\frac{\beta}{\bar\eta_D}.
\end{aligned}$$

(The term-(5) line collects the three off-diagonal pieces of $S_{p_ip_k}$: from $\partial_{p_k}[\omega_i d(u_i)(1-\beta/\mu_i)]$, from $\beta\,p_i^{-1}\partial_{p_k}s_i\cdot S$, and from $\beta g_iS_{p_k}$, using $\partial s_i/\partial p_k=\frac{1-\beta}{p_k}s_i(-s_k)$ and $S/a=1/\bar\eta_D$.) Summing,

$$\Big(\bar\eta_D+1-\tfrac{\bar\eta_D'}{\bar\eta_D}\Big)-\beta-2\bar\eta_D+\Big(2\beta-\tfrac{\beta}{\bar\eta_D}\Big)=1+\beta-\bar\eta_D-\frac{\beta}{\bar\eta_D}-\frac{\bar\eta_D'}{\bar\eta_D}\equiv\bar\chi.$$

Diagonal. The Kronecker pieces of (2) and (5) combine. The δ_{ik} part of (2) contributes $+\delta_{ik}Da\beta\,s_i/p_i^2$, the δ_{ik} parts of (5) from $\partial_{p_i}(1-\beta/\mu_i)=-\beta z_i/p_i^2$ and from the remaining diagonal terms of $S_{p_ip_k}$ combine to $-\delta_{ik}Da\beta\,s_i/p_i^2-\delta_{ik}Da\beta\,z_is_i/p_i^3$; the first cancels the (2) piece, leaving only

$$-\delta_{ik}\,D\,\beta\,a\,z_i\,\frac{s_i}{p_i^3}.$$

Hence

$$\frac{\partial^2\Pi(\bar p,\bar p)}{\partial p_i\partial p_k}=D(1)\,a(\bar p)\,\bar\chi\,\frac{s_is_k}{p_ip_k}-\delta_{ik}\,D\,\beta\,a(\bar p)\,z_i\,\frac{s_i}{p_i^3}.$$

Eliminating z_i. As in Proposition 5, $(p_i-z_i)/p_i=1/\bar\eta_D$ gives $z_i/p_i=\big(\bar\eta_D-1\big)/\bar\eta_D$, so

$\beta\, a\, z_i\, s_i/p_i^3 = \beta\, a\, \frac{\bar\eta_D - 1}{\bar\eta_D}\, s_i/p_i^2$. Therefore, with $g(\bar p) = (s_1/p_1, \ldots, s_n/p_n)^\top$,

$$\frac{\partial^2 \Pi(\bar p, \bar p)}{\partial p\, \partial p} = D(1)\, a(\bar p)\, \bar\chi\, g(\bar p) g(\bar p)^\top - \bar\eta_d\, a(\bar p)\, D(1) \frac{\bar\eta_D - 1}{\bar\eta_D}\, \mathrm{diag}\Big(\frac{s_i(\bar p)}{\bar p_i^2}\Big),$$

as claimed. ■

Proof. (of Proposition 7) At the fixed point the two CES identities $S \equiv \sum_k \omega_k (p_k - z_k) d(p_k/a) = a/\bar\eta_D$ and $S_{p_i} = a_{p_i}$ reduce the five-term Hessian to the rank-one-plus-diagonal form for $\frac{\partial^2 \bar\Pi}{\partial p \partial p}$; the cross matrix $\frac{\partial^2 \bar\Pi}{\partial p \partial P} = \partial^2_{p_i A} \pi \cdot a_{P_k}$ evaluates to $D\,(a/\bar\eta_D)\, \bar\eta'_D\, s_i s_k/(p_i p_k)$. Both share the common prefactor $D\, a\, g g^\top$, so adding the rank-one loadings gives $\bar\chi + \bar\eta'_D/\bar\eta_D = 1 + \bar\eta_d - \bar\eta_D - \bar\eta_d/\bar\eta_D$, which factors as $-(\bar\eta_D - 1)(\bar\eta_D - \bar\eta_d)/\bar\eta_D$. The diagonal cost term comes from $\frac{\partial^2 \bar\Pi}{\partial p \partial p}$ alone.

Lemma 1 *Let $\bar s \in \mathbb{R}^n$ satisfy $\sum_{i=1}^n \bar s_i = 1$ and $\bar s_i > 0$ for all i, let $\bar\eta_d > 0$, and let $\nu \in \mathbb{R}$. Define*

$$\bar Q \;=\; \nu\, \bar s \bar s^\top + \bar\eta_d\, \mathrm{diag}(\bar s).$$

Then $\bar Q$ is positive definite if and only if $\nu + \bar\eta_d > 0$.

