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THE CONCAVE PHILLIPS CURVE

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ABSTRACT

This paper derives the curvature properties of the short-run Phillips curve for a wide class of models with time-dependent pricing frictions. Contrary to conventional thinking, the Phillips curve is asymptotically horizontal for high levels of economic activity and asymptotically vertical for low levels of economic activity. Moreover, it is globally concave under relatively weak conditions that allow real marginal cost to be an unbounded convex function of economic activity. Intuitively, when economic activity is very high (low), substitution effects within the price index imply that inflation behaves as if prices are nearly fully sticky (flexible). Using (conventional) measures of inflation that understate the relevant substitution effects may lead to misleading conclusions about the curvature of the Phillips curve, and to corresponding errors in the formulation of monetary policy.

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1 Introduction

Figure 1 depicts the dominant conventional wisdom about the curvature of the short-run Phillips curve[1], drawn as a positive relationship with a measure of real economic activity on the x-axis and inflation on the y-axis. The curve is globally *convex* - the inflation costs of generating a given increase in economic activity are higher in booms than in recessions. The curve becomes extremely steep for high levels of economic activity, capturing a basic intuition that, even in the short run, there are limits to what monetary policy can accomplish in terms of real stimulus. The curve is flatter for low levels of economic activity, which corresponds to the idea that nominal or real wages are costly to adjust downward.

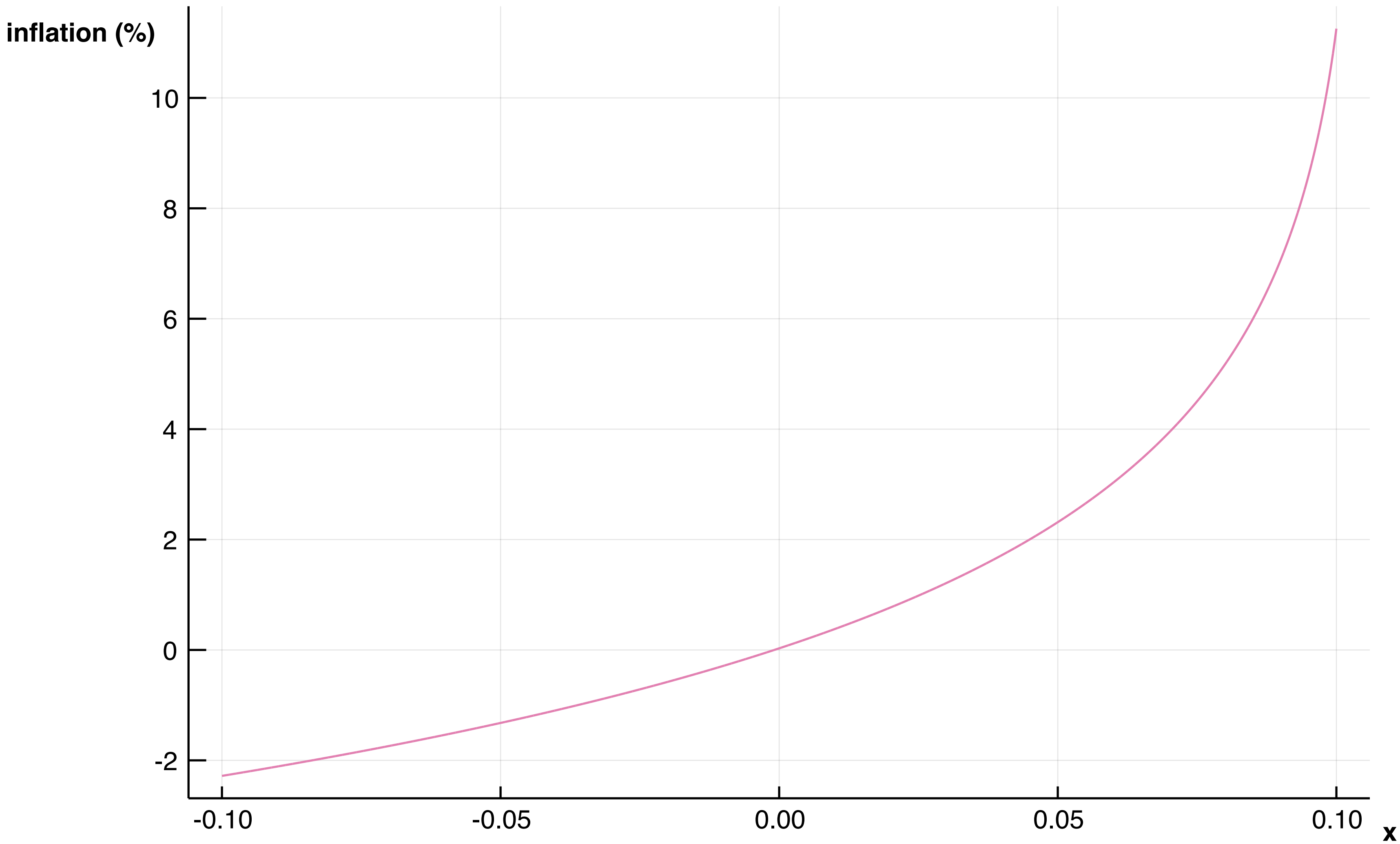
This paper studies the curvature of the short-run Phillips curve through the lens of modern macroeconomic theory. I derive a closed-form representation for the fully nonlinear Phillips curve that is valid for a highly flexible specification of time-dependent pricing frictions (as in Auclert et al. (2024)). I use this representation to show that if the households' common intraperiod elasticity of substitution[2] between goods is larger than 3, then the implied Phillips curve is globally *concave* in a wide class of models (including ones in which the marginal cost of production increases without bound with economic activity). As illustrated in Figure 2, the concavity is in some sense extreme: The curve is asymptotically horizontal for high levels of economic activity and asymptotically vertical for low levels of economic activity.

The intuition behind these results is quite simple. The model-implied Phillips curve uses a measure of inflation that is based on the rate of growth of the true (or Konüs (1924)) price index that fully incorporates substitution effects. When economic activity and inflation are high, consumers allocate almost all of their expenditures to (the cheaper) goods with fixed prices. The Phillips curve looks like one from a world in which prices are constant over time -

[1]By "short-run Phillips curve", I mean to refer, as did Samuelson and Solow (1960), to the locus of possible real economic activity and inflation outcomes achievable by a central bank through different choices of monetary policy.

[2]As in the textbook New Keynesian model (Gali (2015, Chapter 3)), I assume that the elasticity between any two goods is the same. In reality, goods differ in their substitutability. For models with time-dependent pricing frictions, the relevant margin is between goods with prices that changed in a given period and goods with prices that did not change.

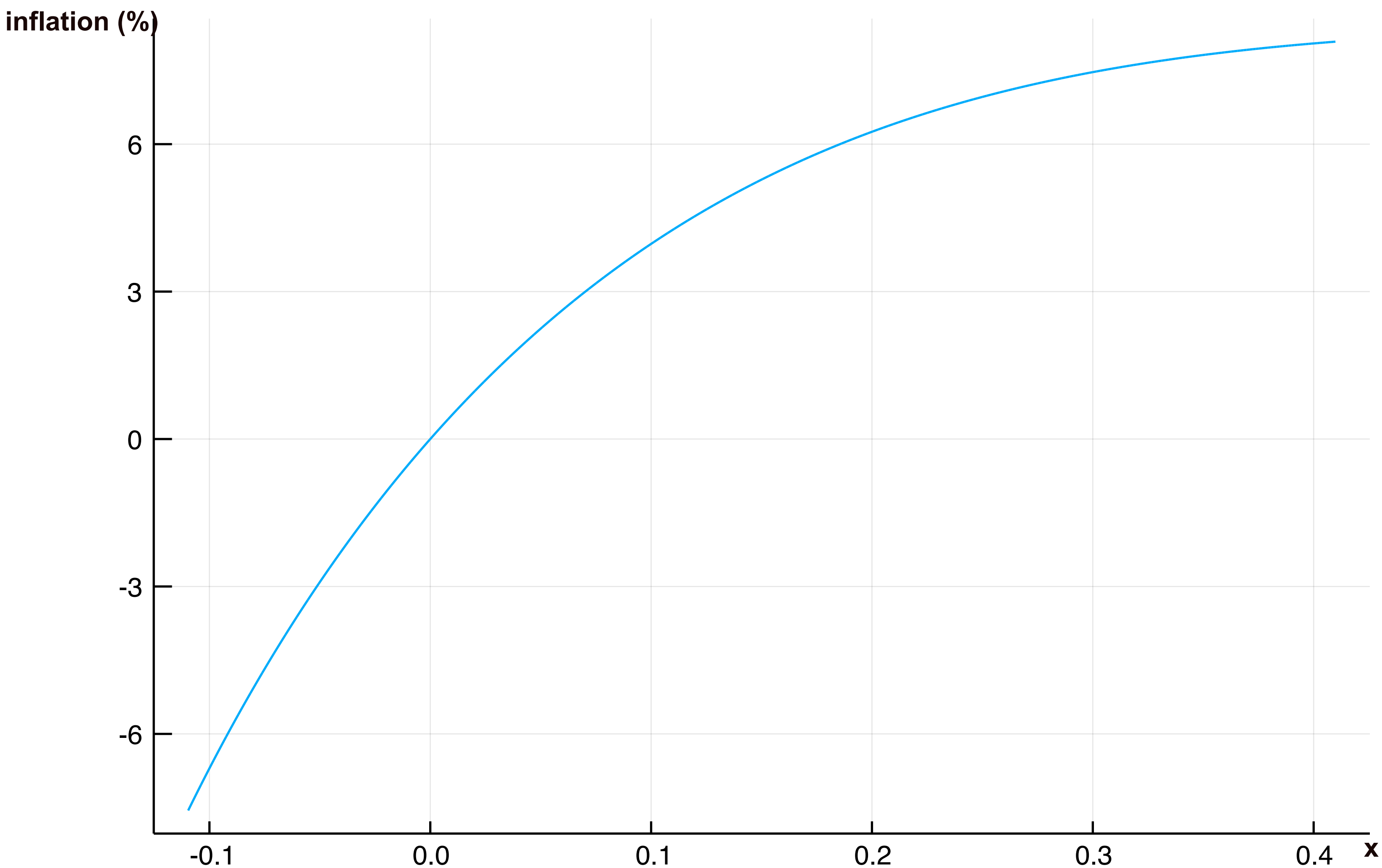
Figure 1: The Conventional Phillips Curve



The conventional Phillips curve represents inflation as a convex function of economic activity (here labelled x).

