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A MODEL OF INFLUENCER ECONOMY

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ABSTRACT

With the rise of social media and streaming platforms, firms and brand-owners increasingly depend on influencers to attract consumers, who care about both common product quality and consumer-influencer interaction. Sellers thus compete in both influencer and product markets. As outreach and distribution technologies improve, influencer payoffs and income inequality change non-monotonically. More powerful influencers sell better-quality products, but pluralism in style mitigates market concentration by effectively differentiating consumer experience. Influencer style dispersion substitutes horizontal product differentiation but serves as either complement (small dispersion) or substitute (large dispersion) to vertical product differentiation. The assortative matching between sellers and influencers remains under endogenous influence-building, with the maximal differentiation principle recovered in the limit of costless style acquisition. Meanwhile, influencers may under-invest in consumer outreach to avoid exacerbating price competition. Finally, while requiring balanced seller-influencer matching can encourage seller competition, uni-directional exclusivity can improve welfare for sufficiently differentiated products and uncrowded influencer markets.

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1 Introduction

The past decade has witnessed the rise of the influencer economy (also known as "Wang Hong economy" and more recently dubbed by the media as the "creator economy") which prominently features large-scale social media marketing (e.g., livestreaming), testimonial endorsements, and precise product placements and distributions by people and organizations who have a purported expertise or influence.[1] From a mere U.S. $1.7 billion in 2016, influencer marketing grew to a size of $9.7 billion in 2020, and was set to reach approximately $13.8 Billion in 2021, with approximately 50M creators, according to SignalFire (Geyser, 2021). In particular, thanks to its exponential growth in Internet adoption and e-commerce, there were already more than 6500 influencer-related companies in 2019 in China and the market size of industries in the influencer economy had exceeded 500 billion yuan by the end of 2021 and is expected to exceed 700 billion yuan by 2024 (iResearch, 2021). The phenomenal growth of the influencer economy is further accelerated by the recent COVID-19 pandemic (e.g., Sinha, 2021). In this new digital economy, influencers come in variety and from diverse background, and include content creators, celebrities and idols, and key opinion leaders (KOL) (Williams, 2016).[2] They manage their own fan base who are drawn to their talent, charisma, wisdom, appearance, etc., and profit by helping brand owners and service providers promote various products to potential consumers.[3]

However, the industrial organization of the influencer economy is little understood. How does technology affect the bargaining between sellers and influencers? How do influencers

[1]The influencer or creator economy generally refers to the independent businesses and side hustles launched by self-employed individuals who make money off of their knowledge, skills, or following. CB Insights (2021) provides an excellent introduction to the industry. Many influencers generate as much as seven-figure incomes. Wei Ya, once the most famous influencer, made a fortune over 1 billion CNY in live streamed pre-sales on Single's day (Nov 11, the black Friday equivalent in China) alone in 2019, as reported at https://wk.askci.com/details/a0f1a24536ab46ac9da04fd494686476/. Famous Instagram influencers like Huda Kattan or Eleonora Pons net up to 6 figures per post. The top writers on Substack can rake in as much as $1M USD annually. Youtube paid out $30B to creators in 2019-2021 (CB Insights, 2021).

[2]Content creators derive from "YouTube stars" marketed by YouTube in as early as 2011 (Lorenz, 2019). Now, it can be anyone who creates any form of content online, including TikTok videos and Clubhouse audios. For instance, on Twitch, daily users can watch live streams video games played by others via Streamlabs, and tips paid out on Twitch alone is estimated to be $141 million. Unlike live stream creators, Internet celebrities on Instagram and the like can post about or live stream special travel or dining experience, or simply routine daily lives. Many rely on physical attributes alone without actively creating content. For instance, Instagram enables brand owners to sell products through idols who attract consumers simply seeking to see them. Similarly, KOLs can target specific demographic in an interactive manner, making product sales more engaging by sharing their own thoughts and ideas.

[3]Influencers touch almost all aspects of life, including entertainment, fashion, food, movies, music, sports, etc., and increasingly utilize short videos (low cost and easy to spread). While there are many ways to monetize the influence, such as compensation for content creation or interaction with fans, influencers' largest income are still from commercials and e-commerce traffic direction.

shape product differentiation and pricing? How are influencers and brand owners matched and how to regulate the process? We answer these questions by developing a novel game-theoretic model in which sellers depend on influencers to acquire customers and compete in both the product market and influencers' labor market, with influencers potentially endogenously choosing the type and power of their influence.

Specifically, we model three important groups of agents, sellers (who are also producers), influencers, and consumers, allowing pair-wise group interactions through the product market, the influencers' labor market, and social media platforms (for influencers to connect with consumers). Sellers or brand owners depend on influencers to sell products to consumers. Consumers are uniformly located on a unit circle in a "type" space $\mathbb{R}^2$ with consumption utilities determined by both the true quality of the product and the style, status, identity, etc.—things that draw people towards influencers selling the product on social media like Instagram, or Alibaba (known as Da Ren). Agents interact in four sequential stages: (i) influencers choose type and influence power, (ii) sellers make production decisions, (iii) sellers hire and match with influencer(s) in the labor market, and finally, (iv) consumers choose which influencer to follow and consume the products the influencer promotes. We solve the model backward and discuss economic insights and predictions in each stage.

Starting with a monopolist seller to abstract from seller competition and seller-influencer matching, we first highlight the impact of general purpose technologies such as digital platforms on influencers' labor market. As such technologies governing marketing outreach improve, non-monotonicities arise in influencer payoff distribution. A cheaper technology helps influencers attract clients, benefiting both the seller and influencers when market coverage competition is still low. However, as the technology gets sufficiently cheap, the market is more saturated and the seller can leverage the fierce influencer competition to gain more bargaining power and pay less to the influencers.

We then consider the setting in which two sellers compete in both the labor market (for two influencers) and the product market. Consistent with the current industry practice, we focus on "balanced matching" or mutual exclusivity contracts in the labor market, in which each seller can hire only one influencer. Assortative matching in which sellers of better products work with more powerful influencers emerge in equilibrium under Nash bargaining, due to the complementarity between influence power and product quality. The additional dimension influencers bring to consumption utility essentially mitigate product competition and influencer competition.

We characterize the price competition equilibrium with heterogeneity in either product quality, influencer power, or influencer style. We also provide general sufficient conditions to prevent any seller from dominating the entire market under multiple-dimension heterogeneity. Sellers enjoy local monopoly power regardless of product quality when influencers' styles are sufficiently distinct. For one seller-influencer group to crowd out the rival group, the style difference between influencers needs to be suffciently small, and both the influencers' power gap and product quality gap must be sufficiently large.

We then move one stage back to endogenize the sellers' production decisions. Specifically, we investigate how influencer heterogeneity affects horizontal and vertical product differentiation, compared to traditional economies. We find it to be a substitute to horizontal product differentiation. When the influencers' style difference is small, sellers differentiate products to reduce competition; when the difference is large, sellers hire influencers and have no incentive to differentiate products because divergent influencers give sellers less elastic demand for their products. This also implies that the well-established principle of maximum differentiation (e.g., d'Aspremont et al., 1979; De Frutos et al., 1999) no longer holds.

When it comes to vertical differentiation, small style differences complement while large differences substitute, mainly due to the incentive to grab the whole market and beat the competitor. Note that when influencers' style difference increases, the return from investing in high quality also increases. When influencers' style difference is sufficiently large, both groups can break even and choose high product quality, giving a minimal vertical differentiation. When the style difference is sufficiently small, both groups choose low product quality, resulting in minimal vertical differentiation again. Only for intermediate style difference, can vertical differentiation be observed because the investment profit is only big enough to support one group investing to break even.

Next, we allow influencers to endogenize their influence. We show that socially inefficient under-investment and over-investment in influence power can arise due to: (i) that influencers ignore their positive externality on consumer welfare in an uncontested influencer market, as well as their negative externality on other influencers in a crowded influencer market, and (ii) that a big endogenous bargaining power encourages acquiring more influence. These two forces jointly determine the direction and magnitude of the sub-optimal acquisition. Meanwhile, under endogenous style selection, assortative matching between sellers and influencers ensues, with the maximum horizontal differentiation principle restored in the limit of costless style selection, given that the seller-influencer group's profit is supermodular in product quality and influencer power when influencers differ significantly in style.

Finally, to better understand the welfare implication of exclusivity contracts in this emerging industry and to guide regulatory policies geared towards balancing the power of influencers and sellers, we extend the analysis to "unbalanced matching." We find that regulations for balanced seller-influencer matching can encourage seller competition under single dimensional seller-influencer heterogeneity, whereas uni-directional exclusivity contracts are welfare-improving for sufficiently differentiated products and uncontested influencer markets.

Literature. Our study adds foremost to the emerging literature on digital platforms and the influencer or creator economy. Previous studies have focused on the relationship between influencers and platforms or Multi-channel networks (MCNs), especially the revenue sharing rules (Bhargava, 2021; Jain and Qian, 2021), disclosure by internet influencers (Mitchell, 2021), search technology and advice transparency (Fainmesser and Galeotti, 2021), influencer cartels (Hinnosaar and Hinnosaar, 2021), and firms' optimal affiliation with influencers (Pei and Mayzlin, 2019). We instead analyze seller competition and seller-influencer matching, which in turn affect product differentiation and endogenous influence acquisition.

Our study is thus related to the broad literature on marketing and industrial organization (e.g., Salop, 1979). We add by analyzing the interaction of the two in the fast-emerging influencer economy. Studies on advertising have focused on the aggregate and cross-sectional levels of advertising and its welfare implications (Becker and Murphy, 1993; Spence and Owen, 1977; Butters, 1978; Dixit and Norman, 1978; Grossman and Shapiro, 1984; Nichols, 1985; Stegeman, 1991; Nelson, 1974; Johnson and Myatt, 2006). Most models assume no media or only focus on the informational effects or nuisance costs on viewers of advertisements (e.g., Johnson, 2013). Moreover, most studies do not endogenize locations of media stations, and the ones that do (e.g., Gal-Or and Dukes, 2003; Dukes, 2004) typically take sellers' product differentiation as exogenous. We study influencers whose matching with sellers is affected by the consumer base, and analyze endogenous product differentiation and influencers' style choices simultaneously. We consider the level of advertising or its informational role in reduced-form through influence, focusing on the complementarity between the multiple dimensions of consumer utility from following influencers and consuming products.

