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**ABSTRACT**

We develop a model of self-fulfilling default cycles with demand externality a la Dixit- Stiglitz to explain the recurrent clustered defaults observed in the data. The literature reports that observable fundamental factors alone are insufficient to explain the cluster. A decline in aggregate output reduces the value of firms and increases their probability of default. As defaults take more firms out of production, aggregate output declines further, creating a positive feedback loop that generates multiple equilibria and self-fulfilling default cycles. Our global analysis using Bogdanov-Takens bifurcation reveals the existence of multiple or even infinite paths that satisfy all equilibrium conditions. Moreover, a family of periodic orbits can emerge in the perfect foresight equilibrium. Our model is consistent with the view that business cycles arise largely because the economy's internal forces tend to endogenously generate cyclical mechanisms (Beaudry et al., 2020).

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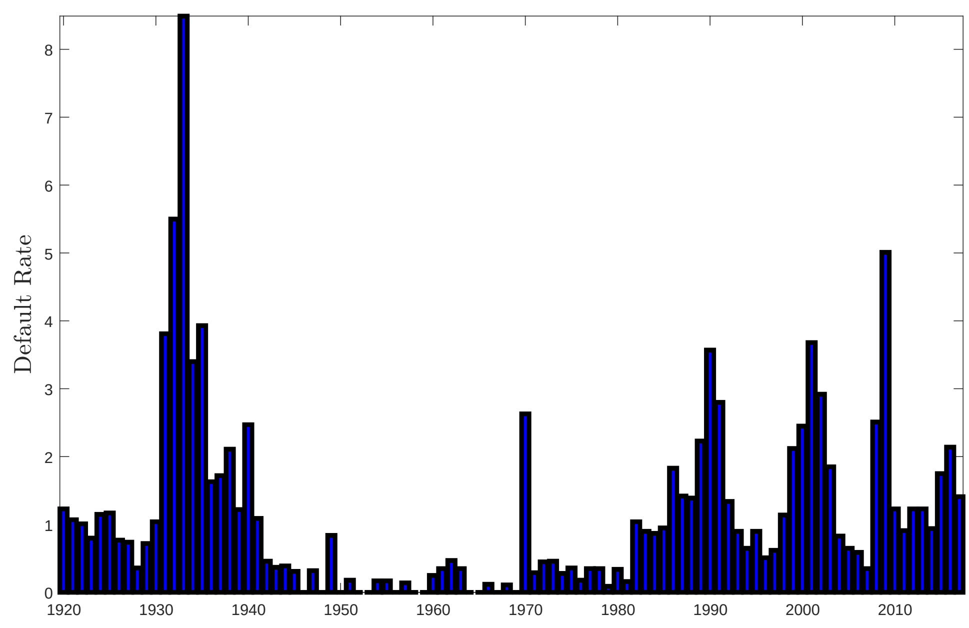
# 1 Introduction

As documented by Giesecke et al. (2011), the corporate bond market has repeatedly suffered clustered defaults, characterized by long periods with few defaults followed by episodes of significant spikes in defaults. For example, while the long-run average default rate during 1866-2008 was only approximately 1.50%, the default rate for the worst three-year period during the Great Depression totaled 12. 88% and reached a disastrous level of 35.80% during the three-year period 1873–1875 after the railroad boom. From these long historical perspectives, the financial turmoil in the corporate bond market during the recent Great Recession is not novel. Figure 1 presents the historical default rate since the 1920s. We observe that the default rate has risen sharply during certain periods. Many studies argue that clustered defaults cannot be fully explained by observable firm-specific or macroeconomic factors but require the consideration of contagion and systematic fragility (Das et al., 2007;Duffie et al., 2009; Azizpour et al., 2018). These facts motivate a theory of self-fulfilling default cycles in this paper to understand the historic recurrence of clustered defaults.

Figure 1: Historical default rate




**Note**: Data source is Moody.

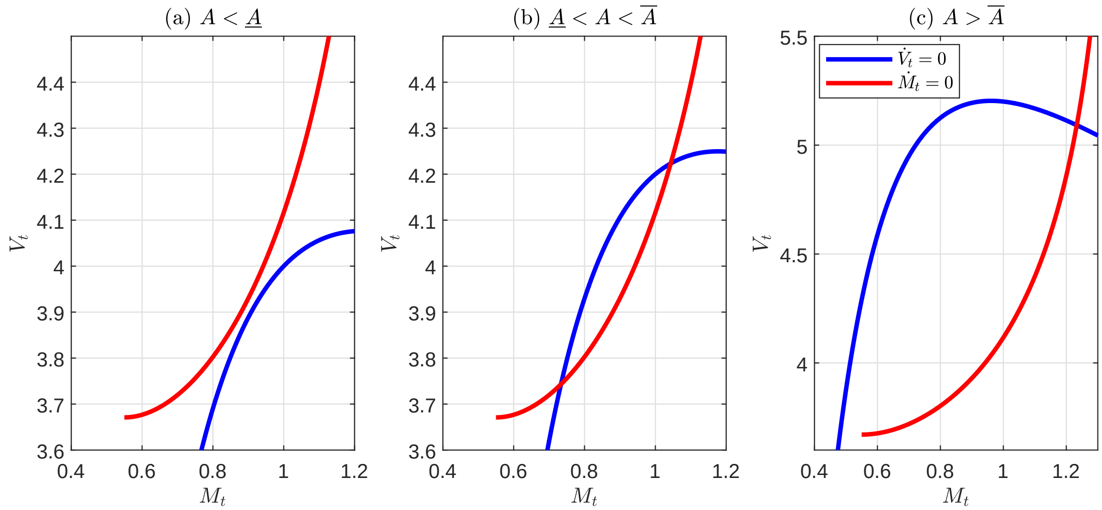
Our theory captures the demand externality. The revenue of individual firms is directly related to overall aggregate output. When aggregate output decreases, individual firms experience decreased revenues and profits. This, in turn, reduces the value of firms and increases their likelihood of default in the face of liquidity shocks. As more firms face production interruption due to default, aggregate output declines further.This creates a positive feedback loop in which firm defaults become contagious, leading to clustered defaults. Therefore, this forms a self-fulfilling prophecy. If firms anticipate that more other firms will default and hence aggregate output will fall, their incentive to default becomes stronger, leading to higher firm default rates.

We formulate our theory in workhorse macroeconomic models with monopolistic competition of the Dixit-Stiglitz type. The CES production structure in the Dixit-Stiglitz framework gives rise to static strategic complementarity of firms' production (Dixit and Stiglitz, 1977; Krugman, 1979; Blanchard and Kiyotaki, 1987; and Romer, 1987). To incorporate dynamic strategic complementarity, we adopt the approach of Jermann and Quadrini (2012) and assume that firms must borrow to finance their working capital. The borrowing capacity is constrained by the firm's value as in Liu and Wang (2014), Miao and Wang (2018), and Lian and Ma (2021). This type of borrowing constraint creates a natural dynamic strategic complementarity between future and current production. Positive expectations regarding future output increase the value of the firm, easing the borrowing constraints for working capital and consequently increasing current production.

To simplify the firms' default problem, we employ a very simply entry and exit mechanism in the model. At each moment, a number of blueprints or projects have the potential to develop into successful varieties of immediate goods if the investment is successful. This investment has an initial cost, which firms need to borrow to cover upon their entry. Having installed the initial investment, the project requires additional investment corresponding to the idiosyncratic liquidity shock in Holmström and Tirole (1998). When the shock exceeds a certain threshold determined by the firm's continuation value, the firm will abandon the project and default on the borrowing for the initial investment. Conversely, if the shock is below the threshold, the firm becomes a monopolist for a new immediate good and enjoys a continuation value through a continuous stream of monopoly profits until its good becomes obsolete. The obsolescence of existing immediate goods follows a Poisson process with a constant arrival rate.

In this environment, an increasing number of firms ($M_t$) increases the profits of firms and therefore increases their value ($V_t$). In turn, $V_t$ is crucial for the dynamics of $M_t$ because it determines the fraction of actual new intermediate goods among potential projects. We show that the equilibrium can be precisely described by two differential equations involving only these two variables, visually represented by a phase diagram (Figure 2), with the vertical axis representing the number of firms ($M_t$) and the horizontal axis representing the value of the firm ($V_t$). Specifically, the locus $\dot{M}_t = 0$ consistently slopes positively as more entrepreneurs are willing to continue the project, but not default, when the value of firms $V_t$ increases. Moreover, the locus has a minimum because the credit market cannot be supported by insufficient firm value. The locus $\dot{V}_t = 0$ is not monotonic. There are two opposing factors. On the one hand,

Figure 2: Number of steady states and the value of $A$




**Note**: Panel (a) shows the case in which productivity is very low ($A < \underline{A}$), Panel (b) shows the case in which productivity is moderate ($\underline{A} < A < \overline{A}$), and Panel (c) shows the case in which productivity is very high ($A > \overline{A}$).

there is a demand externality, and the scale effect means that having a larger number of firms increases the value of all firms. However, on the other hand, having more firms tightens the labor market, which increases costs and lowers firm value. This locus first slopes upward and then downward.

Steady-state equilibria are identified at the intersection of these two loci in the diagram. Given that both loci slope positively, a unique intersection is generally not guaranteed. Our analysis reveals that given other deep parameters, the number of steady states critically depends on productivity $A$. As productivity $A$ increases, the labor output of firms and firm value increases. In other words, the locus $\dot{V}_t = 0$ shifts upward. The graph illustrates that when productivity $A$ is high, only one intersection exits, which is a saddle point. Conversely, when productivity $A$ is insufficient, the locus $\dot{V}_t = 0$ shifts too low, resulting in no intersection. When productivity $A$ is moderate, two intersections emerge. There is a good steady state with a higher firm value and a smaller fraction of firms defaulting and a bad steady state with a smaller firm value and a larger fraction of firms defaulting. The good steady state is a saddle. However, the bad steady state is typically a sink point or a source point, depending on the parameters. The presence of multiple steady states suggests that the system can exhibit self-fulfilling soaring default rates. This potentially explains why observable factors, such as firm characteristics and macroeconomic variables, only partially explain corporate default. Note that a necessary but insufficient condition for multiple steady states is that productivity cannot be too high, that is, $\underline{A} < A < \overline{A}$. High default rates occur only when pessimistic expectations of firms are combined with low productivity. This finding is consistent with the default rate data. A large clustered default rate is more likely to happen when the economy suffers negative shocks, such as during the Great Depression and the Global Financial Crisis.

The equilibrium path and dynamics of large clustered defaults, however, require further analysis beyond local analysis close to steady states. We use the bifurcation theorem to explore the global dynamics of the system, a method similar to the work of Benhabib et al. (2001), Mortensen (1999) and Sniekers (2018). [1] Global dynamics suggest that the system exhibits complex indeterminacy. In addition to the existence of infinite trajectories around the sink steady state, there also exist multiple equilibrium paths diverging from the source steady state and converging to the saddle path. These paths are incorrectly ruled out in local dynamics on the grounds that they violate transversality conditions. Moreover, our model also exhibits periodic orbits that surround the high-default steady state. The existence of endogenous cycles is consistent with the perspective in Beaudry et al. (2020) that business cycles arise largely because the economy's internal forces tend to endogenously generate cyclical mechanisms.

Bifurcation analysis shows that the model has rich dynamic patterns even in the perfect expected equilibrium. Moreover, when there are fundamental or sentiment shocks in the model, the model can effectively simulate the characteristics of the cluster defaults that exist in reality. We assume that productivity follows a two-state Markov process. When productivity is high, the economy has only one saddle point equilibrium (good). When productivity is low, there is one saddle point equilibrium (bad) and one sink point equilibrium (ugly) in the economy. As productivity changes from high to low, it is possible to fall into both equilibria. If firms have more pessimistic expectations, it is possible to fall into a sink point equilibrium (ugly). At this point, the default rate will rise well above the normal level until productivity returns to the high state. Our stochastically simulated time series of default rates exhibit the cluster characteristics documented by Giesecke et al. (2011). Our simulation shows that the features of the clustered default rate can be generated only in the case of global dynamics, and the fluctuation in the default rate will be substantially smaller in the local dynamics without the ugly state. This finding aligns with the conclusions drawn in Azizpour et al. (2018), Duffie et al. (2009) and Das et al. (2007), which suggest that clustered defaults cannot be solely attributed to observable macroeconomic factors.

Based on our findings, we argue that governments can subsidize entrepreneurs without default to eliminate the bad equilibrium. If firms are rewarded for choosing to continue with a project, then even if the number of firms in the economy is low and output is low, firms will still not choose to default and abandon their projects. In this way, the bad steady state will not exist, and the economy only remains in a saddle path solution converging to the good steady

[1] The continuous-time setting turns out to be very useful, providing us with a neat characterization of both local and global dynamics around the steady states.

state that satisfies all requirements of rational expectations equilibria.

Our paper makes the following contributions. In terms of topics, we develop a continuous-time model that incorporates an endogenous default rate. The model yields multiple equilibria, including a steady state with a high default rate and a steady state with a low default rate. The self-fulfilling default rate enables the model to partially account for unexpected changes and fluctuations in default rates and regime switches, which the fundamentals cannot explain. When the economy suffers a negative fundamental shock, the system changes from a unique steady state to multiple steady states. In this case, if pessimistic sentiment spreads, the system will slide into a steady state with a high default rate. Our paper is consistent with many empirical results in the literature. In addition to the high volatility of the default rate, our paper replicates evidence that the low default rate is associated with the high entry rate of firms in Gilchrist and Zakrajšek (2012). Our paper also implies that the measured (gross) markup positively comoves with default rate, and thus the markup measure is predicted to countercyclical, which is consistent with the empirical regularity well documented in the literature (see Bils, 1987 and Rotemberg and Woodford, 1999). Technically, our paper adopts a global dynamics approach and provides both analytical and quantitative illustrations to demonstrate the rich dynamic properties of this economic system. Indeterminacy exists in a wider range of situations than in local analyses. Furthermore, this system can generate endogenous periodic orbits in a perfect foresight equilibrium, which echoes the opinion in Beaudry et al. (2020) that endogenous cyclical forces largely drive economic fluctuations. Our counterfactual analysis also shows that only global analysis can generate a clustered default pattern.

**Related Literature:** Our paper is largely motivated by the literature on limited commitment and equilibrium default. The pioneering literature on default mainly includes Eaton and Gersovitz (1981), Calvo (1988), and Bulow and Rogoff (1989), who are among the first to consider default in the presence of limited commitment. Recently, increasing numbers of studies document the fact that the default is clustered, such as Giesecke et al. (2011) and Berndt et al. (2010). Empirical studies such as Das et al. (2007), Azizpour et al. (2018) and Duffie et al. (2009) argue that the clustered default cannot be totally attributable to observable firm-specific or macroeconomic factors, even though the default rate is countercyclical to some extent. Default contagion and systematic risk are candidate explanations of the cluster pattern. Jorion and Zhang (2007) and Hertzel and Officer (2012) empirically emphasize risk contagion within an industry. Longstaff (2010) empirically shows that contagion spreads primarily through liquidity and risk-premium channels. In terms of theoretical literature, Giesecke (2004) builds a model with asymmetric information on liability where the default of a firm is contagious in this environment. Acemoglu et al. (2015) formulate a theory with the debt network and show that interbank liability can lead to default contagion. Phelan (2017) captures the correlated payoffs of different projects in the model. Although most existing work relies on a shock-amplifier mechanism to explain clustered defaults, our paper takes a different approach by introducing a model with multiple equilibria and self-fulfilling expectations. The key mechanism in our model is demand externality, a feature that is not typically emphasized in the current literature on corporate default. Drawing on the well-established workhorse model of macroeconomics, our model naturally gives rise to multiple default equilibria. Furthermore, the simple structure of our model allows us to globally characterize equilibria. The simulation results demonstrate that global dynamics is essential to capture the default clusters observed in the data.