The curve is near-vertical for high values of x.

Figure 2: The True Phillips Curve



Under the true Phillips curve, inflation is a concave function of economic activity (x).

The curve is asympotically horizontal for large values of x and asymptotically vertical for small values of x.

that is, horizontal. When economic activity and inflation are low, consumers allocate almost all of their expenditures to (the cheaper) goods being sold by firms that can adjust their prices. The Phillips curve looks like one from a world without price-setting frictions - that is, vertical.

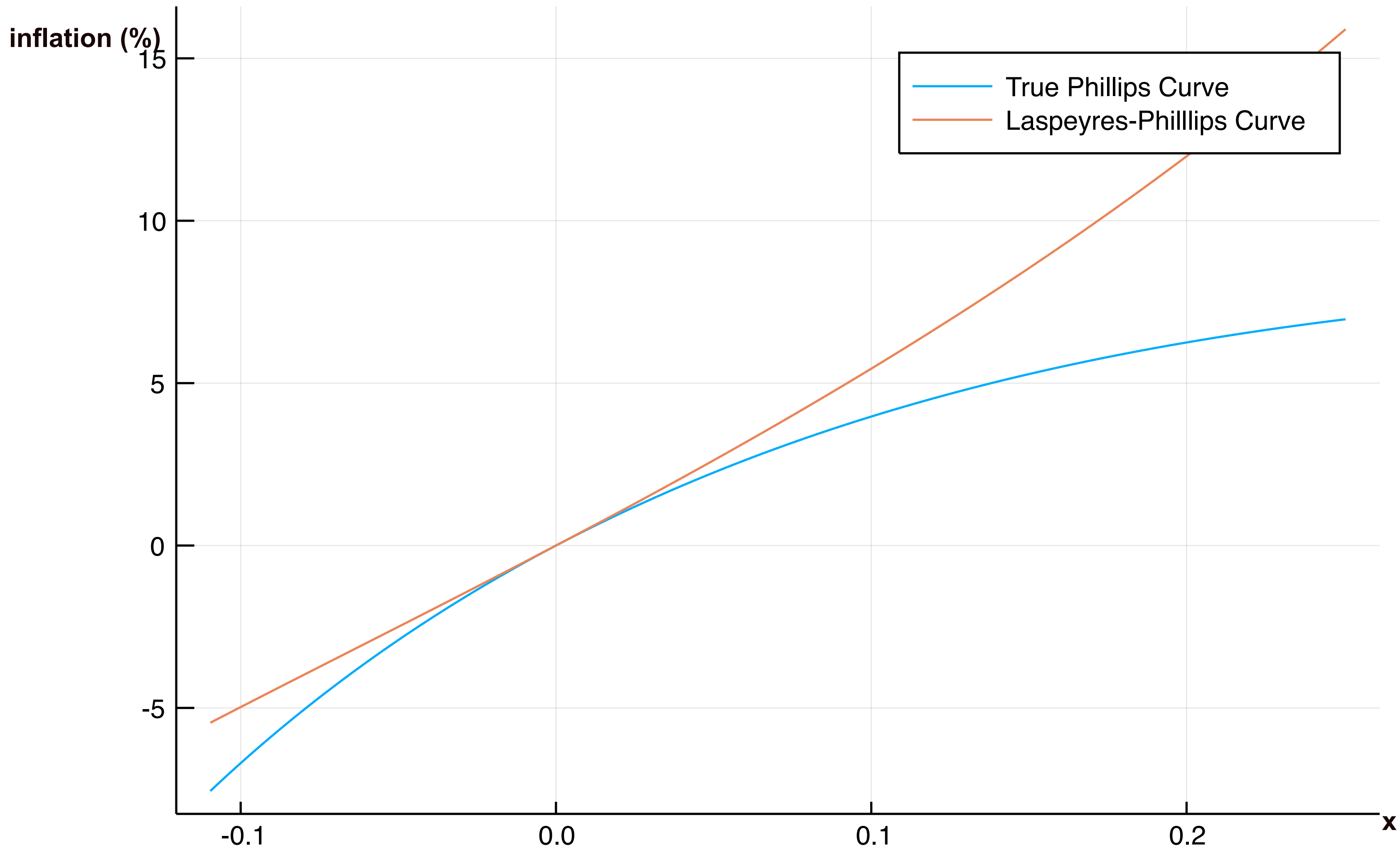
What do the data say about Figure 1 versus Figure 2? The conventional Phillips curve in Figure 1 does have some empirical support. Phillips' (1958) original estimates of the relationship between British nominal wage inflation and unemployment imply that it has the characteristics described in the opening paragraph. (Indeed, my Figure 1 was motivated in large part by Phillips' Figure 1.) Samuelson and Solow's seminal (1960) piece on the relationship between US price inflation and unemployment also describe a convex curve that is near-vertical at high levels of economic activity. More recently, Forbes et al. (2022) have argued that the Phillips curve in the US and other countries does indeed become flatter at low levels of inflation.[3] In a similar vein, Babb and Detmeister (2017) find using data from US cities that the Phillips curve is steeper when the unemployment rate is low.

The concavity in Figure 2 may seem to contravene this statistical evidence. However, there is a critical measurement issue. The estimates in the above paragraph are based on data in which inflation is primarily[4] based on the rate of growth of a Laspeyres price index. In contrast, the theoretical characterizations described above apply to what I will call the *true Phillips curve*, since it is based on the true price index. I also derive the models' implications for the *Laspeyres-Phillips curve*, in which inflation is measured (like the (unchained) Consumer Price Index in the US) using the rate of increase of a Laspeyres price index. I find that the Laspeyres-Phillips curve is typically *concavo-convex*. Like the true

[3]In a (very funny) satirical piece, Smith (2006) highlights this kind of convexity in the Phillips curve for Japan.

[4]Chained inflation is not literally based on a Laspeyres price index. But, given the costs of collecting item-specific expenditure data, the chaining in US data is at a relatively high level. It does not seem well-designed to capture substitution patterns between goods/services produced by firms which have recently changed prices and goods produced by firms which have not recently changed prices. (For details, see https://www.bls.gov/cpi/additional-resources/chained-cpi-questions-and-answers.htm#Question_4. It is worth emphasizing that chaining at a more disaggregated level may be more prone to the well-known of problem of chain drift (see, for example, Diewert (2021)).)

Figure 3: The True Phillips Curve and the Laspeyres-Phillips Curve



Unlike the true Phillips curve, the Laspeyres-Phillips Curve is convex for sufficiently large levels of real economic activity (x) and concave only for sufficiently low levels of economic activity.

Phillips curve, it is asymptotically vertical (and hence concave) for low levels of economic activity and low levels of inflation. But, as depicted in Figure 3, it is also asymptotically vertical (and hence convex) for high levels of economic activity and high levels of inflation.

Intuitively, when real economic activity is high, firms which can adjust their prices respond to high marginal costs by choosing high prices. That translates directly into high overall (Laspeyres) price inflation. This effect is not present with the true inflation rate, because households respond to the adjusting firms' high prices by substituting toward the firms with fixed prices. The paper reports a plausible numerical example in which the Laspeyres-Phillips curve is convex for any non-negative level of inflation.[5]

Thus, the substitution bias in standard measures of inflation mean that estimated Phillips curves may display convexity that is not present in the true Phillips curves. This error in estimation could lead to errors in policy. I consider a central bank without commitment responding optimally to markup shocks. I show that if average inflation is at the central bank's target and the Phillips curve is convex, then average economic activity is high compared to what would prevail in the absence of the markup shocks. But the opposite is true if the Phillips curve is concave. Thus, the sign of the curvature of the Phillips curve affects whether an effective inflation-targeting central bank should run the economy "hot" or "cold" on average.

The rest of the paper is organized as follows. In the next section, I describe the models under study and derive the short-run *marginal cost Phillips curve*, which maps the current real marginal cost of production into the current true inflation rate. In Section 3, I analyze the curvature properties of the marginal cost Phillips curve. When the households' common intraperiod elasticity of substitution is larger than 2, the marginal cost Phillips curve is

[5]To be clear, I do not see the estimates of the curvature of the Laspeyres-Phillips curve reported in the prior paragraph as definitive. Indeed, it could well be argued that the dominant message from the past forty years of research is that even the *first* derivative of the Phillips curve cannot be estimated with precision. See, for example, Mavroedis et al. (2014). Relatedly, McLeay and Tenreyro (2020) have highlighted the challenges associated with estimating the slope of the Phillips curve in the presence of an inflation-targeting central bank. Hazell et al. (2022) use state-level data to sidestep this problem but do not report estimates on the curvature of the Phillips curve.

globally concave, with a slope that nears infinity for low marginal costs and nears zero for high marginal costs. In Section 4, I add an abstract activity cost function that maps the level of economic activity into a corresponding real marginal cost. By combining this function with the marginal cost Phillips curve, I obtain the real activity Phillips curve, which maps the level of economic activity into the true inflation rate. The key result of the paper is that, if the elasticity of substitution is larger than 3 and under relatively weak conditions on the activity cost function, the real activity Phillips curve inherits the global concavity properties of the marginal cost Phillips curve.[6] In Section 5, I explore the properties of the Laspeyres-Phillips curve and show that, unlike the true Phillips curve, it is asymptotically vertical for sufficiently high levels of economic activity. Section 6 discusses policy considerations. Section 7 concludes.

