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SELF-FULFILLING FLUCTUATIONS IN HANK ECONOMIES

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ABSTRACT

We show that in Heterogeneous-Agent New-Keynesian (HANK) economies with countercyclical risk the natural interest rate is endogenous and co-moves with output, leaving the economy susceptible to self-fulfilling fluctuations. Unlike in Representative-Agent New-Keynesian models, the Taylor principle is not sufficient to guarantee uniqueness of equilibrium in HANK if risk is even mildly countercyclical: multiple bounded-equilibria exist, no matter how strongly monetary policy responds to changes in inflation. For an active-monetary policy to eliminate self-fulfilling fluctua-tions, it must stabilize the endogenous natural rate fluctuations. Alternatively, a passive-monetary and active-fiscal regime can also eliminate equilibrium multiplicity.

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Economists have long argued that shifts in expectations can generate fluctuations in output and inflation even absent any change in fundamentals, and that countercyclical stabilization policy plays an important role in mitigating such fluctuations (Keynes, 1936). Left unchecked, these self-fulfilling beliefs can destabilize the economy. Consequently, a central tenet in the science of monetary policy is that when inflation rises, a central bank should increase nominal interest rates aggressively to raise the real interest rate. Without this aggressive response, a self-fulfilling inflationary process may take hold: households' expectations of higher inflation lowers the real interest rate, which in turn stimulates demand and pushes up actual inflation, thereby confirming the initial belief. Raising nominal rates more than one-for-one with inflation nips these non-fundamental beliefs in the bud, stabilizing the economy. Arguably, the determination to keep inflation in check and inflation expectations anchored underlies the steep increase in policy rates around the world in response to the high inflation during the recovery from the COVID-19 recession.

The Representative-Agent New Keynesian (RANK) models, which provide the theoretical backbone for raising nominal rates more than one-for-one with inflation – the Taylor principle – abstract from inequality, market incompleteness and the distributional effects of monetary policy. In recent years, there has been growing interest among both academics and policymakers to understand how these features affect monetary transmission and whether the aforementioned central tenet continues to hold. The fast-growing Heterogeneous Agent New-Keynesian (HANK) literature seeks to address this.[1] We contribute to this literature by showing that no matter how aggressively monetary policy responds to changes in inflation, HANK economies with countercyclical risk are susceptible to self-fulfilling fluctuations or "*endogenous* demand shocks". This is true even when risk is mildly countercyclical. To understand the key force behind this inherent instability, suppose that absent any change in fundamentals, households suddenly entertain the belief that the economy will enter a recession. If risk is countercyclical, this lower level of economic activity implies that households expect to face higher risk in the future, and increase their desired level of precautionary savings to self-insure against the higher probability of future declines in consumption. This higher desire to save puts downward pressure on the *natural* interest rate. If monetary policy keeps policy rates unchanged in the face of a lower natural rate, households reduce their current spending. In the presence of nominal rigidities, this lower spending translates into lower output and below target inflation, pushing the economy into a recession and rendering the initial pessimistic belief self-fulfilling. This is analogous to an "endogenous" negative demand shock, which results in non-fundamental fluctuations in output and inflation.

The reason why HANK economies with countercyclical risk are susceptible to self-fulfilling fluctuations is that the natural rate of interest is *endogenous* and co-moves with output in such economies.[2] Standard monetary policy rules, even if they satisfy the Taylor principle, allow for self-fulfilling fluctuations in the natural rate, which act as "*endogenous* demand shocks", and result in non-fundamental fluctuations in output and inflation. Thus, standard monetary policy rules cannot implement a unique

[1]See, for example, Kaplan et al. (2018); McKay et al. (2016); Auclert et al. (forthcoming); Acharya and Dogra (2020); Ravn and Sterk (2021); Bilbiie (2024); Gornemann et al. (2016); Ahn et al. (2018) among others.

[2]In the RANK literature, the natural rate is typically defined as the notional real interest rate that would arise in an economy where all prices were flexible. Instead, our definition of the natural rate follows Keynes (1936) and is defined as the real interest rate consistent with output remaining constant at some particular level. Importantly, while this definition of the natural rate coincides with the real interest rate in the flexible price limit in RANK economies and HANK economies with acyclical risk, the two concepts diverge when risk is countercyclical. See Section 2.3 for a detailed discussion.

equilibrium if risk is even mildly countercyclical. To ensure a unique equilibrium, monetary policy must act to prevent these endogenous fluctuations in the natural rate. To see this, consider the same example as above, where households believe that the economy is about to enter a recession. The resulting increase in the desired level of precautionary savings pushes down the natural interest rate. Unlike above, suppose that monetary policy instead lowers nominal rates sufficiently in response to the lower natural rate. This discourages households from reducing their current spending, and output from declining, thus preventing the initial beliefs from being confirmed in equilibrium.[3]

More generally, we show that monetary policy rules that do not address the endogenous fluctuations in the natural rate fail to eliminate these non-fundamental fluctuations driven by endogenous demand shocks. While our baseline model shows this in the context of a simple inflation targeting monetary policy rule, this characterization continues to hold even when we allow for other standard monetary policy rules which have been shown to eliminate non-fundamental fluctuations in RANK. In particular, we show that standard rules which adjust the policy rate in response to output fluctuations or rules which display inertial behavior also fail to eliminate self-fulfilling fluctuations in our HANK economy with countercyclical risk. Instead, a rule which can prevent these non-fundamental fluctuations must adjust the nominal rate at least one-for-one with these endogenous fluctuations in the natural rate. In other words, the rule must satisfy a form of the Taylor principle, but for natural rates. In practical terms, this characterization implies that monetary policy must not only respond aggressively to keep the private sector's inflation expectations on target, but must also act as aggressively to keep their expectations about real activity anchored. In fact, a central bank which ignores expectations about real activity may fail at keeping even inflation expectations anchored, as doing so would leave the door open to self-fulfilling beliefs driving non-fundamental fluctuations in output and inflation.

The source of equilibrium multiplicity described above is conceptually distinct from those identified in the literature on liquidity traps. In particular, Benhabib et al. (2001b) show that the possibility of a binding ELB (effective lower bound) on nominal rates results in multiple equilibria and global indeterminacy in RANK.[4] By contrast, our paper purposely abstracts from an ELB, in order to highlight that countercyclical risk is a distinct force driving the global indeterminacy in HANK.

Our analysis goes beyond most existing studies of determinacy in HANK economies by considering *global*, rather than *local* determinacy. The HANK literature has shown that achieving local determinacy in HANK economies can be more demanding relative to RANK economies. Acharya and Dogra (2020); Bilbiie (2024); Auclert et al. (2023); Ravn and Sterk (2021) find that if income risk is countercyclical, then the Taylor principle is not sufficient to ensure local determinacy. However, they also show that while the simple Taylor principle fails, a "cyclical-risk augmented Taylor principle", which demands a stronger response to inflation than in RANK, is sufficient for ensuring *local* determinacy. We show that while a stronger response may be sufficient for local determinacy, it *cannot* rule out *global* indeterminacy if risk is even mildly countercyclical. The only other paper which has studied global indeterminacy in

[3]In the same fashion, if households have optimistic beliefs about the economy, they perceive that they will face lower income risk, causing them to reduce their desired level of precautionary savings. This in turn puts upward pressure on the natural interest rate. If monetary policy does not raise the policy rate sufficiently in response to this upward pressure on the natural rate, the higher spending by households leads to higher output and inflation, confirming the initial optimistic beliefs. This acts like a positive endogenous demand shock.

[4]See also, Heathcote and Perri (2018), who show that in the presence of a binding ELB, the precautionary savings motive of low wealth households can make the expectation of high unemployment a reality.

the context of HANK models with countercyclical risk is Ravn and Sterk (2021).[5] Our characterization of global indeterminacy in the form of two steady states is complementary to theirs. However, unlike Ravn and Sterk (2021), we provide a complete analytical characterization of the dynamics through which countercyclical risk gives rise to global indeterminacy. Moreover, we also describe what kind of policy design can eliminate global indeterminacy. Importantly, we show that designing policies which simply eliminate the untargeted steady state may be unable to deliver global determinacy. In particular, when we calibrate our model to capture an empirically realistic magnitude of countercyclical risk, we show that global indeterminacy manifests not just in the form of multiple steady states, but also as a stable cycle around the targeted steady state, in which the economy can get trapped permanently. We then show that a policy design which ignores the possibility of a cycle, and focuses only on eliminating the untargeted steady state, is unable to eliminate self-fulfilling fluctuations. Thus, we argue that fully characterizing the different ways in which global indeterminacy can arise—not just through an untargeted steady state—is essential for policy design, since only policies that eliminate *all* non-fundamental equilibria can guarantee global determinacy.

Our paper also contributes to the literature which studies how fiscal policy affects equilibrium determinacy in heterogeneous agent economies. Kaplan et al. (2023) study multiplicity of equilibria in a flexible-price heterogeneous-agent incomplete-market economy with nominal government debt. They show that equilibrium multiplicity can emerge if the government runs persistent deficits in their heterogeneous agent economy. Similarly, Bassetto and Cui (2018) and Farmer and Zabczyk (2019) study overlapping generations economies with no nominal rigidities and show that price-level indeterminacy may emerge. Brunnermeier et al. (2020) and Miao and Su (2024) study fiscal rules that can deliver price level determinacy in incomplete markets economies in which agents face rate-of-return risk. In contrast, we study equilibrium determinacy in heterogeneous agent economies with nominal rigidities, and show that a regime with *active* fiscal policy and *passive* monetary policy can eliminate global indeterminacy.

Our paper is also related to the older literature which studied global determinacy in RANK. Benhabib et al. (2001a) study the global determinacy properties of RANK economies under standard monetary policy and fiscal rules. Their findings imply that the Taylor principle delivers global determinacy unless (i) households enjoy utility from holding money, and the cross-partial derivative of the utility function $\frac{\partial^2 u(c,m)}{\partial c \partial m} < 0$, or (ii) if money is an input in the production function. Benhabib and Eusepi (2005) study a RANK economy with physical capital, and show that global indeterminacy in the form of periodic cycles around the targeted steady state may emerge. Global indeterminacy in our HANK economy arises due to conceptually different reasons relative to these papers since (i) our HANK economy abstracts from the presence of money by considering the cashless limit and (ii) global indeterminacy emerges in our economy even when capital is not a factor of production. Instead, global indeterminacy arises in our HANK economy because of the presence of countercyclical risk.

Finally, Beaudry et al. (2020) argue that alongside business cycle shocks, a substantial part of business cycle fluctuations can be explained by deterministic boom-bust cycles. As in our paper, they show that their New Keynesian model with financial frictions and countercyclical risk-premium fea-

[5] Ravn and Sterk (2021) study a HANK economy with search frictions and also find that global indeterminacy can emerge in their economy with countercyclical income risk. They find that a second "unemployment trap" steady state with 100% unemployment may emerge alongside the targeted equilibrium, implying global indeterminacy in their model.

tures deterministic limit cycles via a Hopf-bifurcation, and that this provides a good description of U.S. business cycle data. Instead, our focus is on understanding how countercyclical income risk in a HANK economy can lead to global indeterminacy.

The rest of the paper is organized as follows. Section 1 describes the model environment, while Section 2 characterizes equilibrium. Section 3 studies local and global determinacy under a standard inflation targeting rule. Section 4.1 identifies the root of global indeterminacy and proposes a monetary policy rule which implements a unique equilibrium, and Section 5 concludes.

1 Model

HANK models are typically not analytically tractable because the distribution of wealth is a state variable which evolves endogenously. To make our point in the clearest possible way, we use an analytically tractable HANK model in continuous time. We achieve analytical tractability in our HANK model by assuming that utility is quasi-linear, which ensures that the aggregate dynamics of output and inflation can be characterized independently of the dynamics of the distribution of wealth.[6] For simplicity, we abstract from aggregate risk.

1.1 Households

There is a continuum of households indexed by $j \in [0,1]$. Each household has identical preferences and the expected discounted lifetime utility of household j at date t can be written as

$$V_j(t) = \max_{\{c_{j,\tau}, n_{j,\tau}\}_{\tau=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-\rho(\tau-t)} \left\{ \frac{c_{j,\tau}^{1-\gamma^{-1}}}{1-\gamma^{-1}} - \psi n_{j,\tau} \right\} d\tau,$$

where $c_{j,\tau}$ and $n_{j,\tau}$ denote the household's date τ consumption and hours worked respectively. γ measures the elasticity of intertemporal substitution. The household's choices at all dates must satisfy the budget constraint and borrowing constraint:

$$\frac{da_{j,t}}{dt} = (i_t - \pi_t)a_{j,t} + w_t \xi_{j,t} n_{j,t} + D_t + T_t - c_{j,t} \qquad \text{with} \qquad a_{j,t} \geq -\underline{a} \tag{1}$$

Households can trade a short-term risk-free nominal bond with return i_t. The real value of bond holdings of household j at time t are denoted by $a_{j,t}$. Household j supplies $\xi_{j,t} n_{j,t}$ effective labor hours and earns labor income $w_t \xi_{j,t} n_{j,t}$, where w_t denotes the real wage. Since idiosyncratic productivity $\xi_{j,t}$ is stochastic, households face labor income risk. In particular, we assume that ξ_j follows a 2-state Poisson process, $\xi_j \in \{\xi_l, \xi_h\}$, where $\xi_h > \xi_l$. In addition to labor income, each household also receives dividends from firms. For simplicity, we assume that all households receive an equal share of dividends D_t and transfers from the government T_t. P_t denotes the aggregate price level in the economy. In addition, each household faces a borrowing constraint, which states that their wealth cannot fall below $-\underline{a}$, where $\underline{a} \geq 0$.

[6] Lagos and Wright (2005) also use quasi-linear preferences for tractability in the context of their monetary-search model.

$\lambda_{l,t}$ denotes the rate at which a household with productivity ξ_h switches to productivity ξ_l, while $\lambda_{h,t}$ denotes the rate at which a household with productivity ξ_l switches to a ξ_h at date t. We allow the switching intensities to vary over time to capture the notion that households face different levels of income risk during economic expansions and contractions. In particular, we assume that

$$\lambda_{l,t} = \overline{\lambda}_l y_t^{-\Theta}, \tag{2}$$

where Θ controls the *cyclicality of income risk*. $\Theta > 0$ implies that when output is above its steady state level (an expansion), ξ_h households are less likely to transition to the low productivity state.[7] In what follows, we normalize steady state output in the targeted steady state 1; see Appendix A.2 for details.

Taking some artistic liberty and treating the ξ_l state as "unemployment", the specification above would imply that households face a lower risk of becoming unemployed during economic expansions. In contrast, $\Theta = 0$ corresponds to *acyclical income risk*: the probability of transitioning from the ξ_h to ξ_l state is independent of the level of economic activity.[8]

Since the rate at which households transition from productivity ξ_h to ξ_l depends on y_t, the fraction of households with productivity ξ_l (given by η_t) and with productivity ξ_h (given by $1 - \eta_t$) would change with the level of economic activity if $\lambda_{h,t}$ was constant. To avoid this complication, we assume that the rate $\lambda_{h,t}$ adjusts to satisfy $\lambda_{h,t}\eta_t = \lambda_{l,t}(1 - \eta_t)$, ensuring that $\eta_t = \overline{\eta}$ for all dates t. This assumption implies that $\lambda_{h,t} = \overline{\lambda}_h y_t^{-\Theta}$, where $\overline{\lambda}_h$ is a constant and the fraction of ξ_l households at any date is given by $\overline{\eta} = \overline{\lambda}_l/(\overline{\lambda}_l + \overline{\lambda}_h)$. This assumption is made for expositional clarity; Appendix E.5 relaxes this assumption and shows that allowing η_t to vary over time, does not change our conclusion.

1.2 Firms

There is a continuum of monopolistically competitive firms indexed by $k \in [0, 1]$. At any date t, firm k produces a differentiated intermediate good $y_{k,t}$, which it sells to a representative final-goods firm. The final-goods firm combines the varieties using a CES aggregator to produce the final-good y_t:

$$y_t = \left[\int_0^1 y_{k,t}^{\frac{\varepsilon-1}{\varepsilon}} dk\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{3}$$

which yields the standard demand system for each variety k:

$$y_{k,t} = \left(\frac{P_{k,t}}{P_t}\right)^{-\varepsilon} y_t \tag{4}$$

Each intermediate goods producer uses a linear production function $y_{k,t} = \ell_{k,t}$, where $\ell_{k,t}$ denotes the effective units of labor employed by firm k. Then, the period t profit of firm k can be written as

[7] Our results do not depend on the precise functional form of how this transition rate depends on the level of y_t. We discuss the robustness of our results to the particular functional form in Section 3.3.

[8] While countercyclical income risk is arguably the empirically relevant benchmark , there is no consensus on the exact measure of countercyclicality. For example, Storesletten et al. (2004) find that the standard deviation of persistent shocks to log household income increases from 0.12 to 0.21 as the aggregate economy moves from peak to trough. However, Guvenen et al. (2014) finds that the variance of income is acyclical, but the left-skewness is countercyclical. More recently, Nakajima and Smirnyagin (2019), using a broader definition of income, find that both the variance and left-skewness is countercyclical. Our formulation is consistent with both the variance and left-skewness being countercyclical.

$D_{k,t} = (P_{k,t}/P_t) y_{k,t} - (1-\tau)\, w_t y_{k,t} - \mathcal{T}_{k,t}$, where τ denotes a payroll subsidy, which we set to $\tau = \varepsilon^{-1}$ to eliminate the average monopolistic markup. Furthermore, this payroll subsidy is financed by imposing a lumpsum tax $\mathcal{T}_{k,t} = \tau w_t y_{k,t}$, which the firm treats as given. In symmetric equilibrium, $y_{k,t} = y_t$ and dividends of any firm $k \in [0,1]$ can be written as $D_t = (1 - w_t)\, y_t$. Finally, nominal rigidities are captured by a forward-looking Phillips curve:

$$\dot{\pi}_t = \rho \pi_t - \kappa \left(w_t - 1 \right), \tag{5}$$

where $w_t - 1$ denotes the deviation of marginal cost from the flexible-price benchmark (in which $w_t = 1$ for all t), and $\kappa > 0$ captures the slope of the Phillips curve. While we postulate this simple linear Phillips curve (5) in our baseline model for tractability, Appendix E.3 shows that our results continue to hold if we instead utilize a non-linear microfounded Phillips curve as in Rotemberg (1982).

1.3 Monetary and Fiscal Policy

Monetary Policy Monetary policy sets the nominal rate i_t according to a standard interest rate rule. While we consider other standard monetary policy rules in Appendix E.1 and E.2, in our baseline model, we specify monetary policy as a simple inflation-targeting rule:

$$i_t = \bar{r} + \phi_\pi \pi_t \qquad \text{where} \qquad \phi_\pi > 1, \tag{6}$$

$\phi_\pi > 1$ in (6) denotes how aggressively the central bank raises the nominal rate when inflation is above its steady state level of 0. As is standard in the RANK literature, we set the intercept to $\bar{r}$, which denotes the real interest rate in the flexible-price limit of our economy. As Appendix A.3.1 shows, since we abstract from aggregate shocks, the unique equilibrium in the flexible-price limit features a constant level of output $y_t = 1$ and real interest rate $\bar{r}$ at each date t. Throughout this paper we will require that the real interest rate in the targeted steady state is positive: $\bar{r} > 0$.

Finally, it is important to point out that (6) does not impose an effective-lower bound (ELB) on nominal rates. We purposely make this choice to highlight that the source of multiplicity that we uncover is conceptually different from that in Benhabib et al. (2001b), where multiple equilibria arise due to the presence of an ELB.

Fiscal policy For simplicity, in our baseline model we set government expenditures to zero, and assume that the government runs a balanced budget at each date: $\tau w_t y_t = T_t$, where $\tau w_t y_t$ is the payroll subsidy paid out to firms, and T_t is the lumpsum tax on the same firms. We extend our analysis to include government debt in Section 4.2, where we study non-Ricardian fiscal policy rules.

2 Equilibrium

2.1 Household decisions

Since households have quasi-linear preferences, the date t optimal consumption decision of household j with idiosyncratic productivity ξ_j and wealth a_j can be written as:

$$c_t(a_j, \xi_j) = \left(\frac{\xi_j w_t}{\psi}\right)^{\gamma} \tag{7}$$

Equation (7) shows that the optimal consumption of household j does not depend on their wealth. Consequently, in equilibrium, all households with idiosyncratic productivity ξ_j, $j \in \{l, h\}$ enjoy the same level of consumption. We refer to the date t consumption of a household with productivity ξ_j, $j \in \{l, h\}$ as $c_{j,t}$. Normalizing $\psi = \left[(1-\overline{\eta})\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}\right]^{\frac{1}{\gamma}}$, we have:

$$c_{h,t} = \frac{\xi_h^{\gamma}}{(1-\overline{\eta})\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}} w_t^{\gamma} \qquad \text{and} \qquad c_{l,t} = \frac{\xi_l^{\gamma}}{(1-\overline{\eta})\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}} w_t^{\gamma}, \tag{8}$$

where $c_{h,t} > c_{l,t}$. While the consumption of each household does not depend on their wealth, the amount of leisure they enjoy does depend on how wealthy they are. Comparing two households with the same idiosyncratic productivity, the household with higher wealth tends to enjoy more leisure. Furthermore, Appendix A.1 shows that for a given real interest rate, the expected consumption growth of households with productivity ξ_l is always greater than that of households with productivity ξ_h. Thus, in equilibrium, all households with productivity ξ_l are borrowing constrained. In contrast, ξ_h households are *on* their Euler equation, which can be written as:

$$\frac{\dot{c}_{h,t}}{c_{h,t}} = \underbrace{\gamma\Big(i_t - \pi_t - \rho\Big)}_{\text{intertemporal-substitution}} + \underbrace{\gamma\lambda_{l,t}\left[\left(\frac{c_{l,t}}{c_{h,t}}\right)^{-\frac{1}{\gamma}} - 1\right]}_{\text{precautionary savings}} \tag{9}$$

The first term on the RHS of (9) shows that the consumption growth of unconstrained households depends positively on the real interest rate $r_t = i_t - \pi_t$: the *intertemporal-substitution* channel. A higher real interest rate, holding all else constant, incentivizes the unconstrained households to delay consumption, raising their consumption growth. The second term on the RHS of (9) captures the *precautionary-savings* motive. The larger the drop in consumption when a ξ_h household transitions to the ξ_l state (smaller c_l/c_h), the stronger is this motive. Similarly, holding c_l/c_h fixed, a higher risk aversion (smaller γ) or a higher probability of transitioning from ξ_h to ξ_l (higher $\lambda_{l,t}$) also strengthen this motive and increase consumption growth for any given real interest rate.

2.2 Market Clearing

Goods market clearing requires that the amount of final good produced must be consumed:

$$y_t = (1-\overline{\eta})\, c_{h,t} + \overline{\eta} c_{l,t}, \tag{10}$$

where the RHS of (10) is equal to aggregate consumption. Using (8), Appendix A.2 shows that (10) implies a log-linear relationship between aggregate output and wages:

$$w_t = y_t^{\frac{1}{\gamma}} \tag{11}$$

Since wages $w = 1$ in the targeted steady state, (11) also implies that the steady state level of output in the targeted steady state is $y = 1$. Using (8) and (11) also implies that $c_{h,t}$ and $c_{l,t}$ can be rewritten as:

$$c_{h,t} = \frac{\xi_h^\gamma}{(1-\overline{\eta})\xi_h^\gamma + \overline{\eta}\xi_l^\gamma} y_t \qquad \text{and} \qquad c_{l,t} = \frac{\xi_l^\gamma}{(1-\overline{\eta})\xi_h^\gamma + \overline{\eta}\xi_l^\gamma} y_t \qquad \Rightarrow \qquad \frac{c_{h,t}}{c_{l,t}} = \left(\frac{\xi_h}{\xi_l}\right)^\gamma, \tag{12}$$

While consumption of both ξ_h and ξ_l households co-moves with output, the ratio $c_{h,t}/c_{l,t} > 1$ is constant over time. Finally, at any date t, since all ξ_l households are borrowing constrained and have $a = -\underline{a}$, asset market clearing requires that asset holdings of ξ_h households as a whole is given by $\overline{\eta}\underline{a}$.

2.3 The neutral rate and the natural rate of interest

As is standard in the textbook treatment of the RANK model, the aggregate dynamics of output and inflation in our tractable HANK economy can be also summarized by an IS curve and a Phillips curve. Using (11) in (5), we can express the Phillips curve in terms of output and inflation:

$$\dot{\pi}_t \;=\; \rho\pi_t - \kappa\Big(y_t^{\frac{1}{\gamma}} - 1\Big) \tag{13}$$

Appendix A.3.1 shows that using (12) and (2) in (9), the "IS curve" in our HANK economy is given by:[9]

$$\frac{\dot{y}_t}{y_t} \;=\; \gamma\Big(i_t - \pi_t - r^\star(y_t)\Big) \qquad \text{where} \qquad r^\star(y) = \rho - \sigma y^{-\Theta} \tag{14}$$

As in the textbook 3-equation RANK model, (14) shows that output growth is positive when the monetary policy implements a real interest rate $r_t = i_t - \pi_t$ which is higher than *natural rate of interest* $r^\star(y_t)$. Here, we define $r^\star(y)$ as the real interest rate which sets $\dot{y}_t/y_t = 0$ and $y_t = y$ for all t, i.e., $r^\star(y)$ is the real interest rate which is consistent with output remaining constant at $y_t = y$ for all t.

In the RANK limit of our model ($\overline{\lambda}_l = 0$), since households do not face consumption risk and because we abstract from aggregate shocks, the natural rate is constant over time and simply equal to the discount rate: $r^\star(y) = \rho$. This means that setting $r = r^\star = \rho$ in RANK is consistent with output remaining fixed at any level of output y. The *natural* rate $r^\star(y)$ also coincides with the real interest rate in the flexible-price limit of our economy, $\overline{r} = r^\star(y) = \rho$. The same is true in HANK if risk is acyclical ($\Theta = 0$). The only difference is that now both the natural rate and the flexible-price real interest rate are lower than in RANK, and are given by $r^\star(y) = \overline{r} = \rho - \sigma$, where $\sigma = \overline{\lambda}_l\left(\frac{\xi_h}{\xi_l} - 1\right)$ measures the expected increase in marginal utility when a ξ_h household transitions to lower idiosyncratic productivity ξ_l,

[9] In equilibrium, while the actual value of the debt limit $\underline{a}$ does not affect aggregate dynamics of y_t and π_t, it does matter for the wealth distribution that emerges in equilibrium. However, quasi-linear preferences render our economy block-recursive, allowing us to characterize the dynamics of output and inflation independently of the wealth distribution.

and captures the fact that households face consumption risk. Clearly, $\sigma = 0$ if the probability of transitioning to the low productivity state is zero ($\overline{\lambda}_l = 0$), or if the consumption across the two idiosyncratic productivity levels is the same (which occurs if $\xi_h = \xi_l$). In all other cases, $\sigma > 0$. Since households face consumption risk, they save for precautionary reasons, and this requires a lower interest rate for the asset market to clear.

However, if risk is countercyclical ($\Theta > 0$), the natural rate is *endogenous* and co-moves with output:

$$\frac{dr^\star(y)}{dy} = \sigma\Theta y^{-(1+\Theta)} > 0, \tag{15}$$

i.e., the real interest rate consistent with output being constant at a particular level y, now depends on the level of output itself. Moreover, the natural rate $r^\star(y)$ is an increasing function of y. Thus, in our HANK economy with countercyclical risk, the flexible price real interest rate $\overline{r} = \rho - \sigma$ is, in general, different than the natural rate $r^\star(y)$. The two concepts only coincide in steady state when $y = 1$: $\overline{r} = r^\star(1)$. To see why, recall that if output was lower than $y = 1$, countercyclical risk implies that ξ_h households face greater consumption risk, as now they face a larger chance of switching to the ξ_l state. This causes them to reduce their current consumption demand and increase their precautionary savings. This greater desire to save implies that a lower real interest rate is required to clear asset markets and keep demand constant at that lower level.

Notice that our definition of the natural rate of interest $r^\star(y)$ differs from how the term "natural rate" is used in the New Keynesian literature (see, e.g., Woodford (2003a); Galí (2015)). In the New Keynesian literature, the natural rate is typically defined as the real interest rate which would prevail in the flexible-price limit of the economy, which is equal to $\overline{r} = r^\star(1) = \rho - \sigma$ in our HANK model. This flexible-price real interest rate depends on exogenous parameters, and potentially varies over time only in response to *exogenous* shocks, e.g. shocks to the discount rate ρ. Importantly, it does not depend on *endogenous* variables such as the level of output. Consequently, in RANK and in HANK with acyclical risk, our definition of natural rate coincides with the flexible-price real interest rate. In contrast, when risk is countercyclical $\Theta > 0$, the natural rate $r^\star(y)$, as we define it, *does* depend on *endogenous* variables, specifically output: in a weak economy, where output is below its flexible price level, a lower real interest rate is required to maintain demand, and hence output, at that level. In contrast, even in our economy with $\Theta > 0$, the flexible-price real interest rate does not depend on endogenous variables, $\overline{r} = \rho - \sigma$. Thus, in our HANK economy with countercyclical risk our definition of the natural rate does not always coincide with the flexible-price real interest rate: there are many *natural* rates $r^\star(y)$, one for a given level of y, but there is a unique flexible-price real interest rate $\overline{r}$, which coincides with the natural rate consistent with $y = 1$. Our choice of terminology hearkens back to Keynes (1936) (pp. 242-243):

> *For every rate of interest there is a level of employment for which that rate is the "**natural**" rate, in the sense that the system will be in equilibrium with that rate of interest and that level of employment. ... we might term the **neutral** rate of interest, ... the natural rate in the above sense which is consistent with full employment, given the other parameters of the system.* [emphasis ours]

2.4 Aggregate dynamics and steady states

Given the monetary policy rule (6), Proposition 1 below shows that the aggregate dynamics in our HANK economy are described by the 2-dimensional system of ordinary differential equations (ODEs). However, rather than describing the dynamics of output, it is more convenient to characterize the dynamics of the (scaled) *output-gap*, which We define as the log-deviation of output from its flexible-price level scaled by γ^{-1}: $x = \gamma^{-1}(\ln y - \ln 1)$.

Proposition 1. *Given the interest rate rule (6), the aggregate dynamics of x_t, π_t can be written as:*

$$\dot{x}_t = (\phi_\pi - 1)\,\pi_t - \Big(r^\star(x_t) - \bar{r}\Big) \tag{16a}$$

$$\dot{\pi}_t = \rho\pi_t - \kappa\,(\mathrm{e}^{x_t} - 1) \tag{16b}$$

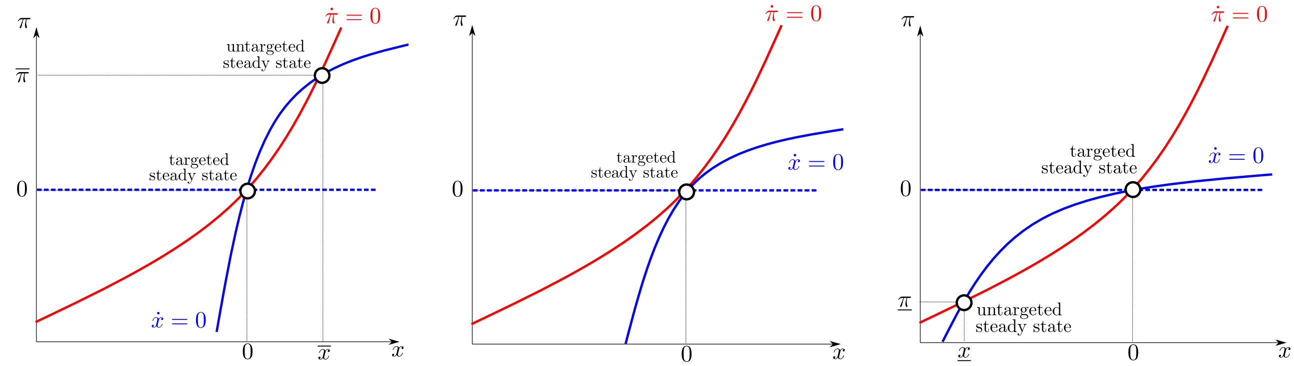
where $r^\star(x_t) - \bar{r} = \sigma\left(1 - \mathrm{e}^{-\gamma\Theta x_t}\right)$ denotes the difference between the natural rate of interest and neutral rate of interest. The target steady state is given by $x = \pi = 0$.

Proof. See Appendix A.3.1. □

Equations (16a)-(16b) nest the RANK benchmark: in RANK $\sigma = 0$ and so, $r^\star(x_t) = \bar{r} = \rho$ for any x, implying that the last two terms on the RHS of (16a) eliminate each other. The same is true in HANK if risk is acyclical ($\Theta = 0$), even though the natural and neutral rates of interest are both lower in this case than in RANK. Thus, (16a) and (16b) show that the global dynamics in the RANK benchmark and in a HANK economy with acyclical ($\Theta = 0$) are identical as long as ϕ_π is the same in both economies. Thus, as has also been pointed out by Werning (2015), simply the presence of risk does not necessarily alter the dynamics of output and inflation. In contrast, when risk is countercyclical, (16a) reveals that an extra force shapes global dynamics relative to the RANK and acyclical risk benchmark. In particular, fluctuations in the output-gap drive an *endogenous* gap between the natural rate and neutral rate of interest, which in turn feeds back into the dynamics of output and inflation.

The three panels of Figure 1 plot the nullclines associated with (16a)-(16b) in (x, π) space for different values of ϕ_π. The solid-red line depicts the $\dot{\pi}_t = 0$ nullcline, which is unaffected by the presence of risk or the cyclicality of risk. In contrast, in the RANK benchmark $\sigma = 0$ (or in HANK with $\Theta = 0$), the $\dot{x} = 0$-nullcline (equivalently the long-run IS) is depicted by the dashed-horizontal curve at $\pi = 0$, while in HANK with $\Theta > 0$, it is depicted by the upward sloping solid-blue curve. The reason that the nullcline is flat in RANK (or in HANK with $\Theta = 0$) is that $r^\star(x)$ is constant, but when risk is countercyclical, $r^\star(x)$ co-moves with output, and is thus upward sloping.

Any intersection of the nullclines constitutes a steady-state. Clearly, both nullclines intersect at $x = \pi = 0$, implying that the *targeted steady state* always exists, in which output equals its flexible-price level ($x = 0$) and inflation is on target ($\pi = 0$). In fact, as Figure 1 shows, this is the only steady state in the RANK limit or when risk is acyclical. In this case, the long-run IS curve is depicted by the horizontal dashed-blue curve and only intersects the $\dot{\pi} = 0-$nullcline at the targeted steady state. However, when risk is countercyclical ($\Theta > 0$), the long-run IS curve intersects the $\dot{\pi} = 0-$nullcline twice, implying that an additional untargeted steady state emerges (see Appendix B.3 for a proof). As we discuss next, whether output is above or below its level in the targeted steady state in this untargeted steady state depends on how aggressive monetary policy is (magnitude of ϕ_π).



(a) $1 \leq \phi_\pi < \varphi(\Theta)$ (b) $\phi_\pi = \varphi(\Theta)$ (c) $\phi_\pi > \varphi(\Theta)$

Figure 1: Multiple steady states with $\Theta > 0$

The emergence of the second steady state is rooted in the fact that, with countercyclical risk, households face greater risk when output is lower. Let's start by considering the case with $1 < \phi_\pi < \varphi(\Theta)$ (depicted graphically in Figure 1a). In this case, the untargeted steady state features a positive output-gap $\overline{x} > 0$, and above-target inflation $\overline{\pi} > 0$. To see why such a steady state emerges, suppose that households believe that the economy will have higher output, $\overline{x} > 0$, forever. Because risk is countercyclical, this belief about higher output also implies that households perceive that they face lower income risk. This causes them to reduce their precautionary savings demand and to increase consumption spending. Owing to the presence of nominal rigidities, this higher spending puts upward pressure on output and inflation. When $1 < \phi_\pi < \varphi(\Theta)$, the monetary policy rule (6) raises nominal rates in response to the higher inflation, but the implied increase in real interest rates is not sufficient to dissipate this higher demand, thus allowing the beliefs of higher output to become self-fulfilling.

Raising ϕ_π towards $\varphi(\Theta)$ induces a larger increase in nominal and hence real interest rates. The higher real rate lowers output in the second steady state, bringing $\overline{x}$ closer to 0. Graphically, a higher ϕ_π shifts the solid-blue long run IS curve lower and makes the untargeted steady state shift lower down on the $\dot{\pi} = 0$-nullcline, closer to $(0, 0)$. In fact, as one increases ϕ_π all the way to $\varphi(\Theta)$, the long run IS curve becomes tangent to the $\dot{\pi} = 0$-nullcine at the targeted steady state. This is depicted graphically in Figure 1b. In this knife-edge case, the only steady state is the targeted steady state $x = \pi = 0$.

Increasing ϕ_π even further ($\phi_\pi > \varphi(\Theta)$), shifts the long-run IS curve down further (depicted in Figure 1c), and multiple steady states emerge again. The untargeted steady state now features lower output $\underline{x} < 0$ and below target inflation $\underline{\pi} < 0$. With countercyclical risk, lower output ($\underline{x} < 0$) implies that households face more risk in this steady state compared to the targeted steady state, prompting them to increase their precautionary savings. This causes them to lower spending, which puts downward pressure on output and inflation. Monetary policy, following the rule (6), lowers the nominal rate in response to the lower inflation, but despite this, the equilibrium real interest rate remains too high to discourage the higher precautionary savings. This reinforces lower household demand, trapping the economy at a lower level of economic activity $\underline{x} < 0$. Appendix B.3 shows that this untargeted steady state persists even if we keep raising ϕ_π further (as long as it remains finite). Increasing ϕ_π further only has the effect of making output in the untargeted steady state even lower.

While the discussion above show that in our HANK economy with countercyclical risk, the untargeted steady state can feature output which is higher, lower or equal to that in the targeted steady state, in what follows, we will focus on the scenario in which the targeted steady state is *locally determinate*. As we show in Section 3, this requires that $\phi_\pi > \varphi(\Theta)$, implying that the untargeted steady state always exists and features lower output and inflation than in the targeted steady state (Figure 1c).

3 Local vs global determinacy of equilibrium

In order to distinguish between local and global determinacy, it is helpful to rewrite (16a)-(16b) as:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \underbrace{A \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}}_{\substack{\text{first-order} \\ \text{terms}}} + \underbrace{\begin{bmatrix} \sigma \sum_{s=2}^{\infty} (-1)^s \frac{\gamma^s \Theta^s}{s!} x^s \\ -\kappa \sum_{s=2}^{\infty} \frac{1}{s!} x^s \end{bmatrix}}_{\substack{\text{higher-order} \\ \text{terms}}} \qquad \text{with} \qquad A = \begin{bmatrix} -\sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}, \tag{17}$$

Rewriting (16a)-(16b) in this way highlights that the dynamics of (x_t, π_t) local to the targeted steady state $(0, 0)$ are dominated by the first-order terms, while the higher-order terms dominate the dynamics when the economy is further away from the targeted steady state.

