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PRECAUTIONARY SAVING AND THE MARGINAL PROPENSITY TO CONSUME

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ABSTRACT

The marginal propensity to consume out of wealth is important for evaluating the effects of taxation on consumption, assessing the possibility of multiple equilibria due to aggregate demand spillovers, and explaining observed variations in consumption. It is also a component of the interest elasticity of consumption and the risk aversion of the value function which gives the expected present value of utility as a function of wealth. This paper analyzes the effect of uncertainty on the marginal propensity to consume within the context of the Permanent Income Hypothesis. Given plausible conditions on the utility function, income risk is found to raise the marginal propensity to consume out of wealth in a multiperiod model with many risky securities. The marginal investment portfolio for additions to wealth is also characterized.

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1. Introduction

One of the most important projects in macroeconomics over the last few decades has been the effort to formally incorporate uncertainty into macroeconomic models. Thomas Sargent, in a wide-ranging set of articles, pioneered the use of linear-quadratic techniques in macroeconomic models. The great virtue *and* the great vice of linear-quadratic assumptions is the implication of certainty equivalence—that decision rules depend only on the expected values of the determining variables. When there is reason to believe that the qualitative effect of uncertainty on the decision rule under consideration is minor compared to other issues at hand, certainty equivalence is a virtue. When there is reason to believe that the qualitative effect of uncertainty on the decision rule under consideration is important, certainty equivalence is a vice.

When the decision rule under consideration is the neoclassical consumption function giving consumption as a function of current wealth and the probability distribution of future income, a growing literature suggests that the qualitative effect of uncertainty on consumption decisions is too important to ignore. Olivier Blanchard and Gregory Mankiw (1988) survey some of this literature. Robert Barsky, Mankiw and Stephen Zeldes (1986) argue that the effect of variations in income uncertainty on the overall level of the consumption function is likely to be important. Zeldes (1989) computes consumption decision rules for particular parameterizations of uncertainty for the intuitively appealing constant relative risk aversion utility functions, finding that uncertainty has an important effect not only on the level but also on the slope of the decision rule giving consumption as a function of current wealth.

The theoretical basis for the effect of income uncertainty on the overall level of the consumption is now fairly well understood. Hayne Leland (1968) shows in a two-period model with no risky assets that a positive third derivative of an additively time-separable utility function leads to precautionary saving—a negative effective of labor income uncertainty on the overall level of consumption. Sandmo (1970) shows that capital income uncertainty has both a precautionary saving effect of the type Leland (1968) found and an effect similar to the effect of reducing the rate of return. David Sibley (1975), Bruce Miller (1976) and Kimball and Mankiw (1989) extend the theoretical analysis of the effect of income uncertainty on the overall level of consumption to multiperiod models.

The theoretical basis for the effect of income uncertainty on the *slope* of the decision rule giving consumption as a function of current wealth is less well understood. Kimball (1990) shows in a two-period model with no risky assets that "decreasing absolute prudence" of an additively time-separable utility function leads to a positive effect of labor income uncertainty on the slope of the neoclassical consumption function, where absolute prudence $\eta(c) = \frac{-u'''(c)}{u''(c)}$ for the second-period utility function $u(c)$ measures the strength of the precautionary saving motive just as absolute risk aversion $a(c) = \frac{-u''(c)}{u'(c)}$ measures the strength of risk aversion. (Sections 2 and 3 review these results.) Intuitively, the reason income uncertainty might raise the slope of the neoclassical consumption function is that an extra dollar in an uncertain world not only loosens an agent's budget constraint, but also makes the agent feel less need for prudence in the allocation of his or her resources, thus encouraging extra consumption.

The aim of this paper is to extend the analysis of the effect of income risk on the slope of the neoclassical consumption function or "marginal propensity to consume out of wealth" to a situation with many risky securities and many periods. The result that the effect of income risk on the marginal propensity to consume out of wealth depends only on whether absolute prudence is increasing or decreasing must be qualified in the presence of either risky securities (section 4), or more than two periods (section 5A). Still, with the help of the result that incomplete markets tend to raise the marginal propensity to consume out of wealth (section 5C), it can be shown that if absolute prudence decreases fast enough compared to changes in risk tolerance then income risk will raise the marginal propensity to consume out of wealth even in the presence of both many risky securities and many periods (section 5D,E). Formally, $-\eta'(c) \geq 2\left(\frac{a'(c)}{a(c)}\right)^2$ (which is satisfied in the case of constant relative risk aversion greater than or equal to one) guarantees that income risk will raise the marginal propensity to consume out of wealth at a given level of initial consumption regardless of the number of periods or risky securities. I argue that decreasing absolute prudence ($-\eta'(c) \geq 0$) and even "strongly decreasing absolute prudence" ($-\eta'(c) \geq 2\left(\frac{a'(c)}{a(c)}\right)^2$) are reasonable conditions to impose on the utility function (section 6). The continuing importance of the simple condition of decreasing absolute prudence ($-\eta'(c) \geq 0$) in the multiperiod case with many risky securities is made clear by the stochastic differential equation for the marginal propensity to consume out of wealth which I derive in order to calibrate the magnitude of the effect of uncertainty on the marginal propensity to consume out of wealth (section 7).

The slope of the neoclassical consumption function or "marginal propensity to consume out of wealth" is important for a number of reasons.

First is the fact that empirical tests of the Permanent Income Hypothesis which seem to find "excess sensitivity" of consumption to current income have been used to call the Permanent Income Hypothesis into question.[1] If income uncertainty tends to raise the marginal propensity to consume out of current resources even for Permanent Income consumers, it becomes less clear that evidence of "excess sensitivity" of consumption to current income contradicts the Permanent Income Hypothesis. Of course, theoretical results about the marginal propensity to consume out of wealth are not directly comparable to empirical claims of "excess sensitivity" of consumption to current income, since a change in current income typically signals a complex change in the probability distribution of future income that is not equivalent to a change in current wealth of the same expected present value. But the Permanent Income Hypothesis' prediction of the response of consumption to a typical income shock cannot be determined without knowing the marginal propensity to consume out of current wealth.

Second is the practical issue of how much the consumption of permanent-income consumers will respond to changes in fiscal policy, whether these are changes in the present value of government spending and taxes, or changes in the timing of taxes in the absence of Ricardian Equivalence. The marginal propensity to

[1] For example, Hall and Mishkin (1982), after taking into account the role of current income in predicting future income but not the effect of interest rate variation, still find excess sensitivity of consumption to current income. They read their estimates as suggesting that 20% of consumption responds to income as if governed by the simple Keynesian consumption function, while the remaining 80% follows the predictions of the life-cycle hypothesis.

consume is only one ingredient in the general equilibrium response of the economy to such changes, but it is the single most important ingredient. Even after considering all general equilibrium effects, a high marginal propensity to consume out of wealth should magnify the effects on aggregate consumption of any change in fiscal policy that has a significant effect on the amount of wealth or close substitutes for current wealth in the hands of consumers.

The third reason to be interested in the marginal propensity to consume out of wealth is that it figures into the interest elasticity of consumption (as a factor in the wealth effect term) and the marginal propensity to consume out of stochastic future income.[2]

Finally, the marginal propensity to consume out of wealth determines the derived risk aversion of the value function in conjunction with the underlying risk aversion of the period utility function. This can be seen as follows. If $u(c)$ is the period utility function and $J(w, \varphi)$ is the value function giving the expected present value of utility as a function of current wealth and a vector of other state variables φ (including any state variables governing labor income) the envelope theorem says that the marginal utility of wealth must be equal to the marginal utility of consumption:

$$J_w(w_t, \varphi_t) = u'(c_t(w_t, \varphi_t)). \tag{1}$$

Taking the derivative of both sides with respect to current wealth reveals that

$$J_{ww}(w_t, \varphi_t) = u''(c_t(w_t, \varphi_t))\frac{\partial c_t}{\partial w_t}. \tag{2}$$

Dividing (2) by (1) and changing signs then yields the following relationship:

$$\frac{-J_{ww}(w_t, \varphi_t)}{J_w(w_t, \varphi_t)} = \frac{-u''(c_t(w_t, \varphi_t))}{u'(c_t(w_t, \varphi_t))}\frac{\partial c_t}{\partial w_t}. \tag{3}$$

In words, the absolute risk aversion of the value function—which is the direct determinant of an agent's optimal level of investment in risky assets that have returns independent of the vector φ of state variables other than current wealth—is the absolute risk aversion of the underlying period utility function, multiplied by the marginal propensity to consume out of wealth. Thus, given the absolute risk aversion of the underlying period utility function (which depends only on current consumption), the greater an agent's marginal propensity to consume out of wealth, the greater the agent's apparent risk aversion will be.[3]

2 Research in progress.

3 In terms of the consumption capital asset pricing model, which states that the mean excess return of any security must equal the product of the underlying risk aversion and the covariance between consumption and that security's return, an increase in the marginal propensity to consume raises the covariance between a security's return and consumption for any given level of investment, implying *ceteris paribus* that this equation will be satisfied at a lower level of investment in that security. See section 6 for more on the connection between the effect of income risk on the marginal propensity to consume and the effect of one risk on the desirability of another independent risk.

2. Prudence and the Precautionary Premium

In this section, I introduce the concepts of prudence and the precautionary premium. These concepts parallel risk aversion and the risk premium, but risk aversion and the risk premium are defined relative to the level of utility, while prudence and the precautionary premium are defined relative to marginal utility. These new concepts, which are controlled by the curvature of marginal utility, govern the comparative statics of the marginal propensity to consume.

There is more than a formal relationship between prudence and risk aversion. Decreasing absolute risk aversion guarantees that prudence is uniformly greater than risk aversion, and that the precautionary premium is greater than the risk premium. This result allows a simple answer to the query of whether there is any *a priori* reason to believe that precautionary saving effects are important. As long as absolute risk aversion is decreasing, precautionary saving effects should be important whenever the effects of risk aversion are important. Though to the discerning reader, this fact is clear from Drèze and Modigliani (1972), a simple demonstration that prudence exceeds risk aversion (and therefore that the precautionary premium exceeds the risk premium) will establish some familiarity with the concepts of prudence and the precautionary premium.

Following Kimball (1990a), consider a two-period model of the consumption/savings decision with additively separable utility $u(c_1) + \mathrm{E}\, v(c_2)$, risky labor income with inelastic labor supply,[4] free borrowing and lending at a fixed risk-free rate, but (for now) no other risky security or contingent claims. Using the fixed risk-free rate to measure everything in present value terms, the consumer's decision problem is

$$\max_c \; u(c) + \mathrm{E}\, v(w_0 - c + y), \tag{4}$$

where $c = c_1$ is the first period consumption, E is an expectation conditional on first-period information, w_0 is the consumer's initial assets plus his first-period labor income (which is received before the first-period consumption decision) and y is second-period labor income. It will be convenient to write $y = \bar{y} + \tilde{y}$, dividing second-period labor income y into its expectation $\bar{y}$ and a zero-mean risky component $\tilde{y}$, and to define $w = w_0 + \bar{y}$, adding mean second-period income to initial assets to get the total of non-human wealth and expected human wealth (which is what determines consumption in the case of certainty equivalence). I will also define for later use $x = w - c$, the amount of "saving" out of non-human and expected 6human wealth.

[4] It is easy to extend the model to allow for a nontrivial labor supply decision in at least one important special case. If the period utility function is homothetic in leisure and goods, there is exact aggregation of leisure and goods into a single composite commodity, which takes the role of consumption in all of the equations below. The endowment shocks can then be interpret as wage shocks to full income which may be correlated with shocks to the exact price index, which is a combination of the price of goods and the price of leisure given by the wage. A real bond is then one that pays an amount denominated in this exact price index. Alternatively, if the period utility function is additively separable in goods and leisure, the utility for goods and the utility for leisure interact in essentially the same way as the utility for consumption at different dates, so the multiperiod results extend to this case with little modification. In each proposition, if the conditions on the period utility functions hold for both the utility for goods and the utility for leisure in each period, holding consumption constant is reinterpreted as holding both consumption and leisure constant, and holding the term structure of real interest rates constant is reinterpreted as holding the current wage as well as the term structures of both wage denominated and goods price denominated real interest rates constant, both the marginal propensity to consume goods and the marginal propensity to consume leisure will be affected in the predicted way.

The consumer's decision problem can then be rewritten:

$$\max_c u(c) + \mathrm{E}\, v(w - c + \tilde{y}). \tag{5}$$

As long as the constraint that the consumer cannot borrow against more than the minimum value of his or her human wealth is never binding, the first-order condition for (5) is

$$u'(c) = \mathrm{E}\, v'(w - c + \tilde{y}). \tag{6}$$

I will always assume that the agent is at such an interior solution, either because of the Inada condition $v'(0) = \infty$ or because of the particular parameter values under consideration.

It is clear from this first-order condition (6) that the risk $\tilde{y}$ in second-period income will affect consumption in the first period only insofar as it affects second-period expected marginal utility. For example, if the risk $\tilde{y}$ were compensated by a change in mean human plus nonhuman wealth from w to $w + \psi^*$ in such a way that

$$v'(w - c) = \mathrm{E}\, v'(w - c + \tilde{y} + \psi^*), \tag{7}$$

then the optimal first-period consumption would be the same with wealth w and no uncertainty as with wealth $w + \psi^*$ and income risk $\tilde{y}$. In such a case, it seems reasonable to call the quantity ψ^* a "precautionary premium,"[5] since it is the quantity that compensates for the effect of uncertainty on consumption, much as a risk premium compensates for the effect of risk on expected utility. The similarity between the two concepts can be seen by comparing (7) to the definition of a risk premium[6] π^*:

$$v(w - c) = \mathrm{E}\, v(w - c + \pi^* + \tilde{y}). \tag{8}$$

Comparing (7) to (8), the only difference between the definition of a precautionary premium and the definition of a risk premium is the substitution of marginal utility v' for total utility v. The analogy is even more precise if one substitutes $-v'$ for v. If v is strictly risk averse, then $(-v')' = -v'' > 0$, so that $-v'$ is a monotonically increasing function to which standard theorems about monotonically increasing utility functions can be applied. For example, Rothschild and Stiglitz' (1970) theorem that a mean-preserving spread in consumption will always reduce the expected utility of a concave utility function can be applied to $-v'$ to yield the result that if v has a positive third derivative, making $-v'$ concave ($(-v')'' < 0$), then a mean-preserving spread in second-period consumption will always increase the expected marginal utility of second-period consumption (because it reduces $\mathrm{E}[-v']$). Less obviously, Pratt's (1964) theorems establishing the usefulness of "absolute risk aversion" $-\frac{v''}{v'}$ as a measure of the curvature of v can be applied to the function

[5] More precisely, ψ^* is a "compensating precautionary premium," which is distinguished in Kimball (1990a) from the "equivalent precautionary premium" ψ satisfying the relation $v'(w - c - \psi) = \mathrm{E}\, v'(w - c + \tilde{y})$. Since the difference between these two concepts is not important for this paper, I will focus exclusively on the compensating precautionary premium.

[6] This is the definition of a "compensating risk premium," as distinct from the "equivalent risk premium" π defined by $v(w - c - \pi) = \mathrm{E}\, v(w - c + \tilde{y})$.

$-v'$. The corresponding measure of the curvature of $-v'$ is $-\frac{(-v')''}{(-v')'} = -\frac{v'''}{v''}$. Pratt's theorem that if there are two utility functions v_1 and v_2 for which $\frac{-v_2''}{v_2'}$ is uniformly greater than $\frac{-v_1''}{v_1'}$, then the risk premium implied by v_2 will be greater than the risk premium implied by v_1 yields the result that if $-\frac{v_2'''}{v_2''}$ is uniformly greater than $-\frac{v_1'''}{v_1''}$, then the precautionary premium implied by v_2 will be greater than the precautionary premium implied by v_1. To see this one need only substitute $-v_1'$ and $-v_2'$ into Pratt's theorem and use the analogy between precautionary premia and risk premia set out above.

