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THE CONCAVE PHILLIPS CURVE

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ABSTRACT

This paper derives the curvature properties of the short-run Phillips curve in a class of canonical models of price-setting frictions. Contrary to conventional thinking, the Phillips curve is asymptotically horizontal for high levels of economic activity and asymptotically vertical for low levels of economic activity. Moreover, it is globally concave for a wide class of models, including many in which average real marginal cost is an unbounded convex function of economic activity. Intuitively, when economic activity is very high (low), substitution effects within the model-implied true price index imply that inflation behaves as if prices are nearly fully sticky (flexible). Using (conventional) measures of inflation that understate the relevant substitution effects may lead to misleading conclusions about the curvature of the Phillips curve, and to corresponding errors in the formulation of monetary policy.

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1 Introduction

Figure 1 depicts (what I perceive to be) the dominant conventional wisdom about the curvature of the short-run Phillips curve[1], drawn as a positive relationship with a measure of real economic activity on the x-axis and inflation on the y-axis. The curve is globally *convex* - the inflation costs of generating a given increase in economic activity are higher in booms than in recessions. The curve becomes extremely steep for high levels of economic activity, capturing a basic intuition that, even in the short run, there are limits to what monetary policy can accomplish in terms of real stimulus. The curve is flatter for low levels of economic activity, which corresponds to the idea that nominal or real wages are costly to adjust downward.

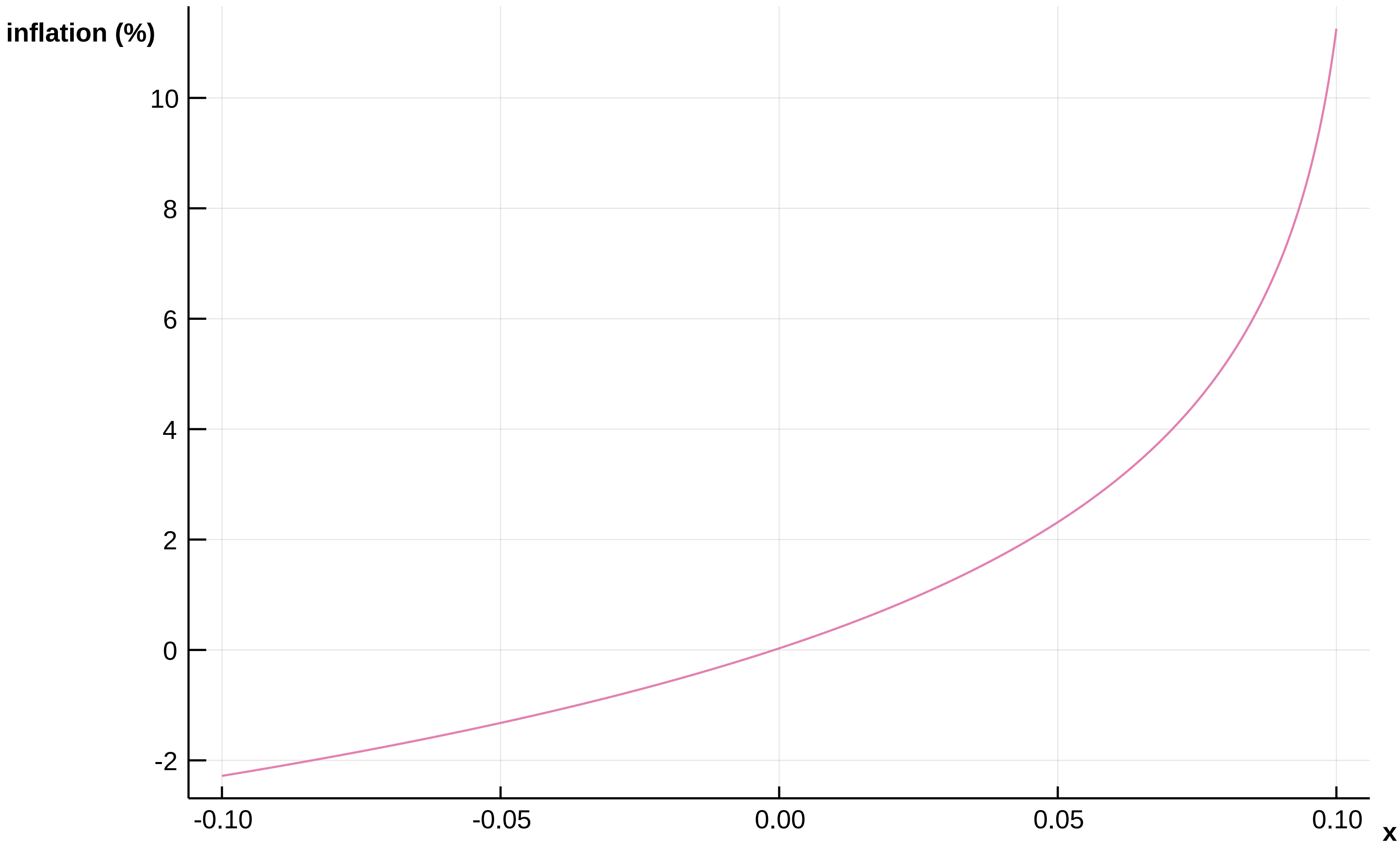
This conventional thinking about the curvature of the Phillips curve enjoys some empirical support. Phillips' (1958) original estimates of the relationship between British nominal wage inflation and unemployment imply that it has the characteristics described in the above paragraph. (Indeed, my Figure 1 was motivated in large part by Phillips' Figure 1.) Samuelson and Solow's seminal (1960) piece on the relationship between US price inflation and unemployment also describe a convex curve that is near-vertical at high levels of economic activity. More recently, Forbes, et al. (2022) have argued that the Phillips curve in the US and other countries does indeed become flatter at low levels of inflation.[2] In a similar vein, Babb and Detmeister (2017) find using data from US cities that the Phillips curve is steeper when the unemployment rate is low. However, I would say that the more dominant perspective from the past forty years of research is that it is hard to estimate even the first derivative of the Phillips curve with precision (see, for example, Mavroeidis, et al. (2014)).[3]

[1]By "short-run Phillips curve", I mean to refer, as did Samuelson and Solow (1960), to the locus of possible real economic activity and inflation outcomes achievable by a central bank through different choices of monetary policy. As is true of almost all important economic relationships, including commodity demand functions (Moore (1914, Chapter V)), shifts due to unobservable variables may mean that the Phillips curve is not immediately apparent in observational data. However, as we shall see, the mathematical properties of this fundamental policy trade-off are readily identifiable within the context of a specific model.

[2]In a (very funny) satirical piece, Smith (2006) highlights this kind of convexity in the Phillips curve for Japan.

[3]Relatedly, McLeay and Tenreyro (2020) have highlighted the challenges associated with estimating the Phillips curve in the presence of an inflation-targeting central bank. Hazell, et al. (2022) use state-level data to sidestep this problem. They find that the slope of the Phillips curve has consistently low (-0.34 when

Figure 1: The Conventional Phillips Curve



The conventional Phillips curve represents inflation as a convex function of economic activity (here labelled x).

The curve is near-vertical for high values of x.

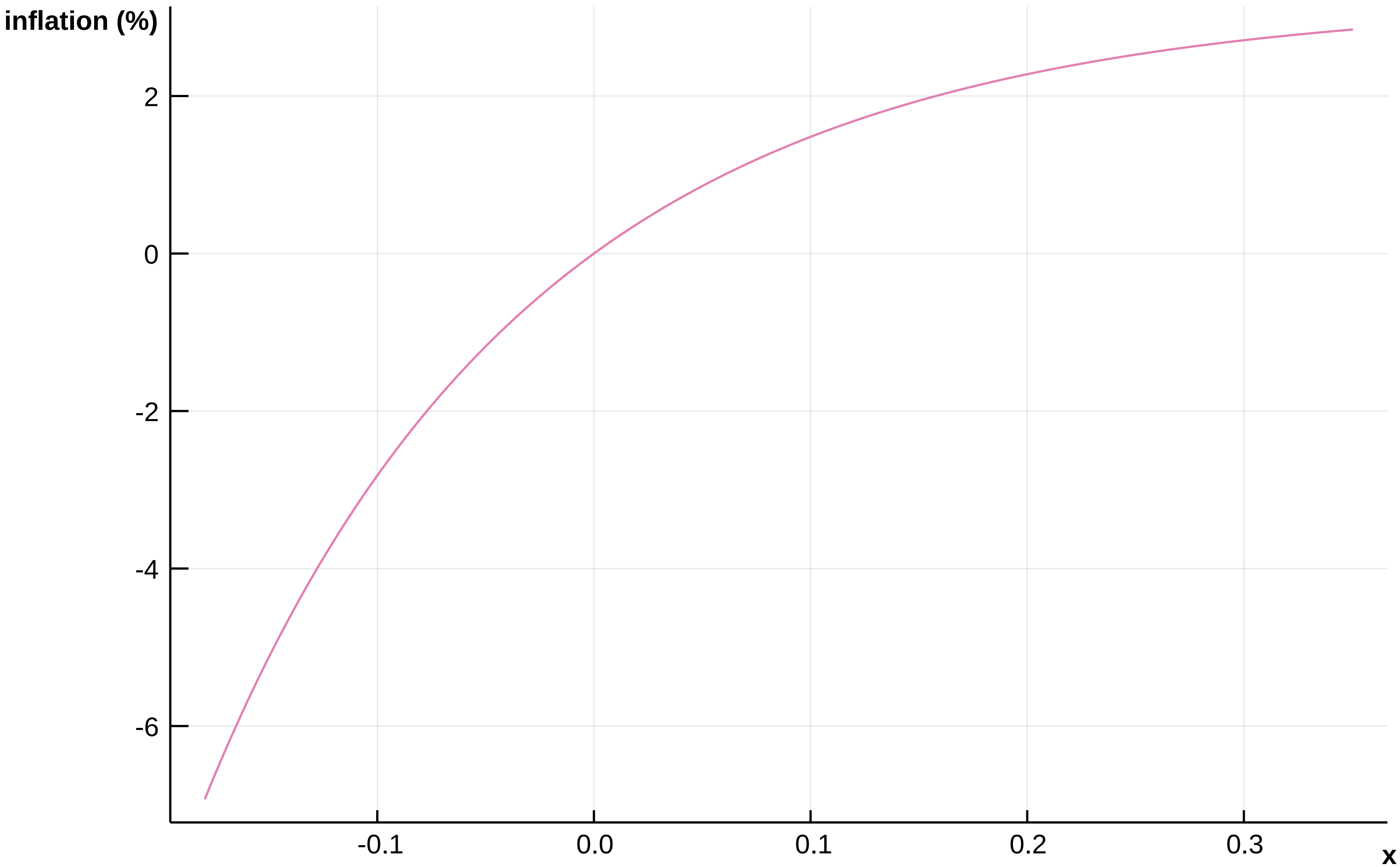
This paper investigates the implications of macroeconomic theory for the curvature properties of the short-run Phillips curve. It derives the full nonlinear Phillips curve in a class of basic canonical models of price-setting frictions in which households have CES preferences over a large number of goods. If the households' common elasticity of substitution is larger than 2, then the implied Phillips curve is globally *concave* in a wide class of models (including ones in which the marginal cost of production increases without bound along with economic activity). As illustrated in Figure 2, the concavity is, in some sense, extreme: The curve is asymptotically horizontal for high levels of economic activity and asymptotically vertical for low levels of economic activity.