Local determinacy requires that any trajectory, other that $(x_t, \pi_t) = (0, 0)$, that starts in a small neighborhood of the targeted-steady state $(0, 0)$ leaves this neighborhood, i.e., all these trajectories *do not remain bounded* inside this neighborhood. The only trajectory that stays bounded in this neighborhood is $(x_t, \pi_t) = (0, 0)$, implying that $(0, 0)$ is the unique *bounded* equilibrium in its neighborhood. In other words, if one limits analysis to rational expectations equilibria in which (x, π) remain forever in a small neighborhood of the targeted-steady state $(0, 0)$, then the only bounded equilibrium is $(0, 0)$. Since the first-order terms dominate the behavior of x_t, π_t in this small neighborhood around the targeted steady state, whether the economy is locally determinate around the targeted steady state depends on the eigenvalues of the matrix A. In particular, we need both eigenvalues of A to *explosive*, i.e., they have positive real parts. Proposition 2 formally describes when this is the case.

Proposition 2 (Local determinacy in HANK with countercyclical risk)**.** *The targeted equilibrium of the economy described by* (16a)-(16b) *is locally determinate if ϕ_π satisfies*

$$\phi_\pi > \varphi(\Theta) \qquad \textit{where} \qquad \varphi(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa}, \tag{18}$$

provided that risk is not too countercyclical $\Theta \in [0, \Theta^\star)$, where $\Theta^\star \equiv \frac{\rho}{\sigma\gamma}$. If $\Theta > \Theta^\star$, then the targeted equilibrium is locally indeterminate for any finite ϕ_π, no matter how large it is.

Proof. See Appendix B.2. □

Equation (18) is the analog of the "cyclical-risk" augmented Taylor principle derived in Acharya and Dogra (2020), Auclert et al. (2023) and Bilbiie (2024), in the context of our model, and states that as long as risk is not too countercyclical, a large enough ϕ_π which satisfies (18) ensures *local* determinacy. This condition simplifies to the standard Taylor principle $\phi_\pi > 1$ in the RANK limit of our model $(\sigma = 0)$ and also when households face risk $(\sigma > 0)$, but this risk is acyclical $(\Theta = 0)$. However, when risk is countercyclical $\Theta > 0$, (18) shows that monetary policy needs to respond more aggressively to changes in inflation, the more countercyclical risk is.

To see why a larger ϕ_π is needed to ensure local determinacy with countercyclical risk, it is useful to first understand why $\phi_\pi > 1$ ensures local determinacy in RANK. As Cochrane (2011) explains: absent the ZLB, the Taylor principle ensures a unique bounded equilibrium in RANK, because off-equilibrium, "*higher inflation leads the Fed to set interest rates in a way that produces even higher future inflation*". In other words, imposing the Taylor principle induces explosive dynamics if the economy is not on the targeted equilibrium, thus leaving the targeted equilibrium as the *unique bounded equilibrium* in RANK. To see this in our model, imposing the RANK limit $\sigma = 0$ in (16a), the IS curve is given by:

$$\dot{x}_t = (\phi_\pi - 1)\pi_t$$

With $\phi_\pi > 1$, the expression above shows that $\dot{x}_t$ is *increasing* in π_t, i.e., any deviation of inflation from its target induces destabilizing dynamics, causing x to change. For example, consider a case in which inflation is below target at date 0, $\pi < 0$. Then, if $\phi_\pi > 1$, the IS curve (16a) implies that x_t must decline. Since the Phillips curve (16b) implies that inflation at any date is the net-present discounted value of future marginal costs, π falls further below target over time.[10] These destabilizing dynamics induced by the Taylor principle ensure that any trajectory which originates at any point other than $(0, 0)$ does not remain bounded, and hence is not a valid equilibrium. In fact, the larger is ϕ_π relative to 1, the more destabilizing these dynamics are. Consequently, $(x, \pi) = (0, 0)$ is the only *locally bounded* equilibrium in RANK if the Taylor principle is satisfied.

In contrast, when risk is countercyclical, the IS equation (16a) can be written as:

$$\dot{x} = \underbrace{(\phi_\pi - 1)\pi - \sigma\gamma\Theta x}_{\text{first-order terms}} + \text{higher-order terms}, \tag{19}$$

and since local determinacy depends on the behavior of (x, π) local to the targeted steady state $(0, 0)$, only the first-order terms of the IS curve matter for local determinacy. When $\phi_\pi > 1$, the first linear-term on the right-hand-side (RHS) of (19) still induces destabilizing dynamics as in RANK. However, in HANK with countercyclical risk, $\Theta > 0$, the natural rate $r^\star(x)$ endogenously *co-moves* with x. This is reflected in the extra first-order term $-\sigma\gamma\Theta x$ on the RHS, which induces *stabilizing* dynamics because the coefficient on x is negative: a positive x causes $\dot{x}$ to become negative, i.e., it causes x to return to 0, given all else. Thus, when risk is countercyclical, ϕ_π need to be larger so that the destabilizing effect overwhelms this stabilizing effect. The cyclical-risk augmented Taylor principle (18) states how large ϕ_π needs to be for this to be the case. Since this stabilizing force is not present in RANK or in HANK with acyclical risk, $\phi_\pi > 1$ suffices.

Global determinacy is more demanding than local determinacy as it requires that any trajectory,

[10] Output declines following date 0 unambiguously, but whether it declines monotonically over time or not depends on the eigenvalues of the system. While the Taylor principle ensures that the 2 eigenvalues are positive, the eigenvalues can either both be real or both complex with positive real parts. When the eigenvalues are real, x continues to decline monotonically towards $-\infty$, while if the two roots are complex, the trajectory of x is oscillatory with ever increasing amplitude. However, in both cases, as long as the Taylor principle is satisfied, the trajectory for x diverges away from its targeted level of 0 over time. Whether inflation diverges from its targeted level monotonically or in an oscillatory fashion depends on whether output declines monotonically or diverges in an oscillatory fashion, which in turn depends on the eigenvalues of the system.

starting at any $(x, \pi) \in (-\infty, \infty)^2$ other than $(x, \pi) = (0, 0)$, *does not remain bounded*. Thus, global determinacy does not just depend on the behavior of x, π local to the targeted steady state. Consequently global determinacy depends not just on the first-order terms but also the higher-order terms in (17). In linear models, local determinacy implies global determinacy as there are no higher-order terms when the model is actually linear. However, in non-linear models local determinacy *need not* imply global determinacy. This is because local determinacy only ensures that the trajectories starting in small neighborhood of the targeted equilibrium diverge away from this equilibrium. It, however, does not ensure that once this trajectory takes the economy further away from the targeted equilibrium, that the higher-order terms prevent this trajectory from growing unbounded. Thus, when one concludes that the equilibrium is unique in a non-linear model by checking for local determinacy, one is implicitly making an assumption that a trajectory which initially diverges from the targeted equilibrium eventually becomes unbounded. However, this assumption may not hold.

Even though our model economy is non-linear in RANK limit ($\sigma = 0$) or in the acyclical risk case ($\sigma > 0, \Theta = 0$), Appendix B.1 shows that the standard Taylor principle $\phi_\pi > 1$, which ensures that the targeted equilibrium is locally determinate, also ensures that it is globally determinate.[11] This is because in the RANK limit or in HANK with acyclical risk, the IS curve does not have any higher-order terms. When risk is countercyclical, $\phi_\pi > \varphi(\Theta)$ still ensures that local to $(0, 0)$, the economy features explosive dynamics.[12] Thus, while $\phi_\pi > \varphi(\Theta)$ ensures that the first-order terms in (19) induce explosive dynamics, it cannot guarantee that the higher-order terms also do so. In fact, as we discuss next, in our HANK economy with countercyclical risk, these higher-order terms induce a stabilizing force which cannot be overwhelmed no matter how large ϕ_π is. Consequently, there exists multiple trajectories $\{x_t, \pi_t\}_{t \geq 0}$ which satisfy all equilibrium conditions and remain bounded, i.e., multiple equilibria exist, implying that there is global indeterminacy. This is formalized in Proposition 3 below.

Proposition 3 (Global Indeterminacy with countercyclical risk)**.** *Consider the economy described in Proposition 1 for any $\Theta > 0$ and assume that $\phi_\pi > \varphi(\Theta)$. Then for any $\Theta > 0$, the equilibrium is globally indeterminate, no matter how large ϕ_π is (as long as it is finite).*

Proof. See Appendix B.5. □

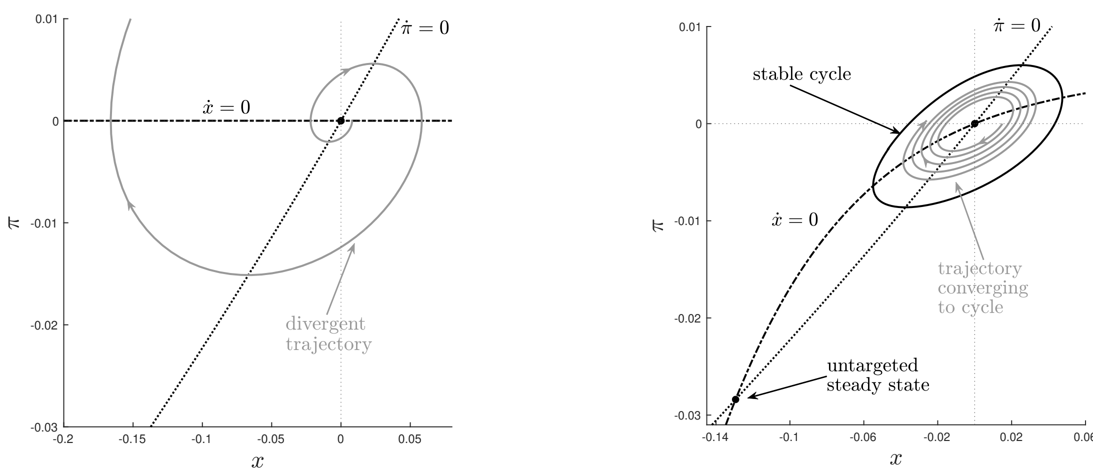
The existence of multiple equilibria opens the door to self-fulfilling beliefs which can drive non-fundamental fluctuations, and can even lead output and inflation to permanently deviate from the targeted steady state. The easiest way to see that multiple equilibria exist is to recall from section 2.4 that when risk is countercyclical, an untargeted steady state (which features a lower level of economic activity), exists alongside the targeted steady state. Thus, there are at least two trajectories, $\{x_t, \pi_t\}_{t>0} = (0, 0)$ and $\{x_t, \pi_t\}_{t>0} = (\underline{x}, \underline{\pi})$, which satisfy all equilibrium conditions and remain bounded, implying that we have global indeterminacy. Even absent the ELB, households' pessimistic self-fulfilling beliefs can cause the HANK economy with countercyclical risk to *jump* from the targeted

[11] Of course, global determinacy would not obtain if we enforced an ELB since Benhabib et al. (2001b) show that imposing an ELB introduces multiple bounded trajectories which are consistent with equilibrium, implying that there can be global indeterminacy even when the target equilibrium is locally determinate. As aforementioned, we purposely do not impose an ELB to highlight that multiplicity of equilibria can emerge in HANK models even absent the ELB.

[12] This is true if risk is not too countercyclical: $\Theta \in (0, \Theta^\star)$. Proposition 2 already shows that when $\Theta > \Theta^\star$, no matter how large ϕ_π is, monetary policy cannot induce destabilizing dynamics if the economy strays from the targeted equilibrium.

to the untargeted steady state. While this is reminiscent of the secular stagnation literature,[13] which showed that the economy can get stuck at low levels of economic activity because monetary policy is stuck at the ELB, our HANK economy stagnates not because of a binding ELB, but because monetary policy fails to fully account for the endogenously lower natural rate.

However the hazards induced by the inflation targeting rule (6) in our HANK economy are not limited to such permanent slumps. Appendix B.5 shows that the inability of monetary policy to fully account for endogenous fluctuations in the natural rate can also lead to other, less abrupt, non-fundamental fluctuations, the precise form of which depends on how countercyclical risk is. In particular, the global dynamics of our economy can be divided into three regimes, which we label (i) mildly countercyclical , (ii) moderately countercyclical and (iii) highly countercyclical. Formally, we deem risk to be highly countercyclical if $\Theta > \Theta^\star$, where $\Theta^\star = \rho/\sigma\gamma$ is the same as in Proposition 2. Risk is said to be moderately countercyclical if $\Theta \in (\Theta^\diamond, \Theta^\star)$, where the threshold $\Theta^\diamond$ lies between 0 and $\Theta^\star$, and is described in Appendix B.5. Finally, we call risk mildly countercyclical when $\Theta \in (0, \Theta^\diamond)$. Proposition 2 already shows that if risk is highly countercyclical, even with $\phi_\pi > \varphi(\Theta)$, the targeted equilibrium is locally *indeterminate*, and thus also globally indeterminate. Thus, in our discussion below, we only focus on the cases in which risk is mildly and moderately countercyclical.



(a) acyclical risk and RANK: ($\Theta = 0$ or $\sigma = 0$)   (b) moderately countercyclical risk ($\Theta^\diamond < \Theta < \Theta^\star$)

Figure 2: **Phase portraits with acyclical risk and countercyclical risk.** In both panels, the dotted curve depicts the $\dot{\pi} = 0$-nullcline, while the dash-dotted curve depicts the $\dot{x} = 0$-nullcline. In panel (a), the gray solid trajectory depicts a representative trajectory which originates near $(0, 0)$ and then grows unbounded. In panel (b), the gray solid trajectory depicts a representative trajectory which originates near $(0, 0)$ and then converges to the stable cycle, which is depicted by the black solid trajectory. Both panels utilize the calibration described in Section 3.1.

When risk is mildly or moderately countercyclical, (18) ensures that the targeted steady state equilibrium is locally determinate. Consequently, any trajectory which starts in the neighborhood of $(0, 0)$ initially diverges away from the targeted steady state. However, as Appendix B.5 shows, not all these initially divergent trajectories grow unbounded eventually. In particular, when risk is mildly countercyclical $\Theta \in (0, \Theta^\diamond)$, the stabilizing influence of the higher-order terms ensures that there exists a *saddle-connection* along which the economy can transition from the neighborhood of the targeted

[13] See, e.g., Benigno and Fornaro (2018); Eggertsson et al. (2019).

steady state to the untargeted steady state and thus remain bounded. Furthermore, any trajectory which originates on this saddle connection also remains bounded. Similarly, when risk is moderately countercyclical, (18) still ensures that the first-order terms cause any trajectory which starts in the neighborhood of $(0, 0)$ to initially diverges away from the targeted steady state (depicted by the gray trajectory in Figure 2b). However, Appendix B.5 shows that *none* of these initially divergent trajectories grow unbounded as the higher-order terms *push back* towards the targeted steady state. Consequently, these trajectories converge to a stable-limit cycle (depicted by the black trajectory in Figure 2b). This shows that even a small shock which dislodges the economy from the targeted steady state can cause the economy to get stuck in a cycle in which the output-gap and inflation are permanently away from their targeted values. In contrast, in the RANK limit of our model ($\sigma = 0$) or if risk is acyclical ($\Theta = 0$), the absence of non-linear terms in the IS equation implies that even with the standard inflation targeting rule (6), the targeted equilibrium $(x_t, \pi_t) = (0, 0)$ for all t is the only bounded trajectory which satisfies all equilibrium conditions. This is depicted in Figure 2a, which plots a trajectory starting away from $(0, 0)$, and shows that it diverges away from the targeted equilibrium and eventually becomes unbounded. Figure 2 only depicts the dynamics in the acyclical risk and moderately countercyclical cases, but Figure 4 in Appendix B.5 describes global dynamics for all values of Θ.

It is useful to point out that, because our baseline model does not feature any predetermined variables, simply the existence of the untargeted steady state (even without characterizing these additional non-fundamental dynamics) is sufficient to establish global indeterminacy. This is because, absent any predetermined variable, the economy can just *jump* from one steady state to another. However, our characterization of the complete set of possible non-fundamental fluctuations is more than a theoretical curiosity. This is because, in Section 4, when we discuss policy design to ensure global determinacy, it will be important to design a rule which can eliminate *all* non-fundamental fluctuations, and not just the untargeted steady state. In particular, we show that simple policies which are designed to only eliminate the untargeted steady state, may fail to eliminate the stable cycle which surrounds the targeted steady state. Consequently, even absent an untargeted steady state, the equilibrium is globally indeterminate because any trajectory starting in the neighborhood of the targeted steady state initially diverges but then converges to the stable cycle, hence remaining bounded.

Moreover, our characterization above helps demonstrate that our findings in the baseline model are robust to when we extend the model to include predetermined variables. Since our baseline model has no predetermined variables, the economy can stochastically jump not just between the targeted steady state and the untargeted steady state, but also between the cyclical trajectories described above in response to *sunspot* shocks. However, such jumps are not possible if the model includes a once a predetermined variable. To show that our characterization of global indeterminacy is robust to the inclusion of predetermined variables, in Appendix E.5, we relax the assumption that the fraction of ξ_l households remains constant. Despite this change, an untargeted steady state with lower economic activity as well as a higher fraction of ξ_l households exists in this extension as long as risk is countercyclical. However, simply the existence of an untargeted steady state no longer implies global indeterminacy since the fraction of ξ_l households at any date is predetermined, and thus cannot *jump*. Despite this caveat, Appendix E.5 shows that we still have global indeterminacy. In fact, the additional bounded trajectories resemble those described in our baseline model. For example, when risk is mildly countercyclical, there exists a *saddle connection* along which the economy can transition from near the targeted steady state to the untargeted steady state. The economy can no longer instantaneously jump from one steady state to another: the transition features a gradual increase in the fraction of ξ_l households alongside a gradual decline in output and inflation. Similarly, when risk is moderately countercyclical, there exists a stable cycle which surrounds the targeted steady state. Appendix E.5 shows that any trajectory starting in the neighborhood of the targeted steady state eventually converges to the stable cycle and remains bounded. Unlike in our baseline model, where the economy could instantaneously jump onto the stable cycle, this transition to the stable cycle is also gradual since the fraction of ξ_l households cannot jump.[14]

3.1 A quantitative perspective on global indeterminacy

While Proposition 3 and the subsequent exposition make clear that the HANK economy with countercyclical risk features global indeterminacy, it is still useful to focus our discussion around the empirically relevant range of Θ. To do so, we calibrate our model to highlight what form global indeterminacy can take once we impose some discipline on the parameters.

Calibration In our preferred calibration, we set the discount rate ρ to be consistent with a real interest rate of 4% in steady state. We set the coefficient of relative risk aversion $\gamma^{-1} = 2$. We set the rate at which ξ_h households transition to ξ_l productivity $\overline{\lambda}_l = 0.013$ based on the estimates of Bilbiie, Primiceri and Tambalotti (2023).[15] Translating their estimates of into continuous time yields a range for Θ between 21.98 and 29.9, with the modal estimate of 28.1. To calibrate the relative differences in idiosyncratic productivity ξ_h/ξ_l, we use estimates of the decline in consumption when a household transitions from employment to unemployment. In particular, we set $c_h/c_l = 1.1$, which is consistent with the empirical estimates of the decline in consumption when a household experiences involuntary unemployment.[16] Equation (12), then implies that $\xi_h/\xi_l = 1.23$. Finally, as is common in the literature, we set $\phi_\pi = 1.5$.

[14]We also introduce predetermined variables to our baseline model in two other ways. First, when we study a backward-looking rule in Appendix E.2, a weighted average of inflation in the past acts as a predetermined variable. Since this lagged inflation measure is predetermined and cannot jump, the existence of the untargeted steady state per se does not imply global indeterminacy. However, Appendix E.2 shows that we still have global indeterminacy and depending on how backward looking the rule is, the additional bounded trajectories take the form of a saddle connection or a stable cycle to which trajectories starting near the targeted steady state converge. Second, we introduce government debt (see Section 4.2). Even with government debt as a predetermined variable, Appendix D.1 shows that the determinacy properties of the economy with government debt mirrors that in our baseline economy, except that the economy can no longer jump between the two steady states. Again, depending on how countercyclical risk is, the additional bounded trajectories take the form of a saddle connection or a stable cycle.

[15]Bilbiie, Primiceri and Tambalotti (2023) postulate that the probability that a unconstrained household stays unconstrained is given by $\ln s_t = \ln s_0 + s_1 \ln y_t$. We translate the probability $1 - s_t$ into an arrival rate by using the conversion formula $1 - s_t = 1 - e^{-\lambda_{l,t}}$, which can be simplified to yield $\ln s_t = -\overline{\lambda}_l y_t^{-\Theta}$. Imposing steady state $y_t = 1$, where $s_t = s_0$ and $\lambda_t = \overline{\lambda}_l$, we can set $\overline{\lambda}_l = -\ln s_0$ and $\frac{d \ln \lambda}{d \ln y} = s_1 = -\overline{\lambda}_l \Theta$. Bilbiie et al. (2023) set $s_0 = 0.987$ and estimate s_1 to lie in the range 0.2880 and 0.3920, which implies $\overline{\lambda}_l = 0.0131$ and $\Theta \in [21.98, 29.9]$.

[16]A 10% decline in consumption is well within the range of empirical estimates, e.g., Cochrane (1991) finds that the consumption growth of households who lost their job was 24-27% lower than households who did not, Ganong and Noel (2019) find that the consumption of households who become unemployed drops by around 11% when unemployment benefits expire, while Gruber (1997) documents that food consumption falls on average by 6.8% after households become unemployed.

Given this calibration, we can now compute the boundaries of the mildly, moderately and highly countercyclical regions. While it is not possible to analytically characterize the value of $\Theta^{\diamond}$, our calibrated model features $\Theta^{\diamond} \approx 15.8$, while $\Theta^{\star} = 31.08$. Thus, the mildly procyclical risk region corresponds to $\Theta \in (0, 15.8)$, the moderately countercyclical region corresponds to $\Theta \in (15.8, 31.08)$ and the highly countercyclical region features $\Theta > 31.08$. Using the estimates of Θ from Bilbiie, Primiceri and Tambalotti (2023) which lie in the range $\Theta \in (21.98, 29.9)$, with a modal estimate of $\Theta = 28.1$, one can see that these estimates comfortably lie within the *moderately countercyclical* region of the parameter space. The characterization above showed that when risk is moderately countercyclical, in addition to stagnating at the untargeted steady state, the economy can also get stuck in a cycle and remain permanently away from the targeted steady state. In Figure 2b –which is plotted with $\Theta = 28.1$– the untargeted steady state features a level of output which is about 6.5 percent lower than in the targeted steady state. While this constitutes a considerably large output-gap, it is comparable to output-gap estimates in the U.S. and the Euro area following the Great Recession (see, e.g. Summers (2016); Jarociński and Lenza (2018)). Furthermore, as discussed earlier, in addition to this large decline in economic activity, self-fulfilling fluctuations can also generate smaller cyclical fluctuations: the stable cycle which surrounds the targeted steady state features output-gap fluctuations with an amplitude of approximately $\pm$ 2.5 percent around the targeted steady state, which is comparable in magnitude to cyclical fluctuations in the U.S. In fact, Beaudry et al. (2020) argue that these deterministic boom-bust cycles account for a substantial part of U.S. business cycle fluctuations. Overall, the analysis above underscores the fact that under a standard Taylor rule, our HANK economy with countercyclical risk is susceptible to self-fulfilling fluctuations with non-fundamental fluctuations of plausible magnitudes.

3.2 Other policy rules

While the analysis in Section 3 is based on a inflation targeting Taylor rule, our conclusions are robust to other standard monetary policy rules which have been studied in the RANK literature studying local determinacy. This literature has found that compared to purely inflation targeting rules, rules which allow the policy rates to display inertial behavior, or to also respond to output-gap fluctuations make local determinacy more likely in RANK economies. Appendix E.1 studies the determinacy properties of our economy with the monetary policy rule $i_t = \phi_\pi \pi_t + \phi_x x_t$, while Appendix E.2 studies the case in which the policy rate exhibits inertial behavior. In particular, it considers a rule $di_t/dt = \alpha\left[i_t - \bar{r} - \phi_\pi \pi_t\right]$, where a smaller α implies a more backward-looking rule. These appendices show that, while a large enough ϕ_x and/or a sufficiently backward-looking rule (small enough α) also make *local* determinacy more likely in our HANK economy with countercyclical risk, they cannot eliminate *global* indeterminacy. In fact, Appendix E.1 shows that for any finite combination of (ϕ_π, ϕ_x) (no matter how large), the equilibrium is still globally indeterminate in our HANK economy with countercyclical risk. Similarly, Appendix E.2 shows that no matter how backward-looking the rule is (however small α is), the equilibrium is globally indeterminate as long as risk is countercyclical.

3.3 A more general specification of countercyclical risk

In our baseline model, we have modeled countercyclical risk by assuming that transition rate at which a ξ_h household switches to ξ_l depends on whether output is above or below its level in the targeted steady state: $\lambda_{l,t} = \overline{\lambda}_l \cdot y_t^{-\Theta}$. Appendix E.4 shows that our characterization of global indeterminacy in our HANK model with countercyclical risk *does not* rely on this precise functional form. In particular, Appendix E.4 considers the case in which the transition rate $\lambda_{l,t}$ is given by:[17]

$$\lambda_{l,t} = \overline{\lambda}_l \cdot \Lambda(\ln y_t) = \overline{\lambda}_l \cdot \Lambda(\gamma x_t) \geq 0,$$

where $\Lambda(\cdot)$ is any analytic function which takes non-negative values, and is weakly decreasing in x.[18] Furthermore, in order to make our analysis comparable with the baseline model, we also make two additional normalizations. First, we normalize $\Lambda(0) = 1$, so that the targeted steady state is the same as in our baseline model with $y = 1 (x = 0)$ and $\pi = 0$. This also ensures that the transition rate from ξ_h to ξ_l in the targeted steady state is given by the constant $\overline{\lambda}_l$ like in our baseline model. Second, we parametrize $\Lambda(x)$ such that $\Lambda'(0) = -\Theta < 0$, i.e., the parameter Θ now only captures the cyclicality of risk *local* to the targeted steady state.

Appendix E.4 shows that our results from the baseline results continue to hold under fairly non-restrictive conditions. In particular, our characterization of global indeterminacy in the form of a stable cycle surrounding the targeted steady state still holds as long as $\Lambda(\cdot)$ is sufficiently convex in x. Intuitively, this means that a 1 percentage point fall in output increases "risk" (the rate at which ξ_h transitions to ξ_l) more than a 1 percentage point increase in output reduces risk. Appendix E.4 also shows that the convexity of $\Lambda(\cdot)$ guarantees the existence of the untargeted steady state. However, Appendix E.4 shows that the untargeted steady state exists even if $\Lambda(\cdot)$ is *linear*. While the model is obviously stylized, there are a few reasons to think why $\Lambda(\cdot)$ may be convex. The easiest way to see this is to start by considering very large fluctuations in output. Since $\Lambda(\cdot)$ is bounded below by zero, but it is unbounded above, the function must be convex at least as y become large. More generally, if we interpret the state ξ_h as employment and ξ_l as unemployment, then $\Lambda(\cdot)$ is proportional to the inflow rate into unemployment. Empirically, this rate *increases sharply* (relative to a simple linear trend) in recessions (i.e., episodes where y falls), but *does not decrease sharply* in expansions (see, for e.g., Figure 1 of Crump et al. (2019)). Furthermore, the business cycle asymmetries in these labor market flow variables, which we have argued are a proxy for idiosyncratic risk faced by households, are more pronounced than the asymmetries, if any, of log GDP (see, e.g., McKay and Reis (2008)). Thus, it is not merely the case that risk is asymmetric over the business cycle because GDP is asymmetric. Rather, if we interpret these cyclicalities in terms of a functional relation between risk and GDP (both measured relative to trend), risk is a convex, decreasing function of (detrended) log GDP. Finally, the convexity of $\Lambda(\cdot)$ is broadly consistent with the anecdotal observation that recessions are periods when households' perception of idiosyncratic risk is extremely high, but expansions are not periods when idiosyncratic risk is extremely low, at least not to the same extent.

[17] This specification implies that the first-derivative of $\Lambda(\cdot)$ with respect to y measures how much the transition rate $\lambda_{l,t}$ changes in response to a 1% increase in output.

[18] $\Lambda(\cdot)$ must be non-negative since it is a component of a transition rate. The assumption that the function is also weakly decreasing is meant to capture the idea that risk is countercyclical: risk is weakly higher when x is lower.

Appendix E.4 also shows that the untargeted steady state exists under even more general conditions. Appendix E.4 shows that while the convexity of $\Lambda(\cdot)$ is sufficient to guarantee the existence of the untargeted steady state, it also exists even if $\Lambda(\gamma x)$ is *linear* in x. Thus, our characterization of globally indeterminacy in our HANK economy with countercyclical risk does not rely on the choice of the precise functional form of $\lambda_{l,t}$ specified in equation (2).

Finally, it is worth pointing out that the conditions which guarantee the existence of the stable cycle and the untargeted steady state *do not* depend on how idiosyncratic risk depends on economic activity far from steady state. This is reassuring because the empirical literature which studies how idiosyncratic risk varies with the business cycle is quite new, and we have only a weak understanding of the connection between the business cycle and idiosyncratic risk. Appendix E.4 shows that these conditions only rely on the behavior of $\Lambda(\gamma x)$ local to $x = 0$. In particular, the earlier condition which required that $\Lambda(\cdot)$ be convex for the existence of the stable cycle only needs to be satisfied locally at $x = 0$. In other words, the existence of the cycle only requires that $\Lambda''(0)$ is sufficiently positive and does not require any additional restrictions on the behavior of $\Lambda(\cdot)$ for any $x \neq 0$. Similarly, the existence of the untargeted steady state only requires that risk is countercyclical local to the targeted steady state, i.e, if $-\Lambda'(0) = \Theta > 0$.[19]

4 Policy design to eliminate self-fulfilling fluctuations

Our analysis has shown that standard monetary policy rules cannot guarantee global determinacy in our HANK economy with countercyclical risk. The key force generating this indeterminacy is that in our economy, the natural rate $r^\star(x)$ is endogenous and co-moves with output, and this opens the economy up to the possibility of self-fulfilling fluctuations. In other words, since standard monetary policy rules fail to fully account for endogenous fluctuations in the natural rate, they leave the economy susceptible to "*endogenous* demand shocks". To see why, suppose that ξ_h households believe that the economy is going to enter a recession, and consequently they face a higher probability of transitioning to the ξ_l state. Holding all else constant, this increases households' desired precautionary savings demand, pushing down the natural rate. If monetary policy does not respond sufficiently to this downward movement in $r^\star(x)$, the real interest rate is higher than the natural rate $r > r^\star$, incentivizing households to reduce spending. This can be seen via IS curve (14), which show that when $r_t > r_t^\star$, output (gap) growth is positive: $\dot{x} = \dot{y}/y > 0$. In other words, given that $r_t > r_t^\star$, households reduce their current consumption, and because of nominal rigidities, this results in lower output, rendering the initial belief self-fulfilling. This acts as a negative endogenous demand shock. We now study policy design which can neutralize these endogenous demand shocks.

4.1 Monetary policy

As discussed above, standard monetary policy rules are unable to prevent endogenous demand shocks from causing fluctuations in the natural rate $r^\star(x)$. A simple monetary policy rule which addresses

[19]While the existence of the cycle and untargeted steady state is guaranteed by the behavior of $\Lambda(\cdot)$ local to the targeted steady state, the magnitude and periodicity of the stable cycle also depend on the shape of the $\Lambda(\cdot)$ away from $x = 0$. Similarly, how low output is in the untargeted steady state also depends on the actual shape of $\Lambda(\cdot)$ away from $x = 0$.

this shortcoming can be written as:

$$i_t = \bar{r} + \phi_\pi \pi_t + \phi_r \Big(r^\star(x_t) - \bar{r} \Big) \tag{20}$$

The key change in a policy rule (20) relative to (6), is that monetary policy now also adjusts the nominal rate in response to endogenous demand shocks which result in deviations of the natural rate $r^\star(x)$ from its steady state value of $\bar{r}$.

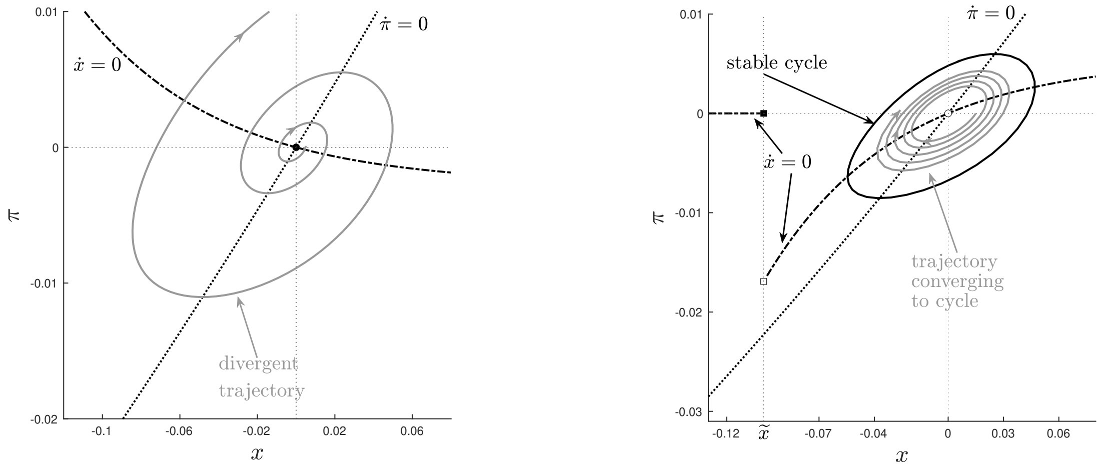
To see how this policy response can eliminate self-fulfilling fluctuations, as before, suppose that ξ_h households believe that the economy is going to enter a recession, and consequently they face a higher probability of transitioning to the ξ_l state. Holding all else constant, this increases households' desired precautionary savings demand, pushing down the natural rate $r^\star$. However, monetary policy now cuts the nominal rate in response to this lower natural rate. In fact, when $\phi_r \geq 1$, monetary policy lowers the policy rate by at least *one-for-one* with $r^\star(x)$. The resulting lower real interest rate undoes the desire to increase precautionary savings, leaving current spending unchanged. Since households do not reduce current consumption, lower output cannot be supported in equilibrium. Hence the initial beliefs about the economy entering a recession cannot be self-fulfilling, and the economy remains at the targeted steady state. Thus, by responding to endogenous fluctuations in $r^\star(x)$ sufficiently strongly, monetary policy neutralizes the endogenous demand shock. Analogous to the standard Taylor principle which rules out self-fulfilling fluctuations in RANK, a sufficiently strong response to endogenous fluctuations in $r^\star(y)$ discourages self-fulfilling beliefs from taking root in our HANK economy with countercyclical risk. In this sense, it can be thought of as a *Taylor principle, but for natural rates*. This *off-equilibrium* commitment to adjust monetary policy ensures that beliefs about higher or lower output cannot be self-fulfilling. Thus, if monetary policy follows the policy rule (20), even though $r^\star(x)$ can deviate from $\bar{r}$, such deviations do not manifest *on-equilibrium*. This idea is formalized in Proposition 4.

Proposition 4. *Suppose that monetary policy is described by* (20)*. Then, for any* $\Theta > 0$*, the targeted equilibrium is globally determinate as long as* $\phi_\pi > 1$ *and* $\phi_r \geq 1$*.*

Proof. See Appendix C. □

Figure 3a depicts the phase portrait under the monetary policy rule (20) with $\phi_r > 1$. The key difference induced by the commitment of monetary policy to respond to any possible fluctuations in $r^\star(x)$ is that, unlike with the standard inflation targeting rule, the long run IS curve (the $\dot{x} = 0$-nullcline) is now *downward sloping*. Consequently, the $\dot{x} = 0$-nullcline and the $\dot{\pi} = 0$-nullcline intersect only once at the targeted steady state, implying that this policy rule does not allow the economy to stagnate at an untargeted steady state with below target inflation and output. Moreover, this commitment to respond to any possible fluctuations in $r^\star(x)$ also eliminates the stable cycle and any trajectory starting away from the targeted equilibrium $(x, \pi) = (0, 0)$ diverges and grows unbounded (as depicted by the gray trajectory in Figure 3a).

Endogenous vs exogenous demand shocks. The reason why the policy rule (20) neutralizes endogenous demand shocks is analogous to the optimal monetary response to *exogenous* demand shocks, which has been studied extensively in the RANK literature. This is easiest to see by setting $\phi_r = 1$ in (20). In this special case, the rule stipulates that holding all else constant, the nominal rate should



(a) Global determinacy with $\phi_r > 1$

(b) unique steady state, but stable cycle persists

Figure 3: **Phase portraits with acyclical risk and countercyclical risk.** In both panels, the dotted curve depicts the $\dot{\pi} = 0$-nullcline, while the dash-dotted curve depicts the $\dot{x} = 0$-nullcline. In panel (a), the gray solid trajectory depicts a representative trajectory which originates near $(0,0)$ and then grows unbounded. In panel (b), the gray solid trajectory depicts a representative trajectory which originates near $(0,0)$ and then converges to the stable cycle, which is depicted by the black solid trajectory.

perfectly track any fluctuations in the natural rate $r^\star(x)$:

$$i_t = r^\star(x_t) + \phi_\pi \pi_t \tag{21}$$

This idea is analogous to the optimal response to *exogenous* demand shocks in the RANK literature. As is well known from RANK, exogenous demand shocks cause fluctuations in the flexible-price real interest rate.[20] In order to neutralize the effect of these exogenous demand shocks on output and inflation, monetary policy should set the nominal rate to perfectly track the flexible-price real interest rate (Galí, 2015). In a similar vein, (21) shows that monetary policy can neutralize *endogenous* demand shocks if the nominal rate tracks the resulting endogenous fluctuations in the natural rate $r^\star(x)$.[21]

Is eliminating the untargeted steady state sufficient? It is worth noting that while the policy rule (20) eliminates both the untargeted steady state and the stable cycle, it is generally not the case that any policy which eliminates the untargeted steady state, necessarily eliminates the stable cycle as well. Figure 3b depicts such a case in which monetary policy follows the simple inflation targeting Taylor

[20]Since our baseline model abstracts from aggregate shocks, the flexible-price real interest rate $\bar{r}_t = \bar{r}$ is constant. More generally, the it would change over time in response to exogenous demand shocks, e.g., a shock to the discount rate ρ_t. But importantly, even when it is time-varying, it does not depend endogenously on the level of output.

[21]The specification of monetary policy in (21) is also closely related to the *robust real interest rate rule* of Holden (2024). Holden (2024) shows that a monetary policy rule of the form $i_t = r_t + \phi_\pi \pi_t$, where r_t denotes the *actual* real interest rate, can deliver a determinate equilibrium in a large class of monetary economies. In fact, it would also deliver a globally determinate equilibrium in our HANK economy with countercyclical risk. To see this, notice that using the IS curve (14), the real interest rate in our economy can be written as $r_t = r^\star(x_t) + \dot{x}_t$, and so the robust real interest rate rule of Holden (2024) can be rewritten as $i_t = r^\star(x_t) + \dot{x}_t + \phi_\pi \pi_t$. This formulation makes it clear that such a rule also delivers global determinacy as it also stipulates that, holding all else constant, the nominal rate should be adjusted to track any potential endogenous fluctuations in the natural rate $r^\star(y)$. However, this rule is not exactly the same as (21), since it requires the nominal interest rate to respond not just to off-equilibrium fluctuations in $r^\star(x_t)$, but additionally to off-equilibrium fluctuations in $\dot{x}_t$ as well.

rule (6) as long as output and inflation stay *close* enough to the targeted steady state, but switches to a strict inflation targeting regime, setting $\pi_t = 0$ at all subsequent dates, if output falls below some threshold level $\tilde{x}$. This formulation of monetary policy is reminiscent of the commonly studied *escape clause*, where deviations of inflation and output outside a specified range prompt the policymaker to depart from a simple linear rule in favor of a policy that directly restores inflation and output to the monitoring range (see, e.g., Christiano and Takahashi (2018)). Figure 3b plots the dynamics under such a rule with $\tilde{x} = -0.1$.[22] Such a policy stance implies that for $x \geq \tilde{x}$, the IS curve is the same as under our baseline, but if $x < \tilde{x}$, the $\dot{x} = 0$-nullcline jumps up to the $\pi_t = 0$ line. Consequently, as long as $\tilde{x} > \underline{x}$ (where $\underline{x}$ denotes the output in the untargeted steady state for a given Θ and ϕ_π in our baseline economy), there is only one intersection between the $\dot{x} = 0$-nullcline and the $\dot{\pi} = 0$-nullcline, implying that such a policy stance eliminates the untargeted steady state from emerging in equilibrium (see Appendix C.1 for a proof). However, Figure 3b shows that such a policy stance *does not* rule out global indeterminacy as the stable cycle which surrounds the targeted steady state still exists, and so any trajectory originating in the neighborhood of the targeted steady state initially diverges (since the targeted steady state is locally determinate) but eventually converges to the stable cycle, thus remaining bounded. This shows that simple *escape clauses* (which have often been used as a means of implementing a unique equilibrium), may eliminate some of the multiple equilibria but not all of them. This points to the importance of identifying *all* the forms in which global indeterminacy manifests (as we did in Section 3), so that one can design policy which neutralizes all of them.

