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ONETARY ECONOMICS AT 30: A REEXAMINATION OF THE RELEVANCE OF MONEY IN CASHLESS LIMITING MONETARY ECONOMIES

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ABSTRACT

The well-known cashless-limiting result in Woodford (1998) has become the theoretical foundation for a large body of work that treats the costs and benefits of holding money as irrelevant for monetary transmission. I reexamine this result and find that it relies on a peculiar credit-market structure consisting of perfectly competitive, zero-interest deferred payment arrangements. I show that the result breaks down when the microstructure is generalized to allow for an endogenous interest rate and market power in credit intermediation. The tenuousness of this influential result should give pause to the widespread practice of basing monetary policy advice on models without money.

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1 Introduction

Over the past thirty years, a professional consensus has emerged among academics and policymakers that artificial economies without money provide an adequate framework for analyzing monetary policy.

This view has been popularized by textbook treatments of the New Keynesian model:

> *The modeling approach favored in much of the recent monetary literature, and the one adopted in this book [...] does not incorporate money explicitly in the analysis. [...] Such model economies can be viewed as a limiting case (the cashless limit) of an economy in which money is valued and held by households. Woodford (2003) provides a detailed discussion and a forceful defense of that approach.*
>
> —Galí (2008, p. 34)

Woodford's (2003) "forceful defense" hinges on the claim that abstracting from money entails no loss of generality, because the microeconomic frictions that give rise to money demand, the costs and benefits of holding money, and liquidity or medium-of-exchange considerations more broadly, are inessential to the transmission of monetary policy:

> *The basic model (developed beginning in Chapter 2) is one that abstracts from monetary frictions, in order to focus attention on more essential aspects of the monetary transmission mechanism...*
>
> —Woodford (2003, p. 32)

Generations of macroeconomists have internalized this view as established wisdom. As a result, applied work in monetary economics now typically abstracts from money altogether, or if money is present, it is treated as a redundant asset, and the incentives that underlie money demand are left out of the analysis.

My objective is to reexamine the cashless-limiting approximation result to which Galí alludes—a result originally derived in Woodford (1998), which has become the theoretical foundation of the moneyless approach to monetary economics. I find that this result relies on a contrived model of credit, popular in the 1980s, consisting of perfectly competitive, zero-interest deferred payment arrangements. I show that the result breaks down when the market microstructure is generalized to allow for an endogenous interest rate and market power in credit intermediation.

The implication is that this result is too narrow to justify the widespread use of moneyless models to guide monetary policy.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the stationary monetary equilibrium. Section 4 characterizes the equilibrium of the nonmonetary economy. Section 5 reviews the pure-credit/cashless-limiting result in Woodford (1998). Section 6 reexamines that result. Section 7 studies social welfare in the monetary economy, the nonmonetary economy, and their respective pure-credit/cashless limits. Section 8 reflects on the importance of market microstructure for monetary economics. Section 9 concludes. Appendix A contains all proofs. Appendix B answers frequently asked questions.

2 Model

Time is represented by a sequence of periods indexed by $t \in \{0, 1, ...\}$, each divided into two stages. There are three classes of infinitely lived agents: *bankers, consumers*, and *producers*, denoted B, C, and P, respectively. An agent in class $i \in \{B, C, P\}$ is represented with a point in the set $\mathcal{I}_i = [0, 1]$. In the first stage, each producer has a labor endowment and a linear technology to transform labor into good 1, which only consumers consume. In the second stage, all agents have a labor endowment and a linear technology to transform labor into good 2, which is consumed by all agents. Goods are nonstorable across stages.

A government issues *money*—a durable, intrinsically useless financial security (i.e., it is neither an input nor an actionable claim to inputs of utility or production functions). The supply of money at the beginning of period t is denoted M_t, with $M_0 \in \mathbb{R}_{++}$ given, and distributed uniformly among consumers. In the second stage of every period t, the government pays nominal interest $r_t^M \in \mathbb{R}_+$ on agents' money holdings, and injects or withdraws money via lump-sum transfers, $T_t^M \in \mathbb{R}$, to consumers. The government budget constraint is $M_{t+1} - M_t = r_t^M M_t + T_t^M$ for all t, where $\{M_t\}_{t=0}^{\infty}$ is a sequence of money supplies, $\{r_t^M\}_{t=0}^{\infty}$ is a sequence of interest rates on money holdings, and $\{T_t^M\}_{t=0}^{\infty}$ is a sequence of monetary lump-sum transfers to consumers. These sequences constitute the monetary-fiscal policy, which is determined exogenously by the government and taken as given by agents.

The market structure is as follows. In the second stage there is a spot Walrasian market for good 2, labor, and money. In the first stage, three distinct spot Walrasian markets operate contemporaneously: a *cash-goods market* where good 1 can be exchanged for money, a *loan market* where money can be exchanged for bonds (privately issued loans), and a *credit-goods* *market* where good 1 can be exchanged for money or deferred-payment claims.

I make the following assumptions on agents' abilities to commit and enforce agreements. At the beginning of every period, consumers and producers are randomly partitioned (uniformly and independently over time) into two *trust types*, indexed by $k \in \{T, N\}$. Consumers and producers of trust type T are *on-trust agents*: on-trust consumers can commit to honoring their deferred-payment promises to on-trust producers. Consumers and producers of trust type N are *no-trust agents*: no-trust consumers cannot commit to honoring their deferred-payment promises to no-trust producers. The fraction of agents in class $i \in \{C, P\}$ of trust type $k \in \{T, N\}$ is denoted $\alpha_i^k \in [0, 1]$, with $\sum_{k \in \{T,N\}} \alpha_i^k = 1$. Bankers can commit to honoring their promises to creditors and enforce promises from their debtors.[1]

Market participation in the first stage is heterogeneous across agents, with the following structure. All on-trust agents have access to the credit-goods market, and are labeled *credit consumers*, or *credit producers*. Since on-trust agents are the only ones who access the credit-goods market (and no other market), they constitute a *market-access type*, denoted T. All bankers have access to the loan market (and no other market). All no-trust agents have access to the cash-goods market, no access to the credit-goods market, and are randomly partitioned (uniformly and independently over time) into two additional market-access types, denoted B or M, depending on their ability to access the loan market. No-trust agents of market-access type M cannot access the loan market; they are *cash consumers*, or *cash producers*. No-trust agents of market-access type B can access the loan market; they are *banked consumers*, or *banked producers*. The fraction of agents in class $i \in \{C, P\}$ of market-access type $j \in \{B, M, T\} \equiv \mathbb{A}$, denoted $\alpha_i^j \in [0, 1]$, is α_i^T for $j = T$, and $\left(1 - \alpha_i^T\right) \alpha_i^j$ for $j \in \{B, M\}$, with $\sum_{j \in \{B,M\}} \alpha_i^j = 1$. In the second stage, all agents have access to the spot Walrasian market where they can trade good 2, labor, and money competitively.

The terms of trade in the first stage are determined as follows. The cash-goods market and the credit-goods market are perfectly competitive. Banked consumers can trade competitively in the loan market, while banked producers can only trade in the loan market indirectly, through bilateral trade with bankers. Each producer-banker pair negotiates the quantities of bonds and money that the banker will buy or sell in the competitive loan market on behalf of the producer,

[1] This ability to trust debtors and be trusted by creditors allows bankers to act as intermediaries between no-trust consumers and no-trust producers. In the first subperiod, no-trust consumers can sell bonds to no-trust producers *through bankers*. In the second subperiod, no-trust consumers repay their debts to bankers, who forward these payments to the bondholders. Without bankers as intermediaries, no-trust consumers would have to rely exclusively on money accumulated in the previous period to purchase good 1 from no-trust producers.

and an intermediation fee for the banker's service. The banker's fee is expressed in terms of good 2, and paid in the second stage. The terms of this bilateral trade are determined by Nash bargaining, where the producer has bargaining power $\theta \in [0, 1]$.

The timeline in a typical period consists of the following sequence of events. Agents enter the first stage with money accumulated in the previous period's second stage. At the beginning of the first stage, agents in class $i \in \{C, P\}$ are assigned a trust type $k \in \{T, N\}$. Next, producers produce good 1. Then, no-trust consumers and producers (trust-type N) are assigned a market-access type (M or B). After this, agents trade in the first-stage markets to which they have access, followed by consumption of good 1. Agents then enter the second stage with portfolios of money and claims to good 2; they settle their debts, consume good 2, and choose their money holdings for the following period.

The preferences of an agent in class $i \in \{B, C, P\}$ are represented by

$$\mathbb{E}_0^i \sum_{t=0}^{\infty} \beta^t \left[u\left(y_{it}\right) \mathbb{I}_{\{i=C\}} - \kappa y_{it} \mathbb{I}_{\{i=P\}} + x_t \right],$$

where the expectation operator, $\mathbb{E}_0^i$, is with respect to the probability measure induced by the random market access in the first stage; $\beta \in (0, 1)$ is the discount factor; $u : \mathbb{R}_+ \to \mathbb{R}$ is the consumer's utility function for good 1; $\mathbb{I}_{\{\cdot\}}$ is an indicator function that equals 1 if the condition in the subscript is satisfied, and 0 otherwise; $\kappa \in \mathbb{R}_{++}$ is the producer's marginal (disutility) cost of producing good 1; $y_{it} \in \mathbb{R}_+$ is the agent's consumption (if $i = C$) or production (if $i = P$) of good 1 in period t; and $x_t \in \mathbb{R}$ is the agent's consumption of good 2 (or production, if negative) in the second stage of period t. Assume u is twice continuously differentiable, with $u'' < u(0) = 0 < u'$, and that there exists $\Upsilon(\kappa) \in \mathbb{R}_{++}$ such that $u'(\Upsilon(\kappa)) = \kappa$. Let $\{y_{Ct}^*, y_{Pt}^*\}_{t=0}^{\infty}$ denote efficient allocation that solves the problem of a social planner who maximizes the equally weighted sum of all agents' expected discounted utilities, where y_{Ct}^* is the individual consumption of good 1, and y_{Pt}^* is the individual production of good 1. Then, it is immediate that $y_{Ct}^* = y_{Pt}^* = \Upsilon(\kappa)$.

2.1 Motivation, Interpretation, and Canonical Monetary Frameworks

In this section I explain the rationale for the key modeling assumptions, provide interpretations, and draw connections with canonical frameworks in monetary economics.

The market structure is general enough to allow buyers to purchase on credit as well as with money, in a way that nests canonical frameworks of the transactions demand for money as special cases. For example, if we shut down the bond and credit-goods markets (by setting $\alpha_i^B = \alpha_i^T = 0$

for $i \in \{C, P\}$), stage 1 reduces to the goods-trading stage in the cash-in-advance framework of Lucas (1980). If we shut down the bond market (by setting $\alpha_i^B = 0$ for $i \in \{C, P\}$), stage 1 becomes an explicit market microstructure for the goods-trading stage in the cash/credit-goods generalization of the cash-in-advance framework proposed by Lucas and Stokey (1983). If we shut down the credit-goods market (by setting $\alpha_i^T = 0$ for $i \in \{C, P\}$) and assume all consumers have access to the loan market (i.e., $\alpha_C^M = 0$), the model specializes to Lagos and Zhang (2022). Stage 2 is the counterpart to the "securities-trading stage" in the canonical cash-in-advance and cash/credit-goods frameworks, and delivers analytical tractability by making agents' portfolio choices independent of their trading histories, as in the literature that builds on Lagos and Wright (2005).[2]

As in most explicit models of transactions demand, I motivate the role of money by introducing a double-coincidence-of-wants problem—specifically, between consumers and producers of good 1 in the cash-goods market. The claims to good 2 exchanged in the credit-goods market represent the familiar form of "credit" in cash/credit-goods models: a zero-interest, deferred-payment trade-credit arrangement between trusted buyers and sellers (see Lucas and Stokey (1983, pp. 78–79) and Lucas (1984, pp. 20–24) for similar interpretations). Given the model's abstract structure, the claims to good 2 traded in the bond market admit multiple interpretations. One is literal: these are bonds issued by consumers, ultimately purchased by producers but intermediated by bankers who act as brokers and collect fees from bond buyers—a market structure reminiscent of Duffie et al. (2005). Bankers could also be viewed as rudimentary credit-card companies. Alternatively, bankers may represent deposit-taking institutions that accept deposits from producers, issue loans to consumers, trade competitively in an interbank market, and earn fees from depositors—implying deposit rates below the interbank rate.

The notion of money in the model also admits several interpretations. When $r_t^M = 0$, money can be viewed as a conventional non-interest-bearing physical currency. When $r_t^M > 0$, it resembles a nominal, one-period, risk-free government bond that is perfectly liquid in the sense that—if valued—it can be used to purchase goods in both trading stages. Alternatively, money in the model could be interpreted as interest-bearing currency, potentially implemented

[2]Strictly speaking, the environment in this paper does not nest the canonical Lagos–Wright model, as the cash-goods market here is competitive rather than bilateral (i.e., mediated by search with terms of trade determined by bargaining). I adopt a competitive cash-goods market to highlight that the main results do not rely on search and bargaining—features typically assumed in the literature on the microfoundations of money demand. That said, there are well-known versions of the Lagos–Wright framework with competitive pricing (see, e.g., Rocheteau and Wright (2005)), which this model does nest.

by a Central Bank Digital Currency (CBDC) infrastructure.[3]

Monetary policy in the model is characterized by two instruments: the path of the money supply, $\{M_t\}_{t=0}^{\infty}$, and the path of the interest rate on money, $\{r_t^M\}_{t=0}^{\infty}$. The former can be interpreted as resulting from conventional open-market operations, which, in this simple model, take the form of lump-sum transfers to or from consumers. The latter can be viewed as a rudimentary approximation of a monetary operating framework with interest-bearing reserves, akin to the contemporary system employed by the Federal Reserve in the United States.

2.2 Individual Optimization

Let $W_{it}(a_t^m, a_t^g)$ denote the maximum expected discounted payoff of an agent in class $i \in \{B, C, P\}$ upon entering the second stage of period t with $a_t^m \in \mathbb{R}_+$ dollars, and claims to $a_t^g \in \mathbb{R}$ units of good 2. Let $\bar{V}_{Ct}(a_t^m)$ denote a consumer's maximum expected discounted payoff upon entering the first stage of period t with a_t^m dollars. Then,

$$W_{Ct}(a_t^m, a_t^g) = \max_{(a_{t+1}^m, x_t) \in \mathbb{R}_+ \times \mathbb{R}} \left[x_t + \beta \bar{V}_{Ct+1}\left(a_{t+1}^m\right)\right] \quad (1)$$
$$\text{s.t. } p_{2t} x_t + a_{t+1}^m \leq p_{2t} a_t^g + R_t^M a_t^m + T_t^M,$$

where p_{2t} is the nominal price of good 2 in the second stage, and $R_t^M \equiv 1 + r_t^M$. Let m_{t+1} denote the end-of-period-t money holding that solves (1). Let $\bar{V}_{it}$ denote the maximum expected discounted payoff of an agent in class $i \in \{B, P\}$ upon entering the first stage of period t.[4] Then,

$$W_{it}(a_t^m, a_t^g) = a_t^g + \frac{1}{p_{2t}} R_t^M a_t^m + \beta \bar{V}_{it+1}, \quad (2)$$

for $i \in \{B, P\}$.

Next, consider the individual optimization problems that each consumer and producer faces in the first stage of period t, conditional on the realization of their market-access type. A *cash consumer* with money holding a_t^m, solves

$$V_{Ct}^M(a_t^m) = \max_{(\hat{y}_t, \hat{a}_t^m) \in \mathbb{R}_+^2} \left[u(\hat{y}_t) + W_{Ct}(\hat{a}_t^m, 0)\right] \quad (3)$$
$$\text{s.t. } p_{1t}^N \hat{y}_t + \hat{a}_t^m \leq a_t^m,$$

[3]The model is too abstract to distinguish between paper money (conventional currency, or *cash*), digital money (such as a CBDC), or book-entry money (e.g., a government-issued bond). All that is implicitly assumed in the equations is that M_t is outside money (i.e., not a claim on another agent), that ownership is commonly agreed upon, and that agents cannot double-spend or forge it.

[4]To simplify the exposition, I assume bankers and producers do not hold money between time periods. This is without loss of generality since their marginal value of a dollar in the second stage is lower than a consumer's.

where p_{1t}^{N} is the nominal price of good 1 in the cash-goods market (i.e., the nominal price faced by no-trust consumers), $\hat{y}_t$ is consumption of good 1, and $\hat{a}_t^m$ is the post-trade nominal money balance. Let $\left(y_{Ct}^{M}(\cdot), m_{Ct}^{M}(\cdot)\right)$ be the trading rules that solve (3). A *cash producer* with inventory y_t of good 1, solves

$$V_{Pt}^{M}(y_t) = \max_{\hat{a}_t^m \in \mathbb{R}_+} W_{Pt}(\hat{a}_t^m, 0) \qquad (4)$$
$$\text{s.t. } \hat{a}_t^m \leq p_{1t}^{N} y_t.$$

Let $m_{Pt}^{M}(\cdot)$ be the trading rule that solves (4).

A *banked consumer* with money holding a_t^m, solves

$$V_{Ct}^{B}(a_t^m) = \max_{\left(\bar{y}_t, \bar{a}_t^m, \bar{a}_t^b\right) \in \mathbb{R}_+^2 \times \mathbb{R}} \left[u(\bar{y}_t) + W_{Ct}\left(\bar{a}_t^m, \bar{a}_t^b\right)\right] \qquad (5)$$
$$\text{s.t. } p_{1t}^{N} \bar{y}_t + \bar{a}_t^m + q_t^B \bar{a}_t^b \leq a_t^m,$$

where $\bar{y}_t$ is consumption of good 1, $\bar{a}_t^m$ is the post-trade nominal money balance, $\bar{a}_t^b$ is the quantity of claims to good 2 (to be redeemed in the next stage) that the banked consumer buys (if positive) or issues (if negative), and q_t^B is the nominal price of these claims in the loan market. Let $\left(y_{Ct}^{B}(\cdot), m_{Ct}^{B}(\cdot), b_{Ct}^{B}(\cdot)\right)$ be the trading rules that solve (5). A *banked producer* with inventory y_t of good 1 bargains with a banker over his post-trade portfolio of money and bonds, $\left(m_{Pt}^{B}(y_t), b_{Pt}^{B}(y_t)\right)$, as well as the service fee that the producer pays the banker, $\phi_{Pt}^{B}(y_t)$. The bargaining outcome, $\left(\phi_{Pt}^{B}(y_t), m_{Pt}^{B}(y_t), b_{Pt}^{B}(y_t)\right)$, is the solution to

$$\max_{\left(\phi_t, \bar{a}_t^m, \bar{a}_t^b\right) \in \mathbb{R}_+^2 \times \mathbb{R}} \left[W_{Pt}\left(\bar{a}_t^m, \bar{a}_t^b - \phi_t\right) - \underline{W}_{PT}(y_t)\right]^{\theta} \left[W_{Bt}(0, \phi_t) - W_{Bt}(0, 0)\right]^{1-\theta} \qquad (6)$$
$$\text{s.t. } \bar{a}_t^m + q_t^B \bar{a}_t^b \leq p_{1t}^{N} y_t \quad \text{ and } \quad \underline{W}_{Pt}(y_t) \leq W_{Pt}\left(\bar{a}_t^m, \bar{a}_t^b - \phi_t\right),$$

with $\underline{W}_{PT}(y_t) \equiv W_{Pt}\left(m_{Pt}^{M}(y_t), 0\right)$. The last constraint ensures the banked producer's payoff from bargaining is at least as large as his outside option, which in a monetary economy equals the payoff from mimicking the optimal trade of a cash producer.[5] Thus, the first-stage value of a banked producer with inventory y_t of good 1, is

$$V_{Pt}^{B}(y_t) = W_{Pt}\left(m_{Pt}^{B}(y_t), b_{Pt}^{B}(y_t) - \phi_{Pt}^{B}(y_t)\right).$$

In what follows, let

$$r_t^B \equiv \frac{p_{2t}}{q_t^B} - 1$$

[5] To simplify the exposition, I am anticipating bankers do not carry money into the second stage.

denote the nominal interest rate implicit in a bond issued in the first-stage loan market.[6]

A *credit consumer* with money holding a_t^m, solves

$$V_{Ct}^T(a_t^m) = \max_{(\tilde{y}_t, \tilde{a}_t^m, \tilde{a}_t^b) \in \mathbb{R}_+^2 \times \mathbb{R}} \left[u(\tilde{y}_t) + W_{Ct}\left(\tilde{a}_t^m, \tilde{a}_t^b\right) \right] \tag{7}$$
$$\text{s.t. } p_{1t}^T \tilde{y}_t + \tilde{a}_t^m + q_t^T \tilde{a}_t^b \leq a_t^m,$$

where p_{1t}^T is the nominal price of good 1 in the credit-goods market (i.e., the nominal price faced by on-trust consumers), $\tilde{y}_t$ is consumption of good 1, $\tilde{a}_t^m$ is the post-trade nominal money balance, $\tilde{a}_t^b$ is the quantity of deferred-payment claims to good 2 (to be redeemed in the next stage) that the credit consumer buys (if positive) or issues (if negative), and q_t^T is the nominal price at which these claims are exchanged in the credit-goods market. Let $\left(y_{Ct}^T(\cdot), m_{Ct}^T(\cdot), b_{Ct}^T(\cdot)\right)$ be the trading rules that solve (7). A *credit producer* with inventory y_t of good 1, solves

$$V_{Pt}^T(y_t) = \max_{(\tilde{a}_t^m, \tilde{a}_t^b) \in \mathbb{R}_+ \times \mathbb{R}} W_{Pt}\left(\tilde{a}_t^m, \tilde{a}_t^b\right) \tag{8}$$
$$\text{s.t. } \tilde{a}_t^m + q_t^T \tilde{a}_t^b \leq p_{1t}^T y_t.$$

Let $\left(m_{Pt}^T(\cdot), b_{Pt}^T(\cdot)\right)$ be the trading rules that solve (8). In what follows, let

$$r_t^T \equiv \frac{p_{2t}}{q_t^T} - 1$$

denote the nominal interest rate implicit in a claim to good 2 issued in the credit-goods market.[7]

Finally, the beginning-of-period values of bankers, consumers, and producers, are respectively,

$$\bar{V}_{Bt} = \left(1 - \alpha_P^T\right) \alpha_P^B W_{Bt}\left(0, \phi_{Pt}^B(y_t)\right) + \left[1 - \left(1 - \alpha_P^T\right) \alpha_P^B\right] W_{Bt}(0, 0)$$
$$\bar{V}_{Ct}(a_t^m) = \alpha_C^T V_{Ct}^T(a_t^m) + \left(1 - \alpha_C^T\right) \sum_{j \in \{B, M\}} \alpha_C^j V_{Ct}^j(a_t^m),$$

[6]One dollar invested in these first-stage claims delivers $\frac{p_{2t}}{q_t^B}$ dollars in the following stage (because a dollar buys $\frac{1}{q_t^B}$ claims, and each claim delivers one unit of good 2 in the following stage, and at that point each unit of good 2 is worth p_{2t} dollars). Thus, $\frac{p_{2t}}{q_t^B} \equiv 1 + r_t^B$ is the gross nominal interest rate implicit in a claim to good 2 issued in the first-stage loan market.

[7]One dollar invested in these first-stage claims delivers $\frac{p_{2t}}{q_t^T}$ dollars in the following stage (because a dollar buys $\frac{1}{q_t^T}$ claims, and each claim delivers one unit of good 2 in the following stage, and at that point each unit of good 2 is worth p_{2t} dollars). Thus, $\frac{p_{2t}}{q_t^T} \equiv 1 + r_t^T$ is the gross nominal interest rate implicit in a claim to good 2 issued in the first-stage credit-goods market.

and $\bar{V}_{Pt} = \alpha_P^T \Pi_{Pt}^T + \left(1 - \alpha_P^T\right) \Pi_{Pt}^N$, with

$$\Pi_{Pt}^T \equiv \max_{y \in \mathbb{R}_+} \left[-\kappa y + V_{Pt}^T (y)\right] \tag{9}$$

$$\Pi_{Pt}^N \equiv \max_{y \in \mathbb{R}_+} \left[-\kappa y + \sum_{j \in \{B,M\}} \alpha_P^j V_{Pt}^j (y)\right]. \tag{10}$$

Let y_{Pt}^T and y_{Pt}^N be the solutions to (9), and (10), respectively.

For equilibrium analysis it is convenient to define some auxiliary variables. Let $\phi_t^k \equiv \frac{p_{1t}^k}{p_{2t}}$ for $k \in \{T, N\}$, $\varphi_t^T \equiv \left(1 + r_t^T\right) \phi_t^T$, and $\varphi_t^j \equiv \left(1 + r_t^j\right) \phi_t^N$ for $j \in \{B, M\}$. Intuitively, ϕ_t^T and ϕ_t^N are the *bare relative prices* of good 1 in terms of good 2 in the credit-goods and cash-goods markets, respectively. The variables φ_t^M, φ_t^B, and φ_t^T, are the bare relative prices augmented by the financial (opportunity) cost of purchasing good 1. This cost equals: the foregone interest rate on money provided by the government, r_t^M, for cash consumers; the borrowing rate in the loan market, r_t^B, for banked consumers; or the deferred-payment interest rate, r_t^T, for credit consumers. Put differently, φ_t^j is the *cum-interest relative price* of good 1 (in terms of good 2) that is relevant for the first-stage consumption decisions of consumers of market-access type $j \in \mathbb{A}$. For any $x \in \mathbb{R}_+$, let $\Upsilon(x) \equiv u'^{-1}(x)$ denote the demand for good 1 by a (financially unconstrained) consumer whose cum-interest relative price of good 1 (in terms of good 2) is equal to x. Finally, define the *nominal risk-free illiquid bond rate*,

$$r_{t+1}^I \equiv \frac{1}{\beta} \frac{p_{2t+1}}{p_{2t}} - 1.$$

Intuitively, r_{t+1}^I is the interest rate on an *illiquid*, one-period, pure-discount nominal bond issued in the second stage of period t and redeemed in the second stage of $t + 1$.[8]

2.3 Definition of Equilibrium

Let $\boldsymbol{p}_t \equiv \left(p_{1t}^N, p_{1t}^T, p_{2t}\right)$ denote the nominal prices of consumption goods in period t. Let $\boldsymbol{q}_t \equiv \left(q_t^B, q_t^T\right)$ denote the vector of nominal prices of claims to good 2 issued in the first stage of period t, where q_t^B and q_t^T are the prices of claims issued in the loan market, and the credit-goods market, respectively.

Let $\boldsymbol{y}_{Pt} \equiv \left(y_{Pt}^T, y_{Pt}^N\right)$, where y_{Pt}^k is the period-t production of good 1 chosen by a typical producer of trust type $k \in \{T, N\}$. Let $\boldsymbol{y}_{Ct}(m_t) \equiv \left(y_{Ct}^j (m_t)\right)_{j \in \mathbb{A}}$, where $y_{Ct}^j (m_t)$ is the period-t

[8] The bond is *illiquid* in the sense that it cannot be traded in the first stage of period $t + 1$.

consumption of good 1 chosen by a typical consumer of market-access type $j \in \mathbb{A}$ who enters the period with m_t dollars.

Let $\boldsymbol{a}_t^m(m_t, \boldsymbol{y}_{Pt}) \equiv (\boldsymbol{m}_{Ct}(m_t), \boldsymbol{m}_{Pt}(\boldsymbol{y}_{Pt}))$, with $\boldsymbol{m}_{Ct}(m_t) \equiv \left(m_{Ct}^j(m_t)\right)_{j\in\mathbb{A}}$ and $\boldsymbol{m}_{Pt}(\boldsymbol{y}_{Pt}) \equiv \left(m_{Pt}^j\left(y_{Pt}^N\right), m_{Pt}^T\left(y_{Pt}^T\right)\right)_{j\in\{B,M\}}$, where $m_{Ct}^j(m_t)$ is the money holding chosen at the end of the first stage of period t by a typical consumer of market-access type $j \in \mathbb{A}$ who enters the period with m_t dollars; and $m_{Pt}^j\left(y_{Pt}^k\right)$ is the money holding chosen at the end of the first stage of period t by a typical producer of market-access type $j \in \mathbb{A}$ who brings to the market inventory y_{Pt}^k of good 1.

Let $\boldsymbol{a}_t^b(m_t, \boldsymbol{y}_{Pt}) \equiv (\boldsymbol{b}_{Ct}(m_t), \boldsymbol{b}_{Pt}(\boldsymbol{y}_{Pt}))$, where $\boldsymbol{b}_{Ct}(m_t) \equiv \left(b_{Ct}^B(m_t), b_{Ct}^T(m_t)\right)$ is the lending position of banked consumers and credit consumers, respectively, at the end of the first stage of period t, and $\boldsymbol{b}_{Pt}(\boldsymbol{y}_{Pt}) \equiv \left(b_{Pt}^B\left(y_{Pt}^N\right), b_{Pt}^T\left(y_{Pt}^T\right)\right)$ is the lending position of banked producers and credit producers, respectively, at the end of the first stage of period t.

Definition 1. Given a government policy, $\left\{M_t, R_t^M, T_t^M\right\}_{t=0}^{\infty}$, an equilibrium is a sequence of prices, $\{\boldsymbol{p}_t, \boldsymbol{q}_t\}_{t=0}^{\infty}$, and individual decisions, including second-stage money demand, $\{m_{t+1}\}_{t=0}^{\infty}$, and first-stage choices of production, consumption, holdings of money and financial claims, and banking fees, $\left\{\boldsymbol{y}_{Pt}, \boldsymbol{y}_{Ct}(m_t), \boldsymbol{a}_t^m(m_t, \boldsymbol{y}_{Pt}), \boldsymbol{a}_t^b(m_t, \boldsymbol{y}_{Pt}), \phi_{Pt}^B\left(y_{Pt}^N\right)\right\}_{t=0}^{\infty}$, such that for all t: *(i)* Individual decisions are optimal given prices: m_{t+1} is the end-of-period consumer money demand that solves (1), $\left(y_{Ct}^M(\cdot), m_{Ct}^M(\cdot)\right)$ solves (3), $m_{Pt}^M(\cdot)$ solves (4), $\left(y_{Ct}^B(\cdot), m_{Ct}^B(\cdot), b_{Ct}^B(\cdot)\right)$ solves (5), $\left(m_{Pt}^B(\cdot), b_{Pt}^B(\cdot), \phi_{Pt}^B(\cdot)\right)$ solves (6), $\left(y_{Ct}^T(\cdot), m_{Ct}^T(\cdot), b_{Ct}^T(\cdot)\right)$ solves (7), $\left(m_{Pt}^T(\cdot), b_{Pt}^T(\cdot)\right)$ solves (8), y_{Pt}^T solves (9), and y_{Pt}^N solves (10); and *(ii)* Prices are such that all markets clear: $m_{t+1} = M_{t+1}$ (second-stage money demand equals money supply), $\left(1-\alpha_C^T\right)\sum_{j\in\{B,M\}}\alpha_C^j y_{Ct}^j(M_t) = \left(1-\alpha_P^T\right)y_{Pt}^N$ (demand for good 1 equals supply in the cash-goods market), $\alpha_C^T y_{Ct}^T(M_t) = \alpha_P^T y_{Pt}^T$ (demand for good 1 equals supply in the credit-goods market), $\left(1-\alpha_C^T\right)\alpha_C^B b_{Ct}^B(M_t) + \left(1-\alpha_P^T\right)\alpha_P^B b_{Pt}^B\left(y_{Pt}^N\right) = 0$ (net supply of claims to good 2 equals zero for banked consumers and producers), $\alpha_C^T b_{Ct}^T(M_t) + \alpha_P^T b_{Pt}^T\left(y_{Pt}^T\right) = 0$ (net supply of claims to good 2 equals zero for credit consumers and producers). An equilibrium is (*permanently*) *monetary* if $\boldsymbol{p}_t < \infty$ for all t.

For each time t, Propositions 20-22 (Appendix A.1) describe the agents' optimal allocations and payoffs as functions of six prices: $\left(\left(\phi_t^k, p_{1t}^k\right)_{k\in\{N,T\}}, \varphi_t^B, \varphi_t^T\right)$.[9] Thus, an equilibrium

[9] All the other prices and interest rates that appear in the allocations and payoffs described in these propositions, i.e., $\left(p_{2t}, \left(r_t^j, q_t^j\right)_{j\in\{B,T\}}, \varphi_t^M\right)$, are implied by $\left(\left(\phi_t^k, p_{1t}^k\right)_{k\in\{N,T\}}, \varphi_t^B, \varphi_t^T\right)$, as follows. Since $\phi_t^k \equiv \frac{p_{1t}^k}{p_{2t}}$, p_{2t} is implied by ϕ_t^k and p_{1t}^k for $k \in \{N,T\}$. Since $\varphi_t^j \equiv \left(1+r_t^j\right)\left(\phi_t^T \mathbb{I}_{\{j=T\}} + \phi_t^N \mathbb{I}_{\{j\in\{B,M\}\}}\right)$ for $j \in \{B,M,T\}$,

can be summarized by a path, $\left\{\left(\phi_t^k, p_{1t}^k\right)_{k\in\{N,T\}}, r_t^B, r_t^T\right\}_{t=0}^{\infty}$: Given an equilibrium path for these six prices, the equilibrium allocations and payoffs follow from Propositions 20-22, and the equilibrium prices, $\{\varphi_t^B, \varphi_t^T\}_{t=0}^{\infty}$ and $\{\boldsymbol{p}_t, \boldsymbol{q}_t\}_{t=0}^{\infty}$, follow from the definitions $\varphi_t^B \equiv \left(1+r_t^B\right)\phi_t^N$; $\varphi_t^T \equiv \left(1+r_t^T\right)\phi_t^T$; $p_{2t} \equiv \frac{p_{1t}^k}{\phi_t^k}$ for $k \in \{N,T\}$; $q_t^B \equiv \frac{p_{1t}^N}{\varphi_t^B}$, and $q_t^T \equiv \frac{p_{1t}^T}{\varphi_t^T}$.

3 Stationary Monetary Equilibrium

To streamline the exposition and focus on the core mechanisms driving the main results, I henceforth specialize the analysis to the class of *stationary* monetary equilibria.[10]

Definition 2. Assume monetary policy is *time-invariant*, that is: (i) $\frac{M_{t+1}}{M_t} = \mu$ for all t; and (ii) $r_t^M = r^M$ for all t. A *stationary monetary equilibrium* is a monetary equilibrium that satisfies, for all t: (a) $0 < \frac{M_t}{p_{2t}} \equiv z_2$ and $0 < \frac{M_t}{p_{1t}^k} \equiv z_1^k$ for $k \in \{N,T\}$; and (b) $r_t^j = r^j$ for $j \in \{B,T\}$. In turn, conditions (i), (ii), (a), and (b) imply, for all t: $\phi_t^k = \phi^k$ for $k \in \{N,T\}$, $\varphi_t^j = \varphi^j$ for $j \in \mathbb{A}$, and $r_t^I = \frac{\mu-\beta}{\beta} \equiv r^I$.

Since an equilibrium can be summarized by a path, $\left\{\left(\phi_t^k, p_{1t}^k\right)_{k\in\{N,T\}}, r_t^B, r_t^T\right\}_{t=0}^{\infty}$, a stationary monetary equilibrium can be summarized by a vector, $\left(\left(\phi^k, z_1^k\right)_{k\in\{N,T\}}, r^B, r^T\right)$. The corresponding nominal prices, $\{\boldsymbol{p}_t, \boldsymbol{q}_t\}_{t=0}^{\infty}$, follow from the definitions: $z_1^k \equiv \frac{M_t}{p_{1t}^k}$, $\phi_t^k \equiv \frac{p_{1t}^k}{p_{2t}}$, and $1 + r_t^j \equiv \frac{p_{2t}}{q_t^j}$, which in a stationary monetary equilibrium, imply: $p_{1t}^k = \frac{M_t}{z_1^k}$, $p_{2t} = \frac{p_{1t}^k}{\phi^k}$, and $q_t^j = \frac{p_{2t}}{1+r^j}$ for $j \in \{B,T\}$ and $k \in \{N,T\}$. The following proposition presents the conditions that characterize a stationary monetary equilibrium for economies with $\alpha_j^i \in (0,1)$ for $i \in \{B,M,T\}$ and $j \in \{C,P\}$.[11]

Proposition 3. *A stationary monetary equilibrium with positive production of good 1 by on-trust and no-trust producers is characterized by a vector,* $\left(\left(\phi^k, z_1^k\right)_{k\in\{N,T\}}, r^B, r^T\right)$, *with* $0 < z_1^k$ *for* $k \in \{N,T\}$, *that satisfies the following conditions:*

1. *Market clearing for claims traded by consumers and producers of market-access type* T*:*

$$r^M - r^T = 0 \le z_1^T, \tag{11}$$

then for $j \in \{B,T\}$, r_t^j is implied by $\left(\phi_t^k\right)_{k\in\{N,T\}}$ and φ_t^j. Given p_{2t} and $\left(r_t^B, r_t^T\right)$, the bond price q_t^j is implied by $1 + r_t^j \equiv \frac{p_{2t}}{q_t^j}$ for $j \in \{B,T\}$. Also, $\varphi_t^M \equiv \left(1+r_t^M\right)\phi_t^N$, so φ_t^M is implied by ϕ_t^N and the policy parameter r_t^M.