Our emphasis on multiple equilibria and demand externality also connects our paper to the literature on multiple equilibria, especially the indeterminacy literature concerning demand externality. Benhabib and Farmer (1994) first demonstrated that multiple equilibria (a continuum of equilibria) exist in a standard macroeconomic model with externalities. They assume that the productivity of each firm is related to aggregate output. Wen (1998) uses a similar model to study utilization. The following work endogenizes the externality. Liu and Wang (2014) introduces the credit constraint and fixed cost so that the economy has increasing returns. Schaal and Taschereau-Dumouchel (2015, 2016) employ CES (constant elasticity of substitution) aggregation so that complementarities exist between different varieties. Benhabib and Wang (2013) combine borrowing constraints with CES aggregation and induce countercyclical markups and local indeterminacy around the steady state.Kaplan and Menzio (2016) innovatively introduces a shopping externality, stating that one firm hiring labor to promote its consumption benefits other firms. Benhabib et al. (2018), on the other hand, highlights the externality of adverse selection whereby if more honest firms are willing to produce and borrow, then lenders will give them lower interest rates. [2] Our paper differs from these studies in two key aspects. First, our focus is on explaining the clustered default patterns observed in empirical data, whereas most of the literature does not feature equilibrium default. Second, our analytical approach employs global methods and standard bifurcation analysis, leading to the emergence of more complex indeterminacy and endogenous cycles in the system. While the theoretical analysis of these papers in the aforementioned literature primarily rely on local dynamics methods, with Kaplan and Menzio (2016) being an exception. Our use of global analysis allows for a more comprehensive understanding of the system, capturing richer indeterminacy than local analysis techniques. This approach enables us to better characterize clustered default patterns than would be possible with local analysis alone.

[2] In addition to demand externality, many other mechanisms can generate multiple equilibria. After the financial crisis, economists realized that a crisis may signal a regime switch rather than fluctuations around a steady state, which requires multiple equilibrium models. The literature is too vast to review. Notable examples include Alvarez and Jermann (2000), Gu et al. (2013), Bocola and Dovis (2019), Krueger and Uhlig (2022), Bethune et al. (2018) and Gorton and Ordonez (2014).

Our paper also contributes to the global dynamics and endogenous cycles of the economic system. Complex patterns of dynamics occur when there are multiple equilibria in the economy, which cannot be discovered by only local analysis. Studies such as Benhabib et al. (2001) and Mortensen (1999) utilize the Bogdanov-Takens bifurcation and saddle-loop bifurcation to study the issues of the Taylor rule and labor market respectively. The latter was also extended by Sniekers (2018), who also established a standard Bogdanov-Takens bifurcation analysis on labor search. Similarly, Antoci et al. (2011), Brito and Venditti (2010), Banerjee et al. (2011), and Antoci et al. (2021) also use the bifurcation method to study various topics in macroeconomics. Apart from the above studies, Howitt and McAfee (1988) and Kaplan and Menzio (2016) also implicitly imply Bogdanov-Takens bifurcation. Periodic orbits can emerge in the bifurcation analysis and this result is consistent with the evidence from spectral analysis in Beaudry et al. (2020) that there are internal forces that endogenously generate cyclical mechanisms. [3] Our paper also uses the global method and Bogdanov-Takens bifurcation. Global indeterminacy and periodic orbits arise in our model. We also discuss the interaction effect of fundamental and sentiment shocks, which is not covered in the above literature.

*The most related works* include Azariadis et al. (2016) and Cui and Kaas (2021). The empirical analysis of Azariadis et al. (2016) reveals the importance of unsecured debt for business cycles. Then, they develop a tractable dynamic general equilibrium model to illustrate that unsecured firm credit can arise from self-enforcing borrowing constraints. The key difference is that there is no equilibrium default in Azariadis et al. (2016), while we focus on self-fulfilling default in the dynamic general equilibrium environment.

Based on Azariadis et al. (2016), Cui and Kaas (2021) show that default cycles can stem from self-fulfilling beliefs about credit market conditions. Both Cui and Kaas (2021) and our paper demonstrate that equilibrium default can be belief driven in a dynamic environment. The difference between Cui and Kaas (2021) and our paper is mainly threefold. First, our setups are complementary in the sense that firms are fully competitive in Cui and Kaas (2021) while they engage in monopolistic competition and consumer preferences are CES in our model. That is, the main transmission mechanism of Cui and Kaas (2021) lies in the default coordination of the firms themselves between the current and the future. We not only consider the default decision in a conventional intertemporal way but also show the novel mechanism of default contagion across firms. That is, even if in the static setup, we can still obtain default contagion in the presence of CES. Furthermore, we complement their study by examining the joint role of external factors, aggregate demand externalities, and internal sentiment. We show that multi-

[3]Apart from the periodic orbits in the continuous-time model, endogenous economic cycles can also emerge through other methods, such as Matsuyama (1999), Benhabib et al. (2002),Chousakos et al. (2023), and Rocheteau and Wang (2023). All these works indicate that local analysis is not sufficient and that the economy exhibits endogenous cycles.

ple equilibria occur only in the presence of poor productivity, indicating that the existence of multiple equilibria depends not only on structural parameters but also on external fundamentals. Moreover, we consider a continuous-time setup in the fully fledged dynamic model and reduce the dynamic system to a two-dimensional system. This allows us to fully characterize the conditions under which indeterminacy arises, so that self-fulfilling beliefs matter in the dynamic environment. The continuous-time framework enables the examination of both local and global dynamics with an analytical phase diagram. Finally, we initiate an empirical analysis of regime switching and use our model to perform regime switching simulation driven by sunspots in two steady states, while Cui and Kaas (2021) focus on the dynamics around one steady state.

The remainder of the paper proceeds as follows. Section 2 initiates the regime switching analysis of firm default using time series data between 1951 and 2017. Section 3 establishes a continuous-time model to illustrate self-fulfilling default. Section 4 discusses the multiple steady states and local dynamics of the system. Section 5 theoretically and numerically discusses global dynamics. Section 6 replicates the cluster pattern of the default rate. Section 7 discusses the potential policies to rule out the bad steady state. Section 8 concludes the paper.

## 2 Regime Switching Analysis of Firm Default

Motivated by Figure 1, we follow Hamilton (1989) by adding Markov switching into the following regression model to capture the potential regime changes in the default rate:

$$DR_t = \mu\left(s_t\right) + \rho\left(s_t\right) DR_{t-1} + \varepsilon_t,\ \varepsilon_t \sim N\left(0, \sigma^2\left(s_t\right)\right), \tag{1}$$

where $DR_t$ is the default rate and the unobservable realization of regime $s_t$ is governed by a discrete-time and two-state Markov stochastic process, which is defined by the transition probabilities:

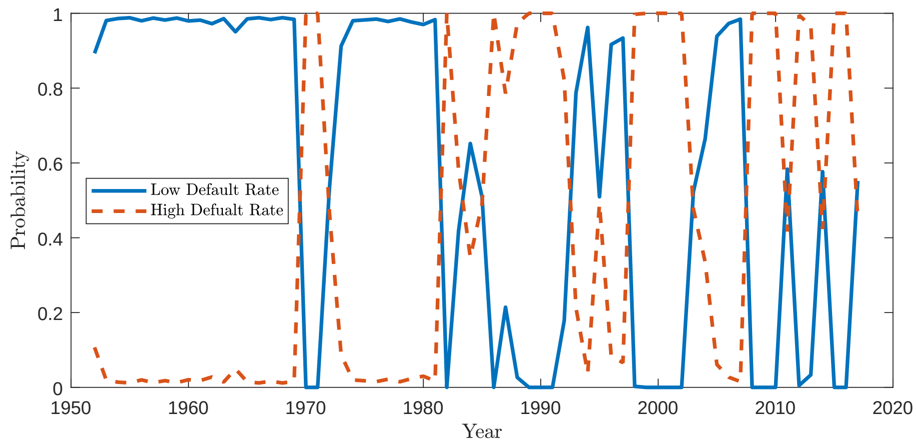
$$P = \begin{bmatrix} p_{11} & 1 - p_{22} \\ 1 - p_{11} & p_{22} \end{bmatrix}.$$

In addition, $\mu\left(s_t\right)$, $\rho\left(s_t\right)$ and $\sigma\left(s_t\right)$ denote the regime-dependent parameters whose values depend on the realization of state $s_t$. To test whether there exist regime changes in the default rate, we compare the above regime switching regression in Equation (1) with the following linear model without regime changes:

$$DR_t = \mu + \rho DR_{t-1} + \varepsilon_t,\ \varepsilon_t \sim N\left(0, \sigma^2\right). \tag{2}$$

We use the post-war default rate data since the 1950s to test the regime switch. Our estima-

Figure 3: The smoothed regime probabilities for the default rate




**Note**: The orange line shows the probability that the economy is in a high default regime, while the blue line shows the probability that the economy is in a low default regime.

tion applies the BFGS-MLM (modified Levenberg-Marquardt) algorithm in the optimization process, and the initial regime probabilities are set to ergodic solutions. Table 1 presents the estimated value of the log likelihood and information criterion for the models in (1) and (2), respectively.

Table 1: The relative fit of the estimated model

| | Default Rate | | | |
|---|---|---|---|---|
| | Log likelihood | AIC | HQ | SC |
| Regime Switch | 256.76 | -3.65 | -3.72 | -3.38 |
| No Switch | 224.31 | -3.31 | -3.33 | -3.21 |

**Note**: We apply the BFGS-MLM (modified Levenberg-Marquardt) algorithm in the optimization process and the initial regime probabilities are set to ergodic solutions. AIC is the Akaike information criterion, SC is the Schwarz criterion, and HQ is the Hannan Quinn criterion. For all information criteria, the smaller they are, the better the fit of the model.

Based on logarithmic likelihoods and several information criteria, it is clear from Table 1 that the regression model without regime switches is strongly rejected by the data, regardless of which proxy variable we use. The log likelihood, AIC, HQ and SC of the models without regime switches are worse than those of the regime switching models. This indicates that there are regime changes in the default rate. Table 2 shows the estimation result for the regime switching models in Equation (1):

Table 2: Parameters estimated from the regime switching models

| | Default Rate | |
|---|---|---|
| parameter | $s_t = 1$ | $s_t = 2$ |
| $\mu$ | $0.014^{**}$ | $0.001^{**}$ |
| | (0.01) | (0.01) |
| $\rho$ | 0.274 | $0.560^{***}$ |
| | (0.17) | (0.00) |
| $\sigma^2$ | $1.04 \times 10^{-4***}$ | $3 \times 10^{-6***}$ |
| | (0.00) | (0.00) |
| $p_{11}$ | $0.83^{**}$ | |
| | (0.02) | |
| $p_{22}$ | | $0.87^{***}$ |
| | | (0.00) |
| Duration | 5.81 | 7.52 |

**Note**: The estimated parameters are provided in the top row, and the p-value is in parentheses. *, ** and *** indicate significance at the 10%, 5% and 1% levels (two-tailed test).

Note that the regime changes in the default rate are reflected in the mean $\mu/(1-\rho)$ and standard deviation $\sigma$. The mean in the first regime (1.93 %) is almost 8 times that in the second regime (0.23 %), and the standard deviation in the first regime is nearly 6 times that in the second regime. Thus, we can refer to the first regime as the high default rate regime, while the second regime is the low default rate regime. The probability of staying in the first regime is approximately 0.83, so the duration of the first regime is approximately 5.81 years, while the probability of remaining in the second regime is approximately 0.87, which lasts for approximately 7.52 years, so the low default rate regime occurs more frequently. Figure 3 shows the smoothed regime probabilities for the default rate.

From the above analysis, we know that the model with regime switching can better characterize the time series of default rates than the linear model. This result also confirms that the defaults are clustered, as mentioned in Giesecke et al. (2011). When the system is in a bad state, the default rate increases sharply.

## 3 Model

In this section, we build a standard continuous-time general equilibrium model. Time is denoted by $t = 0, \Delta, 2\Delta, 3\Delta, ....$ The length of a time period is $\Delta$. We take the limit of this discrete-time economy as $\Delta$ goes to zero. It is more convenient to analyze both local and global dynamics around a steady state in continuous time. There are two sectors in the baseline model, households and firms. Households are risk neutral and hold firms. They derive their utility from consumption and leisure. Households are also entrepreneurs. In each period, they invent a certain number of new ideas. Realizing each idea requires borrowing money to pay for the investment. After investing, new firms face an idiosyncratic liquidity shock, and those with large shocks will abandon the project, leading to default. The other new firms choose to continue the project and become a new firm in the economy. Each firm produces one variety of the product, and different varieties are not completely substitutable for each other. In each period, the firms have to borrow working capital to pay wages, which is subject to a borrowing constraint. In each period, a firm has an exogenous probability of exiting the economy.

## 3.1 Households

The infinitely lived representative household derives utility from consumption and leisure according to the following utility function:

$$\sum_{s=0}^{\infty} e^{-\rho s\Delta} \left[ C_{t+s\Delta} + \psi \log(1 - N_{t+s\Delta}) \right] \Delta, \tag{3}$$

where $C_t$ represents consumption, $N_t$ is hours worked, $e^{-\rho}$ is the discount rate and $\psi$ is the utility weight for leisure. Households face the following budget constraint:

$$C_t\Delta + Z_t = w_t N_t \Delta + e^{\rho\Delta} Z_{t-\Delta},$$

where $w_t$ is the competitive wage rate and $Z_t$ denotes the total assets of the representative household, which is a representative portfolio of all firms. Note that since households are risk neutral, the gross return of the asset ($e^{\rho\Delta}$) should be equal to the inverse of the discount rate ($e^{-\rho\Delta}$). The worker chooses consumption $C_t$ and labor $N_t$ to maximize the utility function (3), taking as given the wage rate $w_t$. The first-order condition of labor supply is given by

$$w_t = \frac{\psi}{1 - N_t}, \tag{4}$$

which states that optimal labor supply depends only on the wage $w_t$. The independence of the labor supply curve from consumption simplifies the system.

## 3.2 Firms

We assume that each firm produces a type of intermediate variety. Different varieties cannot substitute for each other, and firms engage in monopolistic competition. Following Dixit and

Stiglitz (1977), we assume that the final output of the economy $Y_t$ is given by

$$Y_t = \left[ \int_0^{M_t} y_{jt}^{\frac{\sigma-1}{\sigma}} dj \right]^{\frac{\sigma}{\sigma-1}}, \tag{5}$$

where $\sigma$ is the elasticity of substitution between different varieties, $y_{jt}$ is the intermediate good of variety $j$, and $M_t$ is the number of firms (varieties) in the economy.

The demand for the intermediate good $y_{jt}$ is given by

$$y_{jt} = Y_t \left( \frac{p_{jt}}{P_t} \right)^{-\sigma} = Y_t p_{jt}^{-\sigma}, \tag{6}$$

where $p_{jt}$ is the price of the intermediate good $j$ and $P_t$ is the final price, which we normalize to 1. Then, the price is given by $p_{jt} = P_t \left( Y_t / y_{jt} \right)^{\frac{1}{\sigma}} = \left( Y_t / y_{jt} \right)^{\frac{1}{\sigma}}$, and the gross revenue is given by

$$p_{jt} y_{jt} = y_{jt}^{1-\frac{1}{\sigma}} Y_t^{\frac{1}{\sigma}} \tag{7}$$

Equation (7) indicates that the system has demand externality. A firm will take the aggregate output $Y_t$ as given and choose $y_{it}$, but the aggregation of $y_{it}$ decides the aggregate output $Y_t$. Such an externality is widely captured in the literature (see Benhabib and Farmer, 1994; Schaal and Taschereau-Dumouchel, 2015; and Kaplan and Menzio, 2016 ). It may arise from the existence of complementarities between products or from agglomeration due to urbanization (Davis et al., 2014).