All proofs are in the Appendix.[7]

2 The Marginal Cost Phillips Curve

This section defines the short-run *marginal cost Phillips curve*. The idea is that a policymaker can vary the real marginal cost of production m_t through different policy choices at date t. As long as this choice does not affect firms' beliefs about future monetary policy, there is a policy-invariant function that maps m_t into the realized inflation rate. It is this relationship that will be termed the marginal cost Phillips curve.

[6] One of these "relatively weak" conditions is differentiability. Benigno and Eggertsson (2023) present a New Keynesian model in which the Phillips curve is nonlinear in a *first-order* approximation around the steady-state. The non-differentiability emerges because of a kink in the aggregate labor supply curve driven by what they term "wage norms" that prevent wages from adjusting upward in response to excess labor demand (or under-supply). Importantly, these norms apply to all workers in all occupations. Hence, there is no way for the economy to substitute to "non-kinked" forms of labor (such as, for example, so-called travel nurses). Benigno and Eggertsson do not study the curvature properties of the Phillips curve in their model away from the steady-state.

[7] The proofs have some similarity to the analysis of Dew-Becker (2023) in a distinct context.

2.1 Households and the True Price Index

This subsection describes household preferences and the implied true price index. These elements are similar to those in Gali (2015, Chapter 3).

Time is indexed by the natural numbers. There is a large number of identical households that live forever. The households consume a unit measure of goods at each date and sell labor to firms. Their objective functions over consumption and labor processes are given by the expectation of:

$$(\sum_{t=1}^{\infty}\beta^{t-1}(ln(C_t)-v(N_t)), 0<\beta<1. \qquad (1)$$

where v is a strictly increasing and convex function. Households have Dixit-Stiglitz (1977) preferences, so that:

$$C_t \equiv (\int_0^1 c_{jt}^{1-1/\eta} dj)^{\frac{\eta}{\eta-1}}, \eta > 1$$

where c_{jt} is the consumption of good j in period t.

The household preferences imply that the true (or Konüs (1924)) price index in any period $t \in \mathbb{N}$ is defined to be:

$$P_t \equiv (\int_0^{\infty} p^{1-\eta} dF_t(p))^{\frac{1}{1-\eta}}$$

Here, F_t is the cross-firm distribution of period t prices. As above, the parameter η represents the typical household's elasticity of substitution across the various goods in the economy. The incorporation of substitution effects in the definition of P_t is critical in what follows.

2.2 Time-Dependent Firm Pricing

At each date t, each good j is produced by a monopolistically competitive firm. All firms have an output subsidy given by:

$$(\frac{1}{\eta-1})$$

which serves to correct the distortion created by monopolistic competition. The firms have identical CRS production functions that include labor as an input (but may include other factors).

In a *time-dependent* pricing model,[8] at any date t, each firm has a state that lies in the set $\{0,1\}$. A firm in state 0 is *sticky*: it is required to keep its price the same as in the prior period $(t-1)$. A firm in state 1 is *flexible*: it is allowed to choose its price optimally. The states are determined by draws that are stochastically independent across firms, and are also stochastically independent of all macroeconomic variables. In words, the probability of being flexible at a given date only depends on how much time has passed since the firm was last able to change its price. Mathematically, the probability structure is fully determined by a vector $\{q_\tau\}_{\tau=0}^{\infty} \in (0,\bar{q}]^{\infty},$ where $\bar{q} < 1$. Regardless of its history, a firm that is in state 1 in period t transits to state 0 in period $(t+1)$ with probability q_0. A firm that is in state 0 in period t and was last in state 1 in period $(t-\tau), \tau \geq 1$, transits to state 0 in period $(t+1)$ with probability q_τ. The most familiar example of this general structure, of course, is the Calvo (1983) model in which $q_\tau = q_0$ for all $\tau \geq 0$.

This specification of pricing frictions implies that at any date t, some fraction $\theta_t > 0$ of the firms are sticky and are required to keep their prices the same as in period $(t-1)$. The remaining measure $(1-\theta_t)$ of firms are free to adjust their prices as they wish. In doing so, they take into account the impact of their decisions on future profits when they cannot change their prices. At any date $(t+s)$, $s \geq 0$, a firm which charges price p_t has nominal profit given by:

$$(\frac{\eta}{\eta-1}p_t - M_{t+s}) \times (\frac{p_t}{P_{t+s}})^{-\eta} C_{t+s} \tag{2}$$

Here, M_{t+s} is the nominal marginal cost in period $(t+s)$ and C_{t+s} is per capita consumption. (Since firms' production is constant returns to scale, they have identical nominal marginal costs.) A firm at date t discounts these future profits using the nominal stochastic discount

[8] The frameworks described in this paragraph are essentially discrete-time versions of the time-dependent (TD) models in Auclert et al. (2024).

factor, which (from the logarithmic utility function (1)) takes the form:

$$(\frac{\beta^s C_t P_t}{C_{t+s} P_{t+s}})_{s=1}^{\infty}$$

Hence, at date t, a firm that can choose its price does so by solving the problem (Gali (2015, p. 56)):

$$\max_{p_t} E_t \left(\sum_{s=0}^{\infty} \lambda_s (\beta^s \frac{C_t P_t}{C_{t+s} P_{t+s}})(\frac{p_t}{P_{t+s}})^{-\eta} C_{t+s} (\frac{\eta}{\eta-1} p_t - M_{t+s}) \right)$$

In this problem, the terms $(\lambda_s)_{s=0}^{\infty}$ are defined as:

$$\lambda_s \equiv \prod_{\tau=1}^{s} q_{\tau-1}, s \geq 1$$

$$\lambda_0 = 1$$

and so λ_s represents the probability that the firm is required to charge price p_t through period $(t+s)$. The firm's solution to this problem is given by the first order condition:

$$(\eta-1)\frac{\eta}{\eta-1}(p_t^*)^{-\eta} \sum_{s=0}^{\infty} \lambda_s (\beta^s \frac{C_t P_t}{C_{t+s} P_{t+s}})(\frac{1}{P_{t+s}})^{-\eta} C_{t+s}$$
$$= \eta (p_t^*)^{-\eta-1} \sum_{s=0}^{\infty} \lambda_s (\beta^s \frac{C_t P_t}{C_{t+s} P_{t+s}})(\frac{1}{P_{t+s}})^{-\eta} C_{t+s} M_{t+s}$$

which implies in turn that:

$$p_t^* = \frac{E_t \sum_{s=0}^{\infty} \beta^s \lambda_s (\frac{1}{P_{t+s}})^{-\eta} \frac{M_{t+s}}{P_{t+s}}}{E_t \sum_{s=0}^{\infty} \beta^s \lambda_s (\frac{1}{P_{t+s}})^{-\eta} \frac{1}{P_{t+s}}}. \quad (3)$$

Note the cancellation of consumptions implied by log utility.

All firms that can adjust their prices in period t choose the same price p_t^* defined in (3). Define:

$$m_{t+s} = \frac{M_{t+s}}{P_{t+s}}, s \geq 0$$

to be the *real* marginal cost in the current date and all future dates. Multiplying the numerator and denominator by $P_t^{1-\eta}$, we can rewrite (3) as:

$$p_t^* = \frac{m_t P_t + P_t E_t \sum_{s=1}^{\infty} \lambda_s \beta^s (\frac{P_t}{P_{t+s}})^{-\eta} m_{t+s}}{E_t \sum_{s=0}^{\infty} \lambda_s \beta^s (\frac{P_t}{P_{t+s}})^{-\eta} \frac{P_t}{P_{t+s}}} \tag{4}$$

I make the following assumption that serves to create a well-defined notion of a short-run Phillips curve in period t. .

Assumption 1: The central bank can vary m_t without affecting the period t price-setting firms' identical beliefs about the future joint evolution of real marginal costs and inflation:

$$(m_{t+s}, \frac{P_{t+s}}{P_t})_{s=1}^{\infty}$$

The premise of Assumption 1 is that the joint process $(m_{t+s}, \frac{P_{t+s}}{P_t})_{s=1}^{\infty}$ is determined by the central bank's future monetary policy strategy in conjunction with a variety of exogenous shocks. With that in mind, Assumption 1 asserts that the firms' beliefs about the central bank's future reaction function are not affected by its current setting of m_t.

Given Assumption 1, we can rewrite (4) as:

$$p_t^* = P_t(\alpha_t m_t + \chi_t)$$

where:

$$\alpha_t = \frac{1}{E_t \sum_{s=0}^{\infty} \lambda_s \beta^s (\frac{P_t}{P_{t+s}})^{-\eta} \frac{P_t}{P_{t+s}}}$$
$$\chi_t = \frac{E_t \sum_{s=1}^{\infty} \lambda_s \beta^s (\frac{P_t}{P_{t+s}})^{-\eta} m_{t+s}}{E_t \sum_{s=0}^{\infty} \lambda_s \beta^s (\frac{P_t}{P_{t+s}})^{-\eta} \frac{P_t}{P_{t+s}}}.$$

Note that (α_t, χ_t) are potentially state-dependent, as they may depend on any information that help firms predict future inflation and real marginal cost.