The intuition behind these characterizations is quite simple. The model-implied Phillips curve uses a measure of inflation that is based on the rate of growth of the true (or Konüs (1924)) price index that fully incorporates substitution effects. When economic activity and inflation are high, consumers allocate almost all of their spending to (the cheaper) goods with sticky prices. The Phillips curve looks like one from a world in which prices are constant over time - that is, horizontal. When economic activity and inflation are low, consumers allocate almost all of their expenditures to (the cheaper) goods with flexible prices. The Phillips curve looks like one from a world without nominal rigidities - that is, vertical.

The concavity may seem to contravene the (limited) statistical evidence noted above. However, there is a critical measurement issue. The theoretical characterizations described in the preceding paragraph apply to what I will call the *true Phillips curve*, since it is based on the true price index. I also derive the models' implications for the *Laspeyres-Phillips curve*, in which inflation is measured (like the (unchained) Consumer Price Index in the US) using the rate of increase of a Laspeyres price index. I find that the Laspeyres-Phillips curve is typically *concavo-convex* (and, in a plausible numerical example, convex whenever inflation is positive). Like the true Phillips curve, it is asymptotically vertical for low levels of economic activity and low levels of inflation. But, as depicted in Figure 3, it is also asymptotically vertical for

unemployment is used as the measure of economic activity) for over forty years.

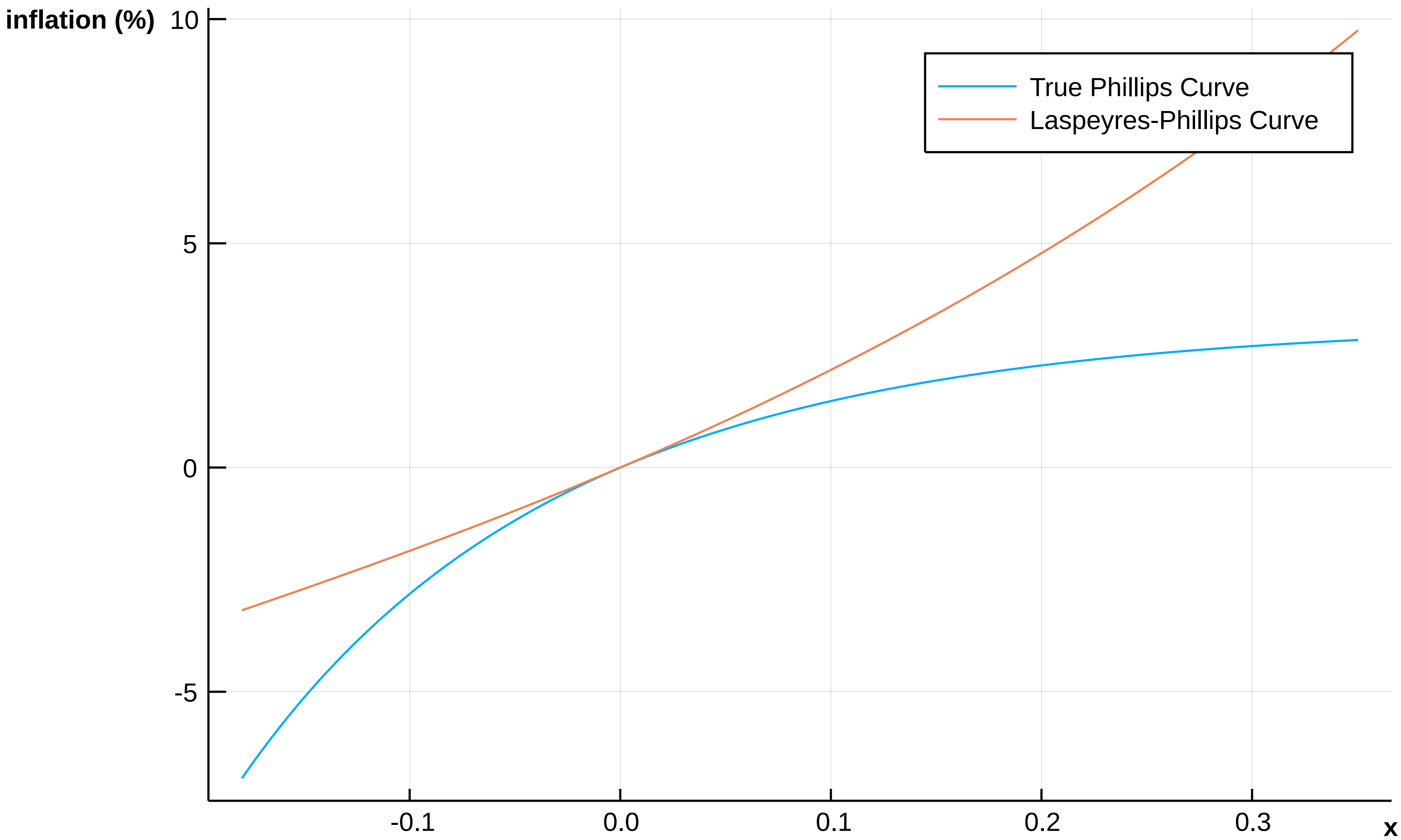
Figure 2: The True Phillips Curve



Under the true Phillips curve, inflation is a concave increasing function of real economic activity (labelled x).

Its slope converges to zero as x grows to infinity, and to infinity as x falls to its lowest level.

Figure 3: The True Phillips Curve and the Laspeyres-Phillips Curve



Unlike the true Phillips curve, the Laspeyres-Phillips Curve is convex for sufficiently large levels of real economic activity (x) and concave only for sufficiently low levels of economic activity.

high levels of economic activity and high levels of inflation. Intuitively, when real economic activity is high, high flexible-goods inflation translates into high overall (Laspeyres) price inflation. This effect is not present with the true inflation rate, because households respond to high flexible firm prices by substituting toward sticky firms with low prices.

Thus, the substitution bias in standard measures of inflation mean that estimated Phillips curves may display convexity that is not present in the true Phillips curves. This error in estimation could lead to errors in policy. Yellen and Akerlof (2006) point out that, with a convex Phillips curve, it is optimal to run the economy "hot" on average. I show that with a concave Phillips curve, their argument is reversed. The central bank can only offset an adverse shock to economic activity by incurring a large increase in inflation. As a result, it is optimal for the average level of economic activity to be low relative to what would prevail in a world without pricing frictions.

The rest of the paper is organized as follows. In the next section, I set forth a simple model of pricing rigidities in which, a la Dixit-Stiglitz (1977), households have common CES preferences over a large number of goods. I use the model to define the *marginal cost Phillips curve*, which maps the average real marginal cost of production into the true inflation rate. (As is true throughout the paper, I focus on the *short-run* or within-period Phillips curve.) In Section 3, I analyze the curvature properties of the marginal cost Phillips curve. When the households' elasticity of substitution is larger than 2, the marginal cost Phillips curve is concave, with a slope that nears infinity for low marginal costs and nears zero for high marginal costs. In Section 4, I add an abstract activity cost function that maps the level of economic activity into a corresponding real marginal cost. By combining this function with the marginal cost Phillips curve, I obtain the real activity Phillips curve, which maps the level of economic activity into the true inflation rate. A key result is that, under relatively weak conditions on the activity cost function, the real activity Phillips curve inherits the concavity properties of the marginal cost Phillips curve.[4] In Section 5, I explore the properties of

[4]One of these "relatively weak" conditions is differentiability. Benigno and Eggertsson (2023) present a New Keynesian model in which the Phillips curve is nonlinear in a *first-order* approximation around the

the Laspeyres-Phillips curve and show that, under the same conditions on the activity cost function, it is asymptotically vertical for sufficiently high levels of economic activity. Section 6 discusses policy considerations. Section 7 concludes.