In Kimball (1990a), the quantity $-\frac{v'''(x)}{v''(x)}$ is given the name "absolute prudence" by analogy to "absolute risk aversion." Similarly $-\frac{xv'''(x)}{v''(x)}$ is termed "relative prudence" by analogy to "relative risk aversion." The word "prudence" seems an appropriate word for a measure of the strength of the precautionary saving motive, which induces individuals to prepare and forearm themselves against uncertainty they cannot avoid—in contrast to "risk aversion," which is how much agents dislike uncertainty and want to avoid it. The difference between prudence and risk aversion is perhaps clearest in the the case of quadratic utility, which displays risk aversion, but no prudence. It is well known that despite being concave and therefore risk averse, quadratic utility yields "certainty equivalence"—that is, to no effect of unavoidable uncertainty on optimal decision rules. (In fact, certainty equivalence is the main attraction of quadratic utility for many applications).

For any particular parameterization of the utility function, absolute prudence is easy to calculate.[7] One quickly finds that for utility functions conforming to the standard requirement of decreasing absolute risk aversion,[8] prudence is always greater than risk aversion. Analytically, this must be so, since if the absolute risk aversion of v is positive, but decreasing, then

$$\frac{-v'''(x)}{v''(x)} - \left(\frac{-v''(x)}{v(x)}\right) = -\frac{d}{dx}\ln\left(\frac{-v''(x)}{v'(x)}\right) > 0 \qquad \forall x. \tag{9}$$

An immediate consequence of prudence being greater than risk aversion is that the precautionary premium ψ^* for a given risk will exceed the risk premium π^* for that risk.[9]

3. The Marginal Propensity to Consume

With the concepts of prudence and the precautionary premium in hand, I am now in a position to retrace the two-period results of Kimball (1990a) that one might hope to extend to the multiperiod case.

The fact that a uniformly more prudent utility function will always have uniformly greater precautionary premia has the important implication that the precautionary premium will be decreasing in wealth if absolute prudence is. Decreasing absolute prudence implies that $v(x)$ will be globally more prudent than $v(x+\epsilon)$ for any positive ϵ, and therefore that the precautionary premium will be smaller for a higher level of initial wealth. The fact that x is not initial wealth but rather the expected amount of wealth available for the

[7] Table 1 offers several examples.

[8] Decreasing absolute risk aversion is a reasonable property to require since, among other things, it is implied by a positive wealth elasticity of risky investment.

[9] To see this, one need only apply Pratt's above-mentioned theorem about global risk aversion comparisons to the functions $-v'$ and v. Decreasing absolute risk aversion implies (9), which in turn implies that $-v'$ is a concave function of v, and therefore that marginal utility is more sensitive to risk than utility is.

second period does not complicate matters much, since as long as second-period consumption is a normal good (which additively separable utility guarantees) $x = w_0 + \bar{y} - c$ will be monotonically increasing in w_0.

Since the precautionary premium is the rightward shift in the consumption function due to income risk, if it is decreasing in wealth, the marginal propensity to consume will be higher at any given level of consumption.[10] This is illustrated by figures 1–3. Figures 1 and 2 show that if the consumption function is initially linear or concave, this increase in the marginal propensity to consume at any given level of consumption implies an increase in the marginal propensity to consume at a given level of wealth as well. Figure 3 shows, however, that if the consumption function is initially convex, this increase in the marginal propensity to consume at any given level of consumption can coincide with a *decrease* in the marginal propensity to consume at particular levels of wealth, since the general rightward shift of the curve tends to carry portions of the curve with a small slope rightward to higher levels of wealth.

The principle that marginal propensities to consume are most easily compared at points with equal levels of initial consumption carries over to the multiperiod case.[11] For some applications it would be more convenient if one could make an unambiguous statement about the effect of income risk on the slope of the consumption function at a given level of wealth, but for others, it is better to have a proposition about the effect of income risk on the slope of the consumption function at a given level of initial consumption. Since income risk changes the position (and shape) of the consumption function, one should not expect to get both types of results. One advantage of being able to compare points with equal levels of initial consumption is that it allows one to sidestep the difficulty that would arise in trying to quantify human wealth in the face of incomplete markets in order to make a fair comparison of the marginal propensity to consume at "points with equal levels of wealth."

It will be argued in section 6 that absolute prudence is, indeed, likely to be decreasing in wealth. However, it is easy to state what will happen if it is not. If absolute prudence is *increasing* in wealth, the precautionary premium will also be increasing in wealth, and income risk will cause the marginal propensity to consume to fall at any given level of consumption. If absolute prudence is *constant*, the precautionary premium will be constant, and uncertainty will cause a parallel shift in the consumption function to the right, leaving the marginal propensity to consume at any given level of consumption unchanged.

Table 1 presents examples of utility functions having decreasing, increasing, and constant absolute prudence, with the absolute risk aversion also shown for comparison. The prime examples of utility function having decreasing absolute prudence are those with constant relative risk aversion. However, the class of utility functions having decreasing absolute prudence is much broader. Most commonly employed utility functions that have decreasing absolute risk aversion also have decreasing absolute prudence, though it is not difficult to construct examples of utility functions such as $x - (4+x)e^{-x}$ which have decreasing absolute

[10] A more formal demonstration can be found in Kimball (1990a)

[11] For one result (about the effect of incomplete markets on the marginal propensity to consume) the entire stochastic pattern of consumption must be the same in order to make a clean comparison, while for another result (about the effect of income risk on the marginal propensity to consume) only the current level of consumption needs to be the same.

risk aversion, but increasing absolute prudence over some range.[12] As for utility functions that have both increasing absolute risk aversion and increasing absolute prudence, there are examples in the hyperbolic absolute risk aversion (HARA) class as well as the normal integral given in Table 1. Of utility functions having constant absolute prudence, there is only a limited group. It can be shown that aside from quadratic utility, for which absolute prudence is constant at zero, the only type of utility functions that have constant absolute prudence are linear combinations of exponential and linear utility—that is, utility functions of the form $\alpha x - \beta e^{-\gamma x}$[13]

The classification of utility functions into those with decreasing, increasing or constant absolute prudence casts light on several previous results reported by other authors. First, as mentioned above, Zeldes (1986) reports computer simulations using constant relative risk aversion utility functions in which income uncertainty increases the marginal propensity to consume. Here, his result can be seen as a special case of income uncertainty increasing the marginal propensity to consume for utility functions with decreasing absolute prudence. Merton (1971), Cantor (1985) and Kimball and Mankiw (1989) find that the assumption of exponential utility is especially conducive to obtaining explicit solutions for optimal consumption under additive income uncertainty. The fact that with absolute prudence constant, income risk does not affect the marginal propensity to consume (bringing about a parallel shift of the consumption function) is no doubt part of the reason why exponential utility yields particularly simple solutions to such problems.[14] The very phenomenon we are studying—the effects of income risk on the marginal propensity to consume—is likely to make it difficult to derive explicit solutions for the optimal consumption rules under additive income uncertainty for utility functions that do not have constant absolute prudence.

4. Investment in a Risky Asset and the Marginal Propensity to Consume in the Two-Period Case

So far, I have discussed the effect of labor income uncertainty on the marginal propensity to consume in a model where agents could only save through a riskless asset. The opportunity to invest some portion of savings in a risky security complicates the story. First, given the opportunity to invest in the risky security, an agent will in general choose to alter the amount of risk he or she faces. For example, if the returns for the risky security are correlated with second-period income, an agent may use either a long or short position in the risky security to hedge against income risk. On the other hand, if the returns for the risky security are independent of second-period income, or if the correlation with second-period income is small in relation to the mean return of the security, the agent may choose to face more total risk than he or she would in the absence of the risky security. These changes in the overall level of risk will affect the marginal propensity

[12] This example is from Elmendorf and Kimball (1988), Appendix B.

[13] To insure that a function of this form is monotonically increasing and concave, α, β and γ should be nonnegative.

[14] Of course, the easiest case to solve explicitly is that of quadratic utility, in which zero prudence implies that neither the level of consumption nor the marginal propensity to consume are affected by income risk.

to consume in the same way that any change in the overall level of risk does.[15] Second, the optimal level of risky investment will depend on an agent's wealth, and this endogenous adjustment in risky investment can affect the marginal propensity to consume.

To understand the effect of endogenous adjustment of risky security holding on the marginal propensity to consume, consider first the envelope theorem. The envelope theorem says that *on the margin* the endogenous adjustment of risky security holding has no effect on expected *utility*. But expected *marginal utility*, which is what matters for the consumption/savings decision, *will* in general be affected by this endogenous adjustment. If absolute risk aversion is decreasing, marginal utility is more sensitive to risk than is utility; therefore, an increase in risky investment that leaves expected utility unchanged on the margin will tend to increase expected marginal utility in the second-period and thereby increase precautionary saving and reduce first-period consumption. Since decreasing absolute risk aversion also ensures that the optimal level of risky investment will be increasing in wealth, the endogenous adjustment of risky security holding will tend to reduce the marginal propensity to consume.[16]

In some cases, this effect can even outweigh the direct effect of risk on the marginal propensity to consume. For example, Merton (1971) works out an explicit solution for optimal consumption when an agent has constant relative risk aversion and all of the uncertainty he or she faces is due to holdings of the risky security. (Merton's solutions are for a continuous time problem, but the solutions for a two-period problem with lognormal return distributions are very similar.) If relative risk aversion is less than one, then the marginal propensity to consume (which in this case is equal to the average propensity to consume) will decline with risk because of the endogenous adjustment of risky security holding, while if relative risk aversion is greater than one, the marginal propensity to consume rises with risk despite the endogenous adjustment of risky security holding.

5. The Marginal Propensity to Consume and the Marginal Portfolio in the Multiperiod Case

The two-period model presented so far highlights some of the reasons income risk might affect the marginal propensity to consume. But these results would be of little relevance to real economies if they did not extend to the multiperiod case. Two important results can be established for the multiperiod case which mirror results for the two-period case. First, adding degrees of freedom for endogenous portfolio adjustment reduces the marginal propensity to consume in the multiperiod case as well. Second, strongly decreasing

[15] Kimball (1990a) presents a discussion of theorems about the effect of incremental additions to risk on the marginal propensity to consume. The main result is that if absolute prudence is constant or decreasing and incremental additions to risk are independent of preexisting risk, then the effects of the additional risk are exactly the same as the effects of risk when starting from a no-risk situation.

[16] When absolute risk aversion is increasing, a small increase in risky investment, beginning at the optimum, tends to *reduce* expected second-period marginal utility since the marginal utility function is then *less* sensitive to risk than the utility function. Thus, a small increase in risky investment will tend to reduce saving and raise first-period consumption. However, increasing absolute risk aversion implies that risky investment will *decrease* with wealth, so that the endogenous adjustment of risky security holding tends to reduce the marginal propensity to consume in this case as well. In the case of constant absolute risk aversion, the effect of endogenous adjustment of risky investment on the marginal propensity to consume is zero, both because a change in the level of risky investment would have no effect on the marginal propensity to consume, and because the optimal quantity of risky investment is invariant to the level of wealth.

absolute prudence—that is, the property $-\eta'(c) \geq 2\left(\frac{a'(c)}{a(c)}\right)^2$—which is a generalization of constant relative risk aversion with relative risk aversion greater than 1—ensures that income risk raises the marginal propensity to consume at a given level of current consumption regardless of the number of periods and the range of different securities available to an agent. Other results do not carry over as well to the multiperiod case. In particular, the result that in the absence of risky assets decreasing absolute prudence alone is sufficient to guarantee that income risk raises the marginal propensity to consume out of wealth at a given level of current consumption does not carry over to the multiperiod case.

I show by example in subsection A that decreasing absolute prudence alone is not enough to guarantee that income risk raises the marginal propensity in the multiperiod case. In subsection B, I derive recursive equations for the marginal propensity to consume and the marginal portfolio which are used in subsection C to show that incomplete markets tend to raise the marginal propensity to consume. In subsection D, I derive a formula for the marginal propensity to consume in the presence of complete markets which is used in subsection E to show that strongly decreasing absolute prudence is enough to guarantee that income risk raises the marginal propensity to consume even when markets are complete, and *a fortiori* that income risk raises the marginal propensity to consume when markets are incomplete.

A. The Non-Inheritance of Decreasing Absolute Prudence by the Value Function

There are two ways of trying to examine the effects of uncertainty on the marginal propensity to consume in the multiperiod case. One is to look at the properties of the value function (the function giving the expected present value of utility as a function of current wealth) that are induced by various properties of the underlying utility function. Unfortunately, while decreasing absolute prudence is preserved under mixture (which is important when taking expectations over uninsured background risks) but not under intertemporal aggregation, as the following example indicates.[17] Consider a three-period model with second-period utility function $u_2(c) = -2e^{-c/2}$ and third-period utility function $u_3(c) = \frac{c}{2} - e^{-c}$ and a real interest rate of zero. Both u_2 and u_3 have nonincreasing absolute prudence, but u_3 violates the condition $-\eta'(c) \geq 2\left(\frac{a'(c)}{a(c)}\right)^2$. If there is no uncertainty between the second and third periods, marginal utility in the second and third periods will be equated so that the second period value function is given implicitly by

$$J_w^{-1}(x) = u_2'^{-1}(x) + u_3'^{-1}(x), \tag{10}$$

where x represents the common value of marginal utility and the inverse marginal utility functions $u_2'^{-1}(x)$, $u_3'^{-1}(x)$, and $J_w^{-1}(x)$ indicate respectively the second-period, third-period and combined second- and third-period expenditures corresponding to a given level of marginal utility. Writing $H(x) = J_w^{-1}(x) = -2ln(x) - ln(x - \frac{1}{2})$, it is easy to verify that $J(w_2)$ has nonincreasing absolute prudence only if

$$H'''(x)H'(x) \leq 2(H''(x))^2, \tag{11}$$

[17] Strangely enough, decreasing absolute prudence is preserved under mixture, while *increasing* absolute prudence is preserved under intertemporal aggregation. In between these two cases, it *is* possible to insure constant absolute prudence of the value function for any income process simply by assuming an underlying exponential utility function, but this assumption is very restrictive, and has the unrealistic implication of a zero wealth elasticity of risky investment.

a condition that is violated at $x = 1$. Therefore, $J(w_2)$ has locally increasing absolute prudence fact that $J(w_2)$ has locally increasing absolute prudence at $w_2 = H(1) = ln(2)$, which guarantees that small risks resolved at the beginning of the second period will raise the marginal propensity to consume out of wealth at levels of first-period consumption consistent with w_2 being in the neighborhood of $ln(2)$.