More recently, Amaldoss and He (2010) study how firms strategically target consumers to avoid intense price competition. Several studies in marketing analyze how firms compete in the effort of hiring influencers, including advertising intensity, competitive targeting of influencers in a network, and the network structure and its influence on prices, firm profits, and consumer surplus (Galeotti and Goyal, 2009; Katona, 2018). In particular, Fainmesser and Galeotti (2021) analyze search quality, advice transparency, and influencer strategy in the market for online influence. We differ by analyzing the interaction of seller-influencer matching and product market competition. In addition, we add to the discussion on exclusivity contracting and the link between uni-directional exclusivity contracts and bargaining (e.g., Gal-Or, 1997; Dukes and Gal-Or, 2003) by contrasting uni-directional with mutual exclusivity contracts in their impact on welfare in the influencer economy.

2 Model Setup

Two risk-neutral, profit-maximizing sellers indexed by $k \in \{1, 2\}$ each sells a product of a common baseline consumption value y_k. The sellers traditionally use advertisements to market their products; in an influencer economy, they work with influencers for interactive marketing and outreach (with direct sales). We denote the kth seller's utility by U_k.

Two representative influencers each has "style" $\theta_j \in \mathbf{S}^1$, where $j \in \{1, 2\}$ indexes them and $\mathbf{S}^1 := \{s \in \mathbb{R}^2 : s_1^2 + s_2^2 = 1\}$. Style could refer to identity, fashion taste, and other things that draw people towards influencers on Instagram, Tiktok, Alibaba, etc., among the recent proliferation of social networks, digital platforms, and broadcasting channels. We denote the jth influencer's power of influence using $I_j \in \mathbb{R}_+$, which governs her outreach. Naturally, I_j is affected by contemporaneous general purpose technologies that determine the cost of marketing, which are common across influencers. Traditional advertising channels through TVs, newspapers, etc., can be viewed as having a large technology cost since the production of TV commercials, for example, entails limited airtime, expensive fee charged by celebrities, and capital-intensive outreach. In contrast, Internet-based livestreaming and social platforms dramatically reduces the outreach cost for each influencer. Heterogeneous influence power also reflects distinctions among celebrities, macro-influencers, and micro-influencers. In equilibrium, the influencer-specific attributes and the common technology jointly determine the effective consumer base.

A continuum of consumers of measure 2π are uniformly located on $\mathbf{S}^1$. They derive utilities from the quality of the goods (i.e., y) as well as from having similar styles as certain influencers.[4] A greater style affinity can be interpreted as how effectively an influencer helps

[4]See, e.g., `https://wearesocial.com/blog/2020/01/the-dawn-of-a-new-influencer-economy` for

a consumer understand, select, and enjoy the product purchased, potentially due to trust built over time or the influencer's expertise or influencer-specific product demo. $\forall x_1, x_2 \in \mathbf{S}^1$, we define $\|x_1 - x_2\|$ to be the distance along the short arc on the unit circle.

The ith consumer's utility is:

$$u_i(x_i, y) = \begin{cases} y * (1 - \|x_i - \theta\|/I) - p, & \text{if a unit good is consumed,} \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where p is the unit price charged for the consumption good. Here, brand owners and product sellers depend on influencers to sell the goods, with only consumers having $\|x_i - \theta\| \leq I$ entering the aggregate demand.

Importantly, consumer's utility depends on the style affinity $\|x_i - \theta\|$. Our specification aims to adequately capture the typical reasons for engaging celebrities or influencers in advertising campaigns: grabbing attention, persuasion through expertise, and global outreach (Moeran, 2003).[5] I reflects direct attention grabbing, either through vacuous "human pseudo-events" in the words of American historian Daniel Boorstin or through skills or performance unrelated to the products; expertise and global, cross-cultural outreach can manifest through the combination of location θ and power I.

Timeline. Influencers' type and power are the accumulation of knowledge and skills since childhood and are therefore set first. Sellers then decide on the products and subsequently hire influencers. Finally, the consumers choose which influencer to follow and consume the products offered. In Sections 3 and 4, we take the influencers' type and power, as well as the sellers' products as given, in order to focus on the sellers' hiring of influencers and influencers' impact on consumption. In Section 5, we allow the sellers to endogenize the products for sale. Our main findings are robust to having product decisions following seller-influencer matching. In practice, firms decide on their business operations before exploring marketing channels, which is what our setup captures. Finally, in Section 6, we endogenize influencers' power (potentially interpreted as skill training over the intermediate term) and

further institutional background.

[5] Advertising through conventional technology, e.g., through TV/newspaper, are often extremely costly. We are cognizant that such a cost is used to indicate the existence of a price premium that can assure contractual performance in competitive equilibrium (Klein and Leffler, 1981). However, in an influencer economy with digital platforms and the proliferation of social-commercial network apps, the cost is relatively low and would not serve such a function for disclosing the presence of a large sunk "selling" costs.

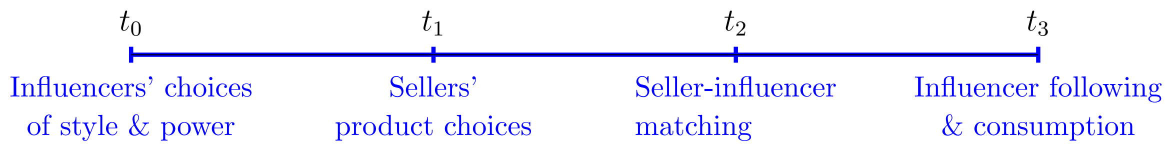

Figure 1: Timeline

type (potentially interpreted as culture, talent, or interest cultivated over the long run).

Matching and bargaining protocols. We use a general (bilateral) Nash bargaining protocol for the negotiation once influencers are hired by sellers.[6] Specifically, γ and $(1-\gamma)$ denote the bargaining power assigned to the seller and the influencer respectively. Once sellers and influencers are matched, they have exogenous options outside the match, e.g., from revisiting the influencer market, which we normalize to zero. Anticipating such bargaining processes, sellers and influencers endogenously match. Our baseline setup focuses on one-to-one match, which can be interpreted as that in practice, the seller-influencer contracts either feature mutual exclusivity clauses or they are all allowed to have multiple relationships so that the matching is balanced. This negotiation-based approach is realistic and is popular in setting advertising price in the media industry (Dukes and Gal-Or, 2003; Gal-Or, 1997).

The joint matching and bargaining problem is non-trivial. In specifying the protocols, we strive to balance tractability, transparency, convention in the literature, coherence with our non-repeated game set-up, and realism. In fact, many key results are independent on how the surplus is divided between matched sellers and influencers, as long as they care about group surplus. We discuss unbalanced matching and the welfare implications of contract exclusivity in Section 7 where sellers can require exclusive relationship and impose non-compete clauses, as seen in many nascent markets for influencers.

3 Influencer-Induced Consumption and Technology

We start by considering a monopolist seller offering a homogenous unit-consumption product, which is marketed by influencer(s) and sold to consumers. The abstraction from seller competition and seller-influencer matching in this subsection allows us to create a benchmark and illustrate the effect of technology on the influencer economy.

[6] Gal-Or (1999) and Dukes and Gal-Or (2003) discuss the advantages of this modeling approach, especially in the commercial media and healthcare industries.

We denote by $\Pi_{\mathcal{J}}$, $\mathcal{J} \subseteq \{1,2\}$, the profit when influencers in $\mathcal{J}$ are hired before advertising costs are deducted. For instance, Π_1 is the monopoly profit when only Influencer 1 is hired, and $\Pi_{\{1,2\}}$ the profit when both Influencers 1 and 2 are hired. We denote by w_j the wage for Influencer j, and by U_k the kth seller's profit. Without loss of generality, we set $I_1 \geq I_2$ and denote by $\beta := \|\theta_1 - \theta_2\|$ the style dispersion between the two influencers. Note that the seller always hires both influencers because she can always let one influencer, say i, effectively serve no clients by charging a price at y to augment her bargaining power, and share with the other influencer only the net profit increment $\Pi_{\{1,2\}} - \Pi_i$.

3.1 The Single Influencer Benchmark

We first consider the case with a single monopolist seller and a single influencer. When $I < \pi$, not all consumers are within the reach of the influencer. Fix the price p, only consumers with a non-negative utility are served, which determines the demand $D(p) = 2(1 - p/y)I$. When $I \geq \pi$, the influencer can reach all consumers. Specifically, if p is sufficiently low such that $p \leq y(1 - \pi/I)$, $D(p) = 2\pi$; otherwise, $D(p) = 2(1 - p/y)I$. We can further analyze the monopolist pricing strategy and the profit as below.

The case of a single influencer. For any given values of (I, y), the optimal pricing strategy p^* and the resulting profit Π for a monopolist seller can be derived (see Appendix A.1):

$$p^* = \begin{cases} \frac{y}{2}, & \text{if } I < 2\pi \\ y\left(1 - \frac{\pi}{I}\right), & \text{if } I \geq 2\pi \end{cases} \quad \textit{and} \quad \Pi = \begin{cases} \frac{yI}{2}, & \text{if } I < 2\pi \\ 2\pi y\left(1 - \frac{\pi}{I}\right), & \text{if } I \geq 2\pi \end{cases} \tag{2}$$

The seller's payoff and the influencer's wage are given by $U_1 = \gamma\Pi$ and $w_1 = (1-\gamma)\Pi$.

Depending on I, a monopolist seller may choose to target a subpopulation or the whole demographic of consumers. The seller only enters the market when the revenue is high enough, which limits the quantity of service. Figures 2 and 3 illustrate the potential (red) and actual (blue) consumer base and monopolist pricing in equation (2) for an influencer located at θ. The vertical axis corresponds to consumer utility with the optimal price p^* in red dashed line and consumers denoted by the thick blue line.