The firm employs labor to produce the intermediate good. We assume that the production function is linear, which is given by

$$y_{jt} = \hat{A} n_{jt} \tag{8}$$

where $n_{jt}$ denotes the factor's input, its price is given by $w_t$, and $\hat{A}$ is productivity. The firm needs to repay its factor payment after production, so it has to borrow an intratemporal loan to pay wages. The borrowing of intratemporal loans is subject to the borrowing constraint. We assume that

$$w_t n_{jt} \leq \theta V_{jt} \tag{9}$$

where $V_{jt}$ is the firm value of the monopoly competitive firm $j$ and $\theta$ is the collateral ratio. $V_{jt}$ must satisfy the Bellman equation that

$$V_{jt} = \pi_{jt} \Delta + e^{-\rho \Delta} (1 - \delta \Delta) \mathbb{E} V_{j,t+\Delta}, \tag{10}$$

where

$$\pi_{jt} = y_{jt}^{1-\frac{1}{\sigma}} Y_t^{\frac{1}{\sigma}} - w_t \frac{y_{jt}}{\hat{A}}.$$

Here, $\delta$ is the exogenous exit rate. The first term $\pi_{jt}\Delta$ of Equation (10) denotes the net profit of firms from $t$ to $t+\Delta$. The second term of Equation (10) is the discounted value of the present value of the firm in the future.

We assume that $\theta$ is small enough that the borrowing constraint is binding. Assuming that all firms are symmetric, we then have the flow of profits given by

$$\pi_t = y_t M_t^{\frac{1}{\sigma-1}} - w_t \frac{y_t}{\hat{A}} \tag{11}$$

where we use lowercase letters without subscripts to represent the symmetric variables for each firm. Equation (11) shows that an increase in the number of firms may increase the profits of each firm through the effect of demand externalities. On the other hand, more firms will reduce the profit of firms through the labor market. In the symmetric case, Equation (4) becomes $w_t = \frac{\psi}{1-n_t M_t}$. Combining it with Equation (9), we have

$$w_t = \psi + \theta V_t M_t,$$

Then, we have the output of each firm

$$y_t = \hat{A} n_t = \frac{\hat{A}\theta V_t}{w_t} = \frac{\hat{A}\theta V_t}{\psi + \theta M_t V_t}. \tag{12}$$

Equation (12) shows that the output of each firm decreases when $M_t$ increases. The reason is that the entry of more firms raises labor demand and the equilibrium wage. The higher cost of the firms lowers the profit of each firm.

Combining Equations (10), (11), and (12), we can rewrite the Bellman equation

$$V_t = \left[ \frac{\hat{A}\theta V_t}{\psi + \theta M_t V_t} M_t^{\frac{1}{\sigma-1}} - \theta V_t \right] \Delta + e^{-\rho\Delta}(1-\delta\Delta)\mathbb{E}V_{t+\Delta} \tag{13}$$

where $\theta V_t = w_t \frac{y_t}{\hat{A}}$ is the borrowing constraint binding condition.

### 3.3 Credit Markets

To ensure that the number of products is stationary, we assume that at the beginning of each period $t$, some entrepreneurs have developed ideas for new products. We assume that there is a total of $\delta\mu\Delta$ new ideas invented by the $\delta\mu\Delta$ entrepreneur. To bring the blueprint to production, these entrepreneurs need to build a project that requires one unit of investment. The entrepreneur has no net worth. To finance the project, entrepreneurs borrow from the competitive financial market at a gross interest rate $R_t$. After the initial investment is made, the entrepreneur is hit by an adverse shock $z$ drawn from the cumulative distribution $G(z)$.

If the entrepreneur exerts effort, the investment is successful, and this project allows the owner to produce a new type of product in each period as long as the product is not obsolete. Hence, she obtains

$$v_t^N(z) = -R_t - z_t + e^{-\rho\Delta}(1-\delta\Delta)\mathbb{E}_t V_{t+\Delta}.$$

The first term $R_t$ is her debt payment to the lender. In the case of success, how the entrepreneur repays the lender does not matter. Entrepreneurs can repay their debt at once by selling the project and obtain $e^{-\rho\Delta}(1-\delta\Delta)\mathbb{E}_t V_{t+\Delta}$. Alternatively, entrepreneurs can specify a flow of debt payments. In that case, the present value of these debt payments will be equal to $R_t$.

If the entrepreneur does not exert effort, the project is liquidated, which yields a liquidation value of $a$ to the lender and a value of 0 to the entrepreneur. This then implies that the entrepreneur will exert effort if and only if $z_t \leq Z_t^*$, which is defined by

$$Z_t^* = e^{-\rho\Delta}(1-\delta\Delta)\mathbb{E}_t V_{t+\Delta} - R_t. \tag{14}$$

Competition in the financial market requires that

$$R_t G(Z_t^*) + a[1 - G(Z_t^*)] = 1$$

where the first part is the return from entrepreneurs who pay their loans while the second term is the return from the entrepreneurs who default. Then, the payment $R_t$ is given by

$$R_t = \frac{1 - a[1 - G(Z_t^*)]}{G(Z_t^*)}. \tag{15}$$

The evolution of $M_t$ is given by

$$M_{t+\Delta} = M_t(1-\delta\Delta) + \delta\mu\Delta G(Z_t^*). \tag{16}$$

$M_{t+\Delta}$ is composed of two parts: a $1-\delta\Delta$ fraction of current products in the next period that are still viable and a $G(Z_t^*)$ fraction of newly invented products that are successfully developed by the entrepreneurs.

### 3.4 Summary of the System

As $\Delta \to 0$, equations (13) to (16) are in continuous time. The system can be expressed as two differential equations, which are summarized in Proposition 1

**Proposition 1.** *In continuous time, the dynamical system on $(M_t, V_t)$ is given by*

$$\dot{V}_t = (\rho + \delta + \theta) V_t [1 - F(V_t, M_t)], \tag{17}$$

$$\dot{M}_t = -\delta M_t + \mu \delta G(Z(V_t)), \tag{18}$$

*where*

$$F(V_t, M_t) = \frac{\hat{A}\theta}{(\psi + \theta M_t V_t)(\rho + \delta + \theta)} M_t^{\frac{1}{\sigma - 1}}$$

*and $Z(\cdot)$ solves*

$$Z(V) + \frac{1 - a[1 - G(Z(V))]}{G(Z(V))} = V. \tag{19}$$

*where Equation (19) comes from the Equations (14) and (15).*

Note that given $V$, there exist multiple $Z$ that solve Equation (19). Only a higher $Z$ is reasonable because firms choose a lower interest rate when both cutoffs satisfy the zero profit condition of financial institutions. In proposition 1, we reduce the entire economic system into two key variables, the value of a firm and the number of firms in the economy. The variable $M_t$ is the state variable, and Equation (18) describes the motion of $M_t$. We can see that a high value of the firm leads more new firms to enter the economy but not default. The variable $V_t$ is the control variable, and Equation (17) describes the motion of $V_t$, which indicates that a large $M_t$ has a scale effect and may increase the profit and value of the firm. The interaction of the two variables gives the economic system a complex character, which we will analyze in the next sections. The two-dimensional dynamical system is convenient for us to mathematically analyze the steady states and dynamical properties. Moreover, the system also creates the condition for us to utilize the Bogdanov-Takens bifurcation theorem for the analysis of global dynamics.

## 4 Steady States and Local Dynamics

This section studies the steady states and local dynamics of the system. There exists a scale effect in the model, so the presence of more varieties increases the value of each firm. In turn, a higher firm value attracts more firms to continue operations but not default after the liquidity shock and enter the economy. Thus, the system may have multiple equilibria due to these mutually reinforcing mechanisms. Generally, a smooth system with multiple steady states has one saddle point, while the other is a sink point or source point, depending on the parameters. In this section, we use $Z_0$ to denote the steady state of variable $Z_t$.

## 4.1 Steady States

To make the analysis more convenient, we use the following parameter transformations for the remainder of the paper. Let $A = \frac{\hat{A}}{\rho+\delta+\theta}$ and $\kappa = \frac{\rho+\theta}{\delta} + 1$. We use the new parameters $A$ and $\kappa$ to replace the old $\hat{A}$ and $\rho$ in the parameter set. Then, the system becomes

$$\dot{V}_t = \kappa\delta V_t[1 - F(V_t, M_t)] \tag{20}$$

$$\dot{M}_t = -\delta[M_t - \mu G(Z(V_t))] \tag{21}$$

where

$$F(V_t, M_t) = \frac{A\theta}{\psi + \theta M_t V_t} M_t^{\frac{1}{\sigma-1}}$$

and $Z(\cdot)$ solves

$$Z(V) + \frac{1 - a[1 - G(Z(V))]}{G(Z(V))} = V$$

Denote $(M_0, V_0)$ as a steady state. The pair $(M_0, V_0)$ should satisfy $F(V_0, M_0) = 1$ and $M_0 = \mu G(Z(V_0))$. Before analyzing the number of steady states, we should verify the assumption that the credit constraint binds, which is equivalent to the condition that the marginal return is greater than the marginal cost

$$(1 - \frac{1}{\sigma})M_t^{\frac{1}{\sigma-1}} - \frac{w_t}{A(\rho + \delta + \theta)} > 0,$$

In any steady state, we have

$$\frac{A\theta}{\psi + \theta M_0 V_0} M_0^{\frac{1}{\sigma-1}} = \frac{A\theta}{w_0} M_0^{\frac{1}{\sigma-1}} = 1.$$

Therefore, we only require that

$$\frac{\rho + \delta}{\rho + \delta + \theta} = 1 - \frac{\theta}{\kappa\delta} > \frac{1}{\sigma} \tag{22}$$

We have the following proposition 2 to show the conditions that the credit constraint binds.

**Proposition 2.** *Given $\sigma$, $\kappa$ and $\theta$, the steady states satisfy the binding condition if $\delta > \frac{\theta\sigma}{(\sigma-1)\kappa}$.*

*Proof.* See Appendix A for details. ∎

In Proposition 2, we find that whether the credit constraint binds is not related to the steady state $(M_0, V_0)$ but depends only on the parameters. We can choose a large enough $\delta$ to ensure that the credit constraint binds. The question is whether the parameter $\delta$ influences the steady states or dynamics of the system in turn. From Equations (20) and (21), we can see that varying $\delta$ causes the motion speed of $V_t$ and $M_t$ to change by the same proportion, but the trajectories will not change. Furthermore, the function $F(V_t, M_t)$ and $G(Z(V_t))$ does not contain $\delta$ or $\kappa$, so the steady states are independent of these two parameters. We thus obtain Proposition 3.

**Proposition 3.** *Given other parameters $\sigma$, $\theta$, $\mu$, $\psi$, $A$, $a$ and the distribution $G(Z)$, the steady states in the system are independent of parameters $\kappa$ and $\delta$. The motion trajectory of the system is independent of $\delta$.*

Proposition 3 allows us to choose a suitable $\delta$ to satisfy the binding condition without worrying about the influence on the steady states or the paths in the phase diagram. In the rest of the paper, we always assume that $\delta$ is large enough that the credit constraint binds. Moreover, Proposition 3 also demonstrates the independence of the steady states of parameter $\kappa$, which allows us to analyze the dynamics of the system by varying $\kappa$ while fixing the steady states. This property also simplifies our later analysis of local and global dynamics.

To facilitate the mathematical analysis, we make common assumptions about the distribution function of the liquidity shock $G(Z)$ and its density function $g(Z)$

**Assumption 1.** *The distribution function $G(Z)$ and density function $g(z)$ satisfy*
*(1) $G(Z)$ is continuous and strictly increasing between $(Z_{\min}, +\infty)$.*
*(2) $g(Z)/G(Z)$ is strictly decreasing.*
*(3) $G(Z) < 1$ for any $Z \in (Z_{\min}, +\infty)$.*
*(4) $\lim_{Z\to+\infty} G(Z) = 1$ and $G(Z_{\min}) = 0$.*

The above assumptions are not very restrictive. A number of common distribution functions satisfy Assumption 1. It is very easy to verify that both the Pareto and exponential distributions satisfy the above properties. This assumption ensures that the distribution function is concave and smooth in $[Z_{\min}, \infty]$. We do not have to worry that $G_t$ reaches the upper limit 1 when $V_t$ is very high.

Before we analyze the existence and number of steady states, we need to discuss the relationship between $V^*$ and $Z^*$. Equation (19) and its differential are

$$V^* = Z^* + \frac{1-a}{G(Z^*)} + a$$

$$\frac{dV^*}{dZ^*} = 1 - \frac{1-a}{G(Z^*)}\frac{g(Z^*)}{G(Z^*)}$$

$dV^*/dZ^*$ is not always larger than zero, which implies that $V^*$ is not monotonically increasing with $Z^*$ and that a single $V^*$ may correspond to multiple $Z^*$ values. According to Assumption 1, $dV^*/dZ^*$ is increasing monotonically from $-\infty$ to $+\infty$. There exists $z_m$ such that $dV^*/dZ^* = 0$ and $V^*(z_m)$ take the minimum value.

Therefore, it seems that any $V^* > V^*(z_m)$ may correspond to two $Z^*$ values. However, only the higher value, which is larger than $z_m$, is reasonable. Given a $V_t$, suppose that both $Z_t^1$ and $Z_t^2$ solve the zero-profit condition of financial institutions. Recall Equation (15): the interest rate is $R_t = (1-a)/G(Z_t^*) + a$ . We have $R_t^1 > R_t^2$. In a competitive financial market, financial institutions have to offer a cheaper interest rate when both interest rates make the profit zero. The lower $Z^*$ (the higher $R$) is not an equilibrium because the firms can optimally choose which cutoff to maximize the value of the firm. The economic meaning of $V^*(z_m)$ is the minimum firm value that supports the credit market. When the value of the firm falls below this threshold, no interest rate allows the return of the financial institution to cover its cost.

When we restrict $Z > z_m$, we also have the lower bound number of firms $M_m = \mu G(z_m)$ because $G(Z)$ is a strictly increasing function. Moreover, as $G(Z) < 1$, we have the upper-bound number of firms $\mu$. Then, in steady state, $M_0$ lies in $[M_m, \mu)$. The number of steady states in the system depends on the number of intersections of the curve $F(V_0, M_0) = 1$ and $M_0 = \mu G(Z(V_0))$ when $M \in [M_m, +\mu)$. Multiple equilibria may exist when the parameters satisfy certain conditions, which is given by Proposition 4

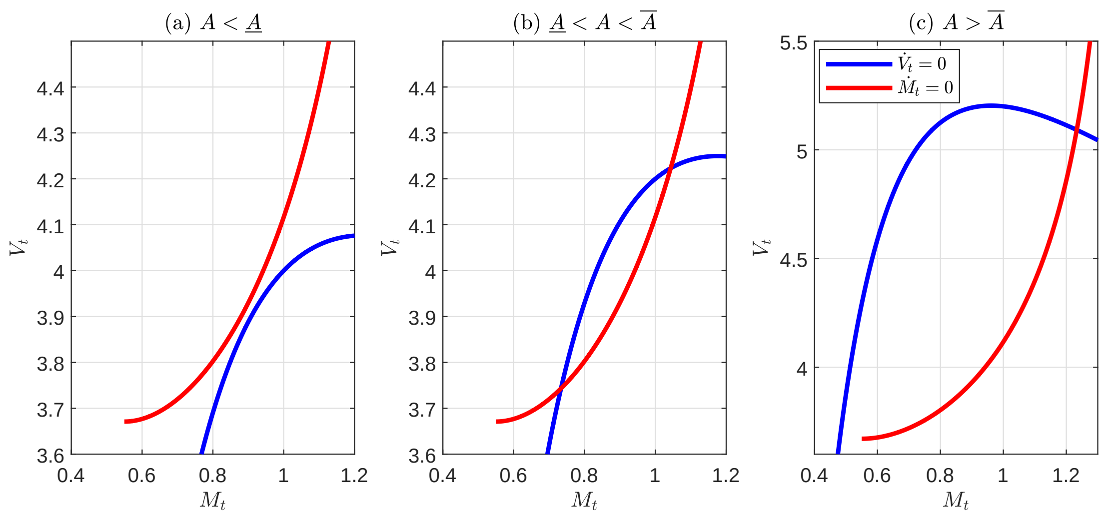
**Proposition 4.** *Given other parameters $\delta$, $\kappa$, $\theta$, $\mu$, $\psi$ and $\sigma < \overline{\sigma}$, there exists $\underline{A}$ and $\overline{A}$ such that the two-dimensional system has multiple equilibria if $\underline{A} < A < \overline{A}$, where $\underline{A}$, $\overline{A}$ and $\overline{\sigma}$ are given by Equations (A.3), (A.4) and (A.5), respectively.*

*Proof.* See Appendix A for details. ∎

Proposition 4 shows that there may exist multiple equilibria in the dynamical system. The mechanism behind this is that firm production has a positive externality on other firms; thus, the system has scale effect. When there are more operating firms (variety of products), the scale effect between different products increases the profits of each firm and the value of the firm. In turn, the increase in firm value reduces the probability of corporate default. This mutually reinforcing cycle leads to multiple equilibria in the economy.