2.3 Deriving the Marginal Cost Phillips Curve

In this subsection, we use the price index to derive a relationship, called the marginal cost Phillips curve, between m_t and inflation in period t. The price index in any period t was defined as:

$$P_t = (\int_0^\infty p^{1-\eta} dF_t(p))^{\frac{1}{1-\eta}}.$$

where F_t is the distribution of prices across firms in period t. We can split the integral into two pieces:

$$P_t^{1-\eta} = \theta_t \int_0^\infty p^{1-\eta} dF_t^{sticky}(p) + (1-\theta_t) \int_0^\infty p^{1-\eta} dF_t^{flex}(p).$$

Here, F_t^{sticky} is the distribution of prices across the firms which cannot change their prices and F_t^{flex} is the distribution of prices across firms that can change their prices. It is helpful to define the (gross) inflation rate for sticky firms as:

$$\bar{\pi}_t \equiv \frac{(\int_0^\infty p^{1-\eta} dF_t^{sticky}(p))^{1/(1-\eta)}}{P_{t-1}}.$$

In the Calvo model, $F_t^{sticky} = F_{t-1}$ and $\bar{\pi}_t = 1$. More generally, though, $\bar{\pi}_t$ may be larger (smaller) than one if the firms that are sticky in period t were ones that charged high (low) prices in period $(t-1)$.

Assumption 1 implies that F_t^{flex} is a point mass on $P_t(\alpha_t m_t + \chi_t)$ and that:

$$\int_0^\infty p^{1-\eta} dF_t^{flex}(p)$$
$$= P_t^{1-\eta}(\alpha_t m_t + \chi_t)^{1-\eta}.$$

where m_t is defined as in the prior subsection to be the real marginal cost in period t. It follows that we can rewrite the price index as:

$$P_t^{1-\eta} = \theta_t \bar{\pi}_t^{1-\eta} P_{t-1}^{1-\eta} + (1-\theta_t) P_t^{1-\eta} (\alpha_t m_t + \chi_t)^{1-\eta} \tag{5}$$

Let the gross inflation rate be defined as:

$$\pi_t \equiv (\frac{P_t}{P_{t-1}})$$

Dividing through (5) by $P_{t-1}^{1-\eta}$, we get:

$$\pi_t^{1-\eta} = \theta_t \bar{\pi}_t^{1-\eta} + (1-\theta) \pi_t^{1-\eta} (\alpha_t m_t + \chi_t)^{1-\eta}$$

We can then re-arrange to obtain the *marginal cost Phillips curve.*

$$\pi_t = (\frac{1 - (1-\theta_t)(\alpha_t m_t + \chi_t)^{1-\eta}}{\theta_t})^{1/(\eta-1)} \bar{\pi}_t \tag{6}$$

The marginal cost Phillips curve (9) is shaped by firm beliefs about future real marginal cost and inflation, as encoded in the variables (α_t, χ_t). The expression for the curve can be simplified under the following natural specification of firm beliefs. Suppose that firms believe that, after date t, the central bank will pursue a monetary policy under which both gross inflation and real marginal costs are equal to one:

$$P_{t+s}/P_{t+s-1} = 1, s \geq 1$$
$$m_{t+s} = 1, s \geq 1.$$

Given these *optimal monetary policy beliefs*, then the marginal cost Phillips curve is timeinvariant. The variables (α_t, χ_t) are constants given by:

$$\alpha_t = \bar{\alpha} = \frac{1}{\sum_{s=0}^{\infty} \beta^s \lambda_s}$$
$$\chi_t = (1 - \bar{\alpha})$$

and the curve[9] is:

$$\pi_t = (\frac{1 - (1 - \theta_t)(\bar{\alpha} m_t + (1 - \bar{\alpha}))^{1-\eta}}{\theta_t})^{1/(\eta-1)} \bar{\pi}_t \tag{7}$$

Note that under optimal monetary policy beliefs, $\pi_t = 1$ if and only if $m_t = 1$.

2.4 Why the Log Utility Assumption Matters

This subsection discusses the impact of assuming logarithmic intertemporal preferences.

Suppose the nominal stochastic discount factor process took the the more general form:

$$\frac{P_t C_t^{\sigma}}{P_{t+s} C_{t+s}^{\sigma}}, \sigma > 0.$$

Then, a firm that can choose its price at date t will set p_t^* so as to satisfy:

$$p_t^* = \frac{m_t P_t C_t^{1-\sigma} + P_t E_t \sum_{s=1}^{\infty} \lambda_s \beta^s (\frac{P_t}{P_{t+s}})^{-\eta} C_{t+s}^{1-\sigma} m_{t+s}}{C_t^{1-\sigma} + E_t \sum_{s=1}^{\infty} \lambda_s \beta^s (\frac{P_t}{P_{t+s}})^{-\eta} \frac{P_t}{P_{t+s}} C_{t+s}^{1-\sigma}}. \tag{8}$$

Here, we could extend Assumption 1 by assuming that the central bank can vary m_t without affecting the firms' beliefs about the joint process $(C_{t+s}, m_{t+s}, \frac{P_{t+s}}{P_t})_{s=1}^{\infty}$. This assumption

[9]In the Calvo model, $\lambda_s = \theta^s$ and $\bar{\alpha} = (1 - \beta\theta)$. The curve (7) becomes:

$$\pi = (\frac{1 - (1 - \theta)((1 - \beta\theta)m + \beta\theta)^{1-\eta}}{\theta})^{1/(\eta-1)}.$$

Define $\hat{\pi} = ln(\pi)$ and $\hat{m} = ln(m)$. In logs, we can rewrite as:

$$\hat{\pi} = \frac{1}{\eta - 1} ln(\frac{1 - (1 - \theta)((1 - \beta\theta)exp(\hat{m}) + \beta\theta)^{1-\eta}}{\theta}).$$

The derivative $d\hat{\pi}/d\hat{m} = (1 - \beta\theta)(1 - \theta)/\theta$, which (not surprisingly) is the same as the slope coefficient on current real marginal cost in the textbook version of the log-linearized New Keynesian Phillips Curve (Gali (2015, Chapter 3)).

would imply that the terms:

$$E_t \sum_{s=1}^{\infty} \beta^s \lambda_s (\frac{P_t}{P_{t+s}})^{-\eta} C_{t+s}^{1-\sigma} m_{t+s}$$
$$E_t \sum_{s=1}^{\infty} \beta^s \lambda_s (\frac{P_t}{P_{t+s}})^{-\eta} \frac{P_t}{P_{t+s}} C_{t+s}^{1-\sigma}$$

in (8) are independent of m_t. However, different choices of m_t by the central bank will necessarily impact the level of current economic activity and so change $C_t^{\sigma-1}$ if $\sigma \neq 1$.

What is the intuition behind the term $C_t^{1-\sigma}$? Suppose the central bank lowers interest rates to raise m_t. Then the firm's price choice will:

- put less weight on current m_t because future profits are discounted less (through higher $C_t^{-\sigma}$).

- put more weight on current m_t because overall period t demand is lower (through lower C_t).

The term $C_t^{1-\sigma}$ combines these two effects. They are exactly offsetting when $\sigma = 1$ (the log utility case).

3 Curvature of the Marginal Cost Phillips Curve

This section develops the curvature properties of the marginal cost Phillips curve derived in the prior section:

$$\Gamma(m) \equiv (\frac{1 - (1-\theta)(\alpha m + \chi)^{1-\eta}}{\theta})^{\frac{1}{\eta-1}} \bar{\pi} \tag{9}$$

Note that there are no time subscripts in (9). I drop these in the remainder of this section as the past and future plays no (further) role in the analysis.

Throughout, I impose the following restriction on (θ, η, χ) :

$$(1-\theta)\chi^{1-\eta} > 1. \tag{10}$$

This restriction in turn implies that the marginal cost Phillips curve is only well-defined if m is bounded from below by m_{LB} :

$$m_{LB} \equiv \frac{(1-\theta)^{\frac{1}{\eta-1}} - \chi}{\alpha}.$$

The condition (10) requires that prices are sufficiently flexible (as that translates into lower θ and lower χ). By creating a positive lower bound on m, it simplifies the analysis of the left tail of the Phillips curve. But the condition plays no role in the results for high values of m.

3.1 High and Low Levels of Real Marginal Cost

The first two propositions concern the properties of the marginal cost Phillips curve when real marginal cost is near its lowest possible level and when real marginal cost is very high. They establish that, in these contexts, the marginal cost Phillips curve is "very" concave, in the sense that its slope is infinite when (gross) inflation is zero and its slope is near zero when inflation is near its maximal level.

Proposition 1. *If $\eta > 2$, the slope of the marginal cost Phillips curve approaches infinity as m nears m_{LB}.*

If the real marginal cost is low, then the firms which are flexible in period t set relatively low prices compared to the firms that cannot adjust in period $(t-1)$. In the extreme, consumers buy goods only from the firms that are adjusting their prices. The economy acts as if prices are fully flexible and so the marginal cost Phillips curve is vertical.

The next proposition shows that the marginal cost Phillips curve has a finite horizontal asymptote for large values of m. Unlike Proposition 1, it does not require η to be larger than 2.

Proposition 2. *As m nears infinity, then $\Gamma(m)$ approaches an upper bound:*

$$\pi^{max} \equiv (\frac{1}{\theta})^{1/(\eta-1)}\bar{\pi}.$$

and its slope nears zero:

$$lim_{m \to \infty} \Gamma'(m) = 0$$

When real marginal costs are high, the adjusting firms' prices are high relative to those of the firms with fixed prices. The consumers switch to buying from the latter, and the economy behaves as if all prices are fixed. As a result, inflation is independent of cost conditions, and the Phillips curve becomes horizontal.

3.2 Global Properties

We have seen that the marginal cost Phillips curve is increasing and concave when the real marginal cost is near its lowest level and when it is near infinity. The following proposition proves that, if $\eta \geq 2$, these properties are valid for the entire domain of the curve.

Proposition 3. *The marginal cost Phillips curve is strictly increasing and, if* $\eta \geq 2$, *strictly concave over its domain* (m_{LB}, ∞).

As m rises, the firms with the ability to adjust their prices make higher choices and so inflation rises. But this effect is dampened by their loss of market share to the monopolists with fixed prices. If $\eta \geq 2$, the loss of market share is sufficiently large to ensure that the curve is globally concave.