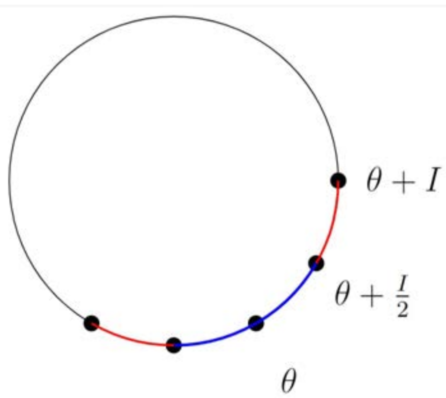

Figure 2: The circular market

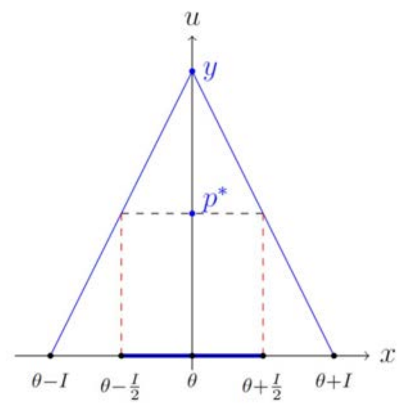

Figure 3: The monopolist pricing

3.2 Macro- and Micro-Influencers

In practice, influencers include celebrities, macro-influencers (with a couple hundred thousand followers with diverse background), micro-influencers (with 3,000-100,000 followers) and nano-influencers (with several hundred homogeneous followers), and micro-influencers are playing an increasingly important role in marketing (Fainmesser and Galeotti, 2021). We demonstrate how both macro-influencers and micro-influencers can emerge in equilibrium. Without loss of generality, suppose $\theta_1 = 0$ and that θ_2 sits to the right of θ_1 along the short arc.

Lemma 1. *When $\beta < \min\left\{2\pi - I_1 - I_2, \frac{1}{3}(I_1 - I_2)\right\}$, $\exists\ 0 < x_1^* < x_2^* < x_3^* < \pi$ such that: (i) Influencer 2 only serves consumers with $x \in (x_2^*, x_3^*)$; and (ii) Influencer 1 serves consumers with $x \in [-x_3^*, x_1^*] \cup [x_2^*, x_3^*]$.*

Here, $I_1 + I_2 < 2\pi - \beta$ rules out competition between influencers along the long arc, which is an artifact of the circular economy that offers no additional insight. $\beta < \frac{1}{3}(I_1 - I_2)$ ensures there is competition along the short arc to avoid the trivial outcome that the influencers are pure local monopolies.

Figure 4 illustrates two interesting patterns emerging from the lemma. First, there exist two style cutoffs for which the cutoff consumer is indifferent between following either influencer. Specifically, the blue solid line is Influencer 1's consumer base and the red solid line is Influencer 2's, that is, Influencer 1 attracts consumers in $[0, x_1^*]$ (i.e., sufficiently loyal to influencer 1) and $[x_2^*, x_3*]$ (i.e., far away from Influencer 1 but not served by Influencer 2). Naturally, Influencer 1 appears as a macro-influencer, and Influencer 2 as a micro-influencer.

Influencer 1 charges a monopolist price, $p_1^* = \frac{y}{2}$, to serve a broad consumer base, and

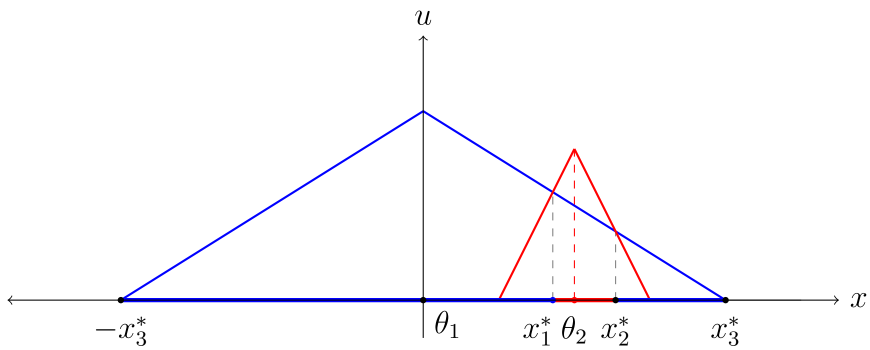

Figure 4: Macro-influencers and micro-influencers

Influencer 2 charges a higher price at $p_2^* = p_1^* + \frac{\beta}{2I_1}y$ to exploit the local affinity.[7] This is consistent with the empirical "rise of the micro-influencers" (Fainmesser and Galeotti, 2021), who remain credible in the eyes of their followers and charge more for the endorsements.

3.3 The Role of General-Purpose Technology

Next, we examine how general-purpose technologies such as digital social platforms and the Internet impact influencers' labor market. Specifically, we model technological advances as exogenous shocks which increase influence power I_j for all j. The seller's payoff and wages for influencers under bilateral Nash bargaining are:

$$w_1 = (1-\gamma)(\Pi_{\{1,2\}} - \Pi_{\{2\}}), \quad w_2 = (1-\gamma)(\Pi_{\{1,2\}} - \Pi_{\{1\}}), \tag{3}$$

and

$$U_1 = \Pi_{\{1,2\}} - w_1 - w_2 = (2\gamma - 1)\Pi_{\{1,2\}} - (1-\gamma)(\Pi_{\{1\}} + \Pi_{\{2\}}). \tag{4}$$

Technological advances can initially benefit influencers, but may have adverse effects when influencers grow too powerful.

Proposition 1 (Technological advances and Influencer Labor Market)**.** *Fix arbitrary $\gamma < 1$, $\beta \geq 0$, and $I_1 > I_2$. When technology advances, we have:*
(i) The total payoffs for influencers is non-monotonic (i.e., $w_1 + w_2 \uparrow$ for small I_2, and $w_1 + w_2 \to 0$ for large I_2);
(ii) The wage gap between influencers is non-monotonic (i.e., $w_1 - w_2 \uparrow$ for small I_2, and $w_1 - w_2 \to 0$ for large I_2).

[7] Lemma 1 in the Appendix contains the derivation of equilibrium prices.

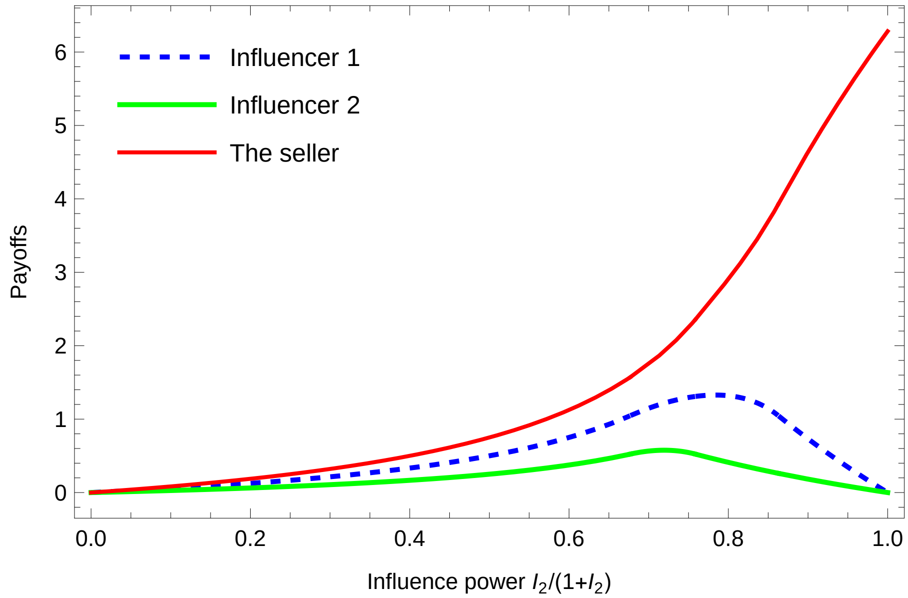

Figure 5: **Non-Monotonicities in Influencers' Wages**

The horizontal axis corresponds to (normalized) influencers' power, and the vertical axis corresponds to payoffs. The blue dashed line and the green line depict wages paid to Influencer 1 and Influencer 2 (i.e., equation (3)) and the red line depicts the payoff to the seller (i.e., equation (4)). Parameters: $y = 1$, $\beta = \pi$, $I_1 = 2I_2$ and $\gamma = 0.5$.

Proposition 1 is best illustrated in Figure 5 (Π_j and $\Pi_{\{1,2\}}$ are derived in the appendix.) Specifically, the blue dashed line, the green line, and the red line correspond to Influencer 1, Influencer 2, and the seller's payoffs. As technology improves (I_1 and I_2 increase proportionally), we see a non-monotonic pattern for both influencers' total wages and the income gap between influencers, that is, these two measures both first increase and then decrease.

Intuitively, when it is very costly to turn on influence, technological advances create a "win-win" situation for both the seller and influencers, since the newly emerged general-purpose technology helps influencers attract more clients and the competition among influencers is still minimal. This leads to an increase in total revenue, which in turn increases both the seller's payoff and influencers' wages. However, when the general technology is sufficiently cheap and influencers' powers are big, even one influencer can produce a big revenue if hired. This also implies that influencers become more of substitutes, and by hiring both influencers, the seller can use fierce competition between influencers for gaining bargaining power and thus pay minimal wages to both influencers. Hence, we only see a big distributional inequality for intermediate influence powers. Note that the non-monotonicity in concentration is related to the long-term sustainability and the unequal income in influencer marketing emphasized in the report by CB Insights (2021).

In short, technology always benefits the seller but can hurt influencers when it is sufficiently cheap.[8] On the one hand, a better technology improves the interaction between influencers and followers and thus attracts more consumers. On the other hand, it can intensify the competition between the influencers when it no longer attracts new consumers. For the seller, these two forces are in the same direction. In contrast, the second effect can dominate when the market for influencers becomes too crowded.

4 Seller Competition and Influencer Hiring

We now consider seller competition in both influencers' labor market and the product market. A seller-influencer group is characterized by the 3-tuple (y_m, θ_m, I_m) for $m = 1, 2$, where $I_m = I$ in the baseline. Without loss of generality, we assume $y_1 \geq y_2$ and $I_1 \geq I_2$.

4.1 Positive Assortative Matching

With exogenous style locations for influencers, a positive assortative matching typically ensues in equilibrium. Denote by $k(j)$ the matched seller identity for influencer $j = 1, 2$.