All proofs are in the Appendix.

2 The Marginal Cost Phillips Curve

This section defines the *marginal cost Phillips curve.* The idea is that a policymaker can vary the average *real* marginal cost of production m_t through different policy choices at date t. Given two key assumptions on firm pricing behavior, there is a policy-invariant function that maps m_t into the realized inflation rate. It is this relationship that will be termed the marginal cost Phillips curve.

2.1 Firm Pricing

I focus on two periods indexed by $\{t-1,t\}$, although the economy may well last indefinitely. There is a fixed population of firms (and so there is no entry or exit). Let the distribution of prices across firms in period $\tau \in \{t-1,t\}$ be given by F_τ. Then, the price index in period τ is defined to be:

$$P_\tau \equiv (\int_0^\infty p^{1-\eta} dF_\tau(p))^{\frac{1}{1-\eta}}, \eta > 1. \tag{1}$$

This is the true (or Konüs (1924)) price index in a Dixit-Stiglitz (1977) model economy in which households have CES preferences over a continuum of goods. Here, the parameter η represents the typical household's elasticity of substitution across the various goods in the economy. The incorporation of substitution effects in the definition of P_τ is critical in what

steady-state. The non-differentiability emerges because of a kink in the aggregate labor supply curve driven by what they term "wage norms" that prevent wages from adjusting upward in response to excess labor demand (or under-supply). Importantly, these norms apply to all workers in all occupations. Hence, there is no way for the economy to substitute to "non-kinked" forms of labor (such as, for example, so-called travel nurses). Benigno and Eggertsson do not study the curvature properties of the Phillips curve in their model away from the steady-state.

follows.

In period t, a fraction θ_t of firms are *sticky* and a fraction $(1 - \theta_t)$ of firms are *flexible.* Each sticky firm sets its period t price equal to its period $(t - 1)$ price. The prices among the sticky firms conform to the following assumption.

Assumption 1: The distribution of period $(t - 1)$ prices among the sticky firms is F_{t-1} - that is, the same as the overall distribution of prices in period $(t - 1)$.

Assumption 1 is exactly satisfied in the Calvo (1983) model, in which firms are randomly assigned to being sticky or flexible, or in the generalizations of that model described most recently by Auclert, et al. (2024, p. 131-32). It would be approximately satisfied in menu cost models a la Gertler and Leahy (2008) or Midrigan (2011) in which firms' decisions about whether to change prices is shaped primarily by large idiosyncratic shocks, as opposed to current aggregate conditions.[5]

The flexible firms set their period t prices as follows. Let M_t be the average nominal marginal cost across all firms. Consider a flexible firm ω. The firm ω sets its period t price equal to:

$$p_{\omega t} = M_t \xi_{\omega t} \tag{2}$$

where the cross-firm distribution of $\xi_{\omega t}$, conditional on the firm being flexible, is given by G_t. The pricing behavior in (2) is readily justified as being optimal if the firm is myopic. More generally, though, the firm may anticipate that its price will remain fixed in future periods. In that context, the pricing behavior (2) is optimal if the firm expects that average nominal marginal cost will grow at a constant rate over time. The term $\xi_{\omega t}$ can then be viewed as capturing a host of firm-specific factors, including its prediction of the growth rate of average nominal marginal cost, its current/expected relative productivities, its demand elasticity, and any output subsidies (being used to offset monopolistic distortions).

[5] Conversely, Assumption 1 is not a good description of what happens in the models of Caplin and Spulber (1989) or Golosov and Lucas (2007). Those models imply (arguably counterfactually) that the Phillips curve is vertical (in the case of Caplin and Spulber), or nearly so (in the case of Golosov and Lucas).

Define the *average real marginal cost* in the economy in period t as:

$$m_t \equiv \frac{M_t}{P_t}.$$

where P_t is the price index in period t as defined above. Then, I assume that G_t and θ_t are exogenous in the following sense.

Assumption 2: It is possible for policymakers to make choices that vary average real marginal cost m_t without affecting the fraction θ_t or the distribution G_t.

Assumption 2 says that there are variations in (monetary) policy that impact overall period t aggregate demand and thereby m_t. But those policy choices do not affect the fraction of flexible firms or the firm-specific factors that influence their period t pricing relative to the current level of average nominal marginal cost M_t.

2.2 Deriving the Marginal Cost Phillips Curve

In this subsection, we use the price index to derive a relationship, called the marginal cost Phillips curve, between m_t and inflation in period t. The restriction is invariant to any policy changes that satisfy Assumption 2.

The price index in period t was defined as:

$$P_t = (\int_0^\infty p^{1-\eta} dF_t(p))^{\frac{1}{1-\eta}}.$$

where F_t is the distribution of prices across firms in period t. We can split the integral into two pieces:

$$P_t^{1-\eta} = \theta_t \int_0^\infty p^{1-\eta} dF_t^{sticky}(p) + (1-\theta_t) \int_0^\infty p^{1-\eta} dF_t^{flex}(p).$$

Here, F_t^{sticky} is the distribution of prices across sticky firms, and F_t^{flex} is the distribution of prices across flexible firms.

From Assumption 1, we know that $F_t^{sticky} = F_{t-1}$ and so:

$$\int_0^\infty p^{1-\eta} dF_t^{sticky}(p) = \int_0^\infty p^{1-\eta} dF_{t-1}(p) = P_{t-1}^{1-\eta}.$$

From Assumption 2, we know that:

$$\begin{aligned}\int_0^\infty p^{1-\eta} dF_t^{flex}(p) &= M_t^{1-\eta} \int_0^\infty \xi^{1-\eta} dG_t(\xi) \\ &= P_t^{1-\eta} m_t^{1-\eta} \bar{\xi}_{\eta t}^{1-\eta}\end{aligned}$$

where m_t was defined above to be the average real marginal cost $\frac{M_t}{P_t}$. Here, the final term $\bar{\xi}_{\eta t}$ summarizes the impact of the exogenous distribution G_t on prices:

$$\bar{\xi}_{\eta t} \equiv (\int_0^\infty \xi^{1-\eta} dG_t(\xi))^{\frac{1}{1-\eta}}.$$

It follows from these observations that we can rewrite the price index in period t as:

$$P_t^{1-\eta} = \theta_t P_{t-1}^{1-\eta} + (1-\theta_t) P_t^{1-\eta} m_t^{1-\eta} \bar{\xi}_{\eta t}^{1-\eta} \tag{3}$$

Let the gross inflation rate be defined as:

$$\pi_t \equiv (\frac{P_t}{P_{t-1}})$$

Dividing through (3) by $P_{t-1}^{1-\eta}$, we get:

$$\pi_t^{1-\eta} = \theta_t + (1-\theta_t) \pi_t^{1-\eta} m_t^{1-\eta} \bar{\xi}_{\eta t}^{1-\eta}.$$

We can then re-arrange to obtain the *marginal cost Phillips curve.*

$$\pi_t = (\frac{1-(1-\theta)\bar{\xi}_{\eta t}^{1-\eta} m_t^{1-\eta}}{\theta})^{1/(\eta-1)} \tag{4}$$

2.3 Meaning of the Marginal Cost Phillips Curve

Why is the marginal cost Phillips curve important? Recall that the variable m_t is the average *real* marginal cost of production:

$$m_t = \frac{M_t}{P_t}.$$

The variable m_t is real because its units are in terms of consumption goods not dollars. Hence, the marginal cost Phillips curve forges a (non-classical) connection between the real side of the economy and inflation.

The marginal cost Phillips curve is entirely a product of price stickiness. In particular, suppose θ_t were to equal zero. Then (3) becomes:

$$P_t^{1-\eta} = P_t^{1-\eta} m_t^{1-\eta} \bar{\xi}_{\eta t}^{1-\eta}$$

The frictionless monopolistic competition among firms then pins down average real marginal cost to have the same value:

$$1/\bar{\xi}_{\eta t}$$

regardless of inflation. Accordingly, we shall refer to:

$$m_t^{flex} \equiv \frac{1}{\bar{\xi}_{\eta t}} \tag{5}$$

as the *flexible benchmark* value for m_t.

3 Curvature of the Marginal Cost Phillips Curve

This section develops the curvature properties of the marginal cost Phillips curve derived in the prior section:

$$\Gamma(m) \equiv \left(\frac{1 - (1-\theta) m^{1-\eta} \bar{\xi}_{\eta}^{1-\eta}}{\theta}\right)^{\frac{1}{\eta-1}}. \tag{6}$$

Note that there are no time subscripts in (6). I drop these in the remainder of this section as the past and future play no (further) role in the analysis.

3.1 High and Low Levels of Real Marginal Cost

The first two propositions concern the properties of the marginal cost Phillips curve when real marginal cost is near its lowest possible level and when real marginal cost is very high. They establish that, in these contexts, the marginal cost Phillips curve is "very" concave, in the sense that its slope is infinite when (gross) inflation is zero and its slope is near zero when inflation is near its maximal level.

Proposition 1. *The marginal cost Phillips curve* Γ *implies that* m *must satisfy a lower bound:*

$$m \geq m^{LB}(\theta) \equiv \frac{(1-\theta)^{1/(\eta-1)}}{\bar{\xi}_\eta}.$$

If $\eta > 2$*, the slope of the marginal cost Phillips curve approaches infinity as* m *nears* $m^{LB}(\theta)$*.*

If the real marginal cost is low, then the flexible firms set relatively low prices compared to the sticky firms. In the extreme, consumers buy goods only from the flexible firms. The economy acts as if prices are fully flexible and so the marginal cost Phillips curve is vertical. Note that $m_{LB}(\theta)$ is close to m^{flex} when θ is small.