4 Curvature of the Real Activity Phillips Curve

The previous section describes the curvature properties of the marginal cost Phillips curve. However, it is more typical to think of the Phillips curve as describing a relationship between measures of *real economic activity* and inflation. This section uses the notion of a policy-invariant *activity cost function* to translate the results about the real marginal cost Phillips curve into characterizations of the *real activity Phillips curve.*

4.1 Building Up the Real Activity Phillips Curve

Let x_t be a measure of real activity in period t that (like an employment gap) takes on values over the entire real line. There is an *activity cost* function:

$$\Phi_t : \mathbb{R} \to (0, \infty)$$

that maps real activity x_t into real marginal cost m_t. I make the following exogeneity assumption about Φ_t :

Assumption 2: It is possible for policymakers to make choices that vary x_t without affecting the activity cost function Φ_t, or firm beliefs about the joint behavior of future real marginal costs and inflation.

Mathematically, I assume the activity cost function Φ_t is twice differentiable over its domain and that its derivative is everywhere positive (although it can become vanishingly small for low or high values of x_t). I assume too that Φ_t is onto, meaning that for any $m_t \in (0, \infty)$, there exists $x_t \in \mathbb{R}$ such that $m_t = \Phi_t(x_t)$. There are no restrictions on the second derivative of Φ_t - the activity cost function can be convex or concave.

Here's one example of how Φ_t works. Consider an economy in which labor markets are competitive. Suppose each firm has a single-input production function that translates n_t units of labor into n_t units of output, for any $n_t \geq 0$. Suppose too that households' disutility of labor takes the form:

$$v(N) = \frac{N^{\gamma+1}}{1+\gamma}, \gamma > 0$$

Then household optimality implies that:

$$W_t/P_t = N_t^{\gamma} C_t.$$

Given this condition, real marginal cost satisfies:

$$m_t = \frac{W_t/P_t}{Y_t/N_t} = N_t^{1+\gamma}\frac{C_t}{Y_t} = N_t^{1+\gamma}.$$

where the last step makes use of market-clearing and the assumption that labor is the sole input. If we let $n_t = ln(N_t)$, then:

$$m_t = \bar{\Phi}(n_t), \text{ where } \bar{\Phi}(n_t) \equiv exp((\gamma+1)n_t).$$

The function $\bar{\Phi}$ satisfies the conditions on Φ_t described above. Note that its slope is unbounded from above and is arbitrarily near zero for large negative values of n.

I next define the real activity Phillips curve PC. (As in Section 3, I suppress the time subscripts in the remainder of this section as inessential.) Recall that the real marginal cost cannot fall below:

$$m_{LB} \equiv \frac{(1-\theta)^{1/(\eta-1)} - \chi}{\alpha} > 0.$$

Define:

$$x_{LB} \equiv \Phi^{-1}(m_{LB})$$

to be the level of economic activity associated with the lowest real marginal cost. (Note that x_{LB} is well-defined because Φ is onto.)

Then, the real activity Phillips curve is defined as the composition of the activity cost function Φ and the marginal cost Phillips curve:

$$PC : (x_{LB}, \infty) \to \mathbb{R}_+$$
$$PC(x) = \Gamma(\Phi(x))$$

The idea here is that Φ maps a level of economic activity x into real marginal cost m, and that translates into inflation via Γ. As noted above, real activity is meant to proxy for the (log of the) host of possible quantity variables that are typically used in Phillips curves. The goal in what follows is to learn when (that is, under what condition on Φ) the prior results about Γ translate into similar characterizations about PC.

4.2 Low and High Levels of Economic Activity

This section uses Propositions 1 and 2 to show that, regardless of the curvature of the real activity function Φ, the real activity Phillips curve is "very" concave at high and low levels of economic activity.

Proposition 4 proves that, under the conditions used in Proposition 1, there is a lower bound on real activity and the Phillips curve is close to vertical when real activity is near that lower bound.

Proposition 4. *Suppose $\eta > 2$. The real activity Phillips curve PC satisfies $PC(x_{LB}) = 0$ and the derivative of the Phillips curve approaches infinity as x nears x_{LB}.*

What about for high levels of economic activity? Regardless of how convex Φ is, the real activity Phillips curve is asymptotically horizontal when x is large. Like Proposition 2, it does not require η to be larger than 2.

Proposition 5. *Suppose that $lim_{x\to\infty}\Phi'(x)$ exists as an element of the extended reals (so it is possibly infinite). Then:*

$$lim_{x\to\infty}PC(x) = \pi^{max}.$$
$$lim_{x\to\infty}PC'(x) = 0$$

Thus, like the marginal cost Phillips curve, the real activity Phillips curve is nearly vertical for low levels of economic activity and nearly horizontal for high levels of economic activity.

4.3 Global Curvature Properties

The real activity Phillips curve is strictly increasing, because it is the composition of two strictly increasing functions. The following proposition provides sufficient conditions on the activity cost function Φ for the real activity Phillips curve PC to inherit the concavity of the marginal cost Phillips curve Γ.

Proposition 6. *Suppose $\eta > 3$ and that:*

$$\frac{\chi}{(1-\theta)^{1/(\eta-1)}} < \left(\frac{\eta-1}{\eta-2}\right)\left(\frac{\eta-3}{\eta-2}\right)^{\frac{\eta-2}{\eta-1}} \tag{11}$$

Suppose too that:

$$\frac{\Phi''(x)}{\Phi'(x)^2} \leq \frac{1}{\Phi(x)}. \tag{12}$$

for all $x \in (0, \infty)$. Then the real activity Phillips curve PC is strictly concave over its domain (x_{LB}, ∞).

Recall that earlier we imposed the restriction (10) on (χ, η, θ),which implies that:

$$\frac{\chi}{(1-\theta)^{1/(\eta-1)}} < 1.$$

The following Table shows that the condition (11) on (η, χ, θ) in Proposition 6 is only a modest tightening of (10) if $\eta \geq 3.5$.

Table 1: The Condition in Proposition 6

η	RHS of (11)
3.5	0.862
4	0.945
5	0.984
6	0.993
7	0.996

The condition in Proposition 6 on the function Φ is satisfied by a wide range of (highly) convex functions, including the following.

Corollary 1. *Suppose* $\Phi(x) = Dexp((\gamma+1)x), \gamma > 0, D > 0$. *Then:*

$$\frac{\Phi''(x)}{\Phi'(x)^2} = \frac{1}{\Phi(x)}.$$

5 The Laspeyres-Phillips Curve

The argument in the last two sections relies on the impact of substitution effects within the price index. But those effects are imperfectly measured in the official data. This section considers the curvature of a (real activity) *Laspeyres-Phillips* curve. For this curve, the inflation measure is constructed using a Laspeyres price index that systematically understates substitution effects. Unlike the true real activity Phillips curve, the real activity Laspeyres-Phillips curve is near-vertical for high levels of economic activity.

Throughout this section, I restrict attention to the Calvo case in which $q_\tau = \theta \in (0,1)$ for all $\tau \geq 0$. The key result in this section (Proposition 7) can be established without this restriction. However, doing so complicates the notation without adding much in the way of insight.

5.1 Laspeyres Inflation

As in Section 2, we let F_τ be the cross-firm distribution of prices in period τ, where $\tau \in \{t-1, t\}$. We define the Laspeyres inflation rate using period $(t-1)$ quantities as the relevant bundle. Hence, the period $(t-1)$ Laspeyres price index is the same as the true price index in period $(t-1)$:

$$
\begin{aligned}
P_{t-1}^L &= (\int_0^\infty (\frac{p}{P_{t-1}})^{-\eta} p dF_{t-1}(p)) \\
&= \frac{\int_0^\infty p^{1-\eta} dF_{t-1}(p)}{P_{t-1}^{-\eta}} \\
&= P_{t-1}.
\end{aligned}
$$

As in Section 2, and under the Calvo assumption imposed above, a measure θ of the firms are sticky in period t and the remaining measure $(1-\theta)$ are flexible. The sticky firms set their prices to be the same as in the prior period, and the remaining fraction $(1-\theta)$ of firms in period t set their prices optimally to be equal to:

$$
\alpha_t M_t + \chi_t P_t
$$

where P_t represents the true price index in period t, not the Laspeyres price index. The independence embedded in the Calvo assumption means that the distribution of period $(t-1)$ prices over the the period t sticky (or period t flexible) firms is given by F_{t-1}. Hence, the Laspeyres price index at date t takes the form:

$$
\begin{aligned}
P_t^L &= \theta \int_0^\infty \frac{p^{1-\eta}}{P_{t-1}^{-\eta}} dF_{t-1}(p) + (1-\theta)(\alpha_t M_t + \chi_t P_t) \int_0^\infty (\frac{p}{P_{t-1}})^{-\eta} dF_{t-1}(p) \\
&= \theta P_{t-1} + (1-\theta)(\alpha_t M_t + \chi_t P_t)
\end{aligned}
$$

It follows that. since $P_{t-1}^L = P_{t-1}$, we can define the (gross) Laspeyres inflation rate as:

$$
\pi_t^L \equiv \frac{P_t^L}{P_{t-1}^L} = \theta + (1-\theta)\frac{\alpha_t M_t + \chi_t P_t}{P_{t-1}} \tag{13}
$$

We can rewrite the last term as:

$$(\alpha_t m_t + \chi_t)\frac{P_t}{P_{t-1}}$$

Then the Laspeyres inflation rate π_t^L satisfies:

$$\pi_t^L = \theta + (1-\theta)(\alpha_t m_t + \chi_t)\pi_t \tag{14}$$

As in Section 3, the variable m_t is the real marginal cost in period t, and the term π_t is the *true* (gross) rate of inflation.

5.2 The Laspeyres-Phillips Curve

In this subsection, I provide a formal definition and characterization of the Laspeyres-Phillips Curve. Throughout the remainder of Section 5, I drop the time subscripts as they are inessential.