While at first glance, the specification of monetary policy required to eliminate global indeterminacy might seem abstract, the prescriptions do not constitute a large deviation from the way in which most central banks conduct monetary policy. Following the centrality of inflation expectations for determinacy in the RANK framework, most central banks vigilantly monitor measures of long-run inflation expectations and stand ready to tighten policy if inflation expectations start to drift above the inflation target. In the same way, our HANK economy with countercyclical risk provides a rationale for central banks to pay equal attention to private sector beliefs about real activity. Just as central banks monitor and react to inflation expectations, if measures of confidence in the real economy (such as consumer confidence, households' perceived probability of job loss etc.) begin to drift down or up, monetary policy should act aggressively to reverse such beliefs. Simply trying to keep inflation expectations on target, while ignoring expectations about real activity, can enable self-fulfilling beliefs which lead to non-fundamental fluctuations in output and inflation.

4.2 Fiscal policy

Next, we study whether the global indeterminacy in our HANK economy can be eliminated using fiscal policy. One reason to do so follows from the criticism of the Taylor principle in Cochrane (2011), who argues that the Taylor principle achieves a unique equilibrium by the restricting the definition of equilibria to those which feature bounded inflation. Cochrane argues that this restriction, which rules out explosive paths of inflation, is ad-hoc and not based on economic theory. Instead, in the context of RANK economies, Cochrane argues that *non-Ricardian* fiscal policy can implement an unique

[22] $\tilde{x} = -0.1$ implies that monetary policy switches to the strict inflation targeting regime setting $\pi_t = 0$ at all subsequent dates if output were to decline more than 5% below its targeted level.

equilibrium without the need for such assumptions. In what follows, we show that such a policy also eliminates the global indeterminacy in our HANK economy with countercyclical risk.

In order to study non-Ricardian fiscal policy, we introduce government debt into our baseline economy. With non-zero debt, the government budget constraint in nominal terms can be written as

$$\dot{B}_t \quad = \quad i_t B_t + P_t(g - T_t), \tag{22}$$

where g denotes government expenditures (we normalize $g = 0$ for simplicity), and T_t denotes lump sum taxes/transfers. As is standard, we assume that T_t is determined by the fiscal rule:

$$T_t = \bar{r}b^\star + \bar{r}\phi_b(b_t - b^\star), \tag{23}$$

where $b_t = B_t/P_t$ denotes the stock of outstanding real government debt at date t, and $b^\star > 0$ is the level of government debt in the targeted steady state,[23] and ϕ_b controls how aggressively the fiscal authority raises taxes when debt is above its targeted level. Similar to Leeper (1991), we describe fiscal policy as *passive* when $\phi_b > 1$ and *active* when $\phi_b \in (0,1)$. As in our baseline model, we assume that monetary policy is still described by the inflation targeting rule (6). However, unlike the baseline model, we do not restrict $\phi_\pi > 1$. Instead we require $\phi_\pi \geq 0$, which allows for the possibility that the monetary policy can be *passive* if $\phi_\pi \in [0,1]$, and *active* if $\phi_\pi > 1$.

Given our assumption of quasi-linear preferences, Appendix A.3 shows that we can still summarize household decisions in terms of a single IS curve which is identical to (16a) in the baseline model, and so, the dynamics of the output-gap, inflation and (real) government debt can then be described by the IS equation (16a), the Phillips curve (16b) and the government budget constraint (in real terms)

$$\dot{b}_t = (i_t - \pi_t)b_t - T_t, \tag{24}$$

where i_t is given by the inflation targeting rule (6) and T_t is given be the fiscal rule (23).

Active Monetary, Passive Fiscal Simply adding government debt to the baseline model does not eliminate global indeterminacy. As long as fiscal policy is passive ($\phi_b > 1$) and monetary policy is sufficiently active $\phi_\pi > \varphi(\Theta)$, the determinacy properties of the economy are the same as those of our baseline economy. In fact, Appendix D.1 shows that in this setting, if risk is not highly countercyclical $\Theta \in (0, \Theta^\star)$ and monetary policy is *active enough*, i.e., $\phi_\pi > \varphi(\Theta)$, then the targeted steady state is still locally determinate as in our baseline model. But, we still have global indeterminacy and the global dynamics of output and inflation mirror those in our baseline economy. In addition to the targeted steady state, there still exists an untargeted steady state with lower economic activity. But since government debt is a predetermined variable, the economy can no longer simply *jump* between steady states. However, we still have global *indeterminacy*: Appendix D.1 shows that when risk is mildly countercyclical, pessimistic beliefs can still cause the economy to slowly transition from near

[23] In our formulation, the assumption that $b^\star > 0$ ensures that primary surplus is positive in steady state. We choose to focus on the case with positive real interest rates and positive primary surpluses as it distinguishes the source of equilibrium multiplicity in our framework from that in Kaplan et al. (2023), who show that in economies with incomplete markets, the fiscal theory of the price level may not yield a unique equilibrium when governments run persistent deficits.

the targeted steady state to the untargeted steady state along a *saddle-connection*. Similarly, when risk is moderately countercyclical, self-fulfilling beliefs can still cause the economy to permanently move away from the targeted steady state, and to converge over time to a stable cycle which surrounds the targeted steady state.[24]

Passive Monetary, Active Fiscal In contrast, Appendix D.2 shows that a regime of active fiscal policy alongside passive monetary policy *does* eliminate all the manifestations of global indeterminacy which appear in our baseline model (or in the active monetary, passive fiscal regime). In particular, an *active fiscal, passive monetary* regime eliminates the existence of the untargeted steady state, as well as the stable cycle for any $\Theta > 0$, i.e., it prevents households' beliefs about the economy going down such trajectories from become self-fulfilling not matter how countercyclical risk is. This is formalized in Proposition 5 below.

Proposition 5 (Global determinacy with passive monetary and active fiscal policy)**.** *Suppose that $\phi_\pi < 1$ and $\phi_b \in [0, 1)$. Then, the economy has a unique steady state with output and inflation at their targeted values and real government debt equal to $b^\star$. Furthermore, the equilibrium is globally determinate: for any given level of government debt b_0, there exists a unique (x_0, π_0) s.t. only the trajectory originating at (x_0, π_0, b_0) remains bounded. Furthermore, this trajectory converges to the targeted steady state in which $x = \pi = 0$ and $b = b^\star$.*

Proof. See Appendix D.2 □

The fact that the passive monetary, active fiscal regime delivers global determinacy in our HANK economy follows from the standard logic of the *Fiscal Theory of the Price Level*. When fiscal policy is active ($0 \leq \phi_b < 1$), the fiscal rule (23) along with the government budget constraint (24) implies that even when the gap between outstanding government debt and its targeted level is getting wider, fiscal policy does not raise taxes sufficiently to narrow this gap. Thus, for a given level of nominal government liabilities, the price must adjust to ensure that in equilibrium, the real value of government debt equals the net present value of future primary surpluses. Consequently, the price level is pinned down uniquely to ensure that the government is solvent (Cochrane, 2011).

To see how this logic rules out the existence of the untargeted steady state, suppose that as before, ξ_h households start to believe that the economy is about to permanently enter a recession: output and inflation will remain permanently below their targeted level. Along with the fact that monetary policy is passive, this deflationary path implies that the interest liability of the government is now higher and consequently, the real value of outstanding government debt would increase over time unless the government raises taxes sufficiently. However, with $\phi_b \in [0, 1)$, the fiscal rule (23) implies that taxes do not increase sufficiently, despite the growing level of real government debt. Thus, if pessimistic beliefs were to actually push the economy into a permanent recession, the government would eventually become insolvent, and the government budget constraint would be violated. This prevents such pessimistic beliefs from becoming self-fulfilling, and hence the untargeted steady state cannot exist in a passive monetary, active fiscal regime. Similarly, with active fiscal policy, households'

[24] Furthermore, Appendix D.1 also shows that the untargeted steady state is *locally indeterminate* which implies that in the neighborhood of the untargeted steady state, for a given level of government debt, there are multiple combinations of (x, π) starting from which the economy converges to the untargeted steady state, implying global indeterminacy.

beliefs of the economy being permanently trapped in a cycle around the targeted steady state cannot be confirmed in equilibrium, because along the trajectory of output, inflation and real government debt implied by such a belief would result in the government budget constraint being violated.[25]

5 Conclusion

We have shown that if risk is even mildly countercyclical, HANK economies can feature self-fulfilling fluctuation under standard monetary policy rules. This is because in HANK economies with countercyclical risk, the natural interest rate is endogenous and co-moves with output, leaving the economy susceptible to endogenous demand shocks. In order to neutralize these endogenous demand shocks, monetary policy needs to commit to adjusting nominal rates one-for-one with any endogenous fluctuations in the natural rate. Doing so implements a unique equilibrium by ensuring that non-fundamental beliefs cannot be self-fulfilling. If doing so is not feasible, a regime with passive monetary policy coupled with an active fiscal regime can also prevent these self-fulfilling beliefs from taking root.

Importantly, since the multiplicity of equilibria does not stem from the presence of the ELB, it can plague the economy even during a tightening cycle. Moreover, our analysis stresses that large and aggressive rate hikes in response to higher inflation *do not* ensure that inflation expectations will remain anchored around its target level. Instead, our framework suggests that, just as central banks monitor and react to inflation expectations, if measures of confidence in the real economy (such as consumer confidence, households' perceived probability of job loss etc.) begin to drift down or up, monetary policy must act aggressively to reverse such beliefs, if it hopes to keep expectations anchored.

Finally, our findings show that local stability analysis can provide a misleading picture regarding the performance of policy rules: in our economy, even when the targeted equilibrium is locally determinate, multiple bounded equilibria exist. This suggests that researchers using HANK models need to be more vigilant regarding the possibility of multiple equilibria. Furthermore, the diverging conclusions based on local vs global determinacy above have important implications regarding the design of monetary policy, which are particularly relevant for the recent post-COVID inflation surge witnessed globally. For example, based on local determinacy one would conclude that an aggressive enough response of monetary policy to higher inflation (high enough ϕ_π) should ensure that the target equilibrium is determinate, or in other words, that inflation expectations remain anchored around

[25] Finally, it is also worth commenting on the relation between our paper and a series of papers by Marcus Hagedorn (Hagedorn, 2016; Hagedorn et al., 2019; Hagedorn, 2024), which study nominal determinacy in a class of incomplete markets models. These papers exploit the non-Ricardian features of many HANK models which result in a relationship between the real interest rate and the steady state level of government debt. In particular, these papers show that if one specifies fiscal policy as a rule for the level (or more generally, the growth rate) of nominal debt, such a rule delivers a *locally*-determinate equilibrium in this class of models.

However, such a specification of policy *does not* rule out global indeterminacy in our HANK economy with countercyclical risk. In fact, it cannot even eliminate local indeterminacy in our framework. The reason behind the inability of such policies in affecting the determinacy properties in our HANK model is because in our model, the steady state real interest rate is independent of the level of real government debt owing to the assumption of quasi-linear preferences: the steady state real interest is given by $\bar{r} = \rho - \overline{\lambda}_l \left(\frac{\xi_h}{\xi_l} - 1 \right)$ and is independent of the level of government debt. The inability of such rules to eliminate equilibrium multiplicity in our economy highlights the fact that multiple equilibria are not driven by the non-Ricardian features of many HANK models which impart a relationship between the steady state real interest rate and the level of real government debt. Consequently, the types of policy rules which eliminate this indeterminacy are conceptually different than those identified in Hagedorn (2016); Hagedorn et al. (2019); Hagedorn (2024).

the targeted level of inflation. However, this conclusion would be incorrect since our global analysis reveals that this prescription (no matter how aggressively monetary policy responds to the higher inflation) can still lead to the economy getting trapped in a situation where inflation expectations become permanently unanchored.

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Appendix

A Derivation of IS curve

A.1 Household problem

Instead of directly solving the household problem in continuous time, we solve it in the discrete time limit in which each period is Δ units of time long. Then we derive the optimal decisions in continuous time by taking limits as $\Delta \to 0$. In discrete time, the problem of the household can be written as, where we have discarded the j subscript for convenience:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} e^{-\rho\Delta t} \left[\frac{c_{t\Delta}^{1-\gamma^{-1}}}{1-\gamma^{-1}} - \psi n_{t\Delta} \right] \Delta$$

s.t.

$$a_{t+\Delta} - a_t = (1 + r_t\Delta) (\xi_h w_t n_t + D_t - c_t) \Delta + r_{t+\Delta}\Delta a_t \qquad \text{and} \qquad a_{t+\Delta} \geq -\underline{a}$$

This problem can be formulated as a Bellman equation. The value function of a household with idiosyncratic productivity ξ_h and wealth a_t can be written as:

$$V(a_t, \xi_h) = \left[\frac{c_t^{1-\gamma^{-1}}}{1-\gamma^{-1}} - \psi n_t \right] \Delta + (1-\rho\Delta) \left[(1 - \lambda_{l,t+\Delta}\Delta) V(a_{t+\Delta}, \xi_h) + \lambda_{l,t+\Delta}\Delta V(a_{t+\Delta}, \xi_l) \right], \tag{a.1}$$

where $a_{t+\Delta}$ is given by:

$$a_{t+\Delta} = (1 + r_t\Delta) \left[(\xi_h w_t n_t + D_t - c_t) \Delta + a_t \right] \qquad \text{and} \qquad a_{t+\Delta} \geq -\underline{a}, \tag{a.2}$$

and we have used the fact that for small Δ, $e^{-\rho\Delta} = 1 - \rho\Delta$. Similarly, the value function for a household with idiosyncratic productivity ξ_l and wealth a_t can be written as:

$$V(a_t, \xi_l) = \left[\frac{c_t^{1-\gamma^{-1}}}{1-\gamma^{-1}} - \psi n_t \right] \Delta + (1-\rho\Delta) \left[(1 - \lambda_{h,t+\Delta}\Delta) V(a_{t+\Delta}, \xi_l) + \lambda_{h,t+\Delta}\Delta V(a_{t+\Delta}, \xi_h) \right],$$

where $a_{t+\Delta}$ is given by:

$$a_{t+\Delta} = (1 + r_t\Delta) \left[(\xi_l w_t n_t + D_t - c_t) \Delta + a_t \right] \qquad \text{and} \qquad a_{t+\Delta} \geq -\underline{a},$$

The optimal choice of hours worked n_t by a household with idiosyncratic productivity ξ_h is given by:

$$\frac{\psi}{\xi_h w_t} = (1 + r_t\Delta)(1-\rho\Delta) \left[(1 - \lambda_{l,t+\Delta}\Delta) V_a(a_{t+\Delta}, \xi_h) + \lambda_{l,t+\Delta}\Delta V_a(a_{t+\Delta}, \xi_l) \right], \tag{a.3}$$

and by a household with idiosyncratic productivity ξ_l is given by:

$$\frac{\psi}{\xi_l w_t} = (1 + r_t\Delta)(1-\rho\Delta) \left[(1 - \lambda_{h,t+\Delta}\Delta) V_a(a_{t+\Delta}, \xi_l) + \lambda_{h,t+\Delta}\Delta V_a(a_{t+\Delta}, \xi_h) \right], \tag{a.4}$$

The optimal choice of consumption for a household with productivity ξ_h and ξ_h can be written as:

$$
\begin{aligned}
c_t(a,\xi_h) &\leq \left\{(1+r_t\Delta)(1-\rho\Delta)\left[(1-\lambda_{l,t+\Delta}\Delta)V_a(a_{t+\Delta},\xi_h)+\lambda_{l,t+\Delta}\Delta V_a(a_{t+\Delta},\xi_l)\right]\right\}^{-\gamma} \quad \text{(a.5)}\\
c_t(a,\xi_l) &\leq \left\{(1+r_t\Delta)(1-\rho\Delta)\left[(1-\lambda_{h,t+\Delta}\Delta)V_a(a_{t+\Delta},\xi_l)+\lambda_{h,t+\Delta}\Delta V_a(a_{t+\Delta},\xi_h)\right]\right\}^{-\gamma}, \quad \text{(a.6)}
\end{aligned}
$$

where we have divided both sides of each equation by Δ. The inequality captures the fact that households may be borrowing constrained. Next, the envelope conditions for ξ_h and ξ_l households are

$$
\begin{aligned}
V_a(a_t,\xi_h) &= (1+r_t\Delta)(1-\rho\Delta)\left[(1-\lambda_{l,t+\Delta}\Delta)V_a(a_{t+\Delta},\xi_h)+\lambda_{l,t+\Delta}\Delta V_a(a_{t+\Delta},\xi_l)\right] \quad \text{(a.7)}\\
V_a(a_t,\xi_l) &= (1+r_t\Delta)(1-\rho\Delta)\left[(1-\lambda_{h,t+\Delta}\Delta)V_a(a_{t+\Delta},\xi_l)+\lambda_{h,t+\Delta}\Delta V_a(a_{t+\Delta},\xi_h)\right] \quad \text{(a.8)}
\end{aligned}
$$

Using the envelope conditions along with (a.3)–(a.6), we have:

$$
c_{h,t} \equiv c_t(a,\xi_h) = \left(\frac{\xi_h w_t}{\psi}\right)^{\gamma} \qquad \text{and} \qquad c_{l,t} \equiv c_t(a,\xi_l) = \left(\frac{\xi_l w_t}{\psi}\right)^{\gamma}, \quad \text{(a.9)}
$$

which shows that the consumption of all households with the same idiosyncratic productivity ξ_j is the same, regardless of their financial wealth.

Next, it is easy to see that for any real interest rate r_t, the expected consumption growth of ξ_l households is larger than that of ξ_h households. The expected consumption growth between dates $t+\Delta$ and t, for a household with productivity ξ_l at date t can be written as:

$$
(1-\lambda_{h,t+\Delta}\Delta)\frac{c_{l,t+\Delta}}{c_{l,t}}+\lambda_{h,t+\Delta}\Delta\frac{c_{h,t+\Delta}}{c_{l,t}} = \left\{1+\lambda_{h,t+\Delta}\Delta\left[\left(\frac{\xi_h}{\xi_l}\right)^{\gamma}-1\right]\right\}\left(\frac{w_{t+\Delta}}{w_t}\right)^{\gamma},
$$

while the expected consumption growth of a household with productivity ξ_h at date t can be written as $\left\{1-\lambda_{l,t+\Delta}\Delta\left[\frac{\xi_h^{\gamma}-\xi_l^{\gamma}}{\xi_h^{\gamma}}\right]\right\}\left(\frac{w_{t+\Delta}}{w_t}\right)^{\gamma}$. Thus, for any path of aggregate variables, the expected consumption growth is higher for ξ_l households. Hence, in equilibrium, at any date t, all ξ_l households must be borrowing constrained, and choose $a_{t+\Delta} = -\underline{a}$. In contrast, the ξ_h households are on their Euler equation, which can be derived by rearranging the envelope condition for ξ_h households as:

$$
\frac{\rho - r_t(1-\rho\Delta)}{(1+r_t\Delta)(1-\rho\Delta)}V_a(a_t,\xi_h) = \left[\frac{V_a(a_{t+\Delta},\xi_h)-V_a(a_t,\xi_h)}{\Delta}+\lambda_{l,t+\Delta}\left\{V_a(a_{t+\Delta},\xi_l)-V_a(a_{t+\Delta},\xi_h)\right\}\right]
$$

Taking the limit of this equation as $\Delta \to 0$, we get:

$$
(\rho - r_t)V_a(a_t,\xi_h) = \dot{V}_a(a_t,\xi_h)+\lambda_{l,t}\left\{V_a(a_t,\xi_l)-V_a(a_t,\xi_h)\right\} \quad \text{(a.10)}
$$

Next, using (a.3)-(a.6), we have:

$$
V_a(a_t,\xi_h) = c_{h,t}^{-\gamma^{-1}} \qquad , V_a(a_t,\xi_l) = c_{l,t}^{-\gamma^{-1}} \qquad \text{and} \qquad \dot{V}_a(a_t,\xi_h) = c_{h,t}^{-\gamma^{-1}}\frac{\dot{c}_{h,t}}{c_{h,t}}
$$

Using this in (a.10), we have the Euler equation:

$$\frac{\dot{c}_{h,t}}{c_{h,t}} = \gamma\left(r_t - \rho\right) + \gamma\lambda_{l,t}\left[\left(\frac{c_{l,t}}{c_{h,t}}\right)^{-\frac{1}{\gamma}} - 1\right],$$

which is the same as (9) in the main text. □

A.2 Government debt and asset market clearing

In our baseline model, we assume that there is zero government debt issued $B_t = 0$ at each date. Since at any date t, all η of the ξ_l households are borrowing constrained and thus, in net, they borrow $\eta\underline{a}$, where η is the constant fraction of ξ_l households. Consequently, asset market clearing implies that ξ_h households at date t must be net savers and hold $\eta\bar{a}$ as a group.

Allowing for non-zero government debt does not affect the savings decision of ξ_l households and at any date t, they are still borrowing constrained and borrow $\eta\underline{a}$ as a group. However, now that there is a non-zero amount of government debt in the economy, asset market clearing implies that all ξ_h households as a whole must save $\eta\underline{a} + b_t$ at date t, where b_t denotes real government debt at date t (see below).

Notice that despite changing the equilibrium asset holdings of ξ_h households, adding non-zero government debt into our economy does not affect the consumption of each type of households, which is still given by (a.9). This is because quasi-linear preferences imply that consumption of each household is independent of their wealth. Consequently, because we always normalize government expenditures to 0, with or without government debt, the goods market clearing condition can be written as:

$$y_t \quad = \quad (1-\eta)c_{h,t} + \eta c_{l,t}, \tag{a.11}$$

Using the expressions for c_h and c_l in (a.9), (a.11) can be re-written as:

$$y_t \quad = \quad \left[(1-\eta)\left(\frac{\xi_h}{\psi}\right)^{\gamma} + \eta\left(\frac{\xi_l}{\psi}\right)^{\gamma}\right] w_t^{\gamma}$$

Normalizing $\psi = \left[(1-\eta)\xi_h^{\gamma} + \eta\xi_l^{\gamma}\right]^{\frac{1}{\gamma}}$, we can rewrite this as:

$$w_t \quad = \quad y_t^{\frac{1}{\gamma}}, \tag{a.12}$$

which is the same as (11) in the main text. Furthermore, setting $\dot{\pi} = 0$ and $\pi = 0$ in the Phillips curve (5) implies that real wages in the steady state with on-target inflation is $w = 1$. Consequently, (a.12) implies that output in the targeted steady state is $y = 1$. Finally, using this relationship in (a.9), we can express the per-capita consumption of households with skill ξ in terms of output at any date t:

$$c_{h,t} = \frac{\xi_h^{\gamma}}{(1-\overline{\eta})\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}} y_t > \frac{\xi_l^{\gamma}}{(1-\overline{\eta})\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}} y_t = c_{l,t} \tag{a.13}$$

Government budget constraint The government budget constraint be written in nominal terms as:

$$\dot{B}_t = i_t B_t - P_t T_t$$

where B_t denotes the stock of nominal government debt at date t, T_t denote taxes net of transfers, and we have normalized government expenditures $g = 0$. Defining real debt as $b_t = B_t / P_t$, we can rewrite the government budget constraint in real terms as

$$\dot{b}_t = (i_t - \pi_t) b_t - T_t \tag{a.14}$$

In our baseline model (Section 3), we restrict attention to the case in which $T_t = b_t = 0$ at all dates. However, in Section 4.2, we assume that fiscal policy sets T_t according to the following rule:

$$T_t = \overline{T} + \overline{r}\phi_b b_t \tag{a.15}$$

Combining (a.14) and (a.15), the evolution of real government debt can be written as:

$$\dot{b}_t = \left(i_t - \pi_t - \overline{r}\phi_b\right) b_t - \overline{T}, \tag{a.16}$$

Defining $b^\star$ as the level of real government debt in the targeted steady state, imposing $\dot{b} = 0$ and $b = b^\star$ in (a.16) implies that

$$\overline{T} = \overline{r}\left(1 - \phi_b\right) b^\star$$

Using this, we can rewrite (a.16) as:

$$\dot{b}_t^g = \left(r_t - \overline{r}\phi_b\right) b_t - \overline{r}\left(1 - \phi_b\right) b^\star, \tag{a.17}$$

where $r_t = i_t - \pi_t$ is the real interest rate at date t. Rather than working with (a.17), it is more convenient to characterize the dynamics of $b_t^g = b_t - b^\star$, i.e, the gap between the actual level of government debt and its level in the targeted steady state. We can then express (a.17) in terms of b_t^g as:

$$\dot{b}_t \quad = \quad \left(r_t - \overline{r}\right) b^\star + \left(r_t - \overline{r}\phi_b\right) b_t^g \tag{a.18}$$

A.3 Aggregate dynamics

The aggregate dynamics of y_t, π_t and b_t^g are described by the the IS curve, the Phillips curve and the government budget constraint. We start by deriving the IS curve. Taking the time-derivative of expression for $c_{h,t}$ in (a.13) yields $\frac{\dot{c}_{h,t}}{c_{h,t}} = \frac{\dot{y}_t}{y_t}$. We can then rewrite the Euler equation of the ξ_h household (9) as:

$$\frac{\dot{y}_t}{y_t} \quad = \quad \gamma\Big(i_t - \pi_t - \rho\Big) + \gamma\overline{\lambda}_l\left(\frac{\xi_h}{\xi_l} - 1\right) y_t^{-\Theta}$$

Next, defining $x_t = \frac{1}{\gamma} \ln y_t$, we can rewrite the above as:

$$\dot{x}_t = i_t - \pi_t - \rho + \overline{\lambda}_l \left(\frac{\xi_h}{\xi_l} - 1 \right) e^{-\gamma \Theta x_t}$$

Substituting out $i_t = \overline{r} + \phi_\pi \pi_t$ using the monetary policy rule (6), we have

$$\dot{x}_t = \overline{r} + (\phi_\pi - 1)\pi_t - \rho + \sigma e^{-\gamma \Theta x_t}, \tag{a.19}$$

where $\sigma = \overline{\lambda}_l \left(\frac{\xi_h}{\xi_l} - 1 \right)$, and $\overline{r} = \rho - \sigma$ is the real interest rate in the targeted steady state (and also the intercept in the monetary policy rule). Similarly, rewriting the Phillips curve (5) in terms of x_t, we have

$$\dot{\pi}_t = \rho \pi_t - \kappa \left(e^{x_t} - 1 \right) \tag{a.20}$$

Finally, by substituting $i_t = \overline{r} + \phi_\pi \pi_t$ using the monetary policy rule (6), into the government budget constraint (a.18), we can rewrite (a.18) as:

$$\dot{b}_t = (\phi_\pi - 1) b^\star \pi_t + \overline{r} (1 - \phi_b) b_t^g + (\phi_\pi - 1) \pi_t b_t^g, \tag{a.21}$$

which is the same as equation (24) in the main text.

Flexible-price limit Since we abstract from aggregate shocks, in the flexible-price limit of our economy ($\kappa \to \infty$), the Phillips curve implies that at any date t, $x_t = \pi_t = 0$. Using this in (a.19) and rearranging, the real interest rate in the flexible price limit $\overline{r}$ is given by $\overline{r} = \rho - \sigma$, where $\sigma = \overline{\lambda}_l \left(\frac{\xi_h}{\xi_l} - 1 \right) \geq 0$ captures the effect of consumption risk faced by households in steady state.

Using the fact that $\overline{r} = \rho - \sigma$, the equilibrium dynamics of x_t, π_t, b_t^g are given by the three dimensional system of non-linear ODEs:

$$\dot{x}_t = (\phi_\pi - 1) \pi_t + \sigma \left(\mathrm{e}^{-\gamma \Theta x_t} - 1 \right) \tag{a.22}$$

$$\dot{\pi}_t = \rho \pi_t - \kappa \left(\mathrm{e}^{x_t} - 1 \right) \tag{a.23}$$

$$\dot{b}_t^g = (\phi_\pi - 1) b^\star \pi_t + \overline{r} (1 - \phi_b) b_t^g + (\phi_\pi - 1) \pi_t b_t^g \tag{a.24}$$

A.3.1 Aggregate dynamics in the baseline model with zero government debt

In our baseline model with zero government debt, which implies that $b_t = b^\star = b_t^g = \dot{b}_t^g = 0$ at all dates. Thus, (a.24) simply states that $0 = 0$, and so the aggregate dynamics of x_t, π_t in our baseline economy with zero government debt are fully described by the IS curve (a.22) and Phillips curve (a.23).

In the flexible price limit of the baseline economy $\kappa \to \infty$, as previously mentioned, the Phillips curve implies that $x_t = \pi_t = 0$ is the only bounded solution which satisfies both the IS and Phillips curves, and is thus the unique bounded equilibrium. □

B Baseline model with inflation targeting rule

This section contains the proof of claims relating to the baseline model described by (16a)-(16b).

B.1 Global determinacy in RANK and in HANK with acyclical risk

In the RANK limit $(\sigma = 0)/$ with acyclical risk $(\sigma > 0, \Theta = 0)$, aggregate dynamics are given by:

$$\dot{x}_t = (\phi_\pi - 1)\pi_t$$
$$\dot{\pi}_t = \rho\pi_t - \kappa(e^{x_t} - 1)$$

The Jacobian of the system evaluated at any (x, π) can be written as:

$$\begin{bmatrix} 0 & \phi_\pi - 1 \\ -\kappa e^x & \rho \end{bmatrix}$$

To show that $\phi_\pi > 1$ delivers global determinacy, we can invoke the Bendixson–Dulac theorem (Bendixson, 1901; Dulac, 1937),[26] which states if the trace does not change sign anywhere in the domain, then there are no non-constant periodic solutions lying entirely within $(x, \pi) \in (-\infty, \infty)^2$. This is true by inspection since the trace is given by $\rho > 0$. Thus, there are no non-constant periodic solutions.

Next, with $\phi_\pi > 1$, the determinant of the Jacobian evaluated at the targeted steady state $(x, \pi) = (0, 0)$ is given by $\kappa(\phi_\pi - 1) > 0$. Together with the fact that the trace is always positive $\rho > 0$, this implies that both eigenvalues have positive real parts. Thus, the targeted steady state $(0, 0)$ is unstable, and hence the only bounded equilibrium is given by the trajectory $(x_t, \pi_t) = (0, 0)$ for all t. □

B.2 Proof of Proposition 2

Close to the targeted steady state $(0, 0)$, the dynamics of the system (16a)-(16b) are governed by:

$$\begin{bmatrix} \dot{x} \\ \dot{\pi}_t \end{bmatrix} = A \begin{bmatrix} x \\ \pi \end{bmatrix} + \mathcal{O}\left(x^2\right) \qquad \text{for} \qquad (x, \pi) \to (0, 0),$$

where A is given by

$$A = \begin{bmatrix} -\sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix},$$

Since both x and π are "jump" variables, local determinacy requires that both eigenvalues of A have a positive real part. As is well known, the sum of the two eigenvalues of A, denoted by z_1 and z_2, is given by the trace of A, while their product is given by the determinant of A:

$$\begin{aligned} z_1 + z_2 &= \rho - \sigma\gamma\Theta, \\ z_1 \times z_2 &= \kappa(\phi_\pi - 1) - \rho\sigma\gamma\Theta \end{aligned}$$

[26] See Theorem 4.1 on page 39 in Verhulst (1990) for a simple statement of the Bendixson-Dulac theorem in English.

Since this is a two dimensional system, either both z_1 and z_2 are real, or they are complex conjugates. Thus, for z_1 and z_2 to both have positive real parts, it is sufficient that both the sum and product of z_1, z_2 be positive. In other words, as long as $\Theta < \Theta^* \equiv \frac{\rho}{\sigma\gamma}$, a sufficient condition for local determinacy is that

$$\phi_\pi > 1 + \frac{\rho\sigma\gamma\Theta}{\kappa},$$

which is the same condition as in Proposition 2. Finally, for $\Theta > \Theta^*$, the sum of the two eigenvalues $z_1 + z_2 < 0$ regardless of the magnitude of ϕ_π, implying that at least one of the eigenvalues must have a negative real part, i.e., regardless of the magnitude of ϕ_π, the equilibrium is locally indeterminate. □

B.3 Multiple Steady States

For any $\Theta > 0$, our baseline HANK economy has two steady states (except in a knife edge case). The $\dot{x} = 0$ and $\dot{\pi}_t = 0$ nullclines imply that in any steady state, (x, π) must satisfy:

$$\begin{aligned} 0 &= (\phi_\pi - 1)\,\pi + \sigma\left(\mathrm{e}^{-\gamma\Theta x} - 1\right) \\ 0 &= \rho\pi - \kappa\left(\mathrm{e}^{x} - 1\right) \end{aligned}$$

Clearly, $(0, 0)$ always satisfies both equations. To see that there is generically another steady state, combine the two equations to eliminate π, to get an expression exclusively in terms of x:

$$F(x) = \frac{\kappa\,(\phi_\pi - 1)}{\rho}\left(\mathrm{e}^{x} - 1\right) + \sigma\left(\mathrm{e}^{-\gamma\Theta x} - 1\right), \tag{b.1}$$

and any x which satisfies $F(x) = 0$ constitutes a steady state. Again, clearly $x = 0$ solves this equation. The derivative of $F(x)$ is given by:

$$F'(x) = \frac{\kappa\,(\phi_\pi - 1)}{\rho}\mathrm{e}^{x} - \sigma\gamma\Theta\mathrm{e}^{-\gamma\Theta x},$$

which, evaluated at $x = 0$ yields

$$F'(0) = \frac{\kappa}{\rho}\left(\phi_\pi - \varphi(\Theta)\right) \qquad \text{where} \qquad \varphi(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa},$$

If $\phi_\pi = \varphi(\Theta)$, then $F'(0) = 0$ and $F(x)$ is tangent to the x-axis at $x = 0$, implying that it is the only zero of $F(x)$ since $F(x)$ is declining in the region $x = 0$ and increasing in the region $x > 0$. This is the knife edge case in which there is a unique steady state. If instead, $\phi_\pi > \varphi(\Theta)$, then $F'(0) > 0$. Since $\lim_{x \to -\infty} F(x) \to \infty$, there must be at least one intersection with $\underline{x} < 0$ and $F'(\underline{x}) < 0$. Since $F(x)$ is strictly convex, this intersection is unique. Further, note that $dF(x)/d\phi_\pi < 0$ for $x < 0$ by inspection. Thus, by the implicit function theorem, we have $d\underline{x}/d\phi_\pi < 0$.

Instead if $1 < \phi_\pi < \varphi(\Theta)$, then $F(x)$ intersects the x axis twice, one of which is $x = 0$. We also know that in this case $F'(0) < 0$ and that $F(x) \to \infty$ as $x \to \infty$, implying that there is at least one intersection at $\overline{x} > 0$ with $F'(\overline{x}) > 0$. Since $F(x)$ is convex as long as $\phi_\pi > 1$, this intersection is the only other intersection except $x = 0$. Furthermore, the implicit function theorem implies that a smaller ϕ_π implies a larger $\overline{x}$ as long as $\phi_\pi > 1$.

Case with $\phi_\pi = 1$ Finally, it is worth mentioning that with $\phi_\pi = 1$, only the targeted steady state exists. If we impose $\phi_\pi = 1$, (b.1) simplifies to:

$$F(x) = \sigma\left(e^{-\gamma\Theta x} - 1\right),$$

which is clearly a monotonic function of x, and so $x = 0$ is the only solution to the equation $F(x) = 0$. Thus, it follows that with $\phi_\pi = 1$, only the targeted steady state exists.

However, there is still both local and global determinacy as there are multiple bounded trajectories $\{x_t, \pi_t\}$ which originate away from the targeted equilibrium $(0, 0)$ and still stay bounded. The easiest way to see this is to check the condition for local determinacy. This requires that that that the eigenvalues of the matrix A, defined in equation (17) in Section 3 of the main paper has two explosive roots. Setting $\phi_\pi = 1$, we can write the matrix A as:

$$A = \begin{bmatrix} -\sigma\gamma\Theta & 0 \\ -\kappa & \rho \end{bmatrix}$$

The product of the two eigenvalues of A is given by the determinant of A, which is given by $-\rho\sigma\gamma\Theta$. This is clearly negative as long as risk is countercyclical $\Theta > 0$. Since the product of the eigenvalues is negative, it is automatic that the eigenvalues are real, and one is positive while the other is negative. This structure of eigenvalues implies that there is a 1-dimensional stable manifold around the targeted steady state, and any trajectory which begins on this manifold converges to the targeted steady state and hence remains bounded. Thus, with $\phi_\pi = 1$, as in RANK, when risk is countercyclical the targeted equilibrium is locally and hence globally indeterminate, and hence, it is without loss of generality that we focus on the case with $\phi_\pi > 1$ in our baseline.