The next proposition shows that the marginal cost Phillips curve has a finite horizontal asymptote for large values of m.

Proposition 2. *As* m *nears infinity, then* $\Gamma(m)$ *approaches an upper bound:*

$$\pi^{max}(\theta) \equiv (\frac{1}{\theta})^{1/(\eta-1)}$$

and its slope nears zero:

$$lim_{m\to\infty}\Gamma'(m) = 0$$

The upper bound $\pi^{max}(\theta)$ converges to infinity as θ approaches zero (that is, for economies that are nearly fully flexible).

When real marginal costs are high, the flexible firms' prices are high relative to those of the sticky firms. The consumers switch to buying from the latter, and the economy behaves as if all prices are sticky. As a result, inflation is independent of cost conditions, and the Phillips curve becomes horizontal.

3.2 Global Properties

We have seen that the marginal cost Phillips curve is increasing and concave when the average real marginal cost is near its lowest level and when it is near infinity. This subsection proves that, if $\eta \geq 2$, these properties are valid for the entire domain of the curve.

Proposition 3. *The marginal cost Phillips curve is strictly increasing and, if $\eta \geq 2$, strictly concave over its domain $(m_{LB}(\theta), \infty)$.*

As m rises, the flexible monopolists set higher prices and so inflation rises. But this effect is dampened by the flexible monopolists' loss of market share to the monopolists with fixed prices. If $\eta \geq 2$, the loss of market share is sufficiently large to ensure that the curve is globally concave.

3.3 Local to a Flexible Benchmark

Recall from (5) that $m = m^{flex} = \bar{\xi}_\eta^{-1}$ when prices are flexible. This subsection sets forth the properties of the marginal cost Phillips curve in the neighborhood of this flexible price benchmark. This local characterization is analogous to the textbook linearization of the New Keynesian Phillips curve.

Proposition 4. *The marginal cost Phillips curve implies that the gross inflation rate is 1 if $m = m^{flex}$. When $m = m^{flex}$ and $\bar{\xi}_\eta = 1$, the slope of the marginal cost Phillips curve is $\Gamma'(m^{flex}) = \frac{(1-\theta)}{\theta}$.*

As in the standard New Keynesian model, the (net) inflation rate is zero when real marginal cost is the same as in a fully flexible world. The slope of the Phillips curve is an increasing function of price flexibility, and is close to infinity when the economy is nearly fully flexible.

4 Curvature of the Real Activity Phillips Curve

The previous section describes the curvature properties of the marginal cost Phillips curve. However, it is more typical to think of the Phillips curve as describing a relationship between measures of *real economic activity* and inflation. This section uses the notion of a policy-invariant *activity cost function* to translate the results about the real marginal cost Phillips curve into characterizations of the *real activity Phillips curve*.

4.1 Building Up the Real Activity Phillips Curve

Let x_t be a measure of real activity in period t that (like an employment gap) takes on values over the entire real line. There is an *activity cost* function:

$$\Phi_t : \mathbb{R} \to (0, \infty)$$

that maps real activity x_t into average real marginal cost m_t. I make the following exogeneity assumption about Φ_t :

Assumption 3: It is possible for policymakers to make choices that vary x_t without affecting the activity cost function Φ_t, the fraction θ_t of sticky firms, or the distribution G_t of firm-specific pricing factors.

Mathematically, I assume the activity cost function Φ_t is twice differentiable over its domain and that its derivative is everywhere positive (although it can become vanishingly small for low or high values of x_t). I assume too that Φ_t is onto, meaning that for any

$m_t \in (0, \infty)$, there exists $x_t \in \mathbb{R}$ such that $m_t = \Phi_t(x_t)$. There are no restrictions on the second derivative of Φ_t - the activity cost function can be convex or concave.[6]

Here's one example of how Φ_t works. Consider an economy in which labor markets are competitive. Let production be such that N_t units of labor translate into N_t units of output, and that there is a representative agent whose disutility of labor takes the form:

$$v(N) = \frac{N^{\gamma+1}}{1+\gamma}, \gamma > 0$$

Then, assuming that there are no income effects, optimality in a representative household setting implies that:

$$\frac{W_t}{P_t} = N_t^{\gamma}$$

Let $n_t = ln(N_t)$. Then, the function $\Phi(n_t) = exp(\gamma n_t)$ satisfies the conditions described above. Note that its slope is unbounded from above and is arbitrarily near zero for large negative values of n.

I next define the real activity Phillips curve PC. (As in Section 3, I suppress the time subscripts in the remainder of this section as inessential.) Recall that $m^{flex} = \bar{\xi}_{\eta}^{-1}$ is the real marginal cost when all firms are flexible and that for any θ, the real marginal cost cannot fall below:

$$m_{LB}(\theta) \equiv \frac{(1-\theta)^{1/(\eta-1)}}{\bar{\xi}_{\eta}}.$$

Define:

$$x_{LB}(\theta) \equiv \Phi^{-1}(m_{LB}(\theta))$$

to be the level of economic activity associated with the lowest real marginal cost. (Note that $x_{LB}(\theta)$ is well-defined because Φ is onto.)

Then, the real activity Phillips curve is defined as the composition of the activity cost

[6] I abstract from the possible effects of inflation on this mapping.

function Φ and the marginal cost Phillips curve:

$$PC : (x_{LB}(\theta), \infty) \to \mathbb{R}_+$$
$$PC(x) = \Gamma(\Phi(x))$$

The idea here is that Φ maps a level of economic activity x into real marginal cost m, and that translates into inflation via Γ. As noted above, real activity is meant to proxy for the (log of the) host of possible quantity variables that are typically used in Phillips curves. The goal in what follows is to learn when (that is, under what condition on Φ) the prior results about Γ translate into similar characterizations about PC.

4.2 Low and High Levels of Economic Activity

This section uses Propositions 1 and 2 to show that the real activity Phillips curve is "very" concave at high and low levels of economic activity.

Proposition 5 proves that there is a lower bound on real activity, and that the Phillips curve is close to vertical when real activity is near that lower bound.

Proposition 5. *The real activity Phillips curve PC satisfies $PC(x_{LB}(\theta)) = 0$ and if $\eta > 2$, the derivative of the Phillips curve approaches infinity as x nears $x_{LB}(\theta)$.*

What about for high levels of economic activity? Regardless of how convex Φ is, the real activity Phillips curve is asymptotically horizontal when x is large.

Proposition 6. *Suppose that $lim_{x\to\infty}\Phi'(x)$ exists as an element of the extended reals (so it is possibly infinite). Then:*

$$lim_{x\to\infty} PC(x) = \pi^{max}(\theta)$$
$$lim_{x\to\infty} PC'(x) = 0$$

Thus, like the marginal cost Phillips curve, the real activity Phillips curve is nearly vertical for low levels of economic activity and nearly horizontal for high levels of economic activity.

4.3 Global Curvature Properties

The real activity Phillips curve is strictly increasing, because it is the composition of two strictly increasing functions. The following proposition provides a sufficient condition on the activity cost function Φ for the real activity Phillips curve PC to inherit the concavity of the marginal cost Phillips curve Γ.

Proposition 7. *Suppose* $\eta \geq 2$ *and that:*

$$\frac{\Phi''(x)}{\Phi'(x)} < 2\frac{\Phi'(x)}{\Phi(x)} \tag{7}$$

for all $x \in (0, \infty)$*. Then the real activity Phillips curve* PC *is concave over its domain* $(x_{LB}(\theta), \infty)$*.*

The condition in Proposition 7 applies to a wide range of functions, including the following.

Corollary 1. *Suppose* $\Phi(x) = Cexp(\gamma x), \gamma > 0, C > 0$*. Then:*

$$\frac{\Phi''(x)}{\Phi'(x)} < 2\frac{\Phi'(x)}{\Phi(x)}.$$

5 The Laspeyres-Phillips Curve

The argument in the last two section relies on the impact of substitution effects within the price index. But those effects are imperfectly measured in the official data. This section considers the curvature of a (real activity) *Laspeyres-Phillips* curve. For this curve, the inflation measure is constructed using a Laspeyres price index that systematically understates substitution effects. Unlike the true real activity Phillips curve, the real activity Laspeyres-Phillips curve is near-vertical for high levels of economic activity.

I simplify the analysis by assuming that all flexible firms in period t set prices so that their markups are equal to zero, which means that that the distribution G_t of firm pricing factors (the ξ's) assigns a unit mass to 1. This restriction is readily relaxed, but doing so adds notation without adding insight.

5.1 Laspeyres Inflation

We return to the same pricing model described in Section 2. As before, let F_τ be the cross-firm distribution of prices in period τ, where $\tau \in \{t-1, t\}$. In addition, let $\hat{F}_t$ be the cross-firm joint distribution of (p_{t-1}, p_t).

We define the Laspeyres inflation rate using period $(t-1)$ quantities as the relevant bundle. Hence, the period $(t-1)$ Laspeyres price index is the same as the true price index (1) in period $(t-1)$:

$$
\begin{aligned}
P_{t-1}^L &= (\int_0^\infty (\frac{p}{P_{t-1}})^{-\eta} p dF_{t-1}(p)) \\
&= \frac{\int_0^\infty p^{1-\eta} dF_{t-1}(p)}{P_{t-1}^{-\eta}} \\
&= P_{t-1}.
\end{aligned}
$$

The period t Laspeyres price index is then given by:

$$
P_t^L = \int_0^\infty \int_0^\infty (\frac{p}{P_{t-1}})^{-\eta} p' d\hat{F}_t(p, p').
$$

Here, we need to use the joint distribution of prices, because the period t price index is calculated using period $(t-1)$ quantities.