If we treat the productivity $A$ as an exogenous shock, we can consider the number of steady states to be state dependent. Economic fundamentals are better when $A$ is particularly high and there is only one steady state with a low default rate. Conversely, when economic fundamentals are worse, there may be two steady states, one with a high default rate and the other with a low default rate. This characterization is consistent with the real economy. When the economy

Figure 4: Number of steady states and the value $A$




**Note**: Panel (a) shows the case in which $A < \underline{A}$, Panel (b) shows the case in which $\underline{A} < A < \overline{A}$, and panel (c) shows the case in which $A > \overline{A}$.

is in a recession, the financial system can be unstable. Firms are more likely to simultaneously form pessimistic expectations and choose default coordination.

Proposition 4 also implies that a necessary condition for multiple equilibria is that $\sigma$ cannot be too large. When $\sigma$ decreases, the externality becomes stronger, and the default decision of the individual firm depends more on the conditions of the aggregate economy. When the externality is strong, the revenue of entrepreneurs is affected more by aggregate output; hence, they will be more likely to default when expecting more other entrepreneurs to default. Therefore, there can be another equilibrium with a high default rate. Note that $\sigma$ is not very restrictive. According to Equation (A.5), when $\mu$ is small enough, $\overline{\sigma}$ can be any large value.

We illustrate the range of $A$ in Figure 4. The red line refers to $\dot{M}_t = 0$, while the blue line refers to $\dot{V}_t = 0$. The red line increases monotonically from $M_m$, implying that as the value of firms increases, more firms choose to enter the market, and the number of firms increases. The blue line, on the other hand, rises and then falls. There are two opposing factors behind this. On the one hand, there is an externality and scale effect of the rising number of firms, which increases the value of all firms, and on the other hand, the rising number of firms makes the labor market tighter, which increases the costs of firms. Therefore, the blue line does not increase monotonically. Panel (a) shows the case of $A < \underline{A}$. In this scenario, productivity is too low, and therefore more firms are exiting than entering. Therefore, the system cannot reach any steady state. Panel (b) shows the case of $\underline{A} < A < \overline{A}$. In this scenario, there are multiple steady states in the system. Panel (c) shows the other case of $A > \overline{A}$, where only a single steady state exits. Productivity is too high in this case, and the value of the firms is very high despite that the number of the firms is relatively low. The high-default equilibrium cannot exist.

## 4.2 Local Dynamics

This section discusses the local dynamics around the two steady states when they exist. We explore the possibility of a continuum of equilibria under local indeterminacy around the steady state. The mechanism generating indeterminacy in our model results from increasing returns to scale, which may give rise to locally indeterminate steady states. The global dynamics will be explored in the next section.

First, we transform the system into logarithmic form. In a steady state $(M_0, V_0)$, the values of $M_0, V_0$ satisfy that $F(V_0, M_0) = 1$ and $M_0 = \mu G(Z(V_0))$. Then, Equations (17) and (18) can be expressed as

$$\dot{v}_t = \delta\kappa[1 - f(v_t, m_t)] \tag{23}$$

$$\dot{m}_t = -\delta[1 - g(v_t)\exp(-m_t)] \tag{24}$$

where $m_t = \log(M_t/M_0)$ and $v_t = \log(V_t/V_0)$ and

$$f(v_t, m_t) = F(V_0 \exp(v_t), M_0 \exp(m_t))/F(V_0, M_0)$$

$$g(v_t) = G(Z(V_0 \exp(v_t)))/G(Z(V_0))$$

In the next section discussing global dynamics, we will also use this transformation. By the Taylor approximation, we can linearize the system around the steady state $v_t = m_t = 0$ as

$$\begin{pmatrix} \dot{v}_t \\ \dot{m}_t \end{pmatrix} \approx J \begin{pmatrix} v_t \\ m_t \end{pmatrix}$$

where

$$J = \delta \begin{bmatrix} -\kappa f_1 & -\kappa f_2 \\ g_1 & -1 \end{bmatrix}$$

Local dynamics around steady state depend on the determinant and trace of Jacobian matrix $J$. Note that given other parameters, the sign of the determinant is the same as $f_1 + f_2 g_1$, which is only dependent on the steady state. In the Appendix, we prove that the sign of the determinant is decided by the pattern of the curves $\dot{V}_t = 0$ and $\dot{M}_t = 0$. If the first curve crosses the second curve downward, then the determinant is smaller than zero and vice versa. Moreover, the sign of the trace depends on $-\kappa f_1 - 1$, which is highly dependent on parameter

$\kappa$ when the steady states are fixed. Note that $f_1$ can be expressed as

$$f_1 = -\frac{\theta M_0 V_0}{\psi + \theta M_0 V_0}$$

in the steady state, which is independent of parameter $\kappa$ according to Proposition 3. The local dynamics around the steady states can be summarized as follows

**Proposition 5.** *When there exist two steady states, the good steady state $(M_0^g, V_0^g)$ (low default rate) is a saddle point. For the bad steady state $(M_0^b, V_0^b)$ (high default rate), if $1 < \kappa < 1 + \frac{\psi}{\theta M_0^b V_0^b}$, it is a sink point with one order of indeterminacy. If $\kappa > 1 + \frac{\psi}{\theta M_0^b V_0^b}$, it is a source point.*

*Proof.* See Appendix A for details. ∎

Proposition 5 describes that the good steady state (low default rate) is always a saddle point, which guarantees that there exists a unique path that converges to that steady state given a state $M_t$ close to $M_0^g$.

Furthermore, the other steady state (high default rate) is a sink point or a source point. When $\kappa > 1 + \frac{\psi}{\theta M^b V^b}$, the trace of the Jacobian matrix is larger than zero, and thus the bad steady state (high default rate) is a source point. It is an unstable equilibrium. Suppose that the economy stays at $(M_0^b, V_0^b)$ and there is a shock to the system; then, the economy will deviate and diverge away from the bad steady state. When $1 < \kappa < 1 + \frac{\psi}{\theta M^b V^b}$, the trace is smaller than zero, and the bad steady state is a sink point. Given $M_t$ close to the bad steady state, there exist infinite trajectories converging to the bad steady state. Recall that $V_t$ is a control variable and $M_t$ is a state variable. The initial $V_t$ cannot be uniquely determined in this case. The bad steady state exhibits local indeterminacy. We define the critical value $\kappa_{Hopf} = 1 + \frac{\psi}{\theta M^b V^b}$ because it is related to Hopf bifurcation, which will be further explained in Section 5.

Note that when there is local indeterminacy, we can introduce the sentiment shock in the system. Recall Equation (13)

$$V_t = \left[(\rho + \delta + \theta)F(V_t, M_t) - \theta\right] V_t \Delta + e^{-r\Delta}(1 - \delta\Delta)\mathbb{E}V_{t+\Delta},$$

As $V_t$ is a control variable and is determined based on expectations, indeterminacy allows us to rewrite the system as

$$V_{t+\Delta} = \frac{V_t - \left[(\rho + \delta + \theta)F(V_t, M_t) - \theta\right] V_t \Delta}{e^{-r\Delta}(1 - \delta\Delta)} + \varepsilon_{t+\Delta},$$

where $\varepsilon_{t+\Delta}$ is an independently and identically distributed random variable with zero mean.

When $\Delta \to 0$, the system can be expressed as a stochastic process

$$dV_t = \kappa\delta V_t[1 - F(V_t, M_t)]dt + \Omega_t dz_t,$$

where $z_t$ is a standard Wiener process and $\Omega_t$ is the standard deviation. Note that the above process is mean-reserved and does not diverge. The presence of sentiment shock gives the economy higher volatility in the high-default regime.

# 5 Global Dynamics

In this section, we move from local dynamics around steady states to global dynamics, which reveals a richer set of not only multiple equilibria but also of periodic equilibria and stable cycles not detectable by local analysis around steady states. We show that global indeterminacy exists in this economic system even when the steady states are locally determinate. The dynamic system of our model exhibits stable periodic trajectories for parameter values that are in empirically relevant regions. As discussed in the introduction, such cycles can help explain empirically observed clustered defaults and may provide insights into endogenous business cycle dynamics explored in Beaudry et al. (2020).

The term local determinacy refers to the unique convergent trajectory when the initial state variable ($M_t$) is close to the steady state. However, even if a system is locally determined, it can also present global indeterminacy. For example, a source-saddle system is considered determined locally. However, there also exist multiple trajectories that start close to the source point and converge to the saddle point. Moreover, the two-dimensional system can also generate periodic orbits under certain circumstances, which indicate that business cycles can emerge endogenously. The global method can cover such issues, while the local method cannot.

Analyzing global dynamics is not an easy task and requires mathematical tools for bifurcation. Following economic applications in Benhabib et al. (2001) and Sniekers (2018), we utilize the Bogdanov-Takens bifurcation theorem for theoretical analysis and then illustrate different cases of indeterminacy through numerical examples.

## 5.1 Theoretical Analysis

In this section, we study the global dynamics of the system theoretically. Our analysis utilizes the theory of Bogdanov-Takens bifurcation (see Chapter 8.4 of Kuznetsov, 2004). The normal form of this theorem starts from a system $\dot{x}_t = f(x_t;\alpha)$, where $x_t \in R^2$, $\alpha \in R^2$ and the Jacobian matrix of $f(x_0;\alpha_0)$ has two zero-eigenvalues. We refer to $\alpha_0$ as the Bogdanov-Takens point. In our model, $x_t$ is $(v_t, m_t)$ and $\alpha$ is $(\hat{\kappa}, \hat{A})$, which will be defined in detail later. If some regular conditions are satisfied, the system exhibits rich dynamic patterns when the $\alpha$ parameters are close to the Bogdanov-Takens point. We first adjust the parameters $A$ and $\kappa$ to the Bogdanov-Takens point so that the system satisfies the normal form and then prove that the regular conditions are generally satisfied. Bogdanov-Takens bifurcation indicates that the system presents different kinds of indeterminacy and periodic trajectories.

Define $\underline{M}, \underline{V}$ as the steady state when $\kappa = \underline{\kappa}$, $A = \underline{A}$. $\underline{\kappa}$ and $\underline{A}$ should ensure that the two eigenvalues are zero in steady state. Define $m_t = \log(M_t/\underline{M})$, $v_t = \log(V_t/\underline{V})$, $\hat{\kappa} = \kappa/\underline{\kappa}$, $\hat{A} = A/\underline{A}$ and

$$f(v_t, m_t) = F(\underline{V}\exp(v_t), \underline{M}\exp(m_t))/F(\underline{V}, \underline{M})$$

$$g(v_t) = G(Z(\underline{V}\exp(v_t)))/G(Z(\underline{V}))$$

To ensure that $(\underline{M}, \underline{V})$ is the steady state, we require that $F(\underline{V}, \underline{M}) = 1$ and $\underline{M} = \mu G(Z(\underline{V}))$. Then, the system can be transformed as

$$\dot{v}_t = \hat{\kappa}\delta\underline{\kappa}[1 - \hat{A}f(v_t, m_t)] \tag{25}$$

$$\dot{m}_t = -\delta[1 - g(v_t)\exp(-m_t)] \tag{26}$$

The Jacobian matrix at $v_t = m_t = 0$, $\hat{\kappa} = \hat{A} = 1$ is

$$J = \delta \begin{bmatrix} -\underline{\kappa}f_1 & -\underline{\kappa}f_2 \\ g_1 & -1 \end{bmatrix},$$

To ensure that the two eigenvalues are zero, we have

$$\underline{\kappa}f_1 + 1 = 0,$$

$$f_1 + f_2 g_1 = 0$$

We impose the condition that $\underline{\kappa} = -1/f_1$ so that the first equation is satisfied, and the value of $\underline{A}$ has already made the second equation satisfied due to the unique steady state. The Bogdanov-Takens bifurcation theorem allows us to analyze the global dynamics when the parameters $\kappa$ and $A$ are close to the Bogdanov-Takens point $(\underline{\kappa}, \underline{A})$. The main result of the analysis is that in this economy, given a starting state variable $M_t$, there may exist infinite or multiple equilibrium trajectories originating with different $V_t$ that converge to the low default steady state or high-default steady state. Sometimes, the system can also exhibit periodic orbits. All these trajectories satisfy the transversality conditions and other equilibrium conditions. We formalize this result in the following proposition.

**Proposition 6.** *For parameter specifications $(\kappa, A)$ sufficiently close to $(\underline{\kappa}, \underline{A})$, given an $A$ such that there exist two steady states, the economy exhibits various types of indeterminacy depending on parameter $\kappa$:*

*(a) There exists a $\kappa_{SL}$ under which the system has saddle-loop bifurcation if $\kappa = \kappa_{SL}$. There exists a homoclinic orbit that connects the saddle point to itself.*

*(b) If $\kappa < \min(\kappa_{Hopf}, \kappa_{SL})$, the economy is a saddle-sink system without periodic orbits. Given an initial $M_t$ close to the bad steady state $M_0^b$, there exist infinite equilibrium trajectories converging to the bad steady state $(M_0^b, V_0^b)$.*

*(c) If $\kappa$ is between $\kappa_{Hopf}$ and $\kappa_{SL}$, the system has periodic orbits. The cycle is stable if $S < 0$ or unstable if $S > 0$ where $S$ is given by (A.13).*

*(d) If $\kappa > \max(\kappa_{Hopf}, \kappa_{SL})$, the economy is a saddle-source system without periodic orbits. Given an initial $M_t$ close to the bad steady state $M_0^b$, there exist multiple equilibrium trajectories that converge to the good steady state $(M_0^g, V_0^g)$.*

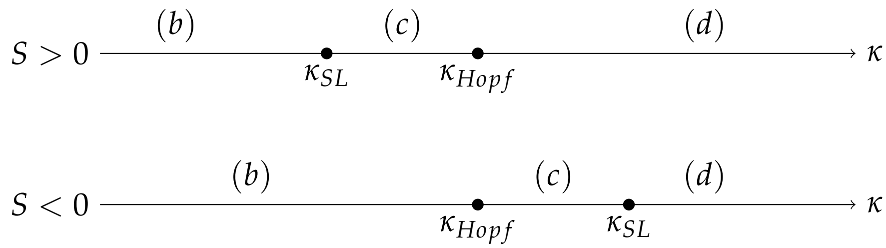
*Proof.* See Appendix A for details. ∎

Proposition 6 shows that the system has abundant dynamic patterns depending on the parameter $\kappa$. Recall that $\kappa$ will not influence the steady states but only the motion of the system. The relationship between the bifurcation and parameter $\kappa$ can be further illustrated by Figure 5. There are two critical values $\kappa_{Hopf}$ and $\kappa_{SL}$. The first $\kappa_{Hopf}$ decides whether the bad steady state is a sink point or a source point. When $\kappa$ goes through $\kappa_{Hopf}$, Hopf bifurcation exists. The stability of the bad steady state changes, and the system exhibits a family of periodic orbits on one side. The second $\kappa_{SL}$ determines the other boundary of the limit cycle. The largest possible cycle around the bad steady state is that passing the good steady state, which is called a homoclinic orbit. The trajectory diverging from the saddle point will also converge to the saddle point. Periodic orbits can emerge if and only if the parameter $\kappa$ is between $\kappa_{Hopf}$ and $\kappa_{SL}$. In the case of $S > 0$, $\kappa_{Hopf} > \kappa_{SL}$, which indicates that the periodic orbit encompasses a sink point and the periodic orbit is unstable. Furthermore, if $S < 0$, $\kappa_{Hopf} < \kappa_{SL}$, which indicates that the periodic orbit encompasses a source point and the periodic orbit is stable. [4]

The various dynamic patterns of the above cases will be described in detail in the next section with numerical examples and diagrams. We will also explain the economic implications behind the trajectories.