□

B.4 Local stability of the untargeted steady state

For any $\Theta > 0$, we focus of the case in which $\phi_\pi > \varphi(\Theta)$, which is a necessary (but not sufficient) condition for the targeted equilibrium to be locally determinate (see Proposition 2). We now show that whenever this condition is satisfied, the untargeted steady state, which features lower output $\underline{x}$. At the untargeted steady state, the Jacobian of the system (16a)-(16b) can be written as:

$$A_{\underline{x}} = \begin{bmatrix} -\sigma\gamma\Theta e^{-\gamma\Theta\underline{x}} & \phi_\pi - 1 \\ -\kappa e^{\underline{x}} & \rho \end{bmatrix} \tag{b.2}$$

Thus, up to first-order, the dynamics around the untargeted steady state are identical to the local dynamics around the *targeted* steady state of an alternate economy with more cyclical risk: $\Theta' = \Theta e^{-\gamma\Theta\underline{x}} > \Theta$ and a flatter Phillips curve with slope $\kappa' = \kappa e^{\underline{x}} < \kappa$ (since $\underline{x} < 0$, and so $e^{-\gamma\Theta\underline{x}} > 1$ and $e^{\underline{x}} < 1$). Consequently, we can apply Proposition 2 to conclude that for the untargeted steady state

$(\underline{x}, \underline{\pi}^g)$ to be unstable (locally determinate), we need:

$$\phi_\pi > 1 + \frac{\rho\sigma\gamma\Theta'}{\kappa'} = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa} e^{-(\gamma\Theta+1)\underline{x}}$$

However, for a given Θ and ϕ_π, this can never be satisfied as long as $\phi_\pi > \varphi(\Theta)$. To see why, recall from Appendix B.3 that we can use (b.1) to write $d\underline{x}/d\phi_\pi$ as:

$$\frac{d\underline{x}}{d\phi_\pi} = \left(\frac{e^{\underline{x}}}{1 - e^{\underline{x}}}\right)\left[\phi_\pi - 1 - \frac{\rho\sigma\gamma\Theta}{\kappa} e^{-(1+\gamma\Theta)\underline{x}}\right]$$

We know that this expression is negative as long as $\phi_\pi > 1 + \frac{\rho\sigma\gamma\Theta}{\kappa}$. Since $\underline{x} < 0$, this implies that

$$\phi_\pi < 1 + \frac{\rho\sigma\gamma\Theta}{\kappa} e^{-(1+\gamma\Theta)\underline{x}}, \tag{b.3}$$

i.e., the untargeted steady state $(\underline{x}, \underline{\pi}^g)$ is stable (locally indeterminate), if $\phi_\pi > 1 + \frac{\rho\sigma\gamma\Theta}{\kappa}$. □

B.5 Proof of Proposition 3

Given the interest rate rule $i_t = \bar{r} + \phi_\pi \pi_t$, the aggregate dynamics of the output-gap x_t and inflation π_t is given by the following 2 dimensional system of ordinary differential equations:

$$\begin{aligned} \dot{x} &= (\phi_\pi - 1)\,\pi + \sigma\left(e^{-\gamma\Theta x} - 1\right) \\ \dot{\pi} &= \rho\pi - \kappa\left(e^{x} - 1\right) \end{aligned}$$

We can rewrite this system in matrix form as:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \underbrace{\begin{bmatrix} -\sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}}_{A} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma\left(e^{-\gamma\Theta x_t} - 1 + \gamma\Theta x_t\right) \\ -\kappa\left(e^{x} - 1 - x_t\right) \end{bmatrix} \tag{b.4}$$

Next, for a given $\Theta > 0$ and assuming that (18) is satisfied, we show that the global dynamics of (x_t, π_t) can be split into 3 broad regions depending on the magnitude of Θ. This result is formally presented in Proposition 6 below.

Proposition 6 (Global dynamics)**.** *Consider the economy described in Proposition 1 for a given $\Theta > 0$ and assume that (18) is satisfied. Then the global dynamics depend on the magnitude of the cyclicality of risk Θ, and can be split into 3 broad regions.*

1. ***Mildly countercyclical risk*** $\Theta \in \left(0, \Theta^\diamond\right)$**:** *$\exists \Theta^\diamond > 0$, such that for any $\Theta \in (0, \Theta^\diamond)$, there exists a* saddle-connection, *along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. In this region, the targeted equilibrium is locally determinate, but there is global indeterminacy, as any trajectory starting on the saddle connection also remains bounded forever, while satisfying all equilibrium conditions. Figure 4a depicts this case.*

2. ***Moderately countercyclical risk*** $\Theta \in [\Theta^\diamond, \Theta^\star]$**:** *For any $\Theta \in [\Theta^\diamond, \Theta^\star)$, trajectories originating in the*

neighborhood of the targeted steady state $(0,0)$ *initially diverge away from the steady state but eventually converge to a* super-critical limit-cycle *surrounding* $(0,0)$*, and thus remain bounded and imply the equilibrium is globally indeterminate, even though the targeted steady state is locally determinate. Moreover, the periodicity of the periodic solutions is a decreasing function of* Θ *in the range* $(\Theta^\diamond, \Theta^\star)$*. Figure 4c depicts the stable cycle.*

In the knife edge case with $\Theta = \Theta^\diamond$*, there exists a* homoclinic orbit*, which is a stable trajectory which connects the untargeted steady state* $(\underline{x}, \underline{\pi})$ *to itself, and lies on the boundary of the periodic solutions described above. Again, this implies global indeterminacy, even though* $(0,0)$ *is locally determinate.*

Finally, if $\Theta = \Theta^\star$*, the limit-cycles collapse onto the steady state* $(0,0)$*. The equilibrium is globally indeterminate since the higher-order terms ensure that any trajectory starting in the neighborhood of the targeted steady state converges back to* $(0,0)$*. This case is depicted by Figure 4b*

3. ***Highly countercyclical risk*** $\Theta > \Theta^\star$***:*** *For any* $\Theta > \Theta^\star$*, the targeted steady state* $(0,0)$ *is locally indeterminate even if (18) is satisfied, i.e., there exists multiple bounded trajectories in the neighborhood of* $(0,0)$*, which converge to the targeted steady state. Since the equilibrium is locally indeterminate, it is also globally indeterminate. In addition to the multiple bounded trajectories which start near the targeted steady state, there also exists a* saddle-connection*, along which, the economy converges to the targeted steady state even if it starts near the untargeted steady state. This is depicted by Figure 4d*

Overall, for any $\Theta > 0$*, i.e, if risk is even mildly countercyclical, there is global indeterminacy. For any finite* ϕ_π*, the inflation targeting rule (6) fails at eliminating the existence of multiple bounded equilibria.*

We will prove Proposition 6 by using Theorem 7.1 in Kopell and Howard (1975), and the Hopf bifurcation theorem (Marsden and McCracken, 1976). We present these theorems here for convenience.

Theorem 1 (Hopf Bifurcation Theorem)**.** *Consider a two-dimensional system*

$$\dot{\mathbf{x}} = F_\mu(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2, \mu \in \mathbb{R}$$

with smooth F, which for all sufficiently small $|\mu|$ *has the equilibrium* $\mathbf{x} = (0,0)$*, and the Jacobian* $\mathrm{D}_\mathbf{x} F_\mu(0,0)$ *has eigenvalues*

$$\lambda_{1,2}(\mu) = \Omega(\mu) \pm \mathtt{i}\omega(\mu) \qquad \textit{where} \qquad \mathtt{i} = \sqrt{-1},$$

Then, if the following conditions are satisfied:

1. *At* $\mu = 0$*, there exists a purely imaginary set of eigenvalues:* $\Omega(0) = 0$ *and* $\omega(0) > 0$
2. *The eigenvalues cross the imaginary axis with non-zero speed:* $\frac{d\Omega(0)}{d\mu} \neq 0$
3. *The first Lyapunov coefficient of the system* $\ell_1(0) \neq 0$*,*

there exists a family of real periodic solutions $\mathbf{x} = \mathbf{x}(t,\epsilon)$*,* $\mu(\epsilon)$ *which has properties* $\mu(0) = 0$*,* $\mathbf{x}(t,0) = (0,0)$*, but* $\mathbf{x}(t,\epsilon) \neq (0,0)$ *for sufficiently small* ϵ*. The same holds for the period* $T(\epsilon)$ *and* $T(0) = 2\pi / |\omega(0)|$*.*

Theorem 2 (Theorem 7.1 in Kopell and Howard (1975))**.** *Let* $\dot{\mathbf{x}} = F_{\mu,\nu}(\mathbf{x})$ *be a two parameter family of ODEs on* $\mathbb{R}^2$*, F smooth in all of its four arguments, such that* $F_{\mu,\nu}(0,0) = \mathbf{0}$*. Also assume:*

1. *$dF_{0,0}(0,0)$ has a double zero eigenvalue and a single eigenvector* $\mathbf{v}$.

2. *The mapping* $(\mu, \nu) \to (det(dF_{0,0}(0,0), tr(dF_{0,0}(0,0))$ *has a nonzero Jacobian at* $(\mu, \nu) = (0,0)$.

3. *Let* $Q(\mathbf{x}, \mathbf{x})$ *be the* 2×1 *vector containing the terms quadratic in* $\mathbf{x}$ *and independent of* (μ, ν) *in a Taylor series expansion of* $F_{\mu,\nu}(\mathbf{x})$ *around 0. Then* $[dF_{0,0}(0,0), Q(\mathbf{v}, \mathbf{v})]$ *has rank 2.*

Then there is a curve $f(\mu, \nu) = 0$ *such that if* $f(\mu_0, \nu_0) = 0$*, then* $\dot{\mathbf{x}} = F_{\mu_0,\nu_0}(\mathbf{x})$ *has a homoclinic orbit. This one-parameter family of homoclinic orbits (in* (X, μ, ν) *space) is on the boundary of a two-parameter family of periodic solutions. For all* $|\mu|, |\nu|$ *sufficiently small, if* $\dot{\mathbf{x}} = F_{\mu,\nu}(\mathbf{x})$ *has neither a homoclinic orbit nor a periodic solution, there is a unique trajectory joining the critical points.*

We first use Theorem 1 to prove point 2 of Proposition 6. We use the cyclicality of risk Θ as the bifurcation parameter (which plays the role of μ in Theorem 1 above). Define $\Theta^\star = \frac{\rho}{\sigma\gamma}$. Imposing $\Theta = \Theta^\star$ in (b.4) yields

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \underbrace{\begin{bmatrix} -\rho & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}}_{A^\star} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma\left(e^{-\frac{\rho}{\sigma}x_t} - 1 + \frac{\rho}{\sigma}x_t\right) \\ -\kappa\left(e^{x} - 1 - x_t\right) \end{bmatrix}, \tag{b.5}$$

It is clear by inspection that the trace of the matrix $A^\star$ is equal to zero, which implies that the two eigenvalues sum to 0. Also, the determinant of $A^\star$ is given by

$$\text{Det}(A^\star) = \kappa(\phi_\pi - 1) - \rho^2$$

Next, recall that the augmented Taylor principle (18) requires that $\phi_\pi > 1 + \frac{\rho\sigma\gamma\Theta}{\kappa}$ for local determinacy. Evaluating this expression at $\Theta = \Theta^\star$, we have $\phi_\pi > 1 + \frac{\rho^2}{\kappa}$, which in turn implies that $\text{Det}(A^\star) > 0$. Consequently, at $\Theta = \Theta^\star$ the eigenvalues of $A^\star$ are purely imaginary and given by $\pm\omega\mathtt{i}$, where $\mathtt{i} = \sqrt{-1}$ and $\omega = \sqrt{\kappa(\phi_\pi - 1) - \rho^2}$. Thus, requirement 1 of Theorem 1 is satisfied at $\Theta = \Theta^\star$ (this corresponds to the $\mu = 0$ in the statement of the theorem).

Next, it is clear by inspection that the eigenvalues of the matrix A in (b.4) change smoothly in Θ, which implies that condition 2 of Theorem 1 is also satisfied. Thus, the only other condition we need to check is that the first Lyapunov coefficient (evaluated at the bifurcation point) is not 0. To check this, we first need to transform the system (b.5) into normal-form, for which we diagonalize $A^\star$ as:[27]

$$A^\star = PDP^{-1},$$

where

$$D = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} \rho & \omega \\ \kappa & 0 \end{bmatrix}$$

[27]The normal form involves a change of variables so that the first-order accurate system can be written in the well known "decoupled" system. Diagonalizing a matrix with real eigenvalues leads to a decoupled system with entries only on the main diagonal. However, since we are diagonalizing a 2x2 matrix with purely complex roots, the diagonalized matrix features non-zero entries only on the anti-diagonal. See the matrix in equation (b.6).

Next, we can pre-multiply both sides of (b.5) by P^{-1} to express the system in normal form:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} f(u,v) \\ g(u,v) \end{bmatrix}, \tag{b.6}$$

where

$$\begin{bmatrix} u \\ v \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ \pi \end{bmatrix},$$

$$\begin{aligned} f(u,v) &= -e^{\rho u+\omega v} + 1 + \rho u + \omega v \\ g(u,v) &= \frac{\rho e^{\rho u+\omega v} + \sigma e^{-\frac{\rho}{\sigma}(\rho u+\omega v)} - (\rho+\sigma)}{\omega} \end{aligned}$$

Finally, the first Lyapunov coefficient at the bifurcation point $\Theta = \Theta^\star$ is given by:

$$\begin{aligned} \ell_1(0) &= f_{uuu}(0,0) + f_{uvv}(0,0) + g_{uuv}(0,0) + g_{vvv}(0,0) \\ &\quad + \frac{1}{\omega}\Big[f_{uv}(0,0)\left(f_{uu}(0,0) + f_{vv}(0,0)\right) - g_{uv}(0,0)\left(g_{uu}(0,0) + g_{vv}(0,0)\right) - f_{uu}(0,0)\, g_{uu}(0,0) \\ &\quad + f_{vv}(0,0)\, g_{vv}(0,0) \Big] \\ &= -(\rho+\sigma)\frac{\rho^2\kappa^2(\phi_\pi - 1)^2}{\sigma^2\omega^2} < 0, \end{aligned}$$

which is non-zero, for any $\phi_\pi > 0$. Thus, condition 3 of Theorem 1 is also satisfied, and the system (b.4) undergoes a Hopf bifurcation at $\Theta = \Theta^\star$. Furthermore, since $\ell_1(0)$ regardless of the value of $\phi_\pi > 1$, the Hopf bifurcation is always *supercritical*, i.e. the higher-order terms of the system (b.4), push $\mathbf{x}$ in towards the equilibrium $(0,0)$.

In terms of our HANK model, this means that there exists $\Theta^\diamond \in (0, \Theta^\star)$ such that for any Θ in the neighborhood $\Theta \in (\Theta^\diamond, \Theta^\star)$, starting from any initial condition in the neighborhood $(x,\pi) = (0,0)$, the system converges to a stable cycle around the targeted steady state. In the main text, if cyclicality of risk is such that $\Theta^\diamond < \Theta < \Theta^\star$, we say that risk is "moderately countercyclical". Thus, for any Θ in the moderately countercyclical region, even though the targeted equilibrium is locally determinate, the equilibrium is globally indeterminate since all trajectories starting near the targeted steady state initially diverge but then converge to a cycle, remaining bounded.

Theorem 1 also implies that the amplitude and periodicity of the stable cycles are decreasing functions of Θ in the moderately countercyclical range, which in turn depends on the inverse of ω (the imaginary part of the eigenvalue). In particular, at $\Theta = \Theta^\diamond$, Proposition 6 shows that there exists a homoclinic orbit (we prove this using Theorem 2 below). The homoclinic orbit has infinite periodicity, which means that $\Theta^\diamond$ is implicitly defined by the Θ for which the cycle has infinite periodicity. While this cannot be characterized analytically, we can numerically compute the value of $\Theta^\diamond$, and this value is presented in the main text. Thus, as we increase Θ from $\Theta^\diamond$ towards $\Theta^\star$, the amplitude and periodicity of the cycles decrease. Finally, at $\Theta = \Theta^\star$ the periodicity of the cycle is 0 and the cycle collapses on to the targeted steady state itself. However, as we show next, the equilibrium is still indeterminate. Since the eigenvalues of the Jacobian at $\Theta = \Theta^\star$ are purely imaginary, the first-order terms do not move the system away from or towards $(0,0)$. However, since the Hopf bifurcation is supercritical, the higher-order terms push in towards the origin, implying that all trajectories which start in the neighborhood of the targeted steady state $(0,0)$ remain bounded, implying global indeterminacy.

As mentioned above, to prove the existence of the homoclinic orbit at $\Theta = \Theta^{\diamond}$, we need to use Theorem 2, which states that as long the conditions 1,2,3 are satisfied, then (b.4) has a homoclinic orbit on the boundary of the stable cycles we described above. While Theorem 1 only required conditions on one parameter Θ, Theorem 2 requires imposing some additional conditions on a second parameter. For us, it is most convenient to choose ϕ_π as the second parameter. First, notice that for $\Theta = \Theta^{\star}$ and $\phi_\pi = \varphi(\Theta^{\star}) = 1 + \frac{\rho^2}{\kappa}$, the Jacobian of (b.4) is:

$$A^{\diamond} = \begin{bmatrix} -\rho & \frac{\rho^2}{\kappa} \\ -\kappa & \rho \end{bmatrix}, \tag{b.7}$$

which has both trace and determinant equal to 0, implying that both eigenvalues are 0, satisfying condition 1 of Theorem 2. Also, since the eigenvalues repeat, it is easy to check that the matrix has eigenvector $\mathbf{v} = \begin{bmatrix} \rho \\ \kappa \end{bmatrix}$, and a generalized eigenvector $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Next, we show that condition 2 is also satisfied. Recall that the Jacobian of (b.4) is given by the matrix A

$$A(\Theta, \phi_\pi) = \begin{bmatrix} -\sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix},$$

and the trace of $A(\Theta, \phi_\pi)$ is $Tr_A = \rho - \sigma\gamma\Theta$, while the determinant is $Det_A = \kappa(\phi_\pi - 1) - \rho\sigma\gamma\Theta$. Then the Jacobian of $[Tr_A, Det_A]$ is given by:

$$J = \begin{bmatrix} -\sigma\gamma & -\rho\sigma\gamma \\ 0 & \kappa \end{bmatrix}$$

The determinant of this matrix is non-zero as long as $\kappa\sigma\gamma \neq 0$. This confirms that condition 2 of Theorem 2 is satisfied. Next, to check condition 3, we need to construct the matrix $Q(\mathbf{x}, \mathbf{x})$, which is a 2×1 vector that contains terms quadratic in (x, π) and independent of (Θ, ϕ_π) in a Taylor series expansion of $F\Big(x, \pi, \Theta^{\star}, \varphi(\Theta^{\star})\Big)$ around $(x, \pi) = (0, 0)$. Since (b.4) has no higher-order terms in π, it is easy to see that $Q(\mathbf{x}, \mathbf{x})$ can be written as:

$$Q = \begin{bmatrix} 0 \\ -0.5\kappa x^2 \end{bmatrix}$$

Evaluating this at the eigenvector $\mathbf{v}$, we have:

$$Q(\mathbf{v}, \mathbf{v}) = \begin{bmatrix} 0 \\ -0.5\kappa\rho^2 \end{bmatrix}$$

Then, clearly the condition 3 of Theorem 2 is satisfied since

$$\text{rank}\begin{bmatrix} -\rho & \kappa^{-1}\rho^2 & 0 \\ -\kappa & \rho & -0.5\kappa\rho^2 \end{bmatrix} = 2,$$

as long as $\rho \neq 0$. Thus, the conditions for Theorem 2 are satisfied. Consequently, Theorem 2 states that at $\Theta = \Theta^{\diamond}$, a homoclinic orbit emerges, which completes the proof of point 2 of Proposition 6.

Next, we prove point 1 of Proposition 6. Point 2 of Proposition 6 established that stable cycles exist for $\Theta \in (\Theta^{\diamond}, \Theta^{\star})$ and a homoclinic orbit exists at $\Theta = \Theta^{\diamond}$. Then Theorem 2 implies that since there is no cycle or homoclinic orbit, there exists a saddle connection along which the economy moves from the targeted to the untargeted steady state for $0 < \Theta < \Theta^{\diamond}$. Any trajectory which originates on this saddle connection remain bounded. Hence, there is global indeterminacy even when $0 < \Theta < \Theta^{\diamond}$. Finally, point 3 of Proposition 6 is also true because of similar reasons. When $\Theta > \Theta^{\star}$, the Hopf bifurcation theorem implies that there are no cycles in this part of the parameter space. Consequently, Theorem 2 implies that there must be a saddle connection from the untargeted to targeted steady state. Since the targeted equilibrium is already locally indeterminate for $\Theta > \Theta^{\star}$, there is also global indeterminacy.

□

C Proof of Proposition 4

The monetary policy rule which implements a unique equilibrium is given by

$$i_t = \bar{r} + \phi_\pi \pi_t + \phi_r \left(r^*(x_t) - \bar{r}\right) \qquad \text{with} \qquad \phi_\pi > 1 \text{ and } \phi_r \geq 1 \tag{c.1}$$

Such a rule ensures the existence of a unique bounded equilibrium in which the economy remains that the targeted steady state $(x, \pi) = (0, 0)$ at all dates. To see this, we can plug in the policy rule (c.1) into the IS equation (14). Then, the global dynamics of (x_t, π_t) are described by the following system of ODEs:

$$\dot{x}_t = (\phi_\pi - 1)\pi_t + (1 - \phi_r)\sigma\left(e^{-\gamma\Theta x_t} - 1\right) \tag{c.2}$$

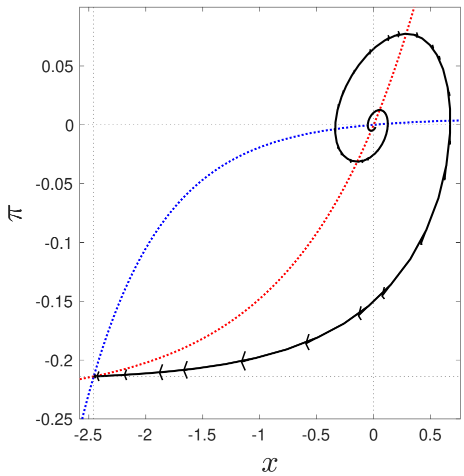
$$\dot{\pi}_t = \rho\pi_t + \kappa\left(e^{x_t} - 1\right) \tag{c.3}$$

This system nests our baseline economy if we set $\phi_r = 0$. First, note that $x_t = \pi_t = 0$ clearly satisfies the system of equations above, thus proving that the targeted steady state $x = \pi = 0$ is a valid equilibrium with the policy rule (c.1).

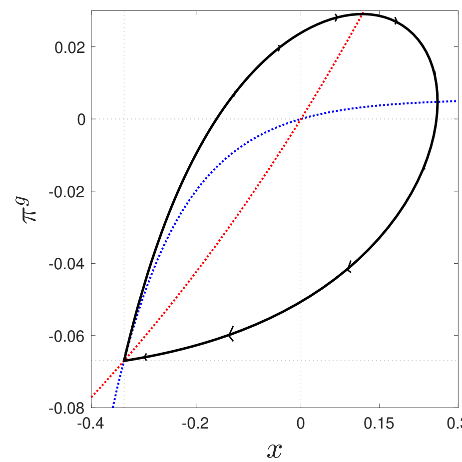
First, note that the Jacobian of the system at any (x, π) can be written as:

$$J(x, \pi) = \begin{bmatrix} (\phi_r - 1)\gamma\Theta e^{-\gamma\Theta x} & \phi_\pi - 1 \\ -\kappa e^{x} & \rho \end{bmatrix}$$

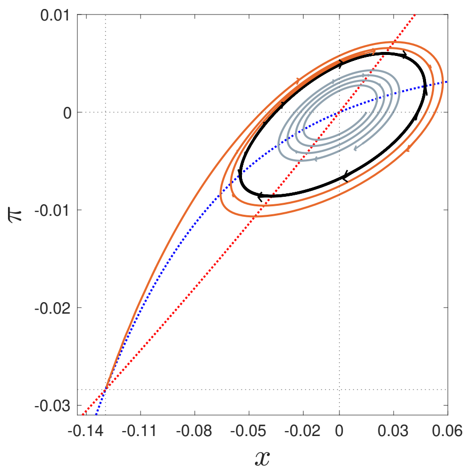
The local determinacy property of the targeted equilibrium $x = \pi = 0$ is determined by the eigenvalues of the matrix $J(0, 0)$. The sum of the eigenvalues of $J(0, 0)$ is given by $\rho + (\phi_r - 1)\gamma\Theta$ and the product of the eigenvalues is given by the determinant of $\rho(\phi_r - 1)\gamma\Theta + \kappa(\phi_\pi - 1)$. A sufficient condition for



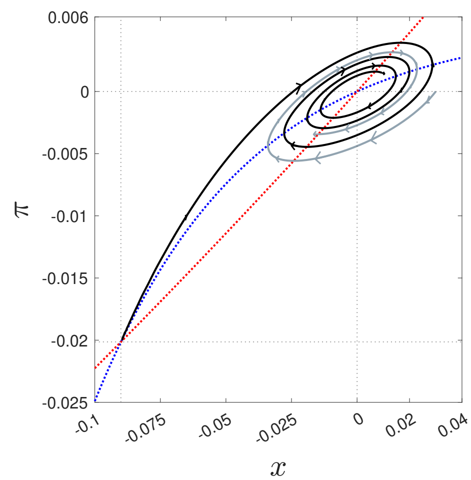
(a) $\Theta \in (0, \overline{\Theta})$



(b) $\Theta = \overline{\Theta}$



(c) $\Theta \in (\overline{\Theta}, \Theta^{\star})$



(d) $\Theta > \Theta^{\star}$

Figure 4: Global dynamics depending on magnitude of Θ

both the sum and product of roots to be positive is that $\phi_\pi > 1$ and $\phi_r \geq 1$. If this condition is satisfied, then for any $\Theta > 0$, $J(0, 0)$ has two eigenvalues with positive real parts, ensuring local determinacy of the targeted equilibrium.

Second, notice that the trace of $J(x, \pi)$ is given by $\rho + (\phi_r - 1)\gamma\Theta e^{-\gamma\Theta x}$, which is positive for any $(x, \pi) \in (-\infty, \infty)^2$ as long as $\phi_r \geq 1$. Since the trace of the Jacobian does not change sign in the entire domain, the Bendixson–Dulac theorem (Bendixson, 1901; Dulac, 1937) implies that there are no non-constant periodic solutions lying entirely within $(x, \pi) \in (-\infty, \infty) \times (-\infty, \infty)$. Thus, we have global determinacy as long as there is a unique constant solution.

We already know that $x = \pi = 0$ is a constant solution. Next, we show that this is the only constant solution. To characterize all constant solutions, we set $\dot{x} = \dot{\pi} = 0$ in (c.2) and (c.3). This yields:

$$0 = (\phi_\pi - 1)\pi + (1 - \phi_r)\sigma\left(e^{-\gamma\Theta x} - 1\right) \qquad \text{and} \qquad \pi \;=\; \frac{\kappa}{\rho}(e^x - 1)$$

We can use the second expression to substitute out π from the first. Doing so yields a single equation in x, which describes all the steady state values of x:

$$F(x) = \frac{\rho(\phi_\pi - 1)}{\kappa}(\mathrm{e}^x - 1) + (1 - \phi_r)\sigma\left(\mathrm{e}^{-\gamma\Theta x} - 1\right),$$

and $x = 0$ is a solution to $F(x) = 0$. To see that $F(x)$ has no other zeros, note that the first derivative of $F(x)$ can be written as:

$$F'(x) = \frac{\rho(\phi_\pi - 1)}{\kappa}\mathrm{e}^x + (\phi_r - 1)\gamma\sigma\Theta\mathrm{e}^{-\gamma\Theta x}$$

A sufficient condition for this expression to be strictly positive for any $x \in (-\infty, \infty)$ is that (i) $\phi_\pi > 1$, and (ii) $\phi_r \geq 1$. Thus, as long as $\phi_\pi > 1$ and $\phi_r \geq 1$, the only zero of $F(x)$ is at $x = 0$, i.e., the targeted steady state is the unique steady state. Consequently, $x = \pi = 0$ is the only bounded solution with the policy rule (c.1). □

C.1 Escape clause

In this section, we provide details of the escape clause policy described in Section 4.1. In particular, we assume that the monetary policy is described by the simple inflation targeting rule $i_t = \bar{r} + \phi_\pi \pi_t$ as long as output is above some threshold level $\widetilde{x}$. However, when output is below this threshold $x_t < \widetilde{x}$, monetary policy switches to a strict-inflation targeting rule $\pi_t = 0$ from that date on. In what follows, we will assume that $\underline{x} < 0$, i.e,, the threshold level of output lies below output in the targeted equilibrium $(x = 0)$. We will also assume that for a given Θ and ϕ_π, $\widetilde{x}$ is marginally higher than the output in the untargeted steady state that would exist if monetary policy was described as in our baseline model.

The strict-inflation targeting rule $\pi_t = 0$ can be interpreted as a whatever-it-takes stance which rules out any equilibria along which at any date the economy passes through a point (x_t, π_t) such that $x_t < \widetilde{x}$ and $\pi_t \neq 0$. The first thing to notice is that $\pi_t = 0$ is not an instrument rule like the policy rule (6); it is instead a *targeting rule* and can be interpreted as a commitment by the central bank that it will set the path of nominal interest rates to whatever level is required to ensure that $\pi_t = 0$ at all dates from then on. In terms of an instrument rule, this targeting rule is often expressed as the limit of a standard inflation targeting Taylor rule $i_t = \bar{r} + \phi_\pi \pi_t$ with ϕ_π set to ∞.

To see how such a rule can rule out any equilibria along which at any date the economy passes through a point (x_t, π_t) such that $x_t < \widetilde{x}$ and $\pi_t \neq 0$, suppose that at date t, the economy features a level of output with some finite x_t, which satisfies $x_t < \widetilde{x}$. Notice that the IS curve (with the inflation targeting rule plugged in) and the Phillips curve in the context of our baseline model is given by:

$$\dot{x}_t = (\phi_\pi - 1)\pi_t + \sigma(e^{-\gamma\Theta x_t} - 1)$$

With ϕ_π to ∞, then for $\dot{x}_t$ to be finite, we need that $\pi_t = 0$. Otherwise, if $\pi_t > 0$, we would have $\dot{x}_t = +\infty$ and if $\pi_t < 0$, we would have $\dot{x}_t = -\infty$ In other words, if $\pi_t \neq 0$ at all dates, x_t will grow or shrink infinitely fast and become unbounded. Thus, the only bounded solution which satisfies the IS and the Phillips curve is $x_t = \pi_t = 0$ at all dates. Thus, the combination of the IS curve and the policy rule imply that $\pi_t = \dot{\pi}_t$ must equal 0 at all dates from then on. However, since $x_t < \widetilde{x}$, with $\pi_t 0$ set to

0, the date t Phillips curve implies that $\dot{\pi}_t = -\kappa x_t > 0$, which contradicts the fact that $\pi_t = \dot{\pi}_t = 0$. This shows that the strict inflation targeting rule ensures that there exists no combination of (x, π) with $x < \tilde{x}$ which satisfies both the IS curve and the Phillips curve, and thus does not constitute an equilibrium. Thus, as long as $\tilde{x} > \underline{x}$, it follows that the untargeted steady state which features an output-gap of $\underline{x} < 0$ and inflation $\underline{\pi} < 0$ cannot emerge in equilibrium, given monetary policy which switches to a strict-inflation targeting rule $\pi_t = 0$ if output falls below $\tilde{x}$.

However, for $\tilde{x}$ sufficiently close to $\underline{x}$, such a policy stance need not rule out the stable cycle. To see this, suppose that risk is moderately countercyclical $\Theta \in (\Theta^\diamond, \Theta^\star)$. Then since monetary policy still follows the same rule as in our baseline as long as the economy is near the untargeted steady state, we can still use the same proof strategy as in Appendix B.5 to prove the existence of the stable cycle. The reason behind this is that the proof of the existence of the stable cycle in Appendix B.5 only uses the properties of the ODEs describing the economy *local* to the targeted steady state. Since the escape clause only kicks in once the economy is far away from the targeted steady state, the behavior of the economy local to the targeted steady state is unchanged. Consequently, as long as $\tilde{x}$ is less than the smallest value of output observed on the stable cycle, the escape clause cannot rule out the economy from converging to a stable cycle surrounding the targeted steady state. Figure 3b presents an example of such a situation in our calibrated economy when the escape clause is triggered if output declines more than 5% below its targeted value. Since the untargeted steady state features a level of output about 6% below its targeted level, but the maximum amplitude of output on the stable cycle is about ±2.5%, the policy rules out the untargeted steady state but fails to eliminate the stable cycle.

D Monetary-fiscal interaction

As described in Appendix A.3, the dynamics of x_t, π_t, b_t^g are given by the three dimensional system of non-linear ODEs:

$$\dot{x}_t = (\phi_\pi - 1)\,\pi_t + \sigma\left(e^{-\gamma\Theta x_t} - 1\right) \tag{d.1}$$

$$\dot{\pi}_t = \rho\pi_t - \kappa\left(e^{x_t} - 1\right) \tag{d.2}$$

$$\dot{b}_t^g = (\phi_\pi - 1)\,b^\star\pi_t + \bar{r}\,(1-\phi_b)\,b_t^g + (\phi_\pi - 1)\,\pi_t b_t^g \tag{d.3}$$

Separating the first-order terms which govern dynamics local to the steady state $(x, \pi, b^g) = (0, 0, 0)$ and higher-order terms which dominate dynamics away from $(0, 0, 0)$, we can equivalently rewrite the 3 dimensional system above as:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \\ \dot{b}_t^g \end{bmatrix} = A \begin{bmatrix} x_t \\ \pi_t \\ b_t^g \end{bmatrix} + \begin{bmatrix} \sigma\left(e^{-\gamma\Theta x_t} - 1 + \gamma\sigma\Theta x_t\right) \\ -\kappa\left(e^{x_t} - 1 - x_t\right) \\ (\phi_\pi - 1)\,\pi_t b_t^g \end{bmatrix}, \tag{d.4}$$

where the matrix A is given by:

$$A = \begin{bmatrix} -\gamma\sigma\Theta & \phi_\pi - 1 & 0 \\ -\kappa & \rho & 0 \\ 0 & (\phi_\pi - 1)\, b^\star & \bar{r}(1-\phi_b) \end{bmatrix} \tag{d.5}$$

The determinacy properties of the economy with government debt depend on the eigenvalues of the matrix A in (d.5). Since we have one predetermined variable b^g, and two jump variables x, π, we need 2 eigenvalues with positive real parts and one negative eigenvalue for local determinacy. The characteristic polynomial associated with the matrix A can be written as:

$$\mathcal{P}(z) = [\bar{r}(1-\phi_b) - z] \cdot Det(A^\dagger - zI) = 0, \tag{d.6}$$

where $A^\dagger$ is the leading principal minor of order 2 of the matrix A, and is given by

$$A^\dagger = \begin{bmatrix} -\gamma\sigma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix} \tag{d.7}$$

Then, we know that one of the eigenvalues of the system is given by $z_1 = \bar{r}(\phi_b - 1)$, while the two remaining eigenvalues are the two roots of the quadratic equation $Det(A^\dagger - zI) = 0$. Furthermore, we know that

$$Tr(A^\dagger) = \gamma\sigma\left(\Theta^\star - \Theta\right) \qquad \text{and} \qquad Det(A^\dagger) = \kappa\left[\phi_\pi - \varphi(\Theta)\right], \tag{d.8}$$

where

$$\Theta^\star = \frac{\rho}{\sigma\gamma} \qquad \text{and} \qquad \varphi(\Theta) = 1 + \frac{\rho\gamma\sigma\Theta}{\kappa}$$

are the same as in our baseline model. In what follows, and we follow the nomenclature of Leeper (1991) and consider two regimes based on the magnitudes of ϕ_π and ϕ_b: (i) Active-Monetary, Passive-Fiscal in which $\phi_\pi > 1$ and $\phi_b > 1$, and (ii) Passive-Monetary, Active-Fiscal in which $\phi_\pi \in [0,1]$ and $\phi_b \in [0,1)$.[28]

D.1 Active-Monetary, Passive-Fiscal (AMPF)

It is worth noting that our baseline model is just a special case of this AMPF regime with $\phi_b \to \infty$ and $\bar{b} = 0$. As such, the determinacy properties of the AMPF regime with $\phi_b < \infty$ and $\bar{b} \neq 0$ are identical to that of our baseline economy without government debt. We show this next.

D.1.1 Local Determinacy

Passive fiscal policy ($\phi_b > 1$) implies that the eigenvalue $z_1 = \bar{r}(1-\phi_b) < 0$. So local determinacy requires that the other two eigenvalues have positive real parts. For this, we need the trace and

[28]It is easy to see that with $\phi_b = 1$, the eigenvalue $z_1 = 0$ and thus, we cannot have local determinacy in the knife-edge case of $\phi_b = 1$. Thus, we define the regions to exclude the case with $\phi_b = 1$.

determinant of the matrix $A^{\dagger}$ be positive. We divide our analysis into three cases depending on how countercyclical risk is and how aggressively monetary policy responds to inflation.

Case I: Mildly or moderately countercyclical risk $0 < \Theta < \Theta^{\star}$ **and** $1 < \phi_{\pi} \leq \varphi(\Theta)$ First, consider the case in which risk is not highly countercyclical $\Theta < \Theta^{\star}$, and monetary policy is active but not too aggressive $1 < \phi_{\pi} < \varphi(\Theta)$. Then, from (d.8), we know that even though the trace of $A^{\dagger}$ is positive, the determinant of $A^{\dagger}$ is negative, implying that we have 2 negative and one positive root. Thus, the targeted steady state has a 2-dimensional stable manifold around it, and hence, is *locally indeterminate* in this configuration under the AMPF regime.

Case II: Mildly or moderately countercyclical risk $0 < \Theta < \Theta^{\star}$**, and** $\phi_{\pi} > \varphi(\Theta)$ Next, consider the case in which risk is mildly or moderately countercyclical $0 < \Theta < \Theta^{*})$ and that monetary policy is sufficiently active $\phi_{\pi} > \varphi(\Theta)$. With $\phi_{\pi} > \varphi(\Theta)$, (d.8) implies that both the determinant and trace of $A^{\dagger}$ are positive, implying that there are two positive and one negative root. Thus, the AMPF regime delivers local determinacy if risk is not highly countercyclical and monetary policy responds sufficiently aggressively to changes in inflation.

Case III: Highly countercyclical risk $\Theta > \Theta^{\star}$ Finally, when risk is highly countercyclical ($\Theta > \Theta^{\star}$), equation (d.8) shows that the trace of $A^{\dagger}$ is negative. Consequently, as in our baseline model, local determinacy is not possible under the AMPF regime if risk is highly countercyclical $\Theta > \Theta^{\star}$.

Thus, as in our baseline model, in the AMPF regime, a large enough ϕ_{π} can deliver local determinacy as long as risk is not highly countercyclical. Next, we show that as in our baseline, no matter how large ϕ_{π} is, the AMPF regime cannot deliver global determinacy.

D.1.2 Global indeterminacy

Since the AMPF regime does not deliver local determinacy if risk is highly countercyclical ($\Theta > \Theta^{\star}$) (Case III above), it follows that the equilibrium also features global indeterminacy. Similarly, if $\Theta < \Theta^{\star}$, but $1 < \phi_{\pi} < \varphi(\Theta)$ (Case I), the AMPF regime also does not deliver global determinacy. Thus, the remaining case to consider is when risk is not highly countercyclical, $\Theta < \Theta^{\star}$, and monetary policy is sufficiently aggressive, $\phi_{\pi} > \varphi(\Theta)$ (Case II). In this case the target equilibrium is locally determinate. However, as in our baseline model, this does not translate into global determinacy. We start with the case of moderately countercyclical risk.

Stable cycle (Moderately countercyclical risk) Similar to Appendix B.5, in the AMPF regime, the economy undergoes a Hopf bifurcation at $\Theta = \Theta^{\star} = \rho/\sigma\gamma$, i.e. if $\Theta = \Theta^{\star}$, the eigenvalues of the matrix A described in (d.5) has two purely imaginary roots. Then, the three dimensional generalization of Theorem 1 guarantees that there exists $\overline{\Theta} < \Theta^{\star}$ such that for any $\Theta \in (\overline{\Theta}, \Theta^{\star})$, any trajectory (except on the one dimensional stable manifold) which starts in the neighborhood of $(x, \pi, b^{g}) = (0, 0, 0)$ converge to a stable cycle along which the economy oscillates around the targeted steady state, remaining bounded forever. We present this Theorem next for convenience.