[10] This involves no meaningful loss of generality, as the main results hold for dynamic and stochastic sunspot equilibria (see Lagos and Zhang (2022, Appendix C)).

[11] The cashless-limiting economies that obtain when $\alpha_j^M \to 0$ or $\alpha_j^T \to 1$ for $j \in \{C,P\}$, are treated in subsequent sections.

where $z_1^T = \frac{\phi^N}{\phi^T} z_1^N$.

2. *Market clearing for claims traded by consumers and producers of market-access type B:*

$$r^M - r^B \leq 0 \leq \mathrm{S}\left(r^B\right) - \mathrm{D}\left(r^B\right), \tag{12}$$

where the second "$\leq$" holds as "$=$" if $r^M < r^B$, *with*

$$\begin{aligned}\mathrm{D}\left(r^B\right) &\equiv \alpha_C^B \left(1-\alpha_C^T\right)\left[\mathrm{Y}\left(\left(1+r^B\right)\phi^N\right) - z_1^N\right]\\ \mathrm{S}\left(r^B\right) &\equiv \alpha_P^B \left(1-\alpha_C^T\right)\left[\alpha_C^M \min\left\{\mathrm{Y}\left(\left(1+r^M\right)\phi^N\right), z_1^N\right\} + \alpha_C^B \mathrm{Y}\left(\left(1+r^B\right)\phi^N\right)\right].\end{aligned}$$

3. *Profit maximization of no-trust producers:*

$$\phi^N = \frac{\kappa}{1 + r^M + \alpha_P^B \theta\left(r^B - r^M\right)}. \tag{13}$$

4. *Profit maximization of on-trust producers:*

$$\phi^T = \frac{\kappa}{1 + r^M}. \tag{14}$$

5. *Euler equation for money holdings:*

$$\begin{aligned} r^I - r^M = \left(1-\alpha_C^T\right)\Bigg[&\alpha_C^B\left(r^B - r^M\right)\\ &+ \alpha_C^M \max\left\{\left[u'\left(z_1^N\right) - \left(1+r^M\right)\phi^N\right]\frac{1}{\phi^N}, 0\right\}\Bigg]. \end{aligned} \tag{15}$$

Real consumption of good 1 is $\min\left\{\mathrm{Y}\left(\left(1+r^M\right)\phi^N\right), z_1^N\right\}$ *for a cash consumer;* $\mathrm{Y}\left(\left(1+r^B\right)\phi^N\right)$ *for a banked consumer, and* $\mathrm{Y}\left(\left(1+r^T\right)\phi^T\right)$ *for a credit consumer. Total real production of good 1 by all on-trust producers is* $\alpha_C^T \mathrm{Y}\left(\left(1+r^T\right)\phi^T\right)$, *and total real production of good 1 by all no-trust producers is* $\left(1-\alpha_C^T\right)\left[\alpha_C^M \min\left\{\mathrm{Y}\left(\left(1+r^M\right)\phi^N\right), z_1^N\right\} + \alpha_C^B \mathrm{Y}\left(\left(1+r^B\right)\phi^N\right)\right]$.

Part 1 of Proposition 3 establishes that in a stationary monetary equilibrium, the nominal interest on a claim to good 2 traded in the credit-goods market must equal the administered interest rate on money. The logic is straightforward: no on-trust agent would be willing to hold money if $0 < r^T - r^M$, since money would then be strictly dominated in terms of its stage-1 rate of return.

Part 2 consists of two conditions. The first inequality states that the equilibrium bond rate must be at least as large as the interest rate on money, a standard no-arbitrage implication. The second inequality is the bond market-clearing condition: $\mathrm{S}\left(r^B\right)$ denotes the supply of credit (bond demand by banked producers), and $\mathrm{D}\left(r^B\right)$ denotes the demand for credit (bond supply by banked consumers). If the parameters support a stationary monetary equilibrium with $r^M < r^B$, then market clearing requires $\mathrm{D}\left(r^B\right) = \mathrm{S}\left(r^B\right)$. If instead $\mathrm{D}\left(r^M\right) < \mathrm{S}\left(r^M\right)$, then the stationary monetary equilibrium has $r^B = r^M$.

Parts 3 and 4 use the expected profit per unit of good 1 for no-trust and on-trust producers, respectively, i.e., $\left[1 + r^M + \alpha_P^B \theta \left(r^B - r^M\right)\right] \phi^N - \kappa$, and $\left(1 + r^T\right) \phi^T - \kappa$, to characterize the bare relative prices of good 1 in terms of good 2: ϕ^N in the cash-goods market, and ϕ^T in the credit-goods market.

Finally, Part 5 states the first-order condition for a consumer's money-demand problem at the end of stage 2. On the left side of (15) is the financial return on the illiquid bond in excess of the financial return on money, i.e., the spread $r^I - r^M$, which represents the opportunity cost of holding money. On the right side is the marginal liquidity return on money in excess of the liquidity return on the illiquid bond (which equals zero). This marginal liquidity return consists of two terms. The first term is the interest saved by spending a dollar held in advance rather than borrowing it—relevant if the consumer is a banked consumer, which occurs with probability $\left(1 - \alpha_C^T\right) \alpha_C^B$. The second term is the marginal return from using a dollar to purchase good 1—relevant if the consumer is a cash consumer, which occurs with probability $\left(1 - \alpha_C^T\right) \alpha_C^M$. In conventional cash-in-advance and search-based models (e.g., Lucas (1980); Lagos and Wright (2005)), the Euler equation for money includes only the second term, since they abstract from stage-1 credit. By contrast, only the first term appears in Lagos and Zhang (2022), since they abstract from cash-only consumers.

The next theorem establishes existence and uniqueness of a stationary monetary equilibrium, and describes the equilibrium prices and allocations. Let

$$\omega \equiv \frac{r^I - r^M}{1 + r^M} \tag{16}$$

and

$$\bar{\omega} \equiv \left(1 - \alpha_C^T\right) \alpha_C^M \left[\frac{u'\left(\underline{z}\left(r^M\right)\right)}{u'\left(\bar{z}\left(r^M\right)\right)} - 1\right],$$

where the functions $\underline{z}(\cdot) : [r^M, \infty) \to \mathbb{R}$, and $\bar{z}(\cdot) : [r^M, \infty) \to \mathbb{R}$, are defined by

$$\underline{z}(r) \equiv \frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B}\mathrm{Y}\left(\frac{1+r}{1+r^M+\alpha_P^B\theta\left(r-r^M\right)}\kappa\right),$$

and

$$\bar{z}(r) \equiv \mathrm{Y}\left(\frac{1+r^M}{1+r^M+\alpha_P^B\theta\left(r-r^M\right)}\kappa\right).$$

The parameter ω denotes the spread between the interest rate on the illiquid risk-free nominal bond and the interest rate on money (expressed as a proportion of the gross return on money); it is determined by the monetary-policy variables μ and r^M, capturing the opportunity cost of holding a dollar. The function $\underline{z}(r)$ denotes the demand for good 1 chosen by a liquidity-constrained cash consumer when the bond rate is equal to $r \in \left(r^M, \infty\right)$. The function $\bar{z}(r)$ is the demand for good 1 that would be chosen by a liquidity-unconstrained cash consumer when the bond rate equals $r \in \left[r^M, \infty\right)$.

Theorem 4. *There exists a stationary monetary equilibrium,* $\left(\left(\phi^k, z_1^k\right)_{k\in\{N,T\}}, r^B, r^T\right)$*, for any* $\omega \geq 0$*, and the equilibrium is unique if* $\omega > 0$*.*

1. *The interest rate on bonds traded by no-trust, banked consumers and producers in the loan market is*

$$r^B = \begin{cases} r^{B*} & \text{if } \bar{\omega} < \omega \\ r^M & \text{if } 0 \leq \omega \leq \bar{\omega}, \end{cases}$$

where $r^{B*} \in \left(r^M, \infty\right)$ *is the unique solution to*

$$\omega = \left(1-\alpha_C^T\right)\left\{\alpha_C^M\left[\frac{u'\left(\underline{z}\left(r^{B*}\right)\right)}{u'\left(\bar{z}\left(r^{B*}\right)\right)}-1\right]+\alpha_C^B\frac{r^{B*}-r^M}{1+r^M}\right\}.$$

The interest rate on deferred-payment claims traded by on-trust consumers and producers in the credit-goods market is $r^T = r^M$.

2. *The bare relative price of good 1 in terms of good 2 in the cash-goods market is*

$$\phi^N = \frac{\kappa}{1+r^M+\alpha_P^B\theta\left(r^B-r^M\right)},$$

and the bare relative price of good 1 in terms of good 2 in the credit-goods market is

$$\phi^T = \frac{\kappa}{1+r^M}.$$

3. *The real value of the money supply in terms of cash goods is*

$$z_1^N \begin{cases} = \frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B} Y\left(\frac{1+r^{B*}}{1+r^M+\alpha_P^B\theta\left(r^{B*}-r^M\right)}\kappa\right) & \text{if } \bar{\omega} < \omega \\ = Y\left(\left(1+\frac{\omega}{\left(1-\alpha_C^T\right)\alpha_C^M}\right)\kappa\right) & \text{if } 0 < \omega \leq \bar{\omega} \\ \in \left[Y\left(\kappa\right), \infty\right) & \text{if } 0 = \omega. \end{cases}$$

The real value of the money supply in terms of credit goods is $z_1^T = \frac{\phi^N}{\phi^T} z_1^N$*, and the real value of the money supply in terms of good 2 is* $z_2 \equiv \frac{M_t}{p_{2t}} = \phi^N z_1^N$.

4. *The nominal price of good 1 in the cash-goods market is* $p_{1t}^N \equiv \frac{M_t}{z_1^N}$*, the nominal price of good 1 in the credit-goods market is* $p_{1t}^T \equiv \frac{M_t}{z_1^T} = \frac{\phi^T}{\phi^N} p_{1t}^N$*, and the nominal price of good 2 is* $p_{2t} = \frac{1}{\phi^N} p_{1t}^N$.

5. *Real aggregate consumption of good 1 is*

$$Y_C \equiv \alpha_C^T y_C^T + \left(1-\alpha_C^T\right) \sum_{j\in\{B,M\}} \alpha_C^j y_C^j,$$

where y_C^j *is real consumption of good 1 of an individual consumer of market-access type* $j \in \{B, M, T\}$*, with*

$$y_C^B = Y\left(\frac{1+r^B}{1+r^M+\alpha_P^B\theta\left(r^B-r^M\right)}\kappa\right)$$
$$y_C^M = \begin{cases} \frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B} Y\left(\frac{1+r^{B*}}{1+r^M+\alpha_P^B\theta\left(r^{B*}-r^M\right)}\kappa\right) & \text{if } \bar{\omega} < \omega \\ Y\left(\left(1+\frac{\omega}{\left(1-\alpha_C^T\right)\alpha_C^M}\right)\kappa\right) & \text{if } 0 \leq \omega \leq \bar{\omega} \end{cases}$$
$$y_C^T = Y\left(\kappa\right).$$

6. *Velocity of money in stage 1 is*

$$v \equiv \frac{p_{1t}^T \alpha_C^T y_C^T + p_{1t}^N \left(1-\alpha_C^T\right) \sum_{j\in\{B,M\}} \alpha_C^j y_C^j}{M_t}$$
$$\begin{cases} = \alpha_C^T \frac{y_C^B}{z_1^N} + \left(1-\alpha_C^T\right)\left(\alpha_C^B \frac{y_C^B}{z_1^N} + \alpha_C^M\right) & \text{if } 0 < \omega \\ \in (0, 1] & \text{if } 0 = \omega. \end{cases}$$

7. *As* $\omega \to 0$,

   (a) $r^B \to r^M$.

(b) $y_C^j \to \mathrm{Y}(\kappa)$ for $j \in \{B, M\}$, and $Y_C \to \mathrm{Y}(\kappa)$.

(c) $z_1^k \to \mathrm{Y}(\kappa)$ for $k \in \{N, T\}$, $z_2 \to \frac{\kappa}{1+r^M}\mathrm{Y}(\kappa)$, $p_{1t}^k \to \frac{M_t}{\mathrm{Y}(\kappa)}$ for $k \in \{N, T\}$, $p_{2t} \to \frac{1+r^M}{\kappa}\frac{M_t}{\mathrm{Y}(\kappa)}$, and $v \to 1$.

8. As $\omega \to \infty$,

(a) $r^B \to \infty$.

(b) $y_C^B \to \mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)$, $y_C^M \to \frac{\alpha_C^B(1-\alpha_P^B)}{\alpha_C^B(1-\alpha_P^B)+\alpha_P^B}\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)$, and

$$Y_C \to \alpha_C^T \mathrm{Y}(\kappa) + \left(1-\alpha_C^T\right)\frac{\alpha_C^B}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B}\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right).$$

(c) $z_1^N \to \frac{\alpha_C^B(1-\alpha_P^B)}{\alpha_C^B(1-\alpha_P^B)+\alpha_P^B}\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)$, $z_1^T \to 0$, $z_2 \to 0$, $p_{1t}^N \to \frac{\alpha_C^B(1-\alpha_P^B)+\alpha_P^B}{\alpha_C^B(1-\alpha_P^B)}\frac{M_t}{\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)}$, $p_{1t}^T \to \infty$, $p_{2t} \to \infty$, and $v \to \infty$.

Theorem 4 provides a detailed characterization of the stationary monetary equilibrium. Part 1 describes the spread between the bond-market rate, r^B, and the administered interest rate on money, r^M: when the opportunity cost of holding money, ω, exceeds the threshold $\bar{\omega}$, then $r^B > r^M$; otherwise, $r^B = r^M$. Part 2 characterizes the relative prices of good 1 in terms of good 2 in the credit-goods and the cash-goods market. Part 3 describes real money balances. Part 4 describes nominal prices. Part 5 characterizes the consumption allocation, and Part 6 gives the velocity of money in stage 1. Part 7 generalizes a standard result in classical monetary models: as $\omega \to 0$, the excess return on bonds over money, $r^B - r^M$, vanishes (item (a)), and the equilibrium allocation converges to the efficient allocation (items (b) and (c)). Finally, Part 8 shows that as $\omega \to \infty$, both the bond rate (item (a)) and velocity (item (c)) diverge, while consumption (item (b)) remains strictly positive for all market-access types.

The stationary monetary equilibrium is determined by a system of two equations in the two unknowns $\left(z_1^N, r^B\right)$, as illustrated in Figures 1-3. The upward-sloping curve labeled $r^B = \mathscr{M}\left(z_1^N\right)$ represents combinations of real money balances and bond rates that satisfy the consumer's Euler equation for money (equation (15)). The nonincreasing curve labeled $r^B = \mathscr{B}\left(z_1^N\right)$ captures combinations consistent with bond-market clearing. Graphically, the stationary monetary equilibrium corresponds to the unique point of intersection of these two curves. The region to the left of the dotted boundary labeled $r^B = \mathscr{L}\left(z_1^N\right)$ consists of the pairs $\left(z_1^N, r^B\right)$ for which the cash consumer is liquidity constrained—that is, their cash-in-advance constraint binds. Figures 1-3 correspond to an economy with $\bar{\omega} < \omega$, $0 < \omega \leq \bar{\omega}$, and $0 = \omega$, respectively. These graphs illustrate that $\frac{\partial z_1^N}{\partial \omega} < 0 \leq \frac{\partial r^B}{\partial \omega}$, with "=" holding only if $0 < \omega \leq \bar{\omega}$. Intuitively, real money balances decline with the opportunity cost of holding money, and in parametrizations where the bond rate is strictly positive, the resulting rise in credit demand by banked consumers bids up the equilibrium bond rate (see Corollary 29).

4 Equilibrium of the Nonmonetary Economy

In this section I consider an economy without money, where credit arrangements are the only means to settle purchases of good 1 in stage 1. In this context, consumers and producers who draw market-access type M remain in autarky in stage 1. Thus, at time t, the first-stage payoff of a cash consumer is $\mathcal{V}_{Ct}^M = \beta \bar{\mathcal{V}}_{Ct+1}$, and the first-stage payoff of a cash producer with inventory y_t, is $\mathcal{V}_{Pt}^M(y_t) = \beta \bar{\mathcal{V}}_{Pt+1}$, where $\bar{\mathcal{V}}_{it+1}$ denotes the maximum expected discounted payoff of an agent in class $i \in \{B, C, P\}$ upon entering the first stage of period $t+1$. Let $\mathcal{W}_{it}(b_t)$ denote the maximum expected discounted payoff of an agent in class $i \in \{B, C, P\}$ who enters the second stage of period t with claims to $b_t \in \mathbb{R}$ units of good 2. Then, $\mathcal{W}_{it}(b_t) = b_t + \beta \bar{\mathcal{V}}_{it+1}$.

Consider the individual optimization problems that each consumer and producer faces in the first stage of period t, conditional on the realization of their market-access type $j \in \{B, T\}$. A *banked consumer* solves

$$\mathcal{V}_{Ct}^B = \max_{(y_t, b_t) \in \mathbb{R}_+ \times \mathbb{R}} \left[u(y_t) + \mathcal{W}_{Ct}(b_t) \right] \tag{17}$$
$$\text{s.t. } \tilde{\varphi}_t^B y_t + b_t \leq 0,$$

where y_t is consumption of good 1, b_t is the quantity of claims to good 2 (to be redeemed in the next stage) that the banked consumer buys (if positive) or issues (if negative), and $\tilde{\varphi}_t^B$ is the relative price of good 1 in terms of these claims to good 2. The solution to (17) is: $\mathrm{y}_{Ct}^B = \mathrm{Y}\left(\tilde{\varphi}_t^B\right)$ and $\mathrm{b}_{Ct}^B = -\tilde{\varphi}_t^B \mathrm{Y}\left(\tilde{\varphi}_t^B\right)$. A *banked producer* with inventory y_t of good 1 bargains with a banker over his post-trade portfolio of bonds, $\mathrm{b}_{Pt}^B(y_t)$, as well as the service fee that the producer pays the banker, $\psi_{Pt}^B(y_t)$. The bargaining outcome, $\left(\psi_{Pt}^B(y_t), \mathrm{b}_{Pt}^B(y_t)\right)$, is the solution to

$$\max_{(\psi_t, b_t) \in \mathbb{R}_+ \times \mathbb{R}} \left[\mathcal{W}_{Pt}(b_t - \psi_t) - \mathcal{W}_{Pt}(0) \right]^\theta [\psi_t]^{1-\theta} \tag{18}$$
$$\text{s.t. } b_t \leq \tilde{\varphi}_t^B y_t \quad \text{and} \quad \mathcal{W}_{Pt}(0) \leq \mathcal{W}_{Pt}(b_t - \psi_t).$$

Recall that in the monetary economy, the banked producer's outside option when bargaining with a banker was equal to the value of mimicking the optimal behavior of a cash producer. In the economy without money, monetary settlement is not an option so the banked producer's outside option when bargaining with a banker is autarky in the current period, i.e., $\mathcal{W}_{Pt}(0)$. The solution to (18) is $\mathrm{b}_{Pt}^{B}(y_t)=\tilde{\varphi}_t^{B}y_t$ and $\psi_{Pt}^{B}(y_t)=(1-\theta)\,\tilde{\varphi}_t^{B}y_t$. Thus, the first-stage value of a banked producer with inventory y_t of good 1, is

$$\mathcal{V}_{Pt}^{B}(y_t)=\mathcal{W}_{Pt}\left(\theta\tilde{\varphi}_t^{B}y_t\right)=\theta\tilde{\varphi}_t^{B}y_t+\beta\bar{\mathcal{V}}_{Pt+1}.$$

A *credit consumer* solves

$$\begin{aligned}\mathcal{V}_{Ct}^{T}=&\max_{(y_t,b_t)\in\mathbb{R}_+\times\mathbb{R}}\left[u(y_t)+\mathcal{W}_{Ct}(b_t)\right]\\&\text{s.t. }\tilde{\varphi}_t^{T}y_t+b_t\leq 0,\end{aligned}\tag{19}$$

where y_t is consumption of good 1, b_t is the quantity of claims to good 2 (to be redeemed in the next stage) that the credit consumer buys (if positive) or issues (if negative), and $\tilde{\varphi}_t^{T}$ is the relative price of good 1 in terms of these claims issued in the credit-goods market. The solution to (19) is: $\mathrm{y}_{Ct}^{T}=\mathrm{Y}\left(\tilde{\varphi}_t^{T}\right)$ and $\mathrm{b}_{Ct}^{T}=-\tilde{\varphi}_t^{T}\mathrm{Y}\left(\tilde{\varphi}_t^{T}\right)$. The first-stage payoff of a *credit producer* with inventory y_t of good 1, is:

$$\mathcal{V}_{Pt}^{T}(y_t)=\mathrm{b}_{Pt}^{T}(y_t)+\beta\bar{\mathcal{V}}_{Pt+1},\text{ with }\mathrm{b}_{Pt}^{T}(y_t)=\tilde{\varphi}_t^{T}y_t.\tag{20}$$

Finally, the beginning-of-period values of bankers, consumers, and producers, are respectively,

$$\begin{aligned}\bar{\mathcal{V}}_{Bt}&=\left(1-\alpha_P^{T}\right)\alpha_P^{B}\mathcal{W}_{Bt}\left(\psi_{Pt}^{B}(y_t)\right)+\left[1-\left(1-\alpha_P^{T}\right)\alpha_P^{B}\right]\mathcal{W}_{Bt}(0)\\\bar{\mathcal{V}}_{Ct}&=\alpha_C^{T}\mathcal{V}_{Ct}^{T}+\left(1-\alpha_C^{T}\right)\sum_{j\in\{B,M\}}\alpha_C^{j}\mathcal{V}_{Ct}^{j},\end{aligned}$$

$\bar{\mathcal{V}}_{Pt}=\alpha_P^{T}\tilde{\Pi}_{Pt}^{T}+\left(1-\alpha_P^{T}\right)\tilde{\Pi}_{Pt}^{N}$, with

$$\tilde{\Pi}_{Pt}^{T}\equiv\max_{y\in\mathbb{R}_+}\left[-\kappa y+\mathcal{V}_{Pt}^{T}(y)\right]\tag{21}$$

$$\tilde{\Pi}_{Pt}^{N}\equiv\max_{y\in\mathbb{R}_+}\left[-\kappa y+\sum_{j\in\{B,M\}}\alpha_P^{j}\mathcal{V}_{Pt}^{j}(y)\right].\tag{22}$$

Let y_{Pt}^{T} and y_{Pt}^{N} denote the solutions to (21) and (22), respectively.

Let $\tilde{\boldsymbol{\varphi}}_t\equiv\left(\tilde{\varphi}_t^{N},\tilde{\varphi}_t^{T}\right)$ denote the vector of relative prices of good 1 in terms of claims to good 2 issued in the first stage of period t. Let $\mathbf{y}_{Ct}\equiv\left(\mathrm{y}_{Ct}^{B},\mathrm{y}_{Ct}^{T}\right)$, where y_{Ct}^{j} is the period-t consumption of good 1 chosen by a typical consumer of market-access type $j\in\{B,T\}$.

Let $\mathbf{y}_{Pt} \equiv \left(y_{Pt}^N, y_{Pt}^T\right)$, where y_{Pt}^k is the period-t production of good 1 chosen by a typical producer of trust type $k \in \{T, N\}$. Let $\mathbf{b}_{Ct} \equiv \left(b_{Ct}^B, b_{Ct}^T\right)$ denote the lending position of banked consumers and credit consumers, respectively, at the end of the first stage of period t. Let $\mathbf{b}_{Pt}\left(\mathbf{y}_{Pt}\right) \equiv \left(b_{Pt}^B\left(y_{Pt}^N\right), b_{Pt}^T\left(y_{Pt}^T\right)\right)$ denote the lending position of banked producers and credit producers, respectively, at the end of the first stage of period t.

Definition 5. An equilibrium of the nonmonetary economy is a sequence of relative prices, $\{\tilde{\boldsymbol{\varphi}}_t\}_{t=0}^{\infty}$, and first-stage allocations of production and consumption, holdings financial claims, and banking fees, $\left\{\mathbf{y}_{Pt}, \mathbf{y}_{Ct}, \mathbf{b}_{Ct}, \mathbf{b}_{Pt}\left(\mathbf{y}_{Pt}\right), \psi_{Pt}^B\left(y_{Pt}^N\right)\right\}_{t=0}^{\infty}$, such that for all t: (i) All individual decisions are optimal given prices: $\left(y_{Ct}^B, b_{Ct}^B\right)$ solves (17), $b_{Pt}^B(\cdot)$ and $\psi_{Pt}^B(\cdot)$ solve (18), $\left(y_{Ct}^T, b_{Ct}^T\right)$ solves (19), $b_{Pt}^T(\cdot)$ is given by (20), y_{Pt}^T solves (21), and y_{Pt}^N solves (22); and (ii) Prices are such that all markets clear: $\left(1-\alpha_C^T\right)\alpha_C^B y_{Ct}^B = \left(1-\alpha_P^T\right)\alpha_P^B y_{Pt}^N$ (demand equals supply in the market for good 1 between banked consumers and producers); and $\alpha_C^T y_{Ct}^T = \alpha_P^T y_{Pt}^T$ (demand equals supply in the market for good 1 between credit consumers and producers).

Theorem 6. *There exists a unique equilibrium of the nonmonetary economy.*

1. *The relative price of good 1 in terms of claims to good 2 issued in the credit-goods market is $\tilde{\varphi}_t^T = \kappa$ for all t. The relative price of good 1 in terms of claims to good 2 traded between banked consumers and producers is $\tilde{\varphi}_t^B = \frac{\kappa}{\alpha_P^B \theta}$ for all t.*

2. *In every period t, real consumption of good 1 of an individual consumer of market-access type $j \in \{B, T\}$, is*

$$y_{Ct}^j = \begin{cases} \mathrm{Y}(\kappa) & \text{for } j = T \\ \mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right) & \text{for } j = B, \end{cases}$$

and real aggregate consumption of good 1 is $\tilde{\mathrm{Y}}_C \equiv \alpha_C^T \mathrm{Y}(\kappa) + \left(1-\alpha_C^T\right)\alpha_C^B \mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)$.

3. *As $\alpha_j^B \to 1$ for $j \in \{C, P\}$, $\tilde{\mathrm{Y}}_C \to \alpha_C^T \mathrm{Y}(\kappa) + \left(1-\alpha_C^T\right)\mathrm{Y}\left(\frac{\kappa}{\theta}\right)$.*

Theorem 6 characterizes the equilibrium of the nonmonetary economy. Part 1 describes the relevant relative prices of good 1 in terms of claims to good 2, both in the credit-goods market and the markets where banked consumers trade with banked producers. Part 2 characterizes the consumption allocation: it is efficient for on-trust consumers, inefficiently low for banked consumers (as long as $\alpha_P^B \theta < 1$), and zero for unbanked no-trust consumers. Part 3 shows that, as long as bankers have market power (i.e., $\theta < 1$), the consumption of no-trust consumers remains inefficiently low even in pure-credit limit in which all no-trust consumers are banked.

5 Cashless Limits 101: Foundations of the M-irrelevance Canon

This section reviews the widely cited cashless-limiting approximation result in Woodford (1998), which forms the theoretical starting point for the moneyless approach to monetary economics. Woodford (1998) assumes a variant of the cash/credit-goods version of the cash-in-advance formulation of Lucas and Stokey (1983), which as discussed in Section 2, corresponds to the special case of our model in which nobody has access to the bond market; that is, $\alpha_i^B = 1-\alpha_i^M = 0$ for $i \in \{C, P\}$.[12] In this case, Proposition 3 implies the following result:

Corollary 7. *Assume $\alpha_i^B = 0$ for $i \in \{C, P\}$, and define $L(z) \equiv \max\left\{u'(z)\frac{1}{\kappa} - 1, 0\right\}$ for any $z \in \mathbb{R}_+$. There exists a stationary monetary equilibrium, $\left(\left(\phi^k, z_1^k\right)_{k\in\{N,T\}}, r^T\right)$, for any $\omega \in \left[0, \left(1-\alpha_C^T\right)L(0)\right)$, and the equilibrium is unique if $\omega > 0$.*

1. *The interest rate on deferred-payment claims traded by on-trust consumers and producers in the credit-goods market is $r^T = r^M$.*

2. *The bare relative price of good 1 in terms of good 2 is the same in the cash-goods and in the credit-goods market: $\phi^N = \phi^T = \frac{\kappa}{1+r^M} \equiv \phi$.*

3. *The real value of the money supply in terms of cash goods, z_1^N, is given by the Euler equation for money holdings, i.e.,*

$$\omega \geq \left(1-\alpha_C^T\right)L\left(z_1^N\right),$$

with "=" if $0 < z_1^N$, which implies

$$z_1^N \begin{cases} = 0 & \text{if } \left(1-\alpha_C^T\right)L(0) \leq \omega \\ = Y\left(\left(1+\frac{\omega}{1-\alpha_C^T}\right)\kappa\right) & \text{if } 0 < \omega < \left(1-\alpha_C^T\right)L(0) \\ \in \left[Y(\kappa), \infty\right) & \text{if } \omega = 0. \end{cases}$$

The real value of the money supply in terms of credit goods is $z_1^T = z_1^N \equiv z_1$, and the real value of the money supply in terms of good 2 is $z_2 \equiv \frac{M_t}{p_{2t}} = \phi z_1$.

[12] Woodford's variant differs from Lucas and Stokey (1983) in two ways. While Stokey and Lucas assume there are two goods—a "credit good" and a "cash good"—Woodford assumes each household consumes a set consisting of a continuum of goods, which is exogenously partitioned into a subset of "cash goods" and a subset of "credit goods". Woodford's motivation for this extension is to introduce a parameter that measures the size of the set of cash goods in order to study the equilibrium as this parameter converges to zero—what he refers to as the "cashless limit" of the monetary economy. The second difference from Lucas and Stokey (1983) is that Woodford assumes a more elaborate timing for the settlement of cash-goods purchases, with the purpose of ensuring that the model admits a specific money-in-the-utility-function representation that he had used in prior work.

4. *The nominal price of good 1 in the cash-goods market is* $p_{1t}^N \equiv \frac{M_t}{z_1}$, *the nominal price of good 1 in the credit-goods market is* $p_{1t}^T = p_{1t}^N \equiv p_{1t}$, *and the nominal price of good 2 is* $p_{2t} = \frac{1}{\phi} p_{1t}$.

5. *Real aggregate consumption of good 1 is* $Y_C \equiv \alpha_C^T y_C^T + \left(1 - \alpha_C^T\right) y_C^M$, *where* y_C^j *is real consumption of good 1 of an individual consumer of market-access type* $j \in \{M, T\}$, *with*

$$y_C^M = \begin{cases} 0 & \text{if } \left(1-\alpha_C^T\right) L(0) \le \omega \\ \textsc{y}\left(\left(1 + \frac{\omega}{1-\alpha_C^T}\right)\kappa\right) & \text{if } 0 \le \omega < \left(1-\alpha_C^T\right) L(0) \end{cases}$$
$$y_C^T = \textsc{y}(\kappa).$$

6. *Velocity of money in stage 1, i.e.,* $v \equiv \frac{p_{1t}\left[\alpha_C^T y_C^T + \left(1-\alpha_C^T\right) y_C^M\right]}{M_t}$, *is*

$$v \begin{cases} = \infty & \text{if } \left(1-\alpha_C^T\right) L(0) \le \omega \\ = \alpha_C^T \frac{\textsc{y}(\kappa)}{\textsc{y}\left(\left(1+\frac{\omega}{1-\alpha_C^T}\right)\kappa\right)} + 1 - \alpha_C^T & \text{if } 0 < \omega < \left(1-\alpha_C^T\right) L(0) \\ \in (0, 1] & \text{if } 0 = \omega. \end{cases}$$

7. *As* $\omega \to 0$,

   (a) $y_C^M \to \textsc{y}(\kappa)$, *and* $Y_C \to \textsc{y}(\kappa)$.

   (b) $z_1 \to \textsc{y}(\kappa)$, $z_2 \to \frac{\kappa}{1+r^M}\textsc{y}(\kappa)$, $p_{1t} \to \frac{M_t}{\textsc{y}(\kappa)}$, $p_{2t} \to \frac{1+r^M}{\kappa}\left[\textsc{y}(\kappa)\right]^{-1} M_t$, *and* $v \to 1$.

8. *As* $\omega \to \left(1-\alpha_C^T\right) L(0)$,

   (a) $y_C^M \to 0$, *and* $Y_C \to \alpha_C^T \textsc{y}(\kappa)$.

   (b) $z_i \to 0$ *and* $p_{it} \to \infty$ *for* $i \in \{1, 2\}$, *and* $v \to \infty$.

9. *If* $\omega > 0$, *then as* $\alpha_C^T \to 1$,

   (a) $y_C^M \to 0$, *and* $Y_C \to \textsc{y}(\kappa)$.

   (b) $z_i \to 0$ *and* $p_{it} \to \infty$ *for* $i \in \{1, 2\}$, *and* $v \to \infty$.

Woodford (1998) is interested in the limiting case in which the fraction of purchases of good 1 that must be settled with cash converges to zero. This corresponds to Part 9 of Corollary 7.[13] As $\alpha_C^T \to 1$, the fraction of cash consumers converges to zero, and the monetary economy

[13] The parameter α_C^T corresponds to "α" in Woodford (1998, p. 180). Woodford (1998) allows α to be a deterministic function of time in Section 3, which turns out to be immaterial to the main results.

approaches a cashless limiting economy in which all consumers are credit consumers who finance their purchases of good 1 using Lucas-Stokey trade credit. Together with Part 2 of Theorem 6, Part 9(a) of Corollary 7 shows that this limiting economy corresponds to the limit of the nonmonetary economy as $\alpha_C^T \to 1$, which in turn corresponds to the Arrow-Debreu frictionless (moneyless) benchmark in which all consumers can afford, and therefore consume $\Upsilon(\kappa)$—the efficient consumption allocation. This observation is the foundational approximation result for the moneyless approach to Monetary Economics:

- ***Result 1***: *As velocity diverges, the equilibrium allocation of an economy where money is used as a medium of exchange converges to that of the corresponding nonmonetary economy.*

Part 9(b) of Corollary 7 confirms that $\alpha_C^T \to 1$ indeed leads to a "cashless" limit: real money balances go to zero, while nominal prices and velocity diverge. As Woodford (1998, fn. 13) recognizes, this is the familiar result that an equilibrium with valued fiat money is impossible in an Arrow-Debreu economy. Here, $\alpha_C^T < 1$ amounts to imposing a "cash-in-advance" constraint on the Arrow-Debreu economy, which distorts the Arrow-Debreu allocation whenever the constraint binds (in this case, when $0 < \omega$). Taking $\alpha_C^T \to 1$ effectively undoes the cash-in-advance constraint and restores the efficient Arrow-Debreu allocation. The price level goes to infinity (the real value of money converges to zero), reflecting that agents are unwilling to give up intrinsically valuable goods for money that is useless in the Arrow-Debreu economy.[14] In textbook treatments of the moneyless approach (e.g., Woodford (2003), Galí (2008)), Result 1 is used to justify ignoring the role of money in exchange.

5.1 Price-Level Determination in Cashless Limiting Economies

Familiar and intuitive as Part 9(b) of Corollary 7 may be, much of the analysis in Woodford (1998) is devoted to amending the elementary result that the price level has an infinite discontinuity in the $\alpha_C^T \to 1$ limit. Woodford's motivation is twofold. First, he appears dissatisfied with the fact that this infinite discontinuity implies one cannot approximate the equilibrium price level for the cashless-limiting case with the equilibrium price level for economies with small

[14] Woodford (1998, 2003) and Galí (2008) popularized the term "cashless" to describe the $\alpha_C^T \to 1$ limit, and I adopt the same term here to emphasize that the same limit is being considered. In this context, however, I prefer "moneyless" over "cashless" as it avoids the potential misinterpretation—by readers unfamiliar with Woodford's terminology—that the cashless limit refers to an economy without paper currency but possibly with some other form of fiat money. (As mentioned in footnote 3, the variable M_t is defined abstractly in this model.)

enough $1-\alpha_C^T$. Second, the conclusion that the price level must necessarily be infinite (or that it must "cease to be defined" (Woodford (1998, p. 193))) in the cashless limit, conflicts with his goal of establishing that a central bank can control the price level in a model without money—or more precisely, in an economy in which money serves no purpose.