As in Section 2, a fraction θ_t of firms in period t set their prices to be the same as in the prior period. Since G_t is a point mass at 1, the remaining fraction $(1-\theta_t)$ of firms in period

t set their prices equal to (nominal) marginal cost M_t. Hence:

$$P_t^L = \theta_t \int_0^{\infty} (\frac{p}{P_{t-1}})^{-\eta} p dF_{t-1}(p) + (1-\theta_t)M_t$$
$$= \theta_t P_{t-1} + (1-\theta_t)M_t.$$

It follows that we can define the (gross) Laspeyres inflation rate as:

$$\pi_t^L \equiv \frac{P_t^L}{P_{t-1}^L} = \theta_t + (1-\theta_t)\frac{M_t}{P_{t-1}} \tag{8}$$

We can rewrite the last term of (8) as:

$$\frac{M_t}{P_t}\frac{P_t}{P_{t-1}}.$$

Critically, P_t represents the true price index described in Section 2, not the Laspeyres price index (there is no difference between the two in period $(t-1)$). Then, the gross Laspeyres inflation rate satisfies:

$$\pi_t^L = \theta_t + (1-\theta_t)m_t\pi_t \tag{9}$$

As in Section 3, the variable m_t is the real marginal cost in period t, and the term π_t is the *true* (gross) rate of inflation.

5.2 The Laspeyres-Phillips Curve

In this subsection, I provide a formal definition and characterization of the Laspeyres-Phillips Curve. Throughout the remainder of Section 5, I drop the time subscripts as they are inessential.

From Section 3, we know that the true inflation rate π satisfies:

$$\pi = \Gamma(m)$$

where:

$$\Gamma(m) = (\frac{(1-(1-\theta)m^{1-\eta})}{\theta})^{\frac{1}{\eta-1}}$$

is the marginal cost Phillips curve. It follows that the Laspeyres inflation rate can be written as:

$$\pi^L = \Gamma^L(m) \equiv \theta + (1-\theta)m\Gamma(m)$$

Intuitively, the gross Laspeyres inflation rate is defined in (9) as a weighted average of 1 (the inflation rate for sticky price goods) and the inflation rate ($m\Gamma(m)$) for flexible price goods. The weights are, by definition, fixed. Hence, the substitution bias in Laspeyres inflation implies that it behaves like a product of real marginal cost and true inflation. When m_t is near its lower bound, this product is highly sensitive to m_t (just like true inflation). However, when real marginal cost is large, the product is a linear function of m_t. This linearity means that, unlike the true inflation rate, the Laspeyres inflation rate is unbounded from above.

We now turn to the construction of a real activity Phillips curve defined in terms of the Laspeyres inflation rate. Because the average real marginal cost m is defined in terms of the true (not Laspeyres) price index, the same policy-invariant activity cost function Φ as in Section 4 maps real economic activity x into average real marginal cost m. Then, we define the Laspeyres-Phillips curve as the composition of Γ^L and Φ:

$$PC^L : (x_{LB}(\theta), \infty) \rightarrow (0, \infty)$$
$$PC^L(x) = \Gamma^L(\Phi(x))$$

where:

$$x_{LB}(\theta) \equiv \Phi^{-1}(m_{LB}(\theta)) = \Phi^{-1}((1-\theta)^{\frac{1}{\eta-1}}).$$

The domain of PC^L is the same as the domain of the true real activity Phillips curve PC. What differs is how the Laspeyres-Phillips curve PC^L maps real economic activity into an inflation rate with substitution bias.

The following proposition establishes that the Laspeyres-Phillips curve and the true Phillips curve are the same up to first order in a neighborhood of a flexible price benchmark. It justifies the typical practice of treating the two as equivalent.

Proposition 8. *Let $x^{flex} = \Phi^{-1}(m^{flex})$, where $m^{flex} = 1$ (since $\bar{\xi}_\eta$ was assumed to equal 1 in this section). Then:*

$$PC^L(x^{flex}) = PC(x^{flex}) = 1$$
$$PC^{L\prime}(x^{flex}) = PC'(x^{flex}) = \frac{(1-\theta)}{\theta}\Phi'(x^{flex})$$

Hence, the properties of the curves are the same for x sufficiently close to x^{flex}. However, the substitution bias in PC^L matters considerably for other values of x. Proposition 6 showed that the true Phillips curve is near-horizontal for high levels of economic activity, even if the activity cost function Φ is nearly vertical. But the result is a consequence of the upper bound on true inflation derived in Proposition 2. As we have seen, there is no such upper bound on Laspeyres inflation, and, as the following proposition[7] shows, the Laspeyres-Phillips curve inherits the verticality of Φ for large values of x.

Proposition 9. *Suppose that $\eta > 2$ and that:*

$$lim_{x\to\infty}\Phi'(x) = \infty.$$

Then, the Laspeyres-Phillips curve PC^L is near-vertical when economic activity is high or

[7] In a recent paper, Blanco, Boar, Jones, and Midrigan (2024) report that the Phillips curve is convex in a parameterized version of a sophisticated menu cost multi-product model. Their intuition for their result is that more firms change prices in response to larger monetary shocks, which is a force that I do not capture in my analysis. Importantly, though, they use a Laspeyres measure of inflation, rather the one based on the true price index. The substitution effects that I emphasize would tend to concavify the Phillips curve in their model relative to, for example, what they plot on page 38.

when it is low:

$$lim_{x \to \infty} PC^{L\prime}(x) = \infty$$

$$lim_{x \to x_{LB}(\theta)} PC^{L\prime}(x) = \infty$$

The following proposition considers the same class of convex activity cost functions as in Corollary 1. In combination with Proposition 7, that corollary showed that the true Phillips curve PC is globally concave for any such Φ. In contrast, Proposition 9 implies that the (mismeasured) Laspeyres-Phillips curve PC^L is necessarily (highly) convex at high levels of economic activity. Proposition 10 demonstrates that the area of convexity includes x^{flex} (the level of economic activity in a fully flexible economy) if:

- goods demand is relatively inelastic (η sufficiently small)
- prices are not all that flexible (θ is sufficiently large).

Proposition 10. *Suppose* $\Phi(x) = exp(\gamma x), \gamma > 0$. *The Laspeyres-Phillips curve* PC^L *is convex at* $x^{flex}(= 0)$ *if* (η, θ) *satisfy:*

$$1 > (\eta - 2)\frac{(1-\theta)}{\theta}.$$

5.3 Numerical Example

The following numerical example illustrate the differences between the true (real activity) Phillips curve and the mismeasured Laspeyres-Phillips curve. Consider a parameterization in which:

$$\eta = 6$$

$$\Phi(x) = exp(2x)$$

The setting of η corresponds to a markup of 20%. If we interpret $exp(x)$ as labor input, then the specification of Φ translates into a Frisch elasticity equal to 1/2.

It remains to parameterize θ. At the flexible benchmark value of $x^{flex} = 0$, the slope of either Phillips curve is:

$$2\frac{(1-\theta)}{\theta}$$

Using vastly different methods, Blanchard, et al. (2015) and Hazell, et al. (2022) both estimate that the slope of the Phillips curve in the US is around -0.3 with respect to unemployment. Given a labor force participation rate of around 0.65, this unemployment slope translates into an employment slope of about 0.46. Accordingly, I set $\theta = 0.81$.

Figure 4 shows the difference between the true Phillips curve and the Laspeyres-Phillips curve. As suggested by Proposition 10, the Laspeyres-Phillips curve is convex whenever inflation is positive (since $1 > (\eta - 2)\frac{(1-\theta)}{\theta} = 0.92$). The true Phillips curve is concave over its full domain. Figure 5 shows that, as indicated by Proposition 9, there is also a region of concavity for the Laspeyres-Phillips curve for low levels of economic activity.