Theorem 3 (Hopf Bifurcation in n-dimensions)**.** *Consider a $n \geq 2$ dimensional system*

$$\dot{\mathbf{x}} = F_\mu(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \mu \in \mathbb{R}$$

with smooth F, which for all sufficiently small $|\mu|$ has the equilibrium $\mathbf{x} = \mathbf{0}$, and the Jacobian $\mathrm{D}_{\mathbf{x}} F_\mu(\mathbf{0})$ has eigenvalues given by $\Omega(\mu) + \omega(\mu)\mathtt{i}$ where $\mathtt{I} = \sqrt{-1}$. Then, if the following conditions are satisfied:

1. *At $\mu = 0$, $\mathrm{D}_{\mathbf{x}} F_{\mu=0}(\mathbf{0})$ has two purely imaginary eigenvalues $\pm\bar{\omega}_0$ and no other eigenvalue of $\mathrm{D}_{\mathbf{x}} F_{\mu=0}(\mathbf{0})$ is an integral multiple of $\bar{\omega}_0\mathtt{i}$.*

2. *Without loss of generality, for ease of exposition, assume that the first two out of the n eigenvalues are the purely imaginary ones at $\mu = 0$, and their continuation is given by:*

$$\lambda_{1,2}(\mu) = \Omega(\mu) \pm \omega(\mu)\mathtt{i}$$

Then, these eigenvalues cross the imaginary axis with non-zero speed: $d\Omega(0)/d\mu \neq 0$.

Under the above conditions there exist a family of real periodic solutions $\mathbf{x} = \mathbf{x}(t, \epsilon)$, $\mu(\epsilon)$ which has the properties $\mu(0) = 0$, $\mathbf{x}(t, 0) = \mathbf{0}$, but $\mathbf{x}(t, \epsilon) \neq \mathbf{0}$ for sufficiently small ϵ. The same holds for the period $T(\epsilon)$ and $T(0) = 2\pi/|\omega(0)|$.

To see that Theorem 3 applies, imposing $\Theta = \Theta^\star = \rho/\sigma\gamma$ in (d.5), we can write A as:

$$A(\Theta^\star) = \begin{bmatrix} -\rho & \phi_\pi - 1 & 0 \\ -\kappa & \rho & 0 \\ 0 & (\phi_\pi - 1)\,\bar{b} & \bar{r}(1 - \phi_b) \end{bmatrix}$$

We know from the characteristic polynomial of A (equation (d.6)) that one of the eigenvalues of $A(\Theta^\star)$ is given by $z_1 = \bar{r}(1 - \phi_b)$, which is negative in the AMPF regime since $\phi_b > 1$. Then the sum of the other two roots is the trace of the principal minor of order 2 associated with $A(\Theta^\star)$, which is given by $-\rho + \rho = 0$. The zero trace means that the eigenvalues z_2 and z_3 cancel each other out. Furthermore, the determinant of the principal minor of order 2 associated with $A(\Theta^\star)$ is given by

$$\kappa\,[\phi_\pi - \varphi(\Theta^\star)] \qquad \text{where} \qquad \varphi(\Theta^\star) = 1 + \frac{\rho^2}{\gamma\sigma},$$

which is positive since we are considering the case with $\phi_\pi > \varphi(\Theta^\star)$. Together these properties of the trace and determinant imply that the other two eigenvalues are purely imaginary, and thus condition 1 is satisfied. Furthermore, it is clear from inspection that the eigenvalues change smoothly in Θ, and thus condition 2 of the Theorem is satisfied. This shows that, as in our baseline model, the economy with non-zero government debt undergoes a Hopf bifurcation at $\Theta = \Theta^\star$ in the AMPF regime.

Finally, to show that the cycle is stable in the economy with government debt, as in our baseline model, we need to show that the first Lyapunov coefficient is negative. While it is analytically cumbersome to compute the first Lyapunov coefficient and show that it is negative, Figure 5 shows that under our calibrated value of $\Theta = 28.1$ (which corresponds to the moderately countercyclical risk case, i.e., $\Theta^\diamond < \Theta < \Theta^\star$) , any trajectory originating in the neighborhood of $(x, \pi, b^g) = (0, 0, 0)$ (depicted by the

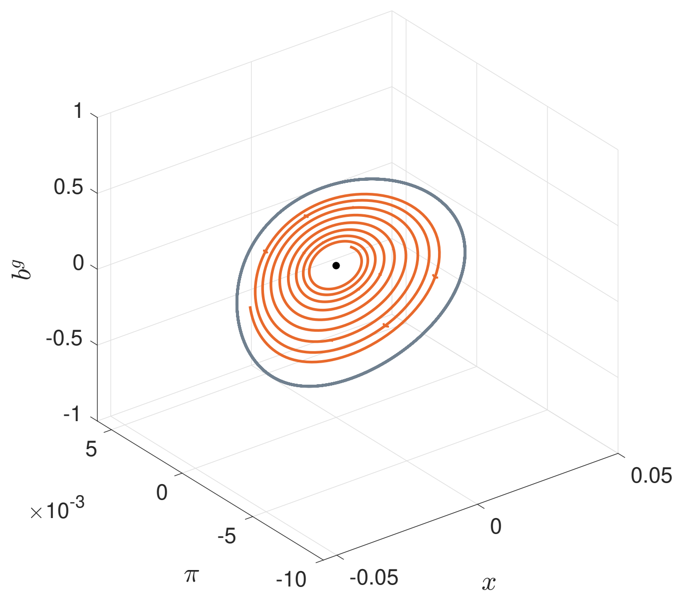


Figure 5: Stable cycle with moderately countercyclical risk in AMPF

orange trajectory) eventually converges to a stable limit-cycle (depicted by the gray trajectory) around the targeted steady state when risk is moderately countercyclical $\Theta \in (\overline{\Theta}, \Theta^{\star})$. Thus, if $\phi_\pi > \varphi(\Theta)$, and risk is moderately countercyclical $\overline{\Theta} < \Theta < \Theta^{\star}$, the AMPF regime delivers local but not global determinacy, since along with the targeted equilibrium $(x, \pi, b^g) = (0, 0, 0)$, the stable cycle and all the trajectories converging to the stable cycle also constitute bounded sequences which satisfy all equilibrium conditions.

Saddle Connection (Mildly countercyclical risk) Next, we consider the case in which risk is mildly countercyclical $\Theta \in (0, \overline{\Theta})$. First, notice that for any $\Theta > 0$, as in our baseline model, there still exists a untargeted steady state alongside the targeted steady state $(x, \pi, b^g) = (0, 0, 0)$. This can be seen by setting $\dot{x} = \dot{\pi} = \dot{b}^g = 0$ in (a.22)-(a.24), which yields:

$$\begin{aligned} 0 &= (\phi_\pi - 1)\,\pi + \sigma\left(e^{-\gamma\Theta x} - 1\right) \\ \pi &= \frac{\kappa}{\rho}\left(e^{x} - 1\right) \\ b^g &= \frac{(\phi_\pi - 1)\,\overline{b}}{\overline{r}\,(\phi_b - 1) - (\phi_\pi - 1)\,\pi}\pi \end{aligned}$$

As in Appendix B.3, combining the first two of these equations allows us to characterize the output levels in all steady states as a solution to one equation in one unknown:

$$F(x) = \frac{\kappa\,(\phi_\pi - 1)}{\rho}\left(e^{x} - 1\right) + \sigma\left(e^{-\gamma\Theta x} - 1\right)$$

Since AMPF implies that $\phi_\pi > 1$, the same argument as in Appendix B.3 applies, and hence $F(x)$ has two zeros as long as $\Theta > 0$: one targeted with $x = \pi = b^g = 0$ and one untargeted steady state. Since we are looking at the case with $\phi_\pi > \varphi(\Theta)$, as Appendix B.3 shows in the context of our baseline model, the untargeted steady state has lower output and inflation than the targeted steady state. However, unlike in our baseline model (which featured only 2 jump variables), the existence of a second steady state by itself does not automatically imply indeterminacy because now we have a predetermined variable b_t^g, and to prove global indeterminacy, we need to show that for at least some initial value of b_0^g, there exist at least two bounded sequences $\{x_t, \pi_t, b_t^g\}_{t=0}^{\infty}$ and $\{\widetilde{x}_t, \widetilde{\pi}_t, \widetilde{b}_t^g\}_{t=0}^{\infty}$ where $\widetilde{b}_0^g = b_0^g$.

To show this, we can invoke the 3-dimensional version of the Theorem 2 (which corresponds to Corollary 7.1 in Kopell and Howard (1975)) to prove the existence of a saddle connection along which the economy can transition from near the targeted steady state to the untargeted steady state. We present this Theorem next:

Theorem 4 (Corollary 7.1 in Kopell and Howard (1975)). *Let $\dot{\mathbf{x}} = F_{\mu,\nu}(\mathbf{x})$ be a two parameter family of ODEs on $\mathbb{R}^n$, $n \geq 2$, with F smooth in all of its four arguments, such that $F_{\mu,\nu}(0) = \mathbf{0}$. Also assume:*

1. *$dF_{0,0}(0)$ has rank $n-1$ and a zero eigenvalue with multiplicity 2 with $\mathbf{v}$ being the right-eigenvector associated with the zero eigenvalue.*

2. *The mapping $(\mu,\nu) \to \big(det(dF_{\mu,\nu}(0), \sigma_{n-1} dF_{\mu,\nu}(0)\big)$ has a nonzero Jacobian at $(\mu,\nu) = (0,0)$, where $\sigma_{n-1} dF_{\mu,\nu}(0)$ is the coefficient on the linear term in the characteristic polynomial of $dF_{\mu,\nu}(0)$.*

3. *Let $Q(\mathbf{x},\mathbf{x})$ be the $n \times 1$ vector containing the terms quadratic in $\mathbf{x}$ and independent of (μ,ν) in a Taylor series expansion of $F_{\mu,\nu}(\mathbf{x})$ around 0. Then $[dF_{0,0}(0), Q(\mathbf{v},\mathbf{v})]$ has rank $n-1$.*

Then there is a curve $f(\mu,\nu) = 0$ such that if $f(\mu_0,\nu_0) = 0$, then $\dot{\mathbf{x}} = F_{\mu_0,\nu_0}(\mathbf{x})$ has a homoclinic orbit. This one-parameter family of homoclinic orbits (in (X,μ,ν) space) is on the boundary of a two-parameter family of periodic solutions. For all $|\mu|, |\nu|$ sufficiently small, if $\dot{\mathbf{x}} = F_{\mu,\nu}(\mathbf{x})$ has neither a homoclinic orbit nor a periodic solution, there is a unique trajectory joining the critical points.

As in our baseline model, we use Θ and ϕ_π as the bifurcation parameters. Condition 1 is clearly satisfied since $dF(0)$ is simply the A matrix in d.5. To see that point 1 is satisfied, imposing $\Theta = \Theta^\star$ and $\phi_\pi = \varphi(\Theta^\star)$ we can rewrite A in (d.5) as:

$$\begin{bmatrix} -\rho & \frac{\rho^2}{\kappa} & 0 \\ -\kappa & \rho & 0 \\ 0 & \frac{\rho^2}{\kappa}\bar{b} & \bar{r}(1-\phi_b) \end{bmatrix},$$

which clearly has rank 2, since the first 2 rows are linearly dependent. Also, the principal minor of order two has trace and determinant zero, implying that the roots of the system are $\bar{r}(1-\phi_b), 0, 0$. Furthermore, one can compute the right-eigenvector associated with 0 to be:

$$\mathbf{v} = \begin{bmatrix} 1 \\ \frac{\kappa}{\rho} \\ -\frac{\rho\bar{b}}{\bar{r}(1-\phi_b)} \end{bmatrix} \tag{d.9}$$

Next, the regularity condition 2 is satisfied since

$$\begin{vmatrix} -\bar{r}\left(1-\phi_b\right)\rho\gamma\sigma & \left[\bar{r}\left(1-\phi_b\right)+\rho\right]\gamma\sigma \\ \kappa\bar{r}\left(1-\phi_b\right) & -\kappa \end{vmatrix} = -\kappa\gamma\sigma\bar{r}^2\left(1-\phi_b\right)^2 \neq 0$$

Finally, taking a second order Taylor approximation of the 3 ODEs (a.22)-(a.24) and imposing $\Theta = \Theta^\star$ and $\phi_\pi = \varphi(\Theta^\star)$, we have

$$Q\left(\mathbf{x},\mathbf{x}\right) = \begin{bmatrix} \frac{\rho^2}{2\sigma}x^2 \\ -\kappa x^2 \\ \frac{\rho^2}{\kappa}\pi b^g \end{bmatrix}$$

Evaluating at $\mathbf{x} = \mathbf{v}$ (which is defined in (d.9), we get

$$Q\left(\mathbf{v},\mathbf{v}\right) = \begin{bmatrix} \frac{\rho^2}{2\sigma} \\ -\kappa \\ -\frac{\rho^2\bar{b}}{\bar{r}(1-\phi_b)} \end{bmatrix}$$

and we can finally verify that

$$\text{rank}\begin{bmatrix} -\rho & \frac{\rho^2}{\kappa} & 0 & \frac{\rho^2}{2\sigma} \\ -\kappa & \rho & 0 & -\kappa \\ 0 & \frac{\rho^2}{\kappa}\bar{b} & \bar{r}\left(1-\phi_b\right) & -\frac{\rho^2\bar{b}}{\bar{r}(1-\phi_b)} \end{bmatrix} = 3,$$

which implies that all three conditions for Theorem 4 are satisfied. Since we know that a stable cycle only exists if risk is moderately countercyclical ($\overline{\Theta} < \Theta < \Theta^\star$), Theorem 4 implies that when risk is mildly countercyclical ($0 < \Theta < \overline{\Theta}$), then there must be a saddle connection along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. As in the baseline economy, any trajectory originating on this saddle connection remains bounded and hence there are multiple bounded sequences which satisfy equilibrium. Thus, even when risk is mildly countercyclical, there is global indeterminacy in the AMPF regime.

D.2 Passive-Monetary, Active-Fiscal (PMAF)

Active fiscal policy ($0 \leq \phi_b < 1$) implies that the eigenvalue $z_1 = \bar{r}(1-\phi_b) > 0$. Then local determinacy requires that one of the eigenvalues of $A^\dagger$ be positive and the other negative.

D.2.1 Local Determinacy

Equation (d.8) shows that since $\phi_\pi < 1 < \varphi(\Theta)$, the determinant of $A^\dagger$ is negative for any $\Theta > 0$, implying that the product of the two remaining roots is negative, which means that one of the roots of $A^\dagger$ is negative and one is positive. Thus, the PMAF regime delivers local determinacy of the targeted equilibrium $(x, \pi, b^g) = (0, 0, 0)$ for any Θ, i.e., regardless of whether risk is mildly, moderately or highly countercyclical.

D.2.2 Global Determinacy

Next, we show that the PMAF regime eliminates the ways in which global determinacy manifests in the AMPF regime. We start by explicitly showing that the PMAF regime eliminates the untargeted steady state as well as the stable cycle. Following this demonstration, we then prove that the PMAF regime delivers global determinacy: for any b_0^g, there is a unique x_0, π_0 such that the trajectory starting at (x_0, π_0, b_0^g) is the only trajectory which remains bounded. Moreover, this trajectory converges to the targeted steady state $(0, 0, 0)$ asymptotically.

Eliminating the stable cycle At $\Theta = \Theta^\star = \rho/\sigma\gamma$, the trace of $A^\dagger$ equals zero (see equation (d.7)), so the two eigenvalues $z_2 = -z_3$. Additionally, since the determinant of $A^\dagger$ is negative, it means that all three eigenvalues are real, and that the absolute value of z_2 and z_3 is identical, but the signs are opposite. Thus, unlike in the AMPF regime, the system does not undergo a Hopf bifurcation since the roots are not complex. Thus, the PMAF regime eliminates the stable cycle, which was one of the manifestations of the global indeterminacy in the baseline model and the AMPF regime.

Eliminating the second steady state and saddle connection The PMAF steady state also prevents the untargeted steady state from existing; only the targeted steady state survives. To see this, we can set $\dot{x} = \dot{\pi} = \dot{b}^g = 0$ in (a.22)-(a.24), to get:

$$
\begin{aligned}
0 &= (\phi_\pi - 1)\,\pi + \sigma\left(\mathrm{e}^{-\gamma\Theta x} - 1\right) \\
\pi &= \frac{\kappa}{\rho}\left(\mathrm{e}^{x} - 1\right) \\
b^g &= \frac{(\phi_\pi - 1)\,\bar{b}}{\bar{r}\,(\phi_b - 1) - (\phi_\pi - 1)\,\pi}\,\pi
\end{aligned}
$$

As in Appendix B.3, combining the first two of these equations allows us to characterize the output levels in all steady states as a solution to one equation in one unknown:

$$
F(x) = \frac{\kappa\,(\phi_\pi - 1)}{\rho}\left(\mathrm{e}^{x} - 1\right) + \sigma\left(\mathrm{e}^{-\gamma\Theta x} - 1\right)
$$

Taking the first derivative of this function:

$$
F'(x) = -\frac{\kappa\,(1 - \phi_\pi)}{\rho}\mathrm{e}^{x} - \gamma\sigma\Theta\mathrm{e}^{-\gamma\Theta x} < 0
$$

Since $\phi_\pi \leq 1$ in the PMAF regime, $F'(x) < 0$ for all $x \in (-\infty, \infty)$. This means that the only zero of $F(x)$ is at $x = 0$, i.e., the targeted steady state. Hence, the PMAF regime also eliminates the untargeted steady state, which was a manifestation of the global indeterminacy. Furthermore, since the untargeted steady state does not exist in the PMAF regime, neither can the saddle connection along which the economy can travel from the targeted to the untargeted steady state, as in the AMPF regime. Thus, the PMAF regime eliminates all three manifestations of global indeterminacy which appear in our baseline model and AMPF regime.

Next, we can now prove that the PMAF regime delivers global determinacy. Notice that in the system of 3 ODEs (d.4), the subsystem which describes the dynamics of x_t, π_t does not explicitly depend on b^g. This is easily seen by computing the Jacobian of (d.4) at any (x, π, b^g) and noticing that the entries in the first two rows of the 3rd column are zero:

$$J(x,\pi,b^g) = \begin{bmatrix} -\sigma\gamma\Theta e^{-\gamma\Theta x} & \phi_\pi - 1 & 0 \\ -\kappa e^x & \rho & 0 \\ 0 & (\phi_\pi - 1)(b^\star + b^g) & \bar{r}(1-\phi_b) + (\phi_\pi - 1)\pi \end{bmatrix}$$

Thus, we can study the dynamics of (x, π) separately by focusing on the principal co-minor of the Jacobian above, which we will refer to as $J^\diamond(x, \pi)$:

$$J^\diamond(x,\pi) = \begin{bmatrix} -\sigma\gamma\Theta e^{-\gamma\Theta x} & \phi_\pi - 1 \\ -\kappa e^x & \rho \end{bmatrix}$$

We have already established that the determinant of the Jacobian evaluated at the unique steady state $x, \pi = 0, 0$ is negative in the AMPF regime, which implies that $A^\diamond = J^\diamond(0,0)$ has one negative and one positive eigenvalue. It follows that local to $(x,\pi) = (0,0)$, there exists a one dimensional stable manifold which can be written as $x = \Psi(\pi)$ such that only trajectories for any π close to 0, only trajectories beginning at $\Big((\Psi(\pi), \pi\Big)$ converge to $(0,0)$, while others diverge locally. However, we can extend this argument for the existence of a 1-dimensional stable manifold $x = \Psi(\pi)$ because in the PMAF regime, the off-diagonal elements of $J^\diamond(x,\pi)$ are always strictly negative for any $(x,\pi) \in \mathbb{R}^2$, which implies that the subsystem which describes the dynamics of (x, π) is a *competitive ODE system*. The existence of the 1-dimensional stable manifold $x = \Psi(\pi)$, along which $\{x_t, \pi_t\} \to 0, 0$, follows from Hirsch's theorem for competitive ODE systems (Hirsch, 1982).[29] Furthermore, Hirsch (1982) also shows that $\frac{d\Psi(\pi)}{d\pi} < 0$, which implies that the stable manifold is strictly monotone and that all trajectories which start off this stable manifold become unbounded.

Next, notice that the dynamics of b^g, which are described by (d.3) only depend on b^g and π_t and can be written as:

$$\dot{b}^g_t = (\phi_\pi - 1)\, b^\star \pi_t + \bar{r}\,(1-\phi_b)\, b^g_t + (\phi_\pi - 1)\, \pi_t b^g_t$$

Since the dynamics of π_t are determined separately from b^g_t, this ODE is simply a linear non homogeneous ordinary differential equation in b^g, and for a given initial condition b^g_0 admits the solution:

$$b^g_t = b^g_0 e^{\int_0^t \mu_s ds} + \int_0^t \nu_s e^{\int_s^t \mu_\tau d\tau} ds$$

where $\mu_t = \bar{r}(1-\phi_b) + (\phi_\pi - 1)\pi_t$ and $\nu_t = (\phi_\pi - 1) b^\star \pi_t$. Next, as we take the limit $t \to \infty$, we know from the x, π subsystem that along the stable manifold $(\Psi(\pi), \pi)$, we have $\lim_{t\to\infty} \pi_t \to 0$. Then to

[29] We thank Greg Kaplan for suggesting the use of Hirsch's theorem.

ensure that $\lim_{t\to\infty} b_t^g \to 0$, we have:

$$b_0^g = -(\phi_\pi - 1)b^\star \int_0^\infty \pi_s e^{-\int_0^s [\bar{r}(1-\phi_b)+(\phi_\pi-1)\pi_\tau]d\tau} ds$$

This expression above can be interpreted as a mapping from π_0 to b_0^g. Also, since the x, π subsystem is competitive, it follows from Hirsch's theorem that a higher π_0 increases π_t for all $t > 0$ and hence the condition above implies a strictly positive relationship between b_0^g and the π_0 which is needed to ensure that b_0^g converges to 0 asymptotically. Thus, for a given b_0^g there exists a unique π_0 for which b_0^g does not grow unbounded and instead converges to 0 asymptotically. We can describe this relation as $\pi_0 = \Lambda(b_0^g)$. Together, the characterization above implies that for any given b_0^g, only the trajectory originating at $(\Psi(\Lambda(b_0^g)), \Lambda(b_0^g), b_0^g)$ converges to $(0, 0, 0)$ asymptotically, while all other trajectories grow unbounded. Thus, in the PMAF regime, we have global determinacy. □

E Extensions of the baseline model

E.1 Monetary policy rule with output-gap stabilization

We first consider the case in which monetary policy also responds to output-gap fluctuations. The augmented rule can be written as:

$$i_t = \bar{r} + \phi_\pi \pi_t + \phi_x x_t, \tag{e.1}$$

which nests the inflation targeting rule (6) if we set $\phi_x = 0$. Given the interest rate rule (e.1), dynamics of the output-gap x_t, and inflation π_t are described by:

$$\dot{x}_t = (\phi_\pi - 1)\,\pi_t + \phi_x x_t - \Big(r^\star(x_t) - \bar{r}\Big) \tag{e.2a}$$

$$\dot{\pi}_t = \rho\pi_t - \kappa\,(\mathrm{e}^{x_t} - 1) \tag{e.2b}$$

How does ϕ_x affect local determinacy? As is well known in the RANK literature, allowing the nominal rate to respond to output-gap fluctuations in addition to inflation reduces the burden on ϕ_π to ensure that the targeted equilibrium is locally determinate (see, e.g. Bullard and Mitra (2002)). The same is true in the context of our HANK economy with countercyclical risk. Recall that in the inflation targeting rule (6) (with $\phi_x = 0$), a higher ϕ_π could only guarantee local determinacy only if risk was mildly or moderately countercyclical ($0 < \Theta < \Theta^\star$). However, if risk is highly countercyclical ($\Theta \geq \Theta^\star$), no finite ϕ_π can deliver local determinacy. However, as Proposition 7 shows, a monetary policy rule which responds to output-gap fluctuations can *always* guarantee local determinacy, no matter how countercyclical risk is.

Proposition 7 (Output-gap stabilization)**.** *For any $\Theta \geq 0$, the combination of a large enough ϕ_x and ϕ_π is sufficient for local determinacy of the targeted steady state. In particular, local determinacy requires:*

$$\phi_\pi > \varphi(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa} \qquad \textit{and} \qquad \phi_x > \sigma\gamma\,(\Theta - \Theta^\star) \tag{e.3}$$

Proof. With $\phi_x \neq 0$, for (x, π) close to the targeted steady state $(0,0)$, the dynamics of the system (16a)-(16b) are governed by the following system:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = A \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \mathcal{O}\left(x^2\right) \qquad \text{for}(x,\pi) \to (0,0),$$

where A is given by

$$A = \begin{bmatrix} \phi_x - \sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix},$$

Since both x and π are jump-variables, local determinacy requires that both eigenvalues of A have a positive real part. As is well known, the sum of the two eigenvalues of A, denoted by z_1 and z_2, is given by the trace of A, while their product is given by the determinant of A:

$$\begin{aligned} z_1 + z_2 &= \rho + \phi_x - \sigma\gamma\Theta, \\ z_1 \times z_2 &= \kappa(\phi_\pi - 1) + \rho\phi_x - \rho\sigma\gamma\Theta \end{aligned}$$

Since this is a two dimensional system, either both z_1 and z_2 are real, or they are complex conjugates. Thus, for z_1 and z_2 to both have positive real parts, it is sufficient that both the sum and product of z_1, z_2 be positive. In other words, a sufficient condition for local determinacy is

$$\phi_\pi + \frac{\rho}{\kappa}\phi_x > \varphi(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa} \qquad \text{and} \qquad \phi_x > \sigma\gamma\left(\Theta - \Theta^\star\right) \tag{e.4}$$

This condition is satisfied if (ϕ_π, ϕ_x) jointly satisfy the following condition

$$\phi_\pi > \varphi(\Theta) \qquad \text{and} \qquad \phi_x > \sigma\gamma\left(\Theta - \Theta^\star\right),$$

which is the same condition as in Proposition 7. For mildly or moderately countercyclical risk $0 < \Theta < \Theta^\star$, the above expression shows that setting $\phi_x = 0$ and $\phi_\pi > \varphi(\Theta)$ is sufficient for local determinacy. This is the same condition as in Proposition 2. However, when risk is highly countercyclical $\Theta \geq \Theta^\star$, setting $\phi_x = 0$ is no longer enough for local determinacy. Local determinacy now requires a large enough $\phi_x > \sigma\gamma(\Theta - \Theta^\star)$ alongside $\phi_\pi > \varphi(\Theta)$. □

Consistent with Proposition 2, (e.3) shows that if risk is for mild or moderately countercyclical, $\Theta \in (0, \Theta^\star]$, then $\phi_x = 0$ is sufficient for local determinacy as long as ϕ_π is large enough: $\phi_\pi > \varphi(\Theta)$. However, if risk is highly countercyclical $(\Theta > \Theta^\star)$, setting ϕ_x slightly above 0 is not sufficient, and local determinacy requires $\phi_x > \sigma\gamma(\Theta - \Theta^\star) > 0$, alongside a large enough ϕ_π. Overall, (e.3) shows that the more countercyclical risk, the higher ϕ_π and ϕ_x need to be to ensure local determinacy; raising only one of the two is not sufficient. To understand why a high enough ϕ_x can guarantee local determinacy, it is again useful to concentrate on the IS curve. With $\phi_x \neq 0$, the IS curve (e.2a) can be written as:

$$\dot{x}_t = \underbrace{(\phi_\pi - 1)\,\pi_t + \phi_x x_t}_{\text{destabilizing}} - \sigma\gamma\Theta x_t + \mathcal{O}(x^2) \qquad for \qquad (x,\pi) \to (0,0)$$

Relative to the case with $\phi_x = 0$, the expression above shows that in addition to the $(\phi_\pi - 1)\pi$ term which supplies the destabilizing forces, now an additional term, $\phi_x x$, also generates destabilizing dynamics. When x is away from its steady state value, say $x > 0$, this term makes $\dot{x}$ more positive, thus pushing x further away from 0. The stabilizing force, provided by the term, $-\sigma\gamma\Theta x$ is the same as in the case with $\phi_x = 0$. Thus, as (e.3) shows, a large enough ϕ_π and ϕ_x make it easier for the destabilizing forces to overwhelm the stabilizing force, generating local determinacy.

While the combination of a high enough ϕ_π and ϕ_x helps resolve the problem of *local* determinacy, it does little to resolve the problem of global indeterminacy. All the forces which contributed to the existence of multiple equilibria in the baseline still continue to operate regardless of how large ϕ_x is.

First, with $\phi_x > 0$, no matter how large, the untargeted steady state $(\underline{x}, \underline{\pi}^g)$ continues to exist, and features lower output than the targeted steady state as well as below target inflation. In fact, the larger the $\phi_x > 0$, the lower are output and inflation in the untargeted steady state. To see this, note that the $\dot{x} = 0$ and $\dot{\pi}_t = 0$ nullclines with $\phi_x \neq 0$, imply that in any steady state, (x, π) must satisfy:

$$
\begin{aligned}
0 &= (\phi_\pi - 1)\,\pi + \phi_x x + \sigma\left(e^{-\gamma\Theta x} - 1\right) \\
0 &= \rho\pi - \kappa\,(e^x - 1)
\end{aligned}
$$

Clearly, $(0, 0)$ still satisfies both equations. To see that the untargeted steady state still exists, combine the two equations to eliminate π, to get an expression exclusively in terms of x:

$$
F(x) = \frac{\kappa\,(\phi_\pi - 1)}{\rho}(e^x - 1) + \phi_x x + \sigma\left(e^{-\gamma\Theta x} - 1\right),
$$

and any x which solves $F(x) = 0$ constitutes a steady state. Again, clearly $x = 0$ solves this equation. The derivative of $F(x)$ is given by:

$$
F'(x) = \frac{\kappa\,(\phi_\pi - 1)}{\rho} e^x + \phi_x - \sigma\gamma\Theta e^{-\gamma\Theta x},
$$

which, evaluated at $x = 0$ yields

$$
F'(0) = \frac{\kappa}{\rho}\left(\phi_\pi + \frac{\rho}{\kappa}\phi_x - \varphi\,(\Theta)\right) \qquad \text{where} \qquad \varphi(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa},
$$

If $\phi_\pi + \frac{\rho}{\kappa} = \varphi(\Theta)$, then $F'(0) = 0$ and $F(x)$ is tangent to the x-axis at $x = 0$, implying that it is the only zero of $F(x)$ since $F(x)$ is declining in the region $x = 0$ and increasing in the region $x > 0$. This is the knife edge case in which there is a unique steady state. If instead, we impose the condition for local determinacy of the targeted equilibrium, $\phi_\pi + \frac{\rho}{\kappa}\phi_x > \varphi(\Theta)$, then $F'(0) > 0$. Since $\lim_{x\to-\infty} F(x) \to \infty$, there must be at least one intersection with $\underline{x} < 0$ and $F'(\underline{x}) < 0$. Since $F(x)$ is strictly convex, this intersection is unique. Further, note that $dF(x)/d\phi_\pi < 0$ for $x < 0$ by inspection. Thus, by the implicit function theorem, we have $d\underline{x}/d\phi_\pi < 0$.

The existence of the second steady state again implies that the equilibrium is *globally* indeterminate, because the two steady states imply that there are at least two bounded trajectories which satisfy all equilibrium conditions: $(x_t, \pi_t) = (0, 0)$ and $(x_t, \pi_t) = (\underline{x}, \underline{\pi})$. However, separate from the second steady state, as in the baseline, there are other equilibria as well, and as with $\phi_x = 0$, the untargeted steady state is not the only other bounded trajectory. Proposition 8 provides an exhaustive characterization of global dynamics for a given cyclicality Θ as a function of ϕ_x.

Proposition 8. *For a given $\Theta > 0$, assume that $\phi_\pi > \varphi(\Theta)$. Then, defining $\phi_x^\star = \sigma\gamma(\Theta - \Theta^\star)$, the global dynamics of the economy with monetary policy rule (e.1) can be split into 3 regions:*

1. *$\phi_x < \phi_x^\star$ **(small ϕ_x)**: If $\phi_x < \phi_x^\star$, trajectories which start in the neighborhood of $(0,0)$ converge to the the targeted steady state, and thus, the targeted equilibrium is both locally and globally indeterminate. In addition to the multiple trajectories originating near the targeted steady state which remain bounded, there also exists a* saddle-connection *along which the economy can transition from the neighborhood of the untargeted steady state to the targeted steady state. All trajectories which originate at any point on this saddle connection also remain bounded.*

2. *$\phi_x \in [\phi_x^\star, \overline{\phi}_x]$ **(medium ϕ_x)**: $\exists \overline{\phi}_x > \phi_x^\star$, such that for any $\phi_x \in [\phi_x^\star, \overline{\phi}_x]$, the targeted equilibrium is locally determinate, but globally indeterminate. Trajectories which start in the neighborhood of the targeted steady state initially diverge away from it, but eventually remain bounded and converge to a* stable limit-cycle *surrounding the targeted steady state. The amplitude and periodicity of the cycles is increasing in ϕ_x in this range. At the upper limit of this interval, $\phi_x = \overline{\phi}_x$, the limit-cycles are absorbed into a* homoclinic orbit*. At the other end of this range, $\phi_x = \phi_x^\star$, the limit-cycles collapse onto the targeted steady state. The equilibrium is still globally indeterminate since the higher-order terms ensure that any trajectory starting in the neighborhood of $(0,0)$ converge back to $(0,0)$.*

3. *$\phi_x > \overline{\phi}_x$ **(large ϕ_x)**: For $\phi_x > \overline{\phi}_x$, the targeted equilibrium is locally determinate but still globally indeterminate. There are no stable limit-cycles in this range of ϕ_x, but the equilibrium is still globally indeterminate, owing to the existence of a saddle-connection, along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. Any trajectory which originates on this saddle connection also remains bounded.*

No matter how large ϕ_π and ϕ_x are, the equilibrium is still globally indeterminate if risk is even mildly countercyclical.

Proof. Given the interest rate rule $i_t = \bar{r} + \phi_\pi \pi_t + \phi_x x_t$, the dynamics of x_t and π_t are given by the ODEs:

$$\begin{aligned} \dot{x} &= (\phi_\pi - 1)\,\pi + \phi_x x_t + \sigma\left(e^{-\gamma\Theta x} - 1\right) \\ \dot{\pi} &= \rho\pi - \kappa\,(e^x - 1) \end{aligned}$$

We can rewrite this system in matrix form as:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \underbrace{\begin{bmatrix} \phi_x - \sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}}_{A} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma\left(e^{-\gamma\Theta x_t} - 1 + \gamma\Theta x_t\right) \\ -\kappa\left(e^x - 1 - x_t\right) \end{bmatrix} \tag{e.5}$$

Notice that the matrix A has has trace equal to zero at $\phi_x = \phi_x^\star = \sigma\gamma\Theta - \rho = \sigma\gamma(\Theta - \Theta^\star)$, where $\Theta^\star = \frac{\rho}{\sigma\gamma}$. Evaluating (e.5) at $\phi_x = \phi_x^\star$, we have:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \underbrace{\begin{bmatrix} -\rho & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}}_{A^\star} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma\left(e^{-\frac{\rho}{\sigma}x_t} - 1 + \frac{\rho}{\sigma}x_t\right) \\ -\kappa\left(e^x - 1 - x_t\right) \end{bmatrix}, \tag{e.6}$$

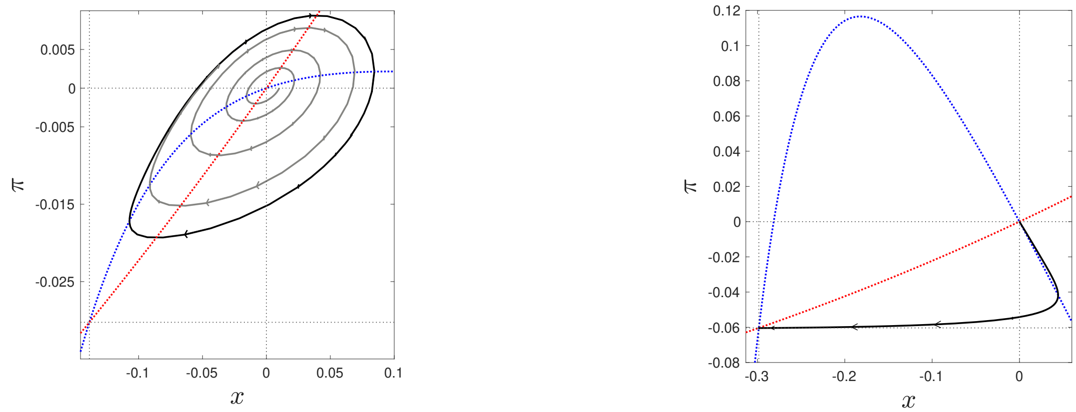
which is identical to (b.5) in the model with $\phi_x = 0$ (See Appendix B.5 for details). Consequently, all the conditions for Theorem 1 are satisfied, and the system undergoes a Hopf bifurcation at $\phi_x = \phi_x^\star$. Consequently, $\exists\overline{\phi}_x > \phi_x^\star$ such that for $\phi_x \in (\phi_x^\star, \overline{\phi}_x)$, trajectories starting in the neighborhood of $(0,0)$ initially diverge, but then converge to a stable cycle which surrounds the targeted steady state. Since all these trajectories remain bounded, in this region $\phi_x \in (\phi_x^\star, \overline{\phi}_x)$, the equilibrium is globally indeterminate, even though the targeted steady state is unstable (locally determinate). For $\phi_x = \phi_x^\star$, the stable cycles collapse onto $(0,0)$. The equilibrium in this case is still globally indeterminate because the higher-order terms of the system push any trajectory originating near the targeted steady state towards $(0,0)$. Next, evaluating $A^\star$ in (e.6) at $\phi_\pi = \varphi(\Theta)$ yields the same matrix as $A^\diamond$ in (b.7) in Appendix B.5. Furthermore, since setting $\phi_x \neq 0$ does not change the higher-order terms of the system, by the same reasoning as in Appendix B.5, all the conditions of Theorem 2 are also satisfied with $\phi_x \neq 0$. Thus, it follows that for $\phi_x = \overline{\phi}_x$, the stable cycles get absorbed into a homoclinic orbit. This proves point 2 of Proposition 8.

For $\phi_x > \overline{\phi}_x$, while the stable cycles disappear but the equilibrium is still globally indeterminate. This is because Theorem 2 guarantees the existence of a saddle connection along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. All trajectories beginning from any point on this saddle connection always remain bounded, thus proving the existence of multiple bounded trajectories which satisfy all equilibrium conditions. Thus even though for a large ϕ_x, the targeted equilibrium is locally determinate, there is global indeterminacy. This proves point 3 of Proposition 8.