[4] Imagine that when the trajectory deviates the cycle inward, the trajectory will return to the orbit if the center is a source but move to the center if it is a sink.

Figure 5: Relationship between bifurcations and the parameter $\kappa$.




**Note**: This figure shows the relationship between bifurcations and parameter $\kappa$. $\kappa_{Hopf}$ is the critical point between the source and the sink point of the bad steady state. $\kappa_{SL}$ is the value where the system presents saddle-loop bifurcation. (b) and (d) are the intervals of saddle-sink and saddle-source systems, respectively, while (c) is the interval where a stable ($S < 0$) or unstable ($S > 0$) periodic orbit emerges.

## 5.2 Numerical Examples

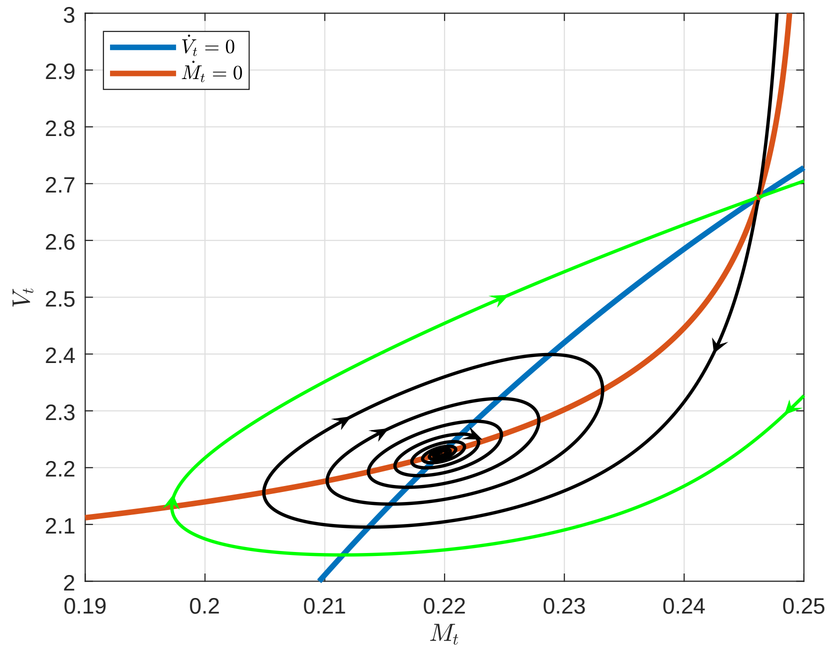
In this section, we illustrate the global dynamics of the system with numerical examples. We assume that the distribution of $G(Z)$ follows the Pareto distribution.

$$G(Z) = 1 - (\frac{Z}{Z_{\min}})^{-\eta}$$

Let a period equal one quarter. We choose $\sigma = 6$ as in Christiano et al. (2014). With the historical median PE and PS of the SP500 at 15 and 1.55, respectively, we can calculate that the ratio of the quarterly cost to market value $\theta$ is approximately 0.145. We normalize $\psi = 1$ and $\mu = 0.25$ to indicate that there is one unit of potential projects each year in the economy. We take $a = 0.95$, which reflects a 5% liquid cost. The default rate is very low in the real world. We choose a combination of $A = 10$, $\eta = 6.5$, and $Z_{\min} = 0.88$. Then, $\underline{A} < A < \overline{A}$, and there are two steady states. Moreover, the low default rate is 1.5%, and the high default rate is between 12% and 13%. The remaining parameter is $\kappa$. From Proposition 3, we know that $\kappa$ does not influence the steady states but only the motion of the system. Under this parameterization, $S$ in Proposition 6 is larger than zero. We change $\kappa$ from a low value to a high value.

If $\kappa = 12$, we have case (b) of Proposition 6. Figure 6 shows that the low steady state is a sink point, while the other steady state is a saddle point. Given an initial $M_0$, equilibrium trajectories satisfying the differential equations and the transversality conditions are infinite. The initial $V_0$ corresponding to $M_0$ can be in the area surrounded by the green saddle path. In this case, the economy would converge to a bad steady state with a high default rate. Note that when the system approaches the bad steady state, the variables within the system converge in oscillations toward the steady state, resulting in endogenous fluctuations. Additionally, given an initial $M_0$, the corresponding $V_0$ could potentially reside on the green saddle path, and the

Figure 6: Convergence to the bad steady state



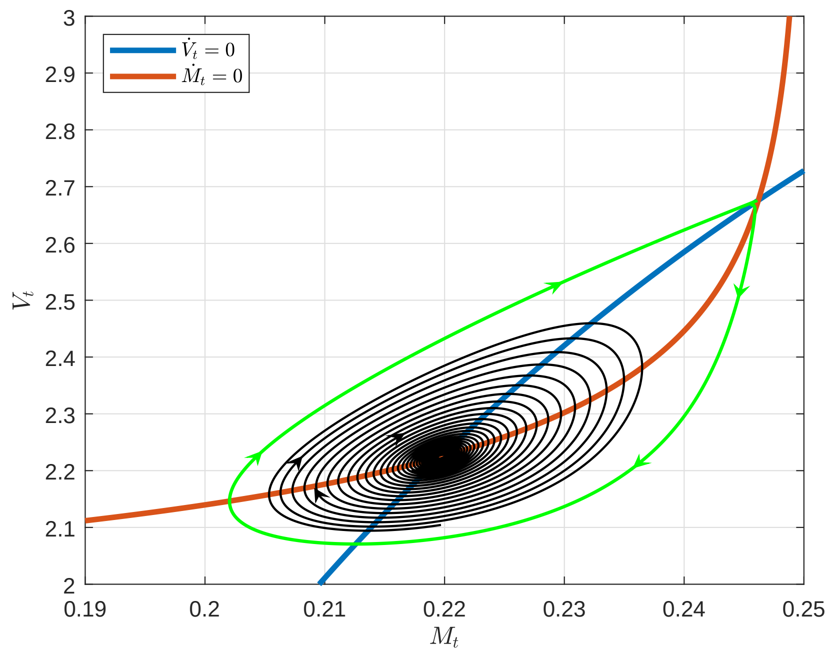

**Note**: We choose $\psi = 1$, $A = 10$, $\sigma = 6$, $\theta = 0.145$, $\eta = 6.5$, $a = 0.95$, $\mu = 0.25$, $Z_{\text{min}} = 0.88$, and $\kappa = 12$. The green line is the saddle path, and the black line is the path that diverges from the saddle point and converges to the sink point.

economy would converge to the good steady state (exhibiting a low default rate) along this path. Note that there are multiple trajectories in this scenario. For example, if $M_t = 0.23$, there are two possible choices for $V_t$ on the saddle path, specifically 2.09 and 3.55. Thus, the system remains indeterminate even if the endpoint is a saddle point. This abundant indeterminacy implies that when a specific number of companies exist in the economy, households and companies may shape an array of rational expectations. These expectations affect the current value of companies and the movement of the economy. Ultimately, they become self-fulfilling prophecies. Indeterminacy exists in both the endpoint and trajectory choices.

When parameter $\kappa$ increases, there exists a critical point where the periodic orbit emerges and the system displays saddle-loop bifurcation. We find that when $\kappa = \kappa_{SL} \approx 14.25$, the system presents saddle-loop bifurcation as in Case (a) of Proposition 6. There is a unique homoclinic orbit that diverges from the saddle point and then converges to itself. Moreover, any trajectories starting from a point within the loop will end up in the bad steady state. Note that the emergence of saddle-loop bifurcation requires that $\kappa$ be exactly equal to $\kappa_{SL}$, the measure of which is zero in the parameter space.

We display the periodic orbit in Case (c) of Proposition 6 by increasing $\kappa$ to 14.9. The bad steady state is a sink point, whereas the other steady state is a saddle point. Figure 8 shows

Figure 7: Saddle-loop bifurcation and homoclinic orbit

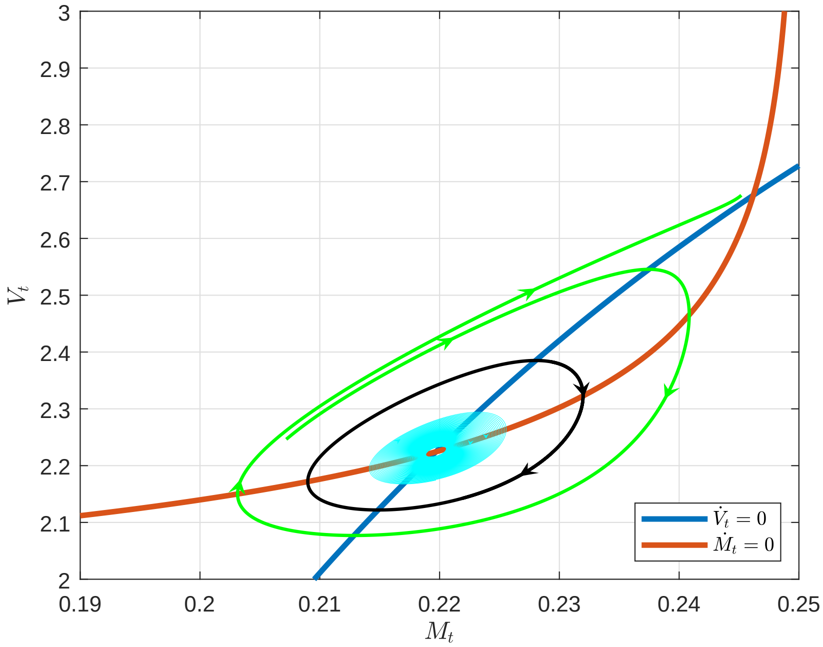



**Note**: We choose $\psi = 1$, $A = 10$, $\sigma = 6$, $\theta = 0.145$, $\eta = 6.5$, $a = 0.95$, $\mu = 0.25$, $Z_{\min} = 0.88$, and $\kappa \approx 14.25$. The green line is the unique homoclinic orbit. The black line converges to the bad steady state.

that when $M_t$ is initially close to the bad steady state, there are two corresponding $V_t$ values that make the economy periodically move around the bad steady state. This pattern indicates endogenous cycles in the economy, in which economic variables oscillate constantly without decay. Note that the cycle is not a stable periodic orbit since $S > 0$ under these parameters. However, the cycle remains a viable option since $V_t$ functions as a control variable, similar to the saddle path. Any value of $V_t$ that falls within the area surrounded by the cycle will spiral toward the bad steady state. Furthermore, the cycle is also accompanied by a saddle path that diverges from it, converging to the saddle point. In summary, there is abundant indeterminacy in the economy. The emergence of periodic orbits has important economic implications. It reveals that the economy can generate endogenous fluctuations in a perfect foresight equilibrium, without external shocks or internal sentiment shocks. This gives us a new perspective to understand business and financial cycles. Furthermore, the finding mirrors the view in Beaudry et al. (2020) that business cycles largely arise because some internal forces tend to endogenously generate cyclical mechanisms. Note that limit cycles can only appear in a global analysis and cannot be constructed in a local analysis, which is one of the purposes of the global analysis chosen for this paper.

If $\kappa$ increases to 16, Case (d) is obtained. The two steady states consist of a source point and

Figure 8: Periodic orbit

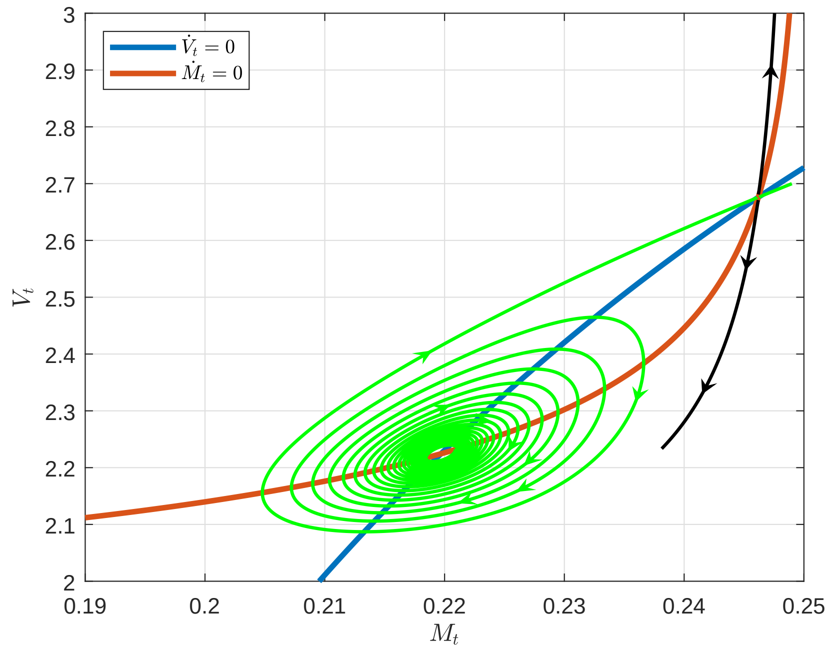



**Note**: We take $\psi = 1$, $A = 10$, $\sigma = 6$, $\theta = 0.145$, $\eta = 6.5$, $a = 0.95$, $\mu = 0.25$, $Z_{\text{min}} = 0.88$, and $\kappa = 14.9$. The black line is the unique periodic orbit. The green line diverges from the periodic orbit and converges to the good steady state. The blue line converges to the sink point.

a saddle point, as illustrated in Figure 9. The system diverges from the bad steady state due to its source nature. Additionally, there exists a saddle path that diverges from the source point and converges to the saddle point. To ensure that the transversality condition is satisfied, it is essential that all trajectories are on the saddle path. Note that there is still indeterminacy in this scenario. When $M_t$ starts near the lower steady state, various options for $V_t$ enable the system to reach the good steady state via a saddle path.

In this section, we explore uncertainty in the economy both theoretically and numerically. Our system has two steady states, one of which is a saddle point, and the other can be either a sink or a source point. We find that in both cases, the system is subject to indeterminacy. Given an initial state variable, there is more than one path that meets all the equilibrium conditions, including the transversality condition. More interesting, the system exhibits periodic paths when the parameters lie in certain intervals, suggesting that the economy may have endogenous cycles.

Figure 9: Divergence from the bad steady state




**Note**: We take $\psi = 1$, $A = 10$, $\sigma = 6$, $\theta = 0.145$, $\eta = 6.5$, $a = 0.95$, $\mu = 0.25$, $Z_{\text{min}} = 0.88$, and $\kappa = 16$. The green line is the path that diverges from the bad steady state and converges to the good steady state. The black line is the path that diverges from the saddle point.

## 6 Clustered Default Rates

In Section 2, we document the cluster pattern of default rates. The series have two regimes, a high default rate regime and a low default rate regime. The economy switches between the two regimes. When the system enters the high default regime, the default rate rises sharply and thus presents a cluster pattern. Our model can replicate the same pattern numerically. Suppose that the economy has two stationary points, and the bad steady state is a sink point, while the good steady state is a saddle point. The default rate will increase much higher than the normal level when the system diverges from the saddle point and slides to the sink point. The high default period ends when the economy returns to the saddle path. We construct the Markov solutions of our model to replicate the cluster pattern in this section. In the first exercise, we show that a negative fundamental shock combined with pessimistic sentiment may lead to a surging default rate. In the second exercise, we show that when productivity is low, a change in sentiment can also present clustering default cycles.