B. Recursive Equations for the Marginal Propensity to Consume and the Marginal Portfolio

Another approach to analyzing the effect of income risk on the marginal propensity to consume in the multiperiod case is to concentrate instead on the first-order condition (or "Euler equation") relating consumption in successive periods. Consider an agent, small enough to take distributions of security returns as exogenous, who wishes to maximize an additively time-separable, von Neumann-Morgenstern utility function of the form $\mathrm{E}_t \sum_{j=0}^{\tau-1} \beta^j u(c_{t+j})$, where $u(\cdot)$ is a (possibly time and state dependent) single-period utility function with $u'(\cdot) > 0$ and $u''(\cdot) < 0$, and τ can be either finite or infinite. If Z^i_{t+1} is the gross real return between t and $t+1$ on any security i, the first-order condition for optimal investment in security i is

$$u'(c_t) = \beta \mathrm{E}_t\, Z^i_{t+1} u'(c_{t+1}). \tag{12}$$

Using the notation $b_t = \frac{\partial c_t}{\partial w_t}$ for the marginal propensity to consume out of wealth (the letter "b" chosen in honor of the marginal propensity to consume in the textbook Keynesian consumption function $C = a + bY$), consider the impact on the agent's choices of an extra increment of wealth, dw_t. Since the first-order condition (12) must hold for any level of wealth, (12) can be totally differentiated to obtain

$$u''(c_t) b_t dw_t = \beta \mathrm{E}_t\, Z^i_{t+1} u''(c_{t+1}) b_{t+1} dw_{t+1}. \tag{13}$$

In words, the change in current marginal utility stemming from the extra consumption due to the increment of wealth dw_t must be equal to the change in discounted expected marginal utility next period stemming from the extra consumption next period that will result from whatever extra wealth dw_{t+1} the consumer ends up with in period $t+1$. To know dw_{t+1}, one needs to know how the consumer would invest an extra dollar of savings. Denoting the gross return on such a marginal portfolio of investments by Z^m_{t+1}, then

$$dw_{t+1} = (1 - b_t) Z^m_{t+1} dw_t, \tag{14}$$

since out of any extra money the agent will save a fraction $1 - b_t$ of his or her extra wealth and invest that fraction in the marginal portfolio. Substituting (12) into (11), and dividing both sides of the equation by $\beta u''(c_t)(1 - b_t) dw_t$, one finds that

$$\frac{b_t}{\beta(1 - b_t)} = \mathrm{E}_t\, \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z^i_{t+1} Z^m_{t+1} \tag{15}$$

for any security i.

Defining Z_{t+1} as the vector of gross returns for a minimal set of securities spanning the space of available security returns between t and $t+1$, I can write

$$Z^m_{t+1} = Z^T_{t+1} \theta_t, \tag{16}$$

where θ_t is the vector of portfolio weights for the marginal portfolio, and Z_{t+1}^T is the transpose of Z_{t+1}. Since, these portfolio weights must sum to one,

$$\iota^T \theta_t = 1, \tag{17}$$

where ι represents a vector of ones. Using this notation, (13) yields the matrix equation

$$\frac{b_t}{\beta(1-b_t)}\iota = \left[\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T\right] \theta_t. \tag{18}$$

The pair of equations (17) and (18) can be solved for b_t and θ_t in terms of the other variables, leading to a recursive expression for the current marginal propensity to consume and marginal portfolio in terms of current consumption and the pattern of next period's consumption, assets returns and marginal propensity to consume. Clearly, $\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T$ is positive semi-definite. If this matrix is positive definite in the strict sense—and therefore invertible—it is easy to show[18] that

$$b_t = \frac{1}{1 + \beta^{-1}\iota^T \left[\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T\right]^{-1} \iota} \tag{19}$$

$$\theta_t = \frac{\left[\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T\right]^{-1} \iota}{\iota^T \left[\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T\right]^{-1} \iota}. \tag{20}$$

More generally, defining the function f from the set of positive semi-definite matrices (and other matrices for which it is well-defined) to the set of real numbers by

$$f(A) = \min_\theta \theta^T A \theta \quad s.t. \;\; \iota^T\theta = 1 \tag{21}$$

and the function g from the set of positive semi-definite matrices (and other matrices for which the minimum in (21) exists) to the set of conformable vectors by

$$g(A) = \arg\min_\theta \theta^T A \theta \quad s.t. \;\; \iota^T\theta = 1, \tag{22}$$

the solution to (17) and (18) is

$$b_t = \frac{\beta f(\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T)}{1 + \beta f(\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T)} \tag{23}$$

and

$$\theta_t = g(\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T) \tag{24}$$

[18] Solve (18) for θ_t, then substitute into (17) to obtain an expression for $\frac{b_t}{\beta(1-b_t)}$ that can be solved for b_t to obtain (19), and substituted into the expression for θ_t to obtain (20).

as a consequence of the following Lemma.

Lemma 1: *Given any positive semi-definite matrix A,*

$$\iota^T\theta = 1 \qquad (i)$$

and

$$A\theta = \lambda\iota \qquad (ii)$$

for some real number λ if and only if θ is a[19] *solution to*

$$\min_{\theta} \theta^T A\theta \qquad (iii)$$
$$s.t. \;\; \iota^T\theta = 1.$$

Furthermore, if (i) and (ii) are satisfied, λ equals the minimum value achieved in (iii).

Proof: See Appendix A.

Equations (23) and (24) result from applying Lemma 1 to (17) and (18) with A equal to $\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T$ and λ equal to $\frac{b_t}{\beta(1-b_t)}$. Lemma 1 has the obvious corollary that whenever the matrix A is invertible, $f(A) = \frac{1}{\iota^T A^{-1}\iota}$ and $g(A) = \frac{A^{-1}\iota}{\iota^T A^{-1}\iota}$, which links (17) and (20) to (23) and (24).

One can interpret what is being minimized in (21) and (22) as the curvature of expected utility in the direction of the marginal portfolio. By investing in the direction of minimum curvature, the agent causes marginal utility to fall as little as possible, which must be the result of trying to transfer resources to the highest marginal utility states first.

In a model with a finite terminal date τ, at which $b_\tau = 1$ in every state of nature, the backward recursion given by (23) uniquely determines the marginal propensity to consume in each state in all previous periods. The marginal portfolio is given as a byproduct of this recursion. When the horizon is infinite, I will avoid technical complications by looking at the limit of finite-horizon solutions for the marginal propensity to consume and the marginal portfolio, whenever this limit exists. Similarly, a continuous-time stochastic differential equation for the marginal propensity to consume and the marginal portfolio can be found by taking the limit as the length of a period goes to zero.

C. Incomplete Markets and the Marginal Propensity to Consume

One of the strongest implications of the recursive relationship represented by (23) is that *reducing the set of available security returns while holding the initial pattern of consumption fixed will always raise the marginal propensity to consume.* In view of (23), what is needed to prove this is to show that $f(\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T$ always rises when securities are deleted.

[19] Note that θ may in general be one of several solutions, making $g(A)$ set-valued. This is not a problem for the matrices I am concerned with in this paper.

Putting the securities to be deleted at the end of the list, partition Z_{t+1} into the vector Z^I_{t+1} of gross returns for those securities that will remain and the vector Z^{II}_{t+1} of gross returns for those securities to be deleted,

$$Z_{t+1} = \begin{bmatrix} Z^I_{t+1} \\ Z^{II}_{t+1} \end{bmatrix}. \tag{25}$$

Partition the matrix $A = \mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z^T_{t+1}$ conformably:

$$\begin{aligned} A &= \begin{bmatrix} A^{I,I} & A^{I,II} \\ A^{II,I} & A^{II,II} \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z^I_{t+1} (Z^I_{t+1})^T & \mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z^I_{t+1} (Z^{II}_{t+1})^T \\ \mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z^{II}_{t+1} (Z^I_{t+1})^T & \mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z^{II}_{t+1} (Z^{II}_{t+1})^T \end{bmatrix}. \end{aligned} \tag{26}$$

Defining $\hat{\theta} = g(A^{I,I})$ and $\hat{\iota}$ as a vector of ones conformable to $A^{I,I}$, then

$$\begin{aligned} f(A) &= \min_\theta \theta^T A \theta \\ &\qquad s.t.\ \ \iota^T \theta = 1 \\ &\leq \min_{\hat{\theta}} [\hat{\theta}^T \quad 0] A \begin{bmatrix} \hat{\theta} \\ 0 \end{bmatrix} \\ &\qquad s.t.\ \ \iota^T \begin{bmatrix} \hat{\theta} \\ 0 \end{bmatrix} = 1 \\ &= \min_{\hat{\theta}} \hat{\theta}^T A^{I,I} \hat{\theta} \\ &\qquad s.t.\ \ \hat{\iota}^T \hat{\theta} \\ &= f(A^{I,I}). \end{aligned} \tag{27}$$

So far, I have established that reducing the set of securities available at time t while holding the pattern of consumption fixed raises the marginal propensity to consume in that period. It can also be shown that such an experiment also raises the marginal propensity to consume in all previous periods, since (23) implies that, other things being equal, a higher marginal propensity to consume in one period leads to a higher marginal propensity to consume in the immediately preceding period, implying in turn that a higher marginal propensity to consume in one period leads to a higher marginal propensity to consume in all previous periods. By the envelope theorem, if $b_{t+1,s}$ is the marginal propensity to consume in state s at time $t+1$, with corresponding notations for other variables, and $\pi_{t+1,s}$ is the probability of state s given the state at time t,[20]

$$\begin{aligned} \frac{\partial}{\partial b_{t+1,s}} f(\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z^T_{t+1}) &= \frac{\partial}{\partial b_{t+1,s}} \min_\theta \theta^T \left[\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z^T_{t+1} \right] \theta \\ &\qquad s.t.\ \ \iota^T \theta = 1 \\ &= \pi_{t+1,s} \frac{u''(c_{t+1,s})}{u''(c_t)} \theta^T_t Z_{t+1,s} Z^T_{t+1,s} \theta_t \\ &= \pi_{t+1,s} \frac{u''(c_{t+1,s})}{u''(c_t)} (Z^m_{t+1,s})^2 \\ &\geq 0. \end{aligned} \tag{28}$$

[20] A continuum of states can be treated as a limit of finite-state situations.

Equations (28) and (23) imply that

$$\frac{\partial b_t}{\partial b_{t+1,s}} \geq 0. \tag{29}$$

The proposition that, for a fixed pattern of consumption, the more badly incomplete markets are, the higher the marginal propensity to consume, makes intuitive sense when seen as a statement that given more flexibility about how one can transfer resources to the future, one will tend to transfer more of an extra dollar to the future; or stated the other way around, given fewer ways to choose from in transferring resources to the future, one is likely to consume more of an extra dollar immediately. Yet this simple proposition has important consequences. In terms of practical concerns, it implies that, in regard to any transmission mechanism working through wealth effects on consumption, more complete markets will tend to stabilize the economy. Analytically, it means that one can use the marginal propensity to consume under the assumption of complete markets—which is relatively easy to calculate—as a lower bound for the marginal propensity to consume in incomplete markets.

D. The Marginal Portfolio and the Marginal Propensity to Consume in the Presence of Perpetually Complete Markets

The marginal propensity to consume in the presence of perpetually complete markets is particularly important since it is a lower bound for the marginal propensity to consume under incomplete markets. Fortunately, one can derive a particularly simple expression for the marginal propensity to consume in this case.

When markets are perpetually complete, the optimal disposition of an extra $1 of wealth is particularly simple. Because markets are complete, consumption today can be traded off directly against consumption at any future node of the event tree. Therefore, for each future date-event pair there is a first-order condition of the form

$$p_{t+j,s} u'(c_t) = \beta^j \pi_{t+j,s} u'(c_{t+j,s}), \tag{30}$$

where $p_{t+j,s}$ is the Arrow-Debreu price of one unit of consumption in state s at time $t+j$ in terms of current consumption and $\pi_{t+j,s}$ is the probability of state s at time $t+j$ given the information available at time t.[21] Since this set of first-order conditions should hold for any level of current wealth, one can differentiate both sides of this equation to see the effect of an infinitesimal change in current wealth:

$$p_{t+j,s} u''(c_t) dc_t = \beta^j \pi_{t+j,s} u''(c_{t+j,s}) dc_{t+j,s}. \tag{31}$$

As long as $p_{t+j,s}$ and $\pi_{t+j,s}$ are nonzero, one can divide (43) by (42) and change signs to find that

$$a(c_t) dc_t = a(c_{t+j,s}) dc_{t+j,s}, \tag{32}$$

where $a(c) = \frac{-u''(c)}{u'(c)}$ is the underlying absolute risk aversion of the period utility function.

[21] This equation says that at an optimum, an agent should be indifferent to reducing current consumption by $p_{t+j,s}$ and getting one more unit of consumption j periods later in state s, which occurs with probability $\pi_{t+j,s}$.

Such a change in the pattern of consumption due to an infinitesimal change in wealth must also satisfy the differential budget constraint

$$\sum_{j=0}^{\tau-t} \sum_{s \in S_{t+j}} p_{t+j,s} dc_{t+j,s} = dw_t, \tag{33}$$

where S_{t+j} is the set of states that are still possible at time $t+j$, in view of information at time t. Using (30) to obtain an expression for $p_{t+j,s}$ and (32) to obtain an expression for $dc_{t+j,s}$, equation (33) becomes

$$\begin{aligned} dw_t &= \sum_{j=0}^{\tau-t} \sum_{s \in S_{t+j}} \beta^j \pi_{t+j,s} \frac{u'(c_{t+j,s})}{u'(c_t)} \frac{a(c_t)}{a(c_{t+j,s})} dc_t \\ &= dc_t \mathrm{E}_t \sum_{j=0}^{\tau-t} \beta^j \frac{a(c_t)}{a(c_{t+j})} \frac{u'(c_{t+j})}{u'(c_t)}. \end{aligned} \tag{34}$$

Rearranging, I can isolate the marginal propensity to consume under perpetually complete markets, which I will label b_t^{**}:

$$b_t^{**} = \frac{1}{\mathrm{E}_t \sum_{j=0}^{\tau-t} \beta^j \frac{a(c_t)}{a(c_{t+j})} \frac{u'(c_{t+j})}{u'(c_t)}}. \tag{35}$$

Thus, in the case of perpetually complete markets one can find a non-recursive expression for the marginal propensity to consume out of wealth.

In words, the marginal propensity to consume in the case of complete markets is equal to the reciprocal of the price of a security which pays a dividend proportional to the risk tolerance of the period utility function, $\frac{u'(c)}{-u''(c)}$, at every current and future date and begins by paying \$1 now. The marginal portfolio in the case of complete markets consists of exactly this security. The name "perpetually ideal marginal portfolio" is particularly apt for this security since the ideal marginal portfolio at any future date will be simply the tail end of the original security.

As long as the perpetually ideal marginal portfolio is available, the marginal propensity to consume will be given by (35) even if markets are not perpetually complete. This is true because increasing consumption at every date and in every state by a small amount proportional to the risk tolerance of the period utility function—and thereby reducing marginal utility by the same proportion at every date and in every state—will allow even the first-order conditions implied by incomplete markets to continue to be satisfied. If the pertpetually ideal marginal portfolio is not available, the marginal propensity to conume must be greater than the b_t^{**} given by (35). Appendix B shows that if the perpetually ideal marginal portfolio does not exist, that will approximate the ideal marginal portfolio.

There are two special cases in which the perpetually ideal marginal portfolio is particularly easy to characterize. In the case of constant relative risk aversion, an agent who receives a little extra wealth would like to buy a security with a dividend that is in direct proportion to the agent's initial consumption in every period. Therefore, the partial equilibrium marginal propensity to consume out of wealth for one of a continuum of identical individuals in a Lucas (1978) "tree" model is just the ratio between the current dividend and the cum-dividend price of the tree, since the tree has exactly the right pattern of dividends.[22] However, a realistic amount of heterogeneity among consumers, would make it unlikely that an agent could construct a security that would pay in proportion to his or her initial pattern of consumption.

In the case of constant absolute risk aversion, an agent who receives a little extra wealth will want to buy a security that pays an equal amount at every date and in every state of nature; that is, an immortal agent will want to purchase a real consol, while a mortal agent will want to purchase a real annuity. In the limiting case of continuous time, the marginal propensity to consume out of wealth will be just the real consol rate or real annuity rate. In discrete time, if R_t is the real consol rate or the appropriate annuity rate at time t, the marginal propensity to consume at time t will be the ratio between the coupon and the price of the consol with its coupon, $\frac{R_t}{1+R_t}$. As long as a real consol or a real annuity exists, the MPC will be equal to the continuous time real consol rate or to $\frac{R_t}{1+R_t}$ even when interest rates follow a very complicated stochastic process.