For $\phi_x < \phi_x^\star$, Proposition 7 proves that the targeted equilibrium is locally indeterminate. Thus, the equilibrium is also globally indeterminate. In addition, Theorem 2 ensures the existence of a saddle connection along which the economy can converge from the neighborhood of the untargeted steady state to the targeted steady state. This proves point 1 of Proposition 8. □

Proposition 8 shows that no combination of π_π and ϕ_x can ensure global determinacy. Figure 6 depicts the non-fundamental fluctuations that can emerge in or HANK economy. To plot this Figure, we set $\Theta = 28.1$, which is the model estimate of Bilbiie, Primiceri and Tambalotti (2023).[30] Figure 6 plots global dynamics as a function of ϕ_x when risk is moderately countercyclical $\Theta \in (\overline{\Theta}, \Theta^\star)$. The dotted-red curve depicts the $\dot{\pi} = 0$-nullcline and the dotted-blue curve depicts the $\dot{x} = 0$-nullcline. Figure 6a plots global dynamics with $\phi_x = 0.01$, which lies in the range $\phi_x \in (\phi_x^\star, \overline{\phi}_x)$: in this range, trajectories starting in the neighborhood of $(0,0)$ (dark gray trajectory) converges to the stable cycle

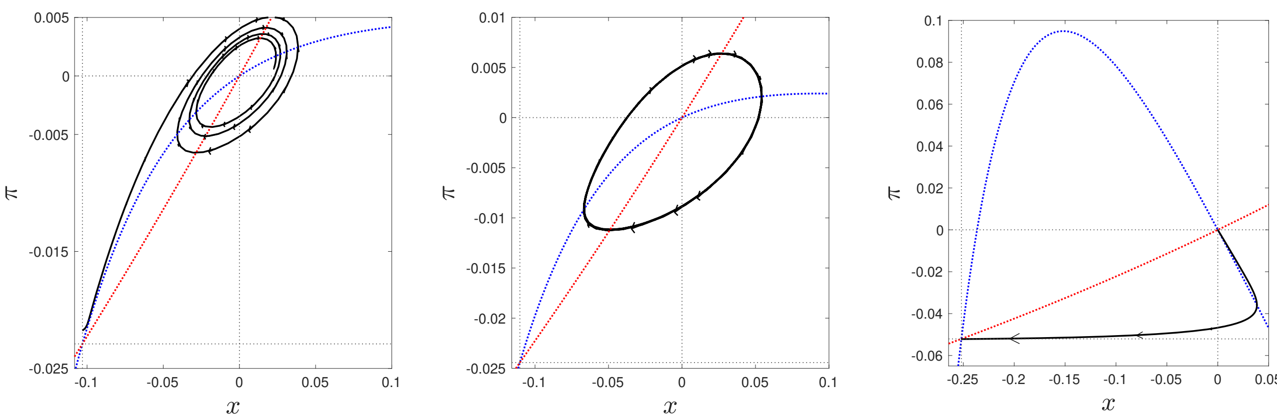
[30]Since $\Theta = 28.1 < 31.1 = \Theta^\star$, we have $\phi_x^\star = \sigma\gamma(\Theta - \Theta^\star) = -0.0041$. In other words, the *small* ϕ_x region corresponds to the interval $\phi_x \in (-\infty, -0.0041)$, which we ignore, because the sensible range of ϕ_x is in the interval $[0, \infty)$. Consequently, Figure 6 only plots dynamics in the *medium* ϕ_x (i.e., $\phi_x \in [\phi_x^\star, \overline{\phi}_x]$) and *large* ϕ_x (i.e., $\phi_x > \overline{\phi}_x$ cases.



(a) $\phi_x \in [\phi_x^\star, \overline{\phi}_x]$ (b) $\phi_x > \overline{\phi}_x$

Figure 6: Global dynamics with $\phi_x > 0$ when $\Theta < \Theta^\star$

(black trajectory).[31] In contrast, Figure 6b features a larger $\phi_x = 0.5$, which is the standard calibration of the Taylor rule (Taylor, 1993) and shows that there exists a saddle connection from the targeted to the untargeted steady state. A trajectory beginning at any point on this saddle-connection remains bounded, and constitutes a bounded sequence which satisfies equilibrium. The same is true for more countercyclical risk, $\Theta > \Theta^\star$.



(a) $\phi_x < \phi_x^\star$ (b) $\phi_x \in [\phi_x^\star, \overline{\phi}_x]$ (c) $\phi_x > \overline{\phi}_x$

Figure 7: Global dynamics as a function of ϕ_x when $\Theta > \Theta^\star$

Figure 7 depicts global dynamics when risk is highly countercyclical risk ($\Theta > \Theta^\star$). In particular, we set $\Theta = 32$, which given our calibration is larger than $\Theta^\star = 31.1$. We impose $\phi_\pi > \varphi(\Theta)$ in all plots. Figure 7a plots dynamics when $\phi_x < \sigma\gamma(\Theta - \Theta^\star)$. As stated in Propositions 7 and 8, in this region of ϕ_x, the equilibrium is both locally and globally indeterminate. The Figure shows a saddle connection along which the economy moves from the neighborhood of the untargeted steady state to the targeted steady state. Figure 7b plots dynamics in the region where $\phi_x \in [\phi_x^\star, \overline{\phi}_x]$. The black trajectory depicts a stable cycle. All trajectories originating near the targeted steady state initially diverge away from it but then converge to the stable cycle and remain bounded. Thus, while the targeted equilibrium is locally

[31] In addition, trajectories originating near the untargeted steady state also converge to the cycle as in Figure 2b, but we omit these trajectories from Figure 6a to avoid clutter.

determinate, it is globally indeterminate. Finally, Figure 7c plots global dynamics in the case with $\phi_x > \overline{\phi}_x$. Propositions 7 and 8 show that in this region the targeted equilibrium is locally determinate but globally indeterminate. The black trajectory depicts a saddle connection along which a trajectory starting near the targeted steady state diverges away from it, only to converge to the untargeted steady state. Thus, Figures 6 and 7 show that for any $\Theta > 0$, the equilibrium is always globally indeterminate, regardless of the combination of ϕ_π, ϕ_x.

E.2 Inertial policy rule

The monetary policy rule we have studied so far only react to changes in current economic conditions. However, empirically, many central banks have been noted to have a tendency to only adjust the policy rate gradually in response to changes in economic conditions. Such inertial rules have been shown to be desirable from the point of view of delivering local determinacy (Bullard and Mitra, 2007), and for various other reasons (Woodford, 2003b). In continuous time, such a rule can be written as:[32]

$$di_t = \alpha \Big[i_t - \bar{r} - \phi_\pi \pi_t \Big] dt, \tag{e.7}$$

where α controls the relative weight on inflation in the past relative to current inflation. A smaller α in (e.8) implies a larger weight on past inflation relative to current inflation, and that the rule is *more backward-looking*. In fact, the limit $\alpha \to 0$ corresponds to the *price-level targeting* (PLT) limit. In the limit $\alpha \to \infty$ converges to the policy rule (6), which only responded to changes in current inflation, and is not backward-looking at all. Instead of working with (e.7), we transform it into the equivalent *average-inflation targeting* (AIT) rule:

$$i_t = \bar{r} + \phi_\pi \pi_t^b, \qquad \text{where} \qquad \pi_t^b = \alpha \int_{-\infty}^{t} e^{-\alpha(t-\tau)} \pi_\tau d\tau \qquad \text{for} \qquad \alpha \in (0, \infty), \tag{e.8}$$

(e.8) shows that the inertial behavior is equivalent to the central bank responding to changes in the weighted-average of past and current inflation (denoted π_t^b), instead of just changes in current inflation. The dynamics of the economy under an AIT rule are described by a 3-dimensional system of ODEs:

$$\dot{x}_t = \phi_\pi \pi_t^b - \pi_t - (r^\star(x_t) - \bar{r}) \tag{e.9}$$

$$\dot{\pi}_t = \rho \pi_t - \kappa \left(e^{x_t} - 1 \right) \tag{e.10}$$

$$\dot{\pi}_t^b = \alpha \left(\pi_t - \pi_t^b \right), \tag{e.11}$$

where (e.11) is derived by taking a time-derivative of π_t^b described in (e.8). Imposing steady state in (e.9)-(e.11), it is easy to see that $x = 0$ and $\pi = \pi^b = 0$ is still a steady state (the targeted steady state).

Compared to the purely inflation targeting rule (6), a sufficiently backward-looking inertial rule (e.8) can guarantee local-determinacy of the targeted steady state, even if risk is highly countercyclical

[32] This rule is analogous to the more familiar discrete-time specification for such rules:

$$1 + i_t = \varrho \left(1 + i_{t-1} \right) + \left(1 - \varrho \right) \left[1 + \bar{r} + \phi_\pi \pi_t \right],$$

where $\varrho \in (0, 1)$ captures the idea that the policy rate at date t displays inertia: $1 + i_t$ depends not just on the deviation of current inflation from target, but also depends on what the policy rate was set to in the past $1 + i_{t-1}$.

$(\Theta > \Theta^{\star})$. This is formalized in Proposition 9.[33]

Proposition 9 (Local determinacy with an inertial rule)**.** *For a given $\Theta > 0$, and any π_0^b in the small neighborhood of the targeted steady state $x = 0$, $\pi = \pi^b = 0$, there exists a unique $\{x_0, \pi_0\}$, such that the trajectory $\{x_t, \pi_t, \pi_t^b\}_{t=0}^{\infty}$ remains bounded inside this neighborhood, as long as α is sufficiently close to 0 and $\phi_\pi > \varphi(\Theta)$. In other words, the targeted equilibrium is locally determinate if*

$$\phi_\pi > \varphi(\Theta) \qquad \text{and} \qquad \alpha \in \Big[0, \alpha^{\star}(\theta)\Big),$$

If risk is mildly or moderately countercyclical $\alpha^{\star}(\Theta) = \infty$, but if risk is highly countercyclical, then $\alpha^{\star}(\Theta) < \infty$ (the exact expression is available in proof below).

Proof. With the inertial rule $\alpha > 0$, for (x, π, π^b) close to the targeted steady state $(0, 0, 0)$, the dynamics of the system (e.9), (e.10) and (e.11) are governed by the following system:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \\ \dot{\pi}_t^b \end{bmatrix} = A \begin{bmatrix} x_t \\ \pi_t \\ \pi_t^b \end{bmatrix} + \mathcal{O}\left(x^2\right) \qquad \text{for}(x, \pi, \pi^b) \to (0,0,0),$$

where A is given by

$$A = \begin{bmatrix} -\sigma\gamma\Theta & -1 & \phi_\pi \\ -\kappa & \rho & 0 \\ 0 & \alpha & -\alpha \end{bmatrix},$$

Since we have one predetermined-variable π^b and two jump-variables x, π, for the targeted steady state to be locally determinate, we need one negative root and two roots with positive real parts. This would ensure that for a given π_0^b in the neighborhood of the targeted steady state, there exists a unique (x_0, π_0) such that the trajectory $\{x_t, \pi_t, \pi_t^b\}_{t=0}^{\infty}$ remains bounded.

To see that this is the case when ϕ_π is large enough and α is small enough, we need to characterize the eigenvalues of A. The characteristic polynomial associated with A can be written as:

$$\mathcal{P}(z) \quad = \quad a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

[33] With an inertial rule, the definition of local determinacy is slightly different relative to our baseline model. This is because our baseline model featured two forward looking variables x, π and so local determinacy required two eigenvalues with positive real parts. In contrast, the inertial rule adds as *predetermined* variable π^b, and so local determinacy requires two eigenvalues with positive real parts, and one negative eigenvalue. In other words, local determinacy of the targeted equilibrium now requires that for a given value of the predetermined variable π_0^b in a small neighborhood of the targeted steady state, there exist a unique (x_0, π_0) starting from which the trajectory $\{x_t, \pi_t, \pi_t^b\}_{t=0}^{\infty}$ satisfies all equilibrium conditions *and* remains bounded. Global determinacy, then, requires that from *any* given value of the predetermined variable π_0^b (not necessarily in the neighborhood of the targeted steady state), there exists a unique (x_0, π_0), starting from which the trajectory $\{x_t, \pi_t, \pi_t^b\}_{t=0}^{\infty}$ remains bounded while satisfying all equilibrium conditions.

where

$$\begin{aligned} a_0 &= -1 \\ a_1 &= -\sigma\gamma\left(\Theta-\Theta^\star\right)-\alpha \\ a_2 &= \kappa\varphi(\Theta)-\alpha\sigma\gamma\left(\Theta-\Theta^\star\right) \\ a_3 &= -\alpha\kappa\Big(\phi_\pi-\varphi(\Theta)\Big), \end{aligned}$$

where

$$\varphi(\Theta)=1+\frac{\rho^2}{\kappa}\left(\frac{\Theta}{\Theta^\star}\right)$$

The stability of the system is governed by the pattern of sign changes in the sequence:

$$a_0, \qquad a_1, \qquad -\frac{a_0a_3-a_1a_2}{a_1}, \qquad a_3$$

For the Jacobian to have one negative root and two roots with positive real parts, we need the sequence to have 2 sign changes. The first term in this sequence is always $-$. Imposing $\phi_\pi>\varphi(\Theta)$ guarantees that the fourth term in the sequence is also $-$. To determine the sign of the other two terms in the sequence, we need to consider two cases:

(i) **Highly countercyclical risk** $(\Theta>\Theta^\star)$: In this case, for any $\alpha>0$, the second term in the sequence a_1 is negative. So the sequence is $-,-,?,-$. Local determinacy then requires that the third term in the sequence be positive (two sign changes) for local determinacy, i.e,

$$\frac{\sigma\gamma\left(\Theta-\Theta^\star\right)\left[\kappa\varphi\left(\Theta\right)-\alpha\sigma\gamma\left(\Theta-\Theta^\star\right)-\alpha^2\right]+\alpha\kappa\phi_\pi}{\sigma\gamma\left(\Theta-\Theta^\star\right)+\alpha}>0,$$

which can be reformulated as:

$$\psi\left(\alpha\right)<\phi_\pi,$$

where

$$\psi(\alpha)=-\frac{\sigma\gamma\left(\Theta-\Theta^\star\right)}{\alpha\kappa}\left[\kappa\varphi\left(\Theta\right)-\alpha\sigma\gamma\left(\Theta-\Theta^\star\right)-\alpha^2\right] \tag{e.12}$$

We know that

$$\psi'\left(\alpha\right)=\frac{\sigma\gamma\left(\Theta-\Theta^\star\right)}{\kappa}\left[1+\frac{\kappa\varphi\left(\Theta\right)}{\alpha^2}\right]>0 \qquad \psi\left(0\right)\to-\infty \qquad \psi\left(\infty\right)\to\infty,$$

which, by the intermediate-value theorem implies that $\exists\alpha^\star(\Theta)\in(0,\infty)$, such that $\psi\Big(\alpha^\star(\Theta)\Big)=\phi_\pi$ and $\psi(\alpha)<\phi_\pi$ for all $\alpha\in\Big(0,\alpha^\star(\Theta)\Big)$. In fact, we can write $\alpha^\star(\Theta)$ as:

$$\alpha^\star(\Theta)=\frac{1}{2}\left\{\frac{\kappa\phi_\pi-\sigma^2\gamma^2\left(\Theta-\Theta^\star\right)^2}{\sigma\gamma\left(\Theta-\Theta^\star\right)}+\sqrt{\frac{\left[\kappa\phi_\pi-\sigma^2\gamma^2\left(\Theta-\Theta^\star\right)^2\right]^2}{\sigma^2\gamma^2\left(\Theta-\Theta^\star\right)^2}+4\kappa\varphi\left(\Theta\right)}\right\}$$

Thus, even when risk is highly countercyclical $\Theta > \Theta^\star$ local determinacy is ensured as long as α is small enough, i.e., the rule is backward-looking enough.

(ii) **Mildly or moderately countercyclical risk** $(\Theta \leq \Theta^\star)$: In this region, we need to check two cases. First consider the case in which α is large: $\alpha \geq \sigma\gamma(\Theta^\star - \Theta)$. In this case, the second term is still negative, and so the sequence is $-, -, ?, -$. Thus, we need the third term in the sequence to be positive. For this to be the case, we need

$$\phi_\pi > \psi(\alpha),$$

where $\psi(\alpha)$ is the same as in (e.12). However, now with $\Theta < \Theta^\star$, we have:

$$\psi'(\alpha) = \frac{\sigma\gamma(\Theta - \Theta^\star)}{\kappa}\left[1 + \frac{\kappa\varphi(\Theta)}{\alpha^2}\right] < 0 \qquad \psi\Big(\sigma\gamma(\Theta^\star - \Theta)\Big) = \varphi(\Theta) \qquad \psi(\infty) \to -\infty,$$

Thus, the third term is always positive in this case. $\alpha \geq \sigma\gamma(\Theta^\star - \Theta)$. Thus, we have local determinacy for any α in this region.

Finally, the remaining case is when α is small: $0 < \alpha < \sigma\gamma(\Theta^\star - \Theta)$. In this case, the second term of the sequence is positive. So the sequence is $-, +, ?, -$. Thus, there are two sign changes regardless of the sign of the third term, and we have local determinacy for any α in this region.

Overall, if $\Theta \in (0, \Theta^\star]$, the targeted equilibrium is locally determinate for any $\alpha \in (0, \infty)$, as long as $\phi_\pi > \varphi(\Theta)$. In other words, for $\Theta \in (0, \Theta^\star]$, we have $\alpha^\star(\Theta) = \infty$. However, if $\Theta > \Theta^\star$, $\phi_\pi > \varphi(\Theta)$ is no longer sufficient for local determinacy. Local determinacy requires $\phi_\pi > \varphi(\Theta)$ alongside a small enough α: $\alpha < \alpha^\star(\Theta)$, i.e., a rule which is also backward-looking enough. □

Proposition 9 shows that when risk is mildly or moderately countercyclical $(\Theta < \Theta^\star)$, any $\alpha \geq 0$ delivers local determinacy, as long as $\phi_\pi > \varphi(\Theta)$. This follows directly from the fact that in the limit as $\alpha \to \infty$, the inertial rule (e.8) converges to the policy rule (6), and we know from Proposition 2 that if $0 < \Theta < \Theta^\star$, then $\phi_\pi > \varphi(\Theta)$ is sufficient for local determinacy. However, even when $\Theta \geq \Theta^\star$, a small enough α ensures that the targeted equilibrium is locally determinate. In fact, a corollary of this result is that in the PLT limit, $\alpha \to 0$, the targeted equilibrium is always locally determinate regardless of how countercyclical risk may be.[34] Thus, a sufficiently backward-looking inertial rule ensures local determinacy of the target equilibrium, no matter how countercyclical risk is.

However, this improved performance in terms of ensuring local determinacy does not translate into global determinacy. In fact, Proposition 11 shows that as long as risk is countercyclical, the equilibrium is always globally indeterminate, no matter how backward-looking the rule. First, as the Proposition below states, inertial rules do not eliminate the second steady state; in fact for a given ϕ_π, the second steady state is the same as in our baseline economy and is unaffected by α (which measures how backward looking the rule is).

[34] Bilbiie (2024) also shows that a PLT rule guarantees local-determinacy in this THANK model. However, that paper does not study whether such a rule ensures global determinacy.

Proposition 10 (Multiple Steady states)**.** *For any $\Theta > 0$, there exist two steady states in which the output-gap and inflation are unaffected by the value of α. The targeted steady state is always one of the steady states. Furthermore, if the targeted steady state is locally determinate, then the untargeted steady state is locally indeterminate and has a negative output gap and inflation below target.*

Proof. The stationary points of the economy are not affected by changing the policy rule from (6) to the AIT policy rule (e.8). With the AIT policy rule, the steady state is represented by three nullclines, which can be written as:

$$\begin{aligned} 0 &= \phi_\pi \pi^b - \pi + \sigma(e^{-\gamma\Theta x} - 1) \\ 0 &= \rho\pi - \kappa\left(e^x - 1\right) \\ 0 &= \alpha\left(\pi - \pi^b\right) \end{aligned}$$

Since the third nullcline implies that $\pi = \pi^b$, the first two nullclines are the same as in our baseline. Thus, the exact value of α does not affect the level of output and inflation in the untargeted steady state, but it does affect the stability properties. The steady state value of x in the untargeted steady state is still defined by the same equation as in the baseline model:

$$F(x) = \frac{\kappa\left(\phi_\pi - 1\right)}{\rho}\left(\mathrm{e}^x - 1\right) + \sigma\left(\mathrm{e}^{-\gamma\Theta x} - 1\right),$$

and the same argument as in Appendix B.3 establishes the existence of the untargeted steady state. Next, we show that if $\phi_\pi > \varphi(\Theta)$, then the untargeted steady state is locally indeterminate.

Suppose that $\phi_\pi > \varphi(\Theta)$. Then, the Jacobian of the system (e.9)-(e.11) evaluated at the untargeted steady state can be written as:

$$A_{\underline{x}} = \begin{bmatrix} -\sigma\gamma\Theta e^{-\gamma\Theta\underline{x}} & -1 & \phi_\pi \\ -\kappa e^{\underline{x}} & \rho & 0 \\ 0 & \alpha & -\alpha \end{bmatrix}$$

Since we have one predetermined-variable π^b and two jump-variables x, π, for the untargeted steady state to be locally determinate, we need one negative root, and two roots with positive real parts. This would then ensure that for a given π_0^b in the neighborhood of the untargeted steady state, there exists a unique (x_0, π_0) such that the trajectory $\{x_t, \pi_t, \pi_t^b\}_{t=0}^\infty$ remains bounded. If, for a given π_0^b, there exist multiple (x_0, π_0) for which the trajectory $\{x_t, \pi_t, \pi_t^b\}_{t=0}^\infty$ remains bounded, then the untargeted steady state is locally indeterminate.

The characteristic polynomial of the $A_{\underline{x}}$ is given by:

$$\mathcal{P}\left(z\right) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

where

$$\begin{aligned}
a_0 &= -1\\
a_1 &= -\sigma\gamma\left(\Theta e^{-\gamma\Theta\underline{x}} - \Theta^\star\right) - \alpha\\
a_2 &= \kappa e^{\underline{x}}\left(1 + \frac{\rho\sigma\gamma\Theta}{\kappa}e^{-(1+\gamma\Theta)\underline{x}}\right) - \alpha\sigma\gamma\left(\Theta e^{-\gamma\Theta\underline{x}} - \Theta^\star\right)\\
a_3 &= -\alpha\kappa e^{\underline{x}}\left[\phi_\pi - 1 - \frac{\rho\sigma\gamma\Theta}{\kappa}e^{-(1+\gamma\Theta)\underline{x}}\right]
\end{aligned}$$

The stability of the system is governed by the pattern of sign changes in the sequence:

$$a_0, \qquad a_1, \qquad -\frac{a_0 a_3 - a_1 a_2}{a_1}, \qquad a_3$$

For the Jacobian to have one negative root and two roots with positive real parts, we need the sequence to have 2 sign changes. Recall from (b.3) in Appendix B.4, that if $\phi_\pi > \varphi(\Theta)$, then we have:

$$\phi_\pi - 1 - \frac{\rho\sigma\gamma\Theta}{\kappa}e^{-(1+\gamma\Theta)\underline{x}} < 0$$

This implies that the fourth term on the sequence is $+$. Clearly, the sign of the first term in the sequence is $-$. If $\alpha > \sigma\gamma\left(\Theta^\star - \Theta e^{-\gamma\Theta\underline{x}}\right)$, then the sign of the second term in the sequence is $-$. So we have: $-,-,?,+$, and no matter what the sign of the third term is, we cannot have two sign changes. Thus, with large α, we have 2 negative and 1 positive root, which implies that for a given π^b, there are multiple bounded trajectories in the neighborhood of the untargeted steady state which converge to it.

Now consider the case in which $\alpha \leq \sigma\gamma\left(\Theta^\star - \Theta e^{-\gamma\Theta\underline{x}}\right)$. With small α, the sequence of signs is now $-,+,?,+$. So if the third term is negative, then a small α can ensure that the untargeted steady state is locally determinate. However, this is not the case, and the third term is positive:

$$\frac{\alpha\kappa e^{\underline{x}}\left(\phi^\diamond - \phi_\pi\right) + \sigma\gamma\left[\Theta^\star - \frac{\alpha}{\sigma\gamma} - \Theta e^{-\gamma\Theta\underline{x}}\right]\left\{\kappa e^{\underline{x}}\phi^\diamond + \alpha\sigma\gamma\left(\Theta^\star - \Theta e^{-\gamma\Theta\underline{x}}\right)\right\}}{\sigma\gamma\left(\Theta^\star - \frac{\alpha}{\sigma\gamma} - \Theta e^{-\gamma\Theta\underline{x}}\right)} > 0,$$

where $\phi^\diamond = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa}e^{-(1+\gamma\Theta)\underline{x}}$, and we have used the fact that if $\phi_\pi > \varphi(\Theta)$, then $\phi_\pi < \phi^\diamond$. Furthermore, we know that $\Theta^\star \geq \Theta e^{-\gamma\Theta\underline{x}} + \frac{\alpha}{\sigma\gamma} > \Theta e^{-\gamma\Theta\underline{x}}$ in this case. So even with small α, the sequence is $-,+,+,+$, which only has one sign change. Thus, regardless of the magnitude of α, the untargeted steady state is locally indeterminate as long as $\Theta > 0$ and $\phi_\pi > \varphi(\Theta)$. □

Proposition 10 showed that as long as $\Theta > 0$, the untargeted steady state always exists, and is locally indeterminate as long as $\phi_\pi > \varphi(\Theta)$, regardless of the magnitude of α. Consequently, there is always global indeterminacy. This is because local determinacy of the untargeted steady state implies that for a given π_0^b in the neighborhood of the untargeted steady state there exists multiple (x_0, π_0) such that there are at least two trajectories $\{x_t, \pi_t, \pi_t^b\}_{t=0}^\infty$ which remain bounded forever. In fact, since the untargeted steady state has two negative and one positive eigenvalue, there exists a 2 dimensional stable manifold containing the untargeted steady state. Any trajectory which originates in this stable manifold remains bounded, in fact it converges to the untargeted steady state. Consequently, there is at least one π_0^b, for which there exists multiple (x_0, π_0) such that the trajectories $\{x_t, \pi_t, \pi_t^b\}_{t=0}^{\infty}$ satisfy equilibrium and always remain bounded. There are even more bounded trajectories which start close to the targeted steady state, as Proposition 11 shows.

Proposition 11. *Suppose that risk is countercyclical* $(\Theta > 0)$ *and monetary policy is described by the inertial rule* (e.8) *satisfying* $\phi_\pi > \varphi(\Theta)$. *Then the equilibrium is always globally indeterminate, regardless of the magnitude of* $\alpha \in [0, \infty)$. *The global dynamics under the inertial rule* (e.8) *can be divided into three regions based on the magnitude of* $\alpha \geq 0$:

1. ***Mildly backward-looking*** $\Big(\alpha > \alpha^\star(\Theta)\Big)$***:*** *For large* α, *the targeted equilibrium is both locally and globally indeterminate. Not only do trajectories which originate in the neighborhood of the targeted steady state converge to it, there also exists a saddle connection along which trajectories which originate near the untargeted steady state converge to the targeted steady state.*

2. ***Moderately backward-looking*** $\Big(\alpha \in [\underline{\alpha}(\Theta), \alpha^\star(\Theta)]\Big)$***:*** $\exists \underline{\alpha}(\Theta) < \alpha^\star(\Theta)$ *such that for any* $\alpha \in (\underline{\alpha}(\Theta), \alpha^\star(\Theta))$, *any trajectory which originates near the targeted steady state initially diverges but then converges to a stable cycle which surrounds the targeted steady state. The amplitude of these cycles is a decreasing function of* α *in this region. Overall, for* $\alpha \in (\underline{\alpha}(\Theta), \alpha^\star(\Theta))$, *the targeted equilibrium is locally determinate but there is global indeterminacy. At the lower boundary* $\alpha = \underline{\alpha}(\Theta)$, *the stable cycles are absorbed into a homoclinic orbit, and at the upper boundary* $\alpha = \alpha^\star(\Theta)$, *the limit cycles are degenerate, but there is still global indeterminacy since the higher-order terms push any trajectory starting near the targeted steady state back towards it.*

3. ***Strongly backward-looking*** $\Big(\alpha < \underline{\alpha}(\Theta)\Big)$***:*** *For small enough* α, *there are no stable cycles. However, there exists a saddle connection along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. Thus, a small* α *ensures local determinacy but not global determinacy.*

Overall, if risk is countercyclical, there is global indeterminacy, no matter how backward-looking the policy rule.

Proof. For a given $\Theta > 0$, the Jacobian of the system (e.9), (e.10), (e.11), evaluated at the targeted steady state, can be written as:

$$A = \begin{bmatrix} -\sigma\gamma\Theta & -1 & \phi_\pi \\ -\kappa & \rho & 0 \\ 0 & \alpha & -\alpha \end{bmatrix}$$

The trace of A can be written as:

$$tr(A) = -\sigma\gamma\Theta + \rho - \alpha = -\sigma\gamma(\Theta - \Theta^\star) - \alpha,$$

where we have used the definition $\Theta^\star = \frac{\rho}{\sigma\gamma}$. Next, the determinant of A is given by:

$$det(A) = -\alpha\kappa\Big(\phi_\pi - \varphi(\Theta)\Big),$$

Since we maintain that $\phi_\pi > \varphi(\Theta)$, we know that $det(A) < 0$ for any $\alpha \in (0, \infty)$. We need to show that $\exists \alpha$ for which the two complex roots have zero real parts. To prove this, we use Orlando's formula, which can be written as:[35]

$$H = -det(A) + tr(A) \times G(A),$$

where $G(A)$ denotes the pairwise product of the eigenvalues of the matrix A and is given by:[36]

$$G(A) = \begin{vmatrix} -\sigma\gamma\Theta & -1 \\ -\kappa & \rho \end{vmatrix} + \begin{vmatrix} -\sigma\gamma\Theta & \phi_\pi \\ 0 & -\alpha \end{vmatrix} + \begin{vmatrix} \rho & 0 \\ \alpha & -\alpha \end{vmatrix} = \alpha\sigma\gamma\left(\Theta - \Theta^\star\right) - \kappa\varphi(\Theta)$$

Using these expressions, we can write H as:

$$H = \alpha\kappa\Big(\phi_\pi - \varphi(\Theta)\Big) + \left[\sigma\gamma\left(\Theta - \Theta^\star\right) + \alpha\right]\left[\kappa\varphi(\Theta) - \alpha\sigma\gamma\left(\Theta - \Theta^\star\right)\right],$$

which can be further simplified to

$$H = \kappa \times \alpha\Big(\phi_\pi - \psi(\alpha)\Big),$$

where $\psi(\alpha)$ is defined in (e.12). Notice that $H = 0$ describes values of α (if one exists) for which two of the eigenvalues cancel each other out. $H = 0$ requires that either (i) two of the eigenvalues are purely imaginary or (ii) two real eigenvalues have the same magnitude but opposite sign. Since Proposition 9 showed that for $\Theta > \Theta^\star$, as long as $\alpha < \alpha^\star(\Theta)$, there are two complex roots with positive real parts and one real negative root. Thus, it follows that at $\alpha = \alpha^\star(\Theta)$, the two complex roots are purely imaginary and thus cancel each other out, leaving the trace to equal the remaining negative root. Thus, a Hopf bifurcation occurs at $\alpha = \alpha^\star(\Theta)$. While verifying that the first Lyapunov coefficient of this 3 dimensional system is possible, it is extremely cumbersome and we verify numerically that it is negative, implying that the Hopf bifurcation is supercritical. Consequently, the Hopf bifurcation theorem implies that for any $\Theta > 0$, $\exists \underline{\alpha}(\Theta) < \alpha^\star(\Theta)$, such that for $\alpha \in (\underline{\alpha}(\Theta), \alpha^\star(\Theta)]$, all trajectories (except those which begin on the one dimensional stable manifold around the targeted steady state), which originate near the targeted steady state converge to a stable cycle. In this region, since $\alpha < \alpha^\star(\Theta)$, Proposition 9 implies that the targeted steady state is locally determinate. However, since for a given π_0^b in the neighborhood of the targeted steady state, there exists multiple (x_0, π_0) such that the trajectories $\{x_t, \pi_t, \pi_t^b\}_{t=0}^\infty$ remain bounded forever, there is global indeterminacy. The convergence to a stable cycle is depicted graphically in Figure 8a, where we have set $\Theta = 28.1$ and $\alpha = 9$. Since $\Theta = 28.1 < \Theta^\star = 31.1$ under our calibration, $\alpha^\star(28.1) = \infty$ and $\underline{\alpha}(28.1) \approx 1.03$ under our calibration. Thus, for any $\alpha > 1.03$, any trajectory originating near the targeted steady state converges to a stable cycle.

Next, Theorem 7.2 and Corollary 7.1 of Kopell and Howard (1975) (which are generalizations of Theorem 2 to $n > 2$ dimensions) imply that for ϕ_π close to $\varphi(\Theta)$, there exists a homoclinic orbit at the boundary of the stable cycles mentioned above. Thus, for $\alpha = \underline{\alpha}(\Theta)$, there exists a homoclinic orbit which cycles around the targeted steady state, while passing through the untargeted steady state. The homoclinic orbit is depicted graphically in Figure 8b. Even in this case, since $\underline{\alpha}(\Theta) < \alpha^\star(\Theta)$, Proposition 9 shows that the targeted equilibrium is locally determinate. However, there is still global

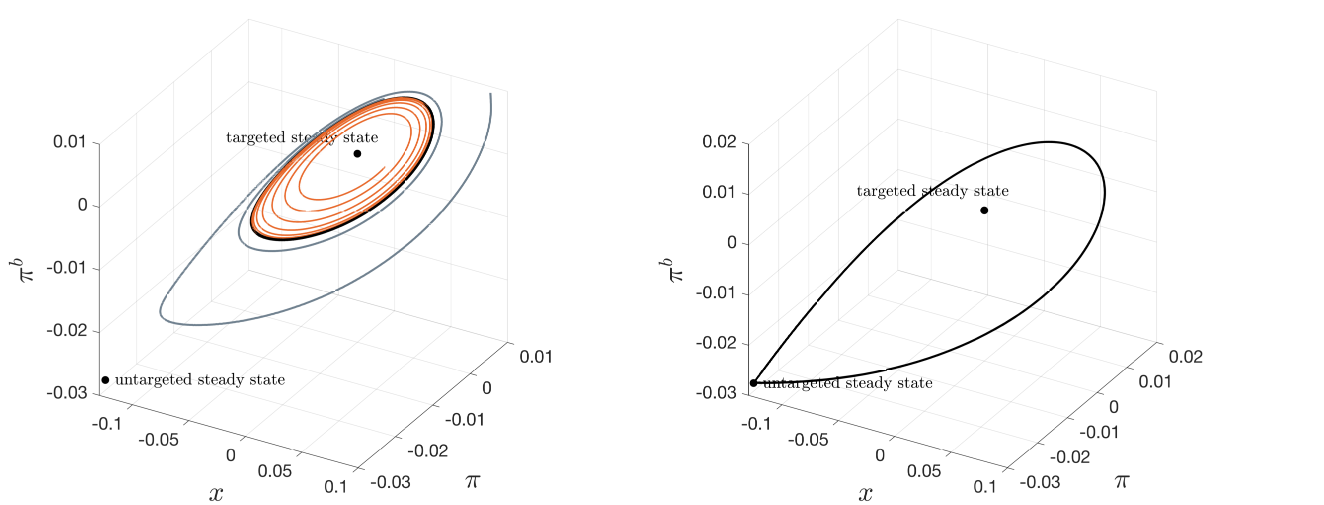
[35] See pp. 196-198 in Chapter XV of Gantmacher (1960).

[36] Let z_1, z_2 and z_3 denote the three eigenvalues of the matrix A. Then $tr(A) = z_1 + z_2 + z_3$, $det(A) = z_1 z_2 z_3$ and $G(A) = z_1 z_2 + z_2 z_3 + z_3 z_1$.

indeterminacy, since for any π_0^b in the neighborhood of the targeted steady state, all combinations of (x_0, π_0) in the neighborhood of the targeted steady state are such that the trajectories $\{x_t, \pi_t, \pi_t^b\}_{t=0}^{\infty}$ remain bounded forever. There is a unique one dimensional manifold around the targeted steady state along which the trajectories converge to the targeted steady state and remain bounded, while all trajectories originating off this stable manifold converge to the homoclinic orbit. At the bifurcation point, $\alpha = \alpha^\star(\Theta)$, for a given π_0^b in the neighborhood of the targeted steady state, while the first-order terms do not move the system towards or away from the targeted steady state, the higher-order terms ensure that all trajectories $\{x_t, \pi_t, \pi_t^b\}_{t=0}^{\infty}$ converge to the targeted steady state, and hence remain bounded, implying global indeterminacy in this case as well.

Finally, for $\alpha < \underline{\alpha}(\Theta)$, there are no stable cycles and Proposition 9 implies that the targeted steady state is locally determinate. However, Theorem 7.2 and Corollary 7.1 of Kopell and Howard (1975) also guarantee that for $\alpha \in (0, \alpha^\star(\Theta))$, there exists a saddle connection along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. Furthermore, any trajectory starting on this saddle connection constitutes a bounded equilibrium since it converges to the untargeted steady state and remains bounded forever. Thus, even in the range $\alpha \in [0, \underline{\alpha}(\Theta))$, the equilibrium is globally indeterminate. □

Proposition 11 characterizes the global dynamics of the economy under the average inflation targeting rule for all degrees of backward-lookingness, and shows that no matter how backward looking a policy rule, it cannot eliminate global indeterminacy. Figure 8 plots global dynamics for $\Theta < \Theta^\star$.



(a) moderately backward-looking

(b) homoclinic orbit ($\alpha = \underline{\alpha}$)

Figure 8: Global dynamics with an inertial rule

When $\Theta = 28.1 < \Theta^\star$, $\alpha^\star(\Theta) = \infty$ and set $\alpha \geq \alpha^\star(\Theta)$ is empty. Thus, the global dynamics in this case are described by points 2 and 3 of Proposition 11. Figure 8a depicts the dynamics for a moderately backward-looking rule, where we have set $\alpha = 9$, which is larger than $\underline{\alpha}(\Theta) = 1.03$ and smaller than $\alpha^\star(\Theta) = \infty$. The figure shows that trajectories which originate near the targeted steady state (orange curves) or even further away from the steady state (grey trajectory), both converge to a stable cycle (black trajectory). Figure 8b depicts the homoclinic orbit which occurs if $\alpha = \underline{\alpha}(\Theta) = 1.03$.[37]

Trajectories which originate inside the homoclinic orbit converge to it and remain bounded. While point 3 of Proposition 11 guarantees that for $\alpha \in [0, \underline{\alpha})$, there exists a saddle connection (along which the economy can move from the neighborhood of the targeted steady state to the untargeted steady state), it is hard to numerically plot this trajectory because it is hard to numerically compute the 1 dimensional stable manifold. Hence we are unable to plot the dynamics described in point 3. Overall, the discussion above shows that as long as risk is countercyclical ($\Theta > 0$), there is global indeterminacy, no matter how backward looking the policy rule is.

E.3 Analysis with a Rotemberg Phillips curve

E.3.1 Deriving the micro-founded Rotemberg Phillips curve

As in Rotemberg (1982), we assume that there each firm k faces a quadratic cost of changing its price at each date, which in our continuous time setting an be written as $\frac{\psi}{2}\left(\frac{\dot{P}_{k,t}}{P_{k,t}}\right)^2 P_t y_t$, where ψ is a constant which scales the cost. We still maintain the assumption that the production function is linear and that the only factor of production is labor. Then, at any date t, firm k's nominal profit net of the price-adjustment cost can be written as:

$$D_{k,t} = \left[P_{j,t}\left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} - \left(1-\varepsilon^{-1}\right) P_t w_t \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} - \frac{\psi}{2}\left(\frac{\dot{P}_{j,t}}{P_{j,t}}\right)^2 P_t \right] y_t,$$

where $(1-\varepsilon^{-1})w_t$ denotes the post-subsidy real wage, where the subsidy is designed to eliminate the monopolistic markup on average. The pricing problem of a firm can be expressed as a Hamiltonian:

$$\mathcal{H} = \left[P_{j,t}\left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} - \left(1-\varepsilon^{-1}\right) P_t w_t \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} - \frac{\psi}{2}\left(\frac{\dot{P}_{j,t}}{P_{j,t}}\right)^2 P_t \right] y_t + \eta_t \dot{P}_{j,t},$$

where η_t denotes the co-state variable. The optimal choice of $\dot{P}_{j,t}$ satisfies

$$\eta_t = \psi \left(\frac{\dot{P}_{j,t}}{P_{j,t}}\right)\left(\frac{P_t}{P_{j,t}}\right) y_t, \tag{e.13}$$

while the evolution of the co-state under the optimal plan can be written as:

$$\dot{\eta}_t = i_t \eta_t - \left[(1-\varepsilon)\left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} + \varepsilon\left(1-\varepsilon^{-1}\right)\left(\frac{P_t}{P_{j,t}}\right) w_t \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} + \psi\left(\frac{\dot{P}_{j,t}}{P_{j,t}}\right)^2\left(\frac{P_t}{P_{j,t}}\right) \right] \tag{e.14}$$

[37] The existence of homoclinic orbits in 3 dimensional systems can also give rise chaotic dynamics, as studied by Shilnikov (See Chapter 6 of Kuznetsov (1998)). Barnett et al. (2022) study an application of Shilnikov chaos to a New Keynesian model.