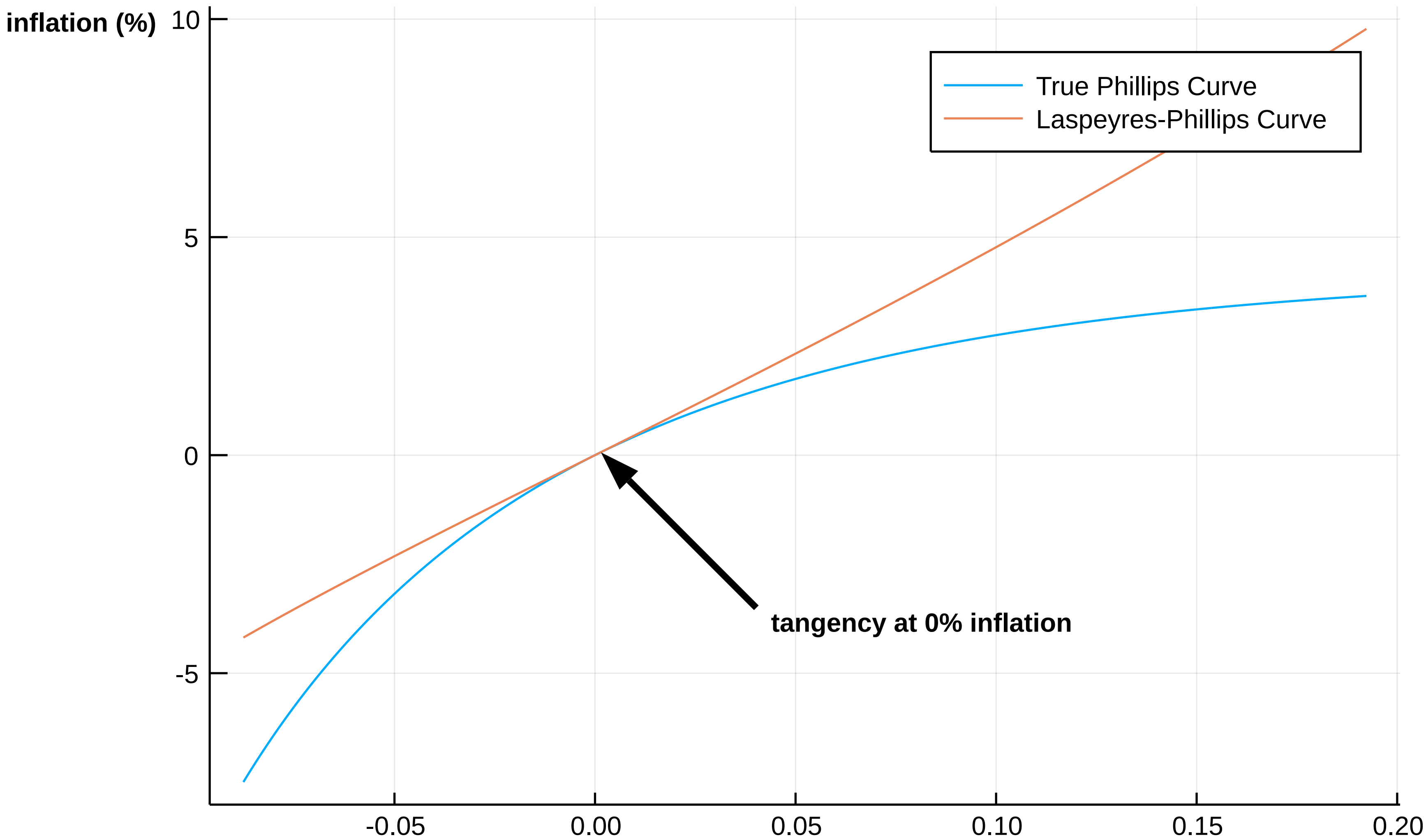
Figure 4 also illustrates that the curvature of the Laspeyres-Phillips curve is not all that pronounced for low levels of inflation. Recall that the slope of the curve is set to be 0.46 when inflation is zero. The curve is still less than 5% inflation when $x = 0.1$, suggesting that its slope is rising only gradually.

In contrast, the curvature of the true Phillips curve is much more striking. Inflation is well under 5% when $x = 0.1$, meaning that the slope of the curve has fallen sharply. In this sense, the main take-away from Figures 4-5 is that, for this parameterization, substitution bias leads the Phillips curve to appear much less concave than it is in reality.

6 Policy Considerations

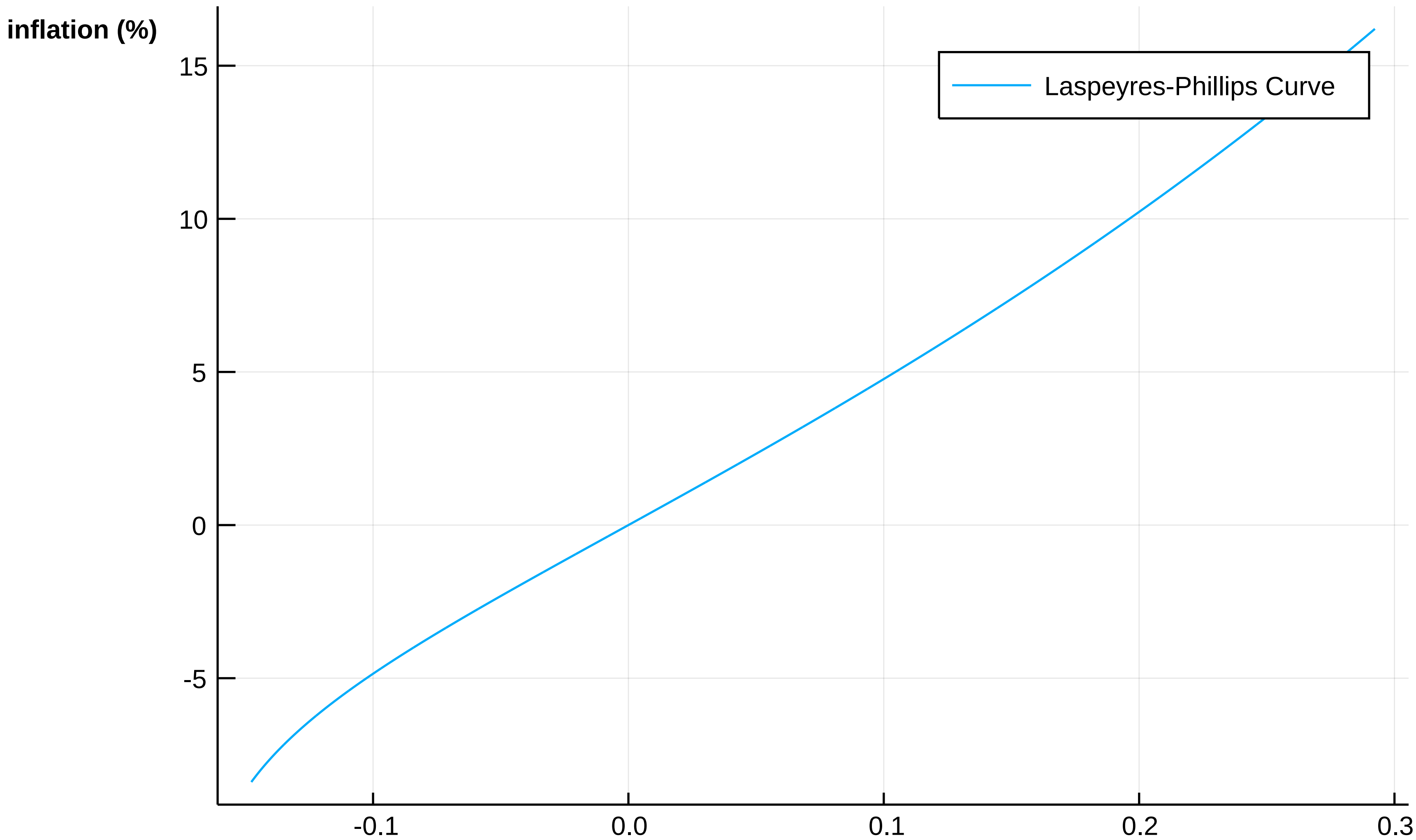
This section discusses why the curvature of the true Phillips curve may (or may not) matter for monetary policy.

Figure 4: The True Phillips and Laspeyres-Phillips Curves



The true Phillips and Laspeyres-Phillips curves when gamma = 2, eta = 6, and theta = 0.81.

Figure 5: The Concavo-Convex Laspeyres-Phillips Curve



The Laspeyres-Phillips curve for eta = 6, theta = 0.81, and gamma = 2.

The curve hits the minimal inflation rate of -18.7% at x = -0.17.

6.1 Optimal Average Slack

This subsection shows that the concavity of the true Phillips curve has an important policy implication: if average inflation equals the central bank's target, the the activity gap is negative on average. The argument is the reverse of one advanced by Yellen and Akerlof (2006) for convex Phillip curves.

Consider a central bank which has a policy instrument with which to vary the level of economic activity x. Suppose too that the (true) Phillips curve PC is affected by additive shocks, so that inflation in any period is given by:

$$\pi = PC(x) + \varepsilon.$$

Here, ε can be positive or negative. The central bank has an inflation target π^{TAR}. It is unable to commit to future policy choices. Hence, it seeks to find a level x of economic activity, conditional on its observation of ε, so as to minimize the quadratic objective:

$$(\pi - \pi^{TAR})^2 + \lambda(x - x^{flex})^2$$

where x^{flex} is the level of real activity in an economy in which all prices are flexible. Given any ε, the central bank chooses $x^*(\varepsilon)$ so as to satisfy the condition:

$$(\pi^*(\varepsilon) - \pi^{TAR}) = \frac{\lambda(x^{flex} - x^*(\varepsilon))}{PC'(x^*(\varepsilon))},$$

where $\pi^*(\varepsilon) = PC(x^*(\varepsilon)) + \varepsilon$. Notice that the central bank's optimality condition implies that inflation is above target if and only if there is a negative activity gap (Qvigstad (2006)).

So far, I have imposed no probability structure on the ε's. However, the following proposition considers an arbitrary sequence $(\varepsilon_1, ..., \varepsilon_N)$ such that the implied average inflation is at target:

$$N^{-1}\sum_{n=1}^{N}\pi^*(\varepsilon_n)=\pi^{TAR}.$$

It shows that for any such sequence, the average activity gap (x^*-x^{flex}) is negative (positive) if PC is concave (convex).

Proposition 11. *Consider* $(\varepsilon_1,...,\varepsilon_N)$ *such that* $(\pi^*(\varepsilon_n),x^*(\varepsilon_n))_{n=1}^{N}$ *has at least two distinct values, satisfies the central bank's optimality condition:*

$$(\pi^*(\varepsilon_n)-\pi^{TAR})=\frac{\lambda(x^{flex}-x^*(\varepsilon_n))}{PC'(x^*(\varepsilon_n))}, n=1,...,N$$

and the average inflation is equal to target:

$$N^{-1}\sum_{n=1}^{N}\pi^*(\varepsilon_n)=\pi^{TAR}.$$

Then $N^{-1}\sum_{n=1}^{N}(x^*(\varepsilon_n)-x^{flex})>0$ *if* $PC''>0$ *and* $N^{-1}\sum_{n=1}^{N}(x^*(\varepsilon_n)-x^{flex})<0$ *if* $PC''<0$.