Proposition 2 (Positive Assortative Matching)**.** *$k(j) = j$ for $j = 1, 2$ when one of the following holds: (i) $\beta \geq \frac{1}{2}I_1 + \frac{1}{2}I_2$, i.e., the market is uncongested, (ii) one seller dominates the entire market, or (iii) influencers are maximally distanced and both sellers get market shares, i.e., $\beta = \pi$ and $I_j \geq \pi$.*

Proposition 2 presents fairly general sufficient conditions for assortative matching to arise. Specifically, (i) shows the emergence of assortative matching when the influence power is relatively small compared to style dispersion. The seller with a more valuable good can offer to hire a more powerful influencer by proposing a higher wage because the seller-influencer group's total profit is supermodular in influencer power and product quality parameter.[9]

[8] In an earlier version of the paper, we model a fixed searching cost $\varepsilon > 0$ to model the searching friction in the influencer market, which generates two new insights. First, when the technology advances (i.e., $k \uparrow$), we see a non-monotonic hiring pattern. Initially, only the strong influencer is hired because only he can help the seller break even. Then, both influencers are hired when they can both break even. Finally, only the strong influencer is hired again because the fixed searching cost dominates the marginal benefits from hiring the weak influencer. Second, endogenous bargaining power building can lead to inefficient hiring. When the seller's bargaining power is small (i.e., low γ), she might also hire the weak influencer even the marginal benefit cannot offset the additional fixed searching cost. This is because, even though it is socially sub-optimal to do so, it allows the seller to more aggressive bargain with the strong seller.

[9] A twice-differentiable function $f : X \times Y \to \mathbb{R}$ is supermodular iff $\frac{\partial^2 f}{\partial x \partial y} \geq 0$ for all $(x, y) \in X \times Y$.

Since influencers do not increase the common value of consumption y, the product with a better quality is always priced higher.

When market competition is intense, as in (ii), a single seller can dominate all other sellers. The intuition is simple. If only a single seller can survive, it has to be the strong seller because consumers sufficiently loyal to her always keep purchasing from her. Furthermore, if the strong seller can defeat the other seller by hiring the relatively weak influencer, it only makes the strong seller more powerful by hiring the strong influencer in the product market. These two facts jointly lead to assortative matching.

Moreover, in (iii), assortative matching also occurs when influencers are maximally distanced (i.e., $\beta = \pi$), which can be an outcome of influencers' endogenous choices of style, as we discuss later. The condition that $I_2 \geq \pi$, combined with $I_1 \geq I_2$ means that the market is crowded and influencers compete on both sides. The intuition remains. The strong seller, by hiring a more powerful influencer, enjoys an advantage in competing against the other seller and getting a bigger consumer share and thus a large joint profit.

4.2 Market Dominance

The presence of influencers can enhance product competition. We establish a set of sufficient conditions under which a single seller-influencer group grabs the entire market. Compared to traditional markets without influencers, it is more difficult to achieve such market dominance, which now requires relatively large gaps in both product quality and influence power, as well as a small style dispersion between influencers.

Proposition 3 (Market Dominance). *Fix $I_1 \leq 2\pi$. If $\frac{\beta}{I_1} \leq \min\{\frac{1}{2} - \frac{y_2}{y_1}, \frac{1}{2} - \frac{I_2}{I_1}\}$, then Seller 1 hires Influencer 1 and sets $p_1^* = \frac{y_1}{2}$ to force Seller 2 out of the market completely. Payoffs for sellers and influencers are $U_1 = \frac{\gamma y_1 I_1}{2}$, $w_1 = \frac{(1-\gamma) y_1 I_1}{2}$, and $U_2 = w_2 = 0$.*

Unlike direct price competition, it is insufficient to just have greater influence power or a much better product to force out the rival in product competition. Suppose Seller 1 has a superior product, Seller 2 can still compete to gain some market share because Influencer 2, who works with Seller 2, creates a sufficiently large "product differentiation" ($\theta_1 \neq \theta_2$).

Here the sufficient condition in Proposition 3 involves multi-dimensional heterogeneity and is reasonably tight. To see this, we consider three examples involving one-dimensional heterogeneity to shed light on how influencers can help sellers gain local monopoly power.

Example 1 (Single-Dimensional Heterogeneity in Product Quality). *Consider* $y_1 \geq y_2$, $\beta = 0$, *and* $I_1 = I_2 = I \leq \pi$. *In equilibrium,* $k(j) = j$ *for* $j = 1, 2$.[10] *After matching, the two seller-influencer groups choose* $(p_1^C, p_2^C) = \left(\frac{2y_1(y_1-y_2)}{4y_1-y_2}, \frac{y_2(y_1-y_2)}{4y_1-y_2}\right)$. *Seller 1 targets consumers with* $\|x-\theta\| \leq \frac{2Iy_1}{4y_1-y_2}$ *and Seller 2 targets those with* $\frac{2Iy_1}{4y_1-y_2} < \|x-\theta\| \leq \frac{3Iy_1}{4y_1-y_2}$. *Furthermore, the group profits are given by:*

$$\Pi_1^C = \frac{8Iy_1^2(y_1 - y_2)}{(4y_1 - y_2)^2}, \quad \text{and} \quad \Pi_2^C = \frac{2Iy_1y_2(y_1 - y_2)}{(4y_1 - y_2)^2}. \tag{5}$$

The payoffs for sellers and influencers are:

$$U_1 = \gamma\Pi_1^C, \ U_2 = \gamma\Pi_2^C, \ w_1 = (1-\gamma)\Pi_1^C, \ \text{and } w_2 = (1-\gamma)\Pi_2^C.$$

Figure 6 illustrates the equilibrium. Specifically, Group 1, who offers the relatively high-quality product, targets consumers with a high loyalty and thus a high willingness to pay, and we illustrate these clients with the blue line. In contrast, Group 2, which offers the relatively low-quality product, attracts those consumers not targeted by Group 1, and we illustrate them with the red line. The intuition is that when $y_1 > y_2$, even if Group 2 sets $p_2^* = 0$, the product offered by Group 1 is still more attractive for consumers with a high loyalty (i.e., $\|x - \theta\| \to 0$). Thus, Group 1 has an incentive to set a positive price, which in turn implies that Group 2 can get a positive profit through attracting consumers not targeted by Group 1, whenever Group 1 decides not to set $p_1^* = 0$.

Finally, when $y_1 \downarrow y_2$, it converges to the Bertrand competition outcome which features $(p_1^C, p_2^C) = (0, 0)$. In contrast, when $y_1 \gg y_2$, it converges to an equilibrium with $(p_1^C, p_2^C) = (y_1/2, y_2/4)$, which means that the high-quality product is priced at its monopoly price, while the low-quality product is priced at a monopoly price in the residual market after removing the market share taken by the strong seller.

Note that the sufficient conditions are violated in Example 1. To appreciate the content of Example 1, we compare it with traditional product competition with vertical differentiation. We compare it with the benchmark in which two sellers with quality indices $y_1 > y_2$ engage in vertical product competition. It is straightforward to check the following equilibrium in traditional product competition without influencers, that is, $p_1 = y_1 - y_2$, $p_2 = 0$ and all consumers purchase the product from Seller 1. Note that a key difference is that Seller 2 has

[10] See Appendix A.6 for a detailed proof.

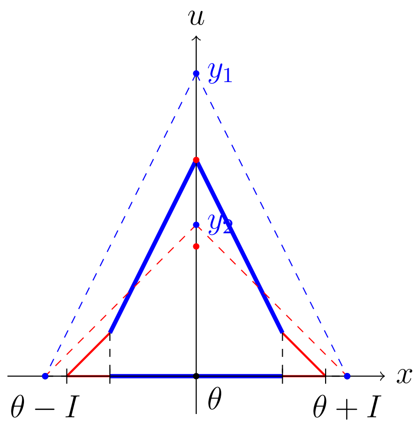

Figure 6: Heterogeneous product quality

zero market share in this benchmark economy. In contrast, Seller 2 has a strictly positive market share, even if there is no horizontal style dispersion between the two influencers.

Without influencers, all consumers are concentrated at a mass, and thus Bertrand price competition always leads to the dominance of the higher-quality product. In contrast, with influencers, even without horizontal differentiation, consumers are heterogeneous and distributed in a non-trivial decentralized way. In particular, consumers' private values (i.e., willingness to pay) for the product differ greatly when located near the influencer's style location, whereas consumers are more similar but less valuable when located away from the center. Hence, the high-quality seller, by targeting the more valuable consumers, can establish a large advantage in revenue through influencer marketing. However, this revenue advantage also deteriorates when it comes to the remote consumers, and creates an incentive conflict with attracting the most valuable consumer for the dominant seller. This counterforce prevents a perfect Bertrand style competition as in traditional economies. Hence, as long as there exists a difference in product quality, both sellers can survive and secure a share of client and thus a positive profit.

Example 2 (One-Dimensional Heterogeneity in Influence Power)**.** *Consider $I_2 \leq I_1 \leq \pi$, $\beta = 0$ and $y_1 = y_2 = y$. Again, $k(j) = j$ for $j = 1, 2$ trivially holds. After matching, prices are set at $(p_1^C, p_2^C) = \left(\frac{2y(I_1 - I_2)}{4I_1 - I_2}, \frac{y(I_1 - I_2)}{4I_1 - I2}\right)$.*[11] *Seller 1 targets consumers whose type x satisfies $\frac{I_1 I_2}{4I_1 - I_2} < \|x - \theta\| \leq \frac{I_1(2I_1 + I_2)}{4I_1 - I_2}$ and Seller 2 targets those such that $\|x - \theta\| \leq \frac{I_1 I_2}{4I_1 - I_2}$. Group profits are given by:*

$$\Pi_1^C = \frac{4I_1^2(I_1 - I_2)y}{(4I_1 - I_2)^2}, \quad \textit{and} \quad \Pi_2^C = \frac{I_1 I_2(I_1 - I_2)y}{(4I_1 - I_2)^2}.$$

[11] See Appendix A.7 for a detailed proof.