From Section 3, we know that the true inflation rate π satisfies:

$$\pi = \Gamma(m)$$

where:

$$\Gamma(m) = (\frac{(1-(1-\theta)(\alpha m + \chi)^{1-\eta})}{\theta})^{\frac{1}{\eta-1}}$$

is the marginal cost Phillips curve. (Here, we exploit the implication of the Calvo model that $\bar{\pi} = 1$.) It follows that the Laspeyres inflation rate can be written as:

$$\pi^L = \Gamma^L(m) \equiv \theta + (1-\theta)\Gamma(m)(\alpha m + \chi)$$

Intuitively, the gross Laspeyres inflation rate is defined in (14) as a weighted average of 1 (the inflation rate for goods with fixed prices) and the inflation rate for firms that can adjust their prices. The weights are, by definition, fixed. Hence, the substitution bias in Laspeyres inflation implies that it behaves like a product of real marginal cost and true inflation. When m_t is near its lower bound, this product is highly sensitive to m_t (just like true inflation). However, when real marginal cost is large, the product is a linear function of m_t. This linearity means that, unlike the true inflation rate, the Laspeyres inflation rate is unbounded from above.

We now turn to the construction of a real activity Phillips curve defined in terms of the Laspeyres inflation rate. Because the real marginal cost m is defined in terms of the true (not Laspeyres) price index, the same policy-invariant activity cost function Φ as in Section 4 maps real economic activity x into real marginal cost m. Then, we define the Laspeyres-Phillips curve as the composition of Γ^L and Φ:

$$PC^L : (x_{LB}, \infty) \rightarrow (0, \infty)$$
$$PC^L(x) = \Gamma^L(\Phi(x))$$

where:

$$x_{LB} \equiv \Phi^{-1}(m_{LB}).$$

The domain of PC^L is the same as the domain of the true real activity Phillips curve PC. What differs is how the Laspeyres-Phillips curve PC^L maps real economic activity into an inflation rate with substitution bias.

The following Proposition shows that substitution bias in the Laspeyres-Phillips curve is spuriously convex for high levels of economic activity, even when the true Phillips curve is not.

Proposition 7. *Suppose that:*

$$lim_{x \rightarrow \infty} \Phi'(x) = \infty.$$

Then, the Laspeyres-Phillips curve PC^L is near-vertical when economic activity is high.

$$lim_{x\to\infty} PC^{L\prime}(x) = \infty.$$

Suppose that $\eta > 2$. Then the Laspeyres-Phillips curve PC^L is near-vertical when economic activity is low:

$$lim_{x\to x_{LB}} PC^{L\prime}(x) = \infty.$$

In combination with Proposition 6, Corollary 1 shows that the true Phillips curve PC is globally concave for any Φ such that:

$$\Phi(x) = Dexp((\gamma + 1)x)$$

In contrast, Proposition 7 implies that for any such Φ, the (mismeasured) Laspeyres-Phillips curve PC^L is necessarily (highly) convex at high levels of economic activity.

5.3 Numerical Example

The following numerical example illustrate the differences between the true (real activity) Phillips curve and the mismeasured Laspeyres-Phillips curve.

Consider a parameterization:

$$\beta = 0.99$$

$$\theta = 2/3$$

$$\eta = 6$$

$$\Phi(x) = exp(3x)$$

The setting of η corresponds to a markup of 20%. If we interpret $exp(x)$ as labor input, then the specification of Φ translates (through the example in Section 5.1) into a Frisch elasticity

$(1/\gamma)$ equal to 1/2. The setting for β is standard, given that a period is interpreted as one quarter. The specification of θ implies that prices stay fixed for an average of three periods, which is again standard for a quarterly model.

Assume that firms have optimal monetary policy beliefs (as defined in Section 2.3). Then:

$$\alpha_t = (1 - \beta\theta) = 0.34$$
$$\chi_t = \beta\theta = 0.66.$$

Note that this parameterization satisfies (10):

$$(1 - \theta)^{1/(\eta-1)} = 0.8 > 0.66 = \chi_t.$$

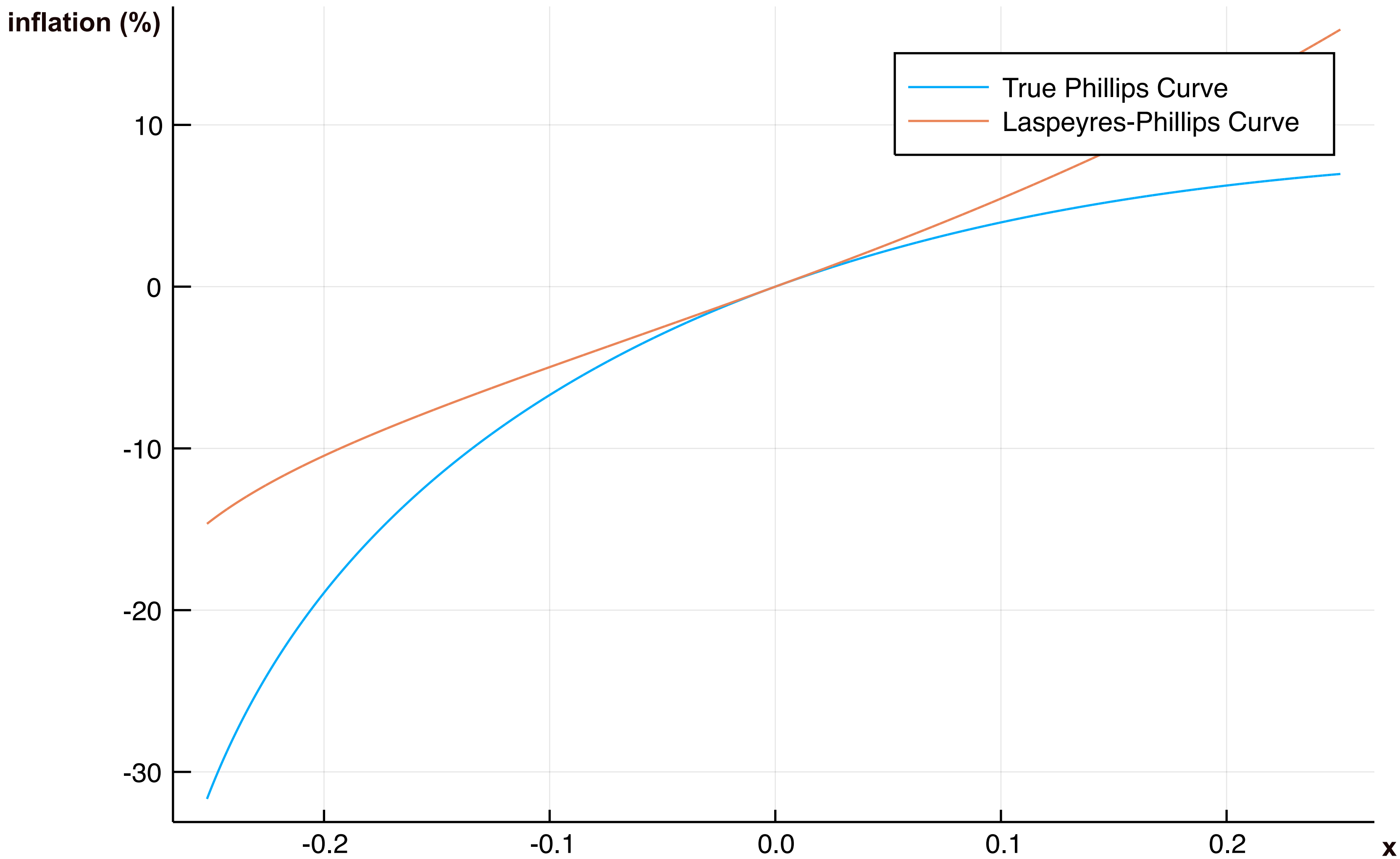
Figure 4 shows the difference between the true Phillips curve and the Laspeyres-Phillips curve for this parameterization. The true Phillips curve is concave over its full domain. The Laspeyres-Phillips curve is convex whenever x is above -0.1 and inflation is above -5%, before switching to being concave for lower values of real activity and inflation. It should not be surprising, given Figure 4, that estimated Laspeyres-Phillips curves are (slightly) convex even though the true Phillips curve is concave.

6 Policy Considerations

This section considers a central bank that looks to trade off economic activity and inflation given markup shocks. It shows that, in this context, the concavity of the true Phillips curve has an important policy implication: if average inflation equals the central bank's target, the activity gap is negative on average.

Consider a central bank which has a policy instrument with which to vary the level of economic activity x. Suppose too that the price-setting firms are affected by shocks to their

Figure 4: The True Phillips and Laspeyres-Phillips Curves



The true Phillips and Laspeyres-Phillips curves for the parameterization described in the text.

markups, so that inflation in any period is given by:

$$\pi = PC(x + \nu).$$

Here, ν can be positive or negative. The central bank has an inflation target π^{TAR}. This inflation target in turn implies a target level $x^{TAR} = PC^{-1}(\pi^{TAR})$ for economic activity associated with the instance in which $\nu = 0$.

I assume that the central bank is unable to commit to future policy choices and so, as in Gali (2015, p. 129), it solves a static problem. Specifically, it seeks to find a level x of economic activity, conditional on its observation of ν, so as to minimize the quadratic objective[10]:

$$(\pi - \pi^{TAR})^2 + \Lambda(x - x^{TAR})^2 \tag{15}$$

where $\Lambda > 0$. Given any ν, the central bank chooses $x^*(\nu)$ so as to satisfy the condition:

$$(\pi^*(\nu) - \pi^{TAR}) = \frac{\Lambda(x^{TAR} - x^*(\nu))}{PC'(x^*(\nu) + \nu)},$$

where $\pi^*(\nu) = PC(x^*(\nu) + \nu)$. Notice that the central bank's optimality condition implies that inflation is above target if and only if there is a negative activity gap (Qvigstad (2006)).