Though real consols do not presently exist, they may in the future, and it may be possible to construct a reasonable approximation to a real consol even now, using long-term indexed bonds. Therefore, if one believed in absolute risk aversion, the prediction that the marginal propensity to consume should be the real consol rate would provide quite a good guide to the marginal propensity to consume. Even if one does not believe in constant absolute risk aversion, this is a useful result for understanding the behavior of theoretical models that assume constant absolute risk aversion for the sake of tractability.

E. Strongly Decreasing Absolute Prudence and the Marginal Propensity to Consume

In order to give a reasonable answer to the question of whether uncertainty raises or lowers the marginal propensity to consume in a multi-period setting, one must examine a situation of consumption under perfect foresight and a situation of consumption under uncertainty that are in some sense comparable. Just as in the second-period case, a useful comparison can be made between a situation of uncertainty (with and without perpetually complete markets) and a situation of perfect foresight that have the same current level of consumption and the same term-structure of risk-free real interest rates (or, for a mortal agent, the same real annuity term-structure).

Since the j-period gross return on a j-period real discount bond must be equal to $\frac{u'(c_t)}{\beta^j \mathrm{E}_t\, u'(c_{t+j})}$ to satisfy the first-order conditions, keeping current consumption and the term structure of risk-free real interest rates (or of real annuities) unchanged requires only that I leave $\mathrm{E}_t\, u'(c_{t+j})$ unchanged at all current and future dates $t+j$. Thus, the question is: what happens to the marginal propensity to consume if the stochastic pattern of consumption at every future date $t+j$ is replaced by a nonstochastic consumption level $\hat{c}_{t+j}$ satisfying $u'(\hat{c}_{t+j}) = \mathrm{E}_t\, u'(c_{t+j})$?

[22] This will not be true in general for utility functions outside the constant relative risk aversion class. The fact that the marginal propensity to consume is equal to the average propensity to consume in this case is also specific to constant relative risk aversion.

I can give a clear answer to this question if the function defined by

$$\phi(u'(c)) = -\frac{u'(c)^2}{u''(c)} \qquad (36a),$$

or equivalently,

$$\phi(x) = -\frac{x^2}{u''(u'^{-1}(x))} \qquad (36b)$$

is concave, and a clear answer for at least the case of perpetually complete markets if this function is convex. By Jensen's inequality, if a random marginal utility $\tilde{x}_{t+j}$ is replaced by its conditional mean $\mathrm{E}_t\, \tilde{x}_{t+j}$,

$$\mathrm{E}_t \frac{a(c_t)}{a(c_{t+j})} \frac{u'(c_{t+j})}{u'(c_t)} = \mathrm{E}_t \frac{\phi(x_{t+j})}{\phi(x_t)} \leq \frac{\phi(\mathrm{E}_t\, x_{t+j})}{\phi(x_t)} = \frac{a(c_t)}{a(\hat{c}_{t+j})} \frac{u'(\hat{c}_{t+j})}{u'(c_t)} \qquad (37)$$

if $\phi(\cdot)$ is concave, with the inequality reversed if $\phi(\cdot)$ is convex. Therefore, if $\phi(\cdot)$ is concave, the marginal propensity to consume under uncertainty is greater than in the cognate perfect-foresight case, even when securities markets are perpetually complete. If markets are incomplete, the contrast will be even greater (as long as the agent can freely borrow and lend in the certainty case), since incomplete markets will only raise the marginal propensity to consume under uncertainty even further.[23] In symbols,

$$b_t^{perfect\ foresight} \leq b_t^{**} \leq b_t. \qquad (38)$$

If $\phi(\cdot)$ is convex, the marginal propensity under uncertainty, is *less than* the marginal propensity to consume in the cognate perfect-foresight case when markets are complete. However, incomplete markets may raise the marginal propensity to consume under uncertainty above what it is in the cognate perfect-foresight case.

Straightforward calculation shows that $\phi(\cdot)$ as defined by (36) will be concave if and only if the condition

$$-\eta'(c) \geq 2\left(\frac{a'(c)}{a(c)}\right)^2 \qquad (39)$$

is satisfied globally; that is, if prudence is falling fast enough to offset the effects of endogenous adjustment of risky investment in determining the impact of uncertainty on the marginal propensity to consume. In the case the familiar constant relative risk aversion utility functions $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, the condition of "strongly decreasing absolute prudence" in the sense of (39) will be satisfied as long as $\gamma \geq 1$. Thus, in a very important range of cases, I can state that uncertainty unambiguously raises the marginal propensity to consume. Conversely, constant relative risk aversion of the period utility function with $\gamma \leq 1$ would make $\phi(\cdot)$ convex, implying that uncertainty can lower the marginal propensity to consume, as the example from Merton (1971) cited in section 4 indicates.

[23] Since in the experiment of deleting securities, the pattern of consumption was left unchanged, neither the implied term-structure of risk free real interest rates nor the current consumption level is changed by adding and subtracting these zero-net-demand securities, so the cognate perfect-foresight case stays the same.

For logarithmic utility—on the border between these two cases—the marginal propensity to consume under perpetually complete markets implied by (47) is simply

$$b_t^{**} = 1 - \beta \tag{40}$$

for an immortal agent, and

$$b_t^{**} = \frac{1}{\sum_{j=0}^{\infty} \pi_{t+j} \beta^j} \tag{41}$$

for a mortal agent, where π_{t+j} is the (exogenous) probability of living to time $t + j$ given all information known at time t. Neither of these depends in any way on the stochastic pattern of consumption, so that in the case of logarithmic utility and perpetually complete markets, uncertainty has no effect on the marginal propensity to consume. However, when markets are not perpetually complete, and an agent cannot construct a security that has a payout proportional to his initial pattern of consumption, uncertainty will raise the marginal propensity to consume even in the case of logarithmic utility, (40) or (41) providing only a lower bound to the marginal propensity to consume out of wealth.

It is interesting that when markets are incomplete the argument that if absolute proudence is strongly decreasing then uncertainty will raise the marginal propensity to consume, relies on comparisons with the intermediate case of uncertainty with complete markets. The experiment of eliminating uncertainty in the absence of complete markets is broken down into the two steps of (1) adding securities in such a way as to achieve complete markets without altering any state prices and then (2) doing away with uncertainty. The intermediate comparison between the incomplete markets case and the corresponding complete markets case is artificial in the sense of holding the entire pattern of consumption fixed, but the overall comparison between the certainty case and the case of uncertainty with incomplete markets is much more natural, holding only the term-structure of real interest rates and the initial level of consumption fixed. The concession of holding the initial level of consumption fixed is not a new one, since it was necessary even in the two-period case. Indeed, the parallel to the results of the two-period case is so close that Figures 1–3 can be applied to the multiperiod case without other than notational modification

6. Characterizing the Utility Function

In section 3 I claimed that decreasing absolute prudence is a reasonable characteristic to expect of a consumer's utility function. The argument for decreasing absolute prudence will be presented in subsection A. Since decreasing absolute prudence is clearly necessary, but not sufficient for strongly decreasing absolute prudence ($-\eta'(c) \geq 2\left(\frac{a'(c)}{a(c)}\right)^2$), subsection B will address the plausibility of this stricter condition. Subsection C addresses the general issue of whether such characteristics of the utility function are knowable in any practical sense.

A. The Argument for Decreasing Absolute Prudence

In the introduction it was argued that income risk should increase the marginal propensity to consume because in the face of income risk, an extra dollar not only loosens a consumer's budget constraint but also makes the consumer feel less need for prudence, so that he or she will be willing to do less precautionary saving and more current consumption. To the extent that this bit of intuition is convincing, it is an argument for decreasing absolute prudence, since that is the analytical condition corresponding to the behavior described. If this story about the effect of risk on the marginal propensity to consume seems unreasonable because it ignores endogenous changes in risky investment, it can still be a convincing demonstration of decreasing absolute prudence, since, as shown above, it is only the partial effect of uncertainty on the marginal propensity to consume holding the quantity of risky securities fixed that is given an unambiguous sign by decreasing absolute prudence.[24]

As a touchstone to test one's intuition, consider a college professor who has \$10,000 in the bank, and a Rockefeller who has a net worth of \$10,000,000, who have the same preferences except for their differences in initial wealth. If each is forced to face a coin toss at the beginning of the next year, with \$5,000 to be gained or lost depending on the outcome, which one will do more extra saving to be ready for the possibility of losing? If one's answer is that the college professor will do more extra saving, it argues for decreasing absolute prudence.[25]

To consider a rather different thought experiment, think of whether moving to a job with a greater risk of being layed off but also greater upside potential would make buying additional fire insurance on one's house seem more attractive. If the extra job risk would make the additional fire insurance more attractive, it argues for decreasing absolute prudence. Kimball (1990b) shows that *standard risk aversion*—which is a formalization of the notion that an increase in one risk will lead an agent to reduce exposure to other risks (even if they are statistically independent)—implies decreasing absolute prudence.

At first thought, it may seem strange that the effect of income risk on the marginal propensity to consume is tied to the effect of one risk on the aversion to other independent risks, but equation (3) indicates the logic behind this connection: a higher marginal propensity to consume tends to raise the covariance of an asset's return with consumption.[26]

One final approach to judging the plausibility of decreasing absolute prudence is to look at its implications for the wealth elasticity of risk tolerance. Using $a(x)$ to denote absolute risk aversion, and $\eta(x)$ for

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[24] Several propositions demonstrated in Kimball (1990a) indicate other kinds of intuitively reasonable behavior that also imply decreasing absolute prudence. For instance, decreasing absolute prudence is a necessary condition for the amount of a risky security with positive mean that can be absorbed without reducing first-period consumption to go up with the amount of saving. As another example, the intuitively reasonable notion that independent income risks added on top of one another should have a rapidly cumulative effect on consumption and saving, when formalized to a stipulation of super-additive precautionary premia, requires properness (in the sense of Pratt and Zeckhauser (1987)) of the negative of the marginal utility (a property discussed in Appendix C and in Kimball (1990)), which in turn implies nonincreasing absolute prudence.

[25] The distinction between decreasing precautionary premia and decreasing precautionary saving is unlikely to be large enough to worry about in this context.

[26] More fundamentally, decreasing absolute prudence guarantees that background risk will make the expected marginal utility over various outcomes of the background risk fall more rapidly with wealth (since the background risk tends to raise expected marginal utility less the higher the level of starting wealth), thereby tending to increase the negative covariance of another asset's return with marginal utility.

absolute prudence as before,

$$\eta(x) = a(x) - \frac{a'(x)}{a(x)} = a(x) + \frac{\varepsilon(x)}{x}, \tag{42}$$

where $\varepsilon(x) = -\frac{x a'(x)}{a(x)}$ is the wealth elasticity of the risk tolerance, which governs the wealth elasticity of risky investment.[27] $\varepsilon(x)$ will always be positive if absolute risk aversion is decreasing. Differentiating (42) reveals that

$$\eta'(x) = a'(x) - \frac{\varepsilon(x)}{x^2} + \frac{\varepsilon'(x)}{x}. \tag{43}$$

Therefore, if absolute risk aversion is decreasing, absolute prudence can only be increasing or constant if

$$\frac{x\varepsilon'(x)}{\varepsilon(x)} > 1 + x a(x). \tag{44}$$

In words, *decreasing* absolute risk aversion and *increasing* or *constant* absolute prudence can only coexist if the wealth elasticity of risk tolerance is so rapidly increasing that *its* wealth elasticity is greater than relative risk aversion plus one. To put it another way, given decreasing absolute risk aversion, approximate constancy of the wealth elasticity of risk-taking is enough to guarantee decreasing absolute prudence.

It should be noted that almost all commonly used parameterizations of the utility function have decreasing absolute prudence. Constant relative risk aversion guarantees decreasing absolute prudence, and even exponential and quadratic utility guarantee at least nonincreasing absolute prudence. Moreover, any positive linear combination of decreasing absolute prudence utility functions has decreasing absolute prudence.[28]

B. The Argument for Strongly Decreasing Absolute Prudence

It is difficult to interpret the condition

$$-\eta'(c) \geq 2\left(\frac{a'(c)}{a(c)}\right)^2$$

without referring to the case of constant relative risk aversion, for which it reduces to the condition that the constant relative risk aversion γ be greater than or equal to one. Accepting that limitation, there is good reason to think that γ should be at least one. First, evidence on the size of the equity premium is difficult enough to square with relative risk aversion less than six, let alone relative risk aversion less than one. [29] Second, thought experiments such as those in Kimball (1988) or Mankiw and Zeldes (1989)—whether presented to undergraduates, graduate students or professional economists—generally yield estimates of risk aversion greater than or equal to one. To give just one example, one's relative risk aversion is less than one only if a 50-50 chance of either doubling or cutting in half one's lifetime resources seems attractive. More

[27] In a simple model of risky investment, if $\varepsilon(x)$ is constant, it will equal the wealth elasticity of risky investment. Also, $\varepsilon(x)$ will be approximately equal to the wealth elasticity of risky investment if the agent initially chooses a small investment in the risky security.

[28] Kimball (1987) gives a simple proof of this proposition.

[29] See Mehra and Prescott (1985) and some of the vast literature this article has spawned, such as Grossman, Melino and Shiller (1987).

abstractly, the notion that a 10% increase in consumption should add at least as much to one's utility when one is starving as when one is well-fed corresponds to relative risk aversion greater than or equal to one.

C. An Answer to a Misgiving

Some readers may object to the foregoing arguments by questioning whether anything definite can be said about the fourth derivative of a utility function. (Positive but decreasing absolute prudence implies that $\frac{-v''''}{v'''} > \frac{-v'''}{v''} > 0$, and therefore that $v'''' < 0$ just as positive, but decreasing absolute risk aversion implies $\frac{-v'''}{v''} > \frac{-v''}{v'} > 0$ and therefore that $v''' > 0$.) But one should not expect our economic intuition always to express itself in mathematical terms. It is only clear that the second derivative of a utility should be negative after a negative second derivative has been given an economic interpretation in terms of diminishing marginal utility or risk aversion. Similarly, a positive third derivative becomes a reasonable assumption only in the light of its interpretation in terms of precautionary saving, or its relationship to decreasing absolute risk aversion (or one might gain an intuition for a positive third derivative by thinking in terms of a desire to push dispersion to higher levels of consumption or to disperse gains in consumption). Decreasing absolute risk aversion is not obvious until one understands the meaning of the Arrow-Pratt measure of risk aversion and the connection of decreasing absolute risk aversion to a positive wealth elasticity of risky investment. Thus, it should not be surprising that it is only after the economic interpretation of prudence is established that one can gain any intuition about the sign of $\left(\frac{-v'''}{v''}\right)'$ and therefore about the sign of the fourth derivative. If economic theory never yielded any surprises, it would not be worth doing.