In a symmetric equilibrium, all firms choose the same $P_{j,t} = P_t$ and we can rewrite (e.13) and (e.14) as:

$$\eta_t = \psi \pi_t y_t \tag{e.15}$$
$$\dot{\eta}_t = i_t \eta_t - \left[(1-\varepsilon) + \varepsilon \left(1 - \varepsilon^{-1}\right) w_t + \psi \pi_t^2 \right] y_t, \tag{e.16}$$

where $\pi_t = \dot{P}_t / P_t$. Next, taking the time derivative of (e.15), we have:

$$\dot{\eta}_t = \psi \dot{\pi}_t y_t + \psi \pi_t \dot{y}_t \tag{e.17}$$

Combining (e.16) and (e.17), we get the Phillips curve:

$$\dot{\pi}_t = \left(r_t - \frac{\dot{y}_t}{y_t} \right) \pi_t - \kappa \left(w_t - 1 \right), \tag{e.18}$$

where $\kappa = \frac{\varepsilon - 1}{\psi}$.

E.3.2 Global indeterminacy with the Rotemberg Phillips curve

As in our baseline, we consider the inflation targeting rule:

$$i_t = \bar{r} + \phi_\pi \pi_t$$

Then, the dynamics of (x_t, π_t) can be written as:

$$\dot{x}_t = (\phi_\pi - 1) \pi_t + \sigma \left(e^{-\gamma \Theta x_t} - 1 \right) \tag{e.19}$$
$$\dot{\pi}_t = (r_t - \gamma \dot{x}_t) \pi_t - \kappa \left(e^{x_t} - 1 \right) \tag{e.20}$$

As before, we can rewrite this system by separating the first-order and higher-order terms:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \underbrace{\begin{bmatrix} -\gamma\sigma\Theta & \phi_\pi - 1 \\ -\kappa & \bar{r} \end{bmatrix}}_{A} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma \left(e^{-\gamma \Theta x_t} - 1 + \gamma \Theta x_t \right) \\ -\kappa \left(e^{x_t} - 1 - x_t \right) - (\gamma - 1)(\phi_\pi - 1) \pi_t^2 - \gamma \sigma \left(e^{-\gamma \Theta x_t} - 1 \right) \pi_t \end{bmatrix}, \tag{e.21}$$

where $\bar{r} = \rho - \sigma$ denotes the real interest rate in the targeted steady state. In what follows, we assume that $\bar{r} > 0$.[38] As in Appendix B, local determinacy in this economy requires that the matrix A have two eigenvalues with positive real parts.

Stable cycle via a Hopf bifurcation Similar to Appendix B, it is clear that the trace of A is 0 at $\Theta = \Theta_R^\star \equiv \frac{\bar{r}}{\sigma\gamma}$. Imposing $\Theta = \Theta_R^\star$ in A, we have:

$$A_R^\star = \begin{bmatrix} -\bar{r} & \phi_\pi - 1 \\ -\kappa & \bar{r} \end{bmatrix},$$

[38] We make this assumption since with $\bar{r} < 0$, the Phillips curve slopes the wrong way and becomes downward sloping.

and the eigenvalues of $A_R^\star$ are given by $\pm\omega\mathtt{i}$ where $\omega = \sqrt{\kappa\left(\phi_\pi - 1\right) - \bar{r}^2}$ and $\mathtt{i} = \sqrt{-1}$. Thus, the system undergoes a Hopf-bifurcation at $\Theta = \Theta_R^\star$.

Next, we can diagonalize the matrix $A_R^\star$ as:

$$A_R^\star = PDP^{-1},$$

where

$$D = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} \bar{r} & \omega \\ \kappa & 0 \end{bmatrix}, P^{-1} = \begin{bmatrix} 0 & \frac{1}{\kappa} \\ \frac{1}{\omega} & -\frac{\bar{r}}{\omega\kappa} \end{bmatrix}$$

Pre-multiplying both sides of (e.21) by P^{-1}, we get:

$$\begin{bmatrix} \dot{u}_t \\ \dot{v}_t \end{bmatrix} = D \begin{bmatrix} u_t \\ v_t \end{bmatrix} + \begin{bmatrix} f\left(u,v\right) \\ g\left(u,v\right) \end{bmatrix},$$

where

$$f\left(u,v\right) = -e^{\bar{r}u_t + \omega v_t} + 1 + \bar{r}u_t + \omega v_t - \sigma\left(e^{-\frac{\bar{r}}{\sigma}\left(\bar{r}u_t + \omega v_t\right)} - 1\right)v_t$$

$$g\left(u,v\right) = \frac{\sigma}{\omega}\left(e^{-\frac{\bar{r}^2}{\sigma}u_t - \frac{\bar{r}\omega}{\sigma}v_t} - 1\right)\left(1 + \bar{r}v_t\right) + \frac{\bar{r}}{\omega}\left(e^{\bar{r}u_t + \omega v_t} - 1\right)$$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = P^{-1}\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \bar{r}u_t + \omega v_t \\ \kappa v_t \end{bmatrix}$$

Finally, the first Lyapunov coefficient at the bifurcation point $\Theta = \Theta_R^\star$ is given by:

$$\begin{aligned} \ell_1(0) \quad = \quad & f_{uuu}\left(0,0\right) + f_{uvv}\left(0,0\right) + g_{uuv}\left(0,0\right) + g_{vvv}\left(0,0\right) \\ & + \frac{1}{\omega}\Big[f_{uv}\left(0,0\right)\left(f_{uu}\left(0,0\right) + f_{vv}\left(0,0\right)\right) - g_{uv}\left(0,0\right)\left(g_{uu}\left(0,0\right) + g_{vv}\left(0,0\right)\right) - f_{uu}\left(0,0\right)g_{uu}\left(0,0\right) \\ & + f_{vv}\left(0,0\right)g_{vv}\left(0,0\right)\Big] \end{aligned}$$

The Lyapunov coefficient is difficult to sign analytically, but is negative under our baseline calibration: $\ell_1(0) = -0.0589$. Thus, even with the Rotemberg Phillips curve, the Hopf bifurcation is *supercritical*, i.e. the higher-order terms of the system (e.21), push x, π in towards the equilibrium $(0,0)$. Overall, this shows that there exists an interval $(\Theta_R^\diamond, \Theta_R^\star)$ for which any trajectories originating in the neighborhood of $(x,\pi) = (0,0)$ initially diverge away from the targeted steady state but then converge to a stable cycle staying bounded forever, implying that there is global indeterminacy. The Hopf bifurcation theorem also ensures that for $\Theta > \Theta_R^\star$, as in the baseline, there are multiple bounded trajectories and hence there is global indeterminacy. Thus, the only effect of the change in the form of the Phillips curve is to redefine the boundaries of the mild, moderate and highly countercyclical risk regions.

Multiple Steady States Furthermore, it is easy to see the second steady state continues to exists even with the Rotemberg Phillips curve for all $\Theta > 0$, which is an additional source of indeterminacy. Imposing $\dot{x} = \dot{\pi}$ in (e.19) and (e.20), and combining them implies that steady state x is implicitly defined by the equation:

$$F(x) = \frac{\kappa(\phi_\pi - 1)}{\bar{r}}(e^x - 1) + \sigma\left(e^{-\gamma\Theta x} - 1\right),$$

which is the same as (b.1) in our baseline model. By the same arguments as in Appendix B.3, this equation also has two zeros, one at $x = 0$ and one at $\underline{x} < 0$ (provided that the $(0,0)$ equilibrium is locally determinate).

Saddle connection from the targeted to the untargeted steady state Finally, to establish the existence of a saddle connection when risk is mildly countercyclical $\Theta \in (0, \Theta_R^\diamond)$, we use Theorem 2. For Theorem 2, we need to consider the case when the matrix A in (e.21) has two zero eigenvalues. As in our baseline, it is easy to see that this happens when $\Theta = \Theta_R^\star$ and $\phi_\pi = 1 + \frac{\bar{r}^2}{\kappa}$, in which case, we can write A in (e.21) as

$$A_R^\diamond = \begin{bmatrix} -\bar{r} & \bar{r}^2/\kappa \\ -\kappa & \bar{r} \end{bmatrix},$$

which ensure that both the trace and determinant are 0. Consequently, $A_R^\diamond$ has eigenvector $e = \begin{bmatrix} \bar{r} \\ \kappa \end{bmatrix}$, and generalized eigenvector $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ Next, for condition 2 of Theorem 2 to be satisfied, we need the Jacobian of $[Tr(A), Det(A)]$ with respect to $[\Theta, \phi_\pi]$ to be non-zero, which is clearly true since the Jacobian can be written as:

$$\begin{bmatrix} -\sigma & -\bar{r}\sigma \\ 0 & \kappa \end{bmatrix}$$

Finally, for condition 3, we need to construct $Q(e, e)$. At the double-zero, the system can be written as:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \begin{bmatrix} -\bar{r} & \frac{\bar{r}^2}{\kappa} \\ -\kappa & \bar{r} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \frac{-\bar{r}^2}{2\sigma} x_t^2 \\ -\frac{\kappa}{2} x_t^2 - (\gamma - 1)\frac{\bar{r}^2}{\kappa}\pi_t^2 + \gamma\bar{r}x_t\pi_t \end{bmatrix} + R_2$$

where R_2 denotes terms with order higher than 2. So, $Q(e, e)$ can be written as:

$$Q(e, e) = \begin{bmatrix} \frac{-\bar{r}^4}{2\sigma} \\ \frac{\bar{r}^2\kappa}{2} \end{bmatrix}$$

Next, we need to check that the following matrix has rank 2:

$$\text{rank} \begin{bmatrix} -\bar{r} & \frac{\bar{r}^2}{\kappa} & -\frac{\bar{r}^4}{2\sigma} \\ -\kappa & \bar{r} & \frac{\bar{r}^2\kappa}{2} \end{bmatrix},$$

which clearly has rank 2, since the second and third columns are not linearly dependent. Thus, Theorem 2 applies to the economy with the Rotemberg Phillips curve as well. Thus, since the Hopf bifurcation theorem established the existence of a stable cycle only for $\Theta \in (\Theta_R^\diamond, \Theta_R^\star)$, then Theorem 2 implies that for $0 < \Theta < \Theta_R^\diamond$, we have a saddle connection, implying that there is also indeterminacy for mildly countercyclical risk. Thus, incorporating a Rotemberg Phillips curve does not affect our main results. □

E.4 Robustness to functional forms

In our baseline model, we assumed a particular functional-form for how the transition rate of switching from ξ_h to ξ_l depends on y_t. In particular, we assumed the functional form:

$$\lambda_{l,t} = \overline{\lambda}_l y_t^{-\Theta} = \overline{\lambda}_l e^{-\gamma\Theta x_t}, \qquad \Theta > 0,$$

where $x_t = \gamma^{-1} \ln y_t$. In this Appendix, we show that our characterization of global indeterminacy does not depend $\lambda_{l,t}$ having this exact functional form. To do so, we allow for the transition rate $\lambda_{l,t}$ to be described by:

$$\lambda_{l,t} = \overline{\lambda}_l \cdot \Lambda\left(\ln y_t\right) = \overline{\lambda}_l \cdot \Lambda\left(\gamma x_t\right), \tag{e.22}$$

where $\Lambda(\cdot)$ is some analytic function which takes non-negative values $\Lambda\left(\gamma x\right) \geq 0$ for any $x \in (-\infty, \infty)$. In addition, we normalize $\Lambda(0) = 1$ so that the transition rate in the targeted steady state with $x = 0$ is still given by $\overline{\lambda}_l$ as in the baseline model. Finally, to make the analysis comparable to the baseline model, we assume that

$$\Lambda'(0) = -\Theta < 0, \tag{e.23}$$

which ensures that risk is countercyclical, with the parameter Θ controlling how countercyclical risk is as in the baseline model. However, it is important to note that (e.23) only makes an assumption about how the transition rate changes *local* to the targeted steady state and *does not* restrict the behavior of how the transition rate responds to changes in x away from $x = 0$.

Keeping everything else the same as in our baseline model except the transition rate, which is now given by (e.22), the IS curve and the Phillips curve can be written as:

$$\begin{aligned} \dot{x}_t &= i_t - \pi_t - \rho + \sigma\Lambda\left(\gamma x_t\right) && \text{(e.24)} \\ \dot{\pi}_t &= \rho\pi_t - \kappa\left(e^{x_t} - 1\right), && \text{(e.25)} \end{aligned}$$

where, as in the baseline model, $\sigma = \overline{\lambda}_l \left(\frac{\xi_h}{\xi_l} - 1\right) > 0$ still captures the average consumption risk faced by ξ_h households in the targeted steady state. Finally, assuming that monetary policy is still described by the policy rule $i_t = \rho - \sigma + \phi_\pi \pi_t$ (which is the same as (6)), the aggregate dynamics of x_t, π_t are fully characterized by the following system of ODEs:

$$\begin{aligned} \dot{x}_t &= \left(\phi_\pi - 1\right)\pi_t + \sigma\Big(\Lambda\left(\gamma x_t\right) - 1\Big) && \text{(e.26)} \\ \dot{\pi}_t &= \rho\pi_t - \kappa\left(e^{x_t} - 1\right) && \text{(e.27)} \end{aligned}$$

As in the main text, we can rewrite the system of ODEs as:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = A \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \underbrace{\begin{bmatrix} \sigma\Big\{\Lambda\left(\gamma x_t\right) - 1 + \gamma\Theta x_t\Big\} \\ -\kappa\left(e^{x_t} - 1 - x_t\right) \end{bmatrix}}_{\text{higher-order terms}}, \tag{e.28}$$

where, as in the baseline model, the properties of matrix A dominate the dynamics local to the targeted steady state $(x, \pi) = (0, 0)$, and the higher-order terms dominate the dynamics. With a general $\Lambda(\cdot)$ function, the matrix A can be written as:

$$A = \begin{bmatrix} \sigma\gamma\Lambda'(0) & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix} = \begin{bmatrix} -\sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix},$$

where the second equality uses (e.23), i.e., local to the steady state, the derivative of $\Lambda(\cdot)$ with respect to x is given by $-\Theta$. Consequently, the A matrix above is the same as in equation (17) in our baseline model, implying that the first-order dynamics of the two economies are identical local to the targeted steady state.

Local Determinacy Since the first-order dynamics of our baseline economy and the economy with a general $\Lambda(\cdot)$ function are identical local to the targeted steady state, it follows that the conditions which guarantee local determinacy of the targeted equilibrium in this economy with the general $\Lambda(\cdot)$ function are the same as in our baseline economy and summarized by Proposition 2, which states that as long as, *local to the targeted steady state*, risk is not too countercyclical: $0 < \Theta < \frac{\rho}{\sigma\gamma}$, local determinacy requires that $\phi_\pi > \varphi(\Theta)$, where $\varphi(\Theta)$ is the same as in (18). Furthermore, as in the baseline model, if *local to the targeted steady state*, risk is too countercyclical $\Theta > \frac{\rho}{\sigma\gamma}$, the targeted equilibrium is locally indeterminate no matter how large ϕ_π, as long as it is finite. Throughout the rest of Appendix E.4, we will maintain the assumption that $\phi_\pi > \varphi(\Theta)$, which is a necessary condition for local determinacy of the targeted equilibrium.

E.4.1 Global Indeterminacy: Existence of a stable cycle

In this section, we show that the existence of the stable cycle *only* depends on the properties of $\Lambda(\cdot)$ local to $x = 0$. As in our baseline model (see Appendix), it is easy to see that $\Theta = \Theta^\star = \frac{\rho}{\sigma\gamma}$ is still the bifurcation point of this general system. To see this, we can evaluate (e.28) at $\Theta = \Theta^\star = \rho/(\sigma\gamma)$ to get:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \underbrace{\begin{bmatrix} -\rho & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}}_{A^\star} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma\left\{\Lambda(\gamma x_t) - 1 + \frac{\rho}{\sigma} x_t\right\} \\ -\kappa\left(e^{x_t} - 1 - x\right) \end{bmatrix}, \tag{e.29}$$

Clearly, as in Appendix B.5 which studies our baseline model, the matrix $A^\star$ in (e.29) has trace 0, which means that the two eigenvalues of the matrix add up to 0. Furthermore, as long as we impose $\phi_\pi > \varphi(\Theta)$, the condition required for local determinacy of the targeted steady state, the determinant of $A^\star$ is positive:

$$Det(A^\star) = \kappa(\phi_\pi - 1) - \rho^2 > 0 \qquad \because \qquad \phi_\pi > \varphi(\Theta^\star) = 1 + \frac{\rho^2}{\kappa}$$

A positive determinant means that the eigenvalues must be purely imaginary, because if the roots were real and canceled each other out, then the determinant would be negative. Thus, as in our baseline model, the economy undergoes a Hopf bifurcation at $\Theta = \Theta^\star$ (see Theorem 1). In fact, at $\Theta = \Theta^\star$, the two eigenvalues can be written as $\pm\omega\mathtt{i}$, where $\omega = \sqrt{\kappa(\phi_\pi - 1) - \rho^2}$. Next, as in Appendix B.5, we can diagonalize $A^\star$ as $A^\star = PDP^{-1}$, where

$$D = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} \rho & \omega \\ \kappa & 0 \end{bmatrix}$$

Next, we can pre-multiply both sides of (e.29) by P^{-1} to express the system in normal form:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = D \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} f(u,v) \\ g(u,v) \end{bmatrix},$$

where

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{\pi}{\kappa} \\ \frac{x}{\omega} - \frac{\rho}{\omega\kappa} \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ \pi \end{bmatrix} = \begin{bmatrix} \rho u + \omega v \\ \kappa u \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} f(u,v) \\ g(u,v) \end{bmatrix} = \begin{bmatrix} -e^{\rho u + \omega v} + 1 + \rho u + \omega v \\ \frac{\rho e^{\rho u + \omega v} + \sigma\Lambda\left(\gamma\rho u + \gamma\omega v\right) - (\rho + \sigma)}{\omega} \end{bmatrix}$$

We can use the normal form to compute the first-Lyapunov coefficient, which can be written as:

$$\begin{aligned} \ell_1(0) &= f_{uuu}(0,0) + f_{uvv}(0,0) + g_{uuv}(0,0) + g_{vvv}(0,0) \\ &\quad + \frac{1}{\omega}\Big[f_{uv}(0,0)\left(f_{uu}(0,0) + f_{vv}(0,0)\right) - g_{uv}(0,0)\left(g_{uu}(0,0) + g_{vv}(0,0)\right) - f_{uu}(0,0)\, g_{uu}(0,0) \\ &\quad + f_{vv}(0,0)\, g_{vv}(0,0)\Big] \\ &= -\frac{\sigma\gamma^2\kappa(\phi_\pi - 1)}{\omega^2}\left[\kappa(\phi_\pi - 1)\Lambda_0'' + \rho\sigma\gamma^2\left(\Lambda_0''\right)^2 - \gamma\omega^2\Lambda_0'''\right], \end{aligned}$$

where

$$\Lambda_0'' = \left.\frac{d^2\Lambda(x)}{dx^2}\right|_{x=0} \qquad \text{and} \qquad \Lambda_0''' = \left.\frac{d^3\Lambda(x)}{dx^3}\right|_{x=0}$$

are the second and third derivatives of the function $\Lambda(x)$ evaluated at $x = 0$. Finally, as in Appendix B.5, the Hopf bifurcation is supercritical as long as $\ell_1(0) < 0$, which requires that

$$\kappa(\phi_\pi - 1)\Lambda_0'' + \rho\sigma\gamma^2\left(\Lambda_0''\right)^2 > \gamma\omega^2\Lambda_0''' \tag{e.30}$$

As long as (e.30) is satisfied, our HANK economy with a general $\Lambda(x)$ function would also feature global indeterminacy, even when the targeted equilibrium is locally determinate. This is because, if (e.30) is satisfied, Theorem 1 guarantees that, as in our baseline model, there exists a $\overline{\Theta} < \Theta^\star$ such that for any $\Theta \in (\overline{\Theta}, \Theta^\star)$, any trajectory (x_t, π_t) originating in the neighborhood of the targeted steady state $(x, \pi) = (0, 0)$ initially diverges away form $(0, 0)$, but coverage to a stable cycle and remain bounded asymptotically, thus implying that multiple bounded trajectories satisfy all equilibrium conditions no matter how large $\phi_\pi > \varphi(\Theta) > 1$ is.

Furthermore, the restriction in (e.30) is not very demanding and is satisfied as long as $\Lambda(\cdot)$ is sufficiently convex local to $x = 0$. This is easiest to see if we assume that $\Lambda_0''' < 0$, in which case (e.30) can be simplified to

$$\kappa(\phi_\pi - 1)\Lambda_0'' + \rho\sigma\gamma^2\left(\Lambda_0''\right)^2 > 0,$$

A sufficient condition for which the above condition is satisfied is that $\Lambda_0'' > 0$, i.e., $\Lambda(\gamma x)$ is a decreasing and convex function local to $x = 0$. Note that the condition (e.30) is satisfied in our baseline specification with $\Lambda(x) = e^{-\gamma\Theta x}$ since it is a decreasing concave function: $\Lambda_0'' = \Theta^2 > 0$ and $\Lambda_0''' = -\Theta^3 < 0$ as long as $\Theta > 0$. Importantly, (e.30) only needs to be satisfied at $x = 0$, and does *not* require the function $\Lambda(x)$ to be concave over the entire domain. In other words, the existence of the stable cycle can be established without restricting the shape of $\Lambda(\gamma x)$ for x far away from its value in the steady state (other than the assumptions about continuity and differentiability); condition (e.30) only depends on how countercyclical risk is in the neighborhood of the targeted steady state ($x = \pi = 0$). A corollary of this is that the existence of the stable cycle *does not* require the existence of an untargeted steady state. It is also useful to note that even the restriction $\Lambda_0'' > 0$ is only a sufficient condition, (e.30) may even be satisfied in $\Lambda(\cdot)$ is concave, but not too concave.

Finally, it is worth pointing out that (e.30) is not satisfied if $\Lambda(\cdot)$ is linear in x since that implies that $\Lambda_0'' = \Lambda_0''' = 0$. However, the argument above makes clear that the case with $\Lambda(x)$ linear is non-generic and constitutes a rather restrictive assumption on the shape of $\Lambda(x)$. (e.30) is satisfied away from this knife-edge case under fairly nonrestrictive assumptions. In fact, in the main text, we argue that it is natural for $\Lambda(x)$ to be a decreasing convex function. Together, these arguments imply that global indeterminacy can easily arise in HANK economies with countercyclical risk as long as standard monetary policy rules are utilized.

However, even if we insist that $\Lambda(\cdot)$ be a linear function in x, then while the stable cycle cannot exist, global indeterminacy can easily arise under some mild conditions which ensure the existence of at least one untargeted steady state alongside the targeted steady state. We describe this scenario next.

E.4.2 Global Indeterminacy: Existence of multiple steady states

In this section, we provide some sufficient conditions on the properties of $\Lambda(\gamma x)$ under which at least one untargeted steady state exists alongside the targeted steady state. As discussed earlier, since our baseline feature no predetermined variables, the existence of two steady states already implies global indeterminacy even if a stable cycle does not exist. To characterize sufficient conditions under which the economy with a general $\Lambda(\cdot)$ function admits multiple steady states, impose $\dot{x} = \dot{\pi} = 0$ in (e.26) and (e.27) to get:

$$\begin{aligned} 0 &= (\phi_\pi - 1)\,\pi + \sigma\Big(\Lambda\,(\gamma x) - 1\Big) \\ 0 &= \rho\pi - \kappa\,(e^x - 1) \end{aligned}$$

Any combination of (x, π) which satisfy the two equations above constitute a steady state of the economy. As in Appendix B.3, we can combine these two equations into one equation, and any x which solves the following equation is steady state of the economy.

$$F\,(x) = \frac{\kappa\,(\phi_\pi - 1)}{\rho}\,(e^x - 1) + \sigma\,(\Lambda\,(\gamma x) - 1) = 0$$

Without further restrictions on the slope of $\Lambda(\cdot)$, it is not possible to ascertain the sign and magnitude of $F(x)$ for any x. However, since we assumed that $\Lambda(0) = 1$, we know that $x = 0$ satisfies $F(0) = 0$.

Thus, $(x, \pi) = (0, 0)$ is still a steady state of the economy. Furthermore, given the assumption (e.23) that $\Lambda'(0) = -\Theta < 0$, i.e., risk is countercyclical local to $x = 0$, we can write the derivative of $F(x)$ at $x = 0$ as:

$$F'(x)\big|_{x=0} = \frac{\kappa(\phi_\pi - 1)}{\rho} + \gamma\sigma\Lambda'(0) = \frac{\kappa}{\rho}\left(\phi_\pi - 1 - \frac{\rho\gamma\sigma\Theta}{\kappa}\right) = \frac{\kappa}{\rho}\left[\phi_\pi - \varphi(\Theta)\right],$$

where the second equality uses (e.23). Given our maintained assumption that $\phi_\pi > \varphi(\Theta)$, we have $F'(0) > 0$, which implies that as we lower x from 0, $F(x)$ declines below 0. This condition is sufficient to ensure that a untargeted steady state exists even when $\Lambda(\gamma x)$ is linear. To see this, given our assumption that $\Lambda'(0) = -\Theta$, when $\Lambda(\gamma x)$ is linear, it can be written as:

$$\Lambda(\gamma x) = \max\{0, 1 - \gamma\Theta x\},$$

where the max operator ensures that the function only takes non-negative values even when x is positive and large. In this case, for $x < 0$, we can rewrite $F(x)$ above as:

$$F(x) = \frac{\kappa(\phi_\pi - 1)}{\rho}(e^x - 1) - \sigma\Theta\gamma x$$

It is true by inspection that as $x \to -\infty$, we have $F(x) \to +\infty$. Since we know that $F(0) = 0$ and $F'(0) > 0$, it must be that for as we lower x starting from 0, $F(x)$ becomes negative. Since $F(x)$ tends to $+\infty$ as $x \to -\infty$, it must be the case that there exists $\underline{x} < 0$ for which $F(\underline{x}) = 0$. Thus, even when $\Lambda(\cdot)$ is linear, the untargeted steady state exists for any $\Theta > 0$, i.e., as long as risk is countercyclical. It follows immediately that even when $\Lambda(\gamma x)$ is non-linear, a sufficient condition for the existence of the untargeted steady state is that $\Lambda(\ln y)$ is convex. Furthermore, under this condition, the existence of this steady state can be guaranteed without making further assumptions about the behavior of $\Lambda(\gamma x)$ away from $x = 0$: convexity of $\Lambda()$ is sufficient to ensure the existence of an untargeted steady state with lower economic activity as long as risk is countercyclical *local* to the targeted steady state $\Lambda'(0) = -\Theta < 0$. Of course, the exact value of output in the untargeted steady state depends on the exact functional form of $\Lambda()$, but we can guarantee existence of such a steady state based purely on the properties of $\Lambda()$ *local* to the targeted steady state.

Thus, our characterization of global determinacy holds under fairly weak restrictions on the shape of the function $\Lambda()$. Furthermore, establishing this only depends on the properties of $\Lambda()$ local to the targeted steady state $x = 0$, and does not require us to take a stance on how cyclical risk behaves as the economy moves far from the steady state. □

E.5 Non-constant fraction of HtM households

In our baseline model we assumed that $\lambda_{h,t}$ adjusts endogenously to ensure that $\lambda_{h,t}\eta = \lambda_{l,t}(1 - \eta)$. This meant that the fraction of ξ_h and ξ_l households stays constant over time. This assumption was made for simplicity. In this appendix, we relax this assumption and allow the fraction of ξ_l households (η_t) to change over time. In particular, if we assume that ξ_l households transition to ξ_h at a constant rate $\overline{\lambda}_h > 0$, we can express the law of motion of the mass of ξ_l households as:

$$\dot{\eta}_t = \lambda_{l,t}\left(1-\eta_t\right) - \overline{\lambda}_h \eta_t, \tag{e.31}$$

where we maintain the assumption that $\lambda_{l,t} = \overline{\lambda}_l y_t^{-\Theta}$. Under this specification, more households transition to ξ_l state and become borrowing constrained when the economy is in a downturn, i.e., when output is below is steady state level.

Despite this change, the rest of our model remains largely unchanged relative to the baseline. Even though η_t cannot change over time, at any date t, the consumption of a ξ_h household and ξ_l household can be written as:

$$c_{h,t} = \left(\frac{\xi_h}{\psi}\right)^{\gamma} w_t^{\gamma} \qquad \text{and} \qquad c_{l,t} = \left(\frac{\xi_l}{\psi}\right)^{\gamma} w_t^{\gamma},$$

respectively. As in the baseline model, we normalize ψ to ensure that output and wages in the targeted steady state equal 1. This now requires:

$$\psi = \left[\left(1-\overline{\eta}\right)\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}\right]^{\frac{1}{\gamma}},$$

where $\overline{\eta} = \frac{\overline{\lambda}_l}{\overline{\lambda}_l + \overline{\lambda}_h}$ denotes the fraction of ξ_l households in the targeted steady state.

However, since η_t can change over time, the goods market clearing condition is now given by:

$$\left(1-\eta_t\right)c_{h,t} + \eta c_{l,t} = y_t \qquad \Rightarrow \qquad \frac{\left(1-\eta_t\right)\xi_h^{\gamma}}{\left(1-\overline{\eta}\right)\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}} w_t^{\gamma} + \frac{\eta_t \xi_l^{\gamma}}{\left(1-\overline{\eta}\right)\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}} w_t^{\gamma} = y_t \tag{e.32}$$

We can rewrite the above equation to derive a relationship between aggregate output and wages:

$$y_t = \mu_t w_t^{\gamma} \qquad \text{where} \qquad \mu_t = \frac{\left(1-\eta_t\right)\xi_h^{\gamma} + \eta_t \xi_l^{\gamma}}{\left(1-\overline{\eta}\right)\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}} \tag{e.33}$$

Equation (e.33) confirms that in the targeted steady state with $\eta_t = \overline{\eta}$, the relationship between wages and output is the same as in our baseline model. However, away from the targeted steady state $\eta_t \neq \overline{\eta}$, μ_t mediates the relationship between wages and output, and captures the fact that ξ_h and ξ_l households work different number of (effective) hours. In fact, there is a one-to-one relationship between μ_t and η_t, which can be written as:

$$\mu_t = 1 - \Gamma(\Delta_c)\cdot\left(\frac{\eta_t - \overline{\eta}}{1-\overline{\eta}}\right) \quad \text{where} \quad \Gamma(\Delta_c) = \frac{(1-\overline{\eta})(\Delta_c - 1)}{1 + \left(1-\overline{\eta}\right)\left(\Delta_c - 1\right)} \in (0,1), \tag{e.34}$$

and $\Delta_c = \frac{c_h}{c_l} = \left(\frac{\xi_h}{\xi_l}\right)^{\gamma} > 1$ denotes the relative consumption of ξ_h and ξ_l households in the targeted steady state.

IS equation Next, we can derive the IS equation by taking the time-derivative of (e.33) (which describes $c_{h,t}$):

$$c_{h,t} = \frac{\xi_h^{\gamma}}{\left(1-\overline{\eta}\right)\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}}\left(\frac{y_t}{\mu_t}\right) \qquad \Rightarrow \qquad \frac{\dot{y}_t}{y_t} = \frac{\dot{c}_{h,t}}{c_{h,t}} + \frac{\dot{\mu}_t}{\mu_t}$$

Then using the Euler equation of ξ_h households (which remains the same as in the baseline and is given by (9)), we can derive the following IS equation:

$$\frac{\dot{y}_t}{y_t} = \gamma\left(i_t - \pi_t - r_t^\star\right), \tag{e.35}$$

where the natural rate is now given by:

$$r_t^\star = \rho - \sigma y_t^{-\Theta} - \frac{1}{\gamma}\frac{\dot{\mu}_t}{\mu_t} \tag{e.36}$$

Equation (e.36) shows that, as in our baseline model, $r_t^\star$ can still endogenously fluctuate in response to contemporaneous output fluctuations. However, now $r^\star$ also responds negatively to increases in the fraction of hand-to-mouth households. This is because a higher fraction of HtM households lowers the desired level of aggregate savings and hence implies a lower real interest rate to clear asset markets and stabilize demand at that particular level of output.

Using the interest rate rule (6) $i_t = \rho - \sigma + \phi_\pi \pi_t$, we can substitute out the nominal interest rate and write the IS equation (e.35) as:

$$\frac{1}{\gamma}\frac{\dot{y}_t}{y_t} = (\phi_\pi - 1)\pi_t + \sigma\left(y_t^{-\Theta} - 1\right) + \frac{1}{\gamma}\frac{\dot{\mu}_t}{\mu_t}, \tag{e.37}$$

Phillips curve We use the same specification of the Phillips curve as in our baseline (given by (5)):

$$\dot{\pi}_t = \rho\pi_t - \kappa(w_t - 1)$$

We can write this in terms of output, rather than wages using (e.33) as:

$$\dot{\pi}_t = \rho\pi_t - \kappa\left[\left(\frac{y_t}{\mu_t}\right)^{\frac{1}{\gamma}} - 1\right] \tag{e.38}$$

Evolution of μ_t Next, differentiating μ_t in (e.33) with respect to time and using the definition of $\dot{\eta}_t$ in (e.31), we can derive the law of motion of $\dot{\mu}_t$:

$$\frac{\dot{\mu}_t}{\mu_t} = -\lambda_l\left[\Gamma\left(\frac{y_t^{-\Theta} - 1}{\mu_t}\right) - \left(y_t^{-\Theta} - 1\right)\left(1 - \mu_t^{-1}\right) - \frac{1}{\eta}\left(1 - \mu_t^{-1}\right)\right]. \tag{e.39}$$

Overall, the three ordinary differential equations (e.37), (e.38) and (e.39) jointly describe the aggregate dynamics of y_t, π_t and μ_t. However, rather than working in terms of y_t and μ_t, it is more convenient to express the dynamics in terms of transformed variables $x_t = \frac{1}{\gamma}\ln y_t$ and $m_t = \frac{1}{\gamma}\ln \mu_t$. As in our baseline, x_t can be interpreted as a scaled version the output-gap and measures the (scaled) percentage deviation of output from its value in the targeted steady state. Similarly, $\mu = 1$ in the targeted steady state, and thus m_t can be interpreted as the (scaled) percentage deviation of μ_t from its value in the targeted steady state. Following the change of variables, we can rewrite (e.37), (e.38) and (e.39) as:

$$\dot{x}_t = (\phi_\pi - 1)\,\pi_t + \sigma\left(e^{-\gamma\Theta x_t} - 1\right) + \dot{m}_t \tag{e.40}$$

$$\dot{\pi}_t = \rho\pi_t - \kappa\left(e^{x_t - m_t} - 1\right) \tag{e.41}$$

$$\dot{m}_t = -\gamma^{-1}\overline{\lambda}_l\left\{\left(e^{-\gamma\Theta x_t} - 1\right)\left[\Gamma + (1-\Gamma)\left(1 - e^{-\gamma m_t}\right)\right] + \frac{1}{\overline{\eta}}\left(1 - e^{-\gamma m_t}\right)\right\} \tag{e.42}$$

As in the baseline, it is useful to separate the terms which dominate aggregate dynamics local to the targeted steady state $x = \pi = m = 0$ and the non-linear terms which dominate dynamics further away from this steady state:

$$\begin{bmatrix}\dot{x}_t\\ \dot{\pi}_t\\ \dot{m}_t\end{bmatrix} = \underbrace{A\begin{bmatrix}x_t\\ \pi_t\\ m_t\end{bmatrix}}_{\text{first-order terms}} + \underbrace{\begin{bmatrix}\overline{\sigma}\left(e^{-\gamma\Theta x_t} - 1 + \gamma\Theta x_t\right) - \gamma^{-1}\overline{\lambda}_l(1-\Gamma)\left(e^{-\gamma\Theta x_t} - 1\right)\left(1 - e^{-\gamma m_t}\right) + \frac{\gamma^{-1}\overline{\lambda}_l}{\overline{\eta}}\left(e^{-\gamma m_t} - 1 + \gamma m_t\right)\\ -\kappa\left[e^{x_t - m_t} - 1 - (x_t - m_t)\right]\\ -\gamma^{-1}\overline{\lambda}_l\left\{\Gamma\left(e^{-\gamma\Theta x_t} - 1 + \gamma\Theta x_t\right) + (1-\Gamma)\left(e^{-\gamma\Theta x_t} - 1\right)\left(e^{-\gamma m_t} - 1\right) - \frac{1}{\overline{\eta}}\left(e^{-\gamma m_t} - 1 + \gamma m_t\right)\right\}\end{bmatrix}}_{\text{higher-order terms}} \tag{e.43}$$

where the matrix A is the Jacobian of the system around the targeted steady state $(x, \pi, m) = (0, 0, 0)$ and is given by

$$A = \begin{bmatrix}-\gamma\overline{\sigma}\Theta & \phi_\pi - 1 & -\overline{\lambda}_l/\overline{\eta}\\ -\kappa & \rho & \kappa\\ \overline{\lambda}_l\Gamma\Theta & 0 & -\overline{\lambda}_l/\overline{\eta}\end{bmatrix} \quad \text{where} \quad \overline{\sigma} = \overline{\lambda}_l\left[(\Delta_c)^{\frac{1}{\gamma}} - 1 - \frac{1}{\gamma}\Gamma(\Delta_c)\right] \tag{e.44}$$

As in our baseline, the first-term on the RHS of (e.43) describes the first-order dynamics of the economy and describes the dynamics of x_t, π_t, m_t local to the targeted steady state $(x, \pi, m) = (0, 0, 0)$, while the second term on the RHS denotes the the higher-order terms which dominate the dynamics of the economy further away from the targeted steady state. Before proceeding, we establish that as long as $\Delta_c > 1$, $\overline{\sigma} > 0$ for any $\gamma > 0$.