So far, I have imposed no probability structure on the ν's. However, the following proposition considers an arbitrary sequence $(\nu_1, ..., \nu_N)$ such that the implied average inflation is at target:

$$N^{-1} \sum_{n=1}^{N} \pi^*(\nu_n) = \pi^{TAR}.$$

It shows that for any such sequence, the average activity gap $(x^* - x^{TAR})$ is negative (positive) if PC is concave (convex).

Proposition 8. *Consider* $(\nu_1, ..., \nu_N)$ *such that* $(\pi^*(\nu_n))_{n=1}^N$ *has at least two distinct values,*

[10]Woodford (2001, p. 22-23) provides an explicit theoretical foundation for the quadratic objective (15). It is important to note that his second-order argument is a justification for targeting true inflation, not Laspeyres inflation.

$(\pi^*(\nu_n), x^*(\nu_n))_{n=1}^N$ *satisfies the central bank's optimality condition:*

$$(\pi^*(\nu_n) - \pi^{TAR}) = \frac{\Lambda(x^{TAR} - x^*(\nu_n))}{PC'(x^*(\nu_n) + \nu_n)}, n = 1, ..., N$$

and the average inflation rate is equal to target:

$$N^{-1} \sum_{n=1}^{N} \pi^*(\nu_n) = \pi^{TAR}.$$

Then $N^{-1} \sum_{n=1}^{N} (x^*(\nu_n) - x^{TAR}) > 0$ *if* $PC'' > 0$ *and* $N^{-1} \sum_{n=1}^{N} (x^*(\nu_n) - x^{TAR}) < 0$ *if* $PC'' < 0$.

The intuition behind the proposition is simple. When the Phillips curve is concave, it is expensive in inflation terms to vary x when it is low and cheap to vary x when it is high. As a consequence, the central bank is willing to tolerate large negative activity gaps, and so x is low on average relative to x^{TAR}. A policymaker who thought that the Phillips curve is convex would run the argument in reverse to reach the opposite conclusion and seek to keep x high on average relative to x^{TAR}.

The above proposition considers a central bank that is responding optimally over time to random shifts in the Phillips curve due to markup shocks. Yellen and Akerlof (2007) instead focus on a central bank facing a fixed Phillips curve. They argue that if the Phillips curve is convex, then reducing the *volatility* of real economic activity, while keeping *average* inflation at target, leads to an increase in the average level of real economic activity. If the Phillips curve is concave, then their argument is reversed: reducing the volatility of real economic activity, while keeping average inflation at target, leads to an *increase* in average real economic activity. However, it can be shown that this benefit in terms of a higher average does not outweigh the cost of a higher variance. Put another way, given a fixed Phillips curve, the central bank with a quadratic objective (15) should seek to eliminate volatility in both inflation and economic activity, regardless of whether the curve is concave or convex.

7 Conclusions

The main contribution of this paper is theoretical. It demonstrates that, from the perspective of time-dependent pricing models, Figure 2 is the correct way to draw the Phillips curve and Figure 1 is not. The paper also shows that substitution biases in official measures of inflation could lead the estimated Phillips curve to look more like Figure 1 rather than the correct Figure 2.

There are a number of ways to build on the results in this paper. From a theoretical perspective, it would seem useful to consider to what extent the results carry over to settings with real rigidities and state-dependent pricing. For real rigidities, the analysis is likely to depend on the specification of higher derivatives of how firm markups respond to their relative demands. For state-dependent pricing, the exact structure of firm-specific idiosyncratic shocks is likely to matter (as Golosov and Lucas (2007) and Midrigan (2011), among others, have shown for the first derivative of the Phillips curve).[11]. From a data perspective, the paper suggests that it may be important to build up model-consistent measures of inflation from (even more) disaggregated data on prices, and use those model-consistent measures as the basis of analysis and policy.

[11] In a recent paper, Blanco, Boar, Jones, and Midrigan (2024) report that the Phillips curve is convex in a parameterized version of a sophisticated menu cost multi-product model. Importantly, though, they use a Laspeyres measure of inflation, rather the one based on the true price index. The substitution effects that I emphasize would tend to concavify the Phillips curve in their model relative to, for example, what they plot on page 38.

Appendix

This appendix contains the proofs of the various propositions.

Proof of Proposition 1

The marginal cost Phillips curve is defined as:

$$\Gamma(m) = (\frac{1-(1-\theta)(\alpha m+\chi)^{1-\eta}}{\theta})^{\frac{1}{\eta-1}}.$$

This is only well-defined if the argument in the parentheses is non-negative so that:

$$(\alpha m+\chi)^{1-\eta} \leq (1-\theta)$$

which immediately implies that:

$$m \geq m_{LB}.$$

The derivative of Γ is:

$$(\frac{1-(1-\theta)(\alpha m+\chi)^{1-\eta})}{\theta})^{1/(\eta-1)-1}\frac{(1-\theta)}{\theta}(\alpha m+\chi)^{-\eta}\alpha$$
$$= \Gamma(m)^{2-\eta}\frac{(1-\theta)}{\theta}\alpha(\alpha m+\chi)^{-\eta}.$$

Since $\eta > 2$, then $lim_{m\to m_{LB}}\Gamma'(m) = \infty$ because $\Gamma(m_{LB}) = 0$.

Proof of Proposition 2

The formula for $\Gamma(m)$ is:

$$\Gamma(m) = (\frac{1-(1-\theta)(\alpha m+\chi)^{1-\eta}}{\theta})^{1/(\eta-1)}\bar{\pi}$$
$$\leq (\frac{1}{\theta})^{1/(\eta-1)}\bar{\pi} = \pi^{max}.$$

It is straightforward to verify that since $\eta > 1$:

$$lim_{m\to\infty}\Gamma(m) = \pi^{max}.$$

The derivative Γ' can be calculated as in the proof of Proposition 1 to be:

$$\Gamma'(m) = \frac{(1-\theta)}{\theta}\Gamma(m)^{2-\eta}\alpha(\alpha m+\chi)^{-\eta}\bar{\pi}^{\eta-1}$$

Hence:

$$lim_{m\to\infty}\Gamma'(m) = 0.$$

Proof of Proposition 3

As in the proof of Proposition 1, the derivative of Γ is given by:

$$\Gamma'(m) = \Gamma(m)^{2-\eta}\frac{(1-\theta)}{\theta}\alpha(\alpha m+\chi)^{-\eta}\bar{\pi}^{\eta-1}$$

This expression is positive for all $m \in (m_{LB}, \infty)$ and so Γ is strictly increasing. Since Γ is strictly increasing, and since $\eta \geq 2$, Γ' is strictly decreasing.

Proof of Proposition 4

It is readily seen that if $m_{LB} > 0$, then:

$$PC(x_{LB}) = \Gamma(\Phi(x_{LB})) = \Gamma(m_{LB}) = 0.$$

The derivative of PC is:

$$PC'(x) = \Gamma'(\Phi(x))\Phi'(x).$$

We know from Proposition 1 that if $\eta > 2$:

$$lim_{x \to x_{LB}} \Gamma'(\Phi(x)) = lim_{m \to m_{LB}} \Gamma'(m) = \infty.$$

Since Φ is continuously differentiable and $\Phi'(x_{LB}) > 0$:

$$lim_{x \to x_{LB}} \Phi'(x) > 0.$$

It follows that:

$$lim_{x \to x_{LB}} PC'(x) = \infty.$$

Proof of Proposition 5

The first limit can be rewritten as:

$$\begin{aligned} lim_{x \to \infty} PC(x) &= lim_{x \to \infty} \Gamma(\Phi(x)) \\ &= lim_{m \to \infty} \Gamma(m) \\ &= \pi^{max}. \end{aligned}$$

The restriction on the limit of Φ' ensures that (since Φ' and $\Gamma'(\Phi)$ are continuous) the limit:

$$lim_{x \to \infty} PC'(x) = lim_{x \to \infty} \Gamma'(\Phi(x)) \Phi'(x)$$

is well-defined as an element of the extended reals. I claim that limit is zero. Suppose it is instead $k > 0$. Then, there exists $x^* > 0$ such that $PC'(x) \geq k/2$ for all $x \geq x^*$. It follows that for any $y > x^*$,$(PC(y) - PC(x^*)) \geq (k/2)(y - x^*)$. But this contradicts the earlier result that:

$$lim_{y \to \infty} PC(y) = \pi^{max}.$$

Proof of Proposition 6

The real activity Phillips curve PC is the composition of Γ and Φ. Hence:

$$PC''(x) = \Gamma''(\Phi(x))\Phi'(x)^2 + \Gamma'(\Phi(x))\Phi''(x)$$

Since the first derivatives are positive, $PC''(x) \leq 0$ iff:

$$\frac{\Phi''(x)}{\Phi'(x)^2} \leq -\frac{\Gamma''(\Phi(x))}{\Gamma'(\Phi(x))}. \tag{16}$$

Our goal then is to show that:

$$-\frac{\Gamma''(m)}{\Gamma'(m)} \geq \frac{1}{m}$$

for all $m \geq m_{LB}$.