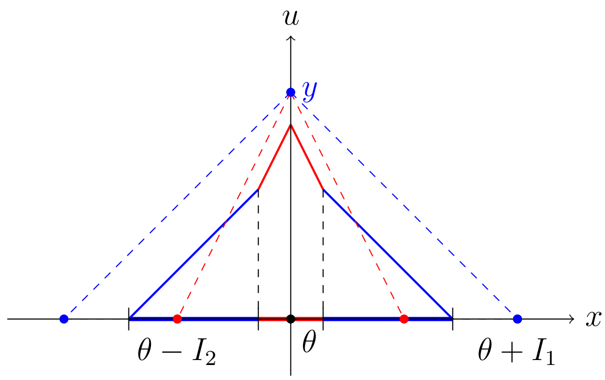

Figure 7: Heterogenous influence power

and payoffs for sellers and influencers are given by

$$U_1 = \gamma \Pi_1^C,\ U_2 = \gamma \Pi_2^C,\ w_1 = (1-\gamma)\Pi_1^C,\ \textit{and } w_2 = (1-\gamma)\Pi_2^C.$$

Figure 7 illustrates the equilibrium outcomes. Specifically, Seller 1, matched with the strong Influencer 1, targets consumers sufficiently distant from the influencer's style, as illustrated with blue lines, mainly to avoid tough price competition in the product market. Indeed, they can afford this because the strong influence power alleviates the utility loss from consumers. In contrast, Seller 2, matched with the weak Influencer 2, targets those consumers sufficiently close to the influencer's style, and we depict it with red lines. For the equilibrium, note that $p_1 \geq p_2$. Otherwise, Seller 2 will be forced out of the market. However, $p_1 = 0$ is sub-optimal for Seller 1 because consumers satisfying $I_2 < \|x - \theta\| \leq I_1$ prefer Seller 1. Thus, Seller 1 can be better off by charging a positive price sufficiently low, which further implies that Seller 2 can also get a positive profit by attracting consumers close to θ.

Finally, when $\frac{I_1}{I_2} \to \infty$, prices satisfy $(p_1^C, p_2^C) \to (\frac{y}{2}, \frac{y}{4})$, and when $\frac{I_1}{I_2} \to 1$, $(p_1^C, p_2^C) \to (0, 0)$, which coincides with the Bertrand price competition outcome.

By comparing Example 1 and 2, we can see the competition mode depends on how heterogeneity arises. Specifically, when it is driven by the consumption value of the product, the stronger group focuses on attracting consumers with a taste similar to that of the influencer. In contrast, when it is linked to how easily the influencer attracts followers, the stronger group focuses on those consumers not reachable by the weaker group and sacrifice the loyal followers in the sense of taste proximity.

Example 3 (Single-Dimensional Heterogeneity in Style)**.** *Consider* $\beta > 0$*,* $I_1 = I_2 = I$ *and*

$y_1 = y_2 = y$.[12] *Define* $\beta_0 := \frac{2}{67}(-7+5\sqrt{10})I \approx 0.263I$. *Again,* $k(j) = j$ *for* $j = 1, 2$. *After matching, the two seller-influencer groups set prices such that*

$$p_1^C = p_2^C = \begin{cases} \frac{y}{5I}(2I+\beta), & \text{if } \beta_0 \le \beta \le \frac{6}{7}I \\ y * \left(1 - \frac{\beta}{2I}\right), & \text{if } \frac{6}{7}I < \beta \le I \\ \frac{y}{2}, & \text{if } \beta > I \end{cases}$$

The two groups' profits, as a function of β*, are given by*

$$\Pi_1^C(\beta) = \Pi_2^C(\beta) = \begin{cases} \frac{3y}{50I} * (2I+\beta)^2, & \text{if } \beta_0 \le \beta \le \frac{6}{7}I \\ y * \beta\left(1 - \frac{\beta}{2I}\right), & \text{if } \frac{6}{7}I < \beta \le I \\ \frac{yI}{2}, & \text{if } \beta > I \end{cases} \tag{6}$$

Furthermore, $U_1 = \gamma\Pi_1^C$, $U_2 = \gamma\Pi_2^C$ $w_1 = (1-\gamma)\Pi_1^C$, *and* $w_2 = (1-\gamma)\Pi_2^C$.

Note that when $\beta < \beta_0$, there exists no pure strategy equilibrium. Once again, the stronger seller cannot dominate the market. To summarize, the three examples above jointly show that it is insufficient to achieve market dominance by introducing one-dimensional heterogeneity in an influencer economy.

5 Style Heterogeneity and Product Differentiations

Having understood how sellers hire influencers and price products, we now endogenize sellers' production stage. Specifically, we investigate how influencer heterogeneity affects horizontal and vertical product differentiations, compared to traditional economies.

5.1 Horizontal Product Differentiation

We investigate how style dispersion affects horizontal product differentiation by the sellers. To this end, we assume identical product quality and identical influence power (i.e., $y_1 = y_2 = y$ and $I_1 = I_2 = I$). Our key observation, perhaps quite intuitively, is that influencers' style difference and horizontal product specialization are substitutes.

To model horizontal differentiation, we stipulate that sellers can either choose to hire an

[12] See Appendix A.8 for a detailed proof.

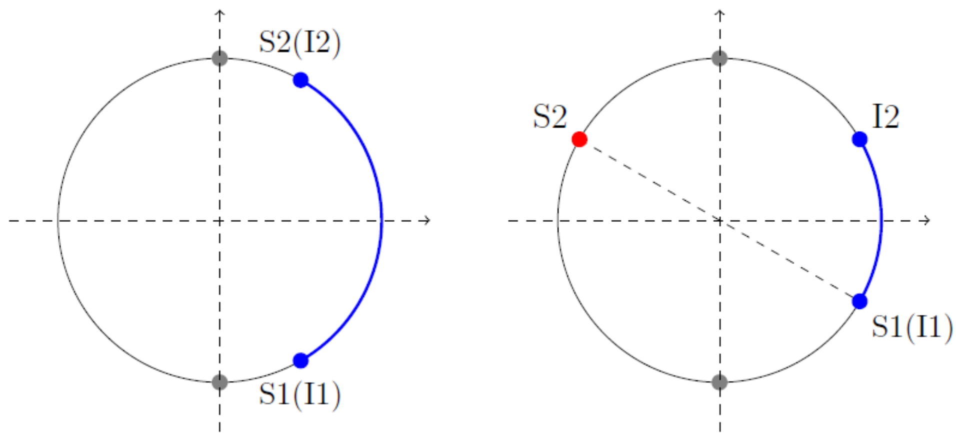

Figure 8: Style dispersion and horizontal differentiation

influencer with a given style, or pay a cost f_E to nurture a new influencer with any style $\theta \in \mathbf{S}^1$.[13] We still focus on the $I \leq \pi$ case as before, although the main insights are general.

To ensure equilibrium existence, we assume that $\beta \geq \beta_0$ and that

$$0 < f_E < \Pi_1^C(I) - \Pi_1^C(\beta_0), \tag{7}$$

which implies that there exists a unique $\tilde{\beta}$ such that $\Pi_1^C(\tilde{\beta}) = \Pi_1^C(I) - f_E$ holds. Denote by α_j the style location chosen by seller $j \in \{1, 2\}$, we have:

Proposition 4 (Horizontal Product Differentiation)**.** *The influencers' style dispersion is a substitute for horizontal product differentiation. Given (7), in equilibrium we have:*
(i) When $\beta \geq \tilde{\beta}$, each seller hires one influencer and accepts his style location.
(ii) When $\beta < \tilde{\beta}$, one seller hires an influencer directly while the other pays f_E and select a location such that $\|\alpha_1 - \alpha_2\| \geq I$.

Figure 8 illustrates the equilibrium in Proposition 4. The left sub-figure entails a case with a large style difference. The style locations for Influencers 1 and 2 are marked in blue, while the two gray nodes illustrate one optimal style separation under the maximum differentiation principle (i.e., $\|\alpha_1 - \alpha_2\| = I \to \pi$). The blue arc corresponds to β. In this case, both sellers hire influencers to save the fixed cost in product differentiation and adapt to the influencers' styles. The well-established principle of maximum differentiation (e.g., d'Aspremont et al., 1979; De Frutos et al., 1999) states that because more differentiated markets yield lower substitutability among products, firms maximally differentiate to reduce

[13] The transportation cost in Salop (1979) without influencers is equivalent to an influencer economy setting with the exogenous influence power given by $\frac{y}{t}$.

competition. The principle fails here because the influencers' style dispersion serves as a substitute for horizontal differentiation. A sufficiently large dispersion, albeit not maximal, removes the incentive to horizontally differentiate.

The right subfigure illustrates the equilibrium when influencers' style dispersion is small that once a seller works with an influencer, the other seller still horizontally differentiates with a style (red dot) to reduce competition, restoring the maximum differentiation principle.

5.2 Vertical Product Differentiation

The relationship between influencers' style dispersion and vertical product differentiation through quality turns out to be less straightforward. Instead of avoiding competition (Shaked and Sutton, 1982), sellers may vertically differentiate to gain a bigger share of consumers, knowing the style dispersion. In general, the endogenous vertical differentiation is non-monotonic in influencers' style dispersion, with a large quality gap only observed for intermediate style dispersion.

To see it, consider two sellers initially making products with identical quality $y_j = \underline{y}$, but $\beta \geq 0$. We further assume that $I_1 = I_2 = 1$ and $\underline{y} = 1$ for simplicity. Suppose influencers can pay a fixed R&D cost F_V to improve the quality to $\bar{y} = y > 1$. We also use "L" and "H" to denote "Low quality" and "High quality". Denote by $V_1(\beta, y)$ the increment profit change from quality investment when the competing group selects low quality, that is,

$$V_1(\beta, y) \equiv \Pi^1_{H,L} - \Pi_{L,L} = \frac{y(1+2y)\big(2+8y+4y^2+\beta(4+3y)\big)^2}{(1+y)\left(8+19y+8y^2\right)^2} - \frac{3}{50}(2I+\beta)^2,$$

and by $V_2(\beta, y)$ the investment benefit when the opponent also selects high quality,

$$V_2(\beta, y) \equiv \Pi_{H,H} - \Pi^1_{L,H} = \frac{3y}{50}(2I+\beta)^2 - \frac{(2+y)\big(4+8y+2y^2+\beta y(3+4y)\big)^2}{(1+y)\left(8+19y+8y^2\right)^2}.$$

Now, define $\bar{\beta}$ and $\overline{\beta}$ such that $V_1(\underline{\beta}, y) = F_V$ and $V_2(\overline{\beta}, y) = F_V$. Note that both $V_1(\beta, y)$ and $V_2(\beta, y)$ are increasing in style dispersion. Thus, $\underline{\beta}$ is the break-even point for quality investment, given the opponent selects low quality. Similarly, $\overline{\beta}$ is the break-even point for investment when the opponent selects high quality. Recall that $\beta_0 = \frac{2}{67}(-7+5\sqrt{10})I \approx 0.263$, a number we use in the figure illustrations.