Lemma 1. *For any $\Delta_c > 1$, we have $\overline{\sigma} > 0$ for all $\gamma > 0$.*

Proof. For this proof, it is convenient to define $g = \gamma^{-1}$ and rewrite $\overline{\sigma}$ in terms of the coefficient of relative risk aversion g:

$$\overline{\sigma}(g) = \overline{\lambda}_l\left[(\Delta_c)^g - 1 - g\Gamma(\Delta_c)\right]$$

First notice that for $g = 0(\gamma = \infty)$, we have:

$$\overline{\sigma}(0) = \overline{\lambda}_l\left[1 - 1 - 0\right] = 0$$

Next, notice that we can write the derivative of $\overline{\sigma}(g)$ with respect to g as:

$$\frac{\partial \overline{\sigma}(g)}{\partial g} = \overline{\lambda}_l \left[(\Delta_c)^g \ln \Delta_c - \Gamma(\Delta_c) \right]$$

Thus, to prove the claim that $\overline{\sigma}(g) > 0$ for all $g \geq 0$, because $\overline{\sigma}(0) = 0$, it is sufficient to show that this derivative is positive for all $g \geq 0$. Start with $g = 0$, in which case the derivative can be written as:

$$\frac{\partial \overline{\sigma}(0)}{\partial g} = \lambda_l \left[(\Delta_c)^0 \ln \Delta_c - \Gamma(\Delta_c) \right] = \ln(\Delta_c) - \Gamma(\Delta_c)$$

Clearly this is positive for $\Delta_c >> 1$ since $\Gamma(\Delta_c)$ is bounded above by 1 while $\log(\Delta_c)$ grows unbounded as Δ_c increases. Thus, we need to show that the derivative is positive for Δ_c close to 1. To do so, consider Δ_c close to 1, which we can equivalently write as $\Delta_c = 1 + \delta$ for some positive δ close to 0. We know that for δ small we can write $\ln(1+\delta)$ as:

$$\ln(1+\delta) = \delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} \cdots$$

In a similar fashion, a power series expansion of $\Gamma(1+\delta)$ around $\delta = 0$ can be written as:

$$\Gamma(1+\delta) = \frac{(1-\overline{\eta})\,\delta}{1+(1-\overline{\eta})\,\delta} = (1-\overline{\eta})\,\delta \left[1 + (1-\overline{\eta})\,\delta + (1-\overline{\eta})\,\delta^2 + \cdots \right] = (1-\overline{\eta})\,\delta - (1-\overline{\eta})^2 \delta^2 + (1-\overline{\eta})^3 \delta^3,$$

and so the leading term of $\ln(1+\delta) - \Gamma(1+\delta)$ is given by:

$$\ln(1+\delta) - \Phi(1+\delta) = \overline{\eta}\delta > 0,$$

which implies that even for Δ_c close to 1, we have $\ln(\Delta_c) - \Gamma(\Delta_c) > 0$ and so for any $\Delta_c > 1$, we have $\frac{\partial \overline{\sigma}(0)}{\partial g} > 0$. Then, since $(\Delta_c)^g > 1$ for all $g > 0$, it must be true for any $\Delta_c > 1$ and $g > 0$ that:

$$\frac{\partial \overline{\sigma}(g)}{\partial g} = \overline{\lambda}_l \left[(\Delta_c)^g \ln \Delta_c - \Gamma(\Delta_c) \right] > 0$$

So, we have $\overline{\sigma} > 0$. □

E.5.1 Local and Global determinacy of the targeted equilibrium

Next, we can derive the conditions for local determinacy of the targeted equilibrium. Since we now have two jump variables x_t, π_t, and one predetermined variable m_t, local determinacy of the targeted equilibrium requires that the matrix A in (e.44) have two eigenvalues with positive real parts and one negative real eigenvalue, which implies that a necessary condition for local determinacy is that the determinant of A must be negative. Before we evaluate the determinant, notice that the trace of A can be written as:

$$\text{tr}(A) = \rho - \frac{\overline{\lambda}_l}{\overline{\eta}} - \gamma \overline{\sigma} \Theta$$

In what follows, we will assume that the fraction of ξ_l households in the targeted steady state (give by $\overline{\eta}$ is not too large. This is formalized in Assumption 1 below.

Assumption 1. *In the targeted steady state with, the fraction of ξ_l households is not too large:*

$$\overline{\eta} < \frac{\overline{\lambda}_l}{\rho} \tag{e.45}$$

Assumption 1 ensures that the trace of A is negative even if risk is acyclical, and ensures that at least one of the roots of A is always negative.

With assumption 1 in place, the determinant of A can be written as:

$$\det(A) = -\frac{\kappa\overline{\lambda}_l}{\overline{\eta}}\left(1-\overline{\eta}\Gamma\Theta\right)\left[\phi_\pi - \varphi_\eta(\Theta)\right], \quad \text{where} \quad \varphi_\eta(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa\left(1-\overline{\eta}\Gamma\Theta\right)}$$

The expression above shows that a necessary condition for local determinacy of the target equilibrium is risk not be too countercyclical and that monetary policy be sufficiently aggressive in response to changes in inflation:

$$\phi_\pi > \varphi_\eta(\Theta) \qquad \text{and} \qquad 0 < \Theta < (\overline{\eta}\Gamma)^{-1} \tag{e.46}$$

Thus, the condition above is analogous to (18) in the baseline model and shows that if risk is acyclical $\Theta = 0$, then $\varphi_\eta(0) = 1$, and the standard Taylor principle $\phi_\pi > 1$ suffices for local determinacy. However, as we increase Θ starting from 0 (which makes risk more countercyclical), the larger ϕ_π must be to ensure local determinacy. This can be seen from the fact that the derivative of $\varphi_\eta(\Theta)$ is an increasing function of Θ.

As in our baseline model (e.46) is only a necessary condition for local determinacy of the target equilibrium. In fact, as we show next, there exists $\Theta^\star \in (0, (\overline{\eta}\Gamma)^{-1})$ such that for $\Theta > \Theta^\star$, the target equilibrium is locally *indeterminate*. In particular, for $0 < \Theta < \Theta^\star$, two of the eigenvalues of A are a pair of complex conjugates with positive real parts, while the third is a negative real root, which implies local determinacy (since we have one predetermined variable and two jump variables). At $\Theta = \Theta^\star$, the pair of complex roots pass become purely imaginary (they have 0 real parts), while the third root stays negative. In other words, the system of ODEs describing aggregate dynamics undergoes a Hopf bifurcation at $\Theta = \Theta^\star$ (we discuss this in detail in the following paragraph). Increasing Θ above $\Theta^\star$ (but keeping it still below $(\overline{\eta}\Gamma)^{-1}$ corresponds to the case in which A now has a negative determinant via three roots with negative real parts, which implies that the targeted equilibrium is locally indeterminate as for a given m, a trajectory starting from any (x, π) close enough to (0,0), now converges to the targeted steady state. Finally, when risk is even more countercyclical, $\Theta > (\overline{\eta}\Gamma)^{-1}$, no matter how large ϕ_π is relative to $\varphi_\eta(\Theta)$, we still have local indeterminacy. This is because for $\Theta > (\overline{\eta}\Gamma)^{-1}$, the determinant is positive in this case, no matter how large ϕ_π is. A positive determinant implies that we cannot have two roots with positive real parts and one negative root and as in our baseline model, even the augmented Taylor principle cannot deliver local determinacy. Thus, the local determinacy properties are similar to those in our baseline model and are qualitatively unaffected by the fact that the fraction of ξ_l households can change over time in this extension.

Hopf Bifurcation and convergence to a stable cycle As mentioned above, the system of ODEs undergoes a Hopf bifurcation at $\Theta = \Theta^{\star}$. As in our baseline model, this Hopf bifurcation means that even when the targeted steady state is locally determinate, there is global indeterminacy because there are multiple bounded trajectories which start off the stable manifold of the targeted steady state, initially diverge away from the targeted steady state, but then converge to a stable cycle surrounding the targeted steady state and remain bounded asymptotically. Next, we show that as long as Assumption 1 holds, and if the targeted steady state is locally determinate, i.e., $\phi_\pi > \varphi_\eta(\Theta)$, then there exists a $\Theta^{\star} \in \left(0, \frac{1}{\overline{\eta}\Gamma}\right)$ such that evaluating the matrix A in (e.44) at $\Theta = \Theta^{\star}$ ensures that two out of its three eigenvalues (denotes by $z_1^\star, z_2^\star, z_3^\star$) are purely imaginary and one is real and negative. As a result, $z_1^\star, z_2^\star, z_3^\star$ satisfy Orlando's formula, which is given by:

$$-S\Big(A(\Theta^\star)\Big) + \frac{\det\Big(A(\Theta^\star)\Big)}{tr\Big(A(\Theta^\star)\Big)} = 0,$$

where $S\Big(A(\Theta^\star)\Big) = z_1^\star z_2^\star + z_2^\star z_3^\star + z_3^\star z_1^\star$, $\det\Big(A(\Theta^\star)\Big) = z_1^\star z_2^\star z_3^\star$ and $\mathrm{tr}\Big(A(\Theta^\star)\Big) = z_1^\star + z_2^\star + z_3^\star$. Since the trace of A is negative, satisfying Orlando's formula is a sufficient condition for two of the roots to be purely imaginary with the third root being negative since the trace and determinant are negative. We prove this claim next.

Lemma 2. *Suppose that $\overline{\eta} \le \overline{\lambda}_l/\rho$, $\phi_\pi > \varphi_\eta(\Theta)$ and $0 \le \Theta < (\overline{\eta}\Gamma)^{-1}$. Then, $\exists \Theta^\star \in \left(0, \frac{1}{\overline{\eta}\Gamma}\right)$, such that*

$$H(\Theta^\star) = -S(\Theta^\star) + \frac{\det(\Theta^\star)}{\mathrm{tr}(\Theta^\star)} = 0$$

Proof. Let's start by calculating $H(\Theta)$ for $0 \le \Theta < (\overline{\eta}\Gamma)^{-1}$. For convenience, we replicate $A(\Theta)$ here:

$$A(\Theta) = \begin{bmatrix} -\gamma\overline{\sigma}\Theta & \phi_\pi - 1 & -\overline{\lambda}_l/\overline{\eta} \\ -\kappa & \rho & \kappa \\ \overline{\lambda}_l\Gamma\Theta & 0 & -\overline{\lambda}_l/\overline{\eta} \end{bmatrix}$$

First, note that following basic linear algebra, we can calculate $S(\Theta)$ as the sum of the principal cominors of A:

$$S(\Theta) = \begin{vmatrix} -\gamma\overline{\sigma}\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{vmatrix} + \begin{vmatrix} \rho & \kappa \\ 0 & -\overline{\lambda}_l/\overline{\eta} \end{vmatrix} + \begin{vmatrix} -\gamma\overline{\sigma}\Theta & -\overline{\lambda}_l/\overline{\eta} \\ \overline{\lambda}_l\Gamma\Theta & -\overline{\lambda}_l/\overline{\eta} \end{vmatrix},$$

which can be simplified to:

$$S(\Theta) = \kappa(\phi_\pi - 1) - \rho\frac{\overline{\lambda}_l}{\overline{\eta}} + \rho\left(\overline{\lambda}_l\Gamma - \gamma\sigma + \frac{\overline{\lambda}_l}{\rho\overline{\eta}}\gamma\sigma\right)\Theta$$

Next, as before, the simplified determinant of A can be written as:

$$\det(\Theta) = \frac{\overline{\lambda}_l}{\overline{\eta}}\left[\rho\gamma\sigma\Theta - \kappa(\phi_\pi - 1)(1 - \overline{\eta}\Gamma\Theta)\right],$$

Since we are imposing conditions so that the targeted steady state is locally determinate ($\phi_\pi > \varphi_\eta(\Theta)$ and $0 \le \Theta < (\overline{\eta}\Gamma)^{-1}$), we know that the determinant is negative for all Θ in the range we are considering. Finally, we can express the trace of A as:

$$\text{tr}(\Theta) = \rho - \frac{\overline{\lambda}_l}{\overline{\eta}} - \gamma\overline{\sigma}\Theta$$

Given our assumption that $\overline{\eta} \le \overline{\lambda}_l/\rho$, the trace is negative even when risk is acyclical ($\Theta = 0$). Furthermore, the more countercyclical the risk (larger Θ), the more negative is the trace. Using these expressions, we can write $H(\Theta)$ as:

$$H(\Theta) = \frac{\mathbb{N}(\Theta)}{\rho - \frac{\overline{\lambda}_l}{\overline{\eta}} - \gamma\overline{\sigma}\Theta},$$

where

$$\mathbb{N}(\Theta) = a_2\Theta^2 + a_1\Theta + a_0,$$

with

$$\begin{aligned}
a_2 &= \rho\gamma\overline{\sigma}\left[\overline{\lambda}_l\Gamma + \left(\frac{\overline{\lambda}_l}{\overline{\eta}} - \rho\right)\frac{\gamma\sigma}{\rho}\right] > 0 \\
a_1 &= \kappa(\phi_\pi - 1)\gamma\sigma + \gamma\sigma\left(\frac{\overline{\lambda}_l}{\overline{\eta}} - \rho\right)^2 + \left(2\frac{\overline{\lambda}_l}{\overline{\eta}} - \rho\right)\rho\overline{\lambda}_l\Gamma > 0 \\
a_0 &= -\rho\left[\frac{\overline{\lambda}_l}{\overline{\eta}}\left(\frac{\overline{\lambda}_l}{\overline{\eta}} - \rho\right) + \kappa(\phi_\pi - 1)\right] < 0
\end{aligned}$$

The signs of a_2 and a_1 are guaranteed by our assumption that $\overline{\eta} \le \overline{\lambda}_l/\rho$, while the sign of a_0 also relies on the fact that we are imposing $\phi_\pi > \varphi_\eta(\Theta) > 1$.

Since the denominator of $H(\Theta)$ is always negative for any $\Theta \ge 0$, in order to show that $H(\cdot)$ has a zero in the interval $0 < \Theta < (\overline{\eta}\Gamma)^{-1}$, we need to show that $\mathbb{N}(\Theta)$ has a zero in the interval $0 < \Theta < (\overline{\eta}\Gamma)^{-1}$. To see that this is in fact the case, notice that

$$\mathbb{N}(0) = a_0 < 0$$

and for any $0 \le \Theta < (\overline{\eta}\Gamma)^{-1}$, we know that

$$\mathbb{N}'(\Theta) = 2a_2\Theta + a_1 > 0$$

Since $\mathbb{N}(0) < 0$ and because $\mathbb{N}(\Theta)$ is an increasing function of Θ in the relevant interval, to prove that $\mathbb{N}(\Theta)$ has a zero in the interval $0 < \Theta < (\overline{\eta}\Gamma)^{-1}$, it is sufficient to show that $\mathbb{N}\Big((\overline{\eta}\Gamma)^{-1}\Big) > 0$. We can write $\mathbb{N}\left(\frac{1}{\overline{\eta}\Gamma}\right)$ as:

$$\begin{aligned}
\mathbb{N}\left(\frac{1}{\overline{\eta}\Gamma}\right) &= \frac{\gamma\overline{\sigma}\rho}{\overline{\eta}^2\Gamma^2}\left[\overline{\lambda}_l\Gamma+\frac{\gamma\sigma}{\rho}\left(\frac{\overline{\lambda}_l}{\overline{\eta}}-\rho\right)\right]+\frac{1}{\overline{\eta}^2\Gamma}\left\{\gamma\sigma\left(\overline{\lambda}_l-\overline{\eta}\rho\right)^2+\left(2\overline{\lambda}_l-\overline{\eta}\rho\right)\rho\overline{\lambda}_l\Gamma\right\}-\rho\frac{\overline{\lambda}_l}{\overline{\eta}}\left(\frac{\overline{\lambda}_l}{\overline{\eta}}-\rho\right) \\
&\quad+\left(\frac{\gamma\sigma}{\overline{\eta}\Gamma}-\rho\right)\kappa\left[\phi_\pi-1\right] \\
&> \frac{\gamma\overline{\sigma}\rho}{\overline{\eta}^2\Gamma^2}\left[\overline{\lambda}_l\Gamma+\frac{\gamma\sigma}{\rho}\left(\frac{\overline{\lambda}_l}{\overline{\eta}}-\rho\right)\right]+\frac{1}{\overline{\eta}^2\Gamma}\left\{\gamma\sigma\left(\overline{\lambda}_l-\overline{\eta}\rho\right)^2+\left(2\overline{\lambda}_l-\overline{\eta}\rho\right)\rho\overline{\lambda}_l\Gamma\right\}-\rho\frac{\overline{\lambda}_l}{\overline{\eta}}\left(\frac{\overline{\lambda}_l}{\overline{\eta}}-\rho\right) \\
&\quad+\left(\frac{\gamma\sigma}{\overline{\eta}\Gamma}-\rho\right)\kappa\left[\varphi_\eta\left(\frac{1}{\overline{\eta}\Gamma}\right)-1\right]
\end{aligned}$$

Since $\varphi_\eta(\Theta)=1+\frac{\rho\sigma\gamma\Theta}{\kappa(1-\overline{\eta}\Gamma\Theta)}$, we know that

$$\lim_{\Theta\to\frac{1}{\overline{\eta}\Gamma}}\varphi_\eta\left(\Theta\right)=\infty$$

So, as long as $\frac{\gamma\sigma}{\overline{\eta}\Gamma}-\rho>0$, we know that $\mathbb{N}\left((\overline{\eta}\Gamma)^{-1}\right)>0$. To see that $\frac{\gamma\sigma}{\overline{\eta}\Gamma}-\rho=\frac{\gamma\sigma-\rho\overline{\eta}\Gamma}{\overline{\eta}\Gamma}>0$, notice that:

$$\begin{aligned}
\gamma\sigma-\rho\overline{\eta}\Gamma &= \gamma\overline{\lambda}_l\left\{\left(\Delta_c^{\frac{1}{\gamma}}-1\right)-\frac{\rho\overline{\eta}}{\overline{\lambda}_l}\frac{1}{\gamma}\frac{\left(1-\overline{\eta}\right)\left(\Delta_c-1\right)}{1+\left(1-\overline{\eta}\right)\left(\Delta_c-1\right)}\right\} \\
&\geq \gamma\overline{\lambda}_l\left\{\Delta_c^{\frac{1}{\gamma}}-1-\frac{1}{\gamma}\frac{\left(1-\overline{\eta}\right)\left(\Delta_c-1\right)}{1+\left(1-\overline{\eta}\right)\left(\Delta_c-1\right)}\right\} \quad \because \quad \overline{\eta}<\overline{\lambda}_l/\rho \\
&= \gamma\overline{\lambda}_l\overline{\sigma} \\
&> 0,
\end{aligned}$$

where, we know that the last term is positive from Lemma 1. Thus, there exists exactly one $\Theta^\star\in\left(0,(\overline{\eta}\Gamma)^{-1}\right)$ for which $H(\Theta)=\mathbb{N}(\Theta)=0$. Thus, a Hopf bifurcation occurs at $\Theta=\Theta^\star$. □

Since a Hopf bifurcation occurs at $\Theta=\Theta^\star$, Theorem 3 ensures that there exists $\Theta^\diamond\in(0,\Theta^\star)$ such that for $\Theta\in(\Theta^\diamond,\Theta^\star)$ trajectories which start off the stable manifold around the targeted steady state initially diverge but then converge to a stable cycle and remain bounded. Thus, for $\Theta\in(\Theta^\diamond,\Theta^\star)$, even though a large enough ϕ_π ensures that the targeted equilibrium is locally determinate, it cannot deliver global determinacy on account of the multiple bounded trajectories which converge to a stable cycle around the targeted steady state and satisfy all equilibrium conditions. In contrast, for $\Theta\in(\Theta^*,(\overline{\eta}\Gamma)^{-1})$, the targeted equilibrium is locally indeterminate, and so we also have global indeterminacy if risk is too countercyclical $\Theta>\Theta^\star$.

Multiple Steady States As in our baseline model, in addition to the global indeterminacy manifesting as the possibility of the economy converging to a stable cycle, global indeterminacy also manifests in the form of an untargeted steady state which exists alongside the targeted steady state. We discuss this next.

As in our baseline model, as long as risk is countercyclical ($\Theta>0$), there are multiple steady states in this extension in which η_t is allowed to vary over time. The targeted steady state with $x=\pi=$

$m = 0$ always exists, but alongside this, there also exists another untargeted steady state in which $x = \underline{x} < 0, \pi = \underline{\pi} < 0$ and $m = \underline{m} < 0$ as long as $\phi_\pi > \varphi_\eta(\Theta)$. In other words, as long as risk is not too countercyclical $\Theta < (\overline{\eta}\Gamma)^{-1}$ and the targeted equilibrium is locally determinate, there always exists an untargeted steady state in which the output-gap is negative, inflation is below target and the fraction of ξ_l households is larger than in the targeted steady state $\eta = \underline{\eta} = \frac{\overline{\lambda}_l}{\overline{\lambda}_l + \overline{\lambda}_h e^{\gamma\Theta\underline{x}}} > \frac{\overline{\lambda}_l}{\overline{\lambda}_l + \overline{\lambda}_h} = \overline{\eta}$. This higher η in the untargeted steady state implies that:

$$\underline{m} = \frac{1}{\gamma} \ln \left[1 - \Gamma \left(\frac{\underline{\eta} - \overline{\eta}}{1 - \overline{\eta}} \right) \right] < 0$$

To see that this untargeted steady state exists, we can set $\dot{x}_t = \dot{\pi}_t = \dot{m}_t = 0$ in (e.40), (e.38) and (e.42) to describe the combinations of x, π, m which constitute a steady state:

$$0 = (\phi_\pi - 1)\pi + \sigma \left(e^{-\gamma\Theta x} - 1 \right) \tag{e.47}$$

$$\pi = \frac{\kappa}{\rho} \left(e^{x-m} - 1 \right) \tag{e.48}$$

$$0 = \left(e^{-\gamma\Theta x} - 1 \right) \left[\Gamma + (1 - \Gamma)\left(1 - e^{-\gamma m}\right) \right] + \frac{1}{\overline{\eta}} \left(1 - e^{-\gamma m} \right) \tag{e.49}$$

Combining (e.47) with (e.48) to eliminate π, we can reduce the problem of solving for three equations in 3 unknowns to 2 equations in 2 unknowns x, m. Thus, in any steady state (x, m) must satisfy:

$$\overline{\eta} \left(e^{-\gamma\Theta x} - 1 \right) \left[\Gamma + (1 - \Gamma)\left(1 - e^{-\gamma m}\right) \right] + \left(1 - e^{-\gamma m} \right) = 0 \tag{e.50}$$

$$\kappa (\phi_\pi - 1) \left(e^{x-m} - 1 \right) + \rho\sigma \left(e^{-\gamma\Theta x} - 1 \right) = 0 \tag{e.51}$$

Next, we can rearrange (e.50) to express x as a function of m:

$$x = \Omega(m) = \ln \left[1 - \frac{1}{\overline{\eta}} \frac{\left(1 - e^{-\gamma m}\right)}{\Gamma + (1 - \Gamma)\left(1 - e^{-\gamma m}\right)} \right]^{-\frac{1}{\gamma\Theta}}$$

Plug this into (e.51) to get one equation in one unknown m:

$$G(m) = \kappa (\phi_\pi - 1) \left[e^{\Omega(m) - m} - 1 \right] - \frac{\rho\sigma}{\overline{\eta}} \frac{\left(1 - e^{-\gamma m}\right)}{\Gamma + (1 - \Gamma)\left(1 - e^{-\gamma m}\right)}$$

Clearly, by inspection, $m = 0$ satisfies $G(0) = 0$. Also, since $\Omega(0) = 0$, $x = \pi = m = 0$ solves the system of equations. Consequently, the targeted steady state always exists.

Next, notice that the derivative of $G(m)$ evaluated at $m = 0$ can be written as:

$$G'(0) = \kappa \left[\frac{(\Gamma\overline{\eta})^{-1} - \Theta}{\gamma\rho\Theta} \right] \left(\phi_\pi - 1 - \frac{\rho\sigma\gamma\Theta}{\kappa(1 - \overline{\eta}\Gamma\Theta)} \right),$$

which is positive as risk is not too countercyclical: $0 < \Theta < (\Gamma\overline{\eta})^{-1}$, and if monetary policy is aggressive enough to ensure that the targeted steady state is locally determinate $\phi_\pi > \varphi_\eta(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa(1-\overline{\eta}\Gamma\Theta)}$. Next, note that as $m \to -\infty$, we have $G(m) \to \infty$. Consequently, it must be the case that as we lower m from

0 towards $-\infty$, $G(m)$ crosses 0 again at least once at some point $\underline{m} < 0$. So, there exists an untargeted steady state $(\underline{x}, \underline{\pi}, \underline{m})$ where $\underline{m} < 0$. Since $\underline{m} < 0$, we have $\underline{x} = \Omega(\underline{m}) < 0$ implying that $\underline{x} < 0$. It remains to show that $\underline{\pi} < 0$.

Recall that $\underline{\pi} = \frac{\kappa}{\rho}(e^{\underline{x}-\underline{m}} - 1)$. To show that $\underline{\pi} < 0$, we need to show that $\underline{x} - \underline{m} < 0$, so that $e^{\underline{x}-\underline{m}} - 1 < 0$. To see that this is in fact the case, we can evaluate (e.51) at $x = \underline{x} < 0$ and $m = \underline{m} < 0$, which we replicate for convenience below:

$$\frac{\kappa\left(\phi_\pi - 1\right)}{\rho}\left(e^{\underline{x}-\underline{m}} - 1\right) = \sigma\left(1 - e^{-\gamma\Theta\underline{x}}\right)$$

We know that since $\underline{x} < 0$, the RHS of this expression is negative since for $\underline{x} < 0$, we have $e^{-\gamma\Theta\underline{x}} > 1$. For a given $\Theta > 0$, under the assumption that the targeted steady state is locally determinate, i.e., $\phi_\pi > \varphi_\eta(\Theta) > 1$, we know that $\phi_\pi - 1 > 0$. So, the only way that the LHS can also be negative is if $e^{\underline{x}-\underline{m}} < 1$, which requires that $\underline{x} < \underline{m}$. Thus, we have $\underline{\pi} < 0$ in the untargeted steady state.

However, since this extension has a predetermined variable (the fraction of ξ_l households), simply the existence of a second steady state does not imply global indeterminacy. However, we next show that for a given Θ, as long as risk is not too countercyclical $0 < \Theta < (\overline{\eta}\Gamma)^{-1}$ and monetary policy is aggressive enough, $\phi_\pi > \varphi_\eta(\Theta)$ (which is necessary for local determinacy of the targeted equilibrium), then the untargeted steady state $(\underline{x}, \underline{\pi}, \underline{m}) < 0$ is locally *indeterminate*. This implies that there exists a neighborhood around $\underline{m}$ around the untargeted steady state such that for each m_0 in this neighborhood, there exist multiple combinations of (x_0, π_0) such that trajectories originating at (x_0, π_0, m_0) remain bounded. This in turn implies that we have global indeterminacy.

To see that the untargeted steady state is locally indeterminate, note that the Jacobian of (e.40)-(e.42) evaluated at the untargeted steady state $(\underline{x}, \underline{\pi}, \underline{m})$ can be written as:

$$\underline{A} = \begin{bmatrix} -\gamma\overline{\sigma}\Theta' & \phi_\pi - 1 & -\overline{\lambda}_l/\eta' \\ -\kappa' & \rho & \kappa' \\ \overline{\lambda}_l\Gamma\Theta' & 0 & -\overline{\lambda}_l/\eta' \end{bmatrix}, \tag{e.52}$$

where

$$\begin{aligned} \Theta' &= \Theta e^{-\gamma\Theta\underline{x}} > \Theta \\ \eta' &= \overline{\eta}e^{\gamma\underline{m}} < \overline{\eta} \\ \kappa' &= \kappa e^{\underline{x}-\underline{m}} < \kappa \end{aligned}$$

We know that for the untargeted steady state to be locally-determinate, $\underline{A}$ must have two roots with positive real parts and one negative root. This requires that the determinant of $\underline{A}$ be negative. However, as we show next, as long as $\phi_\pi > \varphi_\eta(\Theta)$ (which is a necessary condition for the targeted steady state to be locally determinate), the determinant of $\underline{A}$ is positive, which implies that the untargeted steady state is *locally indeterminate*.

Before we evaluate the determinant of $\underline{A}$, notice that the trace of $\underline{A}$ can be written as:

$$\text{tr}(\underline{A}) = \rho - \frac{\overline{\lambda}_l}{\eta'} - \gamma\sigma\Theta'$$

Since $\eta' < \overline{\eta}$, Assumption 1 ensures that the trace of $\overline{A}$ is always negative as long as $\Theta \geq 0$. This ensures that at least one of the eigenvalues of A is negative. Next, note that we can express the determinant of $\underline{A}$ as:

$$det\left(\underline{A}\right) = \frac{\overline{\lambda}_l}{\eta'}\kappa'\left[\frac{\rho\gamma\sigma\Theta'}{\kappa'} - \left(\phi_\pi - 1\right)\left(1 - \eta'\Gamma\Theta'\right)\right] \tag{e.53}$$

To show that this is positive for any Θ as long as $\Theta > \varphi_\eta(\Theta)$, we need to consider two cases: (i) $\eta'\Gamma\Theta' = \overline{\eta}\Gamma\Theta e^{\gamma(\underline{m}-\gamma\Theta\underline{x})} \geq 1$ and (ii) $\eta'\Gamma\Theta' = \overline{\eta}\Gamma\Theta e^{\gamma(\underline{m}-\gamma\Theta\underline{x})} < 1$. First consider the case in which $\eta'\Gamma\Theta' \geq 1$, which implies that $1 - \eta'\Gamma\Theta' \leq 0$. Then, since we are assuming that $\phi_\pi > \varphi_\eta(\Theta) > 1$ (which is a necessary condition for the targeted steady state to be locally determinate), the determinant (e.53) of $\underline{A}$ must be positive for any $\Theta > 0$. Next, consider the case with $\eta'\Gamma\Theta' < 1$. In this case, it is convenient to factorize (e.53) as:

$$\begin{aligned}
\det\left(\underline{A}\right) &= \frac{\overline{\lambda}_l}{\eta'}\left(1-\eta'\Gamma\Theta'\right)\kappa'\left[\frac{\rho\gamma\sigma\Theta'}{\left(1-\eta'\Gamma\Theta'\right)\kappa'} - \left(\phi_\pi - 1\right)\right] \\
&> \frac{\overline{\lambda}_l}{\eta'}\left(1-\eta'\Gamma\Theta'\right)\kappa'\left[\frac{\rho\gamma\sigma\Theta'}{\left(1-\eta'\Gamma\Theta'\right)\kappa'} - \frac{\rho\gamma\sigma\Theta}{\left(1-\overline{\eta}\Gamma\Theta\right)\kappa}\right] \quad \because \quad \phi_\pi - 1 > \frac{\rho\gamma\sigma\Theta}{\left(1-\overline{\eta}\Gamma\Theta\right)\kappa} \\
&= \frac{\overline{\lambda}_l}{\eta'}\left(1-\eta'\Gamma\Theta'\right)\rho\gamma\sigma\left[\frac{\Theta'}{1-\eta'\Gamma\Theta'} - \frac{\Theta}{1-\overline{\eta}\Gamma\Theta}e^{\underline{x}-\underline{m}}\right] \\
&> \frac{\overline{\lambda}_l}{\eta'}\left(1-\eta'\Gamma\Theta'\right)\rho\gamma\sigma\left[\frac{\Theta'}{1-\eta'\Gamma\Theta'} - \frac{\Theta}{1-\overline{\eta}\Gamma\Theta}\right] \qquad \because \qquad \underline{x} - \underline{m} < 0 \\
&= \frac{\overline{\lambda}_l}{\eta'}\left(1-\eta'\Gamma\Theta'\right)\rho\gamma\sigma\left[\frac{\Theta}{1-\eta'\Gamma\Theta'}e^{-\gamma\Theta\underline{x}} - \frac{\Theta}{1-\overline{\eta}\Gamma\Theta}\right] \\
&> \frac{\overline{\lambda}_l}{\eta'}\left(1-\eta'\Gamma\Theta'\right)\rho\gamma\sigma\left[\frac{\Theta}{1-\eta'\Gamma\Theta'} - \frac{\Theta}{1-\overline{\eta}\Gamma\Theta}\right] \quad \because \quad \underline{x} < 0 \\
&= \frac{\overline{\lambda}_l}{\eta'}\left(1-\eta'\Gamma\Theta'\right)\rho\gamma\sigma\Theta\left[\frac{1}{1-\eta'\Gamma\Theta'} - \frac{1}{1-\overline{\eta}\Gamma\Theta}\right]
\end{aligned}$$

The last expression is positive if $1 - \eta'\Gamma\Theta' = 1 - \overline{\eta}\Gamma\Theta e^{\gamma(\underline{m}-\Theta\underline{x})} < 1 - \overline{\eta}\Gamma\Theta$, which in turn is true if $\underline{m} - \Theta\underline{x} > 0$. In order to prove that $\underline{m} - \Theta\underline{x} > 0$ for any $\Theta > 0$ and $\phi_\pi > \varphi_\eta(\Theta)$, we have to use (e.49) evaluated at the untargeted steady state in which $m = \underline{m} < 0$ and $x = \underline{x} < 0$. We replicate this equation below for convenience:

$$\left(e^{-\gamma\Theta\underline{x}} - 1\right)\overline{\eta}\left[\Gamma + (1-\Gamma)\left(1 - e^{-\gamma\underline{m}}\right)\right] = e^{-\gamma\underline{m}} - 1 \tag{e.54}$$

We proceed to prove that $\underline{m} - \Theta\underline{x} > 0$ by contradiction. Suppose, instead that $\underline{m} - \Theta\underline{x} \leq 0$, which in turn implies that:

$$e^{-\gamma\underline{m}} - 1 \geq e^{-\gamma\Theta\underline{x}} - 1$$

Using this, we can rewrite (e.54) as:

$$\left(e^{-\gamma\Theta\underline{x}}-1\right)\overline{\eta}\left[\Gamma+(1-\Gamma)\left(1-e^{-\gamma\underline{m}}\right)\right]=e^{-\gamma\underline{m}}-1\geq e^{-\gamma\Theta\underline{x}}-1$$

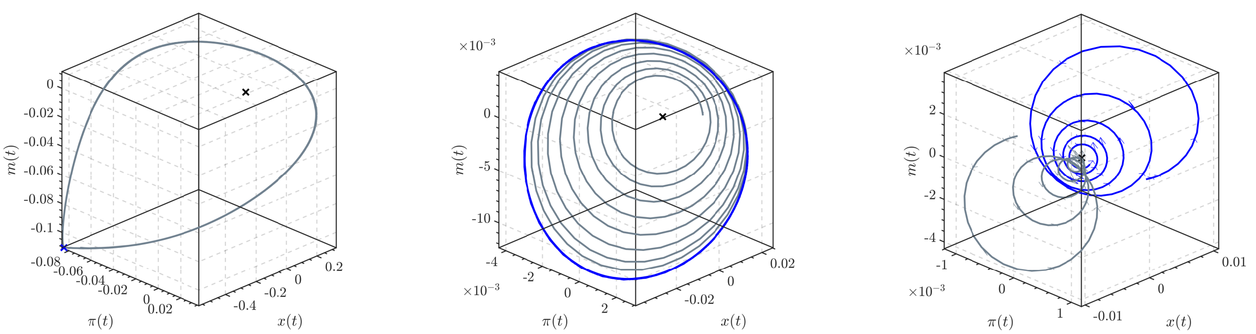
Since, $\underline{x}<0$, we can simplify the above expression by dividing both sides by $e^{-\gamma\Theta\underline{x}}-1$ to get:

$$\overline{\eta}\left[1-(1-\Gamma)\,e^{-\gamma\underline{m}}\right]\geq 1,$$

which is a contradiction since $\overline{\eta}\in(0,1)$ and $1-(1-\Gamma)\,e^{-\gamma\underline{m}}<1$ because $\underline{m}<0$ and $\Gamma\in(0,1)$. Consequently, the LHS is the product of two numbers which are smaller than 1 and their product cannot be larger than 1. Thus, it must be that $\underline{m}>\Theta\underline{x}$, and so even in case (ii), we have $\det(\underline{A})>0$.

Thus, we have shown that for any $\Theta>0$ and $\phi_\pi>\varphi_\eta(\Theta)$, the untargeted steady state is locally indeterminate: a positive determinant and a negative trace imply that the Jacobian of $\underline{A}$ must has two negative roots and one positive root. This means that a 2 dimensional stable manifold (locally) surrounds the untargeted steady state $(\underline{x},\underline{\pi},\underline{m})$: for a given m_0, there exist multiple combinations of x_0,π_0 starting from which (x_t,π_t,m_t) converges to the untargeted steady state $(\underline{x},\underline{\pi},\underline{m})$ and remain bounded. Consequently, we have *global indeterminacy* because there exists a subset of $m_0\in(-\infty,\infty)$ for which there exist multiple combinations of (x_0,π_0,m) starting from which the economy remains bounded.

Overall, the determinacy properties of the economy in which we don't adjust $\lambda_{h,t}$ to keep η_t constant over time are qualitatively similar to those of our baseline model. Thus, even in this extension, the conclusion is that as long as risk is countercyclical, the equilibrium is globally indeterminate as long as monetary policy follows a standard Taylor rule, no matter how aggressively monetary policy responds to changes in inflation.



(a) homoclinic orbit $\overline{\Theta}=9.98$ (b) moderately countercyclical risk $\Theta=21.98\in(\overline{\Theta},\Theta^\star)$ (c) highly countercyclical risk $\Theta=28.1>\Theta^\star=23.98$

Figure 9: Global dynamics as a function of Θ

E.5.2 Calibrated model

In this section, we numerically depict the global dynamics of the economy studied in this section as a function of Θ. In doing so, we maintain the baseline calibration. However, relative to the baseline model, we also need to calibrate $\overline{\eta}$, the fraction of households that are borrowing constrained (or hand-to-mouth) in the targeted steady state.[39] So, following Kaplan et al. (2014), we set the fraction of borrowing constrained households in the targeted steady state $\overline{\eta} = 30\%$.

Given this calibration, the Hopf bifurcation occurs at $\Theta^\star = 23.98$, which is a lower value of $\Theta^\star$ than in our baseline model. More importantly, $\Theta^\star = 23.98$ is lower than the median estimate of $\Theta = 28.1$ from Bilbiie, Primiceri and Tambalotti (2023). However, $\Theta^\star = 23.98$ still lies within the posterior range of $\Theta \in (21.98, 29.9)$. Thus, at Bilbiie, Primiceri and Tambalotti (2023) model estimate of $\Theta = 28.1$, the above analysis implies that the targeted equilibrium is already locally indeterminate and hence we have both local and global indeterminacy. However, in a subset of the posterior range $\Theta \in (21.98, 23.98)$, the targeted equilibrium is locally determinate but we still have global indeterminacy as in our baseline model. Figure 9b considers the case with $\Theta = 21.98$, and depicts (in gray) a sample trajectory which starts in the neighborhood of the targeted steady state $(0, 0, 0)$ (depicted by the black x), diverges away from it, only to converge to the stable cycle (depicted in blue). These dynamics are qualitatively similar for the case in which risk is moderately countercyclical, i.e., when Θ is in the interval $(\Theta^\diamond, \Theta^\star)$, where at $\Theta = \Theta^\diamond$, the stable cycles get absorbed in to a homoclinic orbit (see Figure 9a). Given our calibration, $\Theta^\diamond \approx 9.98$. Finally, Figure 9c depicts local and global indeterminacy when risk is highly countercyclical Θ is between $\Theta^\star$ and $(\overline{\eta}\Gamma)^{-1}$ and shows that for $m_0 = 0$, there are multiple combinations of (x_0, π_0) starting from which the economy converges to the targeted steady state. While we do not plot dynamics in the mildly countercyclical risk case $\Theta \in (0, \Theta^\diamond)$, we have proved analytically that the equilibrium is also globally indeterminate in this range. This is because we showed that as long as the targeted steady state is locally determinate, an untargeted steady state always exists which has $(\underline{x}, \underline{\pi}, \underline{m}) < 0$ and is locally indeterminate. In addition, one can invoke Theorem 4 to establish the existence of a saddle-connection along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. However, numerically finding this saddle connection is quite hard in the context of a 3-dimensional system, so we refrain from doing that. □

[39] Given our simplifying assumption in the baseline model which kept the fraction of ξ_l households constant over time, the precise value of $\overline{\eta}$ did not matter for aggregate dynamics, provided that $\overline{\eta} > 0$.