In fact, speaking heuristically, it is easy to see more than one reasonable economic interpretation of a negative fourth derivative. On one hand, a negative fourth derivative implies that $\frac{d}{dx}v'''(x) < 0$, which suggests the declining precautionary saving motive that results if the fourth derivative is negative enough (*i.e.*, if $v''''(x) < \frac{v'''(x)^2}{v''(x)}$). On the other hand, a negative fourth derivative implies $\frac{d^2}{dx^2}v''(x) < 0$, which suggests the negative effect of one risk on the desirability of another risk discussed above as another consequence of decreasing absolute prudence—that is, as a consequence of the fourth derivative being negative enough.[30]

7. Calibrating the Effect of Income Risk on the Marginal Propensity to Consume.

A. A Stochastic Differential Equation for the Marginal Propensity to Consume

In order to calibrate the effect of income risk on the marginal propensity to consume, I will derive a stochastic differential equation for the marginal propensity to consume. It is not difficult to find closed form solutions to (19) and (20) in discrete time for certain special cases, but the formulas for continuous-time diffusion processes are simpler. To the same end of uncluttered formulas, I will also assume that short-term bonds with the risk-free real rate of return r are always available and I will assume that the period utility

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[30] These two related heuristic interpretations of a negative fourth derivative are akin to the heuristic interpretations of a positive third derivative either as connected to declining risk aversion as suggested by $\frac{d}{dx}v''(x) > 0$ or as having to do with the effect of risk on the desirability of having extra resources available in the second period, as suggested by $\frac{d^2}{dx^2}v'(x) > 0$.

function is time and state independent. Appendix C examines the case in which consumption follows a jump process. The case in which no risk-free security exists is best left as an exercise for the reader.

Most of the difficulty in the following derivation is due to having many risky assets. When there is only one asset all of the quantities in the following equations are scalars. Moreover, for scalar arguments the function $f(\cdot)$ reduces to the identity function. The main reason to go through this derivation in some detail is to clarify why the marginal propensity to consume is reduced by the variance of the marginal portfolio and to clarify the nature of the marginal portfolio.

Let me begin with a rearrangement of (23):

$$\frac{b_t}{\beta(1-b_t)} = f(\mathrm{E}_t\, \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T) \tag{45}$$

Changing the length of a period from one to a small interval of time h alters (45) to

$$\frac{hb_t}{e^{-\rho h}(1-hb_t)} = f\left(\mathrm{E}_t\, \frac{u''(c_{t+h})}{u''(c_t)} hb_{t+h} Z_{t,t+h} Z_{t,t+h}^T\right), \tag{46}$$

where $\beta = e^{-\rho h}$, $Z_{t,t+h}$ is the vector of gross asset returns between t and $t+h$, and b_t is now measured as a rate per unit time rather than as a certain fraction per period.[31] It is obvious from its definition that $f(\cdot)$ is homogenous of degree one as long as the scalar factor multiplying the argument of $f(\cdot)$ is positive. Multiplying both sides of (55) by $\frac{e^{-\rho h}}{b_t h}$ and taking this factor inside the function $f()$ reveals that

$$\frac{1}{1-b_t h} = f\left(\mathrm{E}_t\, e^{-\rho h} \frac{u''(c_{t+h})}{u''(c_t)} \frac{b_{t+h}}{b_t} Z_{t,t+h} Z_{t,t+h}^T\right). \tag{47}$$

Now I will take the limit of both sides of (47) as h goes to zero, after subtracting one from each side and dividing each side by h. On the left,

$$\lim_{h\to 0} h^{-1}\left\{\frac{1}{1-b_t h} - 1\right\} = \lim_{h\to 0} \frac{b_t}{1-b_t h} = b_t. \tag{48}$$

In order to evaluate the corresponding expression on the right, I need the fact that since $\theta^T \iota\, \iota^T \theta = 1$ for any θ satisfying the condition that the portfolio shares must sum to one,

$$\begin{aligned} f(A + \ell\iota\,\iota^T) &= \min_\theta \theta^T (A + \ell\iota\,\iota^T)\theta \\ &\quad s.t.\ \ \iota^T\theta = 1 \\ &= \min_\theta \theta^T A\theta + \ell \\ &\quad s.t.\ \ \iota^T\theta = 1 \\ &= f(A) + \ell. \end{aligned} \tag{49}$$

Using (49) and the continuity of $f(\cdot)$ proved in Appendix A,[32]

$$\begin{aligned} \lim_{h\to 0} h^{-1}\left\{f\left(\mathrm{E}_t\, e^{-\rho h}\frac{u''(c_{t+h})}{u''(c_t)}\frac{b_{t+h}}{b_t} Z_{t,t+h}Z_{t,t+h}^T\right) - 1\right\} &= \lim_{h\to 0} f\left(h^{-1}\left\{\mathrm{E}_t\, e^{-\rho h}\frac{u''(c_{t+h})}{u''(c_t)}\frac{b_{t+h}}{b_t} Z_{t,t+h}Z_{t,t+h}^T - \iota\,\iota^T\right\}\right) \\ &= f\left(\lim_{h\to 0} h^{-1}\left\{\mathrm{E}_t\, e^{-\rho h}\frac{u''(c_{t+h})}{u''(c_t)}\frac{b_{t+h}}{b_t} Z_{t,t+h}Z_{t,t+h}^T - \iota\,\iota^T\right\}\right). \end{aligned} \tag{50}$$

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[31] The quantities c_{t+h} and b_{t+h} are just the values of c and b respectively at time $t+h$.

[32] Appendix A shows that $f(\cdot)$ and $g(\cdot)$ are continuous around any A at which $g(A)$ is single-valued. The single-valuedness of $g(A)$—at least in the limit as $h \to 0$—follows from having chosen a minimal set of securities, as will become clear below.

Assuming everything follows a diffusion process, Itô's Lemma indicates that

$$\lim_{h\to 0} h^{-1}\left\{\mathrm{E}_t\, e^{-\rho h}\frac{u''(c_{t+h})}{u''(c_t)}\frac{b_{t+h}}{b_t}Z_{t,t+h}Z_{t,t+h}^T - \iota\iota^T\right\} = \left(-\rho + \frac{u'''(c_t)}{u''(c_t)}\mu_c + \frac{u''''(c_t)}{u''(c_t)}\frac{\sigma_c^2}{2} + \frac{\mu_b}{b_t} + \frac{u'''(c_t)}{u''(c_t)}\frac{\kappa_{cb}}{b_t}\right)\iota\iota^T + \iota\left[\mu_z + \frac{u'''(c_t)}{u''(c_t)}\kappa_{cz} + \frac{1}{b_t}\kappa_{bz}\right]^T + \left[\mu_z + \frac{u'''(c_t)}{u''(c_t)}\kappa_{cz} + \frac{1}{b_t}\kappa_{bz}\right]\iota^T + \sigma_z^2, \tag{51}$$

where μ_c and μ_b are the instantaneous mean rates of increase in c and b, σ_c^2 is the instantaneous variance of the change in consumption, κ_{cb} is the instantaneous covariance of c and b, μ_z is the vector of instantaneous mean net rates of return, σ_z^2 is the instantaneous variance-covariance matrix of real security returns, and κ_{cz} and κ_{bz} are the vectors of covariances between c and b and the various securities.

If the risk-free short-term real bond is listed as the first security, then

$$\mu_z + \frac{u'''(c_t)}{u''(c_t)}\kappa_{cz} + \frac{1}{b_t}\kappa_{bz} = \begin{bmatrix} r_t \\ q_t \end{bmatrix}. \tag{52}$$

Similarly, σ_z^2 can be written as

$$\sigma_z^2 = \begin{bmatrix} 0 & 0 \\ 0 & \Omega \end{bmatrix}, \tag{53}$$

where Ω is the variance-covariance matrix of the real returns of the risky assets. Combining (50), (51), (52) and (53), and using (49),

$$b_t = -\rho + \frac{u'''(c_t)}{u''(c_t)}\mu_c + \frac{u''''(c_t)}{u''(c_t)}\frac{\sigma_c^2}{2} + \frac{\mu_b}{b_t} + \frac{u'''(c_t)}{u''(c_t)}\frac{\kappa_{cb}}{b_t} + f\left(\iota[r_t \quad q_t^T] + \begin{bmatrix} r_t \\ q_t \end{bmatrix}\iota^T + \begin{bmatrix} 0 & 0 \\ 0 & \Omega \end{bmatrix}\right). \tag{54}$$

In order to evaluate $f(\cdot)$ in (54), let

$$\theta = \begin{bmatrix} 1 - \breve{\iota}^T\breve{\theta} \\ \breve{\theta} \end{bmatrix}, \tag{55}$$

where $\breve{\theta}$ is the vector of shares for *risky* securities and $\breve{\iota}$ is a vector of ones conformable to $\breve{\theta}$. Then

$$f\left(\iota[r_t \quad q_t^T] + \begin{bmatrix} r_t \\ q_t \end{bmatrix}\iota^T + \begin{bmatrix} 0 & 0 \\ 0 & \Omega \end{bmatrix}\right) = \min_{\breve{\theta}} 2r_t(1 - \breve{\iota}^T\breve{\theta}) + 2q_t\breve{\theta} + \breve{\theta}^T\Omega\breve{\theta} \tag{56}$$

The solution $\breve{\theta}_t$ to the minimization in (56), if unique, will be the vector of risky asset portfolio shares for the optimal marginal portfolio at time t, since, as shown in Appendix A, the solution function g is continuous in the neighborhood of any matrix at which it is single-valued, allowing the limit as $h \to 0$ to be passed inside g as it was passed inside f in (50). The first-order condition to the minimization in (56) states that

$$q_t - r_t\breve{\iota} + \Omega\breve{\theta}_t = 0, \tag{57}$$

or since restricting our attention to minimal sets of securities guarantees that Ω is of full rank and therefore invertible, the vector of risky asset shares for the marginal portfolio is uniquely determined as

$$\breve{\theta}_t = -\Omega^{-1}(q_t - r_t\breve{\iota}), \tag{58}$$

and the full vector of asset shares in the marginal portfolio is given by (55) with $\check{\theta} = \check{\theta}_t$. The right-hand side of (56) is best pared down by noting that from (57),

$$2(q_t - r_t\bar{\iota} + \Omega\check{\theta}_t)^T\theta = 0. \tag{59}$$

Adding this nothing to the right-hand side of (57) reveals that

$$\begin{aligned} f\left(\iota[r_t \quad q_t^T] + \begin{bmatrix} r_t \\ q_t \end{bmatrix}\iota^T + \begin{bmatrix} 0 & 0 \\ 0 & \Omega \end{bmatrix}\right) &= 2r_t - \check{\theta}_t^T\Omega\check{\theta}_t \\ &= 2r_t - \sigma_{zm}^2, \end{aligned} \tag{60}$$

where σ_{zm}^2 is the instantaneous variance of the return to the marginal portfolio.

Before substituting (60) into (54), it is helpful to remember that, by a standard result of the Consumption Capital Asset Pricing Model,[33]

$$r_t = \rho - \frac{u''(c_t)}{u'(c_t)}\mu_c - \frac{u'''(c_t)}{u'(c_t)}\frac{\sigma_c^2}{2} \tag{61}$$

if consumption follows a diffusion process. The last term in (61) indicates the effect of precautionary saving in lowering the risk-free rate. In terms of absolute risk aversion $a(c) = -\frac{u''(c)}{u'(c)}$ and absolute prudence $\eta(c) = -\frac{u'''(c)}{u''(c)}$, (61) can be rewritten as

$$r_t = \rho + a(c_t)\left[\mu_c - \eta(c_t)\frac{\sigma^2}{2}\right], \tag{62}$$

confirming the principle that absolute prudence measures the effect of risk relative to the effect of a change in mean. Equation (61) allows us to eliminate the unobservable ρ from (54). Equations (54), (60) and (61) together imply

$$b_t = r_t + \left(\frac{u'''(c_t)}{u''(c_t)} - \frac{u''(c_t)}{u'(c_t)}\right)\mu_c + \frac{\mu_b}{b_t} + \left(\frac{u''''(c_t)}{u''(c_t)} - \frac{u'''(c_t)}{u'(c_t)}\right)\frac{\sigma_c^2}{2} + \frac{u'''(c_t)}{u''(c_t)}\frac{\kappa_{cb}}{b_t} - \sigma_{zm}^2. \tag{63}$$

Equation (63) can be rewritten in terms of absolute risk aversion and absolute prudence as

$$b_t = r_t + \frac{a'(c_t)}{a(c_t)}\mu_c + \frac{\mu_b}{b_t} + \eta(c_t)\left[-\frac{\eta'(c_t)}{\eta(c_t)} - \frac{a'(c_t)}{a(c_t)}\right]\frac{\sigma_c^2}{2} - \eta(c_t)\frac{\kappa_{cb}}{b_t} - \sigma_{zm}^2. \tag{64}$$

Equation (64) is the one I will use for the calibration exercises, since it does not involve the unobservable ρ. However, if one is willing to reintroduce the unobservable ρ, one can simplify further using (62), to obtain

$$b_t = r_t + \left(\frac{-a'(c_t)}{a(c_t)^2}\right)(\rho - r_t) - \eta'(c_t)\frac{\sigma_c^2}{2} + \frac{\mu_b}{b_t} - \eta(c_t)\frac{\kappa_{cb}}{b_t} - \sigma_{zm}^2. \tag{65}$$

The last term in both (64) and (65)—the variance of the marginal portfolio—needs to be analyzed further. The starting point for the analysis of the variance is (58). The factor $q_t - r_t\bar{\iota}$ can be simplified using

[33] See Breeden (1986) or Appendix C for a derivation.

the consumption capital asset pricing model result[34] that expected excess returns are equal to risk aversion times the covariance between asset returns and consumption:

$$\mu_{\tilde{z}} - r_t \tilde{\iota} = -\frac{u''(c_t)}{u'(c_t)} \kappa_{c\tilde{z}}. \tag{66}$$

(The subscript $\tilde{z}$ refers in all cases to the vector of returns of the risky assets.) Substituting from (52) and (66),

$$\begin{aligned} q_t - r_t \tilde{\iota} &= \mu_{\tilde{z}} - r_t \tilde{\iota} + \frac{u'''(c_t)}{u''(c_t)} \kappa_{c\tilde{z}} + \frac{1}{b_t} \kappa_{b\tilde{z}} \\ &= -\frac{u''(c_t)}{u'(c_t)} \kappa_{c\tilde{z}} + \frac{u'''(c_t)}{u''(c_t)} \kappa_{c\tilde{z}} + \frac{1}{b_t} \kappa_{b\tilde{z}} \\ &= \frac{a'(c_t)}{a(c_t)} \kappa_{c\tilde{z}} + \frac{1}{b_t} \kappa_{b\tilde{z}}. \end{aligned} \tag{67}$$

Then substituting from (67) into (58) reveals that

$$\tilde{\theta}_t = -\frac{a'(c_t)c_t}{a(c_t)} \frac{1}{c_t} \Omega^{-1} \kappa_{c\tilde{z}} - \frac{1}{b_t} \Omega^{-1} \kappa_{b\tilde{z}}. \tag{68}$$

This can be interpreted as follows. The vector $\frac{1}{c_t}\Omega^{-1}\kappa_{c\tilde{z}}$ gives the coefficients from a regression of the proportional growth rate in the agent's consumption on the returns of all available securities other than the risk-free bond (which in continuous time adds nothing to the fit of the regression, since the mean growth rate is infinitesimal compared to the standard deviation of the growth rate). The factor $-\frac{a'(c_t)c_t}{a(c_t)}$ is the wealth elasticity of risk tolerance $\varepsilon(c_t)$. Finally, the vector $\frac{1}{b_t}\Omega^{-1}\kappa_{b\tilde{z}}$ gives the coefficients from a regression of the proportional growth rate of the marginal propensity to consume on all available risky securities.