It is unclear why an infinite price level should be a concern in a context where, as $\alpha_C^T \to 1$, all variables relevant for agents' decisions—the interest rate, r^T, the relative price, ϕ, and real money balances, z_1—converge, and the level of every nominal variable becomes irrelevant. Even if one insists on approximating the equilibrium of the cashless-limiting economy with the monetary equilibrium of a small-$\left(1-\alpha_C^T\right)$ economy, this is indeed possible—provided one focuses on real money balances, which are proportional to the reciprocal of the price level.

Having said this, one could entertain the idea that real-world central bankers care about the level of nominal prices (even if agents in the theory do not), and for that reason one may want to complement Result 1 with a similar approximation proposition for the price level.[15] To this end, Woodford (1998) proposes a monetary-fiscal policy regime to determine a price level in the cashless limit. The idea is to assume that the monetary-fiscal policy, which up to now has been represented by a number $\omega \in \mathbb{R}_+$, instead consists of a function of the price level, or equivalently, a function of z_1. Formally, replace ω with $\Omega(z_1)$, where $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a function.[16] Under this policy, the Euler equation in Part 3 of Corollary 7 generalizes to

$$\Omega(z_1) = \left(1-\alpha_C^T\right) L(z_1) \tag{23}$$

(ignoring the complementary slackness, which will not be used in this exercise). The core idea behind the proposal in Woodford (1998, p. 193) for how to implement an arbitrary price level $p_{1t}^\star \in \mathbb{R}_+$ in a limiting cashless economy, is as follows: Assume a policy rule, Ω, that is strictly increasing and continuous, with $\Omega(z_1^\star) = 0$ for $z_1^\star \equiv \frac{M_t}{p_{1t}^\star} \in \mathbb{R}_{++}$. Then, it follows that for any $\alpha_C^T \in [0,1]$ there exists a unique $z_1 \in \mathbb{R}_+$ that solves (23), denoted $z_1^*\left(\alpha_C^T\right)$, and $\lim_{\alpha_C^T \to 1} z_1^*\left(\alpha_C^T\right) = z_1^\star$. To summarize:

- ***Result 2***: *The government can use monetary-fiscal policy to implement any desired equilibrium price level, $p_{1t}^\star \in \mathbb{R}_+$, in the cashless-limiting economy. Moreover, this price level*

[15] In this respect, Woodford follows an approach that is not uncommon in applied monetary economics: Motivated by the fact that the behavior of some variable X garners the attention of a real-world audience of central bankers, he theorizes on the determination of X in the context of a model in which X is irrelevant for agents' decisions and welfare.

[16] The analysis in the previous sections corresponds to the special case with $\Omega(z_1) = \omega \in \mathbb{R}_+$ for all $z_1 \in \mathbb{R}_+$.

approximates the equilibrium price level corresponding to an economy in which transaction velocity is arbitrarily large (i.e., α_C^T *is sufficiently close to* 1*).*

This result plays two roles in Woodford's moneyless approach. First, it delivers a finite price level (or equivalently, nonzero aggregate real money balances), despite money playing no useful role in the cashless-limiting economy. Second, together with Result 1, it is used to support the claim that nominal and real variables in the equilibrium of the cashless-limiting economy provide good approximations to those in modern, high-velocity monetary economies with medium-of-exchange frictions. Therefore—the argument goes—Results 1 and 2 imply that medium-of-exchange frictions can be ignored without meaningful loss. In Section 6, I will show that Results 1 and 2—and therefore this sweeping conclusion—are fragile, as they rely on a peculiar formulation of the market microstructure of the underlying monetary economy.

5.1.1 Monetarism vs. Wicksellianism?

A recurring theme in the moneyless literature is that it is more realistic—and more desirable—to formulate monetary policy in terms of interest-rate rules rather than money-growth rules. Woodford (1998, 2003) labels these approaches *Wicksellian* and *monetarist*, respectively.

The claim that the Wicksellian approach is more realistic rests on the casual observation that, in recent history, several prominent central banks have chosen to communicate their policy stance by announcing a target for a nominal interest rate (typically a short-term interbank rate). The view that the monetarist approach is unrealistic appears to have taken hold despite the fact that, in practice, the quantity of money (e.g., bank reserves) plays a central role in the actual implementation of monetary policy. In a *corridor system* like the one used by the Federal Reserve prior to the Great Financial Crisis, reserves were scarce, and the nominal rate was targeted by adjusting the quantity of reserves—typically via open-market operations. In a *floor system* like the one currently in place in the United States, reserves must be ample enough so that the target nominal rate can be achieved by adjusting administrative rates (e.g., the interest rate paid on reserves, the interest rate on overnight reverse repos, or the discount-window rate), without actively managing the quantity of reserves.

The claim that the Wicksellian approach is more desirable rests on two theoretical observations. First, the monetary/fiscal policy represented by ω (as defined in (16)), which Woodford (1998) refers to as the "quantitative-theoretic monetarist approach," fails to determine a finite price level in the cashless limit—in the sense that the price level diverges, as shown in Part 9(b)

of Corollary 7. Second, a monetary/fiscal policy consisting of a function $\Omega(\cdot)$ (as in (23)), which Woodford (1998) specializes to a "Wicksellian" interest-rate feedback rule, can deliver a finite price level, as shown in the derivation leading to Result 2.

> *The conventional approach to the problem of the determination of the general level of prices in an economy takes as its starting point the quantity theory of money, according to which the equilibrium price level is that value that makes the real purchasing power of the existing money supply equal to the desired level of real money balances. [...] I wish to argue that this entire approach to the problem of price-level determination is neither necessary nor desirable.* (Woodford (1998, p. 175))
>
> *[...] I would argue that it is in fact a great disadvantage of the quantity-theoretic approach to price-level determination that it implies that transactions frictions—obstacles to the execution of mutually beneficial trades—are an essential element in price-level determination. I wish to demonstrate, instead, the possibility of a theory of price-level determination that applies equally to the limiting case in which such frictions become insignificant, and to argue for the usefulness of analysis of that case, as at least a first approximation for most purposes.* (Woodford (1998, p. 176))
>
> *[...] I argue that the neo-Wicksellian interpretation of these equilibrium conditions is a particularly fruitful one, not least because it continues to be possible in the limiting case of a cashless economy.* (Woodford (2003, p. 62))

In the remainder of this section, I show that—for the purpose of determining a finite price level in the cashless limit—Woodford's "Monetarism vs. Wicksellianism" framing misconstrues the two approaches as being fundamentally different.

The key observation is that in order to formulate the policy Ω used to establish Result 2, one does not need to specify whether the policy instrument is the administered interest rate on money, r^M, or the interest rate on the illiquid bond, r^I, which the government can influence with the money-growth rule, μ. Specifically, let the function $\Omega(z_1)$ be formulated as above, i.e., strictly increasing and continuous, with $\Omega(z_1^\star) = 0$ for an arbitrary $z_1^\star \equiv \frac{M_t}{p_{1t}^\star} \in \mathbb{R}_{++}$. Since $\Omega(z_1)$ is just a relabeling of the normalized opportunity cost of holding money, ω, we can write

$$\Omega(z_1) \equiv \frac{r^I(z_1) - r^M(z_1)}{1 + r^M(z_1)},$$

where $r^I(z_1) \equiv \frac{\mu(z_1)-\beta}{\beta} \geq 0$ for all z_1, and $r^M(z_1)$ are arbitrary continuous functions that deliver the properties assumed for the function $\Omega(z_1)$. The function $r^I(\cdot)$ represents the illiquid-bond rate, and the function $r^M(\cdot)$ represents the administered interest rate on money. Clearly, any Ω that delivers Result 2 can be implemented through many combinations of monetarist and Wicksellian policies, represented by the money-growth rule, $\mu(\cdot)$, and the interest-rate rule, $r^M(\cdot)$, respectively. For example, what Woodford refers to as the "Wicksellian regime," is the special case with $r^I(z_1) = \frac{\mu-\beta}{\beta} \equiv r^I > 0$ for all z_1, and

$$r^M(z_1) = \frac{r^I - \Omega(z_1)}{1+\Omega(z_1)}. \tag{24}$$

In this case, to ensure Ω has the properties that deliver Result 2, we need to assume that $r^M(z_1^\star) = r^I$ and that $\frac{\partial r^M(z_1)}{\partial z_1} < 0$. The second property entails an interest-rate policy that prescribes the government to increase the administered nominal interest rate on money when real balances fall (equivalently, when the price level rises), and to decrease it when real balances rise (equivalently, when the price level falls)—a prescription that conforms with Woodford's interpretation of one of Wicksell's monetary-policy proposals.[17]

However, a monetarist regime can also implement Result 2. For example, assume $r^M(z_1) = r^M \in \mathbb{R}_+$ for all z_1, and

$$\mu(z_1) = \beta\left(1+r^M\right)\left[1+\Omega(z_1)\right]. \tag{25}$$

In this case, to ensure Ω has the properties that deliver Result 2, we need to assume $\mu(z_1^\star) = \beta\left(1+r^M\right)$ and $\frac{\partial \mu(z_1)}{\partial z_1} > 0$. The fact that this monetarist regime can deliver Result 2 contradicts the inadequacy of money-growth rules claimed in the foundational New Keynesian moneyless textbooks. For example:

> *Indeed, in a cashless economy, a money-growth target will not succeed in determining an equilibrium price level—at least if such a policy is understood to involve a constant (typically zero) rate of interest on the monetary base. In the case that no interest is paid on money, the following result implies that no equilibrium price level is possible at all.* (Woodford (2003, p. 82))

The "following result" alluded to in this passage, is Proposition 2.4 in Woodford (2003), which reiterates the analysis in Woodford (1998, p. 192), and is a well-known result in classical

[17] See Woodford (1998, p. 193).

monetary economics.[18]

The fundamental reason why the monetarist regime delivers Result 2 here but fails to do so in Woodford (1998) and Woodford (2003), is that those analyses constrain the money-supply process to satisfy an ad hoc restriction, namely, that $\{M_t\}_{t=0}^{\infty}$ must satisfy $M_t \geq \underline{M}$ for some $\underline{M} > 0$, for all t.[19] Notice that if we set $r^M = 0$ as in Woodford's preferred version of the monetarist regime, then in the cashless limit, the monetarist regime (25) prescribes a money-growth rate equal to $\mu(z_1^{\star}) = \beta$, implying $\frac{M_{t+1}}{M_t} = \beta < 1$ for all t, which violates Woodford's assumption that the money supply must be bounded away from zero. This restriction plays no role in Woodford's analysis, other than effectively assuming away money-growth rules such as (25), which can deliver a "monetarist" implementation of Result 2.

The analysis in this section has followed Woodford's suggestion that the determination of the price level in the cashless-limiting economy requires a monetary-fiscal regime in which the monetary policy is formulated as a "feedback rule"—i.e., as a function of endogenous variables, typically the price level or the inflation rate. In Woodford (1998, p. 193) and Woodford (2003, sec. 1.2), for example, the government implements a "Wicksellian" monetary policy by paying interest on money, and this policy rate is a function of the price level, as in (24). In my illustration that a monetarist regime can also deliver Result 2, monetary policy is implemented by (25), a money-growth rule that is a function of the price level. These examples may give the impression that the feedback-rule aspect of the policy is essential for Result 2. It is not. To see this, consider the economy of Corollary 7 under a monetary-fiscal regime that specifies $r^M = 0$ and a money-growth target

$$\mu \equiv \beta + \left(1 - \alpha_C^T\right)(\bar{\mu} - \beta), \tag{26}$$

with $\bar{\mu} \in [\beta, \beta + \beta L(0))$. Under this policy, $\omega = \left(1 - \alpha_C^T\right)\bar{\omega}$, with $\bar{\omega} \equiv \frac{\bar{\mu} - \beta}{\beta}$, and from parts 2-4 of Corollary 7, we get $p_{1t}^N = p_{1t}^T = \kappa p_{2t} = \frac{M_t}{\Upsilon((1+\bar{\omega})\kappa)} \in (0, \infty)$. Thus, even an elementary monetarist money-growth rule—independent of endogenous variables—can deliver Result 2.

The intuitive interpretation of (26) is that, as $\alpha_C^T \to 1$ and the economy becomes cashless, simply allowing for a secular downward drift in the money-growth rule allows for a standard monetarist money-demand-equals-money-supply determination of a finite price level in the cashless limit.[20] As was the case for the policy (25), the monetarist growth-rate policy (26) is implicitly

[18] See Sargent (1987, sec. 4.1).

[19] The example in Sargent (1987, sec. 4.1) satisfies this restriction because it assumes $M_t = M > 0$ for all t.

[20] This kind of downward drift in the money base seems like a natural policy accommodation to the kinds of secular "innovations due to improvements in information processing and to increased creativity in the evasion of

ruled out of the analysis in Woodford (1998) and Woodford (2003) by the ad hoc restriction that the money supply sequence should remain bounded away from zero, even as the volume of trades for which money is required for payment is converging to zero.

To conclude, the claims in Woodford (1998, 2003) that "Wicksellian" policies such as (24) are essential to determine a finite price level in cashless-limiting economies, and that "monetarist" policies are unable to do so, are unwarranted. From a purely theoretical standpoint—for the purpose of determining the price level in the cashless limit—the only reason to favor Wicksellian over monetarist policy is that the former allows proponents of the moneyless approach to formulate and discuss central-bank policy without referencing money demand or supply.

6 Cashless Limits 300: Modern M-relevance Results

This section revisits Result 1 from Section 5—the foundational approximation proposition underlying the moneyless approach to monetary economics. To that end, I provide a more general characterization of cashless limits. The key insight is that Result 1 is highly special: it holds only under restrictive assumptions on the monetary economy's market microstructure.

In the simple cash/credit Lucas–Stokey market microstructure that Woodford (1998) assumed in order to derive his cashless-limiting results, the only path to a cashless economy is to let $\alpha_C^T \to 1$. In contrast, a more general market microstructure—including a bond market intermediated by bankers—allows alternative paths to a cashless economy. For example, for $j \in \{C, P\}$, one could let $\alpha_j^T \to 1$ while holding $\alpha_j^M \in (0,1)$ fixed. This leads to a cashless-limiting economy with a market microstructure in which all consumers are on-trust consumers who consume $\Upsilon(\kappa)$ units of good 1, so Result 1 continues to hold, as in Section 5. Alternatively, for $j \in \{C, P\}$, one could let $\alpha_j^M \to 0$ while holding $\alpha_j^T \in (0,1)$ fixed. This is a more general cashless limit, as the limiting economy features both on-trust consumers and producers who settle purchases of good 1 with zero-interest deferred-payment loans, and no-trust consumers and producers who settle using interest-bearing claims that are intermediated by bankers.

To formalize the possibility that α_C^M and α_P^M may converge to zero at different rates, I replace α_j^M with a strictly decreasing function $\bar{\alpha}_j^M(\cdot) : \mathbb{R}_+ \to \left[0, \alpha_j^M\right]$ that satisfies $\bar{\alpha}_j^M(0) = \alpha_j^M \in (0,1)$ and $\lim_{x\to\infty} \bar{\alpha}_j^M(x) = 0$, and I replace α_j^B with $\bar{\alpha}_j^B(x) \equiv 1 - \bar{\alpha}_j^M(x)$, for $j \in \{C, P\}$.

The main result of this section is stated in the following corollary to Theorem 4. This

the remaining regulatory constraints" that Woodford (1998, p. 174) uses to motivate the practical relevance of cashless-limiting economies.

corollary generalizes Corollary 7 by allowing a market microstructure with banked consumers and producers who borrow and lend through a bond market intermediated by bankers.

Corollary 8. *Consider the stationary monetary equilibrium characterized in Theorem 4, with* $u' < \infty$, $\omega \in (0, \infty)$, $\alpha_j^T \in [0,1)$, *and the probability* α_j^M *generalized to a strictly decreasing function,* $\bar{\alpha}_j^M(\cdot) : \mathbb{R}_+ \to \left[0, \alpha_j^M\right]$, *with* $\bar{\alpha}_j^M(x) = 1 - \bar{\alpha}_j^B(x)$, $\bar{\alpha}_j^M(0) = \alpha_j^M \in (0,1)$, *and* $\lim_{x\to\infty} \bar{\alpha}_j^M(x) = 0$, *for* $j \in \{C, P\}$. *Then, as* $x \to \infty$*:*

1. $1 + r^B \to \left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T}\right)(1 + r^M) \geq 1 + r^M = 1 + r^T$.
2. $\phi^N \to \left(\frac{1-\alpha_C^T}{1-\alpha_C^T+\theta\omega}\right)\frac{\kappa}{1+r^M} \leq \frac{\kappa}{1+r^M} = \phi^T$.
3. $\varphi^B \to \frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa \geq \kappa = \varphi^T$.
4. $y_C^M \to 0$, $y_C^B \to \mathrm{Y}\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)$, $y_C^T = \mathrm{Y}(\kappa)$, *and*
$$Y_C \to \alpha_C^T \mathrm{Y}(\kappa) + \left(1 - \alpha_C^T\right)\mathrm{Y}\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right).$$
5. $z_1^N \to 0$, $z_1^T \to 0$, $z_2 \to 0$, $p_{1t}^N \to \infty$, $p_{1t}^T \to \infty$, $p_{2t} \to \infty$, *and* $v \to \infty$.

Corollary 8 provides a detailed characterization of the cashless limit of the stationary monetary equilibrium. The main finding is that Result 1 breaks down: the limiting economy is indeed cashless (by Part 5 of the corollary), yet its equilibrium allocation differs from that of the nonmonetary economy.

To make this precise, let Y_C^∞ denote aggregate consumption in the pure-credit (cashless) limit of the monetary economy (as characterized in Part 4 of Corollary 8), and let $\tilde{\mathrm{Y}}_C^\infty$ denote aggregate consumption in the corresponding limit of the nonmonetary economy (as characterized in Part 3 of Theorem 6). Then:

$$\mathrm{Y}_C^\infty - \tilde{\mathrm{Y}}_C^\infty = \left(1 - \alpha_C^T\right)\left[\mathrm{Y}\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right) - \mathrm{Y}\left(\frac{\kappa}{\theta}\right)\right] \geq 0,$$

with "=" only if $\alpha_C^T = 1$ (the cashless limit of the Lucas and Stokey (1983) cash/credit economy without a bond market, treated in Corollary 7), or $\omega = \infty$ (the opportunity cost of holding money is prohibitively high and no consumer is willing to hold money along the cashless limit of the monetary economy), or $\theta = 1$ (bankers have no market power). Thus, in economies with

$\alpha_C^T \in [0,1)$, $\theta \in (0,1)$, and $\omega \in [0,\infty)$, output is strictly higher in the pure-credit (cashless) limit of the monetary equilibrium than in the corresponding limit of the cashless economy. The gap $Y_C^\infty - \tilde{\mathrm{Y}}_C^\infty$ is a decreasing function of ω. That is, despite real money balances being virtually zero in the cashless limit, the opportunity cost of holding money still influences the real allocation—much as it does in conventional monetary equilibria outside the cashless limit.

The following result gives an analytical expression for the elasticity of aggregate output with respect to ω in the cashless limit.

Corollary 9. *In the economy of Corollary 8,*

$$\frac{d\mathrm{Y}(\varphi^\infty(\omega))}{d\omega}\frac{\omega}{\mathrm{Y}(\varphi^\infty(\omega))} = \frac{(1-\theta)\left(1-\alpha_C^T\right)\omega}{\left(1-\alpha_C^T+\theta\omega\right)\left(1-\alpha_C^T+\omega\right)}\rho\left(\mathrm{Y}(\varphi^\infty(\omega))\right), \tag{27}$$

where $\varphi^\infty(\omega) \equiv \frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa$, *and* $\rho(y) \equiv \frac{u'(y)}{u''(y)y}$.

The first factor in (27) is the elasticity of aggregate output with respect to the cum-interest relative price, φ^∞; it reflects the market microstructure—specifically, the parameters α_C^T and θ. The second factor, which depends on preferences, is the elasticity of the cum-interest relative price with respect to the opportunity cost of holding money, ω.

The key takeaway from (27) is that even in the cashless limit—when transaction velocity diverges and the volume of monetary trade is virtually zero—conventional monetary forces remain operative. In particular, policy-induced changes in the opportunity cost of money continue to affect real output and consumption. This result fails only in degenerate cases: when the bond market is perfectly competitive ($\theta = 1$), or when all credit takes the form of zero-interest deferred-payment arrangements à la Lucas and Stokey (1983) ($\alpha_C^T = 1$). Thus, from a positive standpoint, it does not follow that money, its opportunity cost, and medium-of-exchange considerations are irrelevant simply because aggregate real money balances are small.

7 Welfare

In this section, I compare social welfare in the equilibrium of the nonmonetary economy with that in the stationary monetary equilibrium. Welfare is defined as the equally weighted sum of all agents' beginning-of-period expected discounted lifetime utilities. As a benchmark, let $\mathscr{W}^*$ denote welfare in the first-best economy, where per-capita consumption and production of good 1 equal $\mathrm{Y}(\kappa)$, i.e., $\mathscr{W}^* = \frac{w^*}{1-\beta}$, with $w^* \equiv u(\mathrm{Y}(\kappa)) - \kappa\mathrm{Y}(\kappa)$.

The following proposition ranks welfare across the equilibrium of the nonmonetary economy, the stationary monetary equilibrium, and the first-best allocation.

Proposition 10. *Welfare in the equilibrium of the nonmonetary economy is* $\tilde{\mathscr{W}} = \frac{\tilde{w}}{1-\beta}$*, where*

$$\tilde{w} \equiv \alpha_C^T \left[u\left(\mathrm{Y}(\kappa)\right) - \kappa \mathrm{Y}(\kappa)\right] + \left(1-\alpha_C^T\right)\alpha_C^B \left[u\left(\mathrm{Y}\left(\tilde{\varphi}^B\right)\right) - \frac{\kappa}{\alpha_P^B}\mathrm{Y}\left(\tilde{\varphi}^B\right)\right],$$

with $\tilde{\varphi}^B$ *as given in Theorem 6. Welfare in the stationary monetary equilibrium is* $\bar{\mathscr{W}} = \frac{\bar{w}}{1-\beta}$*, where*

$$\bar{w} \equiv \alpha_C^T \left[u\left(\mathrm{Y}(\kappa)\right) - \kappa \mathrm{Y}(\kappa)\right] + \left(1-\alpha_C^T\right)\alpha_C^B \left[u\left(\mathrm{Y}\left(\varphi^B\right)\right) - \kappa \mathrm{Y}\left(\varphi^B\right)\right] + \left(1-\alpha_C^T\right)\alpha_C^M \left[u\left(z_1^N\right) - \kappa z_1^N\right],$$

with $\varphi^B \equiv \left(1+r^B\right)\phi^N$*, and* r^B*,* ϕ^N*, and* z_1^N*, as given in Theorem 4.*

If $\alpha_j^T \in [0,1)$ *and* $\alpha_j^M \in (0,1]$ *for* $j \in \{C,P\}$*, then:*

1. $\tilde{\mathscr{W}} < \bar{\mathscr{W}} < \mathscr{W}^*$ *for all* $\omega \in (0,\infty)$*.*
2. $\lim_{\omega\to 0}\bar{\mathscr{W}} = \mathscr{W}^*$*,* $\lim_{\omega\to\infty}\bar{\mathscr{W}} \geq \tilde{\mathscr{W}}$ *(with "=" only if* $\alpha_j^M = 0$ *for* $j \in \{C,P\}$*), and* $\frac{\partial \bar{\mathscr{W}}}{\partial \omega} < 0$ *for* $\omega \in (0,\infty)$*.*

Proposition 10 considers the case of interest in which not all consumers and producers are on-trust agents—that is, the market structure departs from the simple cash/credit cash-in-advance model of Lucas and Stokey (1983)—and not all no-trust consumers and producers are banked. Part 1 establishes that, for any strictly positive opportunity cost of holding money ω, welfare in the stationary monetary equilibrium is higher than in the nonmonetary economy but lower than in the first-best economy. Part 2 shows that welfare in the stationary monetary equilibrium is decreasing in ω: it converges to the first-best level as $\omega \to 0$, and remains higher than in the nonmonetary economy even as $\omega \to \infty$.

The following theorem ranks welfare across the pure-credit limit of the equilibrium of the nonmonetary economy, the first-best economy, and the pure-credit (cashless) limit of the stationary monetary equilibrium.

Theorem 11. *Consider the stationary monetary equilibrium characterized in Theorem 4, with $u' < \infty$, $\omega \in (0, \infty)$, $\alpha_j^T \in [0, 1)$, and the probability α_j^M generalized to a strictly decreasing function, $\bar{\alpha}_j^M(\cdot) : \mathbb{R}_+ \to \left[0, \alpha_j^M\right]$, with $\bar{\alpha}_j^M(x) = 1 - \bar{\alpha}_j^B(x)$, $\bar{\alpha}_j^M(0) = \alpha_j^M \in (0, 1)$, and $\lim_{x\to\infty} \bar{\alpha}_j^M(x) = 0$, for $j \in \{C, P\}$. Let $\tilde{\mathscr{W}}^\infty \equiv \lim_{x\to\infty} \tilde{\mathscr{W}}$ and $\bar{\mathscr{W}}^\infty \equiv \lim_{x\to\infty} \bar{\mathscr{W}}$. Then,*

$$(1-\beta)\,\tilde{\mathscr{W}}^\infty = \alpha_C^T\left[u\left(\mathrm{Y}(\kappa)\right) - \kappa\mathrm{Y}(\kappa)\right] + \left(1-\alpha_C^T\right)\left[u\left(\mathrm{Y}\left(\frac{1}{\theta}\kappa\right)\right) - \kappa\mathrm{Y}\left(\frac{1}{\theta}\kappa\right)\right]$$

and

$$(1-\beta)\,\bar{\mathscr{W}}^\infty = \alpha_C^T\left[u\left(\mathrm{Y}(\kappa)\right) - \kappa\mathrm{Y}(\kappa)\right] + \left(1-\alpha_C^T\right)\left[u\left(\mathrm{Y}\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\right) - \kappa\mathrm{Y}\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\right],$$

with $\tilde{\mathscr{W}}^\infty \le \bar{\mathscr{W}}^\infty \le \mathscr{W}^*$.

Moreover:

1. *If $\alpha_C^T = 1$ or $\theta = 1$, then $\tilde{\mathscr{W}}^\infty = \bar{\mathscr{W}}^\infty = \mathscr{W}^*$.*
2. *If $\alpha_C^T < 1$ and $\theta < 1$, then:*
   (a) $\tilde{\mathscr{W}}^\infty < \bar{\mathscr{W}}^\infty < \mathscr{W}^*$ for all $\omega \in (0, \infty)$.
   (b) $\lim_{\omega\to\infty} \bar{\mathscr{W}}^\infty = \tilde{\mathscr{W}}^\infty$, $\lim_{\omega\to 0} \bar{\mathscr{W}}^\infty = \mathscr{W}^*$, and $\frac{\partial \bar{\mathscr{W}}^\infty}{\partial\omega} < 0$ for $\omega \in (0, \infty)$.

Part 1 of Theorem 11 considers an economy with the market structure of the Lucas and Stokey (1983) cash/credit cash-in-advance model, which Woodford (1998, 2003) adopts to argue for the irrelevance of money. In this special case, the pure-credit limits of both the monetary and nonmonetary equilibrium allocations coincide with the first-best allocation.

Part 2 generalizes Woodford's market structure by allowing that not all consumers are on-trust ($\alpha_C^T < 1$), and that loan markets are not perfectly competitive ($\theta < 1$). In general, for any $\omega \in (0, \infty)$, welfare in the pure-credit (cashless) limit of the stationary monetary equilibrium, $\bar{\mathscr{W}}^\infty$, is strictly higher than in the corresponding limit of the nonmonetary economy—even though real money balances are arbitrarily small in the former. Thus, from a normative standpoint, it does not follow that money, its opportunity cost, and medium-of-exchange considerations are irrelevant simply because aggregate real money balances are close to zero.

As illustrated in Figure 4, $\bar{\mathscr{W}}^{\infty}$ is a decreasing function of ω, converging to welfare in the first-best economy as $\omega \to 0$, and to welfare in the nonmonetary economy as $\omega \to \infty$. The difference in welfare between the cashless limiting economy and the nonmonetary economy satisfies

$$0 \leq \bar{\mathscr{W}}^{\infty}(\omega) - \tilde{\mathscr{W}}^{\infty} \leq \left(1 - \alpha_C^T\right) \frac{u\left(\Upsilon(\kappa)\right) - \kappa \Upsilon(\kappa) - \left[u\left(\Upsilon\left(\frac{\kappa}{\theta}\right)\right) - \kappa \Upsilon\left(\frac{\kappa}{\theta}\right)\right]}{1-\beta},$$

with "=" in the first inequality if $\omega = \infty$, and in the second if $\omega = 0$. Welfare in the cashless limit of the monetary economy is thus bounded below by $\tilde{\mathscr{W}}^{\infty}$ and above by $\mathscr{W}^{*}$. The particular cashless limit analyzed by Woodford (1998) corresponds to letting $\alpha_C^T \to 1$, which causes the lower bound $\tilde{\mathscr{W}}^{\infty}$ to converge to the upper bound $\mathscr{W}^{*}$. In this particular market microstructure, money is *inessential* because it no longer improves upon the allocation implemented by the barter equilibrium of the nonmonetary economy.

The more general market microstructure considered in Part 2 of Theorem 11, however, reveals that the limit $\alpha_C^T \to 1$ is not merely a high-velocity limit reflecting widespread substitution from money toward credit. Rather, it is a high-velocity limit reflecting widespread substitution from money toward *perfectly competitive* credit. In other words, the irrelevance of money in Woodford's cashless limit is not a generic feature of economies where real balances are small because agents rely on liquidity-saving credit arrangements. It is a feature of a special limiting market microstructure in which those credit arrangements are also frictionless. The secular rise in velocity observed through the mid-1990s—often used at the time to motivate the cashless-limiting approach—has no bearing on whether the relevant pure-credit benchmark should assume perfect competition or retain frictions such as market power in credit markets.

8 The Devil is in the Details, but so is Monetary Economics

The Walrasian general equilibrium theory that serves as the all-purpose macroeconomic framework cannot answer the most elementary question in monetary economics: Why does money sell at a positive price? There has long been broad agreement on the basic diagnosis—that trade under the Walrasian paradigm is too seamless for money to play a meaningful role. But there is also long-standing disagreement over which frictions should be introduced, and how they should be modeled. This helps explain why integrating monetary economics into mainstream macroeconomics has proven difficult, and why progress within monetary theory has proceeded along disjoint strands, siloed along methodological lines.

Say I want to study a particular monetary question. Which model should I use? One in the style of Samuelson (1958), Sidrauski (1967), Levhari and Patinkin (1968), Bewley (1980), Lucas (1980), Townsend (1980), Lucas and Stokey (1983), Shi (1997), or Lagos and Wright (2005)? Perhaps *all* of them to ensure robustness to how money is introduced?

This fragmentation is plainly inconvenient for applied macroeconomists, who would prefer to work with a consensus "standard" model rather than risk having their findings dismissed on the grounds that the underlying monetary framework is deemed inadequate. The burden of having to take a stance on the microfoundations of money demand can be a powerful force for coordination. It is perhaps not surprising, then, that Woodford's proposal to abandon money altogether gained traction, despite being grounded solely in the cashless limit of Woodford (1998) and a handful of numerical exercises based on models from the 1960s.

8.1 *From Postal Economics to onetary Economics*

> *Ed Prescott used to like to ask, "Why is monetary economics any more important than postal economics?" Sure, there are frictions involved in making payments, but there are frictions involved in getting goods and documents from one place to another, too. So why should economic theorists lavish so much attention upon the problem of modeling the details of the use of money in making payments, while completely neglecting the role in the economy of the postal system?*
>
> *When I first heard this question—I was at that time quite engaged with the foundations of the demand for money and their implications for the welfare analysis of monetary policy—I regarded it as too frivolous to deserve much thought.[...]*
>
> *But over the years, I have found myself coming around to something like the view that Prescott may have meant to suggest. [...] I have realized that the project of modeling the fine details of the payments system and the sources of money demand is not essential to the explanation of how money prices are determined or to the analysis of the effects of alternative monetary policies.*
>
> —Woodford (1998, pp. 217-218).

Economists pursuing transformative research agendas often downplay competing perspectives. Prescott—who famously quipped about "postal economics"—minimized the role of monetary factors to foreground real disturbances as drivers of business cycles. Similarly, Woodford downplayed the entire classical tradition in monetary economics to elevate the role of price rigidities in macroeconomic stabilization. Results 1 and 2 (Section 5) enabled a delicate balancing act: they allowed money to be abstracted away while preserving inflation dynamics and a role for policy rules that resemble those used by central banks.

There has been significant progress in the theory of the microfoundations of money demand during the thirty years since Woodford's convergence to Prescott's nonmonetary view of macroeconomics. The remainder of this section draws on these advances to explain why Woodford's conclusion—that "the fine details of the payments system and the sources of money demand" are inessential for analyzing monetary policy—is no longer warranted.

8.2 Market Structure and Foundations of Money Demand

It is generally accepted that fiat money is valued because its marketability facilitates indirect exchange—that is, because of its *liquidity*. The Walrasian paradigm is ill-suited to study liquidity because it leaves the exchange process unmodeled. The traditional allusion to an auctioneer who announces prices and coordinates trade—a fiction that personifies the unmodeled exchange process—fills this conceptual void.

The notion of liquidity can only be made precise within a theory that features an explicit *market microstructure* of exchange: one that specifies who can trade with whom and when, as well as the trading mechanisms by which agents interact with markets and other agents. Under appropriate conditions, the portfolio choices of agents who interact within microstructures where money serves a useful role give rise to a demand for money.

The underlying market microstructure shapes the properties of the money-demand relationship. One should therefore be cautious in drawing general conclusions about money demand from simplistic or inadequate microstructures. As Keynes might have phrased it: *Practical monetary economists, who believe themselves to be quite exempt from the fine details of payments and sources of money demand, are usually slaves of some defunct market microstructure.*

The central tenet of the moneyless approach is that there is no meaningful loss in abstracting from the factors that determine the costs and benefits of holding money, and from all the transmission mechanisms that operate through policy-induced changes in these factors. The key result on which this doctrine rests is that the Euler equation for money cannot hold with equality in Woodford's cashless, pure-credit limit. The implication is that the opportunity cost of holding money—through which monetary policy would normally operate—disappears from the set of equilibrium conditions. This result renders all traditional monetarist transmission channels inoperative. The problem with this result is that it is driven by an inadequate credit-market microstructure.

To be more precise, consider the version of the model of Section 2 with $\alpha_i^B = 1 - \alpha_i^M = 0$ for $i \in \{C, P\}$, which corresponds to the credit-market microstructure of Lucas and Stokey (1983) adopted by Woodford (1998). In this model, as $\alpha_C^T \to 1$, the right side of the Euler equation for money, (15), collapses to zero, implying that real money balances, z_1^N, converge to zero. Intuitively, as $\alpha_C^T \to 1$, the marginal value of a dollar in exchange falls to zero, driving money demand to zero. Since the Euler equation for money cannot hold with equality in this cashless limit, the opportunity cost of holding money, $\omega \equiv \frac{r^I - r^M}{1+r^M}$, which appears only in that equation, ceases to influence any other equilibrium condition. The fundamental concern here is that the rudimentary credit-market microstructure of Lucas and Stokey (1983) induces an Euler equation for money that is too fragile under Woodford's pure-credit limit.

In contrast, the Euler equation for money induced by the more general market microstructure studied in Section 6 (i.e., equation (15) with $\alpha_C^T < 1$, as $\alpha_C^M \to 0$) is robust to pure-credit limits, in the sense that it continues to hold with equality for a broad range of monetary policies (represented by ω). This means that the opportunity cost of money, ω, continues to affect the equilibrium in the cashless limit, preserving the traditional monetarist transmission channels. In this model, ω affects the consumption allocation in the cashless limit through its impact on the endogenous bond rate, r^B, faced by banked consumers (see parts 1 and 4 of Corollary 8 for economies with $\theta < 1$).

There is an intuitive explanation for why the Euler equation for money is robust in this more general credit-market microstructure but fragile in the cash/credit formulation of Lucas and Stokey (1983) (the special case with $\alpha_i^B = 1 - \alpha_i^M = 0$ for $i \in \{C, P\}$). In the Lucas-Stokey market structure, bringing an additional dollar into the credit-goods market does not increase a consumer's purchasing capacity, since consumers have unlimited access to zero-interest deferred-payment credit. As a result, the marginal value of carrying money is zero in the pure-credit limit in which all consumers have access to the Lucas-Stokey zero-interest deferred-payment style of credit—so the right side of the Euler equation for money collapses to zero.

The model of Section 2 includes a segment of Lucas-Stokey credit, in which the net interest rate is fixed at zero ($r^T - r^M = 0$). But the market labeled "bond market" represents a more general form of credit: it features an endogenous interest rate ($r^B - r^M$ is not fixed at 0) and allows for market power in financial intermediation. In this more general case with $\alpha_C^T < 1$, the right side of (15) does not collapse to zero as $\alpha_C^M = 1 - \alpha_C^B \to 0$, because the first term remains positive. The reason is that, in the pure-credit limit of this economy, the marginal value of money as a medium of exchange remains equal to the net interest rate cost of borrowing money in stage 1, i.e., $r^B - r^M$, which is positive (or strictly positive away from the Friedman rule).