The intuition behind the proposition is simple. When the Phillips curve is concave, it is expensive in inflation terms to vary x when it is low and cheap to vary x when it is high. As a consequence, the central bank is willing to tolerate large negative activity gaps, and so x is low on average relative to x^{flex}. A policymaker who thought that the Phillips curve is convex would run the argument in reverse to reach the opposite conclusion.

6.2 Measuring Inflation: A Possible Caveat

Economists have suggested a number of costs of inflation. For example, if prices are sticky, non-zero inflation leads to an inefficient dispersion in relative prices (Woodford (2001, p. 22-23)). Shocks to inflation also lead to unanticipated re-allocations of wealth between parties who have agreed to an employment or financial contract that is risk-free in nominal terms.

For these kinds of purely economic costs, the correct measure of inflation is based on the Konüs price index.

However, monetary policymakers are often concerned about the linkage between realized inflation and inflation expectations. Here it is less clear that the relevant inflation measure should be based on the Konüs price index. Thus, Figure 4 illustrates that an economy with relatively similar true inflation rates might have quite different Laspeyres inflation rates. Would agents' inflation expectations be stable across these different inflation states, because the growth rate of their true cost of living is not actually changing all that much? Or would agents' inflation expectations respond more dramatically to the variation in Laspeyres inflation? An affirmative answer to the latter question would presumably provide a justification for policymakers viewing the trade-off between economic activity and inflation through the lens of the Laspeyres-Phillips curve.

7 Conclusions

The main contribution of this paper is theoretical. It demonstrates that, from the perspective of the class of canonical models used in this paper, Figure 2 is the correct way to draw the Phillips Curve and Figure 1 is not. The paper also shows that substitution biases in official measures of inflation could lead the estimated Phillips Curve to look more like Figure 1 rather than the correct Figure 2.

There are at least of couple of ways to build on the results in this paper. The first has to do with data. The paper underscores the importance of incorporating substitution effects between sticky and flexible price goods in building appropriate measures of inflation for use by macroeconomists and policymakers. Chained price indices (reported in the US since the beginning of the century) are a potentially important step forward in this regard. But, given the costs of collecting item-specific expenditure data, the chaining is at a relatively high level[8] and so is not well-designed to capture substitution patterns between goods/services

[8]For details, see https://www.bls.gov/cpi/additional-resources/chained-cpi-questions-and-

produced by sticky versus flexible firms. The paper shows that it may be important to build up model-consistent measures of inflation from disaggregated data on prices, and use those model-consistent measures as the basis of analysis and policy.

The second has to do with theory. The results in this paper are limited to settings in which households have identical CES preferences and (implicitly) consume the same bundle of underlying non-durable goods and services. It would be useful, although far from trivial, to extend the analysis to include other preferences and to allow for taste-driven and wealth-driven differences in consumptions of (potentially durable) goods.

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answers.htm#Question_4. It is worth emphasizing that chaining at a more disaggregated level may be more prone to the well-known of problem of chain drift (see, for example, Diewert (2021)).

Appendix

This appendix contains the proofs of the various propositions.

Proof of Proposition 1

The marginal cost Phillips curve is defined as:

$$\Gamma(m) = (\frac{1-(1-\theta)\bar{\xi}_\eta m^{1-\eta}}{\theta})^{\frac{1}{\eta-1}}.$$

This is only well-defined if the argument in the parentheses is positive, so that:

$$m^{1-\eta} \leq (1-\theta)^{-1}\bar{\xi}_\eta^{-1}$$

or:

$$m \geq m_{LB}(\theta).$$

The derivative of Γ is:

$$\begin{aligned}(\frac{1-m^{1-\eta}\bar{\xi}_\eta^{1-\eta}(1-\theta)}{\theta})^{1/(\eta-1)-1}\frac{(1-\theta)}{\theta}\bar{\xi}_\eta^{1-\eta}m^{-\eta} \\ = \Gamma(m)^{2-\eta}\frac{(1-\theta)}{\theta}\bar{\xi}_\eta^{1-\eta}m^{-\eta}.\end{aligned}$$

Since $\eta > 2$, then $lim_{m\rightarrow m_{LB}(\theta)}\Gamma'(m) = \infty$ because $\Gamma(m_{LB}(\theta)) = 0$.

Proof of Proposition 2

The formula for $\Gamma(m)$ is:

$$\begin{aligned}\Gamma(m) &= (\frac{1-m^{1-\eta}\bar{\xi}_\eta^{1-\eta}(1-\theta)}{\theta})^{1/(\eta-1)} \\ &\leq (\frac{1}{\theta})^{1/(\eta-1)} = \pi^{max}(\theta).\end{aligned}$$

It is straightforward to verify that since $\eta > 1$:

$$lim_{m \to \infty} \Gamma(m) = \pi^{max}(\theta).$$

The derivative Γ' can be calculated as:

$$\Gamma'(m) = \Gamma(m)^{2-\eta} m^{-\eta} \bar{\xi}_\eta (1-\theta)/\theta.$$

Hence:

$$lim_{m \to \infty} \Gamma'(m) = 0.$$

Proof of Proposition 3

As in the proof of Proposition 1, the derivatives of Γ are given by:

$$\Gamma'(m) = \Gamma(m)^{2-\eta} \frac{(1-\theta)}{\theta} \bar{\xi}_\eta^{1-\eta} m^{-\eta}$$

This expression is positive for all $m \in (m_{LB}(\theta), \infty)$ and so Γ is strictly increasing. Since Γ is strictly increasing, and since $\eta \geq 2$, Γ' is strictly decreasing.

Proof of Proposition 4

The formula for the marginal cost Phillips curve is:

$$\Gamma(m) \equiv (\frac{1 - m^{1-\eta} \bar{\xi}_\eta^{1-\eta} (1-\theta)}{\theta})^{1/(\eta-1)}.$$

If we plug $m = m^{flex}$ into this formula, we get $\Gamma(m^{flex}) = 1$.

The derivative of the marginal cost Phillips curve is:

$$\Gamma'(m) = \Gamma(m)^{2-\eta} \frac{(1-\theta)}{\theta} \bar{\xi}_\eta^{1-\eta} m^{-\eta}.$$

Plugging in $m = m^{flex}$and $\bar{\xi}_\eta = 1$ into this formula, we get:

$$\Gamma'(m^{flex}) = \frac{(1-\theta)}{\theta}.$$

Proof of Proposition 5

It is readily seen that:

$$PC(x_{LB}(\theta)) = \Gamma(\Phi(x_{LB}(\theta))) = \Gamma(m_{LB}(\theta)) = 0.$$

The derivative of PC is:

$$PC'(x) = \Gamma'(\Phi(x))\Phi'(x).$$

We know from Proposition 1 that if $\eta > 2$:

$$lim_{x\to x_{LB}(\theta)}\Gamma'(\Phi(x)) = lim_{m\to m_{LB}(\theta)}\Gamma'(m) = \infty.$$

Since Φ is continuously differentiable and $\Phi'(x_{LB}(\theta)) > 0$:

$$lim_{x\to x_{LB}(\theta)}\Phi'(x) > 0.$$

It follows that:

$$lim_{x\to x_{LB}(\theta)}PC'(x) = \infty.$$

Proof of Proposition 6

The first limit can be rewritten as:

$$\begin{aligned} lim_{x\to\infty} PC(x) &= lim_{x\to\infty}\Gamma(\Phi(x)) \\ &= lim_{m\to\infty}\Gamma(m) \\ &= \pi^{max}(\theta). \end{aligned}$$

The restriction on the limit of Φ' ensures that the limit:

$$lim_{x\to\infty} PC'(x) = lim_{x\to\infty}\Gamma'(\Phi(x))\Phi'(x)$$

is well-defined as an element of the extended reals. I claim that limit is zero. Suppose it is instead $k > 0$. Then, there exists $x^* > 0$ such that $PC'(x) \geq k/2$ for all $x \geq x^*$. It follows that for any $y > x^*$,$(PC(y) - PC(x^*)) \geq (k/2)(y - x^*)$. But this contradicts the earlier result that:

$$lim_{y\to\infty} PC(y) = \pi^{max}(\theta).$$

Proof of Proposition 7

The real activity Phillips curve PC is the composition of Γ and Φ. Hence:

$$PC''(x) = \Gamma''(\Phi(x))\Phi'(x)^2 + \Gamma'(\Phi(x))\Phi''(x)$$

Since the first derivatives are positive, $PC''(x) \leq 0$ iff:

$$\frac{\Phi''(x)}{\Phi'(x)} \leq -\frac{\Gamma''(\Phi(x))}{\Gamma'(\Phi(x))}\Phi'(x). \tag{10}$$

We now show that the inequality (7) implies the inequality (10). We have seen in the proof of Proposition 1 that the derivative $\Gamma'(m)$ is given by:

$$\Gamma'(m) = \Gamma(m)^{2-\eta}\frac{(1-\theta)}{\theta}\bar{\xi}_\eta^{1-\eta}m^{-\eta}$$

and so:

$$\begin{aligned}\Gamma''(m) = &(2-\eta)\Gamma(m)^{1-\eta}\Gamma'(m)\frac{(1-\theta)}{\theta}\bar{\xi}_\eta^{1-\eta}m^{-\eta}\\ &-\eta\Gamma(m)^{2-\eta}\frac{(1-\theta)}{\theta}\bar{\xi}_\eta^{1-\eta}m^{-\eta-1}.\end{aligned}$$

The right hand side of (10) is then bounded below by:

$$\begin{aligned}-\frac{\Gamma''(\Phi(x))}{\Gamma'(\Phi(x))}\Phi'(x) &= (\eta-2)\Phi'(x)\Gamma(\Phi(x))^{1-\eta}\frac{(1-\theta)}{\theta}\bar{\xi}_\eta^{1-\eta}\Phi(x)^{-\eta}\\ &+\eta\Phi(x)^{-1}\Phi'(x)\\ &= \Phi'(x)[(\eta-2)\Gamma(\Phi(x))^{1-\eta}\frac{(1-\theta)}{\theta}\bar{\xi}_\eta^{1-\eta}\Phi(x)^{-\eta}+\frac{\eta}{\Phi(x)}]\\ &\geq \frac{2\Phi'(x)}{\Phi(x)} \text{ since } \eta \geq 2\\ &> \frac{\Phi''(x)}{\Phi'(x)} \text{ from (7)}\end{aligned}$$

Hence, we have verified (10).