B. Calibrating the Constant Relative Risk Aversion Case

In the special case of constant relative risk aversion and consumption following a geometric random walk with no jumps, (64) has a solution with a constant marginal propensity to consume. For this case (64) simplifies to

$$b = r - \frac{\mu_c}{c} + (\gamma + 1)\frac{\sigma_c^2}{c^2} - \sigma_{zm}^2, \tag{69}$$

where γ is the relative risk aversion. The marginal propensity to consume is associated positively with the interest rate, negatively with the mean growth rate of consumption, positively with the variance of consumption and negatively with the variance of the marginal portfolio. Equivalently, (65) simplifies in this case to

$$b = (1 - \frac{1}{\gamma})r + \frac{\rho}{\gamma} + \frac{\gamma + 1}{2}\frac{\sigma_c^2}{c^2} - \sigma_{zm}^2. \tag{70}$$

The comparison between (70) and (69) indicates that the term in (69) involving μ_c reflects the influence of the rate of time preference ρ, which becomes observable through its effect on the relation between the mean

[34] See Breeden (1986) or Appendix C.

and variance of the growth rate of consumption and the interest rate. The term $(\gamma+1)\frac{\sigma_c^2}{c^2}$ in (69) includes both the effect of consumption variance on the marginal propensity to consume and the contribution of consumption variance to raising our estimate of ρ. The term $\frac{\gamma+1}{2}\frac{\sigma_c^2}{c^2}$ in (70) is the effect of income uncertainty (as reflected in consumption variance) that has been the main subject of this paper. The last term in both equations—minus the variance of the marginal portfolio—is the effect of endogenous adjustment of risky asset holding.

Ignoring for the moment the variance of the marginal portfolio, which I will argue below to be relatively small, I can roughly calibrate (69). The real interest rate over the last 50 years has been about 1%, while the rate of growth of consumption has been about 2%. A relative risk aversion of 4 seems reasonable in the light of estimates on either side of this value and the introspective implausibility of extremely high values of γ.[35] Finally, Hall and Mishkin (1978) estimate the standard deviation of permanent income over a year's time at approximately 10% of the mean (see also Barsky, Mankiw and Zeldes (1986)). Assuming that consumption approximately tracks permanent income, one can then estimate a 10% standard deviation of consumption over a year or a figure of .01 annually for $\frac{\sigma_c^2}{c^2}$. This set of figures is supported by the fact that by (61) they imply

$$\rho = r - \gamma\frac{\mu_c}{c} + \frac{\gamma(\gamma+1)}{2}\frac{\sigma_c^2}{c^2} \approx 1\% - 8\% + 10\% = 3\%,$$

which is reasonable. Substituting the set of figures given above into (73), I find that

$$b \approx 1\% - 2\% + 5\% - \sigma_{zm}^2 = 4\% - \sigma_{zm}^2$$

which is substantially higher than the real interest rate, unless the variance of the marginal portfolio is large.

Ideally, one would use direct empirical evidence on the variance of the marginal portfolio. However, in the absence of direct data on the variance of the marginal portfolio, one can use a theoretical prediction about the marginal portfolio based on the characterization of the marginal portfolio in (68). In the case of constant relative risk aversion and a geometric random walk of consumption, the elasticity of risk tolerance $\varepsilon(c_t) = -\frac{a'(c_t)c_t}{a(c_t)}$ is equal to one and the marginal propensity to consume is constant, so the marginal portfolio looks like a regression of proportional changes in consumption on the returns of all available securities. In other words, in the case of constant relative risk aversion, the consumer chooses a marginal portfolio as much as possible like the "total portfolio" he or she holds, as that "total portfolio" (including human capital) is reflected in consumption. With complete markets, the instantaneous variance of the marginal portfolio would therefore be equal to the instantaneous variance of consumption in proportional terms–perhaps .01 annually. However, with incomplete markets the variance of the marginal portfolio could be much less. If,

[35] Hansen and Singleton (1983) obtain an estimate near 1, Friend and Blume (1975) suggest a value above 2, Grossman and Shiller (1981) recommend a value of 4, and Friend and Hasbrouck (1982) offer 6 as their estimate. Presenting thought experiments such as the one in Table 2 of Kimball (1988) to undergraduate and graduate students tends to lead to introspective estimates of relative risk aversion between 2 and 6. Presenting students with thought experiments designed to elicit the resistance to intertemporal substitution yields numbers in a similar range, so that introspective tests as yet give little reason to abandon the assumption of additively time-separable Von-Neumann Morgenstern utility on grounds of a difference between relative risk aversion and the resistance to intertemporal substitution.

for instance, available securities only allow agents to match the variation of aggregate consumption, the variance of the marginal portfolio would be closer to .001 annually.[36] The actual variance of the marginal portfolio is probably somewhere between these two figures.[37]

The variance of the marginal portfolio should be small compared to the effect of infrequent large falls in consumption in raising the marginal propensity to consume. It is difficult to assess the contribution of such risks on the marginal propensity to consume with any precision, but Appendix C shows that with a constant relative risk aversion of four, a one in a thousand chance per year of a sudden 50% drop in consumption would add 1.425 % to the annual marginal propensity to consume.

C. Other Cases

Three other special cases are of interest. For quadratic utility and diffusion processes, (64) and (65) reduce to

$$b_t = r_t + \frac{\mu_c}{c_{bliss} - c_t} + \frac{\mu_b}{b_t} - \sigma^2_{zm} = 2r_t - \rho + \frac{\mu_b}{b_t} - \sigma^2_{zm}. \tag{71}$$

For constant absolute risk aversion utility ($u(c) = \frac{e^{-ac}}{-a}$) and diffusion processes, (64) indicates that

$$b_t = r_t + \frac{\mu_b}{b_t} - a\frac{\kappa_{cb}}{b_t}. \tag{72}$$

This is exactly the stochastic differential equation for the real consol rate, confirming the result in Section 6 that the marginal propensity to consume is equal to the real consol rate in the case of constant absolute risk aversion.

Finally, it is instructive to look at the differential equation for the marginal propensity to consume under certainty. With all of the stochastic terms deleted, (64) becomes

$$b_t = r_t + \frac{a'(c_t)}{a(c_t)}\dot{c}_t + \frac{\dot{b}_t}{b_t}. \tag{73}$$

Rewriting (73) in terms of $\frac{1}{b_t}$ allows it to be integrated using the terminal condition $\lim_{t \to \tau} b_t = +\infty$, revealing that in the certainty case,

$$b_t = \frac{1}{\int_t^\tau e^{-\int_t^{t_1}[r_{t_2} + \frac{a'(c_{t_2})}{a(c_{t_2})}\dot{c}_{t_2}]dt_2} dt_1}. \tag{74}$$

Letting τ go to infinity yields the infinite-horizon solution.[38] Equations (73) and (74) relate b_t and $r_t + \frac{a'(c_t)}{a(c_t)}\dot{c}$ in exactly the same way the rate R_t on the long-term bond with maturity $\tau - t$ is related to the short-term rate r_t. Thus, in the case of certainty, the term $\frac{\dot{b}}{b}$ in (73) indicates that b is closer to an average value of $r_t + \frac{a'(c_t)}{a(c_t)}\dot{c}_t$ in the future than it is to the current value. The term $\frac{\mu_b}{b_t}$ should have the same kind of averaging effect in the general case given by (64).

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[36] I use Mankiw's (1986) number for the variance of aggregate consumption growth.

[37] For comparison, the standard deviation of stock returns is about 20% annually, and for long-term bonds about 10%, so that the annual variances are .04 and .01 respectively.

[38] For the case of constant relative risk aversion, in which $\frac{a'(c_t)}{a(c_t)}\mu_c = \frac{\rho - r_t}{\gamma}$, this is essentially identical to an equation in Blanchard (1985).

8. Conclusion

This paper has argued that given a reasonable condition on the form of the utility function, income risk will tend to raise the marginal propensity to consume out of wealth. Several interesting empirical predictions can be made on the basis of this theoretical proposition. The first and most important prediction is that even in the absence of liquidity constraints the marginal propensity to consume out of wealth should be considerably higher than would be predicted by the Permanent Income Hypothesis under certainty. Using an infinite-horizon continuous time model with constant relative risk aversion and a lognormal diffusion process for consumption to estimate the magnitude of the effect of income risk on the marginal propensity to consume indicates that idiosyncratic income risk might raise the marginal propensity to consume out of wealth by several percent per year, which could more than double the very low marginal propensity to consume predicted by the Permanent Income Hypothesis in the absence of uncertainty.[39] Thus, the effect of uncertainty on the marginal propensity to consume can be substantial. This conclusion about the magnitude of the effect does, however, depend in an important way on the assumption that idiosyncratic income risk is mostly uninsured. If almost all idiosyncratic income risk is insured, it is the much smaller variance of changes in aggregate consumption that best measures the magnitude of the uncertainty that would affect the marginal propensity to consume.

Second, to the extent that income risk influences the marginal propensity to consume, individuals facing greater risks to permanent income should have higher marginal propensities to consume out of wealth. It should be possible to test this prediction of cross-sectional variation in the marginal propensity to consume. Not only do some individuals face more risk to permanent income than others for idiosyncratic reasons, but certain professions, certain nations and regions, and certain stages of life are associated with higher risk to permanent income. If one could accurately measure the marginal propensity to consume out of wealth for these different groups, and measure the variance of permanent income for individuals in each group, this prediction could be put to the test.

In a sense, such a test would turn the work of Friedman (1957) on its head. Friedman assumed a constant marginal propensity to consume across various groups of people and looked for different reactions to movements in income depending on whether movements in income were mostly transitory or mostly permanent for each group. The test suggested above would assume that observed changes in income could be converted econometrically to a common denominator of changes in permanent income or human wealth which then might have different effects on consumption per dollar from group to group because of differing marginal propensities to consume out of wealth.

Third, the aggregate consumption function should show a greater marginal propensity to consume in

[39] As indicated by (53), in the absence of uncertainty, and with a constant growth rate of consumption over an infinite horizon, the marginal propensity to consume out of wealth for constant relative risk aversion should be the real interest rate minus the growth rate of consumption. This is likely to be a very small number. It may even be negative, hinting at some of the difficulty in fitting even this basic data on real interest rates and consumption growth rates in the context of the Permanent Income Hypothesis and constant relative risk aversion without taking into account precautionary saving effects.

periods of time when the economy as a whole is facing greater income risk. For example, consumption should have been more sensitive to income during the Great Depression. It is possible that the implied *interaction* between low income and high uncertainty could have contributed to the persistence of the Great Depression. As another example, one could predict that the marginal propensity to consume should have been higher in the 70's, when there was great uncertainty about the future rate of productivity growth.

Stepping back, one can see that the inquiry I have made into the effect of uncertainty on the marginal propensity to consume falls under the more general heading of studying the effects of uncertainty on behavioral elasticities. The marginal propensity to invest in various securities, in the form of the marginal portfolio, has already come to our attention and deserves further study. The effect of uncertainty on the overall interest elasticity of consumption is also of interest and the response of current consumption to uncertain additions to future income on current consumption are also of obvious importance. Finally, the effect of uncertainty on overall labor supply elasticities seems a fit subject for research. Studying the effect of uncertainty on the marginal propensity to consume out of wealth is a good place to start, both for technical reasons and because of the intrinsic importance of the marginal propensity to consume.

In studying the effect of uncertainty on the marginal propensity to consume and on other characteristics of consumption and investment decision rules, the key issue is not only the response of consumption and investment to policy changes, but also the true meaning of the Permanent Income Hypothesis. The intricate interaction of uncertainty with the precautionary saving motive considered here indicates that the simple version of the Permanent Income Hypothesis with quadratic utility may be a straw man. Given more plausible utility functions, the Permanent Income Hypothesis may be able to explain empirical observations which, in the past, have caused economists to question the Permanent Income Hypothesis.

Appendix A

The Functions $f(\cdot)$ and $g(\cdot)$.

Proof of Lemma 1.

First, suppose that

$$\iota^T \theta = 1 \tag{A.1}$$

and

$$A\theta = \lambda\iota \tag{A.2}$$

for some real number λ. If another vector $\theta + \zeta$ satisfies

$$\iota^T(\theta + \zeta) = 1 \tag{A.3}$$

then

$$\iota^T \zeta = 0 \tag{A.4}$$

As a consequence,

$$\begin{aligned}(\theta+\zeta)^T A(\theta+\zeta) &= \theta^T A\theta + \theta^T A\zeta + \zeta^T A\theta + \zeta^T A\zeta \\ &= \theta^T A\theta + \lambda\iota^T\zeta + \lambda\zeta^T\iota + \zeta^T A\zeta \\ &= \theta^T A\theta + \zeta^T A\zeta \\ &\geq \theta^T A\theta.\end{aligned} \tag{A.5}$$

The last inequality in (A.5) is due to the positive semi-definiteness of A.

Second, suppose θ is a solution to the minimization problem given in (iii). By definition, θ must satisfy (A.1). Furthermore, a necessary condition for a minimum is that the derivative of the Lagrangian $\theta^T A\theta + \xi(1 - \iota^T\theta)$ with respect to θ be zero. Formally, it must be true that

$$2A\theta - \xi\iota = 0 \tag{A.6}$$

for some real number ξ. Writing $\lambda = \frac{\xi}{2}$, this means that

$$A\theta = \lambda\iota \tag{A.7}$$

for some real number λ.

Finally, (A.7) and (A.1) imply that

$$\theta^T A\theta = \theta^T(\lambda\iota) = \lambda(\theta^T\iota) = \lambda. \tag{A.8}$$

Proof of (39).

Apply (A.5) with

$$\zeta = \begin{bmatrix} \hat{\theta} \\ 0 \end{bmatrix} - \theta_t \tag{A.9}$$

to find that

$$\begin{aligned} \arg\min_{\hat{\theta}} [\hat{\theta}^T \quad 0] A \begin{bmatrix} \hat{\theta} \\ 0 \end{bmatrix} &= \arg\min_{\hat{\theta}} \theta_t^T A \theta_t + ([\hat{\theta}^T \quad 0] - \theta_t^T) A \left(\begin{bmatrix} \hat{\theta} \\ 0 \end{bmatrix} - \theta_t \right) \\ s.t. \quad \iota^T \begin{bmatrix} \hat{\theta} \\ 0 \end{bmatrix} = 1 \qquad & s.t. \quad \iota^T \begin{bmatrix} \hat{\theta} \\ 0 \end{bmatrix} = 1 \end{aligned} \tag{A.10}$$

$$\begin{aligned} &= \arg\min_{\hat{\theta}} ([\hat{\theta}^T \quad 0] - \theta^T) A \left(\begin{bmatrix} \hat{\theta} \\ 0 \end{bmatrix} - \theta \right) \\ & s.t. \quad \iota^T \begin{bmatrix} \hat{\theta} \\ 0 \end{bmatrix} = 1. \end{aligned}$$

Substituting in $A = \mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T$ shows that $\hat{\theta}$ solves

$$\min_{\hat{\theta}} \mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} (\hat{Z}_{t+1}^m - Z_{t+1}^m)^2 \tag{A.11}$$

where

$$\hat{Z}_{t+1}^m = ([\hat{\theta}^T \quad 0] - \theta)\, Z_{t+1} \tag{A.12}$$

and

$$Z_{t+1}^m = \theta_t^T Z_{t+1}. \tag{A.13}$$

The Continuity of $f(\cdot)$ and $g(\cdot)$.

Since

$$f(A) = [g(A)]^T A g(A), \tag{A.14}$$

it is clear that f is continuous around any point around which g is continuous. The function g, in turn, is continuous around any point at which it is single-valued.