In the cashless limit, agents know they will never be constrained to pay with dollars. Nevertheless, they are willing to bring dollars into stage 1 because each dollar saves them the interest cost they would otherwise incur by borrowing to purchase good 1. Thus, agents are willing to hold dollars at the end of stage 2 as long as the opportunity cost of holding money equals this borrowing cost—that is, as long as the Euler equation for money holds with equality. In the cashless-limiting equilibrium, the bond rate adjusts to ensure that this condition is met. As a result, ω does not drop out of the equilibrium, and neither do the traditional monetarist transmission channels.

The economic narrative in Woodford (1998) is that the costs and benefits of holding money cannot matter for monetary transmission if the aggregate quantity of real money balances is sufficiently small. However, with a more plausible credit-market microstructure (one with an endogenous interest rate), this conclusion does not follow, because the Euler equation for money imposes restrictions on the equilibrium through the endogenous borrowing rate, even if real money balances are arbitrarily small.

For the last thirty years, the prevailing view in applied monetary economics has been that modeling the details of the payments system and costs and benefits of holding money are unnecessary for understanding price determination or the transmission of real-world monetary policy. The analysis I have presented here suggests that this view is mistaken. Microstructure factors are *foundational*: they even compel us to reassess the robustness of the theorems that have been used to justify doing monetary economics without money.

The challenge of integrating monetary economics into macroeconomics is that macroeconomics thrives on abstraction from micro details, yet micro details are essential to monetary economics. Thirty years ago, when the New Keynesian cashless approach was proposed, it might have seemed infeasible to build into a general-equilibrium macro model an explicit microfoundation of money demand—one grounded in segmented or decentralized exchange, information frictions, lack of commitment, search frictions, or non-Walrasian pricing mechanisms. But today, many tractable approaches are available. We are unlikely to reach general agreement on which is the "right" way to incorporate liquidity into Arrow-Debreu, because—as Bob Lucas put it, invoking *Anna Karenina*—there are many ways to make a happy family unhappy. In each application, the research question and relevant evidence must guide which microstructure features matter most. As for how deep the microfoundations ought to go: one can always go deeper, and some applications will demand it. Knowing when to stop digging is a kind of art. We have to accept that progress in the field will invariably turn the deep microfoundations of today into the reduced forms of tomorrow. We must accept that the Devil is in the market microstructure, but so is monetary economics.

9 Conclusion

The central focus of monetary economics has traditionally been the determination of real money balances—or, equivalently, the price level—through the interaction of money demand and money supply. Canonical monetary frameworks differ in the frictions they incorporate into the Walrasian (e.g., Arrow-Debreu) benchmark in order to induce a demand for fiat money. Some assume Walrasian economies with ad hoc preferences or constraints (such as money-in-the-utility-function models, as in Sidrauski (1967), or cash-in-advance models, as in Clower (1967) and Lucas (1980)); others assume Walrasian economies with incomplete markets (such as the models of Samuelson (1958), Bewley (1980), and Townsend (1980)); and others consider non-Walrasian economies with explicit frictional market microstructures (such as Shi (1997) and Lagos and Wright (2005)). Regardless of the specific monetary framework, a central theme has been to understand the effect of policy-induced changes in the opportunity cost of holding money on the Euler equation for money, and, ultimately, on equilibrium allocations and welfare.

On Saturday, June 29, 1996, in Mexico City, Michael Woodford delivered a keynote address at the Annual Meeting of the Society for Economic Dynamics, titled "Money and Prices in the Theory of Value." The title of the lecture was as bold as it was evocative—a direct allusion to Gérard Debreu's *Theory of Value*, the foundational expression of Arrow-Debreu general equilibrium theory and the cornerstone of modern macroeconomics. A well-known limitation of the Arrow-Debreu paradigm is its omission of fiat money. In his lecture, Woodford explained how nominal prices and monetary policy could be analyzed within the abstract machinery of the *Theory of Value*. The audience was captivated by the boldness of the proposition. I remember it clearly—I was there, a Ph.D. student attending my first conference. The version of the lecture published two years later (Woodford (1998)) was even bolder: it recast the cashless-limiting result presented in Mexico City as a foundation for monetary economics without money—*onetary Economics*, as I like to call it. In time, I came to realize I had witnessed a turning point in the history of monetary economics—one that would shape the field for the next thirty years.

The cashless-limiting result in Woodford (1998) has since become the theoretical foundation for a large literature that invokes the "standard" New Keynesian model to deliver practical monetary-policy recommendations based on models that dispense with money altogether. This consensus did not arise by accident. It was the outgrowth of a research agenda whose explicit objective was to divorce monetary policy from money and focus exclusively on sticky prices.[21]

Corollary 8 and Theorem 11 show that the prevailing view—that money-demand considerations can be ignored without loss in high-velocity economies—only applies to economies with perfectly competitive credit and payment microstructures. Otherwise, policy-induced changes in the opportunity cost of holding money affect real allocations and welfare—even in the "cashless" limit, where the use of money in payments is virtually zero. Intuitively, when credit, settlement, or payment services involve financial intermediaries with market power, the off-equilibrium threat to settle transactions directly with money—a *latent money demand*—strengthens the stance of sellers relative to these intermediaries, and propagates policy-induced changes in the opportunity cost of holding money to the relative prices faced by *all* agents—even those who use credit instead of money to settle trades.

In the special cases where either $\theta = 1$ or $\alpha_C^T = 1$, Corollary 8 and Theorem 11 recover the approximation result underlying the moneyless approach of Woodford (1998) (stated as Result 1 in Section 5). Woodford's approximation result is not *wrong*—at least not in the narrow mathematical sense. The problem, rather, is that the result implicitly presumes a frictionless market microstructure for credit and payments—an assumption conceptually unrelated to how far the economy is along the high-velocity limit. From an applied perspective, the assumption that credit markets are perfectly competitive, which was common in the 1980s and through the early 2000s, is now widely deemed ill-suited for empirical or quantitative applications of macro models with credit.

The foundational cashless-limiting result—and the scaffolding of special assumptions used to derive it—has been accepted at face value for nearly thirty years. I think it is time we asked: Can we be confident that ignoring money entails no significant loss for monetary policy analysis?

[21]This moneyless research agenda is advocated in Woodford (1995), Woodford (1998), Woodford (2000), Woodford (2008), and in the introductory chapters of the two popular New Keynesian textbooks, Woodford (2003) and Galí (2008).

My findings suggest the answer is no. The microstructure for which this result holds is too restrictive—and too detached from our current understanding of credit markets—to justify a three-decade detour of monetary economics away from money.

References

Bewley, T. (1980): "The Optimum Quantity of Money, in "Models of Monetary Economies" (J. Kareken and N. Wallace, Eds.)," *Federal Reserve Bank of Minneapolis.*

Clower, R. (1967): "A Reconsideration of the Microfoundations of Monetary Theory," *Economic Inquiry*, 6, 1–8.

Drechsler, I., A. Savov, and P. Schnabl (2017): "The Deposits Channel of Monetary Policy," *Quarterly Journal of Economics*, 132, 1819–1876.

Duffie, D., N. Gârleanu, and L. H. Pedersen (2005): "Over-the-Counter Markets," *Econometrica*, 73, 1815–1847.

Gale, D. (1982): *Money: In Equilibrium*, vol. 2, Cambridge University Press.

Galí, J. (2008): *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework*, Princeton University Press.

Hahn, F. H. (1987): "The Foundations of Monetary Theory," in *Monetary Theory and Economic Institutions: Proceedings of a Conference held by the International Economic Association at Fiesole, Florence, Italy*, Springer, 21–43.

Hellwig, M. F. (1993): "The Challenge of Monetary Theory," *European Economic Review*, 37, 215–242.

Ireland, P. N. (2001): "Money's Role in the Monetary Business Cycle," .

Jeenas, P. and R. Lagos (2024): "Q-Monetary Transmission," *Journal of Political Economy*, 132, 971–1012.

Kareken, J. H. and N. Wallace (1980): *Models of Monetary Economies: proceedings and contributions from participants of a December 1978 conference sponsored by the Federal Reserve Bank of Minneapolis*, Federal reserve bank of Minneapolis.

Lagos, R. and R. Wright (2005): "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113, 463–484.

Lagos, R. and S. Zhang (2020): "Turnover Liquidity and the Transmission of Monetary Policy," *American Economic Review*, 110, 1635–1672.

——— (2022): "The Limits of onetary Economics: On Money as a Constraint on Market Power," *Econometrica*, 90, 1177–1204.

Levhari, D. and D. Patinkin (1968): "The Role of Money in a Simple Growth Model," *American Economic Review*, 58, 713–753.

Lucas, R. E. (1980): "Equilibrium in a Pure Currency Economy," *Economic Inquiry*, 18, 203–220.

——— (1984): "Money in a Theory of Finance," in *Carnegie-Rochester Conference Series on Public Policy*, vol. 21, 9–46.

Lucas, R. E. and N. L. Stokey (1983): "Optimal Fiscal and Monetary Policy in an Economy Without Capital," *Journal of Monetary Economics*, 12, 55–93.

McCallum, B. T. (2001): "Monetary Policy Analysis in Models Without Money," Working Paper 8174, NBER.

Nagel, S. (2016): "The Liquidity Premium of Near-Money Assets," *Quarterly Journal of Economics*, 131, 1927–1971.

Rocheteau, G. and R. Wright (2005): "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium," *Econometrica*, 73, 175–202.

Samuelson, P. A. (1958): "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *Journal of Political Economy*, 66, 467–482.

Sargent, T. J. (1987): *Dynamic Macroeconomic Theory*, Cambridge MA: Harvard University Press.

Shi, S. (1997): "A Divisible Search Model of Fiat Money," *Econometrica*, 75–102.

Sidrauski, M. (1967): "Inflation and Economic Growth," *Journal of Political Economy*, 75, 796–810.

Townsend, R. M. (1980): "Models of Money with Spatially Separated Agents," *Models of Monetary Economies*, 265–303.

Wallace, N. (2001): "Whither Monetary Economics?" *International Economic Review*, 42, 847–869.

Woodford, M. (1995): "Price-Level Determinacy Without Control of a Monetary Aggregate," *Carnegie-Rochester Conference Series on Public Policy*, 43, 1–46.

——— (1998): "Doing Without Money: Controlling Inflation in a Post-Monetary World," *Review of Economic Dynamics*, 1, 173–219.

——— (2000): "Monetary Policy in a World Without Money," *International Finance*, 3, 229–260.

——— (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press.

——— (2008): "How Important is Money in the Conduct of Monetary Policy?" *Journal of Money, Credit and Banking*, 40, 1561–1598.

A Proofs

A.1 Monetary Economy: Individual Optimization

Lemma 12. *The second-stage value functions can be written as*

$$W_{Ct}\left(a_t^m, a_t^g\right) = a_t^g + \frac{1}{p_{2t}} R_t^M a_t^m + W_{Ct}\left(0,0\right),$$

with

$$W_{Ct}\left(0,0\right) \equiv \frac{1}{p_{2t}} T_t^M + \max_{a_{t+1}^m \in \mathbb{R}_+} \left[-\frac{1}{p_{2t}} a_{t+1}^m + \beta \bar{V}_{Ct+1}\left(a_{t+1}^m\right) \right], \tag{28}$$

and

$$W_{it}\left(a_t^m, a_t^g\right) = a_t^g + \frac{1}{p_{2t}} R_t^M a_t^m + \beta \bar{V}_{it+1} \tag{29}$$

for $i \in \{B, P\}$.

Proof. Immediate from (1) and (2). □

Lemma 13. *The optimal first-stage trading rules of a cash consumer who entered period* t *with* m_t *dollars are:*

$$y_{Ct}^M\left(m_t\right) = \min\left\{ \mathrm{Y}\left(\varphi_t^M\right), \frac{m_t}{p_{1t}^N} \right\}$$
$$m_{Ct}^M\left(m_t\right) = \max\left\{ m_t - p_{1t}^N \mathrm{Y}\left(\varphi_t^M\right), 0 \right\},$$

and

$$V_{Ct}^M\left(m_t\right) = \begin{cases} u\left(\mathrm{Y}\left(\varphi_t^M\right)\right) + \left[m_t - p_{1t}^N \mathrm{Y}\left(\varphi_t^M\right)\right] \frac{R_t^M}{p_{2t}} + W_{Ct}\left(0,0\right) & \text{if } \mathrm{Y}\left(\varphi_t^M\right) \le \frac{m_t}{p_{1t}^N} \\ u\left(\frac{m_t}{p_{1t}^N}\right) + W_{Ct}\left(0,0\right) & \text{if } \frac{m_t}{p_{1t}^N} < \mathrm{Y}\left(\varphi_t^M\right), \end{cases}$$

where $\mathrm{Y}(x) \equiv u'^{-1}(x)$ *for any* $x \in \mathbb{R}_+$.

Proof. With Lemma 12, (3) can be written as

$$V_{Ct}^M\left(m_t\right) = W_{Ct}\left(0,0\right) + \frac{R_t^M}{p_{2t}} m_t + \max_{\hat{y}_t \in \left[0, \frac{m_t}{p_{1t}^N}\right]} \left[u\left(\hat{y}_t\right) - \frac{p_{1t}^N R_t^M}{p_{2t}} \hat{y}_t \right]$$

with

$$\hat{a}_t^m = m_t - p_{1t}^N \hat{y}_t.$$

The first-order condition for this consumer's demand for good 1 is

$$u'\left(\hat{y}_t\right) - \frac{p_{1t}^N R_t^M}{p_{2t}} - \zeta = 0,$$

where ζ is the Lagrange multiplier on the constraint $\hat{y}_t \leq \frac{m_t}{p_{1t}^N}$. If

$$u'^{-1}\left(\frac{p_{1t}^N R_t^M}{p_{2t}}\right) \leq \frac{m_t}{p_{1t}^N},$$

then $\zeta = 0$, $\hat{y}_t = u'^{-1}\left(\frac{p_{1t}^N R_t^M}{p_{2t}}\right)$, and $\hat{a}_t^m = m_t - p_{1t} u'^{-1}\left(\frac{p_{1t}^N R_t^M}{p_{2t}}\right) \geq 0$. If

$$\frac{m_t}{p_{1t}^N} < u'^{-1}\left(\frac{p_{1t}^N R_t^M}{p_{2t}}\right),$$

then $\hat{y}_t = \frac{m_t}{p_{1t}^N}$ and $\hat{a}_t^m = 0 < \zeta = u'\left(\frac{m_t}{p_{1t}^N}\right) - \frac{p_{1t}^N}{p_{2t}} R_t^M$. Hence,

$$y_{Ct}^M\left(m_t\right) = \begin{cases} \mathrm{Y}\left(\frac{p_{1t}^N R_t^M}{p_{2t}}\right) & \text{if } \mathrm{Y}\left(\frac{p_{1t}^N R_t^M}{p_{2t}}\right) \leq \frac{m_t}{p_{1t}^N} \\ \frac{m_t}{p_{1t}^N} & \text{if } \frac{m_t}{p_{1t}^N} < \mathrm{Y}\left(\frac{p_{1t}^N R_t^M}{p_{2t}}\right) \end{cases}$$

$$m_{Ct}^M\left(m_t\right) = \begin{cases} m_t - p_{1t}^N \mathrm{Y}\left(\frac{p_{1t}^N R_t^M}{p_{2t}}\right) & \text{if } \mathrm{Y}\left(\frac{p_{1t}^N R_t^M}{p_{2t}}\right) \leq \frac{m_t}{p_{1t}^N} \\ 0 & \text{if } \frac{m_t}{p_{1t}^N} < \mathrm{Y}\left(\frac{p_{1t}^N R_t^M}{p_{2t}}\right) \end{cases}$$

and

$$V_{Ct}^M\left(m_t\right) = u\left(y_{Ct}^M\left(m_t\right)\right) + \left[m_t - p_{1t}^N y_{Ct}^M\left(m_t\right)\right] \frac{R_t^M}{p_{2t}} + W_{Ct}\left(0,0\right).$$

□

Lemma 14. *The optimal first-stage trading rule of a cash producer who carries inventory y_t of good 1 is*

$$m_{Pt}^M\left(y_t\right) = p_{1t}^N y_t,$$

and

$$V_{Pt}^M\left(y_t\right) = \varphi_t^M y_t + \beta \bar{V}_{Pt+1}.$$

Proof. With Lemma 12, (4) can be written as

$$V_{Pt}^M\left(y_t\right) = \frac{R_t^M}{p_{2t}} \max_{\hat{a}_t^m \in \left[0, p_{1t}^N y_t\right]} \hat{a}_t^m + \beta \bar{V}_{Pt+1},$$

so $m_{Pt}^M\left(y_t\right) = p_{1t}^N y_t$, and

$$V_{Pt}^M\left(y_t\right) = \frac{p_{1t}^N R_t^M}{p_{2t}} y_t + \beta \bar{V}_{Pt+1}.$$

□

Lemma 15. *The optimal first-stage trading rules of a banked consumer who entered period t with m_t dollars are:*

$$y^B_{Ct}(m_t) = \mathrm{Y}\left(\varphi^B_t\right)$$

$$m^B_{Ct}(m_t) \begin{cases} = \infty & \text{if } r^B_t < r^M_t \\ \in [0,\infty] & \text{if } r^B_t = r^M_t \\ = 0 & \text{if } r^M_t < r^B_t \end{cases}$$

$$q^B_t b^B_{Ct}(m_t) = m_t - m^B_{Ct}(m_t) - p^N_{1t} y^B_{Ct}(m_t),$$

where $\mathrm{Y}(x) \equiv u'^{-1}(x)$ *for any* $x \in \mathbb{R}_+$*, and*

$$V^B_{Ct}(m_t) = \frac{m_t}{q^B_t} + V^B_{Ct}(0),$$

with

$$V^B_{Ct}(0) \equiv u\left(\mathrm{Y}\left(\varphi^B_t\right)\right) - \varphi^B_t \mathrm{Y}\left(\varphi^B_t\right) + W_{Ct}(0,0).$$

Proof. With Lemma 12, (5) can be written as

$$V^B_{Ct}(m_t) = \frac{m_t}{q^B_t} + \max_{(\bar{y}_t, \bar{a}^m_t) \in \mathbb{R}^2_+} \left[u(\bar{y}_t) - \frac{p^N_{1t}}{q^B_t}\bar{y}_t + \left(\frac{R^M_t}{p_{2t}} - \frac{1}{q^B_t}\right)\bar{a}^m_t\right] + W_{Ct}(0,0)$$

with

$$\bar{a}^b_t = \frac{1}{q^B_t}\left(m_t - \bar{a}^m_t - p^N_{1t}\bar{y}_t\right). \tag{30}$$

Equivalently, this value function can be written as

$$V^B_{Ct}(m_t) = \frac{m_t}{q^B_t} + \max_{(\bar{y}_t, \bar{a}^m_t) \in \mathbb{R}^2_+} \left[u(\bar{y}_t) - \varphi^B_t \bar{y}_t + \left(r^M_t - r^B_t\right)\frac{\bar{a}^m_t}{p_{2t}}\right] + W_{Ct}(0,0),$$

with (30). The trading rules in the statement of the lemma, $\left(y^B_{Ct}(m_t), m^B_{Ct}(m_t), b^B_{Ct}(m_t)\right)$, follow immediately from this optimization problem, and therefore

$$V^B_{Ct}(m_t) = u\left(\mathrm{Y}\left(\varphi^B_t\right)\right) + \varphi^B_t\left[\frac{m_t}{p^N_{1t}} - \mathrm{Y}\left(\varphi^B_t\right)\right] + \left(r^M_t - r^B_t\right)\frac{m^B_{Ct}(m_t)}{p_{2t}} + W_{Ct}(0,0).$$

The consumer's money-demand problem has no solution if $0 < r^M_t - r^B_t$, and therefore any equilibrium must have $r^M_t - r^B_t \leq 0$, which implies

$$V^B_{Ct}(m_t) = u\left(\mathrm{Y}\left(\varphi^B_t\right)\right) + \varphi^B_t\left[\frac{m_t}{p^N_{1t}} - \mathrm{Y}\left(\varphi^B_t\right)\right] + W_{Ct}(0,0).$$

□

Lemma 16. *The first-stage bargaining outcome between a banker and banked producer with inventory y_t, is*

$$m^B_{Pt}(y_t) = \begin{cases} = \infty & \text{if } r^B_t < r^M_t \\ \in [0,\infty] & \text{if } r^B_t = r^M_t \\ = 0 & \text{if } r^M_t < r^B_t \end{cases}$$
$$p_{2t}\phi^B_{Pt}(y_t) = (1-\theta)\left(r^B_t - r^M_t\right) p^N_{1t} y_t$$
$$q^B_t b^B_{Pt}(y_t) = p^N_{1t} y_t - m^B_{Pt}(y_t),$$

and

$$V^B_{Pt}(y_t) = \left[\theta\varphi^B_t + (1-\theta)\varphi^M_t\right] y_t + \beta \bar{V}_{Pt+1}.$$

Proof. With Lemma 12, (6) can be written as

$$\max_{\left(\phi_t, \bar{a}^m_t, \bar{a}^b_t\right)\in\mathbb{R}^2_+\times\mathbb{R}} \left[\bar{a}^b_t + \frac{R^M_t}{p_{2t}}\bar{a}^m_t - \phi_t - \frac{R^M_t}{p_{2t}} m^M_{Pt}(y_t)\right]^\theta [\phi_t]^{1-\theta}$$
$$\text{s.t. } \bar{a}^m_t + q^B_t \bar{a}^b_t \le p^N_{1t} y_t \quad \text{and} \quad 0 \le \bar{a}^b_t + \frac{R^M_t}{p_{2t}}\bar{a}^m_t - \phi_t - \frac{R^M_t}{p_{2t}} m^M_{Pt}(y_t).$$

With Lemma 14, which gives $m^M_{Pt}(y_t) = p^N_{1t} y_t$, and substituting the producer's budget constraint, the bargaining problem can be written as

$$\max_{(\bar{a}^m_t, \phi_t)\in\mathbb{R}_+\times\mathbb{R}} \left[\left(\frac{1}{q^B_t} - \frac{R^M_t}{p_{2t}}\right)\left(p^N_{1t} y_t - \bar{a}^m_t\right) - \phi_t\right]^\theta [\phi_t]^{1-\theta} \tag{31}$$
$$\text{s.t. } 0 \le \phi_t \le \left(\frac{1}{q^B_t} - \frac{R^M_t}{p_{2t}}\right)\left(p^N_{1t} y_t - \bar{a}^m_t\right),$$

with

$$\bar{a}^b_t = \frac{1}{q^B_t}\left(p^N_{1t} y_t - \bar{a}^m_t\right).$$

After noticing that

$$\frac{1}{q^B_t} - \frac{R^M_t}{p_{2t}} = \left(r^B_t - r^M_t\right)\frac{1}{p_{2t}},$$

it is clear the money demand for the producer that is implied by the optimization problem (31), i.e., $m^B_{Pt}(y_t)$, is as in the statement of the lemma. The first-order condition for ϕ_t (ignoring the constraints on ϕ_t in (31)) implies

$$\phi_t = (1-\theta)\left(\frac{1}{q^B_t} - \frac{R^M_t}{p_{2t}}\right)\left(p^N_{1t} y_t - m^B_{Pt}(y_t)\right),$$

which satisfies the two constraints on ϕ_t in (31). Thus, the solution of the bargaining problem is $\left(m^B_{Pt}(y_t), b^B_{Pt}(y_t), \phi^B_{Pt}(y_t)\right)$, with $m^B_{Pt}(y_t)$ as given in the statement of the lemma, and

$$\phi^B_{Pt}(y_t) = (1-\theta)\left(r^B_t - r^M_t\right)\frac{1}{p_{2t}}\left[p^N_{1t}y_t - m^B_{Pt}(y_t)\right]$$
$$b^B_{Pt}(y_t) = \frac{1}{q^B_t}\left[p^N_{1t}y_t - m^B_{Pt}(y_t)\right].$$

Given this bargaining outcome, use Lemma 12 to write

$$V^B_{Pt}(y_t) = \frac{R^M_t}{p_{2t}}m^B_{Pt}(y_t) + b^B_{Pt}(y_t) - \phi^B_{Pt}(y_t) + \beta\bar{V}_{Pt+1},$$

which with $b^B_{Pt}(y_t) = \frac{1}{q^B_t}\left[p^N_{1t}y_t - m^B_{Pt}(y_t)\right]$, can be written as

$$V^B_{Pt}(y_t) = \frac{p^N_{1t}}{q^B_t}y_t - \left(\frac{1}{q^B_t} - \frac{R^M_t}{p_{2t}}\right)m^B_{Pt}(y_t) - \phi^B_{Pt}(y_t) + \beta\bar{V}_{Pt+1},$$

and with $\phi^B_{Pt}(y_t) = (1-\theta)\left(r^B_t - r^M_t\right)\frac{1}{p_{2t}}\left[p^N_{1t}y_t - m^B_{Pt}(y_t)\right]$, can be written as

$$\begin{aligned} V^B_{Pt}(y_t) &= \left[\theta\frac{1}{q^B_t} + (1-\theta)\frac{R^M_t}{p_{2t}}\right]p^N_{1t}y_t - \theta\left(\frac{1}{q^B_t} - \frac{R^M_t}{p_{2t}}\right)m^B_{Pt}(y_t) + \beta\bar{V}_{Pt+1} \\ &= \left[\theta\frac{1}{q^B_t} + (1-\theta)\frac{R^M_t}{p_{2t}}\right]p^N_{1t}y_t - \theta\left(r^B_t - r^M_t\right)\frac{1}{p_{2t}}m^B_{Pt}(y_t) + \beta\bar{V}_{Pt+1}. \end{aligned}$$

The producer's money-demand problem has no solution if $0 < r^M_t - r^B_t$, and therefore any equilibrium must have $r^M_t - r^B_t \leq 0$, which implies

$$p_{2t}\phi^B_{Pt}(y_t) = (1-\theta)\left(r^B_t - r^M_t\right)p^N_{1t}y_t$$

and

$$V^B_{Pt}(y_t) = \left[\theta\frac{1}{q^B_t} + (1-\theta)\frac{R^M_t}{p_{2t}}\right]p^N_{1t}y_t + \beta\bar{V}_{Pt+1}.$$

□

Lemma 17. *The optimal first-stage trading rules of a credit consumer who entered period t with m_t dollars are:*

$$y^T_{Ct}(m_t) = \mathrm{Y}\left(\varphi^T_t\right)$$
$$m^T_{Ct}(m_t)\begin{cases} = \infty & \text{if } r^T_t < r^M_t \\ \in [0,\infty] & \text{if } r^T_t = r^M_t \\ = 0 & \text{if } r^M_t < r^T_t \end{cases}$$
$$q^T_t b^T_{Ct}(m_t) = m_t - m^T_{Ct}(m_t) - p^T_{1t}y^T_{Ct}(m_t),$$

where $\Upsilon(x) \equiv u'^{-1}(x)$ *for any* $x \in \mathbb{R}_{+}$, *and*

$$V_{Ct}^{T}(m_t) = \frac{m_t}{q_t^T} + V_{Ct}^{T}(0),$$

with

$$V_{Ct}^{T}(0) \equiv u\left(\Upsilon\left(\varphi_t^T\right)\right) - \varphi_t^T \Upsilon\left(\varphi_t^T\right) + W_{Ct}(0,0).$$

Proof. With Lemma 12, (7) can be written as

$$V_{Ct}^{T}(m_t) = \frac{m_t}{q_t^T} + \max_{(\tilde{y}_t, \tilde{a}_t^m) \in \mathbb{R}_+^2} \left[u(\tilde{y}_t) - \frac{p_{1t}^T}{q_t^T}\tilde{y}_t + \left(\frac{R_t^M}{p_{2t}} - \frac{1}{q_t^T}\right)\tilde{a}_t^m\right] + W_{Ct}(0,0)$$

with

$$\tilde{a}_t^b = \frac{1}{q_t^T}\left(m_t - \tilde{a}_t^m - p_{1t}^T \tilde{y}_t\right). \tag{32}$$

Equivalently, this value function can be written as

$$V_{Ct}^{T}(m_t) = \frac{m_t}{q_t^T} + \max_{(\tilde{y}_t, \tilde{a}_t^m) \in \mathbb{R}_+^2} \left[u(\tilde{y}_t) - \varphi_t^T \tilde{y}_t + \left(r_t^M - r_t^T\right)\frac{\tilde{a}_t^m}{p_{2t}}\right] + W_{Ct}(0,0),$$

with (32). The trading rules in the statement of the lemma, $\left(y_{Ct}^T(m_t), m_{Ct}^T(m_t), b_{Ct}^T(m_t)\right)$, follow immediately from this optimization problem, and therefore,

$$V_{Ct}^{T}(m_t) = \frac{m_t}{q_t^T} + u\left(\Upsilon\left(\varphi_t^T\right)\right) - \varphi_t^T \Upsilon\left(\varphi_t^T\right) + \left(r_t^M - r_t^T\right)\frac{m_{Ct}^T(m_t)}{p_{2t}} + W_{Ct}(0,0).$$

The consumer's money-demand problem has no solution if $0 < r_t^M - r_t^T$, and therefore any equilibrium must have $r_t^M - r_t^T \leq 0$, which implies

$$V_{Ct}^{T}(m_t) = \frac{m_t}{q_t^T} + u\left(\Upsilon\left(\varphi_t^T\right)\right) - \varphi_t^T \Upsilon\left(\varphi_t^T\right) + W_{Ct}(0,0).$$

□

Lemma 18. *The optimal first-stage trading rules of a credit producer who carries inventory* y_t *of good 1 are*

$$m_{Pt}^T(y_t) \begin{cases} = \infty & \text{if } r_t^T < r_t^M \\ \in [0,\infty] & \text{if } r_t^T = r_t^M \\ = 0 & \text{if } r_t^M < r_t^T \end{cases}$$

$$q_t^T b_{Pt}^T(y_t) = p_{1t}^T y_t - m_{Pt}^T(y_t),$$

and

$$V_{Pt}^T(y_t) = \varphi_t^T y_t + \beta \bar{V}_{Pt+1}.$$

Proof. With Lemma 12, (8) can be written as

$$V_{Pt}^{T}(y_t) = \frac{p_{1t}^{T}}{q_t^{T}} y_t + \max_{\tilde{a}_t^m \in \mathbb{R}_+} \left(\frac{R_t^M}{p_{2t}} - \frac{1}{q_t^T} \right) \tilde{a}_t^m + \beta \bar{V}_{Pt+1}$$

with

$$\tilde{a}_t^b = \frac{1}{q_t^T} \left(p_{1t}^T y_t - \tilde{a}_t^m \right). \tag{33}$$

Equivalently, this value function can be written as

$$V_{Pt}^{T}(y_t) = \varphi_t^T y_t + \max_{\tilde{a}_t^m \in \mathbb{R}_+} \left(r_t^M - r_t^T \right) \frac{\tilde{a}_t^m}{p_{2t}} + \beta \bar{V}_{Pt+1},$$

with (33). The trading rules in the statement of the lemma, $\left(m_{Pt}^T(y_t), b_{Pt}^T(y_t) \right)$, follow immediately from this optimization problem. The producer's money demand problem has no solution if $0 < r_t^M - r_t^T$, and therefore any equilibrium must have $r_t^M - r_t^T \leq 0$, which implies $V_{Pt}^T(y_t) = \varphi_t^T y_t + \beta \bar{V}_{Pt+1}$. □

Lemma 19. *(a) The optimal production problem of an on-trust producer (i.e., (9)) has no solution if $\kappa < \varphi_t^T$. Otherwise, the solution is:*

$$y_{Pt}^T \begin{cases} = 0 & \text{if } \varphi_t^T < \kappa \\ \in [0, \infty) & \text{if } \varphi_t^T = \kappa. \end{cases}$$

(b) The optimal production problem of a no-trust producer (i.e., (10)) has no solution if $\kappa < \alpha_P^B \left[\theta \varphi_t^B + (1-\theta) \varphi_t^M \right] + \alpha_P^M \varphi_t^M$. Otherwise, the solution is:

$$y_{Pt}^N \begin{cases} = 0 & \text{if } \alpha_P^B \left[\theta \varphi_t^B + (1-\theta) \varphi_t^M \right] + \alpha_P^M \varphi_t^M < \kappa \\ \in [0, \infty) & \text{if } \alpha_P^B \left[\theta \varphi_t^B + (1-\theta) \varphi_t^M \right] + \alpha_P^M \varphi_t^M = \kappa. \end{cases}$$

Proof. From Lemma 14, Lemma 16, and Lemma 18,

$$\begin{aligned} V_{Pt}^M(y_t) &= \varphi_t^M y_t + \beta \bar{V}_{Pt+1} \\ V_{Pt}^B(y_t) &= \left[\theta \varphi_t^B + (1-\theta) \varphi_t^M \right] y_t + \beta \bar{V}_{Pt+1} \\ V_{Pt}^T(y_t) &= \varphi_t^T y_t + \beta \bar{V}_{Pt+1}. \end{aligned}$$

With these expressions, the maximization problems on the right sides of (9) and (10) can be written, respectively, as

$$\max_{y \in \mathbb{R}_+} \left(\varphi_t^T - \kappa \right) y \tag{34}$$

and

$$\max_{y\in\mathbb{R}_+} \left\{\alpha_P^B \left[\theta\varphi_t^B + (1-\theta)\,\varphi_t^M\right] + \alpha_P^M \varphi_t^M - \kappa\right\} y. \tag{35}$$

For Part (a), notice the maximization problem (34) has no solution if $\kappa < \varphi_t^T$, so in any equilibrium it must be that $\varphi_t^T \leq \kappa$, and therefore

$$y_{Pt}^T \equiv \arg\max_{y\in\mathbb{R}_+} \left(\varphi_t^T - \kappa\right) y$$
$$\begin{cases} = 0 & \text{if } \varphi_t^T < \kappa \\ \in [0,\infty) & \text{if } \varphi_t^T = \kappa. \end{cases}$$

For Part (b), notice that the maximization problem (35) has no solution if

$$\kappa < \alpha_P^B \left[\theta\varphi_t^B + (1-\theta)\,\varphi_t^M\right] + \alpha_P^M \varphi_t^M,$$

so in any equilibrium it must be that

$$\alpha_P^B \left[\theta\varphi_t^B + (1-\theta)\,\varphi_t^M\right] + \alpha_P^M \varphi_t^M \leq \kappa,$$

and therefore

$$y_{Pt}^N \equiv \arg\max_{y\in\mathbb{R}_+} \left[\alpha_P^B \left[\theta\varphi_t^B + (1-\theta)\,\varphi_t^M\right] + \alpha_P^M \varphi_t^M - \kappa\right] y$$
$$\begin{cases} = 0 & \text{if } \alpha_P^B \left[\theta\varphi_t^B + (1-\theta)\,\varphi_t^M\right] + \alpha_P^M \varphi_t^M < \kappa \\ \in [0,\infty) & \text{if } \alpha_P^B \left[\theta\varphi_t^B + (1-\theta)\,\varphi_t^M\right] + \alpha_P^M \varphi_t^M = \kappa. \end{cases}$$

□

Next, I state three propositions (Propositions 20-22) whose proofs follow immediately from Lemmas 12-19.

The following proposition reports the optimal first-stage trading rules.