Lemma 2 (Non-Monotonic Vertical Differentiation)**.** *Assume that: (i)* $\beta_0 \leq \beta \leq \frac{5}{6}$*; (ii)*

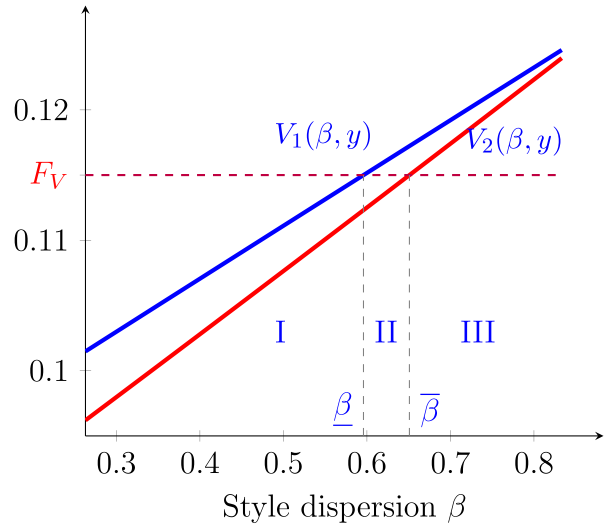

Figure 9: Vertical product differentiation

$\overline{y}(1-\beta_0) \leq \underline{y}$; and (iii) $V_1(\beta_0, y) < F_V < V_2(\frac{5}{6}, y)$. Then, there exists $\overline{\beta}$ and $\underline{\beta}$ such that:

(a) $\beta \geq \overline{\beta}$, there exists one Nash Equilibrium (H, H);

(b) $\underline{\beta} \leq \beta < \overline{\beta}$, there are two asymmetric Nash Equilibrium: (H, L) and (L, H);

(C) $\beta_0 \leq \beta < \underline{\beta}$, there exists one Nash Equilibrium (L, L).

Lemma 2 characterizes a non-monotonic relationship between influencers' style dispersion and vertical product differentiation. Note that (i) ensures the existence of pure strategy equilibrium and simplifies algebra; (ii) ensures that under asymmetric investment (i.e., (H, L)), both groups can survive in the product market; and (iii) focuses on the most interesting cost range in which non-monotonicity arises.[14]

Figure 9 illustrates the equilibrium configuration in Lemma 2 with $\bar{y} = y = 5/4$ and $\underline{y} = 1$. Specifically, the horizontal axis β corresponds to style dispersion. The two functions, $V_1(\beta, y)$ and $V_2(\beta, y)$, corresponding to the blue and red solid line in the figure, measures the profit gap between "High quality" (Investing in R&D) and "Low quality," given the other group's choice. Fix the cost of investment $F_V > 0$ (i.e., the purple dashed line). There are three regions, "I", "II" and "III", separated by two cutoffs, $\overline{\beta}$ and $\underline{\beta}$. In Region III, $V_2(\beta, y) > F_V$, and thus both groups choose high quality. In Region II, $V_2(\beta, y) < F_V$ and $V_1(\beta, y) > F_V$, which means only one group chooses high quality. In Region I, both groups choose low quality.

Lemma 2 implies that, when style dispersion, β, increases from low values, we initially see more vertical differentiation (and thus influencers' style dispersion and vertical product

[14]For $F_V \leq V_1(\beta_0, y)$, (H, H) is the unique Nash equilibrium. Similarly, for $F_V > V_1(\frac{5}{6}, y)$, (L, L) is the unique Nash equilibrium.

differentiation are complements). In contrast, as β further increases when it already takes a large value, we see less vertical differentiation (and thus these two are substitutes). Intuitively, for a small β, the intense competition greatly limits the return from investing in high quality. Thus, both groups choose low quality, and the vertical differentiation is minimal. For an intermediate β, the competition is less intense which improves the investment return for a seller provided that the other seller does not invest. Finally, for a large β, the competition is minimal and even when both groups invest, they can break even. Note that the investment profit is strictly increasing in influencers' style dispersion. Hence, we no longer observe vertical product differentiation. The following proposition summarizes the insights:

Proposition 5. *The influencers' style dispersion, when small, is a complement with vertical product differentiation; when it is large, it becomes a substitute for vertical differentiation.*

6 Endogenous Influence and Welfare Implications

We now move back to time t_0 to allow influencers to endogenize their influence either in power or style. First, we allow endogenous power acquisition and show that socially inefficient under-investment and over-investment in influence can arise due to externalities and incentives for building bargaining power. Second, we show that maximum style differentiation and assortative seller-influencer matching hold in the long run.

6.1 Socially Insufficient Power Acquisition

We now examine the influencers' endogenous influence power acquisition and its welfare implications. Because the utility from consumption is bounded above, which means that many influencers might spend effort to acquire power and too many endogenously become influencers in practice. We show that this concern about the arms race among influencers is not warranted. Instead, socially insufficient power acquisition can arise because of externalities and intense price competition under balanced matching.

For simplicity, consider two maximally distanced influencers with initial power I. Either one can pay a fixed cost $C_P > 0$ to increase influence power to $kI > I$.

Proposition 6 (Socially Insufficient Power Acquisition)**.** *Fix $\beta = \pi$. There exists under-investment in influencer power when: (i) $kI \leq \pi/2$ and $\frac{1}{2}(1-\gamma)(k-1)yI < C_P < 2(k-1)yI$; or (ii) $I \geq \frac{3}{2}\pi$ and $0 \leq C_P < \frac{\pi^2 y(k-1)(7k+2)}{18k(k+1)I}$.*

There are two forces driving this result. First, influencers do not internalize the externality of their power acquisition on consumer welfare. When the market for influencers is not crowded, power acquisition can increase consumer utility and it can exhibit under-investment when the positive externality on consumer welfare is not internalized. The bargaining power clearly matters as a large γ reduces the incentive for power acquisition and causes under-investment because condition (i) is more likely to hold. Second, the intense price competition between the two sellers when the influencer market is congested reduces influencers' incentive to invest in power, because it further intensifies the existing competition in the product market. In particular, any potential gain in consumers is dominated by the lowered product prices, even when power acquisition costs very little.

Remark 1 (Monopolist and Over-investment in Influence)**.** *Endogenous power acquisition also depends on the market structure. In Proposition 6, the two forces both induce under-investment. In contrast, when the market only features a monopolist seller, the intense price competition is absent. However, a third channel arises: influencers exert a negative externality on other influencers, when hired by the same monopoloy seller. Depending on the congestion of influencer market and the bargaining power, these three forces can generate over-investment, efficient investment and under-investment in power acquisition.*[15]

6.2 Style Selection

In this section, we study costless style location selection.[16] Proposition 2 verifies the validity of assortative matching in several scenarios. The intuition is simple. A seller with a more valuable good can offer to hire a more powerful influencer by proposing a higher wage because a stronger seller, when matched with a stronger influencer, can jointly produce a larger profit. Now, we turn to the problem of endogenous style location selection, and we only prove a special case of an uncongested influence economy in which $\frac{I_1+I_2}{2} \leq \pi$. Denote by θ_j^* the optimal style location chosen by influencer $j = 1, 2$. Define $\beta^* = \|\theta_1^* - \theta_2^*\|$.

Proposition 7 (Maximal Style Differentiation)**.** *Assume that $\frac{I_1+I_2}{2} \leq \pi$. When style location selection is costless, the maximum style differentiation holds, that is, $\beta^* \geq \frac{I_1+I_2}{2}$, and there*

[15] See proposition B.1 in the online appendix for more details.

[16] Under a set of mild conditions, influencers always invest in costly style selection as long as maximal differentiation fails initially, but they never achieve maximal style differentiation since the marginal return vanishes as the style dispersion approaches the maximal style differentiation.

exists no overlapping in consumers served. Furthermore, assortative matching ensues under endogenous style location selection.

Proposition 7 states that influencers follow the maximal style differentiation principle whenever possible under both seller competition and balanced matching, because it minimizes the competition between the two influencer-seller groups. Furthermore, assortative matching ensues under endogenous style location selection.

The result is also in stark contrast with Gal-Or and Dukes (2003) that discovers a minimum differentiation in commercial media markets. In that industry, product differentiation is taken as exogenous and thus the substitutability between style differentiation and product differentiation is absent.

Remark 2 (Seller Preferred Style Dispersion). *A related question is that whether it helps to let the seller endogenously decide influencers' style locations since they are more informed about their own products? The answer is no. We illustrate this with a special case in which* $I_1 = I_2 < \pi$. *Denote by* β_S^* *the seller-preferred style dispersion. Then: (i) if her bargaining power is large (i.e.,* $\gamma > \frac{1}{2}$*), the seller prefers maximal style dispersion (i.e.,* $\beta_S^* \geq I$*); (ii) if* $\gamma = \frac{1}{2}$*, she is indifferent among all style dispersion (i.e.,* $\beta_S^* \in [0, \pi]$*); and (iii) if her bargaining power is small (i.e.,* $\gamma < \frac{1}{2}$*) , she prefers minimal style dispersion (*$\beta_S^* = 0$*).*

Due to the endogenous bargaining power building, the seller prefers a minimal style dispersion when her bargaining power is small and vice versa.[17]*. To see it, first consider a small bargaining power parameter for the seller (i.e.,* γ *is small). On the one hand, a large style difference implies a big market coverage and a small over-lapping in consumer base for the two influencers, leading to a large total revenue. However, a small* γ *also implies that only a very limited fraction of total revenue flows to the seller. On the other hand, a small* β *decreases the market coverage and generates limited revenue, but it does lead to intense competition and perfect substitution between the two influencers and leads to a more advantageous bargaining position for the seller under the bilateral Nash bargaining protocol. In the limit case where* $\beta = 0$*, the seller gets the total revenue of working with a sole influencer in the product market. In other words, a small style difference increases the cash flow to the seller although it decreases market coverage and the incentive for bargaining power building*

[17]The proof is simple. First, note that, from equation (4), β affects U_1 only through $\Pi_{\{1,2\}}$. Second, from Lemma A.2 in the Appendix, $\Pi_{\{1,2\}}$ is monotone in β. Thus, when $\gamma > \frac{1}{2}$, by equation (4), U_1 is strictly increasing in $\Pi_{\{1,2\}}$ and thus strictly increasing in β for $\beta \leq I$ (and becomes constant when $\beta > I$), which implies that it achieves maximum for any $\beta \geq I$. The other two cases can be shown similarly.

dominates. In contrast, when the seller enjoys a big bargaining power (i.e., a big γ), the incentive distortion by endogenous bargaining power building is minimal and thus the incentive for revenue generating dominates, and thus more style dispersion leads to a larger client base and an increase in total revenue.