Let A be $n \times n$. Choose a fixed orthonormal matrix M which has $\frac{1}{\sqrt{n}}\iota$ as its first column. Then for any θ satisfying $\iota^T\theta = 1$,

$$M^T\theta = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \nu \end{bmatrix} \tag{A.15}$$

for some $n-1$ vector ν. Let

$$M^T A M = \begin{bmatrix} x & \xi^T \\ \xi & X \end{bmatrix}, \tag{A.16}$$

where x is a scalar, ξ is an $n-1$ vector and X is a $n-1 \times n-1$ matrix. Then

$$\begin{aligned} \theta^T A \theta &= \theta^T M M^T A M M^T \theta \\ &= [\tfrac{1}{\sqrt{n}} \quad \nu^T] \begin{bmatrix} x & \xi^T \\ \xi & X \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \nu \end{bmatrix} \\ &= \frac{x}{n} + 2\xi^T\nu + \nu^T X \nu, \end{aligned} \tag{A.17}$$

so

$$g(A) = M \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \nu^\circ \end{bmatrix} \tag{A.18}$$

where

$$\nu^\circ = \arg\min_{\nu} \; 2\xi^T \nu + \nu^T X \nu. \tag{A.19}$$

The matrix A is only in the domain of f and g if a minimum exists in (A.19). For a minimum exist, the second-order necessary condition says that X must be positive semi-definite. The first-order necessary condition for a minimum in (A.19) is

$$\xi + X\nu^\circ = 0. \tag{A.20}$$

These two conditions are also sufficient for minimum since if $\nu = \nu^\circ + \omega$,

$$\begin{aligned} 2\xi^T \nu + \nu^T X \nu &= 2\xi^T(\nu^\circ + \omega) + (\nu^\circ + \omega)^T X (\nu^\circ + \omega) \\ &= [2\xi^T \nu^\circ + \nu^{\circ T} X \nu^\circ] + 2(\xi + X\nu^\circ)\omega + \omega^T X \omega \\ &= [2\xi^T \nu^\circ + \nu^{\circ T} X \nu^\circ] + \omega^T X \omega. \end{aligned} \tag{A.21}$$

Equation (A.21) also indicates that there is a unique solution to (A.19) if and only if X is strictly positive definite, since otherwise there exists an $\omega \neq 0$ for which $\omega^T X \omega = 0$. If $g(A)$ is single-valued, then there is a unique solution ν°, X is positive definite, $\nu^\circ = X^{-1}\xi$ by (A.20) and

$$g(A) = M \begin{bmatrix} \frac{1}{\sqrt{n}} \\ X^{-1}\xi \end{bmatrix}. \tag{A.22}$$

Since X and ξ are continuous functions of A, and the determinant of X which indicates whether X is invertible is a continuous function of X, (A.22) shows that $g(A)$ is locally continuous around any matrix A for which $g(A)$ is single-valued.

Appendix B

The Marginal Portfolio when Asset Markets are Incomplete

The Marginal Portfolio in the Presence of Currently Complete Markets

It is helpful to characterize the marginal portfolio for the extra saving in response to increased wealth *first* for the special case in which a complete set of securities exists to market all risk that will be realized between t and $t+1$. Such a situation will be called "currently complete markets" to distinguish it from a situation of "perpetually complete markets," in which a complete set of securities exists to market all risks for all time. The marginal portfolio optimal under currently complete markets will be called the "currently ideal marginal portfolio." Its gross return will be denoted Z^*_{t+1} and its portfolio weights will be denoted θ^*_t.

Having currently complete markets means that the set of securities available for investment between t and $t+1$ is equivalent to a set of state-contingent securities, each of which pay $1 in one state defined by information known at time $t+1$ and zero in all other states at time $t+1$. Indexing states at time $t+1$ by s, the first order condition $u'(c_t) = \beta \mathrm{E}_t\, Z_{t+1}^s u'(c_{t+1})$ must hold, where Z_{t+1}^s is the gross return on the security for state s. This implies that

$$Z_{t+1}^s = \frac{u'(c_t)}{\beta \pi_{t+1,s} u'(c_{t+1,s})} \chi_s, \tag{B.1}$$

where $\pi_{t+1,s}$ is the probability of state s conditional on information known at time t, $c_{t+1,s}$ is consumption in state s, and χ_s is a characteristic random variable equal to 1 in state s and zero otherwise.

The returns for the various state-contingent securities can be lined up to form $Z_{t+1} = \sum_s 1_s Z_{t+1}^s$, where 1_s is the vector with 1 in the s-th entry and zero elsewhere. Then one can calculate

$$\begin{aligned} A &= \mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1} Z_{t+1} Z_{t+1}^T \\ &= \sum_s \pi_{t+1,s} b_{t+1,s} \frac{u''(c_{t+1,s})}{u''(c_t)} \frac{u'(c_t)^2}{\beta^2 \pi_{t+1,s}^2 u'(c_{t+1,s})^2} 1_s 1_s^T \\ &= \underset{s}{diag} \left(\frac{b_{t+1,s}}{\beta^2 \pi_{t+1,s}} \frac{u''(c_{t+1,s})}{u''(c_t)} \frac{u'(c_t)^2}{u'(c_{t+1,s})^2} \right). \end{aligned} \tag{B.2}$$

Since A is invertible, it is easiest to use (B.2) in conjunction with (19) and (20) to calculate b_t and θ_t. From (B.2),

$$A^{-1}\iota = \underset{s}{diag} \left(\pi_{t+1,s} \frac{\beta^2}{b_{t+1,s}} \frac{u''(c_t)}{u'(c_t)^2} \frac{u'(c_{t+1,s})^2}{u''(c_{t+1,s})} \right) \tag{B.3}$$

and

$$\begin{aligned} \iota^T A^{-1} \iota &= \beta^2 \frac{u''(c_t)}{u'(c_t)^2} \sum_s \frac{\pi_{t+1,s}}{b_{t+1,s}} \frac{u'(c_t+1,s)^2}{u''(c_{t+1})} \\ &= \beta^2 \frac{u''(c_t)}{u'(c_t)^2} \mathrm{E}_t \frac{1}{b_{t+1}} \frac{u''(c_{t+1})}{u'(c_{t+1})^2}. \end{aligned} \tag{B.4}$$

Substituting these expressions into (17) and (18) yields

$$b_t^* = \frac{1}{1 + \beta \mathrm{E}_t \frac{u''(c_t)}{u'(c_t)^2} \frac{u'(c_{t+1})^2}{u''(c_{t+1}) b_{t+1}}} \tag{B.5}$$

$$\theta_t^* = \frac{1}{\mathrm{E}_t \frac{u''(c_t)}{u'(c_t)^2} \frac{u'(c_{t+1})^2}{u''(c_{t+1}) b_{t+1}}} \underset{s}{column} \left(\pi_{t+1,s} \frac{u''(c_t)}{u'(c_t)^2} \frac{u'(c_{t+1,s})^2}{u''(c_{t+1,s}) b_{t+1,s}} \right) \tag{B.6}$$

Therefore, the gross return on the currently ideal marginal portfolio is:

$$\begin{aligned} Z_{t+1}^* &= Z_{t+1}^T \theta_t \\ &= \sum_s \frac{u'(c_t)}{\beta \pi_{t+1,s} u'(c_{t+1,s})} \chi_s 1_s^T \theta_t^* \\ &= \frac{\frac{1}{a(c_{t+1}) b_{t+1}}}{\beta \mathrm{E}_t \frac{1}{a(c_{t+1}) b_{t+1}} \frac{u'(c_{t+1})}{u'(c_t)}}, \end{aligned} \tag{B.7}$$

where, as before, $a(c) = \frac{-u''(c)}{u'(c)}$ is absolute risk aversion.

In words, the currently ideal marginal portfolio has a payout that is inversely proportional to absolute risk aversion times the marginal propensity to consume in period $t+1$. The denominator indicates the price of such a security.

The product $a(c_{t+1})b_{t+1}$ which appears in (B.7) is more meaningful on remembering that by (3) it is the absolute risk aversion of the value function. Being in inverse proportion to the absolute risk aversion of the value function (or in direct proportion to the *risk tolerance* of the value function), the gross return on the currently ideal marginal portfolio lowers marginal utility by the same proportion in every state of nature, which is what must happen in order to continue to satisfy all of the first order conditions. Indeed, it is easy to calculate from (B.1) and (B.2) that

$$\frac{(1-b_t^*)Z_{t+1}^*}{b_t^*} = \frac{1/a(c_{t+1})b_{t+1}}{1/a(c_t)}, \tag{B.8}$$

so that the extra spending at time t will cause marginal utility in period t to fall by the same proportion as the marginal utility in each state s at time $t+1$.[40]

The Marginal Portfolio in the Absence of Currently Complete Markets

If the currently ideal marginal portfolio is impossible to construct out of available securities, one would expect an agent to construct a marginal portfolio that is, in some sense, as close as possible to the currently ideal marginal portfolio. This is indeed the case. Appendix A shows that in general, if securities are deleted, the new marginal portfolio with portfolio shares $\hat{\theta}$ on the restricted set of securities is the one that solves

$$\min_{\hat{\theta}} \mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1}(\hat{Z}_{t+1}^m - Z_{t+1}^m)^2, \tag{B.9}$$

where Z_{t+1}^m is the gross return vector for the marginal portfolio before any securities are deleted and

$$\hat{Z}_{t+1}^m = \hat{\theta} Z_{t+1}. \tag{B.10}$$

Since any current set of securities can be obtained by starting from currently complete markets and deleting securities, Z_{t+1}^* can take the place of Z_{t+1}^m and Z_{t+1}^m can take the place of $\hat{Z}_{t+1}^m$ in (B.9), indicating that Z_{t+1}^m solves

$$\min_{\theta} \mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1}(Z_{t+1}^m - Z_{t+1}^*)^2. \tag{B.11}$$

Solving this minimization problem amounts to a weighted regression of the gross return on the currently ideal marginal portfolio on the set of available securities. Note that in the continuous time limit, if consumption and the marginal propensity to consume follow a diffusion process with no sudden jumps, the weights in the regression given by (B.11) all converge to the same value, b_t, so that the marginal portfolio is formed according to an ordinary least squares regression of the returns on the currently ideal marginal portfolio on the returns of all available securities.

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[40] More formally, if $dc_t = \frac{k}{a(c_t)}dw_t$ and $dw_{t+1} = \frac{k}{a(c_{t+1})b_{t+1}}dw_t$, where k is a constant, then $d(u'(c_t)) = u''(c_t)\frac{k}{a(c_t)}dw_t = -ku'(c_t)dw_t$, and $d(J_w(w_{t+1},\varphi_{t+1})) = J_{ww}(w_{t+1},\varphi_{t+1})\frac{k}{a(c_{t+1})b_{t+1}}dw_t = -kJ_w(w_{t+1},\varphi_{t+1})dw_t$.

The reason behind (B.11) is that the random variable Z^*_{t+1} globally minimizes $\mathrm{E}_t \frac{u''(c_{t+1})}{u''(c_t)} b_{t+1}(Z^*_{t+1})^2$ among all securities that satisfy the first order condition, while $\hat{Z}^m_{t+1}$ is a constrained minimum. By minimizing the weighted distance to the global minimum, $\hat{Z}^m_{t+1}$ attains the constrained minimum.

Appendix C

Calibrating the Marginal Propensity to Consume when Consumption and Asset Returns follow Jump Processes

In assessing the effect of jumps on the marginal propensity to consume, it is helpful to have in hand the formula relating consumption and asset returns when consumption and asset returns follow jump processes. When a time period has length h, the vector of equations of type (16) becomes

$$0 = h^{-1}\left\{\mathrm{E}_t\, e^{-\rho h}\frac{u'(c_{t+h})}{u'(c_t)} Z_{t,t+h} - \iota\right\} \tag{C.1}$$

after a bit of rearranging. Taking the limit as $h \to 0$ while allowing for the possibility of jumps produces

$$\begin{aligned} 0 = &\left[-\rho + \frac{u''(c_t)}{u'(c_t)}\mu^\bullet_c + \frac{u'''(c_t)}{u'(c_t)}\frac{\sigma^{\bullet 2}}{2}\right]\iota + \mu^\bullet_z + \frac{u''(c_t)}{u'(c_t)}\kappa^\bullet_{cz} \\ &+ \lambda_t \mathrm{E}^\#_t\left\{\frac{u'(c^\#_{t+})}{u'(c_t)} Z^\#_{t,t+} - \iota\right\} \end{aligned} \tag{C.2}$$

where a bullet (•) indicates a mean, variance or covariance conditional on no jumps, λ_t is the instantaneous Poisson probability of a jump, $\mathrm{E}^\#_t$ is an expectation conditional on a jump and a pound sign (#) on a variable indicates the value of that variable after a jump. Defining $z^\#_{t+} = Z^\#_{t,t+} - 1$ as the vector of net returns after a jump, the overall instantaneous means, variances and covariances depend on the means, variances and covariances conditional on no jump or on some jump as follows:

$$\begin{aligned} \mu_c &= \mu^\bullet_c + \lambda_t \mathrm{E}^\#_t\,(c^\#_{t+} - c_t) \\ \mu_z &= \mu^\bullet_z + \lambda_t \mathrm{E}^\#_t\, z^\#_{t+} \\ \sigma^2_c &= \sigma^{\bullet 2}_c + \lambda_t \mathrm{E}^\#_t\,(c^\#_{t+} - c_t)^2 \\ \kappa_{cz} &= \kappa^\bullet_{cz} + \lambda_t \mathrm{E}^\#_t\,(c^\#_{t+} - c_t) z^\#_{t+}. \end{aligned} \tag{C.3}$$

Rewriting (C.2) in terms of these overall instantaneous means, variances and covariances and simplifying yields the following equation:

$$\begin{aligned} \mu_z = &\left[\frac{-u''(c_t)}{u'(c_t)}\mu_c - \frac{u'''(c_t)}{u'(c_t)}\frac{\sigma^2_c}{2}\right. \\ &\left.+\frac{\lambda_t}{u'(c_t)}\mathrm{E}^\#_t\left(u'(c_t) + u''(c_t)(c^\#_{t+} - c_t) + u'''(c_t)\frac{(c^\#_{t+} - c_t)^2}{2} - u'(c^\#_{t+})\right)\right]\iota \\ &+ \frac{-u''(c_t)}{u'(c_t)}\kappa_{cz} + \frac{\lambda_t}{u'(c_t)}\mathrm{E}^\#_t\, z^\#_{t+}[u'(c_t) + u''(c_t)(c^\#_{t+} - c_t) - u'(c^\#_{t+})]. \end{aligned} \tag{C.4}$$

Since, with the risk-free real bond put first, the topmost entry in the other terms on the right is zero, the coefficient of the vector of ones (ι) in (C.4) is the real interest rate. The other terms, therefore, add up to the vector of risk premia for the other securities. Since by the remainder theorem for Taylor expansions,

$$u'(c_t) + u''(c_t)(c^{\#}_{t+} - c_t) + u'''(c_t)\frac{(c^{\#}_{t+} - c_t)^2}{2} - u'(c^{\#}_{t+}) = -u''''(c^{\dagger})\frac{(c^{\#}_{t+} - c_t)^3}{6}$$

for some $c^{\dagger}$ between c_t and $c^{\#}_{t+}$, (C.4) indicates (given the negative fourth derivative guaranteed by decreasing absolute prudence) that the possibility of jumps *down* in consumption tends to *lower* the risk-free rate below what one would expect from looking only at means and variances, while the possibility of jumps *up* in consumption tends to *raise* the risk-free rate above what one would expect from looking only at means and variances. Similarly, since

$$u'(c_t) + u''(c_t)(c^{\#}_{t+} - c_t) - u'(c^{\#}_{t+}) = -u'''(c^{\dagger\dagger})\frac{(c^{\#}_{t+} - c_t)^2}{2},$$

equation (C.4) indicates (given the positive third derivative guaranteed by decreasing absolute risk aversion) that the possibility of a jump (either way) in consumption synchronized with a jump *down* in an asset price tends to *raise* the corresponding risk premium above what one would expect from looking at covariances, while the possibility of a jump in consumption synchronized with a jump *up* in an asset price tends to *lower* the corresponding risk premium below what one would expect from looking only at covariances. The implication of (C.4) about the effect of possible jumps on risk premia confirms the main result of Mankiw (1986) by extending it to the multiperiod continuous-time case. An interesting negative implication of (C.4) is that the possibility of jumps does not affect risk-premia at all unless jumps in consumption and in asset prices are synchronized. Indeed, the aggregation results of Grossman and Shiller (1982) for the diffusion case can be extended to the case in which consumption and asset values jump but never at the same instant.