### 6.1 Exogenous Productivity Shock

In Section 3, we show that there are two steady states when productivity is low, while there is only one good steady state when productivity is high. We assume that there are two productivity rates, $A^h$ and $A^l$. Productivity follows a Markov process. When productivity falls from high to low, the bad steady state and sink path emerge, and there exist multiple equilibria. We call the saddle path with high productivity the good state, the saddle path with low productivity the bad state, and the sink path with low productivity the ugly state. The probabilities of shifting from good to bad, bad to good, good to ugly and ugly to good are $\pi_{GB}dt$, $\pi_{BG}dt$, $\pi_{GU}dt$ and $\pi_{UG}dt$ during $t$ to $t + dt$, respectively. For simplicity, we assume that there is no possibility that the system jumps from ugly to bad. The motion of the model can be written as

$$\dot{V}_t^G = \kappa\delta V_t^G[1 - F(V_t^G, M_t; A^h)] - \pi_{GB}[V_B(M_t) - V_t^G] - \pi_{GU}[V_U(M_t) - V_t^G]$$

$$\dot{V}_t^B = \kappa\delta V_t^B[1 - F(V_t^B, M_t; A^l)] - \pi_{BG}[V_G(M_t) - V_t^B]$$

$$\dot{V}_t^U = \kappa\delta V_t^U[1 - F(V_t^U, M_t; A^l)] - \pi_{UG}[V_G(M_t) - V_t^U]$$

$$\dot{M}_t = -\delta[M_t - \mu G(Z(V_t))]$$

where $V_G(M_t)$ and $V_B(M_t)$ should be the saddle paths in the good and bad states. $V_U(M_t)$ is not unique for indeterminacy.

Let a period equal one quarter. We assume that $V_U(M_t) = (1 - x)V_B(M_t)$. Denote $\pi_{GB} = \kappa\delta p_{GB}$, $\pi_{BG} = \kappa\delta p_{BG}$, $\pi_{GU} = \kappa\delta p_{GU}$, and $\pi_{UG} = \kappa\delta p_{UG}$. We take $\delta = 0.06$ so that approximately 20% of the startups die in the first year, close to the data in Sedláček and Sterk (2017). We take $\kappa = 3.5$ so that the risk-free interest rate is 2% every year. We normalize $\psi = 1$, and $\mu = 0.25$ to indicate that there is one unit of potential projects each year in the economy. We choose $p_{BG} = p_{UG} = 0.2$, $p_{GU} = 0.3$ and $p_{GB} = 0.15$. In this way, the average duration of the ugly time is approximately 6 years and roughly 45% of periods are ugly, which is consistent with our regime switching estimation. We assume that the value drops $x = 1\%$ when the system changes to ugly compared to bad. We still choose $\theta = 0.145$ to match the PE and PS ratio as in Section 5. To ensure that the default rate in good times is low enough, we choose a high enough $\eta = 15$. We take $A^h = 10$ and $A^l = 9.34$ so that the system has one steady state when productivity is high but two steady states when productivity is low. We take $a = 0.9$ to assume that the liquidity cost is 10%. We take $Z_{\min} = 0.2$ so that the highest default rate is approximately 12% as during the Great Depression (Giesecke et al., 2011).

Figure 10 replicates the cluster pattern. When the economy is in high productivity (good), the default rate is very low, close to zero. Moreover, the system has only one state with a low

Table 3: Parameter values

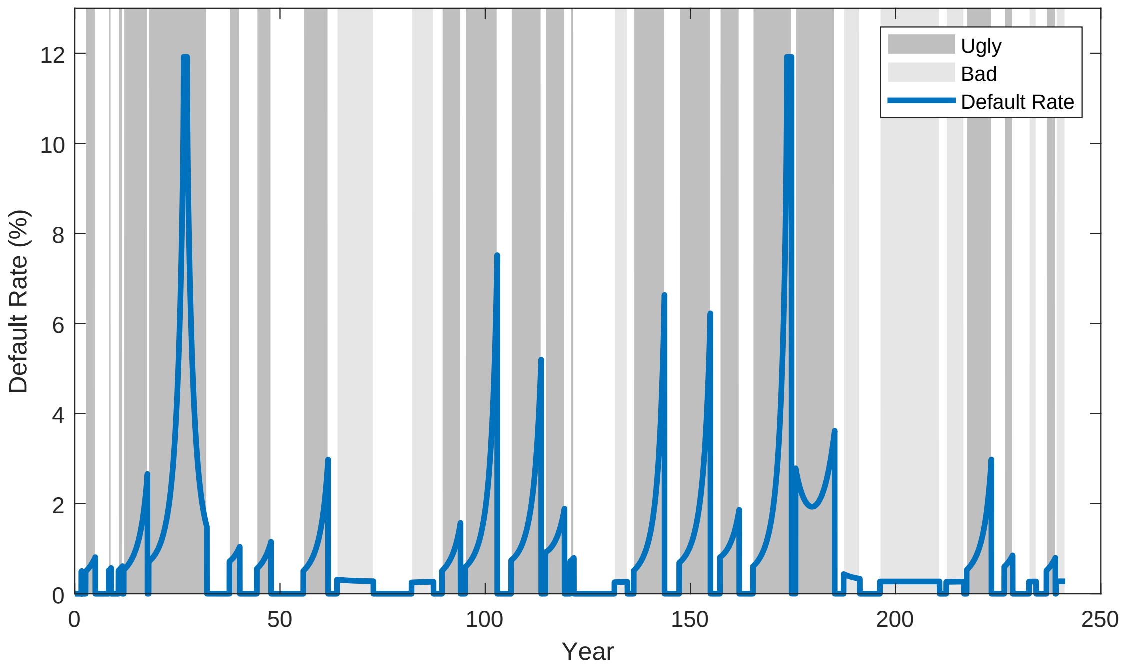
| | | |
|---|---|---|
| $\sigma$ | elasticity of substitution | 6 |
| $\delta$ | exit rate | 0.06 |
| $\rho$ | riskless interest rate | 0.005 |
| $\theta$ | collateral rate | 0.145 |
| $\kappa$ | combination of $\delta$, $\theta$, and $\rho$ | 3.5 |
| $p_{BG}$ | probability parameter from bad to good | 0.2 |
| $p_{UG}$ | probability parameter from ugly to good | 0.2 |
| $p_{GU}$ | probability parameter from good to ugly | 0.3 |
| $p_{GB}$ | probability parameter from good to bad | 0.15 |
| $x$ | value discount in the ugly state | 1% |
| $A^h$ | high productivity | 10 |
| $A^l$ | low productivity | 9.34 |
| $a$ | recovery rate | 0.9 |
| $\eta$ | shape parameter in the Pareto distribution | 15 |
| $Z_{\min}$ | the lowest liquidity shock | 0.2 |
| $\psi$ | weight of leisure in utility function | 1 |
| $\mu$ | number parameter of potential projects | 0.25 |

**Note**: Parameter $\kappa$ equals $1 + (\theta + \rho)/\delta$. The real probability of moving from state $i$ to $j$ is $\pi_{ij} = \kappa\delta p_{ij}$.

default rate. Sentiment has no impact on the equilibrium choice. As productivity changes from high to low, the system becomes unstable. It is possible to fall into two types of equilibria. If firms maintain optimistic sentiment, the economy remains in a saddle path equilibrium, defaults will be modestly elevated, and the overall financial system will remain robust (bad). However, if firms have more pessimistic expectations, it is possible to fall into an equilibrium path toward the sink point (ugly). Here, the default rate will rise well above the normal level until productivity returns to the high state. Our stochastically simulated time series of default rates have the cluster characteristics documented by Giesecke et al. (2011). In contrast, if we never allow the system to go to the ugly state and reset $p_{GB} = 0.45$ and $p_{GU} = 0$, we have Figure 11. The default rate remains at a very low level, and there is no cluster pattern.

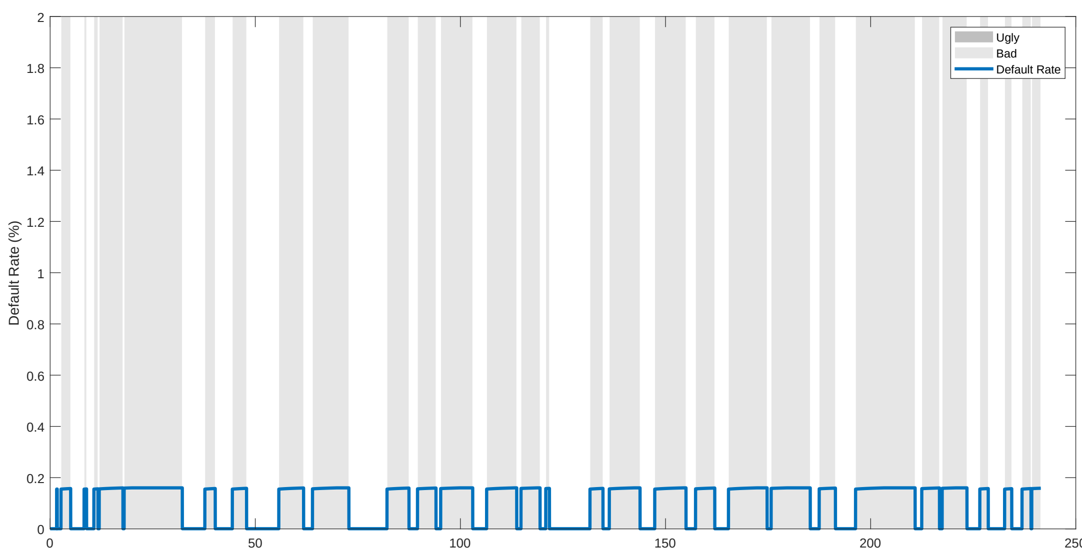
In our framework, the combination of negative fundamental shocks and changes in the expectations of economic agents leads to a significant increase in default rates. Both factors are important. This is consistent with the fact that clustered defaults occur frequently when the economy suffers a negative shock, while is also consistent with the argument in Azizpour et al. (2018), Das et al. (2007) and Duffie et al. (2009) that clustered defaults cannot be purely explained by observable macroeconomic factors.

Figure 10: Default pattern: good, bad, and ugly



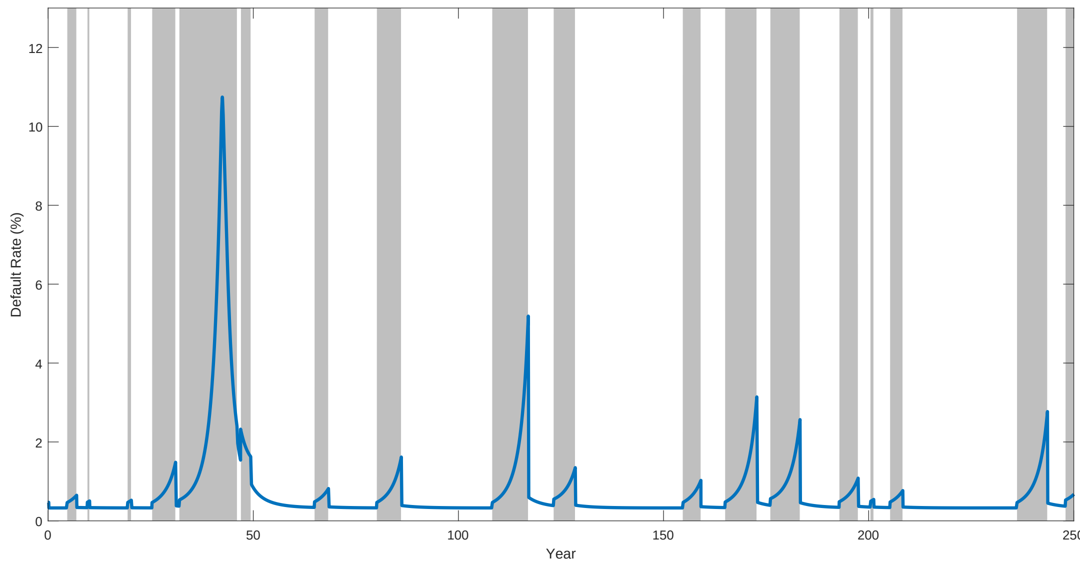

**Note**: Parameter values are summarized in Table 3.

Figure 11: Default pattern: only good and bad




**Note**: Parameter values are summarized in Table 3 except that $p_{GB} = 0.45$, and the ugly state is deleted.

Figure 12: Default pattern: pure sentiment




**Note**: Parameter values are summarized in Table 3 except that $p_{BU} = 0.16$, $p_{UB} = 0.2$, $x = 0.5\%$ and $A = 9.53$. The good state is deleted.

## 6.2 Purely Sentiment-Driven Cycles

In this section, we examine the changes in default rates that occur when only sentiment shocks are present. We fix productivity to a relatively low level. The system can switch between bad and ugly states driven by sentiment.

$$\dot{V}_t^B = \kappa\delta V_t^B[1 - F(V_t^G, M_t)] - \pi_{BU}[V_U(M_t) - V_t^B]$$

$$\dot{V}_t^U = \kappa\delta V_t^U[1 - F(V_t^B, M_t)] - \pi_{UB}[V_B(M_t) - V_t^U]$$

$$\dot{M}_t = -\delta[M_t - \mu G(Z(V_t))]$$

where $V_B(M_t)$ should be the saddle path in the bad state. $V_U(M_t)$ is not unique for indeterminacy. We assume that $V_U(M_t) = (1-x)V_B(M_t)$. We take $x = 0.5\%$, $A = 9.53$, $p_{GU} = 0.16$ and $p_{UG} = 0.2$ so that the durations are consistent with our regime switching estimation.

The result is shown in Figure 12. We find that when productivity is consistently low, the economy itself may move on two equilibrium paths in response to changes in sentiment. When entrepreneurs suddenly develop pessimistic expectations, firm value falls, and the whole system moves toward a steady state with a high default rate and the default rate also increases. When sentiment turns optimistic, the default rate falls to the normal level.

The results of the simulations differ somewhat from those reported in Figure 10. In Figure

12, default rates may spike when regime switches, but in most cases, they rise only slightly. Furthermore, the rise in default rates can no longer be characterized as countercyclical, which is a deviation from the truth. Therefore, we argue that the observed default cluster is the result of a combination of fundamental and sentiment shocks.

## 7 Policy Implications

In this section, we discuss some measures to enhance welfare. When assumption 1 is met, the system may enter a bad steady state with a high default rate. Our policies focus on eliminating this steady state. The main idea of this section is to provide subsidies to firms that continue their projects without defaulting. This measure ensures that a project can continue even in cases where the firm value $V_t$ is not very high and the number of firms $M_t$ is insufficient.

We consider a subsidy policy financed by a lump-sum tax. Recall Figure 4: when the government subsidizes firms that do not default, the curve $\dot{M}_t = 0$ moves downward because entrepreneurs are willing to continue the project with a subsidy, even if the firm value is not very high. When the curve $\dot{M}_t = 0$ falls such that it is below the other curve $\dot{V}_t = 0$ at $M_t = M_m$, there is only a good steady state in the economy. We assume that with the government transfer to firms not defaulting $tr(V_t)$, Equation (18) becomes

$$\dot{M}_t = -\delta M_t + \mu\delta G(Z(V_t + tr(V_t))),$$

then the curve $\dot{M}_t$ will move downward. We assume that $tr(V_t) = \tau(V^G - V_t)$ where $V^G$ is the value when the default rate is high. $\tau$ is positive, so the subsidy is countercyclical. When the economy is in normal condition, the government does not need to provide additional subsidies. When the economy deviates from normal conditions and the value of firms declines, the government needs to subsidize firms that choose not to default. The curve $\dot{M}_t = 0$ presents a linear transformation, and its convexity will not change. To diminish the bad steady state, we only require that the curve $\dot{M}_t = 0$ is higher than the curve $\dot{V}_t = 0$ at $M_m = \mu G(z_m)$. We formalize this result by the following proposition.