Proof of Corollary 1

The left-hand side ratio is:

$$\gamma$$

and the right-hand side ratio is:

$$2\gamma.$$

The right-hand side ratio is larger.

Proof of Proposition 8

From the definition of x^{flex}, $\Phi(x^{flex}) = 1$. Hence:

$$PC(x^{flex}) = \Gamma(1) = 1$$
$$PC^L(x^{flex}) = \theta + (1-\theta)PC(x^{flex})\Phi(x^{flex})$$
$$= 1.$$

The derivative of PC is given by:

$$PC'(x^{flex}) = \Gamma'(\Phi(x^{flex}))\Phi'(x^{flex})$$
$$= \Gamma(\Phi(x^{flex}))^{2-\eta}\Phi(x^{flex})^{-\eta}\frac{1-\theta}{\theta}\Phi'(x^{flex}).$$

But $\Phi(x^{flex}) = 1$ and $\Gamma(1) = 1$.

The derivative of PC^L is given by:

$$PC^{L\prime}(x^{flex}) = (1-\theta)\Phi'(x^{flex})PC(x^{flex}) + (1-\theta)\Phi(x^{flex})PC'(x^{flex})$$
$$= (1-\theta)\Phi'(x^{flex}) + (1-\theta)^2\Phi'(x^{flex})/\theta$$
$$= \frac{(1-\theta)}{\theta}\Phi'(x^{flex}).$$

Proof of Proposition 9

The derivative is:

$$PC^{L\prime}(x) = \Gamma^{L\prime}(\Phi(x))\Phi'(x)$$
$$= \Phi'(x)(1-\theta)(\Gamma(\Phi(x)) + \Phi(x)\Gamma'(\Phi(x)))$$
$$= \Phi'(x)(1-\theta)(\Gamma(\Phi(x)) + \Gamma(\Phi(x))^{2-\eta}\Phi(x)^{1-\eta}\frac{(1-\theta)}{\theta})$$

Since Φ is onto, and $\eta > 2$, $lim_{x\to\infty}\Phi(x)^{1-\eta} = 0$. It follows that:

$$lim_{x\to\infty}(\Gamma(\Phi(x)) + \Phi(x)\Gamma'(\Phi(x))) = \pi^{max}(\theta)$$

Hence:

$$lim_{x\to\infty}PC^{L\prime}(x) = lim_{x\to\infty}\Phi'(x)(1-\theta)\pi^{max}(\theta) = \infty.$$

On the other hand, since $\eta > 2$ and $\Gamma(x_{LB}(\theta)) = 0$:

$$\begin{aligned} &lim_{x\to x_{LB}(\theta)}PC^{L\prime}(x) \\ &= \Phi'(x_{LB}(\theta))lim_{x\to x_{LB}(\theta)}(\Gamma(\Phi(x)) + \Gamma(\Phi(x))^{2-\eta}\Phi(x)^{1-\eta}\frac{(1-\theta)}{\theta}) \\ &= \infty. \end{aligned}$$

Proof of Proposition 10

Recall that PC^L is given by:

$$\theta + (1-\theta)\Gamma(\Phi(x))\Phi(x)$$

The first derivative of PC^L with respect to x is then given by:

$$\Phi'(x)(1-\theta)(\Gamma(\Phi(x)) + \Gamma(\Phi(x))^{2-\eta}\Phi(x)^{1-\eta}\frac{(1-\theta)}{\theta})$$

Hence, the second derivative of PC^L is given by:

$$\begin{aligned} PC^{L\prime\prime}(x) &= \Phi''(x)(1-\theta)(\Gamma(\Phi(x)) + \Gamma(\Phi(x))^{2-\eta}\Phi(x)^{1-\eta}\frac{(1-\theta)}{\theta}) \\ &+ \Phi'(x)(1-\theta)\Gamma'(\Phi(x))\Phi'(x) \\ &+ \Phi'(x)(1-\theta)(2-\eta)\Gamma(\Phi(x))^{1-\eta}\Gamma'(\Phi(x))\Phi'(x)\Phi(x)^{1-\eta}\frac{(1-\theta)}{\theta} \\ &+ \Phi'(x)(1-\theta)(1-\eta)\Gamma(\Phi(x))^{2-\eta}\Phi(x)^{-\eta}\Phi'(x)\frac{(1-\theta)}{\theta}. \end{aligned}$$

Recall from Proposition 4 that:

$$\Gamma(1) = 1$$
$$\Gamma'(1) = \frac{1-\theta}{\theta}.$$

Define $x^{flex} = 0$, so that $\Phi(x^{flex}) = 1$. It follows that:

$$\Phi'(x^{flex}) = \gamma$$
$$\Phi''(x^{flex}) = \gamma^2.$$

Hence:

$$\begin{aligned}
PC^{L''}(x) &= \gamma^2(1-\theta)(1+(1-\theta)/\theta) \\
&+ \gamma^2((1-\theta)/\theta)(1-\theta) \\
&+ \gamma^2\frac{(1-\theta)^2}{\theta}(2-\eta)\frac{(1-\theta)}{\theta} \\
&+ \gamma^2(1-\theta)(1-\eta)\frac{(1-\theta)}{\theta}. \\
&= \frac{\gamma^2(1-\theta)}{\theta}(1+(1-\theta)+\frac{(2-\eta)(1-\theta)^2}{\theta}+(1-\eta)(1-\theta)) \\
&= \frac{\gamma^2(1-\theta)}{\theta}(1+\frac{(2-\eta)(1-\theta)^2}{\theta}+(2-\eta)(1-\theta)) \\
&= \frac{\gamma^2(1-\theta)}{\theta}(1+\frac{(2-\eta)(1-\theta)}{\theta}).
\end{aligned}$$

This is positive under the parametric conditions in the proposition.

Proof of Proposition 11

[I thank Eugenio Gonzalez Flores for his help in simplifying this proof.]

We will prove the proposition for the case in which $PC'' < 0$. Without loss of generality, assume:

$$x^*(\varepsilon_n) \leq x^*(\varepsilon_{n+1}), n = 1, ..., N-1.$$

The two hypotheses imply that:

$$0 = N^{-1} \sum_{n=1}^{N} \frac{(x^{flex} - x^*(\varepsilon_n))}{PC'(x^*(\varepsilon_n))}. \tag{11}$$

Since there are at least two distinct values of $x^*(\varepsilon_n)$, (11) implies that there exists $N^* < N$ such that:

$$x^*(\varepsilon_n) \leq x^{flex}, n \leq N^*$$
$$x^*(\varepsilon_n) > x^{flex}, n > N^*.$$

and so we can rewrite (11) as:

$$0 = N^{-1}[\sum_{n=1}^{N^*} \frac{(x^{flex} - x^*(\varepsilon_n))}{PC'(x^*(\varepsilon_n))} - \sum_{n=N^*+1}^{N} \frac{(x^*(\varepsilon_n) - x^{flex})}{PC'(x^*(\varepsilon_n))}]. \tag{12}$$

where both terms are positive. Since $PC'' < 0$, we know that;

$$PC'(x^{flex}) \leq PC'(x^*(\varepsilon_n)), n \leq N^*$$
$$PC'(x^{flex}) > PC'(x^*(\varepsilon_n)), n > N^*.$$

and so:

$$\sum_{n=1}^{N^*} \frac{(x^{flex} - x^*(\varepsilon_n))}{PC'(x^*(\varepsilon_n))} \leq \sum_{n=1}^{N^*} \frac{(x^{flex} - x^*(\varepsilon_n))}{PC'(x^{flex})}$$
$$\sum_{n=N^*+1}^{N} \frac{(x^*(\varepsilon_n) - x^{flex})}{PC'(x^*(\varepsilon_n))}] > \sum_{n=N^*+1}^{N} \frac{(x^*(\varepsilon_n) - x^{flex})}{PC'(x^{flex})}.$$

Substituting into (12), we obtain:

$$
\begin{aligned}
0 &< N^{-1}[\sum_{n=1}^{N^*} \frac{(x^{flex} - x^*(\varepsilon_n))}{PC'(x^{flex})} - \sum_{n=1}^{N^*} \frac{(x^*(\varepsilon_n) - x^{flex})}{PC'(x^{flex})}] \\
&= \frac{1}{PC'(x^{flex})} N^{-1} \sum_{n=1}^{N} (x^{flex} - x^*(\varepsilon_n))
\end{aligned}
$$

which proves the proposition.

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