To find the formula for the marginal propensity to consume when consumption and asset returns do not follow a diffusion process, begin by modifying (60) to

$$\begin{aligned}
&\lim_{h\to 0} h^{-1}\left\{\mathrm{E}_t\, e^{-\rho h}\frac{u''(c_{t+h})}{u''(c_t)}\frac{b_{t+h}}{b_t} Z_{t,t+h} Z^T_{t,t+h} - \iota\,\iota^T\right\} \qquad (C.5)\\
&= \left(-\rho + \frac{u'''(c_t)}{u''(c_t)}\mu^{\bullet}_c + \frac{u''''(c_t)}{u''(c_t)}\frac{\sigma^{\bullet 2}_c}{2} + \frac{\mu^{\bullet}_b}{b_t}\right.\\
&\quad \left. + \frac{u'''(c_t)}{u''(c_t)}\frac{\kappa^{\bullet}_{cb}}{b_t}\right)\iota\,\iota^T + \iota\left[\mu^{\bullet}_z + \frac{u'''(c_t)}{u''(c_t)}\kappa^{\bullet}_{cz} + \frac{1}{b_t}\kappa^{\bullet}_{bz}\right]^T\\
&\quad + \left[\mu^{\bullet}_z + \frac{u'''(c_t)}{u''(c_t)}\kappa^{\bullet}_{cz} + \frac{1}{b_t}\kappa^{\bullet}_{bz}\right]\iota^T + \sigma^{\bullet 2}_z\\
&\quad + \lambda_t \mathrm{E}^{\#}_t\left[\left(\frac{u''(c^{\#}_{t+})}{u''(c_t)}\frac{b^{\#}_{t+}}{b_t} - 1\right)\iota\iota^T + \frac{u''(c^{\#}_{t+})}{u''(c_t)}\frac{b^{\#}_{t+}}{b_t} z^{\#}_{t+} z^{\#T}_{t+}\right].
\end{aligned}$$

Before rewriting (C.5) in terms of the overall instantaneous means, variances and covariances, add to (C.3) as follows:

$$\mu_b = \mu_b^* + \lambda_t \mathrm{E}_t^{\#}\,(b_{t+}^{\#} - b_t) \tag{C.6}$$

$$\sigma_z^2 = \sigma_z^{*2} + \lambda_t \mathrm{E}_t^{\#}\, z_{t+}^{\#} z_{t+}^{\#T}$$

$$\kappa_{cb} = \kappa_{cb}^* + \lambda_t \mathrm{E}_t^{\#}\,(c_{t+}^{\#} - c_t)(b_{t+}^{\#} - b_t)$$

$$\kappa_{bz} = \kappa_{bz}^* + \lambda_t \mathrm{E}_t^{\#}\,(b_{t+}^{\#} - b_t) z_{t+}^{\#}.$$

Then using (C.3), (C.6), the prediction of (C.4) for the risk-free rate and the first line of (69), calculate that

$$\begin{aligned} b_t = r_t &+ \left(\frac{u'''(c_t)}{u''(c_t)} - \frac{u''(c_t)}{u'(c_t)}\right)\mu_c + \left(\frac{u''''(c_t)}{u''(c_t)} - \frac{u'''(c_t)}{u'(c_t)}\right)\frac{\sigma_c^2}{2} + \frac{\mu_b}{b_t} + \frac{u'''(c_t)}{u''(c_t)b_t}\kappa_{cb} \\ &+ \frac{\lambda_t}{u''(c_t)}\mathrm{E}_t^{\#}\left\{u''(c_{t+}^{\#}) - u''(c_t) - u'''(c_t)(c_{t+}^{\#} - c_t) - u''''(c_t)\frac{(c_{t+}^{\#} - c_t)^2}{2}\right\} \\ &- \frac{\lambda_t}{u'(c_t)}\mathrm{E}_t^{\#}\left\{u'(c_{t+}^{\#}) - u'(c_t) - u''(c_t)(c_{t+}^{\#} - c_t) - u'''(c_t)\frac{(c_{t+}^{\#} - c_t)^2}{2}\right\} \\ &+ \frac{\lambda_t}{u''(c_t)b_t}\mathrm{E}_t^{\#}\left\{[b_{t+}^{\#} - b_t][u''(c_{t+}^{\#}) - u''(c_t) - u'''(c_t)(c_{t+}^{\#} - c_t)]\right\} - \hat{\theta}_t^T \Omega \hat{\theta}_t, \end{aligned} \tag{C.7}$$

where

$$\begin{bmatrix} 0 & 0 \\ 0 & \Omega \end{bmatrix} = \sigma_z^2 + \lambda_t \mathrm{E}_t^{\#}\left(\frac{u''(c_{t+}^{\#})b_{t+}^{\#}}{u''(c_t)b_t} - 1\right) z_{t+}^{\#} z_{t+}^{\#T} \tag{C.8}$$

and

$$\hat{\theta}_t = -\Omega^{-1}(q_t - r_t\hat{\iota}) \tag{C.9}$$

with

$$\begin{aligned} q_t - r\hat{\iota} = &\frac{-a'(c_t)}{a(c_t)}\kappa_{c\hat{z}} + \frac{1}{b_t}\kappa_{b\hat{z}} \\ &+ \frac{\lambda_t}{u''(c_t)}\mathrm{E}_t^{\#}\, z_{t+}^{\#}[u''(c_{t+}^{\#}) - u''(c_t) - u'''(c_t)(c_{t+}^{\#} - c_t)] \\ &+ \frac{\lambda_t}{u''(c_t)b_t}\mathrm{E}_t^{\#}\, z_{t+}^{\#}(b_{t+}^{\#} - b_t)(u''(c_{t+}^{\#}) - u''(c_t)). \end{aligned} \tag{C.10}$$

The terms on the last line of (C.8) are difficult to interpret except to say that the last term should still be fairly small if there is little or no insurance for idiosyncratic income risk, and the term involving jumps in the marginal propensity to consume will be zero when consumption follows a geometric random walk with occasional geometric jumps so that the marginal propensity to consume is constant.

The two new terms on the second and third lines of (C.8) put together indicate that jumps *down* in consumption will tend to *raise* the marginal propensity to consume if to decreasing absolute risk aversion and decreasing absolute prudence I can add the assumption (implied by Pratt and Zeckhauser's (1987) complete properness, which they show is exhibited by most commonly used utility functions, including those with constant relative risk aversion[41]), that $\frac{-u''''(c)}{u'''(c)}$ is decreasing, or equivalently that

$$\frac{-u^{(5)}(c)}{u''''(c)} \geq \frac{u''''(c)}{u'''(c)}.$$

(Decreasing absolute prudence and decreasing $\frac{-u''''(c)}{u'''(c)}$ are the two assumptions needed to guarantee that independent risks have a more-than-additive effect on precautionary premium, since these two assumptions guarantee that the negative of marginal utility displays standard risk aversion.) The direction of the effect of these two terms on the marginal propensity to consume is made clear by the fact that

$$\begin{aligned}
&\frac{1}{u''(c_t)}\left\{u''(c^{\#}_{t+}) - u''(c_t) - u'''(c_t)(c^{\#}_{t+} - c_t) - u''''(c_t)\frac{(c^{\#}_{t+} - c_t)^2}{2}\right\} \qquad (C.11)\\
&\quad - \frac{1}{u'(c_t)}\mathrm{E}^{\#}_t\left\{u'(c^{\#}_{t+}) - u'(c_t) - u''(c_t)(c^{\#}_{t+} - c_t) - u'''(c_t)\frac{(c^{\#}_{t+} - c_t)^2}{2}\right\}\\
&\quad = \frac{1}{u''(c_t)}\int_{c_t}^{c^{\#}_{t+}}\int_{c_t}^{c^{\#}_{t+}}\int_{c_t}^{c^{\#}_{t+}} u^{(5)}(x_3)dx_3dx_2dx_1\\
&\quad - \frac{1}{u'(c_t)}\int_{c_t}^{c^{\#}_{t+}}\int_{c_t}^{c^{\#}_{t+}}\int_{c_t}^{c^{\#}_{t+}} u''''(x_3)dx_3dx_2dx_1\\
&\quad = \int_{c_t}^{c^{\#}_{t+}}\int_{c_t}^{c^{\#}_{t+}}\int_{c_t}^{c^{\#}_{t+}}\left[\frac{u'(x_3)}{u'(c)}\left(\frac{-u''''(x_3)}{u'(x_3)}\right) - \frac{u''(x_3)}{u''(c)}\left(\frac{-u^{(5)}(x_3)}{u''(x_3)}\right)\right]dx_3dx_2dx_1.
\end{aligned}$$

The integrand is negative, making the value of the integral positive, since decreasing absolute risk aversion (with its consequence that $-u'(c)$ is more risk averse then $u(c)$), and the fact that the dummy variable x_3 is always less than c_t if $c^{\#}_{t+} < c_t$, guarantee that

$$\frac{u''(x_3)}{u''(c_t)} \geq \frac{u'(x_3)}{u'(c_t)}, \qquad (C.12)$$

while decreasing absolute risk aversion, decreasing absolute prudence and decreasing $\frac{-u''''(c)}{u'''(c)}$ imply that

$$\begin{aligned}
\frac{-u^{(5)}(x_3)}{u''(x_3)} &= \left(\frac{-u^{(5)}(x_3)}{u''''(x_3)}\right)\left(\frac{-u''''(x_3)}{u'''(x_3)}\right)\left(\frac{-u'''(x_3)}{u''(x_3)}\right) \qquad (C.13)\\
&\geq \left(\frac{-u''''(x_3)}{u'''(x_3)}\right)\left(\frac{-u'''(x_3)}{u''(x_3)}\right)\left(\frac{-u''(x_3)}{u'(x_3)}\right)\\
&= \frac{-u''''(x_3)}{u'(x_3)}
\end{aligned}$$

since each of the three factors on the right of the first line is greater than the corresponding factor on the second line.

Even small probabilities of large jumps down in consumption can have important effects on the marginal propensity to consume. Suppose, for example, that the only possible jump is that each year there is a one

[41] See Kimball (1987) for an extended discussion of the relationship between complete properness and decreasing absolute prudence.

in a thousand chance that consumption will suddenly fall to half of what it was. With constant relative risk aversion of four, one can calculate that

$$\frac{\lambda_t}{u''(c_t)}\mathrm{E}_t^{\#}\left\{u''(c_{t+}^{\#}) - u''(c_t) - u'''(c_t)(c_{t+}^{\#} - c_t) - u''''(c_t)\frac{(c_{t+}^{\#} - c_t)^2}{2}\right\}$$
$$- \frac{\lambda_t}{u'(c_t)}\mathrm{E}_t^{\#}\left\{u'(c_{t+}^{\#}) - u'(c_t) - u''(c_t)(c_{t+}^{\#} - c_t) - u'''(c_t)\frac{(c_{t+}^{\#} - c_t)^2}{2}\right\} = .01425,$$

so that this term adds 1.425 % to the annual marginal propensity to consume. If the Poisson probability were greater than a one in a thousand chance per year by the appropriate factor. If the risk is totally idiosyncratic, so that asset prices do not jump with consumption, the other two terms newly introduced by the possibility of jumps are equal to zero.

The intuition for the effect of jumps downward on the marginal propensity to consume might be approached as follows. Suppose the possibility of large jumps downward in income forces an agent to keep an eye on the neighborhood of an Inada condition at zero, but the probability is small enough that the agent cannot afford to guarantee that he or she will be very far from the Inada condition at zero in the event of a bad shock, then the optimal distance from the Inada condition in the event of a bad shock ought to be fairly inelastic. Given an extra dollar, the agent can spend much of it and still guarantee about the same distance from the Inada condition in the event of a bad shock.

The forgoing intuition corresponds to the term involving only jumps in consumption. The last term in (C.8) hints at how the forgoing intuition might go awry. Suppose that there is insurance for the bad income shock, but that this insurance is very far from being actuarially fair, so that the agent is far from fully insured, although he or she purchases a little of this insurance. Then given an extra dollar, the agent might choose to save most of the dollar and restore something close to the former distance from the Inada condition in the event of a bad shock by reducing his or her costly insurance coverage by a little rather than spend much of the extra dollar.

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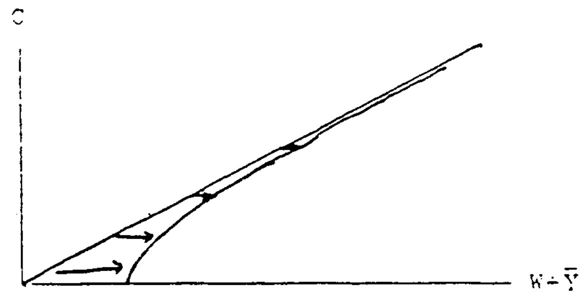

Figure 1

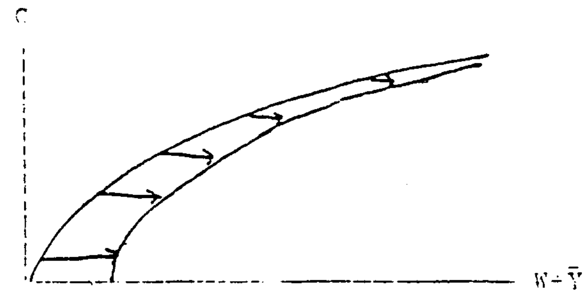

Figure 2

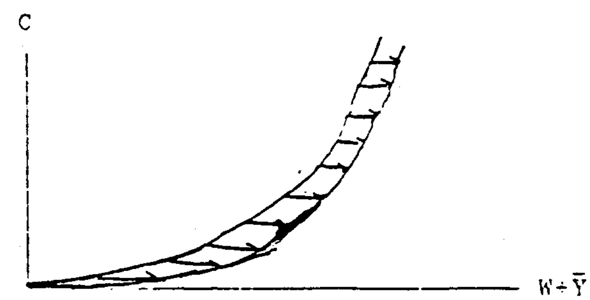

Figure 3

Table 1
Examples of Utility Functions

1. Decreasing Absolute Prudence:

$$v(x) = \frac{(x-\delta)^{1-\gamma}-1}{1-\gamma} \quad (\gamma>0)$$ (HARA with decreasing absolute risk aversion: when $\delta=0$, CRRA)

$$a(x) = \frac{\gamma}{x+\delta} \qquad a'(x) = \frac{-\gamma}{(x+\delta)^2}$$

$$\eta(x) = \frac{\gamma+1}{x+\delta} \qquad \eta'(x) = \frac{-(\gamma+1)}{(x+\delta)^2}$$

2. Increasing Absolute Prudence:

$$v(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-\xi^2}{2}} d\xi \quad (x>0)$$ (the distribution function of a standard normal)

$$a(x) = x \qquad a'(x) = 1$$

$$\eta(x) = x - \frac{1}{x} \qquad \eta'(x) = 1 + \frac{1}{x^2}$$

3. Constant Absolute Prudence

   a. Exponential Utility

$$v(x) = \frac{e^{-ax}}{-a}$$

$$a(x) = a \qquad a'(x) = 0$$

$$\eta(x) = a \qquad \eta'(x) = 0$$

   b. Quadratic Utility

$$v(x) = -(x-x_0)^2$$

$$a(x) = \frac{1}{x_0-x} \qquad a'(x) = \frac{1}{(x_0-x)^2}$$

$$\eta(x) = 0 \qquad \eta'(x) = 0$$