Proposition 20. *In the first stage of period t, the optimal trading rules for consumers and producers of market-access type $j \in \mathbb{A}$, are as follows.*

(a) The optimal first-stage trading rules of a cash consumer with m_t dollars, are:

$$y_{Ct}^M(m_t) = \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{m_t}{p_{1t}^N}\right\}$$
$$m_{Ct}^M(m_t) = \max\left\{m_t - p_{1t}^N \Upsilon\left(\varphi_t^M\right), 0\right\}.$$

The optimal first-stage trading rule of a cash producer with inventory y_t, is:

$$m_{Pt}^{M}(y_t) = p_{1t}^{N} y_t.$$

(b) The first-stage optimization problem of a banked consumer has no solution if $r_t^B < r_t^M$, so any equilibrium must have $r_t^M \leq r_t^B$. The optimal first-stage trading rules of a banked consumer with m_t dollars, are:

$$\begin{aligned}
y_{Ct}^{B} &= \mathrm{Y}\left(\varphi_t^B\right) \\
m_{Ct}^{B} &\begin{cases} \in [0,\infty] & \text{if } r_t^B = r_t^M \\ = 0 & \text{if } r_t^M < r_t^B \end{cases} \\
q_t^B b_{Ct}^{B}(m_t) &= m_t - m_{Ct}^{B} - p_{1t}^{N} \mathrm{Y}\left(\varphi_t^B\right).
\end{aligned}$$

The first-stage bargaining outcome between a banker and banked producer with inventory y_t, is:

$$\begin{aligned}
m_{Pt}^{B} &= \begin{cases} \in [0,\infty] & \text{if } r_t^B = r_t^M \\ = 0 & \text{if } r_t^M < r_t^B \end{cases} \\
q_t^B b_{Pt}^{B}(y_t) &= p_{1t}^{N} y_t - m_{Pt}^{B} \\
p_{2t}\phi_{Pt}^{B}(y_t) &= (1-\theta)\left(r_t^B - r_t^M\right) p_{1t}^{N} y_t.
\end{aligned}$$

(c) The first-stage optimization problem of a credit consumer has no solution if $r_t^T < r_t^M$, so any equilibrium must have $r_t^M \leq r_t^T$. The optimal first-stage trading rules of a credit consumer with m_t dollars, are:

$$\begin{aligned}
y_{Ct}^{T} &= \mathrm{Y}\left(\varphi_t^T\right) \\
m_{Ct}^{T} &\begin{cases} \in [0,\infty] & \text{if } r_t^T = r_t^M \\ = 0 & \text{if } r_t^M < r_t^T \end{cases} \\
q_t^T b_{Ct}^{T}(m_t) &= m_t - m_{Ct}^{T} - p_{1t}^{T} \mathrm{Y}\left(\varphi_t^T\right).
\end{aligned}$$

The optimal first-stage trading rules of a credit producer who carries inventory y_t of good 1, are:

$$\begin{aligned}
m_{Pt}^{T} &\begin{cases} \in [0,\infty] & \text{if } r_t^T = r_t^M \\ = 0 & \text{if } r_t^M < r_t^T \end{cases} \\
q_t^T b_{Pt}^{T}(y_t) &= p_{1t}^{T} y_t - m_{Pt}^{T}.
\end{aligned}$$

The following proposition reports the consumers' beginning-of-period value functions.

Proposition 21. *Let* $W_{Ct}(0,0) \equiv \frac{T_t^M}{p_{2t}} + \max_{m_{t+1}\in\mathbb{R}_+}\left[-\frac{m_{t+1}}{p_{2t}} + \beta\bar{V}_{Ct+1}(m_{t+1})\right]$. *The first-stage pre-trade value functions for consumers of market-access type* $j \in \mathbb{A}$*, are as follows.*

(a) The first-stage pre-trade value of a cash consumer with m_t *dollars, is:*

$$V_{Ct}^M(m_t) = \begin{cases} u\left(\Upsilon\left(\varphi_t^M\right)\right) + \varphi_t^M\left[\frac{m_t}{p_{1t}^N} - \Upsilon\left(\varphi_t^M\right)\right] + W_{Ct}(0,0) & \text{if } \Upsilon\left(\varphi_t^M\right) \leq \frac{m_t}{p_{1t}^N} \\ u\left(\frac{m_t}{p_{1t}^N}\right) + W_{Ct}(0,0) & \text{if } \frac{m_t}{p_{1t}^N} < \Upsilon\left(\varphi_t^M\right). \end{cases}$$

(b) The first-stage pre-trade value of a consumer of market-access type $j \in \{B, T\}$ *with* m_t *dollars, is:*

$$V_{Ct}^j(m_t) = \left(1 + r_t^j\right)\frac{m_t}{p_{2t}} + V_{Ct}^j(0),$$

with $V_{Ct}^j(0) \equiv u\left(\Upsilon\left(\varphi_t^j\right)\right) - \varphi_t^j\Upsilon\left(\varphi_t^j\right) + W_{Ct}(0,0)$.

The following proposition reports the producers' first-stage value functions.

Proposition 22. *The first-stage value functions for producers of market-access type* $j \in \mathbb{A}$*, are as follows.*

(a) The post-production pre-trade value of a producer of market-access type $j \in \{M, T\}$ *with inventory* y_t*, is:*

$$V_{Pt}^j(y_t) = \varphi_t^j y_t + \beta\bar{V}_{Pt+1}.$$

(b) The post-production pre-trade value of a banked producer with inventory y_t*, is:*

$$V_{Pt}^B(y_t) = \left[\theta\varphi_t^B + (1-\theta)\varphi_t^M\right]y_t + \beta\bar{V}_{Pt+1}.$$

(c) The optimal production choice of a producer of trust type $k \in \{T, N\}$*, denoted* y_{Pt}^k*, is:*

$$y_{Pt}^T \begin{cases} = 0 & \text{if } \varphi_t^T < \kappa \\ \in [0,\infty) & \text{if } \varphi_t^T = \kappa. \end{cases}$$

and

$$y_{Pt}^N \begin{cases} = 0 & \text{if } \alpha_P^B\left[\theta\varphi_t^B + (1-\theta)\varphi_t^M\right] + \alpha_P^M\varphi_t^M < \kappa \\ \in [0,\infty) & \text{if } \alpha_P^B\left[\theta\varphi_t^B + (1-\theta)\varphi_t^M\right] + \alpha_P^M\varphi_t^M = \kappa. \end{cases}$$

Corollary 23. *(a) In an equilibrium with positive production of good 1 by on-trust producers,*

$$\varphi_t^T = \kappa, \tag{36}$$

and aggregate production of good 1 by on-trust producers (i.e., total supply of good 1 in the credit-goods market, $\alpha_P^T y_{Pt}^T$) is

$$\alpha_P^T y_{Pt}^T = \alpha_C^T y_{Ct}^T (M_t). \tag{37}$$

(b) In an equilibrium with positive production of good 1 by no-trust producers,

$$\alpha_P^B \left[\theta \varphi_t^B + (1-\theta)\varphi_t^M\right] + \alpha_P^M \varphi_t^M = \kappa, \tag{38}$$

and aggregate production of good 1 by no-trust producers (i.e., total supply of good 1 in the cash-goods market, $\left(1-\alpha_P^T\right) y_{Pt}^N$) is

$$\left(1-\alpha_P^T\right) y_{Pt}^N = \left(1-\alpha_C^T\right) \sum_{j\in\{B,M\}} \alpha_C^j y_{Ct}^j (M_t). \tag{39}$$

Proof. From Part (c) of Proposition 22, (36) must hold in an equilibrium with positive production of good 1 by on-trust producers, and (38) must hold in an equilibrium with positive production of good 1 by no-trust producers. Given (36), on-trust producers are indifferent between choosing any level of production, so the aggregate production they bring to the credit-goods market is determined by (37) (the market-clearing condition for good 1 in the credit-goods market). Given (38), no-trust producers are indifferent between choosing any level of production, so the aggregate production they bring to the cash-goods market is determined by (39) (the market-clearing condition for good 1 in the cash-goods market). □

Lemma 24. *The solution to the consumer's second-stage money-demand problem in period t, i.e., the m_{t+1} that solves the maximization in (1), is characterized by the following first-order condition:*

$$\begin{aligned} r_{t+1}^I - r_{t+1}^M \geq\ & \alpha_C^T\left(r_{t+1}^T - r_{t+1}^M\right) + \left(1-\alpha_C^T\right)\alpha_C^B\left(r_{t+1}^B - r_{t+1}^M\right) \\ & + \left(1-\alpha_C^T\right)\alpha_C^M \max\left\{\left[u'\left(\frac{m_{t+1}}{p_{1t+1}^N}\right) - \varphi_{t+1}^M\right]\frac{1}{\phi_{t+1}^N}, 0\right\}, \end{aligned} \tag{40}$$

with "=" if $m_{t+1} > 0$.

Proof. The first-order condition for a_{t+1}^m in problem (1), or equivalently, (28), is:

$$\frac{1}{p_{2t}} \geq \beta \frac{\partial \bar{V}_{Ct+1}(m_{t+1})}{\partial m_{t+1}}, \tag{41}$$

with "=" if $m_{t+1} > 0$. Recall (from Section 2.2) that

$$\bar{V}_{Ct}(m_t) = \alpha_C^T V_{Ct}^T(m_t) + \left(1 - \alpha_C^T\right) \sum_{j \in \{B,M\}} \alpha_C^j V_{Ct}^j(m_t).$$

The expressions for $\left\{V_{Ct}^j(m_t)\right\}_{j \in \mathbb{A}}$ are given in Proposition 21; specifically,

$$V_{Ct}^M(m_t) = \begin{cases} u\left(\mathrm{Y}\left(\varphi_t^M\right)\right) + \left[m_t - p_{1t}^N \mathrm{Y}\left(\varphi_t^M\right)\right] \frac{1+r_t^M}{p_{2t}} + W_{Ct}(0,0) & \text{if } \mathrm{Y}\left(\varphi_t^M\right) \le \frac{m_t}{p_{1t}^N} \\ u\left(\frac{m_t}{p_{1t}^N}\right) + W_{Ct}(0,0) & \text{if } \frac{m_t}{p_{1t}^N} < \mathrm{Y}\left(\varphi_t^M\right) \end{cases}$$

$$V_{Ct}^j(m_t) = \frac{1 + r_t^j}{p_{2t}} m_t + V_{Ct}^j(0), \text{ for } j \in \{B, T\}.$$

Thus,

$$\begin{aligned} \frac{\partial V_{Ct}^M(m_t)}{\partial m_t} &= \begin{cases} \frac{1+r_t^M}{p_{2t}} & \text{if } \mathrm{Y}\left(\varphi_t^M\right) \le \frac{m_t}{p_{1t}^N} \\ u'\left(\frac{m_t}{p_{1t}^N}\right) \frac{1}{p_{1t}^N} & \text{if } \frac{m_t}{p_{1t}^N} < \mathrm{Y}\left(\varphi_t^M\right) \end{cases} \\ &= \max\left\{u'\left(\frac{m_t}{p_{1t}^N}\right), \varphi_t^M\right\} \frac{1}{p_{1t}^N} \\ \frac{\partial V_{Ct}^j(m_t)}{\partial m_t} &= \frac{1 + r_t^j}{p_{2t}}, \text{ for } j \in \{B, T\}, \end{aligned}$$

and therefore,

$$\begin{aligned} \frac{\partial \bar{V}_{Ct}(m_{t+1})}{\partial m_{t+1}} &= \alpha_C^T \frac{\partial V_{Ct+1}^T(m_{t+1})}{\partial m_{t+1}} + \left(1 - \alpha_C^T\right) \sum_{j \in \{B,M\}} \alpha_C^j \frac{\partial V_{Ct+1}^j(m_{t+1})}{\partial m_{t+1}} \\ &= \alpha_C^T \frac{1 + r_{t+1}^T}{p_{2t+1}} + \left(1 - \alpha_C^T\right) \alpha_C^B \frac{1 + r_{t+1}^B}{p_{2t+1}} \\ &+ \left(1 - \alpha_C^T\right) \alpha_C^M \max\left\{u'\left(\frac{m_{t+1}}{p_{1t+1}^N}\right), \varphi_{t+1}^M\right\} \frac{1}{p_{1t+1}^N} \\ &= \frac{1}{p_{2t+1}} \Bigg\{\alpha_C^T \left(1 + r_{t+1}^T\right) + \left(1 - \alpha_C^T\right) \alpha_C^B \left(1 + r_{t+1}^B\right) \\ &+ \left(1 - \alpha_C^T\right) \alpha_C^M \max\left\{u'\left(\frac{m_{t+1}}{p_{1t+1}^N}\right) \frac{1}{\phi_{t+1}^N}, 1 + r_{t+1}^M\right\}\Bigg\} \\ &= \frac{1}{p_{2t+1}} \Bigg\{1 + r_{t+1}^M + \alpha_C^T \left(r_{t+1}^T - r_{t+1}^M\right) + \left(1 - \alpha_C^T\right) \alpha_C^B \left(r_{t+1}^B - r_{t+1}^M\right) \\ &+ \left(1 - \alpha_C^T\right) \alpha_C^M \max\left\{\left[u'\left(\frac{m_{t+1}}{p_{1t+1}^N}\right) - \left(1 + r_{t+1}^M\right) \phi_{t+1}^N\right] \frac{1}{\phi_{t+1}^N}, 0\right\}\Bigg\}. \end{aligned} \quad (42)$$

With (42), (41) implies

$$\frac{1}{p_{2t}} \geq \beta \frac{1}{p_{2t+1}} \Bigg\{ 1 + r_{t+1}^M + \alpha_C^T \left(r_{t+1}^T - r_{t+1}^M \right) + \left(1 - \alpha_C^T \right) \alpha_C^B \left(r_{t+1}^B - r_{t+1}^M \right)$$

$$+ \left(1 - \alpha_C^T \right) \alpha_C^M \max \left\{ \left[u' \left(\frac{m_{t+1}}{p_{1t+1}^N} \right) - \left(1 + r_{t+1}^M \right) \phi_{t+1}^N \right] \frac{1}{\phi_{t+1}^N}, 0 \right\} \Bigg\},$$

with "=" if $m_{t+1} > 0$. With the definition $r_{t+1}^I \equiv \frac{1}{\beta} \frac{p_{2t+1}}{p_{2t}} - 1$, this condition can be written as (40). □

A.2 Monetary Economy: Equilibrium

The following lemma describes the market-clearing conditions in the first-stage credit markets.

Lemma 25. *(a) The time-t market-clearing condition for the first-stage financial claims traded by consumers and producers of market-access type B, is*

$$0 = \begin{cases} \left(1 - \alpha_C^T \right) \alpha_C^B \hat{b}_{Ct}^B + \left(1 - \alpha_P^T \right) \alpha_P^B \hat{b}_{Pt}^B & \text{if } r_t^B = r_t^M \\ \alpha_C^B \left[\frac{M_t}{p_{1t}^N} - \Upsilon \left(\varphi_t^B \right) \right] + \alpha_P^B \left[\alpha_C^M \min \left\{ \Upsilon \left(\varphi_t^M \right), \frac{M_t}{p_{1t}^N} \right\} + \alpha_C^B \Upsilon \left(\varphi_t^B \right) \right] & \text{if } r_t^M < r_t^B, \end{cases}$$

where

$$\hat{b}_{Ct}^B \in \left(-\infty, \frac{M_t}{p_{1t}^N} - \Upsilon \left(\varphi_t^B \right) \right]$$

and

$$\hat{b}_{Pt}^B \in \left(-\infty, \frac{1 - \alpha_C^T}{1 - \alpha_P^T} \left[\alpha_C^M \min \left\{ \Upsilon \left(\varphi_t^M \right), \frac{M_t}{p_{1t}^N} \right\} + \alpha_C^B \Upsilon \left(\varphi_t^B \right) \right] \right].$$

(b) The time-t market-clearing condition for the first-stage financial claims traded by consumers and producers of market-access type T, is

$$0 = \begin{cases} \alpha_C^T \hat{b}_{Ct}^T + \alpha_P^T \hat{b}_{Pt}^T & \text{if } r_t^T = r_t^M \\ \frac{M_t}{p_{1t}^T} & \text{if } r_t^M < r_t^T, \end{cases}$$

where

$$\hat{b}_{Ct}^T \in \left(-\infty, \frac{M_t}{p_{1t}^T} - \Upsilon \left(\varphi_t^T \right) \right]$$

and

$$\hat{b}_{Pt}^T \in \left(-\infty, \frac{\alpha_C^T}{\alpha_P^T} \Upsilon \left(\varphi_t^T \right) \right].$$

Proof. (a) The market-clearing condition for the first-stage financial claims traded by consumers and producers of market-access type B, is

$$0 = \left(1-\alpha_C^T\right)\alpha_C^B b_{Ct}^B\left(M_t\right) + \left(1-\alpha_P^T\right)\alpha_P^B b_{Pt}^B\left(y_{Pt}^N\right). \tag{43}$$

From Proposition 20,

$$q_t^B b_{Ct}^B\left(M_t\right)\begin{cases}\in\left(-\infty, M_t - p_{1t}^N \Upsilon\left(\varphi_t^B\right)\right] & \text{if } r_t^B = r_t^M\\ = M_t - p_{1t}^N \Upsilon\left(\varphi_t^B\right) & \text{if } r_t^M < r_t^B\end{cases} \tag{44}$$

and

$$q_t^B b_{Pt}^B\left(y_{Pt}^N\right)\begin{cases}\in\left(-\infty, p_{1t}^N y_{Pt}^N\right] & \text{if } r_t^B = r_t^M\\ = p_{1t}^N y_{Pt}^N & \text{if } r_t^M < r_t^B.\end{cases} \tag{45}$$

Corollary 23 implies that in equilibrium, $y_{Pt}^N = \frac{1-\alpha_C^T}{1-\alpha_P^T}\sum_{j\in\{B,M\}}\alpha_C^j y_{Ct}^j\left(M_t\right)$, where (from Proposition 20)

$$y_{Ct}^B\left(M_t\right) = \Upsilon\left(\varphi_t^B\right)$$

and

$$y_{Ct}^M\left(M_t\right) = \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{M_t}{p_{1t}^N}\right\}.$$

Thus,

$$y_{Pt}^N = \frac{1-\alpha_C^T}{1-\alpha_P^T}\left[\alpha_C^M \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{M_t}{p_{1t}^N}\right\} + \alpha_C^B \Upsilon\left(\varphi_t^B\right)\right]. \tag{46}$$

With (46), (44) and (45) imply

$$\frac{1}{p_{1t}^N} q_t^B b_{Ct}^B\left(M_t\right) = \begin{cases}\hat{b}_{Ct}^B & \text{if } r_t^B = r_t^M\\ \frac{M_t}{p_{1t}^N} - \Upsilon\left(\varphi_t^B\right) & \text{if } r_t^M < r_t^B\end{cases} \tag{47}$$

where

$$\hat{b}_{Ct}^B \in \left(-\infty, \frac{M_t}{p_{1t}^N} - \Upsilon\left(\varphi_t^B\right)\right], \tag{48}$$

and

$$\frac{1}{p_{1t}^N} q_t^B b_{Pt}^B\left(y_{Pt}^N\right) = \begin{cases}\hat{b}_{Pt}^B & \text{if } r_t^B = r_t^M\\ \frac{1-\alpha_C^T}{1-\alpha_P^T}\left[\alpha_C^M \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{M_t}{p_{1t}^N}\right\} + \alpha_C^B \Upsilon\left(\varphi_t^B\right)\right] & \text{if } r_t^M < r_t^B,\end{cases} \tag{49}$$

where

$$\hat{b}_{Pt}^B \in \left(-\infty, \frac{1-\alpha_C^T}{1-\alpha_P^T}\left[\alpha_C^M \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{M_t}{p_{1t}^N}\right\} + \alpha_C^B \Upsilon\left(\varphi_t^B\right)\right]\right]. \tag{50}$$

With (47) and (49), the market-clearing condition (43) is equivalent to

$$\begin{aligned} 0 &= \frac{1}{p_{1t}^N} q_t^B \left[\left(1-\alpha_C^T\right) \alpha_C^B b_{Ct}^B \left(M_t\right) + \left(1-\alpha_P^T\right) \alpha_P^B b_{Pt}^B \left(y_{Pt}^N\right) \right] \\ &= \begin{cases} \left(1-\alpha_C^T\right) \alpha_C^B \hat{b}_{Ct}^B + \left(1-\alpha_P^T\right) \alpha_P^B \hat{b}_{Pt}^B & \text{if } r_t^B = r_t^M \\ \alpha_C^B \left[\frac{M_t}{p_{1t}^N} - \Upsilon\left(\varphi_t^B\right) \right] + \alpha_P^B \left[\alpha_C^M \min\left\{ \Upsilon\left(\varphi_t^M\right), \frac{M_t}{p_{1t}^N} \right\} + \alpha_C^B \Upsilon\left(\varphi_t^B\right) \right] & \text{if } r_t^M < r_t^B, \end{cases} \end{aligned}$$

where $\hat{b}_{Ct}^B$ and $\hat{b}_{Pt}^B$ satisfy (48) and (50), respectively.

(b) The market-clearing condition for the first-stage financial claims traded by consumers and producers of market-access type T, is

$$0 = \alpha_C^T b_{Ct}^T \left(M_t\right) + \alpha_P^T b_{Pt}^T \left(y_{Pt}^T\right). \tag{51}$$

From Proposition 20,

$$q_t^T b_{Ct}^T \left(M_t\right) \begin{cases} \in \left(-\infty, M_t - p_{1t}^T \Upsilon\left(\varphi_t^T\right)\right] & \text{if } r_t^T = r_t^M \\ = M_t - p_{1t}^T \Upsilon\left(\varphi_t^T\right) & \text{if } r_t^M < r_t^T \end{cases} \tag{52}$$

and

$$q_t^T b_{Pt}^T \left(y_{Pt}^T\right) \begin{cases} \in \left(-\infty, p_{1t}^T y_{Pt}^T\right] & \text{if } r_t^T = r_t^M \\ = p_{1t}^T y_{Pt}^T & \text{if } r_t^M < r_t^T. \end{cases} \tag{53}$$

Corollary 23 implies that in equilibrium, $y_{Pt}^T = \frac{\alpha_C^T}{\alpha_P^T} y_{Ct}^T \left(M_t\right)$, where (from Proposition 20) $y_{Ct}^T \left(M_t\right) = \Upsilon\left(\varphi_t^T\right)$. Thus,

$$y_{Pt}^T = \frac{\alpha_C^T}{\alpha_P^T} \Upsilon\left(\varphi_t^T\right). \tag{54}$$

With (54), (52) and (53) imply

$$\frac{1}{p_{1t}^T} q_t^T b_{Ct}^T \left(M_t\right) = \begin{cases} \hat{b}_{Ct}^T & \text{if } r_t^T = r_t^M \\ \frac{M_t}{p_{1t}^T} - \Upsilon\left(\varphi_t^T\right) & \text{if } r_t^M < r_t^T \end{cases} \tag{55}$$

where

$$\hat{b}_{Ct}^T \in \left(-\infty, \frac{M_t}{p_{1t}^T} - \Upsilon\left(\varphi_t^T\right)\right], \tag{56}$$

and

$$\frac{1}{p_{1t}^T} q_t^T b_{Pt}^T \left(y_{Pt}^T\right) = \begin{cases} \hat{b}_{Pt}^T & \text{if } r_t^T = r_t^M \\ \frac{\alpha_C^T}{\alpha_P^T} \Upsilon\left(\varphi_t^T\right) & \text{if } r_t^M < r_t^T, \end{cases} \tag{57}$$

where

$$\hat{b}_{Pt}^{T} \in \left(-\infty, \frac{\alpha_C^T}{\alpha_P^T} \Upsilon\left(\varphi_t^T\right)\right]. \tag{58}$$

With (55) and (57), the market-clearing condition (51) is equivalent to

$$\begin{aligned} 0 &= \frac{1}{p_{1t}^T} q_t^T \left[\alpha_C^T b_{Ct}^T \left(M_t\right) + \alpha_P^T b_{Pt}^T \left(y_{Pt}^T\right)\right] \\ &= \begin{cases} \alpha_C^T \hat{b}_{Ct}^T + \alpha_P^T \hat{b}_{Pt}^T & \text{if } r_t^T = r_t^M \\ \frac{M_t}{p_{1t}^T} & \text{if } r_t^M < r_t^T, \end{cases} \end{aligned}$$

where $\hat{b}_{Ct}^T$ and $\hat{b}_{Pt}^T$ satisfy (56) and (58), respectively. □

Corollary 26. *In any monetary equilibrium,* $r_t^M = r_t^T \leq r_t^B$.

Proof. As shown in Part (b) of Proposition 20, the first-stage optimization problem of a banked consumer has no solution if $r_t^B < r_t^M$, so any equilibrium must have $r_t^M \leq r_t^B$. As shown in Part (c) of Proposition 20, the first-stage optimization problem of a credit consumer has no solution if $r_t^T < r_t^M$, so any equilibrium must have $r_t^M \leq r_t^T$. Thus, any equilibrium must have $r_t^M \leq \min\left(r_t^T, r_t^B\right)$. In a monetary equilibrium, $\frac{M_t}{p_{1t}^T} > 0$, so Part (b) of Lemma 25 implies the market-clearing condition for claims in the credit-goods market cannot hold if $r_t^M < r_t^T$. Thus, $r_t^M = r_t^T \leq r_t^B$ in any monetary equilibrium. □

Corollary 27. *In any monetary equilibrium,* $0 < r_{t+1}^I - r_{t+1}^M$ *implies* $\frac{M_{t+1}}{p_{1t+1}^N} < \Upsilon\left(\varphi_{t+1}^M\right)$.

Proof. In a monetary equilibrium, $0 = r_{t+1}^T - r_{t+1}^M$ (by Corollary 26), so the Euler equation for money ((40) in Lemma 24) can be written as

$$\begin{aligned} r_{t+1}^I - r_{t+1}^M = &\left(1-\alpha_C^T\right)\alpha_C^B\left(r_{t+1}^B - r_{t+1}^M\right) \\ &+ \left(1-\alpha_C^T\right)\alpha_C^M \max\left\{\left[u'\left(\frac{M_{t+1}}{p_{1t+1}^N}\right) - \varphi_{t+1}^M\right]\frac{1}{\phi_{t+1}^N}, 0\right\}. \end{aligned} \tag{59}$$

There are two possibilities for any t in a monetary equilibrium: either $r_{t+1}^B - r_{t+1}^M = 0$, or $0 < r_{t+1}^B - r_{t+1}^M$. I want to show that in both cases, $0 < r_{t+1}^I - r_{t+1}^M$ implies $\frac{M_{t+1}}{p_{1t+1}^N} < \Upsilon\left(\varphi_{t+1}^M\right)$.

Case 1. Suppose $r_{t+1}^B - r_{t+1}^M = 0$, then (59) implies

$$0 < r_{t+1}^I - r_{t+1}^M = \left(1-\alpha_C^T\right)\alpha_C^M \max\left\{\left[u'\left(\frac{M_{t+1}}{p_{1t+1}^N}\right) - \varphi_{t+1}^M\right]\frac{1}{\phi_{t+1}^N}, 0\right\},$$

which in turn implies $0 < u'\left(\frac{M_{t+1}}{p_{1t+1}^N}\right) - \varphi_{t+1}^M$, which is equivalent to $\frac{M_{t+1}}{p_{1t+1}^N} < \Upsilon\left(\varphi_{t+1}^M\right)$.

Case 2. Suppose $0 < r_{t+1}^B - r_{t+1}^M$, then from Part (a) of Lemma 25, the market-clearing condition for the first-stage financial claims traded by consumers and producers of market-access type B in period $t+1$ is

$$0 = \alpha_C^B \left[\frac{M_{t+1}}{p_{1t+1}^N} - \Upsilon\left(\varphi_{t+1}^B\right)\right] + \alpha_P^B \left[\alpha_C^M \min\left\{\Upsilon\left(\varphi_{t+1}^M\right), \frac{M_{t+1}}{p_{1t+1}^N}\right\} + \alpha_C^B \Upsilon\left(\varphi_{t+1}^B\right)\right]. \tag{60}$$

Proceed by contradiction, i.e., assume $\Upsilon\left(\varphi_{t+1}^M\right) \leq \frac{M_{t+1}}{p_{1t+1}^N}$. Then, (60) implies

$$0 = \alpha_C^B \left[\frac{M_{t+1}}{p_{1t+1}^N} - \Upsilon\left(\varphi_{t+1}^B\right)\right] + \alpha_P^B \left[\alpha_C^M \Upsilon\left(\varphi_{t+1}^M\right) + \alpha_C^B \Upsilon\left(\varphi_{t+1}^B\right)\right],$$

which is equivalent to

$$\Upsilon\left(\varphi_{t+1}^M\right) - \frac{M_{t+1}}{p_{1t+1}^N} = \frac{\alpha_P^B \left[\alpha_C^M \Upsilon\left(\varphi_{t+1}^M\right) + \alpha_C^B \Upsilon\left(\varphi_{t+1}^B\right)\right] + \alpha_C^B \left[\Upsilon\left(\varphi_{t+1}^M\right) - \Upsilon\left(\varphi_{t+1}^B\right)\right]}{\alpha_C^B} > 0,$$

a contradiction. (The strict inequality follows from the fact that in this case $0 < r_{t+1}^B - r_{t+1}^M$, which implies $0 < \Upsilon\left(\varphi_{t+1}^B\right) < \Upsilon\left(\varphi_{t+1}^M\right)$.) Thus, in this case the monetary equilibrium must have $\frac{M_{t+1}}{p_{1t+1}^N} < \Upsilon\left(\varphi_{t+1}^M\right)$. $\square$

Proposition 28. *A monetary equilibrium with positive production of good 1 by on-trust and no-trust producers is characterized by a path,* $\left\{\left(\phi_t^k, p_{1t}^k\right)_{k\in\{N,T\}}, r_t^B, r_t^T\right\}_{t=0}^{\infty}$, *that satisfies the following conditions:*

1. Market clearing for claims traded by consumers and producers of market-access type T:

$$r_t^M - r_t^T = 0 \leq \frac{M_t}{p_{1t}^T}, \tag{61}$$

   where $p_{1t}^T = \frac{\phi_t^T}{\phi_t^N} p_{1t}^N$.

2. Market clearing for claims traded by consumers and producers of market-access type B:

$$r_t^M - r_t^B \leq 0 \leq \alpha_C^B \left[\frac{M_t}{p_{1t}^N} - \Upsilon\left(\varphi_t^B\right)\right] + \alpha_P^B \left[\alpha_C^M \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{M_t}{p_{1t}^N}\right\} + \alpha_C^B \Upsilon\left(\varphi_t^B\right)\right], \tag{62}$$

   where the second "$\leq$" holds as "$=$" if $r_t^M < r_t^B$, and $\varphi_t^j \equiv \left(1 + r_t^j\right) \phi_t^N$ for $j \in \{B, M\}$.

3. Profit maximization of no-trust producers:

$$\alpha_P^B \left[\theta \varphi_t^B + (1-\theta) \varphi_t^M\right] + \alpha_P^M \varphi_t^M = \kappa. \tag{63}$$

4. Profit maximization of on-trust producers:

$$\varphi_t^T = \kappa, \tag{64}$$

where $\varphi_t^T \equiv \left(1 + r_t^T\right) \phi_t^T$.

5. Euler equation for money holdings:

$$\begin{aligned} r_{t+1}^I - r_{t+1}^M = {} & \left(1-\alpha_C^T\right) \alpha_C^B \left(r_{t+1}^B - r_{t+1}^M\right) \\ & + \left(1-\alpha_C^T\right) \alpha_C^M \max\left\{\left[u'\left(\frac{m_{t+1}}{p_{1t+1}^N}\right) - \varphi_{t+1}^M\right] \frac{1}{\phi_{t+1}^N}, 0\right\}. \end{aligned} \tag{65}$$

Proof. Part 1. From Part (b) of Lemma 25 and Corollary 26, we know that in a monetary equilibrium, $r_t^T = r_t^M$, and the market-clearing condition for claims issued in the credit-goods market is

$$0 = \alpha_C^T \hat{b}_{Ct}^T + \alpha_P^T \hat{b}_{Pt}^T$$

with

$$\hat{b}_{Ct}^T \in \left(-\infty, \frac{M_t}{p_{1t}^T} - \Upsilon\left(\varphi_t^T\right)\right]$$

and

$$\hat{b}_{Pt}^T \in \left(-\infty, \frac{\alpha_C^T}{\alpha_P^T} \Upsilon\left(\varphi_t^T\right)\right].$$

By defining $D_t^T \equiv -\alpha_C^T \hat{b}_{Ct}^T$ and $S_t^T \equiv \alpha_P^T \hat{b}_{Pt}^T$, the market-clearing condition can be written as

$$D_t^T = S_t^T \tag{66}$$

with

$$\begin{aligned} D_t^T &\in \left[\alpha_C^T \left[\Upsilon\left(\varphi_t^T\right) - \frac{M_t}{p_{1t}^T}\right], \infty\right) \equiv \mathcal{D}_t^T \\ S_t^T &\in \left(-\infty, \alpha_C^T \Upsilon\left(\varphi_t^T\right)\right] \equiv \mathcal{S}_t^T. \end{aligned}$$

Notice that (66) can hold if and only if there exists some number $x \in \mathcal{D}_t^T \cap \mathcal{S}_t^T$, i.e., if and only if $\mathcal{D}_t^T \cap \mathcal{S}_t^T \neq \emptyset$, or equivalently, if and only if $0 \leq \frac{M_t}{p_{1t}^T}$. The fact that $p_{1t}^T = \frac{\phi_t^T}{\phi_t^N} p_{1t}^N$ (or equivalently, $z_{1t}^T = \frac{\phi_t^N}{\phi_t^T} z_{1t}^N$) follows from the definitions $\phi_t^k \equiv \frac{p_{1t}^k}{p_{2t}}$ for $k \in \{N, T\}$.

Part 2. From Part (a) of Lemma 25, the market-clearing condition for claims traded between consumers and producers of market-access type B is

$$0 = \begin{cases} \left(1-\alpha_C^T\right)\alpha_C^B \hat{b}_{Ct}^B + \left(1-\alpha_P^T\right)\alpha_P^B \hat{b}_{Pt}^B & \text{if } r_t^B = r_t^M \\ \alpha_C^B \left[\frac{M_t}{p_{1t}^N} - \Upsilon\left(\varphi_t^B\right)\right] + \alpha_P^B \left[\alpha_C^M \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{M_t}{p_{1t}^N}\right\} + \alpha_C^B \Upsilon\left(\varphi_t^B\right)\right] & \text{if } r_t^M < r_t^B, \end{cases}$$

where

$$\hat{b}_{Ct}^B \in \left(-\infty, \frac{M_t}{p_{1t}^N} - \Upsilon\left(\varphi_t^B\right)\right]$$

and

$$\hat{b}_{Pt}^B \in \left(-\infty, \frac{1-\alpha_C^T}{1-\alpha_P^T}\left[\alpha_C^M \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{M_t}{p_{1t}^N}\right\} + \alpha_C^B \Upsilon\left(\varphi_t^B\right)\right]\right].$$

By defining $D_t^B \equiv -\left(1-\alpha_C^T\right)\alpha_C^B \hat{b}_{Ct}^B$ and $S_t^B \equiv \left(1-\alpha_P^T\right)\alpha_P^B \hat{b}_{Pt}^B$, these conditions become

$$0 = \begin{cases} S_t^B - D_t^B & \text{if } r_t^B = r_t^M \\ \alpha_C^B \left[\frac{M_t}{p_{1t}^N} - \Upsilon\left(\varphi_t^B\right)\right] + \alpha_P^B \left[\alpha_C^M \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{M_t}{p_{1t}^N}\right\} + \alpha_C^B \Upsilon\left(\varphi_t^B\right)\right] & \text{if } r_t^M < r_t^B, \end{cases} \tag{67}$$

where

$$D_t^B \in \left[\left(1-\alpha_C^T\right)\alpha_C^B \left[\Upsilon\left(\varphi_t^B\right) - \frac{M_t}{p_{1t}^N}\right], \infty\right) \equiv \mathcal{D}_t^B$$

and

$$S_t^B \in \left(-\infty, \alpha_P^B \left(1-\alpha_C^T\right)\left[\alpha_C^M \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{M_t}{p_{1t}^N}\right\} + \alpha_C^B \Upsilon\left(\varphi_t^B\right)\right]\right] \equiv \mathcal{S}_t^B.$$

Notice that if $r_t^B = r_t^M$, then the market-clearing condition $0 = S_t^B - D_t^B$ can hold if and only if there exists some number $x \in \mathcal{D}_t^B \cap \mathcal{S}_t^B$, i.e., if and only if $\mathcal{D}_t^B \cap \mathcal{S}_t^B \neq \emptyset$, or equivalently, if and only if

$$0 \leq \alpha_C^B \left[\frac{M_t}{p_{1t}^N} - \Upsilon\left(\varphi_t^B\right)\right] + \alpha_P^B \left[\alpha_C^M \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{M_t}{p_{1t}^N}\right\} + \alpha_C^B \Upsilon\left(\varphi_t^B\right)\right].$$

Thus, the market-clearing condition (67) is equivalent to (62).

Part 3. From Part (b) of Corollary 23, in an equilibrium with positive production of good 1 by no-trust producers, their profit-maximization implies (63).

Part 4. From Part (a) of Corollary 23, in an equilibrium with positive production of good 1 by on-trust producers, their profit-maximization implies (64).

Part 5. Follows from Lemma 24 after imposing $r_t^T - r_t^M = 0$. $\square$

A.3 Monetary Economy: Stationary Monetary Equilibrium

Below is the proof of Proposition 3.

Proof of Proposition 3. Parts 1-5 follow from Parts 1-5 of Proposition 28 after imposing stationarity, i.e., $r_t^k = r^k$, $\phi_t^k = \phi^k$, and $z_1^k \equiv \frac{M_t}{p_{1t}^k}$ for $k \in \{N, T\}$, $\varphi_t^T = \varphi^T \equiv \left(1 + r^T\right)\phi^T$, and $\varphi_t^j = \varphi^j \equiv \left(1 + r^j\right)\phi^N$ for $j \in \{B, M\}$. □

Below is the proof of Theorem 4.