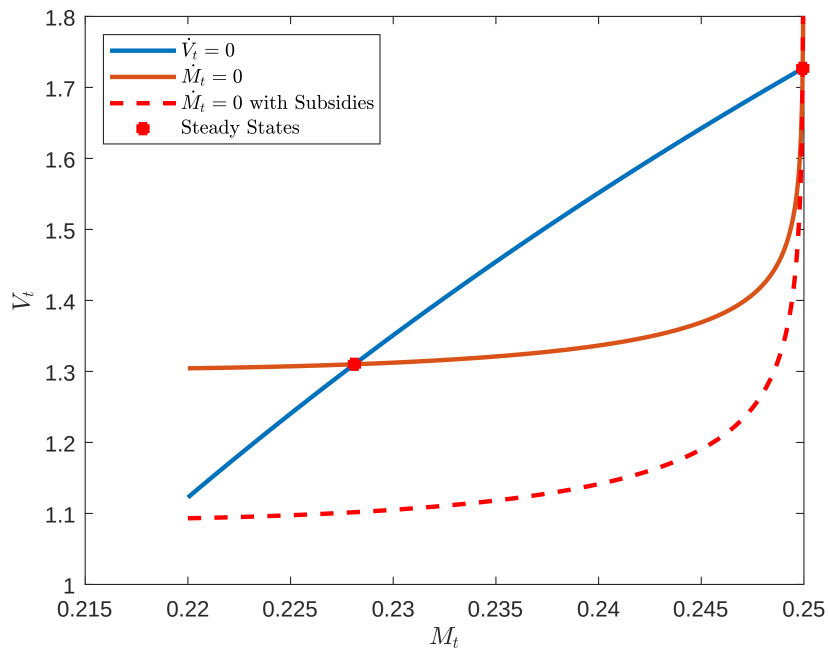
**Proposition 7.** *Suppose that the economy has two steady states. If the government transfers* $\tau > [z_m + \frac{1-a}{G(z_m)} + a - AM_m^{\frac{2-\sigma}{\sigma-1}} + \frac{\psi}{\theta M_m}]/[V^G - AM_m^{\frac{2-\sigma}{\sigma-1}} + \frac{\psi}{\theta M_m}]$ *to firms with successful entry, only a good steady state remains.*

*Proof.* See Appendix A for details. ∎

This proposition describes that if the government subsidy is large enough, the bad steady state diminishes. The subsidy increases the liquidity shock threshold of default $Z_t^*$. In this economy, firm production has a positive externality on other firms, and thus a scale effect exists. When there are fewer firms, the scale effect between different products decreases the profits of each firm and the value of the firm. In turn, it further reduces the probability of corporate default. This loop leads to multiple equilibria in the economy. However, the government subsidy breaks the loop. Even if the number of firms in the economy $M_t$ is relatively small and the firms are less valuable, there are enough new firms entering the economy, and the bad steady state is eliminated.

We provide an example to illustrate how the subsidy works. We use the static model and take the parameters in Section 6 but $A = 9.67$ so that there are two steady states without the subsidy. We subsidize the entrants with the rule that $tr(V_t) = \tau(V^G - V_t)$ where $\tau = 1/3$, the entry threshold is lowered. The $\dot{M}_t = 0$ curve moves downward, and there remains only one steady state.

Figure 13: Subsidize the entrants




**Note**: We take $\sigma = 6$, $\theta = 0.145$, $\eta = 15$, $a = 0.9$, $A = 9.67$, $\psi = 1$ $\mu = 0.25$, $Z_{\min} = 0.2$ and $\kappa = 3.5$. The solid line is the original $\dot{V}_t = 0$ and $\dot{M}_t = 0$, while the new $\dot{M}_t = 0$ curve is when $\tau = 1/3$. The figure shows that when there is a subsidy for non-defaulting firms, the bad steady state may diminish.

## 8 Conclusion

In this paper, we construct a model that features monopolistic competition to examine limited commitment and default. Our model captures the positive feedback loop between high output and a low default rate. In particular, entrepreneurs with potential projects can borrow for investment, but projects may face liquidity shocks before production. Entrepreneurs must choose whether to continue with the project or to default. An aggregate output reduction leads to corresponding declines in individual firms' revenue and profit. This reduces the value of the firm and increases the probability of default in the face of a liquidity shock. As the defaults of more firms impede their production, aggregate output further declines.

The positive feedback loop creates multiple equilibria and self-fulfilling default rates. The system displays abundant dynamic properties. We employ the Bogdanov-Takens bifurcation theorem to demonstrate global indeterminacy. When considering the initial state variable, there might be various trajectories that converge toward different steady states. Even if the economy is determined locally, it can also present global indeterminacy. Specifically, a source-saddle system is considered to be determined locally. However, there exist multiple trajectories that start close to the source point and converge to the saddle point. Moreover, the system also features periodic solutions in perfect expected equilibrium. The emergence of endogenous periodic orbits is in line with the discoveries of Beaudry et al. (2020).

Moreover, our Markov simulation shows that the model can simulate the characteristics of the default cluster that exist in reality, as documented by Giesecke et al. (2011). When productivity suffers a negative shock, the system may have two stationary points. When the shock is combined with a change in sentiment, the economy will diverge from the low default state and enter the high default state. The default rate increases dramatically. Our simulation shows that the features of the clustered default rate can be generated only in the case of global dynamics, and the fluctuation of the default rate will be much smaller in the local dynamics. This finding aligns with the conclusions drawn in Azizpour et al. (2018), Duffie et al. (2009) and Das et al. (2007), which suggest that clustered defaults cannot be solely attributed to observable macroeconomic factors. In terms of policy implications, our results suggest that governments can financially support entrepreneurs without defaulting, thus eradicating the high default equilibrium.

To ensure the tractability and clarity of our model, we have abstracted from the primary features of limit commitment and increasing returns to scale. Our framework has the potential for extension to cover many aspects. In future studies, researchers can analyze an endogenous recovery rate and introduce nominal rigidity to address the implications of monetary policy for debt dilution, thus mitigating default loss and volatility.

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# Appendix

## A Proofs

### Proof of Proposition 2

Each time, a firm wants to maximize its profit

$$\pi_{jt} = y_{jt}^{1-\frac{1}{\sigma}} Y_t^{\frac{1}{\sigma}} - w_t \frac{y_{jt}}{\hat{A}},$$

subject to the constraint that

$$w_t n_{jt} = w_t \frac{y_{jt}}{\hat{A}} \leq \theta V_{jt}.$$

Then the Lagrange equation is

$$\mathcal{L} = y_{jt}^{1-\frac{1}{\sigma}} Y_t^{\frac{1}{\sigma}} - w_t \frac{y_{jt}}{\hat{A}} + \lambda(\theta V_{jt} - w_t \frac{y_{jt}}{\hat{A}}),$$

and the first order condition for $y_{jt}$ is

$$(1 - \frac{1}{\sigma}) y_{jt}^{-\frac{1}{\sigma}} Y_t^{\frac{1}{\sigma}} - \frac{w_t}{\hat{A}} - \lambda \frac{w_t}{\hat{A}} = 0.$$

As long as $\lambda > 0$, we have the constraint $w_t n_{jt} \leq \theta V_{jt}$ binding. Therefore, the sufficient condition is

$$(1 - \frac{1}{\sigma}) y_{jt}^{-\frac{1}{\sigma}} Y_t^{\frac{1}{\sigma}} - \frac{w_t}{\hat{A}} > 0. \tag{A.1}$$

In a symmetric equilibrium, we have $y_{jt} = y_t$ and $Y_t = M_t^{\frac{\sigma}{\sigma-1}} y_t$. Meanwhile, $\hat{A} = A(\delta + \rho + \theta)$. Equation (A.1) can be written as

$$(1 - \frac{1}{\sigma}) M_t^{\frac{1}{\sigma-1}} - \frac{w_t}{A(\rho + \delta + \theta)} > 0,$$

In any steady state, we have

$$\frac{A\theta}{\psi + \theta M_0 V_0} M_0^{\frac{1}{\sigma-1}} = \frac{A\theta}{w_0} M_0^{\frac{1}{\sigma-1}} = 1.$$

Therefore, we only require that

$$\frac{\rho + \delta}{\rho + \delta + \theta} = 1 - \frac{\theta}{\kappa\delta} > \frac{1}{\sigma} \Rightarrow \delta > \frac{\sigma\kappa}{\theta(\sigma - 1)}.$$

## Proof of Proposition 3

It is easy to obtain from Equations (20) and (21).

## Proof of Proposition 4

We first consider equation $M_0 = \mu G(Z(V_0))$. Denote $M_m = \mu G(z_m)$ and $V_1(M_0) = G^{-1}(Z^{-1}(\frac{M_0}{\mu}))$. Note that $V_1(M_0)$ is well defined in $[M_m, \mu)$. $V_1(M_0)$ is monotonically increasing, and $\lim_{M_0 \to \mu} V_1(M_0) = \infty$ due to assumption 1.

Equation $F(V_0, M_0) = 1$ implies that

$$V_0 = V_2(M_0) = AM_0^{\frac{2-\sigma}{\sigma-1}} - \frac{\psi}{\theta M_0} \tag{A.2}$$

When $M = \mu$, $V_2(\mu) < V_1(\mu) = \infty$. If we can construct $V_2(M_m) < V_1(M_m)$ and $V_2(M_1) > V_1(M_1)$ for an $M_1 \in (M_m, \mu)$, then there exist multiple equilibria due to the intermediate theorem. To satisfy the above conditions, we have

$$AM_1^{\frac{2-\sigma}{\sigma-1}} - \frac{\psi}{\theta M_1} > V_1(M) \text{ for a } M_1 \in (M_m, \mu)$$

and

$$AM_m^{\frac{2-\sigma}{\sigma-1}} - \frac{\psi}{\theta M_m} > V_1(M_m)$$

Therefore, we have

$$A > \underline{A} = \inf_{M \in (M_m, \mu)} \frac{V_1(M)M + \psi/\theta}{M^{\frac{1}{\sigma-1}}} \tag{A.3}$$

$$A < \overline{A} = \frac{V_1(M_m)M_m + \psi/\theta}{M_m^{\frac{1}{\sigma-1}}} \tag{A.4}$$

Denote

$$P(M) = \frac{V_1(M)M + \psi/\theta}{M^{\frac{1}{\sigma-1}}} = V_1(M)M^{\frac{\sigma-2}{\sigma-1}} + (\psi/\theta)M^{-\frac{1}{\sigma-1}}$$

We need the difference in $P(M)$ to be smaller than 0 on $M_m$, so that $\underline{A} < \overline{A}$ is satisfied.

$$P'(M_m) = \frac{\sigma-2}{\sigma-1}V_1(M_m)M_m^{-\frac{1}{\sigma-1}} - \frac{1}{\sigma-1}(\psi/\theta)M_m^{-\frac{\sigma}{\sigma-1}}$$

Therefore, $P'(M_m) < 0$ requires that

$$\sigma < \overline{\sigma} = \frac{\psi/\theta}{M_m V_1(M_m)} + 2 = \frac{\psi/\theta}{\mu G(z_m)\left(z_m + \frac{1-a}{G(z_m)} + a\right)} + 2 \tag{A.5}$$

## Proof of Proposition 5

The local dynamics depend on the trace and determinant of the Jacobian matrix

$$J = \delta \begin{bmatrix} -\kappa f_1 & -\kappa f_2 \\ g_1 & -1 \end{bmatrix}$$

The scale of $\delta$ does not influence the signs of the trace and determinant. Without loss of generality, we assume that $\delta = 1$. We have $tr(J) = -1 - \kappa f_1$ and $det(J) = \kappa(f_1 + f_2 g_1)$.

In the good steady state, the slope $dv/dm$ of the curve $f(v_t, m_t) = 1$ is smaller than that of $g(v_t)\exp(-m_t) = 1$. In the other steady state, the relationship is different. For $f(v_t, m_t) = 1$, we have $f_1 dv_t + f_2 dm_t = 0$, and thus $\frac{dv_t}{dm_t} = -\frac{f_2}{f_1}$. For $g(v_t)\exp(-m_t) = 1$, we have $g'(v_t)\exp(-m_t)dv_t - g(v_t)\exp(-m_t)dm_t = 0$, and thus $\frac{dv_t}{dm_t} = \frac{1}{g_1}$. Thus, in the good steady state, we have $-\frac{f_2}{f_1} < \frac{1}{g_1}$, and thus $det(J) = \kappa(f_1 + f_2 g_1) < 0$ (note that $f_1 < 0$ and $\kappa > 1$). In this way, the steady state with a high value is a saddle point. Similarly, $det(J) > 0$ in the other steady state. Thus, the bad steady state is a source or sink point.

Note that

$$f_1 = -\frac{A\theta}{(\psi + \theta M_0 V_0)^2} M_0^{\frac{1}{\sigma-1}} \theta M_0 V_0$$

and

$$tr(J) = \frac{\theta M_0 V_0}{\psi + \theta M_0 V_0}\kappa - 1$$

For the stationary equilibrium with a lower $(M_0, V_0)$, if $1 < \kappa < 1 + \frac{\psi}{\theta M_0 V_0}$, it is a sink point. If $\kappa > 1 + \frac{\psi}{\theta M_0 V_0}$, it is a source point.

## Proof of Proposition 6

To prove this proposition, we first prove the four regular conditions in the normal form of Bogdanov-Takens bifurcation, which indicates that when $(\mu, \kappa)$ are in the vicinity of $(\overline{\mu}, \overline{\kappa})$, the system can present all patterns in Figure 8.8 of Kuznetsov (2004) if $S < 0$ or Figure 7.3.1 of Guckenheimer and Holmes (1983) if $S > 0$. Then, we will discuss how the patterns change with the parameters. We first show the normal-form representation of Bogdanov-Takens bifurcation (Theorem 8.4 of Kuznetsov, 2004)

**Lemma 1.** *Suppose that a planar system*

$$\dot{x} = f(x, \alpha), \quad x \in R^2, \quad \alpha \in R^2,$$

*with smooth $f$, has at $\alpha = 0$ the equilibrium $x = 0$ with a double zero eigenvalue. Via a Taylor series* *expansion around $x = 0$ and transformation of variables, this system can be expressed as*

$$\begin{aligned}\dot{y}_1 =& y_2 + a_{00}(\alpha) + a_{10}(\alpha)y_1 + a_{01}(\alpha)y_2 \\ &+ \frac{1}{2}a_{20}(\alpha)y_1^2 + a_{11}(\alpha)y_1y_2 + \frac{1}{2}a_{02}(\alpha)y_2^2 + P_1(y,\alpha) \\ \dot{y}_2 =& b_{00}(\alpha) + b_{10}(\alpha)y_1 + b_{01}(\alpha)y_2 \\ &+ \frac{1}{2}b_{20}(\alpha)y_1^2 + b_{11}(\alpha)y_1y_2 + \frac{1}{2}b_{02}(\alpha)y_2^2 + P_2(y,\alpha),\end{aligned}$$

*where $a_{kk}(\alpha), b_{lk}(\alpha)$ and $P_{1,2}(y,\alpha) = O(\|y\|)^3$ are smooth functions of their arguments. Assume that*

$$a_{00}(0) = a_{10}(0) = a_{01}(0) = b_{00}(0) = b_{10}(0) = b_{01}(0) = 0$$

*and that the following nondegeneracy conditions are satisfied:*
*(BT.0) the Jacobian matrix $\frac{\partial f}{\partial x}(0,0) \neq 0$;*
*(BT.1) $a_{20}(0) + b_{11}(0) \neq 0$;*
*(BT.2) $b_{20}(0) \neq 0$;*
*(BT.3) the map*

$$(x,\alpha) \mapsto \left(f(x,\alpha), \text{tr}\left(\frac{\partial f(x,\alpha)}{\partial x}\right), \det\left(\frac{\partial f(x,\alpha)}{\partial x}\right)\right)$$