We have seen in the proof of Proposition 1 that the derivative $\Gamma'(m)$ is given by:

$$\Gamma'(m) = \Gamma(m)^{2-\eta}\frac{(1-\theta)}{\theta}(\alpha m + \chi)^{-\eta}\alpha\bar{\pi}^{\eta-1}$$

and so:

$$\begin{aligned}\Gamma''(m) = &(2-\eta)\Gamma(m)^{1-\eta}\Gamma'(m)\frac{(1-\theta)}{\theta}(\alpha m + \chi)^{-\eta}\alpha\bar{\pi}^{\eta-1} \\ &- \eta\Gamma(m)^{2-\eta}\frac{(1-\theta)}{\theta}(\alpha m + \chi)^{-\eta-1}\alpha^2\bar{\pi}^{\eta-1}.\end{aligned}$$

Taking ratios:

$$
\begin{aligned}
-\frac{\Gamma''(m)}{\Gamma'(m)} - \frac{1}{m} &= (\eta-2)\Gamma(m)^{1-\eta}\frac{(1-\theta)}{\theta}(\alpha m+\chi)^{-\eta}\alpha\bar{\pi}^{\eta-1} + \frac{\alpha\eta}{(\alpha m+\chi)} - \frac{1}{m} \\
&= \frac{\alpha(\eta-2)(1-\theta)(\alpha m+\chi)^{-\eta}}{1-(1-\theta)(\alpha m+\chi)^{1-\eta}} + \frac{\alpha\eta}{\alpha m+\chi} - \frac{1}{m} \\
&= \alpha(\eta-2)\frac{(1-\theta)(\alpha m+\chi)^{1-\eta}+1-(1-\theta)(\alpha m+\chi)^{1-\eta}}{\alpha m+\chi-(1-\theta)(\alpha m+\chi)^{2-\eta}} - \frac{1}{m} + \frac{2\alpha}{\alpha m+\chi} \qquad (17) \\
&> \frac{\alpha(\eta-2)m-(\alpha m+\chi)+(1-\theta)(\alpha m+\chi)^{2-\eta}}{\alpha m+\chi-(1-\theta)(\alpha m+\chi)^{2-\eta}} \qquad (18)
\end{aligned}
$$

Note that the denominator of (18) is positive because $m > m_{LB}$.

Now consider the numerator of (18). Its derivative with respect to m is:

$$
\alpha(\eta-3)-(\eta-2)(1-\theta)\alpha(\alpha m+\chi)^{1-\eta}
$$

This derivative is strictly increasing in m. It is $-\alpha$ when m is close to m_{LB} and, since $\eta > 3$, positive when m is large. Hence, there is a unique m that minimizes the numerator of (18). We can solve for that unique m^* as:

$$
\begin{aligned}
(\alpha m^*+\chi)^{1-\eta} &= (\frac{\eta-3}{\eta-2})\frac{1}{(1-\theta)} \\
\Rightarrow (\alpha m^*+\chi) &= (\frac{\eta-2}{\eta-3})^{\frac{1}{\eta-1}}(1-\theta)^{\frac{1}{\eta-1}} \\
\Rightarrow (\alpha m^*+\chi)^{2-\eta} &= (\frac{\eta-2}{\eta-3})^{\frac{2-\eta}{\eta-1}}(1-\theta)^{\frac{2-\eta}{\eta-1}} \\
\Rightarrow (\alpha m^*+\chi)^{2-\eta}(1-\theta) &= (\frac{\eta-2}{\eta-3})^{\frac{2-\eta}{\eta-1}}(1-\theta)^{\frac{1}{\eta-1}}
\end{aligned}
$$

Now plug that minimizing m^* into the numerator of (18). We obtain:

$$\begin{aligned} &\alpha(\eta-2)m^* - (\alpha m^* + \chi) + (1-\theta)(\alpha m^* + \chi)^{2-\eta} \\ &= (\eta-3)(\alpha m^* + \chi) + (1-\theta)(\alpha m^* + \chi)^{2-\eta} - \chi(\eta-2) \\ &= (1-\theta)^{\frac{1}{\eta-1}}(\eta-3)(\frac{\eta-2}{\eta-3})^{\frac{1}{\eta-1}} + (\frac{\eta-2}{\eta-3})^{\frac{2-\eta}{\eta-1}}(1-\theta)^{\frac{1}{\eta-1}} - \chi(\eta-2) \end{aligned}$$

This is positive if:

$$\begin{aligned} \chi &< \frac{(\eta-3)(\frac{\eta-2}{\eta-3})^{1/(\eta-1)} + (\frac{\eta-2}{\eta-3})^{\frac{2-\eta}{\eta-1}}}{\eta-2}(1-\theta)^{\frac{1}{\eta-1}} \\ &= [(\eta-2)^{\frac{2-\eta}{\eta-1}} + (\eta-2)^{\frac{3-2\eta}{\eta-1}}](\eta-3)^{\frac{\eta-2}{\eta-1}}(1-\theta)^{\frac{1}{\eta-1}} \\ &= [1 + (\eta-2)^{\frac{3-2\eta}{\eta-1}+\frac{\eta-2}{\eta-1}}](1-\theta)^{\frac{1}{\eta-1}}(\frac{\eta-3}{\eta-2})^{\frac{\eta-2}{\eta-1}} \\ &= [1 + \frac{1}{\eta-2}](1-\theta)^{\frac{1}{\eta-1}}(\frac{\eta-3}{\eta-2})^{\frac{\eta-2}{\eta-1}} \\ &= (\frac{\eta-1}{\eta-2})(1-\theta)^{\frac{1}{\eta-1}}(\frac{\eta-3}{\eta-2})^{\frac{\eta-2}{\eta-1}} \end{aligned}$$

Hence, under the condition on η, χ, and Φ in the proposition, it follows that:

$$\begin{aligned} -\frac{\Gamma''(\Phi(x))}{\Gamma'(\Phi(x))} &> \frac{1}{\Phi(x)} \\ &\geq \frac{\Phi''(x)}{\Phi'(x)} \end{aligned}$$

Hence, we have verified (16).

Proof of Corollary 1

The left-hand side ratio is:

$$\frac{1}{Dexp(\gamma x)}$$

and the right-hand side ratio is:

$$\frac{1}{Dexp(\gamma x)}.$$

Proof of Proposition 7

Recall that:

$$PC^L(x) = \theta + (1-\theta)(\alpha\Phi(x) + \chi)\Gamma(\Phi(x)).$$

The derivative is:

$$\begin{aligned} PC^{L\prime}(x) &= (1-\theta)\Gamma'(\Phi(x))\Phi'(x)(\alpha\Phi(x)+\chi) + (1-\theta)\alpha\Phi'(x)\Gamma(\Phi(x)) \\ &= \frac{(1-\theta)^2}{\theta}\Gamma(\Phi(x))^{2-\eta}\alpha(\alpha\Phi(x)+\chi)^{1-\eta}\Phi'(x) + (1-\theta)\alpha\Phi'(x)\Gamma(\Phi(x)) \end{aligned}$$

It follows that:

$$lim_{x\to\infty} PC^{L\prime}(x) \geq lim_{x\to\infty}\alpha\Phi'(x)(1-\theta)\pi^{max} = \infty.$$

On the other hand, since $\eta > 2$ and $\Gamma(\Phi(x_{LB})) = 0$:

$$\begin{aligned} & lim_{x\to x_{LB}} PC^{L\prime}(x) \\ \geq \Phi'(x_{LB}) lim_{x\to x_{LB}} & \frac{(1-\theta)^2}{\theta}\Gamma(\Phi(x))^{2-\eta}\alpha(\alpha\Phi(x)+\chi)^{1-\eta}\Phi'(x) \\ & = \infty. \end{aligned}$$

Proof of Proposition 8

[I thank Eugenio Gonzalez Flores for his help in simplifying this proof.]

We will prove the proposition for the case in which $PC'' < 0$. Without loss of generality, assume:

$$\pi^*(\nu_n) \leq \pi^*(\nu_{n+1}), n = 1, ..., N-1.$$

The assumption of optimality implies that:

$$N^{-1}\sum_{n=1}^{N}(x^*(\nu_n)-x^{TAR}) \tag{19}$$

$$=-N^{-1}\sum_{n=1}^{N}\Lambda^{-1}PC'(x^*(\nu_n)+\nu_n)(\pi^*(\nu_n)-\pi^{TAR}) \tag{20}$$

$$=\Lambda^{-1}N^{-1}\sum_{n=1}^{N}PC'(PC^{-1}(\pi^*(\nu_n)))(\pi^{TAR}-\pi^*(\nu_n)) \tag{21}$$

Hence, to prove the proposition, we need only establish that (21) is less than zero.

Since there are at least two distinct values of $\pi^*(\nu_n)$, (19) implies that there exists $N^* < N$ such that:

$$\pi^*(\nu_n) \leq \pi^{TAR}, n \leq N^*$$

$$\pi^*(\nu_n) > \pi^{TAR}, n > N^*.$$

and so we can rewrite (21) as:

$$N^{-1}\Lambda^{-1}\sum_{n=1}^{N^*}PC'(PC^{-1}(\pi^*(\nu_n)))(\pi^{TAR}-\pi^*(\nu_n)) \tag{22}$$

$$-N^{-1}\Lambda^{-1}\sum_{n=N^*+1}^{N}PC'(PC^{-1}(\pi^*(\nu_n)))(\pi^*(\nu_n)-\pi^{TAR}) \tag{23}$$

where both terms are positive. Since $PC' > 0$ and $PC'' < 0$, we know that:

$$PC'(PC^{-1}))$$

is a strictly decreasing function. It follows that:

$$PC'(PC^{-1}(\pi^{TAR})) \leq PC'(PC^{-1}(\pi^*(\nu_n)), n \leq N^*$$

$$PC'(PC^{-1}(\pi^{TAR})) > PC'(PC^{-1}(\pi^*(\nu_n)), n > N^*.$$

We can conclude that (22) is less than:

$$N^{-1}\Lambda^{-1}PC'(PC^{-1}(\pi^{TAR}))\sum_{n=1}^{N^*}(\pi^{TAR}-\pi^*(\nu_n)) \tag{24}$$

$$-N^{-1}\Lambda^{-1}PC'(PC^{-1}(\pi^{TAR}))\sum_{n=N^*+1}^{N}(\pi^*(\nu_n)-\pi^{TAR}) \tag{25}$$

which equals zero because:

$$N^{-1}\sum_{n=1}^{N}(\pi^*(\nu_n)-\pi^{TAR}))=0.$$

The proposition follows.

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