Proof of Theorem 4. A stationary monetary equilibrium is a vector, $\left(\left(\phi^k, z_1^k\right)_{k\in\{N,T\}}, r^B, r^T\right)$, that satisfies the conditions in Parts 1-5 of Proposition 3. Parts 1 and 2 of Proposition 3 establish that r^T and r^B satisfy

$$r^T - r^M = 0 \leq r^B - r^M,$$

and that $z_1^T = \frac{\phi^N}{\phi^T} z_1^N$. Part 4 of Proposition 3 gives ϕ^T in closed form, and Part 3 gives ϕ^N as an explicit function of the equilibrium bond rate, r^B. Thus, substituting the expression for ϕ^N from Part 3 into the Euler equation of Part 5, and into the second inequality of Part 2, reduces the characterization of stationary monetary equilibrium to finding a pair, $\left(z_1^N, r^B\right) \in \mathbb{R}_{++} \times [r^M, \infty)$, that satisfies

$$\omega = \left(1 - \alpha_C^T\right)\left[\alpha_C^B \frac{r^B - r^M}{1 + r^M} + \alpha_C^M \max\left\{\frac{1 + r^M + \alpha_P^B \theta\left(r^B - r^M\right)}{1 + r^M}\frac{1}{\kappa}u'\left(z_1^N\right) - 1, 0\right\}\right], \tag{68}$$

with $\omega \equiv \frac{r^I - r^M}{1 + r^M}$, and

$$0 \leq \alpha_C^B\left[z_1^N - \Upsilon\left(\frac{1 + r^B}{1 + r^M + \alpha_P^B\theta\left(r^B - r^M\right)}\kappa\right)\right] + \alpha_P^B\left[\alpha_C^M \min\left\{\Upsilon\left(\frac{1 + r^M}{1 + r^M + \alpha_P^B\theta\left(r^B - r^M\right)}\kappa\right), z_1^N\right\} + \alpha_C^B \Upsilon\left(\frac{1 + r^B}{1 + r^M + \alpha_P^B\theta\left(r^B - r^M\right)}\kappa\right)\right], \tag{69}$$

with "=" if $r^M < r^B$.

Begin by writing (68) as

$$0 = \left(1 - \alpha_C^T\right)\left[\alpha_C^M \lambda\left(z_1^N, r^B\right) + \alpha_C^B \frac{r^B - r^M}{1 + r^M}\right] - \omega, \tag{70}$$

with

$$\lambda\left(z_1^N, r^B\right) \equiv \max\left\{\frac{1+r^M+\alpha_P^B\theta\left(r^B-r^M\right)}{1+r^M}\frac{1}{\kappa}u'\left(z_1^N\right)-1, 0\right\}$$
$$= \begin{cases} \frac{1+r^M+\alpha_P^B\theta\left(r^B-r^M\right)}{1+r^M}\frac{1}{\kappa}u'\left(z_1^N\right)-1 & \text{if } z_1^N < \bar{z}\left(r^B\right) \\ 0 & \text{if } \bar{z}\left(r^B\right) \leq z_1^N, \end{cases}$$

where, for any $r \in [r^M, \infty)$, the function $\bar{z}(\cdot)$ is defined by:

$$\bar{z}(r) \equiv \mathrm{Y}\left(\frac{1+r^M}{1+r^M+\alpha_P^B\theta\left(r-r^M\right)}\kappa\right).$$

The condition $z_1^N < \bar{z}\left(r^B\right)$ characterizes the set of $\left(z_1^N, r^B\right) \in \mathbb{R}_+ \times \left[r^M, \infty\right)$ for which a cash consumer's liquidity constraint is binding, i.e., for which $\lambda\left(z_1^N, r^B\right) > 0$. To describe the equilibrium graphically, it will be useful to define the function $r^B = \mathscr{L}\left(z_1^N\right) \equiv \bar{z}^{-1}\left(z_1^N\right)$, which describes the boundary (in the z_1^N-r^B coordinate axis) of the set of pairs $\left(z_1^N, r^B\right)$ for which the cash consumer is liquidity constrained. The equilibrium condition (70) defines an implicit function $r^B = \mathscr{M}\left(z_1^N\right)$ with the following properties:

- Property $\mathscr{M}$-1:

$$\mathscr{M}\left(\mathrm{Y}\left(\left(1+\frac{\omega}{\left(1-\alpha_C^T\right)\alpha_C^M}\right)\kappa\right)\right) = r^M \leq r^M + \frac{1+r^M}{\left(1-\alpha_C^T\right)\alpha_C^B}\omega = \mathscr{M}\left(z_1^N\right)$$

  for $z_1^N \geq \bar{z}\left(r^M + \frac{r^I-r^M}{\left(1-\alpha_C^T\right)\alpha_C^B}\right) \equiv \mathrm{Y}\left(\frac{\kappa}{1+\frac{\alpha_P^B\theta}{\left(1-\alpha_C^T\right)\alpha_C^B}\omega}\right)$, with "=" only if $\omega = 0$.

- Property $\mathscr{M}$-2: The equilibrium relationship $r^B = \mathscr{M}\left(z_1^N\right)$ implies $z_1^N \leq \bar{z}\left(r^B\right)$ for all $r^B \in \left[r^M, r^M + \frac{r^I-r^M}{\left(1-\alpha_C^T\right)\alpha_C^B}\right)$, with "=" only if $r^B = r^M + \frac{r^I-r^M}{\left(1-\alpha_C^T\right)\alpha_C^B}$. To show this property, consider the implicit function $r^B = \hat{\mathscr{M}}\left(z_1^N\right)$ defined by

$$0 = \mathcal{E}\left(z_1^N, r^B\right), \tag{71}$$

  with

$$\mathcal{E}\left(z_1^N, r^B\right) \equiv \left(1-\alpha_C^T\right)\left\{\alpha_C^M\left[\frac{1+r^M+\alpha_P^B\theta\left(r^B-r^M\right)}{1+r^M}\frac{1}{\kappa}u'\left(z_1^N\right)-1\right]+\alpha_C^B\frac{r^B-r^M}{1+r^M}\right\}-\omega.$$

  Then,

$$\mathcal{E}\left(\bar{z}\left(r^B\right), r^B\right) = \left(1-\alpha_C^T\right)\alpha_C^B\frac{r^B-r^M}{1+r^M}-\omega \leq 0 \tag{72}$$

for all $r^B \in \left[r^M, r^M + \frac{r^I - r^M}{(1-\alpha_C^T)\alpha_C^B}\right]$, with "=" only if $r^B = r^M + \frac{r^I - r^M}{(1-\alpha_C^T)\alpha_C^B}$ (under the maintained assumption $0 < \omega$). Together with the fact that $\frac{\partial \mathcal{E}(z_1^N, r^B)}{\partial z_1^N} < 0$, (72) implies that an equilibrium pair $\left(z_1^N, r^B\right)$ that satisfies $r^B = \mathscr{M}\left(z_1^N\right)$ must satisfy $z_1^N \leq \bar{z}\left(r^B\right)$ for all $r^B \in \left[r^M, r^M + \frac{r^I - r^M}{(1-\alpha_C^T)\alpha_C^B}\right]$, with "=" only if $r^B = r^M + \frac{r^I - r^M}{(1-\alpha_C^T)\alpha_C^B}$. Equivalently, Property $\mathscr{M}$-2 means $\mathscr{M}\left(z_1^N\right) = \hat{\mathscr{M}}\left(z_1^N\right)$ for all $z_1^N \in \left(\Upsilon\left(\left(1 + \frac{\omega}{(1-\alpha_C^T)\alpha_C^M}\right)\kappa\right), \bar{z}\left(r^M + \frac{r^I - r^M}{(1-\alpha_C^T)\alpha_C^B}\right)\right)$.

- Property $\mathscr{M}$-3: $\mathscr{M}' > 0$ for all $z_1^N \in \left(\Upsilon\left(\left(1 + \frac{\omega}{(1-\alpha_C^T)\alpha_C^M}\right)\kappa\right), \bar{z}\left(r^M + \frac{r^I - r^M}{(1-\alpha_C^T)\alpha_C^B}\right)\right)$. To show this, notice that

$$\hat{\mathscr{M}}'\left(z_1^N\right) = \left.\frac{\partial r^B}{\partial z_1^N}\right|_{r^B = \hat{\mathscr{M}}\left(z_1^N\right)} = -\frac{\alpha_C^M\left[1 + r^M + \alpha_P^B\theta\left(r^B - r^M\right)\right]\frac{1}{\kappa}u''\left(z_1^N\right)}{\alpha_C^M\alpha_P^B\theta\frac{1}{\kappa}u'\left(z_1^N\right) + \alpha_C^B} > 0. \quad (73)$$

From Property $\mathscr{M}$-2, $\mathscr{M}\left(z_1^N\right) = \hat{\mathscr{M}}\left(z_1^N\right)$, and therefore $\mathscr{M}'\left(z_1^N\right) = \hat{\mathscr{M}}'\left(z_1^N\right) > 0$, for all $z_1^N \in \left(\Upsilon\left(\left(1 + \frac{\omega}{(1-\alpha_C^T)\alpha_C^M}\right)\kappa\right), \bar{z}\left(r^M + \frac{r^I - r^M}{(1-\alpha_C^T)\alpha_C^B}\right)\right)$.

- Property $\mathscr{M}$-4: $\mathscr{M}$ is continuous and bounded on $\left[\Upsilon\left(\left(1 + \frac{\omega}{(1-\alpha_C^T)\alpha_C^M}\right)\kappa\right), \infty\right)$.

Next, simplify condition (69) as follows. The equilibrium can have $0 < r^B - r^M$, or $0 = r^B - r^M$, so consider each case in turn. Case 1: In an equilibrium with $0 < r^B - r^M$, (69) holds with "=", and it implies the equilibrium must have $z_1^N < \bar{z}\left(r^B\right)$. To see this, proceed by contradiction; i.e., assume the equilibrium has $\bar{z}\left(r^B\right) \leq z_1^N$. Then, (69) implies

$$\begin{aligned} 0 = {} & \alpha_C^B\left[z_1^N - \Upsilon\left(\frac{1 + r^B}{1 + r^M + \alpha_P^B\theta\left(r^B - r^M\right)}\kappa\right)\right] \\ & + \alpha_P^B\left[\alpha_C^M\Upsilon\left(\frac{1 + r^M}{1 + r^M + \alpha_P^B\theta\left(r^B - r^M\right)}\kappa\right) + \alpha_C^B\Upsilon\left(\frac{1 + r^B}{1 + r^M + \alpha_P^B\theta\left(r^B - r^M\right)}\kappa\right)\right] > 0, \end{aligned}$$

a contradiction. (The strict inequality follows from the fact that $\Upsilon\left(\frac{1+r^M}{1+r^M+\alpha_P^B\theta(r^B-r^M)}\kappa\right) \leq z_1^N$ and $r^M < r^B$ imply $0 \leq z_1^N - \Upsilon\left(\frac{1+r^M}{1+r^M+\alpha_P^B\theta(r^B-r^M)}\kappa\right) < z_1^N - \Upsilon\left(\frac{1+r^B}{1+r^M+\alpha_P^B\theta(r^B-r^M)}\kappa\right)$.) Given that $0 < r^B - r^M$ implies $z_1^N < \Upsilon\left(\frac{1+r^M}{1+r^M+\alpha_P^B\theta(r^B-r^M)}\kappa\right)$, condition (69), which holds with "=", implies

$$z_1^N = \underline{z}\left(r^B\right), \quad (74)$$

where, for any $r \in [r^M, \infty)$, the function $\underline{z}(\cdot)$ is defined by:

$$\underline{z}(r) \equiv \frac{\alpha_C^B \left(1-\alpha_P^B\right)}{\alpha_C^B \left(1-\alpha_P^B\right)+\alpha_P^B} \Upsilon \left(\frac{1+r}{1+r^M+\alpha_P^B \theta \left(r-r^M\right)} \kappa\right).$$

Case 2: In an equilibrium with $0 = r^B - r^M$, the inequality (69) holds if and only if

$$\underline{z}\left(r^M\right) \leq z_1^N. \tag{75}$$

By combining (69) from Case 1 with (75) from Case 2, condition (69) is equivalent to

$$z_1^N \begin{cases} = \underline{z}\left(r^B\right) & \text{if } r^M < r^B \\ \in \left[\underline{z}\left(r^M\right), \infty\right) & \text{if } r^M = r^B. \end{cases} \tag{76}$$

The equilibrium condition (76) defines an implicit function,

$$r^B = \mathscr{B}\left(z_1^N\right) \equiv \begin{cases} \underline{z}^{-1}\left(z_1^N\right) & \text{if } \underline{z}(\infty) < z_1^N < \underline{z}\left(r^M\right) \\ r^M & \text{if } \underline{z}\left(r^M\right) \leq z_1^N, \end{cases}$$

where $\underline{z}(\infty) \equiv \lim_{r \to \infty} \underline{z}(r)$. The key properties of this function are:

- Property $\mathscr{B}$-1: $\lim_{z_1^N \downarrow \underline{z}(\infty)} \mathscr{B}\left(z_1^N\right) = \infty$.
- Property $\mathscr{B}$-2: The function $\mathscr{B}$ is continuous on $\left(\underline{z}(\infty), \infty\right)$, with

$$\mathscr{B}'\left(z_1^N\right) = \left[\frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B} \frac{1}{u''\left(\Upsilon\left(\frac{1+r^B}{1+r^M+\alpha_P^B\theta\left(r^B-r^M\right)}\kappa\right)\right)} \frac{\left(1-\alpha_P^B\theta\right)\left(1+r^M\right)\kappa}{\left[1+r^M+\alpha_P^B\theta\left(r^B-r^M\right)\right]^2}\right]^{-1}$$
$$< 0$$

on $\left(\underline{z}(\infty), \underline{z}\left(r^M\right)\right)$.

Next, use Properties $\mathscr{M}$-1, $\mathscr{M}$-2, $\mathscr{M}$-3, $\mathscr{M}$-4, $\mathscr{B}$-1, and $\mathscr{B}$-2 to establish existence and uniqueness of a stationary monetary equilibrium. Consider three parameter configurations in turn.

Configuration 1: $\omega > \bar{\omega}$, where $\bar{\omega} \in [0, \infty)$ is the unique solution to $\Upsilon\left(\left(1+\frac{\bar{\omega}}{\left(1-\alpha_C^T\right)\alpha_C^M}\right)\kappa\right) = \underline{z}\left(r^M\right)$, i.e.,

$$\bar{\omega} = \left(1-\alpha_C^T\right)\alpha_C^M \left[\frac{u'\left(\underline{z}\left(r^M\right)\right)}{u'\left(\bar{z}\left(r^M\right)\right)} - 1\right].$$

In this configuration, $\Upsilon\left(\left(1+\frac{\omega}{\left(1-\alpha_C^T\right)\alpha_C^M}\right)\kappa\right) < \underline{z}\left(r^M\right)$, but consider two cases depending on how large ω is relative to $\bar{\omega}$. First, suppose ω is such that $\Upsilon\left(\left(1+\frac{\omega}{\left(1-\alpha_C^T\right)\alpha_C^M}\right)\kappa\right) \leq$

$\underline{z}(\infty)$. Then, since $\mathscr{M}$ is bounded (Property $\mathscr{M}$-4), $\mathscr{M}(\underline{z}(\infty)) < \mathscr{B}(\underline{z}(\infty)) = \infty$. Conversely, suppose ω is such that $\underline{z}(\infty) < \Upsilon\left(\left(1 + \frac{\omega}{(1-\alpha_C^T)\alpha_C^M}\right)\kappa\right) < \underline{z}\left(r^M\right)$. Then,

$$\mathscr{M}\left(\Upsilon\left(\left(1 + \frac{\omega}{(1-\alpha_C^T)\alpha_C^M}\right)\kappa\right)\right) = \mathscr{B}\left(\underline{z}\left(r^M\right)\right) = r^M < \mathscr{B}\left(\Upsilon\left(\left(1 + \frac{\omega}{(1-\alpha_C^T)\alpha_C^M}\right)\kappa\right)\right),$$

where the inequality follows from the fact that $\mathscr{B}' < 0$ on $\left(\underline{z}(\infty), \underline{z}\left(r^M\right)\right)$ (Property $\mathscr{B}$-2). Thus, in this configuration,

$$\mathscr{M}\left(\underline{\underline{z}}\right) < \mathscr{B}\left(\underline{\underline{z}}\right), \tag{77}$$

where

$$\underline{\underline{z}} \equiv \max\left\{\underline{z}(\infty), \Upsilon\left(\left(1 + \frac{\omega}{(1-\alpha_C^T)\alpha_C^M}\right)\kappa\right)\right\}.$$

Also, notice that

$$\mathscr{B}\left(\underline{z}\left(r^M\right)\right) = \mathscr{M}\left(\Upsilon\left(\left(1 + \frac{\omega}{\left(1-\alpha_C^T\right)\alpha_C^M}\right)\kappa\right)\right) = r^M < \mathscr{M}\left(\underline{z}\left(r^M\right)\right), \tag{78}$$

where the inequality follows from Property $\mathscr{M}$-3. Finally, the inequalities (77) and (78), together with the fact that $\mathscr{M}$ and $\mathscr{B}$ are continuous, with $\mathscr{M}' > 0$ and $\mathscr{B}' < 0$, on $\left(\underline{\underline{z}}, \underline{z}\left(r^M\right)\right)$, implies there exists a unique stationary monetary equilibrium, $\left(z_1^{N*}, r^{B*}\right)$, with $z_1^{N*} \in \left(\underline{z}(\infty), \underline{z}\left(r^M\right)\right)$ that satisfies $\mathscr{M}\left(z_1^{N*}\right) = \mathscr{B}\left(z_1^{N*}\right) = r^{B*}$. Equivalently, $r^{B*} \in \left(r^M, r^M + \frac{r^I - r^M}{(1-\alpha_C^T)\alpha_C^B}\right)$ is the unique solution to

$$0 = \mathcal{E}\left(\underline{z}\left(r^{B*}\right), r^{B*}\right)$$

on $\left(r^M, r^M + \frac{r^I - r^M}{(1-\alpha_C^T)\alpha_C^B}\right)$, and $z_1^{N*} = \underline{z}\left(r^{B*}\right) \in \left(\underline{z}(\infty), \underline{z}\left(r^M\right)\right)$.

Configuration 2: $\omega \in (0, \bar{\omega}]$. In this configuration we have $\underline{z}\left(r^M\right) \leq \Upsilon\left(\left(1 + \frac{\omega}{(1-\alpha_C^T)\alpha_C^M}\right)\kappa\right)$, which given $\mathscr{M}\left(\Upsilon\left(\left(1 + \frac{\omega}{(1-\alpha_C^T)\alpha_C^M}\right)\kappa\right)\right) = \mathscr{B}\left(z_1^N\right) = r^M$ for all $z_1^N \geq \underline{z}\left(r^M\right)$, and $r^M < \mathscr{M}\left(z_1^N\right)$ for all $z_1^N > \Upsilon\left(\left(1 + \frac{\omega}{(1-\alpha_C^T)\alpha_C^M}\right)\kappa\right)$, implies $\mathscr{B}\left(z_1^N\right)$ and $\mathscr{M}\left(z_1^N\right)$ intersect only once, at the stationary monetary equilibrium,

$$\left(z_1^{N*}, r^{B*}\right) = \left(\Upsilon\left(\left(1 + \frac{\omega}{\left(1-\alpha_C^T\right)\alpha_C^M}\right)\kappa\right), r^M\right).$$

Configuration 3: $\omega = 0$. In this configuration $\mathscr{B}$ is unchanged, but $\mathscr{M}(z) = r^M$ for all z on the domain of $\mathscr{M}$, which is $[\Upsilon(\kappa), \infty)$. Thus, $\mathscr{B}\left(z_1^N\right) = \mathscr{M}\left(z_1^N\right) = r^M$ for all $z_1^N \geq \Upsilon(\kappa)$, and there is a continuum of stationary monetary equilibria, all with $r^{B*} = r^M$, and each indexed by a $z_1^{N*} \in [\Upsilon(\kappa), \infty)$.

The analysis in Configurations 1-3 establishes the existence and uniqueness result. The remaining results follow from combining this analysis with Proposition 3. □

Corollary 29. *In any stationary monetary equilibrium,* for all $\omega \in (0, \infty)$,

1. $\frac{\partial r^B}{\partial \omega} \geq 0$, *with "=" only if* $\omega \in (0, \bar{\omega}]$.
2. $\frac{\partial z_1^N}{\partial \omega} < 0$.

Proof. 1. From Part 1 of Theorem 4,

$$\frac{\partial r^B}{\partial \omega} = \begin{cases} \frac{\partial r^{B*}}{\partial \omega} & \text{if } \bar{\omega} < \omega \\ 0 & \text{if } 0 \leq \omega \leq \bar{\omega}, \end{cases}$$

where $r^{B*} \in \left(r^M, \infty\right)$ is implicitly defined by $T\left(r^{B*}; \omega\right) = 0$, with

$$T(x; \omega) \equiv \left(1 - \alpha_C^T\right) \left\{ \alpha_C^M \left[\frac{u'\left(\frac{\alpha_C^B(1-\alpha_P^B)}{\alpha_C^B(1-\alpha_P^B)+\alpha_P^B} \Upsilon\left(\frac{1+x}{1+r^M+\alpha_P^B\theta(x-r^M)} \kappa \right) \right)}{\frac{1+r^M}{1+r^M+\alpha_P^B\theta(x-r^M)}\kappa} - 1 \right] + \alpha_C^B \frac{x - r^M}{1 + r^M} \right\} - \omega.$$

Hence,

$$\frac{\partial r^{B*}}{\partial \omega} = -\frac{\frac{\partial T\left(r^{B*}; \omega\right)}{\partial \omega}}{\frac{\partial T\left(r^{B*}; \omega\right)}{\partial r^{B*}}},$$

with

$$\begin{aligned} \frac{\partial T(x; \omega)}{\partial x} &= \frac{\left(1 - \alpha_C^T\right) \alpha_C^B}{1 + r^M} \\ &+ \frac{\left(1 - \alpha_C^T\right) \alpha_C^M}{\left(1 + r^M\right) \kappa} \frac{\partial}{\partial x} \left[\frac{u'\left(\frac{\alpha_C^B(1-\alpha_P^B)}{\alpha_C^B(1-\alpha_P^B)+\alpha_P^B} \Upsilon\left(\frac{1+x}{1+r^M+\alpha_P^B\theta(x-r^M)} \kappa \right) \right)}{\left[1 + r^M + \alpha_P^B \theta\left(x - r^M\right)\right]^{-1}} \right], \end{aligned}$$

where

$$\frac{\partial}{\partial x}\left[\frac{u'\left(\frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B}\mathrm{Y}\left(\frac{1+x}{1+r^M+\alpha_P^B\theta(x-r^M)}\kappa\right)\right)}{\left[1+r^M+\alpha_P^B\theta(x-r^M)\right]^{-1}}\right]=$$

$$\alpha_P^B\theta u'\left(\frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B}\mathrm{Y}\left(\frac{1+x}{1+r^M+\alpha_P^B\theta(x-r^M)}\kappa\right)\right)$$

$$+\frac{u''\left(\frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B}\mathrm{Y}\left(\frac{1+x}{1+r^M+\alpha_P^B\theta(x-r^M)}\kappa\right)\right)}{\left[1+r^M+\alpha_P^B\theta(x-r^M)\right]^{-1}}$$

$$\times\frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B}\mathrm{Y}'\left(\frac{1+x}{1+r^M+\alpha_P^B\theta(x-r^M)}\kappa\right)\frac{\left(1-\alpha_P^B\theta\right)\left(1+r^M\right)}{\left[1+r^M+\alpha_P^B\theta(x-r^M)\right]^2}\kappa$$

$$>0.$$

This inequality implies $\frac{\partial T\left(r^{B*};\omega\right)}{\partial r^{B*}}>0$, and since $\frac{\partial T(x;\omega)}{\partial\omega}=-1$, we have

$$\frac{\partial r^{B*}}{\partial\omega}=\frac{1}{\frac{\partial T(r^{B*};\omega)}{\partial r^{B*}}}>0.$$

2. From Part 3 of Theorem 4,

$$\frac{\partial z_1^N}{\partial\omega}=\begin{cases}\frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B}\frac{\left(1-\alpha_P^B\theta\right)\left(1+r^B\right)}{\left[1+r^M+\alpha_P^B\theta(r^B-r^M)\right]^2}\kappa\mathrm{Y}'\left(\frac{1+r^B}{1+r^M+\alpha_P^B\theta(r^B-r^M)}\kappa\right)\frac{\partial r^B}{\partial\omega} & \text{if } \bar{\omega}<\omega\\ \frac{1}{\left(1-\alpha_C^T\right)\alpha_C^M}\kappa\mathrm{Y}'\left(\left(1+\frac{\omega}{\left(1-\alpha_C^T\right)\alpha_C^M}\right)\kappa\right) & \text{if } 0<\omega\leq\bar{\omega}.\end{cases}$$

Since $\frac{\partial r^B}{\partial\omega}>0$ (for $\bar{\omega}<\omega$), and $\mathrm{Y}'(\cdot)<0$, we have $\frac{\partial z_1^N}{\partial\omega}<0$. □

A.4 Nonmonetary Economy

Proof of Theorem 6. Recall that $\mathcal{V}_{Pt}^M(y_t)=\beta\bar{\mathcal{V}}_{Pt+1}$, $\mathcal{V}_{Pt}^B(y_t)=\theta\tilde{\varphi}_t^B y_t+\beta\bar{\mathcal{V}}_{Pt+1}$, and $\mathcal{V}_{Pt}^T(y_t)=\tilde{\varphi}_t^T y_t+\beta\bar{\mathcal{V}}_{Pt+1}$, so the first-order conditions for (21) and (22) imply $\tilde{\varphi}_t^T=\kappa$ and $\tilde{\varphi}_t^B=\frac{\kappa}{\alpha_P^B\theta}$ for all t, respectively. The solution to (17) implies per-capita real consumption of good 1 by a banked consumer is $\mathrm{y}_{Ct}^B=\mathrm{Y}\left(\tilde{\varphi}_t^B\right)=\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B\theta}\right)$. The solution to (19) implies per-capita real consumption of good 1 by a credit consumer is $\mathrm{y}_{Ct}^T=\mathrm{Y}\left(\tilde{\varphi}_t^T\right)=\mathrm{Y}(\kappa)$. Aggregate real consumption is

$$\begin{aligned}\tilde{\mathrm{Y}}_C&=\alpha_C^T\mathrm{y}_{Ct}^T+\left(1-\alpha_C^T\right)\alpha_C^B\mathrm{y}_{Ct}^B\\&=\alpha_C^T\mathrm{Y}(\kappa)+\left(1-\alpha_C^T\right)\alpha_C^B\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B\theta}\right).\end{aligned}$$

As $\alpha_j^B\to1$ for $j\in\{C,P\}$, $\tilde{\mathrm{Y}}_C\to\alpha_C^T\mathrm{Y}(\kappa)+\left(1-\alpha_C^T\right)\mathrm{Y}\left(\frac{\kappa}{\theta}\right)$. □

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Proof of Corollary 7. For an economy without a bond market, i.e., $\alpha_i^B = 1 - \alpha_i^M = 0$ for $i \in \{C, P\}$, Proposition 3 implies a stationary monetary equilibrium with positive production of good 1 by on-trust and no-trust producers is characterized by a vector, $\left(\left(\phi^k, z_1^k\right)_{k\in\{N,T\}}, r^T\right)$, with $0 < z_1^k$ for $k \in \{N, T\}$, that satisfies the following conditions:

1. Market clearing for claims traded by consumers and producers of market-access type T:
$$r^M - r^T = 0 \leq z_1^T. \tag{79}$$

2. Profit maximization of no-trust producers and on-trust producers:
$$\phi^N = \phi^T = \frac{\kappa}{1 + r^M}. \tag{80}$$

3. Euler equation for money holdings:
$$r^I - r^M \geq \left(1 - \alpha_C^T\right) \max\left\{\left[u'\left(z_1^N\right) - \left(1 + r^M\right)\phi^N\right]\frac{1}{\phi^N}, 0\right\}, \tag{81}$$
with "=" if $0 < z_1^N$. Condition (81) can be rewritten as
$$\omega \geq \left(1 - \alpha_C^T\right) \max\left\{u'\left(z_1^N\right)\frac{1}{\kappa} - 1, 0\right\},$$
with "=" if $0 < z_1^N$, which implies
$$z_1^N \begin{cases} = 0 & \text{if } \left(1 - \alpha_C^T\right) L(0) \leq \omega \\ = \Upsilon\left(\left(1 + \frac{\omega}{1-\alpha_C^T}\right)\kappa\right) & \text{if } 0 < \omega < \left(1 - \alpha_C^T\right) L(0) \\ \in \left[\Upsilon(\kappa), \infty\right) & \text{if } \omega = 0. \end{cases}$$

From Proposition 3, real consumption of good 1 is $\min\left\{\Upsilon\left(\left(1 + r^M\right)\phi^N\right), z_1^N\right\}$ for a cash consumer, and $\Upsilon\left(\left(1 + r^T\right)\phi^T\right)$ for a credit consumer. In this case, to:
$$\min\left\{\Upsilon\left(\left(1 + r^M\right)\phi^N\right), z_1^N\right\} = \min\left\{\Upsilon(\kappa), \Upsilon\left(\left(1 + \frac{\omega}{1 - \alpha_C^T}\right)\kappa\right)\right\}$$
$$\Upsilon\left(\left(1 + r^T\right)\phi^T\right) = \Upsilon(\kappa),$$

so aggregate real consumption is equal to
$$\alpha_C^T \Upsilon(\kappa) + \left(1 - \alpha_C^T\right) \min\left\{\Upsilon(\kappa), \Upsilon\left(\left(1 + \frac{\omega}{1 - \alpha_C^T}\right)\kappa\right)\right\}.$$

□

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Proof of Corollary 8. Start with $\bar{\omega}$ as defined in the paragraph leading to Theorem 4, and replace α_j^M with $\alpha_j^M(x)$ for $j \in \{C, P\}$ to obtain the generalized threshold,

$$\bar{\omega}(x) \equiv \left(1-\alpha_C^T\right)\bar{\alpha}_C^M(x)\left[u'\left(\frac{\left[1-\bar{\alpha}_C^M(x)\right]\bar{\alpha}_P^M(x)}{\left[1-\bar{\alpha}_C^M(x)\right]\bar{\alpha}_P^M(x)+1-\bar{\alpha}_P^M(x)}\mathrm{Y}(\kappa)\right)\frac{1}{\kappa}-1\right].$$

Let $\bar{\omega}^\infty \equiv \lim_{x\to\infty}\bar{\omega}(x)$, and notice that

$$\begin{aligned}
\bar{\omega}^\infty &= \lim_{x\to\infty}\left\{\left(1-\alpha_C^T\right)\bar{\alpha}_C^M(x)\left[u'\left(\frac{\left[1-\bar{\alpha}_C^M(x)\right]\bar{\alpha}_P^M(x)}{\left[1-\bar{\alpha}_C^M(x)\right]\bar{\alpha}_P^M(x)+1-\bar{\alpha}_P^M(x)}\mathrm{Y}(\kappa)\right)\frac{1}{\kappa}-1\right]\right\}\\
&= \tfrac{1-\alpha_C^T}{\kappa}\left[\lim_{x\to\infty}\bar{\alpha}_C^M(x)\right]\left[\lim_{x\to\infty}u'\left(\frac{\left[1-\bar{\alpha}_C^M(x)\right]\bar{\alpha}_P^M(x)}{\left[1-\bar{\alpha}_C^M(x)\right]\bar{\alpha}_P^M(x)+1-\bar{\alpha}_P^M(x)}\mathrm{Y}(\kappa)\right)\right]\\
&= 0.
\end{aligned}$$

The claims in the statement of the corollary follow from Theorem 4, after replacing α_j^M with $\lim_{x\to\infty}\bar{\alpha}_j^M(x)$ for $j \in \{C, P\}$, and $\bar{\omega}$ with $\bar{\omega}^\infty = 0$. □

Below, is the proof of Corollary 9.

Proof of Corollary 9. In general, we can write

$$\frac{d\mathrm{Y}(\varphi^\infty(\omega))}{d\omega}\frac{\omega}{\mathrm{Y}(\varphi^\infty(\omega))} = \left(\frac{d\mathrm{Y}(\varphi^\infty(\omega))}{d\varphi^\infty(\omega)}\frac{\varphi^\infty(\omega)}{\mathrm{Y}(\varphi^\infty(\omega))}\right)\left(\frac{d\varphi^\infty(\omega)}{d\omega}\frac{\omega}{\varphi^\infty(\omega)}\right) \tag{82}$$

where $\varphi^\infty(\omega) \equiv \frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa$. The first bracket is the elasticity of demand with respect to the cum-interest relative price, φ^∞. The second bracket is the elasticity of the cum-dividend relative price with respect to ω. The first bracket in (27) can be written as:

$$\begin{aligned}
\frac{d\mathrm{Y}(\varphi^\infty(\omega))}{d\varphi^\infty(\omega)}\frac{\varphi^\infty(\omega)}{\mathrm{Y}(\varphi^\infty(\omega))} &= \frac{\varphi^\infty(\omega)}{u''(\mathrm{Y}(\varphi^\infty(\omega)))\,\mathrm{Y}(\varphi^\infty(\omega))}\\
&= \frac{u'(\mathrm{Y}(\varphi^\infty(\omega)))}{u''(\mathrm{Y}(\varphi^\infty(\omega)))\,\mathrm{Y}(\varphi^\infty(\omega))} \equiv \rho(\mathrm{Y}(\varphi^\infty(\omega))),
\end{aligned}$$

i.e., it is equal to the negative of the reciprocal of the coefficient of relative risk aversion. The second bracket in (27) can be written as:

$$\begin{aligned}\frac{d\varphi^{\infty}(\omega)}{d\omega}\frac{\omega}{\varphi^{\infty}(\omega)} &= \frac{(1-\theta)\left(1-\alpha_C^T\right)}{\left[1-\alpha_C^T+\theta\omega\right]^2}\kappa\frac{\omega}{\varphi^{\infty}(\omega)}\\ &= \frac{(1-\theta)\left(1-\alpha_C^T\right)}{\left[1-\alpha_C^T+\theta\omega\right]^2}\kappa\frac{\omega}{\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa}\\ &= \frac{(1-\theta)\left(1-\alpha_C^T\right)\omega}{\left(1-\alpha_C^T+\theta\omega\right)\left(1-\alpha_C^T+\omega\right)}.\end{aligned}$$

Hence, (82) can be written as (27). □

A.7 Welfare

Proof of Proposition 10. The proof proceeds in four steps. Steps 1 and 2 derive $\tilde{\mathscr{W}}$ and $\bar{\mathscr{W}}$, respectively. Step 3 and 4 establish parts 1 and 2 in the statement of the proposition, respectively.