*is regular at the point $(x,\alpha) = (0,0)$.*

*Then, there exist smooth invertible variable transformations smoothly depending on the parameters, a direction-preserving time reparameterization, and smooth invertible parameter changes, which together reduce the system to*

$$\begin{aligned}\dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \beta_1 + \beta_2\eta_1 + \eta_1^2 + s\eta_1\eta_2 + O\left(\|\eta\|^3\right),\end{aligned}$$

*where $s = \text{sign}\left[b_{20}(0)\left(a_{20}(0) + b_{11}(0)\right)\right] = \pm 1$.*

In our system, $x_t = (v_t, m_t)$ and $\alpha = (\hat{\kappa} - 1, \hat{A} - 1)$. Define $\underline{M}, \underline{V}$ and $\underline{\kappa}, \underline{A}$ so that the dynamic system is in steady state and two eigenvalues are zero at $v_t = m_t = 0$ and $\hat{\kappa} = \hat{A}$=1, where

$$\begin{aligned}m_t &= \log(M_t/\underline{M}) \\ v_t &= \log(V_t/\underline{V}) \\ \hat{\kappa} &= \kappa/\underline{\kappa} \\ \hat{A} &= A/\underline{A}\end{aligned}$$

and

$$\begin{aligned}f(v_t, m_t) &= F(\underline{V}\exp(v_t), \underline{M}\exp(m_t))/F(\underline{V}, \underline{M}) \\ g(v_t) &= G(Z(\underline{V}\exp(v_t)))/G(Z(\underline{V}))\end{aligned}$$

The values of $\underline{M}, \underline{V}, \underline{\kappa}$ and $\underline{A}$ ensure that it is the steady state, and we have

$$F(\underline{V}, \underline{M}) = 1$$

$$\underline{M} = \mu G(Z(\underline{V}))$$

Then, the system can be transformed as

$$\dot{v}_t = \hat{\kappa}\delta\underline{\kappa}[1 - \hat{A}f(v_t, m_t)] \quad \text{(A.6)}$$

$$\dot{m}_t = -\delta[1 - g(v_t)\exp(-m_t)] \quad \text{(A.7)}$$

Note that $\delta$ does not influence the diagram of the system. Without loss of generality, we assume that $\delta = 1$. The Jacobian matrix at $v_t = m_t = 0$, $\hat{\kappa} = \hat{A} = 1$ is

$$J = \begin{bmatrix} -\underline{\kappa}f_1 & -\underline{\kappa}f_2 \\ g_1 & -1 \end{bmatrix},$$

Condition (BT.0) is verified. To ensure that the two eigenvalues are zero, we have

$$\underline{\kappa}f_1 + 1 = 0,$$

$$f_1 + f_2 g_1 = 0$$

Then, Jacobian matrix becomes

$$J = \begin{bmatrix} 1 & -\frac{1}{g_1} \\ g_1 & -1 \end{bmatrix}, \quad \text{(A.8)}$$

The eigenvector is given

$$e_0 = \begin{bmatrix} 1 \\ g_1 \end{bmatrix}$$

and we compute $e_1 = [1, 0]'$ such that $Je_1 = e_0$, and then we have $J'\omega_1 = 0$, $J'\omega_0 = \omega_1$, and

$$\begin{aligned} \omega_1 &= \begin{bmatrix} 1 \\ -\frac{1}{g_1} \end{bmatrix}, \omega_0 = \begin{bmatrix} 0 \\ \frac{1}{g_1} \end{bmatrix} \\ e_0 &= \begin{bmatrix} 1 \\ g_1 \end{bmatrix}, e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

This leads to

$$\begin{aligned} < & \ e_0, \omega_0 >= 1, < e_1, \omega_1 > 1 \\ < & \ e_0, \omega_1 >= 0, < e_1, \omega_0 > 0 \end{aligned}$$

Define $y_t$ as

$$\begin{aligned}
y_{1t} &= \frac{1}{g_1} m_t \\
y_{2t} &= v_t - \frac{m_t}{g_1} \\
v_t &= y_{1t} + y_{2t} \\
m_t &= g_1 y_{1t}
\end{aligned}$$

$$\begin{aligned}
\dot{v}_t &= -\underline{\kappa}[f_1 v_t + f_2 m_t + \frac{1}{2} f_{11} v_t^2 + f_{12} m_t v_t + \frac{1}{2} f_{22} m_t^2] \\
&= y_{2t} - \underline{\kappa}[\frac{1}{2} f_{11}(y_{1t} + y_{2t})^2 + f_{12} g_1 y_{1t}(y_{1t} + y_{2t}) + \frac{1}{2} f_{22} g_1^2 y_{1t}^2] \\
&= y_{2t} - \underline{\kappa}[\frac{1}{2}(f_{11} + 2 f_{12} g_1 + f_{22} g_1^2) y_{1t}^2 + (f_{11} + f_{12} g_1) y_{1t} y_{2t} + \frac{f_{11}}{2} y_{2t}^2]
\end{aligned}$$

$$\begin{aligned}
\dot{m}_t &= \left\{ g_1 v_t - m_t + \frac{1}{2} g_{11} v_t^2 - g_1 m_t v_t + \frac{1}{2} m_t^2 \right\} \\
&= g_1 y_{2t} + \left\{ \frac{1}{2} g_{11} v_t^2 - g_1 m_t v_t + \frac{1}{2} m_t^2 \right\} \\
&= g_1 y_{2t} + \left\{ \frac{1}{2} g_{11}(y_{1t} + y_{2t})^2 - g_1^2 y_{1t}(y_{1t} + y_{2t}) + \frac{1}{2} g_1^2 y_{1t}^2 \right\} \\
&= g_1 y_{2t} + \left\{ (\frac{1}{2} g_{11} - \frac{1}{2} g_1^2) y_{1t}^2 + (g_{11} - g_1^2) y_{1t} y_{2t} + \frac{g_{11}}{2} y_{2t}^2 \right\}
\end{aligned}$$

$$\dot{y}_{1t} = y_{2t} + \frac{1}{g_1} \left\{ (\frac{1}{2} g_{11} - \frac{1}{2} g_1^2) y_{1t}^2 + (g_{11} - g_1^2) y_{1t} y_{2t} + \frac{g_{11}}{2} y_{2t}^2 \right\} \tag{A.9}$$

$$\begin{aligned}
\dot{y}_{2t} = &\tfrac{1}{f_1}[\tfrac{1}{2}(f_{11} + 2 f_{12} g_1 + f_{22} g_1^2) y_{1t}^2 + (f_{11} + f_{12} g_1) y_{1t} y_{2t} + \tfrac{f_{11}}{2} y_{2t}^2] \\
&- \tfrac{1}{g_1} \left\{ (\tfrac{1}{2} g_{11} - \tfrac{1}{2} g_1^2) y_{1t}^2 + (g_{11} - g_1^2) y_{1t} y_{2t} + \tfrac{g_{11}}{2} y_{2t}^2 \right\}
\end{aligned} \tag{A.10}$$

In this way,

$$b_{20} = \frac{1}{f_1}(f_{11} + 2 f_{12} g_1 + f_{22} g_1^2) - \frac{1}{g_1}(g_{11} - g_1^2) \tag{A.11}$$

and

$$a_{20} + b_{11} = \frac{f_{11} + f_{12} g_1}{f_1} = \frac{f_{11}}{f_1} - \frac{f_{12}}{f_2} \tag{A.12}$$

Note that when we fix the function $f(v_t, m_t)$, the other four parameters $\{\mu, a, Z_{\min}, \eta\}$ determine Equation $\dot{m}_t = -\delta[1 - g(v_t)\exp(-m_t)]$. These four parameters only need to satisfy $\underline{M} = \mu G(Z(\underline{V}))$ and $g_1(0) = -f_1/f_2$. There are two degrees of freedom to adjust $g_{11}(0)$ flexibly. From Equation (A.11), we knows that $b_{20}$ should be a constant $+ f_2/f_1 g_{11}(0)$. The measure of parameters where $b_{20} = 0$ is zero in the parameter space. We can say that $b_{20}$ is generally not zero.

For $a_{20} + b_{11}$, we have

$$f(v_t, m_t) = \frac{\underline{A}\theta}{\psi + \theta \underline{MV} \exp(m_t + v_t)} \underline{M}^{\frac{1}{\sigma-1}} \exp(\frac{1}{\sigma - 1} m_t)$$

$$f_1(v_t, m_t) = -\frac{\underline{A}\theta^2}{(\psi + \theta \underline{MV} \exp(m_t + v_t))^2} \underline{M}^{\frac{\sigma}{\sigma-1}} \underline{V} \exp(\frac{\sigma}{\sigma - 1} m_t + v_t)$$

$$a_{20} + b_{11} = \frac{f_{11}}{f_1} - \frac{f_{12}}{f_2} = 1 + 2f_1 - \frac{\frac{\sigma}{\sigma-1} + 2f_1}{\frac{1}{\sigma-1} + f_1} f_1 = 1 + \frac{\frac{2-\sigma}{\sigma-1} f_1}{\frac{1}{\sigma-1} + f_1}$$

which is generally not zero. In this way, (BT.1) and (BT.2) are verified. We define

$$S = b_{20}(a_{20} + b_{11}) \tag{A.13}$$

Now we check that the matrix $(v_t, m_t, \hat{\kappa}, \hat{A}) \rightarrow (f(x, \alpha), \det(\frac{\partial f(x,\alpha)}{\partial x}), tr(\frac{\partial f(x,\alpha)}{\partial x}))$ is regular in steady state. Note that

$$\dot{v}_t = \hat{\kappa}\underline{\kappa}[1 - \hat{A} f(v_t, m_t)]$$

$$\dot{m}_t = -[1 - g(v_t) \exp(-m_t)]$$

Then,

$$J = \begin{bmatrix} -\hat{\kappa}\hat{A}\underline{\kappa} f_1(v_t, m_t) & -\hat{\kappa}\hat{A}\underline{\kappa} f_2(v_t, m_t) \\ g_1(v_t) \exp(-m_t) & -g(v_t) \exp(-m_t) \end{bmatrix}$$

$$\det(J) = \hat{\kappa}\hat{A}\underline{\kappa}[f_1(v_t, m_t) g(v_t) \exp(-m_t) + f_2(v_t, m_t) g_1(v_t) \exp(-m_t)]$$

$$tr(J) = -\hat{\kappa}\hat{A}\underline{\kappa} f_1(v_t, m_t) - g(v_t) \exp(-m_t)$$

Then, the matrix $(v_t, m_t, \hat{\kappa}, \hat{A}) \rightarrow (f(x, \alpha), \det(\frac{\partial f(x,\alpha)}{\partial x}), tr(\frac{\partial f(x,\alpha)}{\partial x}))$ is

$$\mathcal{J} = \begin{bmatrix} -\underline{\kappa} f_1 & -\underline{\kappa} f_2 & 0 & -\underline{\kappa} \\ g_1 & -1 & 0 & 0 \\ \underline{\kappa}[f_{11} g + f_1 g_1 + f_{21} g_1 + f_2 g_{11}] & \underline{\kappa}[f_{12} g + f_{22} g_1] & 0 & 0 \\ -\underline{\kappa} f_{11} - g_1 & -\underline{\kappa} f_{12} + g & -\underline{\kappa} f_1 & -\underline{\kappa} f_1 \end{bmatrix}$$

$$\det(\mathcal{J}) = -\begin{bmatrix} 0 & -\underline{\kappa} f_2 & 1 & -\underline{\kappa} \\ 0 & -1 & g_1 & 0 \\ 0 & \underline{\kappa}[f_{12} + f_{22} g_1] & \underline{\kappa}[f_{11} + f_1 g_1 + f_{21} g_1 + f_2 g_{11}] & 0 \\ 1 & -\underline{\kappa} f_{12} + 1 & -\underline{\kappa} f_{11} - g_1 & 1 \end{bmatrix}$$

$$\det(\mathcal{J}) = \underline{\kappa}^2 [f_{12} g_1 + f_{22} g_1^2] + \underline{\kappa}^2 (f_{11} + f_1 g_1 + f_{21} g_1 + f_2 g_{11}) = -\underline{\kappa} b_{20}$$

Generally, $\det(\mathcal{J})$ is not zero according to the argument of $b_{20}$. All four conditions are verified.

Note that condition (BT.3) guarantees that $\det\left(\frac{\partial\beta}{\partial\alpha}\right) \neq 0$, which means that the continuous mapping of $(A,\kappa) \to (\beta_1,\beta_2)$ is a local homeomorphism around $(\underline{A},\underline{\kappa})$ and $\beta = (0,0)$. Parameter set $(A,\kappa)$ around $(\underline{A},\underline{\kappa})$ can present all patterns in Figure 8.8 of Kuznetsov (2004) if $S < 0$ or Figure 7.3.1 of Guckenheimer and Holmes (1983) if $S > 0$.

Now, we connect parameter $\kappa$ with the different patterns. Without loss of generality, we assume that $S < 0$. Note that there exist two steady states, and then the value of $\kappa$ only determines whether the system is in region 2, 3 or 4 of Figure 8.8 of Kuznetsov (2004). In proposition 5, we have shown that given $A$, $\kappa = \kappa_{Hopf}$ is the critical point of sink or source. Thus, when $\kappa < \kappa_{Hopf}$, the system will exhibit pattern 2. When $\kappa > \kappa_{Hopf}$, the system will exhibit pattern 3 or 4.

According to Lemma 8.7 of Kuznetsov (2004), there exists a unique smooth curve P corresponding to a saddle homoclinic bifurcation that originates at $\beta = 0$. The corresponding $\alpha = (A,\kappa)$ should also be a smooth curve that originates at $(\underline{A},\underline{\kappa})$. Note that there exist two steady states only if $A > \underline{A}$ or $\hat{A} > 1$. The corresponding $A$ changes continuously, remaining larger than $\underline{A}$. Therefore, for any $A$ sufficiently close to the right side of $\underline{A}$, we always have a corresponding $\kappa_{SL}$ that makes the system exhibit a homoclinic orbit.

Due to the smooth change, when $\kappa$ lies between $\kappa_{SL}$ and $\kappa_{Hopf}$, the system lies in region 3 and periodic orbits emerge. When $\kappa > \kappa_{SL}$, the system lies in region 4. Note that when $S < 0$, the periodic orbits are stable and $\kappa_{SL} > \kappa_{Hopf}$. Otherwise, the periodic orbits are unstable and $\kappa_{SL} < \kappa_{Hopf}$. The $S > 0$ case is similar and shown by Figure 7.3.1 of Guckenheimer and Holmes (1983)

## Proof of Proposition 7

Denote $M_m = \mu G(z_m)$, where the curve $\dot{M}_t = 0$ takes the minimum $V_m = [z_m + \frac{1-a}{G(z_m)} + a - \tau V^G]/(1-\tau)$. Moreover, the curve $\dot{V}_t = 0$ takes $V_1(M_m) = AM_m^{\frac{2-\sigma}{\sigma-1}} - \frac{\psi}{\theta M_m}$ at this point. If

$$AM_m^{\frac{2-\sigma}{\sigma-1}} - \frac{\psi}{\theta M_m} = V_1(M_m) > V_m = [z_m + \frac{1-a}{G(z_m)} + a - \tau V^G]/(1-\tau), \tag{A.14}$$

then the curve $\dot{V}_t = 0$ is higher than the curve $\dot{M}_t = 0$, and the bad steady state diminishes. We have

$$\tau > [z_m + \frac{1-a}{G(z_m)} + a - AM_m^{\frac{2-\sigma}{\sigma-1}} + \frac{\psi}{\theta M_m}]/[V^G - AM_m^{\frac{2-\sigma}{\sigma-1}} + \frac{\psi}{\theta M_m}]$$