Proof. For any $x \in \mathbb{R}^n$, the quadratic form associated with $\bar Q$ is

$$x^\top \bar Q x \;=\; \nu\, (\bar s^\top x)^2 \;+\; \bar\eta_d \sum_{i=1}^n \bar s_i x_i^2. \tag{47}$$

Sufficiency. Suppose $\nu + \bar\eta_d > 0$ and take any $x \neq 0$. First note that $\sum_{i=1}^n \bar s_i x_i^2 > 0$, since $\bar s_i > 0$ for all i and $x_i^2 > 0$ for at least one i. If $\nu \geq 0$, both terms in equation (47) are nonnegative and the second is strictly positive, so $x^\top \bar Q x > 0$. Suppose instead $\nu < 0$. Writing $\bar s_i x_i = \sqrt{\bar s_i} \cdot \sqrt{\bar s_i}\, x_i$ and applying the Cauchy–Schwarz inequality,

$$(\bar s^\top x)^2 = \left(\sum_{i=1}^n \sqrt{\bar s_i}\, \big(\sqrt{\bar s_i}\, x_i\big) \right)^2 \;\leq\; \left(\sum_{i=1}^n \bar s_i \right) \left(\sum_{i=1}^n \bar s_i x_i^2 \right) \;=\; \sum_{i=1}^n \bar s_i x_i^2, \tag{48}$$

where the last equality uses $\sum_{i=1}^n \bar s_i = 1$. Since $\nu < 0$, multiplying equation (48) by ν reverses the inequality, and substituting into equation (47) gives

$$x^\top \bar Q x \;\geq\; \nu \sum_{i=1}^n \bar s_i x_i^2 \;+\; \bar\eta_d \sum_{i=1}^n \bar s_i x_i^2 \;=\; \big(\nu + \bar\eta_d\big) \sum_{i=1}^n \bar s_i x_i^2 \;>\; 0.$$

Hence $x^\top \bar Q x > 0$ for all $x \neq 0$, i.e. $\bar Q$ is positive definite.

Necessity. Suppose $\nu + \bar{\eta}_d \leq 0$. Evaluate the quadratic form at $x = \mathbf{1} = (1, \ldots, 1)^\top \neq 0$. Since $\bar{s}^\top \mathbf{1} = \sum_i \bar{s}_i = 1$,

$$\mathbf{1}^\top \bar{Q}\, \mathbf{1} \;=\; \nu\, (\bar{s}^\top \mathbf{1})^2 + \bar{\eta}_d \sum_{i=1}^{n} \bar{s}_i \;=\; \nu + \bar{\eta}_d \;\leq\; 0,$$

so $\bar{Q}$ is not positive definite. This proves the claim. ■

Now we can prove Proposition 8.

Proof. (of Proposition 8) The proof uses that by definition of $\tilde{\Pi}, x$ and X we have:

$$\begin{aligned}
\Pi(\bar{p}, \bar{p})\, \frac{\partial \tilde{\Pi}\,(0,0)}{\partial x} &= \frac{\partial \Pi\,(\bar{p}, \bar{p})}{\partial p}\, \mathrm{diag}\Big(\bar{p}_i\Big) \\
\Pi(\bar{p}, \bar{p})\, Q \equiv -\frac{\partial^2 \tilde{\Pi}\,(0,0)}{\partial x \partial x} &= -\,\mathrm{diag}\Big(\bar{p}_i\Big)\, \frac{\partial^2 \Pi\,(\bar{p}, \bar{p})}{\partial p \partial p}\, \mathrm{diag}\Big(\bar{p}_i\Big) - \frac{\partial \Pi\,(\bar{p}, \bar{p})}{\partial p}\, \mathrm{diag}\Big(\bar{p}_i\Big) \\
&= -\,\mathrm{diag}\Big(\bar{p}_i\Big)\, \frac{\partial^2 \Pi\,(\bar{p}, \bar{p})}{\partial p \partial p}\, \mathrm{diag}\Big(\bar{p}_i\Big) \\
\Pi(\bar{p}, \bar{p})\, \Theta \equiv -\frac{\partial^2 \tilde{\Pi}\,(0,0)}{\partial x \partial X} &= -\,\mathrm{diag}\Big(\bar{p}_i\Big)\, \frac{\partial^2 \Pi\,(\bar{p}, \bar{p})}{\partial p \partial P}\, \mathrm{diag}\Big(\bar{p}_i\Big)
\end{aligned}$$

where the expression for Q uses that the gradient vanishes due to the first order conditions. Using the expressions in Propositions 6 and 7 for the matrices of second derivatives of Π, and of g we obtain the desired results for the expressions.

For the positive definiteness of Q and $Q + \Theta$ we use Lemma 1 as well as the maintained assumptions that $\bar{\eta}_d > 0$ and that $\bar{\eta}_D > 1$. For Q set $\nu = \bar{\eta}_D - \bar{\eta}_d + \frac{\bar{\eta}'_D}{\bar{\eta}_D - 1}$ so $\nu + \bar{\eta}_d > 0$ is equivalent to $\bar{\eta}_D + \frac{\bar{\eta}'_D}{\bar{\eta}_D - 1} > 0$, which under the assumption that $\bar{\eta}_D > 1$ is equivalent to the condition in the proposition. Again using that $\bar{\eta}_D > 1$, Q and $\bar{Q}$ are proportional to each other. For $Q + \Theta$ note that $\nu = \bar{\eta}_D - \bar{\eta}_d$ so $\nu + \bar{\eta}_d > 0$ is the same as $\bar{\eta}_D > 0$. Thus, using again that $\bar{\eta}_D > 1$, we have that $Q + \Theta$ is proportional to $\bar{Q}$.

■