7 Unbalanced Matching, Exclusivity, and Regulation

In practice, a seller sometimes hires multiple influencers and requires them not to advertise rival sellers' products (e.g., Zietek, 2016). For example, a large survey of influencers by Mavrck (Katz, 2019) shows that the majority of influencers (61%) are receiving exclusivity requests from brands. In fact, exclusivity contracts have been prevalent in industries such as healthcare and insurance and have led to many antitrust cases (Gal-Or, 1999). However, policies have been recently introduced to better protect influencers and to reduce market concentration through encouraging competition. The awareness has also grown in the industry that exclusivity should be mutual.[18] This means that either both sides can contract with multiple counterparties or both sides have to exclusively collaborate—exactly what our setting of balanced matching aims to capture.

Nevertheless, to better understand the welfare implication of exclusivity contracts in this emerging industry and to guide regulatory policies geared towards balancing the power of influencers and sellers, we extend the analysis to "unbalanced matching." In such a setting, a seller can hire multiple influencers, but not the other way around, which is consistent with the contracting landscape in the early stages of influencer industry (e.g., Zietek, 2016). We then compare settings with and without balanced matching to derive two key results: First, balanced matching (mutual exclusivity contracting) is optimal under one-dimensional heterogeneity even when we allow sellers to compete for multiple influencers. Second, unbalanced matching (uni-directional exclusivity) can be optimal when influencers' style locations are sufficiently unique in uncongested influencer markets.

[18] Influencers increasingly value long-term partnerships With brands rather than one-off exclusivity requests; they also expect to be compensated more when exclusivity is required. As early as 2008, the entertainment industry began to see the value behind full-time creators building multi-platform brands, and influencers started getting Hollywood agents to help negotiate bilateral exclusive contracts (Collectively, 2020). New York State Regulators, China's State Administration for Market Regulation, and the British Federal Trade Commissions are all increasing regulations regarding influencer contracting. See, e.g., https://www.ftc.gov/news-events/press-releases/2019/11/ftc-releases-advertising-disclosures-guidance-online-influencers; http://www.zhonglun.com/Content/2020/11-11/1619074334.html; and https://wwd.com/business-news/legal/wwd-law-review-noncompetes-influencers-model-protection-11038135/.

As before, without loss of generality, we assume that $y_1 \geq y_2$ and $I_1 \geq I_2$.

Lemma 3 (Equilibrium under unbalanced matching)**.** *Allowing unbalanced matching (i.e., uni-directional exclusivity contracts):*

(i) When there only exists one-dimensional heterogeneity in influencers or sellers, the equilibrium coincides with those in Examples 1, 2, and 3.

(ii) When influencer style dispersion is large (i.e., $\beta \geq \frac{I_1+I_2}{2}$), Seller 1 hires both influencers and offers prices at $p_1^* = p_2^* = \frac{y_1}{2}$. Payoffs for sellers and influencers satisfy:

$$U_1 = \frac{\gamma y_1(I_1+I_2)}{2}, U_2 = 0, \ w_1 = \frac{(1-\gamma)y_1 I_1}{2} \text{ and } w_2 = \frac{(1-\gamma)y_1 I_2}{2}.$$

How do different forms of exclusivity affect welfare? Intuitively, compared to balanced matching, unbalanced matching features a monopolist seller with a more valuable product. On the one hand, it increases welfare by letting the high-quality product dominate the market and its magnitude depends on the quality gap between products. On the other hand, market concentration, combined with monopolist pricing, decreases surplus for consumers attracted, and prices out a large fraction of potential consumers. Given this insight, when the quality gap between two sellers decreases, the former effect vanishes. Thus, unbalanced matching hurts consumers and social welfare decreases when products feature homogenous quality.

Proposition 8 next compares the efficiency between balanced and unbalanced matchings.

Proposition 8 (Exclusivity Contracting and Welfare)**.**

(i) (Congested influencer market or homogeneous product market). Unbalanced matching lowers total welfare under one dimension heterogeneity, including heterogeneous product quality, heterogeneous influencer power and heterogeneous influencers' style locations.

(ii) (Uncongested influencer market). When $\beta > \frac{I_1+I_2}{2}$ and $y_1 > y_2$, unbalanced matching dominates balanced matching in total welfare.

The key messages in Proposition 8 are intuitive. On the one hand, both product quality gap and influencer style dispersion affect the intensity of seller competition. When the market for influencers is not crowded and influencers' styles are distinct, regulation on mutual exclusivity contracting does not help encourage competition because of the inevitable local market power derived from influencer heterogeneity. Given that the economy features monopoly pricing anyway, uni-directional exclusivity is welfare-improving because it allows the high-quality product to dominate. On the other hand, when products are quite homogeneous or influencers are very similar in style, requiring mutual exclusivity and balanced matching can improve consumer welfare. Note that unbalanced matching always features joint profit maximization and the dominance of the higher-quality product, while balanced matching features greater price competition. Simply put, the quality improvement channel is shut down in a homogeneous product market, while regulation can improve competition when the influencer market is crowded.

8 Conclusion

We build a model of the influencer economy in which: (i) sellers produce goods and compete for consumers through influencers, (ii) sellers and influencers are matched in influencers' labor market and engage in Nash bargaining, and (iii) influencers acquire influence to attract consumers who identify with their style in addition to value the products they promote.

We derive five key insights. First, as technologies governing marketing outreach improve, the equilibrium features non-monotonicities in influencer influencer payoffs, and distributional inequality. Second, influencer heterogeneity and horizontal product differentiation are substitutes. Meanwhile, small style differences complement vertical product differentiation while large differences substitute. Third, assortative matching between sellers and influencers occurs under endogenous influence, with the maximum horizontal differentiation principle recovered in the limit of costless style selection. Fourth, sellers' bargaining power counteracts the influencers' tendency to over-invest in influence power and they jointly determine the direction and magnitude of the sub-optimal influence-power acquisition. Fifth, regulations for balanced seller-influencer matching can encourage seller competition under single dimensional seller-influencer heterogeneity. But uni-directional exclusivity contracts are welfare-improving for sufficiently differentiated products and uncongested influencers' markets.

For tractability and to focus on the industrial organization of the influencer economy, we have largely abstracted away from the inner working of platforms and MCNs, which leaves more to be desired: Profit sharing and contracting between influencers and platforms remain a crucial topic in understanding the digital economy. The organization of MCNs such as TikTok and Weibo, and their heterogeneity also constitute interesting future research. In this regard, our findings set initial benchmark results rather than foregone conclusions.

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Appendix

A Relevant Proofs

First, we introduce two auxiliary lemmas as below. In particular, Lemma A.1 presents the monopolist seller's profit when the two influencers are maximally distanced (i.e., $K = 1$, $J = 2$ and $\beta = \pi$). Lemma A.2 presents the monopolist seller's profit when two influencers have identical power for arbitrary style dispersion (i.e., $K = 1$, $J = 2$ and $I_1 = I_2 = I$). The proofs of these two lemmas can be found at the end of this Appendix.

Lemma A.1. *Fix $\beta = \pi$. Then, if $I_1 + I_2 \geq 2\pi$, then $p_1^* = p_2^* = y(1 - \frac{\pi}{I_1+I_2})$, $s_1 = \pi - s_2 = \frac{I_1 \pi}{I_1+I_2}$ and $\Pi_{\{1,2\}} = 2\pi y(1 - \pi/(I_1 + I_2))$.*

Lemma A.2. *Assume $I_1 = I_2 = I \leq \pi$. The joint profit $\Pi_{\{1,2\}}$ is given by: (i) if $\beta \geq I$, $\Pi_{\{1,2\}} = yI$; (ii) if $2I/3 \leq \beta < I$, $\Pi_{\{1,2\}} = (2I - \beta)\beta y/I$; and (iii) if $0 \leq \beta < 2I/3$, $\Pi_{\{1,2\}} = (2I + \beta)^2 y/(8I)$.*

A.1 Derivation of Equation (2)

Proof. First, consider the case that $I < \pi$. In this case, $D(p) \leq 2\pi$ for all $p \geq 0$. Thus, given the price $p \geq 0$, the total demand is given by $D(p) = 2(1 - p/y) * I$, which further implies that $\Pi(p) = D(p) * p$ and thus $p^* = \frac{y}{2}$ and $\Pi(p^*) = \frac{yI}{2}$.

Second, consider the case that $I \geq \pi$. In this case, $\Pi(p) = 2\pi * p$ if $p \leq y(1 - \pi/I)$, and $\Pi(p) = 2p(1-p/y)*I$ if $p > y(1-\pi/I)$. Note that $\Pi(p)$ is continuous at $p = y(1-\pi/I)$. Again, depending on the value of I, there are two cases. One, when $I \in [\pi, 2\pi]$, $y(1 - \pi/I) \leq \frac{y}{2}$. The quadratic term implies that $\Pi(p)$ is strictly increasing for all $p \in [y(1 - \pi/I), y/2]$ and strictly decreasing for $p > y/2$. Hence, $\Pi(p)$ is maximized at $p^* = \frac{y}{2}$, which yields $\Pi(p^*) = \frac{yI}{2}$. Two, when $I > 2\pi$, $y(1 - \pi/I) > \frac{y}{2}$ holds. This implies that $\Pi(p)$ is strictly increasing for $p \leq y(1-\pi/I)$ and strictly decreasing for $p > y(1-\pi/I)$ because the quadratic term is strictly decreasing for all $p > \frac{y}{2}$. Hence, $\Pi(p)$ is maximized at $p = y(1 - \pi/I)$ and $\Pi(p^*) = 2\pi y(1 - \pi/I)$. Finally, note that we can combine the two cases that $I < \pi$ and $I \in [\pi, 2\pi]$ to simplify the formula. □

A.2 Proof of Lemma 1

Proof. First, note that $I_1+I_2 < 2\pi-\beta$ implies no competition along the long arc between the two influencers. Meanwhile, $\beta < \frac{1}{3}(I_1 - I_2)$ implies that $\beta < \frac{1}{2}(I_1 + I_2)$ and thus influencers compete along the short arc. This is because, if there exists a positive measure of consumers not served along the short arc, it must be case that $p_j < \frac{y}{2}$, the monopolist price, and thus it is profitable to increase the price by a small amount. Given these two observations, there exists three cases to consider: i) there exists only one cutoff type consumer, who is indifferent in purchasing between the two influencers, and receives a positive utility; ii) there exists only one cutoff type consumer who receives a zero utility; and iii) there exist two cutoff types indifferent in purchasing from either influencer.