Step 1: Derive $\tilde{\mathscr{W}}$. First, consider consumers. From Section 4, a cash consumer's value is

$$\mathcal{V}_{Ct}^M = \beta\bar{\mathcal{V}}_{Ct+1}, \tag{83}$$

and the value of a consumer of market-access type $j \in \{B, T\}$, is

$$\mathcal{V}_{Ct}^j = u\left(\mathrm{Y}\left(\tilde{\varphi}_t^j\right)\right) - \tilde{\varphi}_t^j\mathrm{Y}\left(\tilde{\varphi}_t^j\right) + \beta\bar{\mathcal{V}}_{Ct+1}. \tag{84}$$

The beginning-of-period value of a consumer in the nonmonetary economy is

$$\bar{\mathcal{V}}_{Ct} \equiv \alpha_C^T\mathcal{V}_{Ct}^T + \left(1-\alpha_C^T\right)\sum_{j\in\{B,M\}}\alpha_C^j\mathcal{V}_{Ct}^j,$$

which with (83) and (84), can be written as

$$\begin{aligned}\bar{\mathcal{V}}_{Ct} - \beta\bar{\mathcal{V}}_{Ct+1} = \alpha_C^T&\left[u\left(\mathrm{Y}\left(\tilde{\varphi}_t^T\right)\right) - \tilde{\varphi}_t^T\mathrm{Y}\left(\tilde{\varphi}_t^T\right)\right]\\ &+\left(1-\alpha_C^T\right)\alpha_C^B\left[u\left(\mathrm{Y}\left(\tilde{\varphi}_t^B\right)\right) - \tilde{\varphi}_t^B\mathrm{Y}\left(\tilde{\varphi}_t^B\right)\right].\end{aligned}$$

In any equilibrium of the nonmonetary economy, $\tilde{\varphi}_t^B = \frac{\kappa}{\alpha_P^B\theta}$, $\tilde{\varphi}_t^T = \kappa$, $\mathrm{y}_{Ct}^B = \mathrm{Y}\left(\frac{\kappa}{\alpha_P^B\theta}\right)$, and $\mathrm{y}_{Ct}^T = \mathrm{Y}(\kappa)$ (from Theorem 6). Thus, $\tilde{w}_C \equiv \bar{\mathcal{V}}_{Ct} - \beta\bar{\mathcal{V}}_{Ct+1}$ is independent of time, and we can write

$$\begin{aligned}\tilde{w}_C = \alpha_C^T&\left[u\left(\mathrm{Y}(\kappa)\right) - \kappa\mathrm{Y}(\kappa)\right]\\ &+\left(1-\alpha_C^T\right)\alpha_C^B\left[u\left(\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B\theta}\right)\right) - \frac{\kappa}{\alpha_P^B\theta}\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B\theta}\right)\right].\end{aligned} \tag{85}$$

Second, consider producers. From Section 4, the first-stage, post-production, pre-trade value of a cash producer who is carrying their optimal production of good 1, y_{Pt}^{N}, is

$$\mathcal{V}_{Pt}^{M}\left(\mathrm{y}_{Pt}^{N}\right) = \beta\bar{\mathcal{V}}_{Pt+1}. \tag{86}$$

The first-stage, post-production, pre-trade value of a banked producer who is carrying their optimal production of good 1, y_{Pt}^{N}, is

$$\mathcal{V}_{Pt}^{B}\left(\mathrm{y}_{Pt}^{N}\right) = \theta\tilde{\varphi}_{t}^{B}\mathrm{y}_{Pt}^{N} + \beta\bar{\mathcal{V}}_{Pt+1}. \tag{87}$$

The first-stage, post-production, pre-trade value of a credit producer who is carrying their optimal production of good 1, y_{Pt}^{T}, is

$$\mathcal{V}_{Pt}^{T}\left(\mathrm{y}_{Pt}^{T}\right) = \tilde{\varphi}_{t}^{T}\mathrm{y}_{Pt}^{T} + \beta\bar{\mathcal{V}}_{Pt+1}. \tag{88}$$

The beginning-of-period value of a producer is $\bar{\mathcal{V}}_{Pt} = \alpha_{P}^{T}\tilde{\Pi}_{Pt}^{T} + \left(1-\alpha_{P}^{T}\right)\tilde{\Pi}_{Pt}^{N}$, with

$$\begin{aligned}
\tilde{\Pi}_{Pt}^{T} &= -\kappa\mathrm{y}_{Pt}^{T} + \mathcal{V}_{Pt}^{T}\left(\mathrm{y}_{Pt}^{T}\right) \\
\tilde{\Pi}_{Pt}^{N} &= -\kappa\mathrm{y}_{Pt}^{N} + \sum_{j\in\{B,M\}}\alpha_{P}^{j}\mathcal{V}_{Pt}^{j}\left(\mathrm{y}_{Pt}^{N}\right),
\end{aligned}$$

i.e.,

$$\begin{aligned}
\bar{\mathcal{V}}_{Pt} = &\left(1-\alpha_{P}^{T}\right)\alpha_{P}^{M}\left[\mathcal{V}_{Pt}^{M}\left(\mathrm{y}_{Pt}^{N}\right) - \kappa\mathrm{y}_{Pt}^{N}\right] \\
&+ \left(1-\alpha_{P}^{T}\right)\alpha_{P}^{B}\left[\mathcal{V}_{Pt}^{B}\left(\mathrm{y}_{Pt}^{N}\right) - \kappa\mathrm{y}_{Pt}^{N}\right] \\
&+ \alpha_{P}^{T}\left[\mathcal{V}_{Pt}^{T}\left(\mathrm{y}_{Pt}^{T}\right) - \kappa\mathrm{y}_{Pt}^{T}\right].
\end{aligned}$$

With (86)-(88), $\bar{\mathcal{V}}_{Pt}$ can be written as

$$\begin{aligned}
\bar{\mathcal{V}}_{Pt} - \beta\bar{\mathcal{V}}_{Pt+1} = &\alpha_{P}^{T}\left(\tilde{\varphi}_{t}^{T} - \kappa\right)\mathrm{y}_{Pt}^{T} \\
&+ \left(1-\alpha_{P}^{T}\right)\left(\alpha_{P}^{B}\theta\tilde{\varphi}_{t}^{B} - \kappa\right)\mathrm{y}_{Pt}^{N}.
\end{aligned}$$

In the equilibrium of the nonmonetary economy, we have $\mathrm{y}_{Pt}^{i} = \mathrm{y}_{P}^{i}$ for $i \in \{N, T\}$, and $\tilde{\varphi}_{t}^{j} = \tilde{\varphi}^{j}$ for $j \in \{B, T\}$, so $\tilde{w}_{P} \equiv \bar{\mathcal{V}}_{Pt} - \beta\bar{\mathcal{V}}_{Pt+1}$ is independent of time, and we can write

$$\begin{aligned}
\tilde{w}_{P} = &\alpha_{P}^{T}\left(\tilde{\varphi}^{T} - \kappa\right)\mathrm{y}_{P}^{T} \\
&+ \left(1-\alpha_{P}^{T}\right)\left(\alpha_{P}^{B}\theta\tilde{\varphi}^{B} - \kappa\right)\mathrm{y}_{P}^{N}.
\end{aligned} \tag{89}$$

Lastly, consider bankers. From Section 4, the beginning-of-period values of bankers is

$$\bar{\mathcal{V}}_{Bt} - \beta \bar{\mathcal{V}}_{Bt+1} = \left(1 - \alpha_P^T\right) \alpha_P^B \left(1 - \theta\right) \tilde{\varphi}_t^B \mathrm{y}_{Pt}^N.$$

Since $\tilde{\varphi}_t^B = \tilde{\varphi}^B$ and $\mathrm{y}_{Pt}^N = \mathrm{y}_P^N$, $\tilde{w}_B \equiv \bar{\mathcal{V}}_{Bt} - \beta \bar{\mathcal{V}}_{Bt+1}$ is independent of time, and therefore

$$\tilde{w}_B = \left(1 - \alpha_P^T\right) \alpha_P^B \left(1 - \theta\right) \tilde{\varphi}^B \mathrm{y}_P^N. \tag{90}$$

Next, define the (equally weighted, flow) welfare function, $\tilde{w} \equiv \sum_{i \in \{B,C,P\}} \tilde{w}_i$, which using (85), (89), and (90), can be written as

$$\begin{aligned}
\tilde{w} = {} & \alpha_C^T \left[u\left(\mathrm{Y}\left(\kappa\right)\right) - \kappa \mathrm{Y}\left(\kappa\right)\right] \\
& + \left(1 - \alpha_C^T\right) \alpha_C^B \left[u\left(\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)\right) - \frac{\kappa}{\alpha_P^B \theta} \mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)\right] \\
& + \alpha_P^T \left(\tilde{\varphi}^T - \kappa\right) \mathrm{y}_P^T \\
& + \left(1 - \alpha_P^T\right) \left(\alpha_P^B \theta \tilde{\varphi}^B - \kappa\right) \mathrm{y}_P^N \\
& + \left(1 - \alpha_P^T\right) \alpha_P^B \left(1 - \theta\right) \tilde{\varphi}^B \mathrm{y}_P^N.
\end{aligned}$$

Combining the last two terms gives

$$\begin{aligned}
\tilde{w} = {} & \alpha_C^T \left[u\left(\mathrm{Y}\left(\kappa\right)\right) - \kappa \mathrm{Y}\left(\kappa\right)\right] \\
& + \left(1 - \alpha_C^T\right) \alpha_C^B \left[u\left(\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)\right) - \frac{\kappa}{\alpha_P^B \theta} \mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)\right] \\
& + \alpha_P^T \left(\tilde{\varphi}^T - \kappa\right) \mathrm{y}_P^T \\
& + \left(1 - \alpha_P^T\right) \left(\alpha_P^B \tilde{\varphi}^B - \kappa\right) \mathrm{y}_P^N.
\end{aligned}$$

In the equilibrium of the nonmonetary economy, market-clearing for good 1 among no-trust consumers and producers implies $\left(1 - \alpha_P^T\right) \alpha_P^B \mathrm{y}_P^N = \left(1 - \alpha_C^T\right) \alpha_C^B \mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)$, and market-clearing for good 1 among on-trust consumers and producers implies $\alpha_P^T \mathrm{y}_P^T = \alpha_C^T \mathrm{Y}\left(\kappa\right)$. Using these two conditions to substitute y_P^N and y_P^T from the last two terms of the previous expression for $\tilde{w}$, gives

$$\begin{aligned}
\tilde{w} = {} & \alpha_C^T \left[u\left(\mathrm{Y}\left(\kappa\right)\right) - \kappa \mathrm{Y}\left(\kappa\right)\right] \\
& + \left(1 - \alpha_C^T\right) \alpha_C^B \left[u\left(\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)\right) - \frac{\kappa}{\alpha_P^B \theta} \mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right)\right] \\
& + \alpha_C^T \left(\tilde{\varphi}^T - \kappa\right) \mathrm{Y}\left(\kappa\right) \\
& + \left(\alpha_P^B \tilde{\varphi}^B - \kappa\right) \frac{1}{\alpha_P^B} \left(1 - \alpha_C^T\right) \alpha_C^B \mathrm{Y}\left(\frac{\kappa}{\alpha_P^B \theta}\right).
\end{aligned}$$

Also, in the equilibrium of the nonmonetary economy, $\tilde{\varphi}^T - \kappa = 0$, so the above expression simplifies to

$$\begin{aligned}\tilde{w} &= \alpha_C^T \left[u\left(\Upsilon\left(\kappa\right)\right) - \kappa\Upsilon\left(\kappa\right)\right] \\ &+ \left(1 - \alpha_C^T\right)\alpha_C^B \left[u\left(\Upsilon\left(\frac{\kappa}{\alpha_P^B\theta}\right)\right) - \frac{\kappa}{\alpha_P^B\theta}\Upsilon\left(\frac{\kappa}{\alpha_P^B\theta}\right)\right] \\ &+ \left(\alpha_P^B\tilde{\varphi}^B - \kappa\right)\frac{1}{\alpha_P^B}\left(1 - \alpha_C^T\right)\alpha_C^B\Upsilon\left(\frac{\kappa}{\alpha_P^B\theta}\right).\end{aligned}$$

Then combine the last two lines to obtain

$$\begin{aligned}\tilde{w} &= \alpha_C^T \left[u\left(\Upsilon\left(\kappa\right)\right) - \kappa\Upsilon\left(\kappa\right)\right] \\ &+ \left(1 - \alpha_C^T\right)\alpha_C^B \left[u\left(\Upsilon\left(\frac{\kappa}{\alpha_P^B\theta}\right)\right) - \left[\frac{\kappa}{\theta} - \left(\alpha_P^B\tilde{\varphi}^B - \kappa\right)\right]\frac{1}{\alpha_P^B}\Upsilon\left(\frac{\kappa}{\alpha_P^B\theta}\right)\right],\end{aligned}$$

which using the fact that in the equilibrium, $\tilde{\varphi}_t^B = \frac{\kappa}{\alpha_P^B\theta}$, simplifies to

$$\begin{aligned}\tilde{w} &= \alpha_C^T \left[u\left(\Upsilon\left(\kappa\right)\right) - \kappa\Upsilon\left(\kappa\right)\right] \\ &+ \left(1 - \alpha_C^T\right)\alpha_C^B \left[u\left(\Upsilon\left(\frac{\kappa}{\alpha_P^B\theta}\right)\right) - \frac{\kappa}{\alpha_P^B}\Upsilon\left(\frac{\kappa}{\alpha_P^B\theta}\right)\right].\end{aligned} \tag{91}$$

Given the equilibrium condition $\tilde{\varphi}^B = \frac{\kappa}{\alpha_P^B\theta}$, this is the same expression for $\tilde{w}$ that appears in the statement of the proposition.

Step 2: Derive $\bar{\mathscr{W}}$. First, consider consumers. From Lemma 13,

$$V_{Ct}^M\left(m_t\right) = u\left(y_{Ct}^M\left(m_t\right)\right) - \varphi_t^M y_{Ct}^M\left(m_t\right) + \left(1 + r_t^M\right)\frac{m_t}{p_{2t}} + W_{Ct}\left(0,0\right), \tag{92}$$

with $y_{Ct}^M\left(m_t\right) = \min\left\{\Upsilon\left(\varphi_t^M\right), \frac{m_t}{p_{1t}^N}\right\}$. From Lemma 15 and Lemma 17,

$$V_{Ct}^j\left(m_t\right) = u\left(\Upsilon\left(\varphi_t^j\right)\right) - \varphi_t^j\Upsilon\left(\varphi_t^j\right) + \left(1 + r_t^j\right)\frac{m_t}{p_{2t}} + W_{Ct}\left(0,0\right), \tag{93}$$

for $j \in \{B, T\}$. With (28), (92), and (93), the beginning-of-period value of a consumer with m_t dollars is

$$\bar{V}_{Ct}\left(m_t\right) \equiv \alpha_C^T V_{Ct}^T\left(m_t\right) + \left(1 - \alpha_C^T\right)\sum_{j\in\{B,M\}}\alpha_C^j V_{Ct}^j\left(m_t\right),$$

which can be written as

$$
\begin{aligned}
\bar{V}_{Ct}(m_t) &= \left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(y_{Ct}^M(m_t)\right)-\varphi_t^M y_{Ct}^M(m_t)+\left(1+r_t^M\right)\frac{m_t}{p_{2t}}\right]\\
&+\left(1-\alpha_C^T\right)\alpha_C^B\left[u\left(\Upsilon\left(\varphi_t^B\right)\right)-\varphi_t^B\Upsilon\left(\varphi_t^B\right)+\left(1+r_t^B\right)\frac{m_t}{p_{2t}}\right]\\
&+\alpha_C^T\left[u\left(\Upsilon\left(\varphi_t^T\right)\right)-\varphi_t^T\Upsilon\left(\varphi_t^T\right)+\left(1+r_t^T\right)\frac{m_t}{p_{2t}}\right]\\
&+\frac{1}{p_{2t}}T_t^M+\max_{m_{t+1}\in\mathbb{R}_+}\left[-\frac{1}{p_{2t}}m_{t+1}+\beta\bar{V}_{Ct+1}(m_{t+1})\right].
\end{aligned}
$$

Substituting the government budget constraint, $T_t^M = M_{t+1}-\left(1+r_t^M\right)M_t$, and imposing the equilibrium condition $m_t = M_t$ for all t, we get

$$
\begin{aligned}
\bar{V}_{Ct}(M_t) &= \left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(y_{Ct}^M(M_t)\right)-\varphi_t^M y_{Ct}^M(M_t)+\left(1+r_t^M\right)\frac{M_t}{p_{2t}}\right]\\
&+\left(1-\alpha_C^T\right)\alpha_C^B\left[u\left(\Upsilon\left(\varphi_t^B\right)\right)-\varphi_t^B\Upsilon\left(\varphi_t^B\right)+\left(1+r_t^B\right)\frac{M_t}{p_{2t}}\right]\\
&+\alpha_C^T\left[u\left(\Upsilon\left(\varphi_t^T\right)\right)-\varphi_t^T\Upsilon\left(\varphi_t^T\right)+\left(1+r_t^T\right)\frac{M_t}{p_{2t}}\right]\\
&+\frac{1}{p_{2t}}\left[M_{t+1}-\left(1+r_t^M\right)M_t\right]-\frac{1}{p_{2t}}M_{t+1}+\beta\bar{V}_{Ct+1}(M_{t+1}),
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\bar{V}_{Ct}(M_t)-\beta\bar{V}_{Ct+1}(M_{t+1}) &= \left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(y_{Ct}^M(M_t)\right)-\varphi_t^M y_{Ct}^M(M_t)\right]\\
&+\left(1-\alpha_C^T\right)\alpha_C^B\left[u\left(\Upsilon\left(\varphi_t^B\right)\right)-\varphi_t^B\Upsilon\left(\varphi_t^B\right)+\left(r_t^B-r_t^M\right)\frac{M_t}{p_{2t}}\right]\\
&+\alpha_C^T\left[u\left(\Upsilon\left(\varphi_t^T\right)\right)-\varphi_t^T\Upsilon\left(\varphi_t^T\right)+\left(r_t^T-r_t^M\right)\frac{M_t}{p_{2t}}\right].
\end{aligned}
$$

It is convenient to rewrite this expression as

$$
\begin{aligned}
\bar{V}_{Ct}(M_t)-\beta\bar{V}_{Ct+1}(M_{t+1}) &= \left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(y_{Ct}^M(M_t)\right)-\varphi_t^M y_{Ct}^M(M_t)\right]\\
&+\left(1-\alpha_C^T\right)\alpha_C^B\left[u\left(\Upsilon\left(\varphi_t^B\right)\right)-\varphi_t^B\Upsilon\left(\varphi_t^B\right)+\left(\varphi_t^B-\varphi_t^M\right)\frac{M_t}{p_{1t}^N}\right]\\
&+\alpha_C^T\left[u\left(\Upsilon\left(\varphi_t^T\right)\right)-\varphi_t^T\Upsilon\left(\varphi_t^T\right)+\left(r_t^T-r_t^M\right)\phi_t^N\frac{M_t}{p_{1t}^N}\right].
\end{aligned}
$$

Along a stationary equilibrium, $\frac{M_t}{p_{1t}^N}=z_1^N$, $\phi_t^N=\phi^N$, $r_t^j=r^j$ and $\varphi_t^j=\varphi^j$ for $j\in\mathbb{A}$, and from parts 3 and 5 of Theorem 4, $y_{Ct}^M(M_t)=z_1^N$.[22] Imposing these equilibrium properties, the

[22]Strictly speaking, in a stationary monetary equilibrium we have $y_{Ct}^M(M_t)=y_C^M\le z_1^N$ for all ω, with "=" unless $\omega=0$. But to simplify the exposition, here I set $y_{Ct}^M(M_t)=z_1^N$, and will consider the case $\omega\to 0$ separately.

previous expression becomes

$$\begin{aligned}\bar{V}_{Ct}(M_t) - \beta\bar{V}_{Ct+1}(M_{t+1}) = &\left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(z_1^N\right) - \varphi^M z_1^N\right] \\ &+ \left(1-\alpha_C^T\right)\alpha_C^B\left[u\left(\mathrm{Y}\left(\varphi^B\right)\right) - \varphi^B\mathrm{Y}\left(\varphi^B\right) + \left(\varphi^B-\varphi^M\right)z_1^N\right] \\ &+ \alpha_C^T\left[u\left(\mathrm{Y}\left(\varphi^T\right)\right) - \varphi^T\mathrm{Y}\left(\varphi^T\right) + \left(r^T - r^M\right)\phi^N z_1^N\right].\end{aligned}$$

Clearly, $\bar{V}_{Ct}(M_t) - \beta\bar{V}_{Ct+1}(M_{t+1})$ is independent of time along a stationary monetary equilibrium, so let $\bar{w}_C \equiv \bar{V}_{Ct}(M_t) - \beta\bar{V}_{Ct+1}(M_{t+1})$, and write it as

$$\begin{aligned}\bar{w}_C = &\left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(z_1^N\right) - \varphi^M z_1^N\right] \\ &+ \left(1-\alpha_C^T\right)\alpha_C^B\left[u\left(\mathrm{Y}\left(\varphi^B\right)\right) - \varphi^B\mathrm{Y}\left(\varphi^B\right) + \left(\varphi^B-\varphi^M\right)z_1^N\right] \\ &+ \alpha_C^T\left[u\left(\mathrm{Y}\left(\varphi^T\right)\right) - \varphi^T\mathrm{Y}\left(\varphi^T\right) + \left(r^T - r^M\right)\phi^N z_1^N\right].\end{aligned} \tag{94}$$

Second, consider producers. From Lemma 14, the first-stage, post-production, pre-trade value of a cash producer who is carrying their optimal production of good 1, y_{Pt}^N, is

$$V_{Pt}^M\left(y_{Pt}^N\right) = \varphi_t^M y_{Pt}^N + \beta\bar{V}_{Pt+1}. \tag{95}$$

From Lemma 16, the first-stage, post-production, pre-trade value of a banked producer who is carrying their optimal production of good 1, y_{Pt}^N, is

$$V_{Pt}^B\left(y_{Pt}^N\right) = \left[\theta\varphi_t^B + (1-\theta)\varphi_t^M\right]y_{Pt}^N + \beta\bar{V}_{Pt+1}. \tag{96}$$

From Lemma 18, the first-stage, post-production, pre-trade value of a trust producer who is carrying their optimal production of good 1, y_{Pt}^T, is

$$V_{Pt}^T\left(y_{Pt}^T\right) = \varphi_t^T y_{Pt}^T + \beta\bar{V}_{Pt+1}. \tag{97}$$

The beginning-of-period value of a producer is $\bar{V}_{Pt} = \alpha_P^T\Pi_{Pt}^T + \left(1-\alpha_P^T\right)\Pi_{Pt}^N$, with

$$\begin{aligned}\Pi_{Pt}^T &\equiv -\kappa y_{Pt}^T + V_{Pt}^T\left(y_{Pt}^T\right) \\ \Pi_{Pt}^N &\equiv -\kappa y_{Pt}^N + \sum_{j\in\{B,M\}}\alpha_P^j V_{Pt}^j\left(y_{Pt}^N\right),\end{aligned}$$

i.e.,

$$\begin{aligned}\bar{V}_{Pt} = &\left(1-\alpha_P^T\right)\alpha_P^M\left[V_{Pt}^M\left(y_{Pt}^N\right) - \kappa y_{Pt}^N\right] \\ &+ \left(1-\alpha_P^T\right)\alpha_P^B\left[V_{Pt}^B\left(y_{Pt}^N\right) - \kappa y_{Pt}^N\right] \\ &+ \alpha_P^T\left[V_{Pt}^T\left(y_{Pt}^T\right) - \kappa y_{Pt}^T\right].\end{aligned}$$

With (95)-(97), $\bar{V}_{Pt}$ can be written as

$$\begin{aligned}\bar{V}_{Pt} - \beta\bar{V}_{Pt+1} = {} & \left(1-\alpha_P^T\right)\left\{\alpha_P^M\varphi_t^M + \alpha_P^B\left[\theta\varphi_t^B + (1-\theta)\,\varphi_t^M\right] - \kappa\right\} y_{Pt}^N \\ & + \alpha_P^T\left(\varphi_t^T - \kappa\right) y_{Pt}^T.\end{aligned}$$

Along a stationary equilibrium, $\varphi_t^j = \varphi^j$ for $j \in \mathbb{A}$, $y_{Pt}^N = y_P^N$, and $y_{Pt}^T = y_P^T$, so

$$\begin{aligned}\bar{V}_{Pt} - \beta\bar{V}_{Pt+1} = {} & \left(1-\alpha_P^T\right)\left\{\alpha_P^M\varphi^M + \alpha_P^B\left[\theta\varphi^B + (1-\theta)\,\varphi^M\right] - \kappa\right\} y_P^N \\ & + \alpha_P^T\left(\varphi^T - \kappa\right) y_P^T.\end{aligned}$$

Clearly, $\bar{V}_{Pt} - \beta\bar{V}_{Pt+1}$ is independent of time along a stationary monetary equilibrium, so let $\bar{w}_P \equiv \bar{V}_{Pt} - \beta\bar{V}_{Pt+1}$, and write it as

$$\begin{aligned}\bar{w}_P = {} & \left(1-\alpha_P^T\right)\left\{\alpha_P^M\varphi^M + \alpha_P^B\left[\varphi^M + \theta\left(\varphi^B - \varphi^M\right)\right] - \kappa\right\} y_P^N \\ & + \alpha_P^T\left(\varphi^T - \kappa\right) y_P^T. \end{aligned} \tag{98}$$

Lastly, consider bankers. Their value at the beginning of the first stage of period t is

$$\bar{V}_{Bt} = \left(1-\alpha_P^T\right)\alpha_P^B W_{Bt}\left(0, \phi_{Pt}^B\left(y_{Pt}^N\right)\right) + \left[1 - \left(1-\alpha_P^T\right)\alpha_P^B\right] W_{Bt}\left(0,0\right),$$

which with (29), can be written as $\bar{V}_{Bt} = \left(1-\alpha_P^T\right)\alpha_P^B\phi_{Pt}^B\left(y_{Pt}^N\right) + \beta\bar{V}_{Bt+1}$. From Lemma 16, we know that $\phi_{Pt}^B(y_t) = (1-\theta)\left(r_t^B - r_t^M\right)\phi_t^N y_t$, so

$$\begin{aligned}\bar{V}_{Bt} - \beta\bar{V}_{Bt+1} &= \left(1-\alpha_P^T\right)\alpha_P^B(1-\theta)\left(r_t^B - r_t^M\right)\phi_t^N y_{Pt}^N \\ &= \left(1-\alpha_P^T\right)\alpha_P^B(1-\theta)\left(\varphi_t^B - \varphi_t^M\right) y_{Pt}^N.\end{aligned}$$

Along a stationary equilibrium, $\varphi_t^j = \varphi^j$ for $j \in \mathbb{A}$, and $y_{Pt}^N = y_P^N$, so

$$\bar{V}_{Bt} - \beta\bar{V}_{Bt+1} = \left(1-\alpha_P^T\right)\alpha_P^B(1-\theta)\left(\varphi^B - \varphi^M\right) y_P^N.$$

Clearly, $\bar{V}_{Bt} - \beta\bar{V}_{Bt+1}$ is independent of time along a stationary monetary equilibrium, so let $\bar{w}_B \equiv \bar{V}_{Bt} - \beta\bar{V}_{Bt+1}$, and write it as

$$\bar{w}_B = \left(1-\alpha_P^T\right)\alpha_P^B(1-\theta)\left(\varphi^B - \varphi^M\right) y_P^N. \tag{99}$$

Next, define the (equally weighted, flow) welfare function, $\bar{w} \equiv \sum_{i \in \{B,C,P\}} \bar{w}_i$, which using (94), (98), and (99), can be written as

$$\begin{aligned}
\bar{w} = & \left(1-\alpha_C^T\right) \alpha_C^M \left[u\left(z_1^N\right) - \varphi^M z_1^N\right] \\
& + \left(1-\alpha_C^T\right) \alpha_C^B \left[u\left(\mathrm{Y}\left(\varphi^B\right)\right) - \varphi^B \mathrm{Y}\left(\varphi^B\right) + \left(\varphi^B - \varphi^M\right) z_1^N\right] \\
& + \alpha_C^T \left[u\left(\mathrm{Y}\left(\varphi^T\right)\right) - \varphi^T \mathrm{Y}\left(\varphi^T\right) + \left(r^T - r^M\right) \phi^N z_1^N\right] \\
& + \left(1-\alpha_P^T\right) \left\{\alpha_P^M \varphi^M + \alpha_P^B \left[\varphi^M + \theta\left(\varphi^B - \varphi^M\right)\right] - \kappa\right\} y_P^N \\
& + \alpha_P^T \left(\varphi^T - \kappa\right) y_P^T \\
& + \left(1-\alpha_P^T\right) \alpha_P^B \left(1-\theta\right) \left(\varphi^B - \varphi^M\right) y_P^N .
\end{aligned}$$

Combining the fourth and sixth terms gives

$$\begin{aligned}
\bar{w} = & \left(1-\alpha_C^T\right) \alpha_C^M \left[u\left(z_1^N\right) - \varphi^M z_1^N\right] \\
& + \left(1-\alpha_C^T\right) \alpha_C^B \left[u\left(\mathrm{Y}\left(\varphi^B\right)\right) - \varphi^B \mathrm{Y}\left(\varphi^B\right) + \left(\varphi^B - \varphi^M\right) z_1^N\right] \\
& + \alpha_C^T \left[u\left(\mathrm{Y}\left(\varphi^T\right)\right) - \varphi^T \mathrm{Y}\left(\varphi^T\right) + \left(r^T - r^M\right) \phi^N z_1^N\right] \\
& + \left(1-\alpha_P^T\right) \left(\alpha_P^M \varphi^M + \alpha_P^B \varphi^B - \kappa\right) y_P^N \\
& + \alpha_P^T \left(\varphi^T - \kappa\right) y_P^T .
\end{aligned}$$

In the stationary monetary equilibrium, market-clearing for good 1 among no-trust consumers and producers implies $\left(1-\alpha_P^T\right) y_P^N = \left(1-\alpha_C^T\right) \left[\alpha_C^B \mathrm{Y}\left(\varphi^B\right) + \alpha_C^M z_1^N\right]$, and market-clearing for good 1 among on-trust consumers and producers implies $\alpha_P^T y_P^T = \alpha_C^T \mathrm{Y}\left(\varphi^T\right)$. Using these two conditions to substitute y_P^N and y_P^T from the last two terms of the previous expression for $\bar{w}$, gives

$$\begin{aligned}
\bar{w} = & \left(1-\alpha_C^T\right) \alpha_C^M \left[u\left(z_1^N\right) - \varphi^M z_1^N\right] \\
& + \left(1-\alpha_C^T\right) \alpha_C^B \left[u\left(\mathrm{Y}\left(\varphi^B\right)\right) - \varphi^B \mathrm{Y}\left(\varphi^B\right) + \left(\varphi^B - \varphi^M\right) z_1^N\right] \\
& + \alpha_C^T \left[u\left(\mathrm{Y}\left(\varphi^T\right)\right) - \varphi^T \mathrm{Y}\left(\varphi^T\right) + \left(r^T - r^M\right) \phi^N z_1^N\right] \\
& + \left(1-\alpha_C^T\right) \left(\alpha_P^M \varphi^M + \alpha_P^B \varphi^B - \kappa\right) \left[\alpha_C^B \mathrm{Y}\left(\varphi^B\right) + \alpha_C^M z_1^N\right] \\
& + \alpha_C^T \left(\varphi^T - \kappa\right) \mathrm{Y}\left(\varphi^T\right) .
\end{aligned}$$

In an equilibrium, we also have $r^T - r^M = \varphi^T - \kappa = 0$ (parts 1 and 2 of Theorem 4), so the previous expression simplifies further, to

$$\begin{aligned}\bar{\omega} &= \left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(z_1^N\right)-\varphi^M z_1^N\right]\\&+\left(1-\alpha_C^T\right)\alpha_C^B\left[u\left(\Upsilon\left(\varphi^B\right)\right)-\varphi^B\Upsilon\left(\varphi^B\right)+\left(\varphi^B-\varphi^M\right)z_1^N\right]\\&+\left(1-\alpha_C^T\right)\left(\alpha_P^M\varphi^M+\alpha_P^B\varphi^B-\kappa\right)\left[\alpha_C^B\Upsilon\left(\varphi^B\right)+\alpha_C^M z_1^N\right]\\&+\alpha_C^T\left[u\left(\Upsilon\left(\kappa\right)\right)-\kappa\Upsilon\left(\kappa\right)\right].\end{aligned}$$

After some calculations, this expression can be rearranged as

$$\begin{aligned}\bar{\omega} &= \left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(z_1^N\right)-\kappa z_1^N\right]\\&+\left(1-\alpha_C^T\right)\alpha_C^B\left[u\left(\Upsilon\left(\varphi^B\right)\right)-\kappa\Upsilon\left(\varphi^B\right)\right]\\&+\alpha_C^T\left[u\left(\Upsilon\left(\kappa\right)\right)-\kappa\Upsilon\left(\kappa\right)\right]\\&+\left(1-\alpha_C^T\right)\left[\left(\alpha_C^B+\alpha_C^M\alpha_P^B\right)z_1^N-\alpha_C^B\alpha_P^M\Upsilon\left(\varphi^B\right)\right]\left(\varphi^B-\varphi^M\right).\end{aligned}\tag{100}$$

If $\omega \leq \bar{\omega}$, the last term of (100) is equal to 0 (because in this case $\varphi^B - \varphi^M = 0$). It is also equal to 0 if $\bar{\omega} < \omega$. To see this, one can use Part 3 of Theorem 4 to replace z_1^N in the the last term of (100) with

$$z_1^N = \frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B}\Upsilon\left(\varphi^B\right),$$

which implies

$$\begin{aligned}\left(\alpha_C^B+\alpha_C^M\alpha_P^B\right)z_1^N-\alpha_C^B\alpha_P^M\Upsilon\left(\varphi^B\right) &= \left(\alpha_C^B+\alpha_C^M\alpha_P^B\right)\frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B}\Upsilon\left(\varphi^B\right)-\alpha_C^B\alpha_P^M\Upsilon\left(\varphi^B\right)\\&= \alpha_C^B\left[\frac{\left(\alpha_C^B+\alpha_C^M\alpha_P^B\right)\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B}-\alpha_P^M\right]\Upsilon\left(\varphi^B\right)\\&= 0.\end{aligned}\tag{101}$$

(The last line requires replacing α_j^B with $1-\alpha_j^M$ for $j \in \{C, P\}$.) With (101), (100) simplifies to:

$$\begin{aligned}\bar{\omega} &= \left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(z_1^N\right)-\kappa z_1^N\right]\\&+\left(1-\alpha_C^T\right)\alpha_C^B\left[u\left(\Upsilon\left(\varphi^B\right)\right)-\kappa\Upsilon\left(\varphi^B\right)\right]\\&+\alpha_C^T\left[u\left(\Upsilon\left(\kappa\right)\right)-\kappa\Upsilon\left(\kappa\right)\right],\end{aligned}\tag{102}$$

which is the same expression for $\bar{\omega}$ that appears in the statement of the proposition.

Step 3: Establish Part 1. From (91) and (102),

$$\begin{aligned}\bar{w}-\tilde{w} &= \left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(z_1^N\right)-\kappa z_1^N\right] \\ &+ \left(1-\alpha_C^T\right)\alpha_C^B\left\{u\left(\mathrm{Y}\left(\varphi^B\right)\right)-\kappa\mathrm{Y}\left(\varphi^B\right)-\left[u\left(\mathrm{Y}\left(\tilde{\varphi}^B\right)\right)-\frac{\kappa}{\alpha_P^B}\mathrm{Y}\left(\tilde{\varphi}^B\right)\right]\right\} \\ &= \left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(z_1^N\right)-\kappa z_1^N\right] \\ &+ \left(1-\alpha_C^T\right)\alpha_C^B\left\{u\left(\mathrm{Y}\left(\varphi^B\right)\right)-\kappa\mathrm{Y}\left(\varphi^B\right)-\left[u\left(\mathrm{Y}\left(\tilde{\varphi}^B\right)\right)-\kappa\mathrm{Y}\left(\tilde{\varphi}^B\right)\right]\right\} \\ &+ \left(1-\alpha_C^T\right)\alpha_C^B\frac{1-\alpha_P^B}{\alpha_P^B}\kappa\mathrm{Y}\left(\tilde{\varphi}^B\right). \end{aligned} \tag{103}$$

Consider the signs of the the three terms on the right side of the second equality. The last term is clearly strictly positive given $\alpha_j^T \in [0,1)$ for $j \in \{C,P\}$. Since the assumptions on u imply $\mathcal{S}(y) \equiv u(y)-\kappa y$ satisfies $\mathcal{S}'' < 0 = \mathcal{S}(0) = 0$, with $\mathrm{Y}(\kappa) = \arg\max_{y\in\mathbb{R}_+}\mathcal{S}(y)$, it follows that

$$0 < \mathcal{S}(y_h) - \mathcal{S}(y_l) \text{ for any } 0 \le y_l < y_h \le \mathrm{Y}(\kappa). \tag{104}$$

Given $0 < \left(1-\alpha_C^T\right)\alpha_C^M$, condition (104) together with the fact that $0 < z_1^N \le \mathrm{Y}(\kappa)$ in any stationary monetary equilibrium, imply the first term is strictly positive. Given $0 < \left(1-\alpha_C^T\right)\alpha_C^B$, condition (104) together with the fact that $\mathrm{Y}\left(\tilde{\varphi}^B\right) = \mathrm{Y}\left(\frac{1}{\alpha_P^B\theta}\kappa\right) < \mathrm{Y}\left(\frac{1+r^B}{1+r^M+\alpha_P^B\theta(r^B-r^M)}\kappa\right) \le \mathrm{Y}(\kappa)$ (because $\alpha_P^B\theta < 1$, and $r^B \in \left[r^M,\infty\right)$ in any stationary monetary equilibrium), imply the first second is strictly positive. Thus, $0 < \bar{w}-\tilde{w}$.

Second, notice that

$$\begin{aligned} w^*-\bar{w} &= \left(1-\alpha_C^T\right)\alpha_C^B\left\{u\left(\mathrm{Y}(\kappa)\right)-\kappa\mathrm{Y}(\kappa)-\left[u\left(\mathrm{Y}\left(\varphi^B\right)\right)-\kappa\mathrm{Y}\left(\varphi^B\right)\right]\right\} \\ &+ \left(1-\alpha_C^T\right)\alpha_C^M\left\{u\left(\mathrm{Y}(\kappa)\right)-\kappa\mathrm{Y}(\kappa)-\left[u\left(z_1^N\right)-\kappa z_1^N\right]\right\}. \end{aligned} \tag{105}$$

Given $0 < 1-\alpha_C^T$, condition (104) together with the fact that $0 \le \mathrm{Y}\left(\varphi^B\right) < \mathrm{Y}(\kappa)$ in any stationary monetary equilibrium with $\omega > 0$, imply the first term is strictly positive. Given $0 < 1-\alpha_C^T$, condition (104) together with the fact that $0 \le z_1^N < \mathrm{Y}(\kappa)$ in any stationary monetary equilibrium with $\omega > 0$, imply the second term is strictly positive. Thus, $0 < w^*-\bar{w}$.

Step 4: Establish Part 2. From Theorem 4, $\lim_{\omega\to 0}\mathrm{Y}\left(\varphi^B\right) = \lim_{\omega\to 0} z_1^N = \mathrm{Y}(\kappa)$, so (105) implies $\lim_{\omega\to 0}\frac{\bar{w}}{1-\beta} = \frac{w^*}{1-\beta}$. Also from Theorem 4, $\lim_{\omega\to\infty} z_1^N = \frac{\alpha_C^B\left(1-\alpha_P^B\right)}{\alpha_C^B\left(1-\alpha_P^B\right)+\alpha_P^B}\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B\theta}\right) \equiv \underline{z}_1^N > 0$, and $\lim_{\omega\to\infty}\varphi^B = \frac{\kappa}{\alpha_P^B\theta} = \tilde{\varphi}^B$, so (103) implies

$$\begin{aligned}\lim_{\omega\to\infty}\bar{w}-\tilde{w} &= \left(1-\alpha_C^T\right)\alpha_C^M\left[u\left(\underline{z}_1^N\right)-\kappa\underline{z}_1^N\right] \\ &+ \left(1-\alpha_C^T\right)\alpha_C^B\frac{1-\alpha_P^B}{\alpha_P^B}\kappa\mathrm{Y}\left(\frac{\kappa}{\alpha_P^B\theta}\right). \end{aligned}$$

Thus, $\lim_{\omega\to\infty} \bar{w} - \tilde{w} \geq 0$, with "=" only if $\alpha_C^M = 1 - \alpha_P^B = 0$. To conclude, I establish $\frac{\partial \bar{\mathscr{W}}}{\partial \omega} < 0$ for $\omega \in (0, \infty)$. From (102),

$$\begin{aligned}
\frac{\partial \bar{w}}{\partial \omega} &= \left(1-\alpha_C^T\right)\alpha_C^M\left[u'\left(z_1^N\right)-\kappa\right]\frac{\partial z_1^N}{\partial \omega} \\
&+ \left(1-\alpha_C^T\right)\alpha_C^B\left[u'\left(\Upsilon\left(\varphi^B\right)\right)-\kappa\right]\Upsilon'\left(\varphi^B\right)\frac{\partial \varphi^B}{\partial \omega} \\
&= \left(1-\alpha_C^T\right)\alpha_C^M\left[u'\left(z_1^N\right)-\kappa\right]\frac{\partial z_1^N}{\partial \omega} \\
&+ \left(1-\alpha_C^T\right)\alpha_C^B\left[u'\left(\Upsilon\left(\varphi^B\right)\right)-\kappa\right]\Upsilon'\left(\varphi^B\right)\frac{\partial}{\partial \omega}\left[\tfrac{1+r^B}{1+r^M+\alpha_P^B\theta(r^B-r^M)}\kappa\right] \\
&= \left(1-\alpha_C^T\right)\alpha_C^M\left[u'\left(z_1^N\right)-\kappa\right]\frac{\partial z_1^N}{\partial \omega} \\
&+ \left(1-\alpha_C^T\right)\alpha_C^B\left[u'\left(\Upsilon\left(\varphi^B\right)\right)-\kappa\right]\Upsilon'\left(\varphi^B\right)\frac{\partial}{\partial \omega}\left[\tfrac{1+r^B}{1+r^M+\alpha_P^B\theta(r^B-r^M)}\kappa\right] \\
&= \left(1-\alpha_C^T\right)\alpha_C^M\left[u'\left(z_1^N\right)-\kappa\right]\frac{\partial z_1^N}{\partial \omega} \\
&+ \frac{\left(1-\alpha_C^T\right)\alpha_C^B\left[u'\left(\Upsilon\left(\varphi^B\right)\right)-\kappa\right]\left(1-\alpha_P^B\theta\right)\left(1+r^M\right)}{\left[1+r^M+\alpha_P^B\theta\left(r^B-r^M\right)\right]^2}\kappa\Upsilon'\left(\varphi^B\right)\frac{\partial r^B}{\partial \omega}.
\end{aligned}$$

From Corollary 29, we know that for any $\omega \in (0, \infty)$, $\frac{\partial r^B}{\partial \omega} \geq 0$ (with "=" only if $\omega \in (0, \bar{\omega}]$), and $\frac{\partial z_1^N}{\partial \omega} < 0$. Thus, since $\Upsilon'(\cdot) < 0$, it follows that $\frac{\partial \bar{w}}{\partial \omega} < 0$. □

Below, is the proof of Theorem 11.

Proof of Theorem 11. Consider the stationary monetary equilibrium characterized in Theorem 4, with $u' < \infty$, $\omega \in (0, \infty)$, $\alpha_j^T \in [0, 1)$, and the probability α_j^M generalized to a strictly decreasing function, $\bar{\alpha}_j^M(\cdot) : \mathbb{R}_+ \to \left[0, \alpha_j^M\right]$, with $\bar{\alpha}_j^M(x) = 1 - \bar{\alpha}_j^B(x)$, $\bar{\alpha}_j^M(0) = \alpha_j^M \in (0, 1)$, and $\lim_{x\to\infty} \bar{\alpha}_j^M(x) = 0$, for $j \in \{C, P\}$. For this economy, the welfare functions in Proposition 10 generalize to $\tilde{\mathscr{W}}(x) = \frac{\tilde{w}(x)}{1-\beta}$, where

$$\begin{aligned}
\tilde{w}(x) &\equiv \alpha_C^T\left[u\left(\Upsilon(\kappa)\right)-\kappa\Upsilon(\kappa)\right] \\
&+ \left(1-\alpha_C^T\right)\bar{\alpha}_C^B(x)\left[u\left(\Upsilon\left(\frac{\kappa}{\bar{\alpha}_P^B(x)\theta}\right)\right)-\frac{\kappa}{\bar{\alpha}_P^B(x)}\Upsilon\left(\frac{\kappa}{\bar{\alpha}_P^B(x)\theta}\right)\right],
\end{aligned}$$

and to $\bar{\mathscr{W}}(x) = \frac{\bar{w}(x)}{1-\beta}$, where

$$\begin{aligned}
\bar{w}(x) &\equiv \alpha_C^T\left[u\left(\Upsilon(\kappa)\right)-\kappa\Upsilon(\kappa)\right] \\
&+ \left(1-\alpha_C^T\right)\bar{\alpha}_C^B(x)\left[u\left(\Upsilon\left(\varphi^B(x)\right)\right)-\kappa\Upsilon\left(\varphi^B(x)\right)\right] \\
&+ \left(1-\alpha_C^T\right)\bar{\alpha}_C^M(x)\left[u\left(z_1^N(x)\right)-\kappa z_1^N(x)\right],
\end{aligned}$$

with $\varphi^B(x) \equiv \left(1 + r^B(x)\right)\phi^N(x)$, and $r^B(x)$, $\phi^N(x)$, and $z_1^N(x)$, as given in Theorem 4.

First, notice that

$$\begin{aligned}\lim_{x\to\infty} \tilde{w}(x) &= \alpha_C^T\left[u(\Upsilon(\kappa)) - \kappa\Upsilon(\kappa)\right] \\ &\quad + \left(1-\alpha_C^T\right)\left[u\left(\Upsilon\left(\frac{\kappa}{\theta}\right)\right) - \kappa\Upsilon\left(\frac{\kappa}{\theta}\right)\right], \\ &= (1-\beta)\tilde{\mathscr{W}}^\infty, \end{aligned} \tag{106}$$

with $\tilde{\mathscr{W}}^\infty$ as defined in the statement of theorem. From Corollary 8, $\lim_{x\to\infty} z_1^N(x) = 0$ and

$$\begin{aligned}\lim_{x\to\infty}\varphi^B(x) &= \lim_{x\to\infty}\left(1 + r^B(x)\right)\phi^N(x) \\ &= \frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{x\to\infty}\bar{w}(x) &= \alpha_C^T\left[u(\Upsilon(\kappa)) - \kappa\Upsilon(\kappa)\right] \\ &\quad + \left(1-\alpha_C^T\right)\left[u\left(\Upsilon\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\right) - \kappa\Upsilon\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\right] \\ &= (1-\beta)\bar{\mathscr{W}}^\infty,\end{aligned} \tag{107}$$

with $\bar{\mathscr{W}}^\infty$ as defined in the statement of theorem. The inequalities $\tilde{\mathscr{W}}^\infty \leq \bar{\mathscr{W}}^\infty \leq \mathscr{W}^*$ are immediate from (106) and (107), together with the fact that

$$\Upsilon\left(\frac{\kappa}{\theta}\right) \leq \Upsilon\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right) \leq \Upsilon(\kappa).$$

If either $\alpha_C^T = 1$ or $\theta = 1$, as in Part 1 in the statement of the theorem, then (106) and (107) specialize to

$$\tilde{\mathscr{W}}^\infty = \bar{\mathscr{W}}^\infty = \mathscr{W}^* = \frac{u(\Upsilon(\kappa)) - \kappa\Upsilon(\kappa)}{1-\beta}.$$

If $\theta\alpha_C^T < 1$, as in Part 2 in the statement of the theorem, then (106) and (107) imply

$$\bar{\mathscr{W}}^\infty - \tilde{\mathscr{W}}^\infty = \left(1-\alpha_C^T\right)\frac{u\left(\Upsilon\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\right) - \kappa\Upsilon\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right) - \left[u\left(\Upsilon\left(\frac{\kappa}{\theta}\right)\right) - \kappa\Upsilon\left(\frac{\kappa}{\theta}\right)\right]}{1-\beta} \tag{108}$$

and

$$\mathscr{W}^* - \bar{\mathscr{W}}^\infty = \left(1-\alpha_C^T\right)\frac{u(\Upsilon(\kappa)) - \kappa\Upsilon(\kappa) - \left[u\left(\Upsilon\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\right) - \kappa\Upsilon\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\right]}{1-\beta}, \tag{109}$$

so clearly, $\tilde{\mathscr{W}}^{\infty} - \bar{\mathscr{W}}^{\infty} < 0 < \mathscr{W}^{*} - \bar{\mathscr{W}}^{\infty}$, which establishes Part (a). Part (b) is immediate from (108), (109), and from (107), which with $\theta\alpha_C^T < 1$ reduces to

$$\begin{aligned}(1-\beta)\,\bar{\mathscr{W}}^{\infty} &= \alpha_C^T\left[u\left(\Upsilon\left(\kappa\right)\right) - \kappa\Upsilon\left(\kappa\right)\right] \\ &\quad + \left(1-\alpha_C^T\right)\left[u\left(\Upsilon\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\right) - \kappa\Upsilon\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\right],\end{aligned}$$

and implies

$$\begin{aligned}\frac{\partial\bar{\mathscr{W}}^{\infty}}{\partial\omega} &= \frac{(1-\theta)\left(1-\alpha_C^T\right)^2}{(1-\beta)\left(1-\alpha_C^T+\theta\omega\right)^2}\left[u'\left(\Upsilon\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\right) - \kappa\right]\Upsilon'\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\kappa \\ &= \frac{(1-\theta)^2\left(1-\alpha_C^T\right)^2\kappa^2}{(1-\beta)\left(1-\alpha_C^T+\theta\omega\right)^3}\Upsilon'\left(\frac{1-\alpha_C^T+\omega}{1-\alpha_C^T+\theta\omega}\kappa\right)\omega \\ &< 0.\end{aligned}$$

□

B Frequently Asked Questions

B.1 Could Result 1 be Rescued in a Broader Class of Monetary Equilibria?

In Section 6, I have shown that in economies with $0 < (1-\theta)\left(1-\alpha_C^T\right)$, as velocity diverges (by letting $\alpha_j^M \to 0$ for $j \in \{C, P\}$), the allocation implemented by the stationary monetary equilibrium converges to a limit that differs from the high-velocity, pure-credit limit of the nonmonetary economy, contradicting Result 1. One may wonder if there is some monetary equilibrium outside the class of stationary equilibria where Result 1 holds. For the special case with $\alpha_C^M = \alpha_j^T = 0$ for $j \in \{C, P\}$, Lagos and Zhang (2022, Appx. C.2.1) show that consumption of good 1 and welfare are lower in the pure-credit limit of the nonmonetary economy than in the pure-credit cashless limit of any *dynamic* monetary equilibrium. For the case with $\alpha_C^M = \alpha_j^T = 0$ for $j \in \{C, P\}$, Lagos and Zhang (2022, Appx. C.3.1) show that a carefully crafted equilibrium selection scheme can produce a *sunspot* monetary equilibrium whose pure-credit cashless limit converges to the pure-credit limit of the nonmonetary economy. The scheme selects an equilibrium in which the probability assigned to a sunspot state where money loses value forever converges to 1 along the high-velocity pure-credit limit. So, while it is theoretically possible to construct nonstationary equilibria for which Result 1 holds, the fact that it requires such a delicate selection from a large set of sunspot equilibria renders the result too fragile to offer a general foundation for a moneyless approach to monetary economics.

B.2 What About Existing Quantitative Validations of Result 1?

To support the claim that medium-of-exchange considerations are immaterial and that money can be ignored in high-velocity economies, textbooks that advocate the moneyless approach, such as Woodford (2003) and Galí (2008), often complement the theoretical Result 1 with numerical exercises presented as quantitative validations.

> *The second part of chapter 2, however, generates a motive to hold money by introducing real balances as an argument of the household's utility function, and examines its implications under the alternative assumptions of separability and nonseparability of real balances. In the latter case, in particular, the result of monetary policy neutrality is shown to break down, even in the absence of nominal rigidities. The resulting non-neutralities, however, are shown to be of limited interest empirically.*
>
> —Galí (2008, p. 10)

At the same time, neither the usefulness nor the validity of the approach proposed here depends on a claim that monetary frictions do not exist in actual present-day economies. After expounding the theory for the cashless case, I show how the framework can easily be generalized to allow for monetary frictions, modeled in one or another of the ways that are common in monetarist models of inflation determination (by including real balances in the utility function or assuming a cash-in-advance constraint). I show in this case that equilibrium relations continue to be obtained that are direct generalizations of those for the cashless economy and that need not even imply results that are too different as a quantitative matter, if the monetary frictions are parameterized in an empirically plausible way. Hence the cashless analysis can be viewed as a useful approximation even in the case of an economy where money balances do facilitate transactions to some extent.

—Woodford (2003, p. 62)

The numerical exercises compare two versions of an economy: one without money, and another in which money enters a utility function (or, in some cases, a transactions-cost function). Parameter values are sometimes arbitrary, and sometimes chosen so that the monetary version generates a velocity (or inverse real balances) consistent with the empirical velocity associated with a narrow monetary aggregate—typically the monetary base, which tended to be high at the time of their writing. The main references are Ireland (2001), McCallum (2001), Woodford (2003, Ch. 2, Sec. 3.4; Ch. 4, Sec. 3.2), and Galí (2008, Ch. 2, Sec. 2.5). These studies find that the responses to various shocks are quantitatively similar with or without money.

Since the reduced-form specifications used to introduce money are often described as "very general"—in terms of interactions between real balances, consumption, and leisure—the conclusion that money is quantitatively irrelevant may appear robust, aside from disagreements over which moments the calibration should target (e.g., broad money aggregates rather than narrow ones). The problem, however, is that these reduced-form approaches are not, in fact, very general: for example, they cannot capture the role of money as a constraint on market power—the mechanism emphasized in this paper. See Lagos and Zhang (2022, Sec. 7.3) for a detailed discussion.

B.3 Are Monetarist Channels Relevant?

This question can only be answered one research paper at a time, and the answer will depend on the data and subject of interest. Still, a growing body of research highlights the empirical and quantitative relevance of monetary transmission channels activated by policy-induced changes in the opportunity cost of holding money.

Lagos and Wright (2005) estimate that the decline in real money balances caused by the higher opportunity cost of holding money under 10 percent inflation entails a welfare loss equivalent to 3 to 5 percent of consumption. Nagel (2016) finds that the opportunity cost of holding money is a dominant driver of the liquidity premia of near-money financial assets (such as T-bills and "on-the-run" Treasury notes) at business-cycle frequencies. Drechsler et al. (2017) document the "deposits channel" of monetary policy, whereby market power in deposit markets leads banks to widen deposit spreads in response to policy-induced increases in depositors' opportunity cost of holding money. Lagos and Zhang (2020) document the "turnover-liquidity channel" of monetary policy, whereby increases in financial investors' opportunity cost of holding money reduce the resalability of equity and depress its price. Jeenas and Lagos (2024) document the "Q-monetary transmission channel," whereby changes in financial investors' opportunity cost of holding money affect corporate investment and capital structure via their effects on Tobin's Q.

B.4 Is this a Critique of *Walrasian*, or of *Cashless* Monetary Models?

Both. The class of Walrasian models with ad hoc preferences or constraints imposed to accommodate a demand for money, such as money-in-the-utility-function models (Sidrauski (1967)), money-in-the-production-function models (Levhari and Patinkin (1968)), or cash-in-advance models (Clower (1967)), represented a step forward relative to the state of the field at the time they were introduced. Since then, however, three conceptual shortcomings have been identified: (i) treatment of money as *inessential*, (ii) reliance on *implicit theorizing*, and (iii) vulnerability to the *Kareken-Wallace-Lucas critique*. The problem for the moneyless New Keynesian approach is that it relies on this unsatisfactory class of Walrasian monetary models to argue for the irrelevance of money. Moreover, the concern is not merely aesthetic—all three of these known shortcomings are essential to Woodford's derivation of Result 1.

B.4.1 (In)Essentiality of Money

Money is said to be *essential* if the set of allocations that can be supported as equilibria with money is larger (and Pareto superior) than without it.[23] Informally, money is essential if the economy functions better with money. In a Walrasian economy, agents can purchase any budget-feasible bundle. Paradoxically, then, imposing an extraneous cash-in-advance constraint as a stand-in for money's role as a medium of exchange results in a model where money *impedes* trade rather than facilitating it.[24] The Walrasian economy with money functions no better—indeed, it functions *worse*—than without introducing money via a cash-in-advance constraint.

Conceptually, Result 1 is very straightforward. Start with an Arrow-Debreu economy where agents are required to satisfy a cash-in-advance constraint for an exogenous share α of their consumption purchases. Naturally, if the constraint binds, the equilibrium allocation differs from the Arrow-Debreu allocation (which is the Pareto-optimal allocation). Woodford's "cashless limit" corresponds to the limit $\alpha \to 0$, which consists of gradually undoing the cash-in-advance constraint, thereby restoring the underlying Arrow-Debreu economy. The allocation of the cash-in-advance economy converges to the Arrow-Debreu allocation, and since the cash-in-advance constraint is absent in the limit, money is no longer valued in the limiting economy—the well-known result that there is no room for valued money in an Arrow-Debreu economy.

The gist of the foundational approximation result of the moneyless approach is the trivial observation that, if the cash-in-advance Walrasian monetary economy is far enough along the $\alpha \to 0$ limit (velocity is very high), the equilibrium allocation of the monetary cash-in-advance economy will be very close to that of the corresponding nonmonetary economy—the Arrow-Debreu economy. Intuitively, money plays no role in Woodford's cashless limit because it is *inessential* in the background economy—the Arrow-Debreu economy on which the extraneous cash-in-advance constraint is first imposed to make it economy, and then removed, by taking $\alpha \to 0$. In contrast, money is *essential* in the background economy of Section 2 (provided $0 < (1 - \theta) \left(1 - \alpha_C^T\right))$, in the precise sense that the set of allocations that can be supported as equilibria is larger (and actually *better*) with money than without it. And as a result, the role of money does not vanish in the cashless limit. Thus, Result 1 leans into the first shortcoming of the Walrasian monetary models and breaks down in the more relevant economic environments

[23]The concept of essentiality of money can be traced back to Hahn (1987). Neil Wallace gave it sustained emphasis, turning it into a methodological imperative (see Wallace (2001)).

[24]This point goes back at least to Gale (1982) and is also emphasized by Hellwig (1993).

where money is essential.

B.4.2 Implicit Theorizing

Implicit theorizing is the practice of assuming, as part of the economic environment, the functions that money is meant to perform, rather than deriving them as equilibrium outcomes. The term describes monetary models where "moneyness" is an *assumption* rather than a *result*. Examples include Walrasian monetary models where money is assumed to enter utility functions, production functions, and shopping technologies, as well as models that impose constraints on patterns of trade involving money. These reduced-form constructs are intended to stand in for the frictions that would, in a more explicit model, give money value in equilibrium—but that model is not specified.[25] Because these models are designed to deviate minimally from the frictionless Walrasian benchmark, they tend to gloss over important elements. In the influential cash/credit cash-in-advance model of Lucas and Stokey (1983), which serves as the foundation for Woodford (1998), "credit" bears little resemblance to an actual credit market. It consists of a zero-interest deferred payment arrangement between a buyer and a seller and lacks elementary features such as financial intermediaries or even an equilibrium interest rate that responds to credit-market conditions. One consequence of the interest rate being fixed at zero is that it cannot respond to changes in the opportunity cost of holding money.

In contrast, the model of Section 2 treats money as intrinsically useless, and buyers are not constrained to hold cash in advance to purchase good 1 (banked consumers can issue as much debt as they wish to finance these purchases). Credit between consumers and producers in the bond market is intermediated by bankers, who may possess market power, and the equilibrium interest rate responds to credit-market conditions. Unlike in Lucas and Stokey (1983), it is not mechanically tied to the administered interest rate on money. Moreover, policy-induced changes in the opportunity cost of holding money influence the bond-market rate, even in the cashless limit, and this mechanism breaks Result 1. In contrast, the cashless-limiting results in Woodford (1998) correspond to a pure-credit economy in which all agents have access to a very peculiar kind of credit. The interest rate on consumption loans is fixed at zero, regardless of the tightness of the credit market, and regardless of the opportunity cost of holding money as an alternative means of payment. Thus, Woodford's derivation of Result 1 leans into the second shortcoming of Walrasian models and breaks down under more plausible credit-market

[25] Concerns about this style of monetary economics date back at least to Kareken and Wallace (1980).

structures.

B.4.3 Kareken-Wallace-Lucas Critique

No one regards the constructs that make money useful in Walrasian monetary models as primitive. The utility functions, production functions, and shopping technologies assumed to depend on money balances are obviously subject to the Kareken-Wallace-Lucas critique: These constructs will in general not be invariant to the very monetary policies that the model is used to analyze. Evident as this critique may be, it is routinely ignored in applied work—perhaps dismissed as a high-level philosophical or theoretical point of little consequence for practical monetary economics.

In a model that is a special case of the model of Section 2, Lagos and Zhang (2022, Sec. 7.3) show that there is a fairly simple reduced form that can represent the equilibrium of the model, but it requires certain modeling ingredients that are rather unorthodox from the standpoint of the reduced-form literature, such as a monopolistically competitive consumption sector where the elasticity of substitution between varieties is a function of the monetary policy rate. The existing quantitative validations of Result 1 are based on Walrasian models that lack these unorthodox ingredients, which makes them inadequate for drawing any general conclusions about the irrelevance of money.

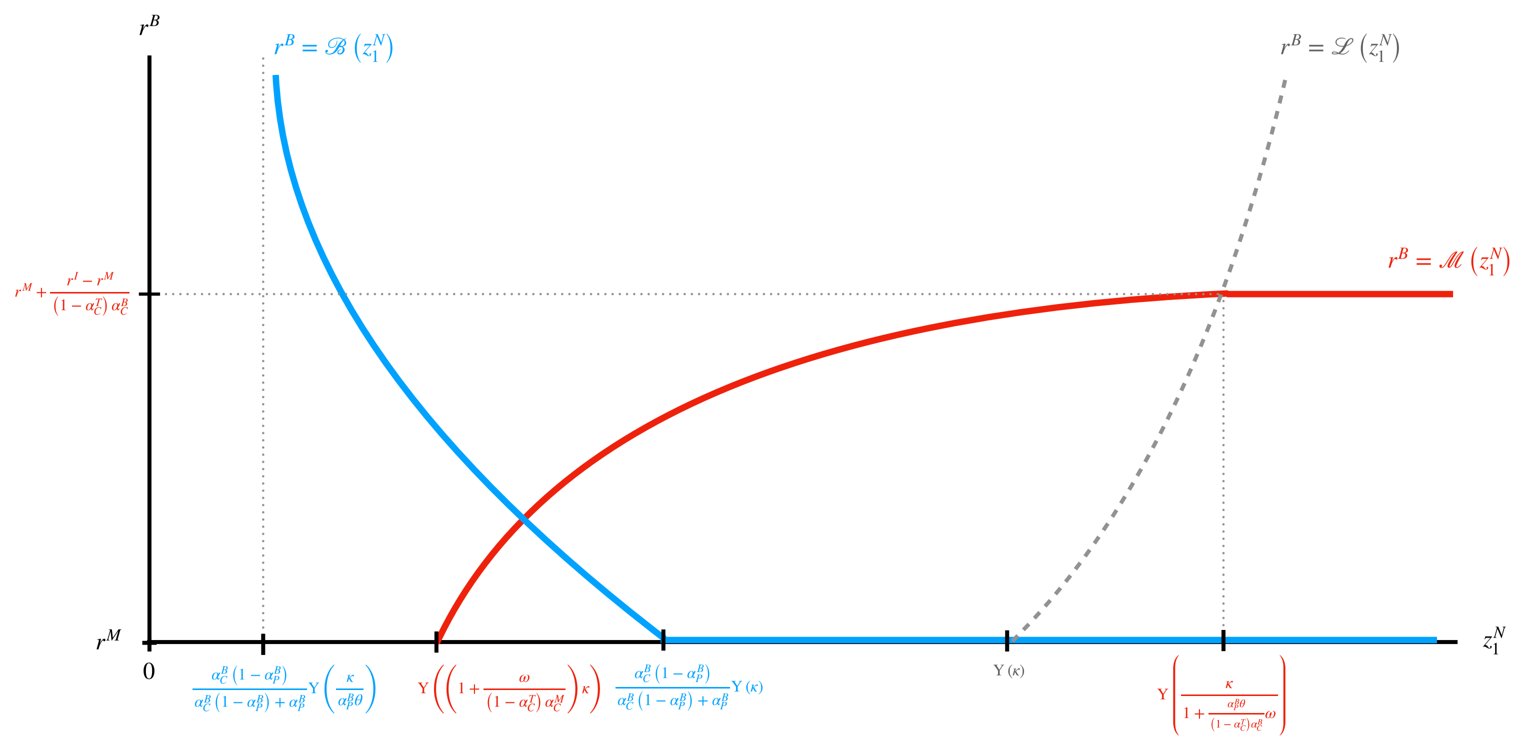

Figure 1: Characterization of the stationary monetary equilibrium for an economy with $\bar{\omega} < \omega$.

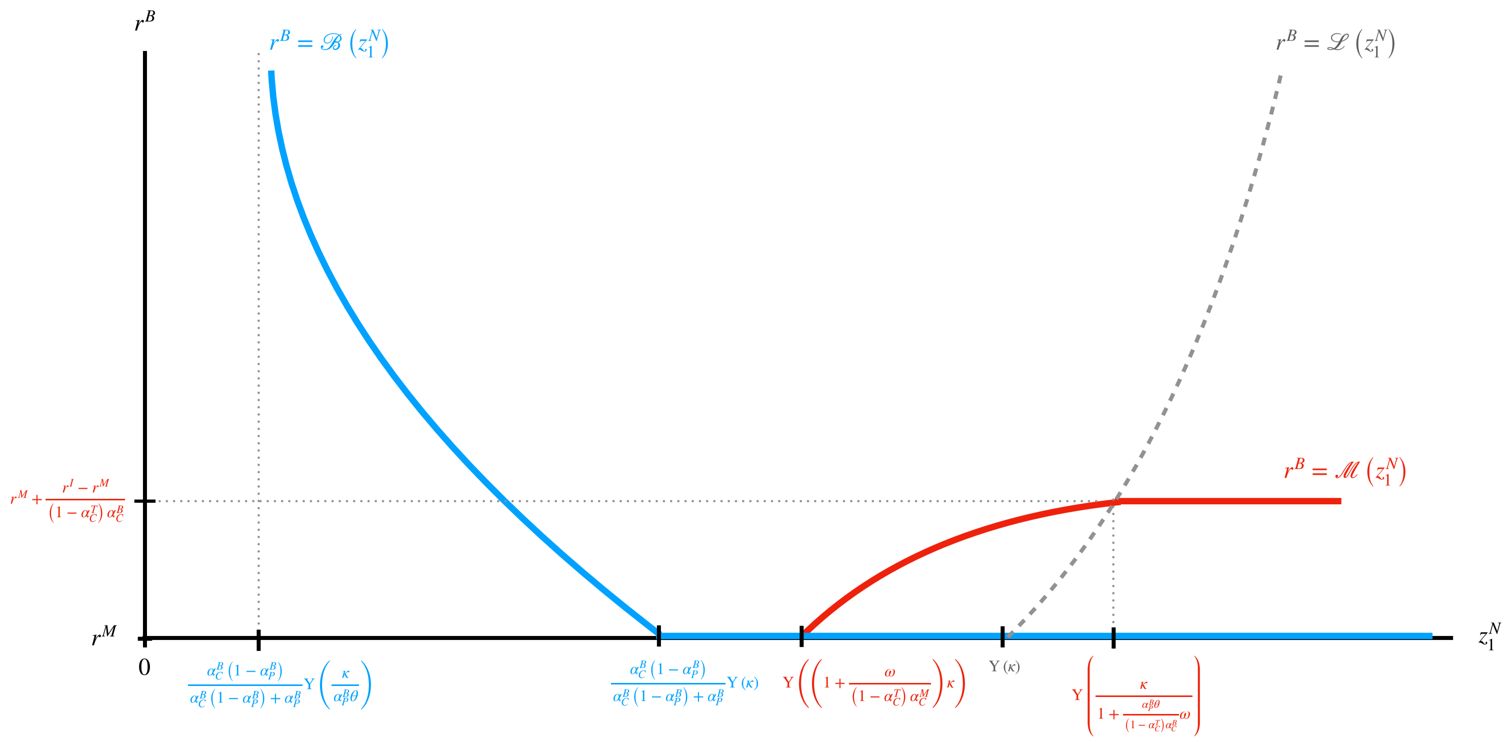

Figure 2: Characterization of the stationary monetary equilibrium for an economy with $\omega \leq \bar{\omega}$.

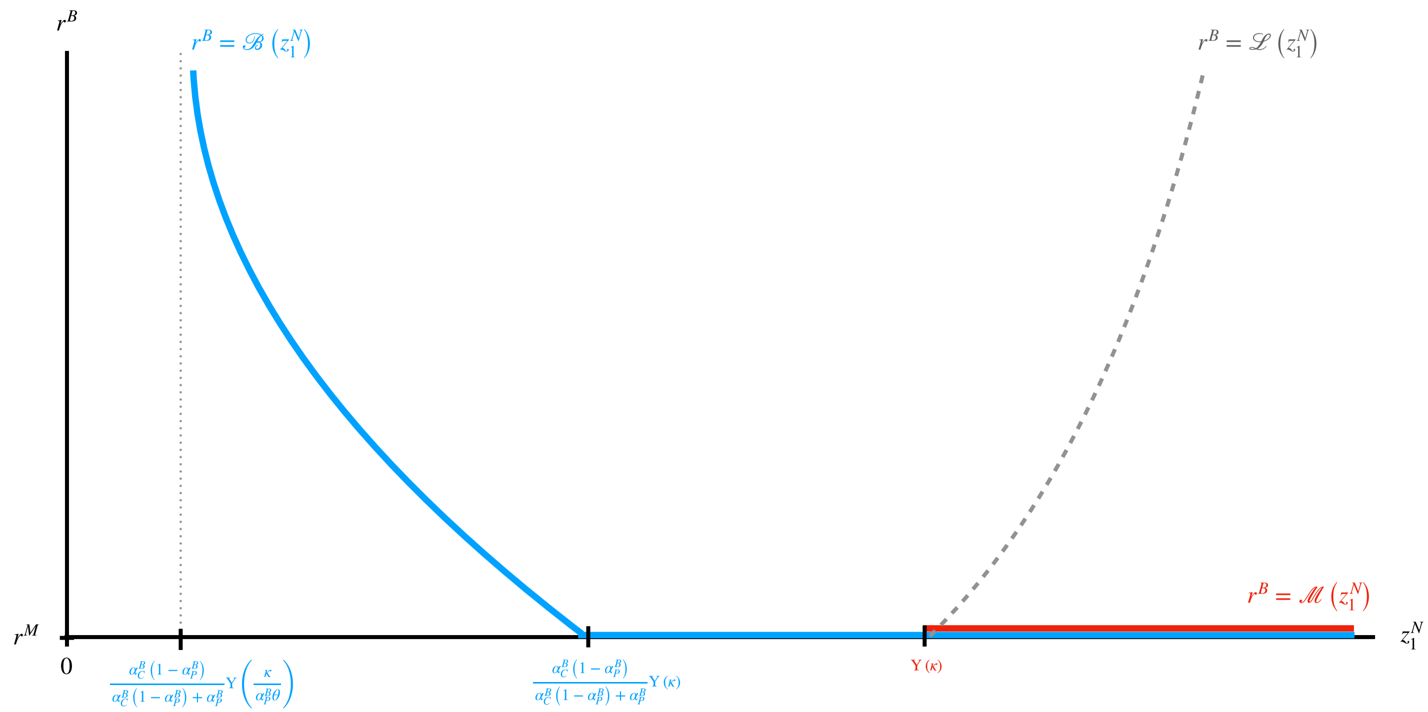

Figure 3: Characterization of the stationary monetary equilibrium for an economy with $\omega = 0$.

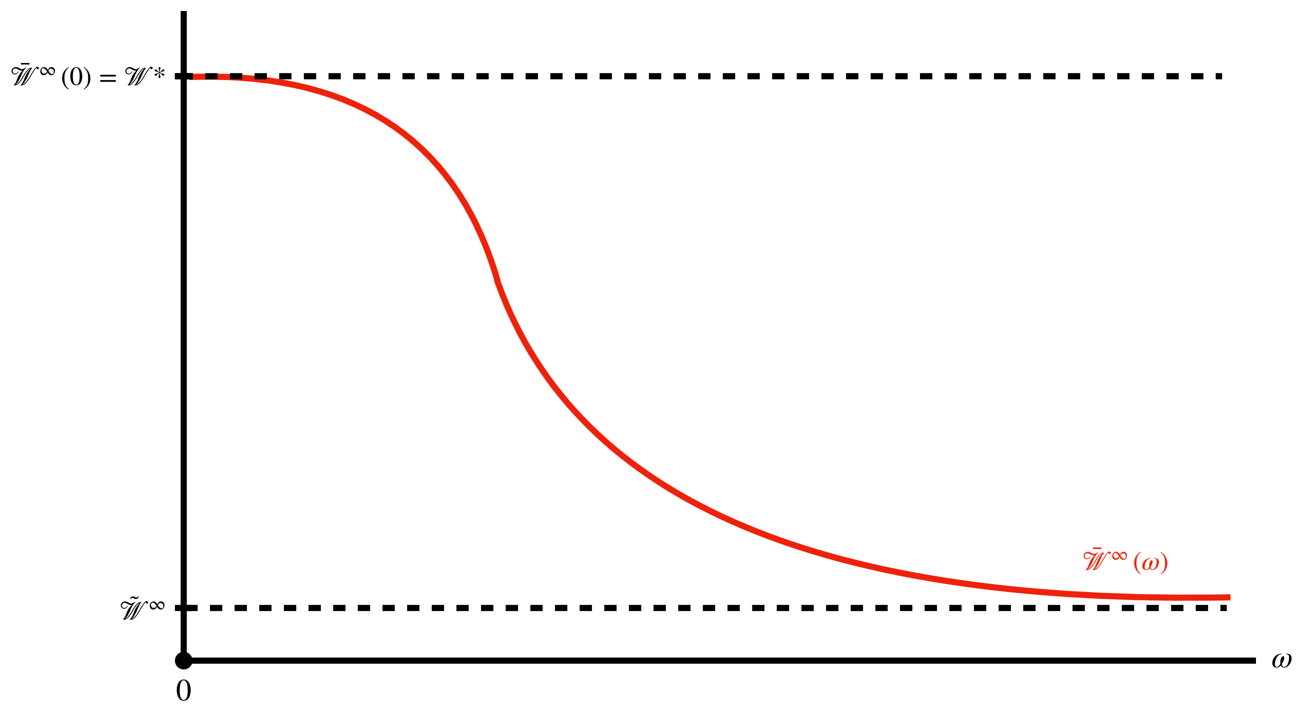

Figure 4: Welfare as a function of the opportunity cost of money in the cashless limit of the stationary monetary equilibrium.