Second, we start from case i) and identify conditions under which case iii) occurs. If there exists only one cutoff type consumer, the indifference condition is defined as equating the utility from purchasing from the two influencers, that is, $u_1 = u_2$, where $u_1 = y(1-s_1/I_1)-p_1$ and $u_2 = y(1-s_2/I_2)-p_2$. Here, s_j is the size of consumers served by influencer $j = 1, 2$ along the short arc. This implies that $s_1 = \frac{I_1(I2(p_2-p_1)+\beta y)}{(I_1+I_2)y}$ and $s_2 = \beta - s_1$. Next, we can compute the total profit $W_1 = p_1 * I_1(1-p_1/y)+p_1s_1+p_2s_2+p_2*I_2(1-p_2/y)$. We can compute partial derivatives $\frac{\partial W_1}{\partial p_1} = \frac{I_1(\beta y+I_1(-2p_1+y)+I_2(-4p_1+2p_2+y))}{(I_1+I_2)y}$ and $\frac{\partial W_1}{\partial p_2} = \frac{I_2(\beta y+I_2(-2p_2+y)+I_1(-4p_2+2p_1+y))}{(I_1+I_2)y}$. It is easy to check the second-order conditions and thus we can use first-order conditions to jointly solve that $p_1^* = p_2^* = \frac{y}{2} + \frac{\beta y}{2(I_1+I_2)}$ and thus $u_1 = \frac{(I_1+I_2-3\beta)y}{2(I_1+I_2)}$. We can further compute the total profit $W_1 = \frac{(\beta+I_1+I_2)^2 y}{4(I_1+I_2)}$.

On the one hand, note that when $\beta < \frac{1}{3}(I_1 + I_2)$, $u_1 < 0$ and case ii) ensues.[19] On the other hand, we can find another cutoff on the right side of θ_2 (recall that $\theta_1 = 0$ and θ_2 sits to the right of θ_1) by equating $\hat{u}_1 = y(1 - (\beta + \hat{s}_2)/I_1) - p_1$ and $\hat{u}_2 = y(1 - \hat{s}_2/I_2) - p_2$, where $\hat{s}_2$ is the distance between the far right cutoff location and θ_2. We can solve it to get $\hat{s}_2 = \frac{I_2(I_1(p_1-p_2)+\beta y)}{(I_1-I_2)y}$. Hence, we can plug in p_j^* to get $\hat{s}_2 = \frac{\beta I_2}{I_1-I_2}$. Meanwhile, note that the distance to influencer 1 from the potential consumer under price p_1^* on the right side is $\hat{s}_1 = I_1(1 - p_1^*/y)$. Hence, case ii) only occurs when $\hat{s}_1 \leq \beta + \hat{s}_2$. In other words, case iii) occurs when $\hat{s}_1 > \beta + \hat{s}_2$, or equivalently, $\hat{s}_1 - \hat{s}_2 - \beta = \frac{1}{2}I_1(1 - \beta(3I_1 + I_2)/(I_1^2 - I_2^2)) \geq 0$. We can further simplify it as $\beta < \frac{I_1^2-I_2^2}{(3I_1+I_2)} \leq \frac{(I_1-I_2)}{3}$. The proof for the lemma concludes here.

Finally, we can compute the equilibrium prices as follows. Since we already know the two cutoffs, s_1 and $\hat{s}_2$, we can define $W_3 = p_1 * 2I_1(1 - p_1/y) + p_2(\beta - s_1 + \hat{s}_2) - p_1(\beta - s_1 + \hat{s}_2)$, the total profit in case iii). We can apply the first order approach to solve it and get $p_1^* = \frac{y}{2}$ and $p_2^* = \frac{y}{2} + \frac{\beta y}{2I_1}$. We can further compute $\hat{s}_1 = \frac{I_1}{2}$ and $\hat{s}_2 = \frac{\beta I_2}{2(I_1-I_2)}$. Indeed, $\hat{s}_1 > \beta + \hat{s}_2$ holds when $\beta < \frac{(I_1-I_2)}{3}$. In this case, $W_3 = \frac{yI_1}{2} + \frac{\beta^2 I_2}{2(I_1^2-I_2^2)}$. □

A.3 Proof of Proposition 1

Proof. First, we prove an auxiliary result as follows.

Lemma A.3. *Fix $\beta > 0$. Then: (i) if $I_1 + I_2 \leq 2\beta$, $\Pi_{\{1,2\}} = \frac{y(I_1+I_2)}{2}$; and (ii) for $I_2 \to \infty$ (and so is I_1), $\Pi_{\{1,2\}} \to 2\pi y$.*

To show claim i), note that under monopolist pricing (i.e., $p_1 = p_2 = \frac{y}{2}$), the j-th influencer attracts consumers $\|x - \theta_j\| \leq \frac{I_j}{2}$. When $I_1 + I_2 \leq 2\beta$ holds, there exists no overlapping in consumer base and thus the total profit $\Pi_{\{1,2\}} = \Pi_1 + \Pi_2$ and thus it follows

[19] In this case, we can set s_1 to be the consumer size for influencer 1 along the short arc, and then express $p_1 = y(1 - s_1/I_1)$ and $p_2 = y(1 - (\beta - s_1)/I_2)$ and further express the total profit as a function of s_1 and optimize over s_1 to get the optimal solution. The optimal price $p_j^* = y - \frac{\beta y}{I_1+I_2}$ and $W_2 = \frac{2\beta y((I_1+I_2)-\beta)}{(I_1+I_2)}$.

from equation (2). Furthermore, to show claim ii), note that $\Pi_{\{1,2\}}$ is bounded below by $\geq \Pi_1$ because the monopolist can always set $p_1 = \frac{y}{2}$ and $p_2 = y$, so that influencer 2 is effectively serving a zero measure of consumers. Also note that the total profit is bounded above by $2\pi y$. To summarize, $\Pi_1 \leq \Pi_{\{1,2\}} \leq 2\pi y$. However, $\Pi_1 \to 2\pi y$ as $I_1 \to \infty$ and by the sandwich rule, $\lim_{I_2 \to \infty} \Pi_{\{1,2\}} = 2\pi y$.

Second, when the two influencers have zero style dispersion ($\beta = 0$), the equilibrium characterization is simple, that is, the monopolist seller sets $p_1 = p_2 = \frac{y}{2}$, which implies that only Influencer 1 effectively serves consumers whose type x satisfies $\|x - \theta_1\| \leq \frac{I_1}{2}$.

Third, we come to show Proposition 1. On the one hand, we show that both $w_1 + w_2$ and $|w_1 - w_2|$ are both strictly increasing in I_2 when $I_1 + I_2 \leq 2\beta$. By equation (3) and Lemma A.3, we get $w_1 = \frac{(1-\gamma)yI_1}{2}$ and $w_2 = \frac{(1-\gamma)yI_2}{2}$, and thus

$$w_1 + w_2 = \frac{(1-\gamma)y(I_1 + I_2)}{2}, \quad \text{and} \quad |w_1 - w_2| = \frac{(1-\gamma)y(I_1 - I_2)}{2}$$

which are obviously both strictly increasing when $I_1 + I_2$ and $I_1 - I_2$ increase.

On the other hand, we show that $\lim_{I_2 \to \infty} w_1 + w_2 = 0$, which implies that $|w_1 - w_2| \leq w_1 + w_2 \to 0$. By equation (2), we have that $\Pi_1 \to 2\pi y$ and that $\Pi_2 \to 2\pi y$. Furthermore, by Lemma A.3, $\lim_{I_2 \to \infty} \Pi_{\{1,2\}} = 2\pi y$. Thus, by equation (3), $\lim_{I_2 \to \infty} w_j = 0$. The proof concludes. □

A.4 Proof of Proposition 2

Proof. Denote by $\Pi(i,j)$ the profit when seller i hires influencer j. Recall that bilateral Nash bargaining implies that

$$U_i(i,j) = \gamma \Pi(i,j) \quad \text{and} \quad w_j(i,j) = (1-\gamma)\Pi(i,j).$$

Thus, it suffices to show that $U_1(1,1) \geq U_1(1,2)$ and $w_1(1,1) \geq w_1(2,1)$ to establish assortative matching, which further reduce to

$$\Pi(1,1) \geq \Pi(1,2) \quad \text{and} \quad \Pi(1,1) \geq \Pi(2,1). \tag{A.1}$$

Indeed, if these conditions are satisfied, then $k(j) = j$ for $j = 1, 2$ is a stable matching because when Seller 1 is matched with Influencers 1, they both have no incentive to deviate. Given this, Seller 2 and Influencer 2 form a match.

Now, we are ready to show it case by case under balanced matching.

Case (i). Given that $\beta \geq \frac{I_1 + I_2}{2}$, independent of the matching outcome, Seller $j \in \{1,2\}$ charge a monopolist price $p_j^* = \frac{y_j}{2}$, which implies that $\Pi(i,j) = \frac{y_i I_j}{2}$. Obviously, equation (A.1) holds.

Case (ii). First, since $y_1 > y_2$, Seller 1 can always get a positive profit since any consumer sufficiently close the influencer hired by Seller 1 always purchases from her. Thus, if a single seller dominates, it must be Seller 1. Second, we show that negative assortative matching (i.e., $k(j) = 3 - j$ for $j = 1, 2$) never happens. If not, suppose that Seller 1 hires Influencer 2 and sets a price p_1 and beats Seller 2 who hires Influencer 1 and sets any arbitrary price p_2. Now, consider the case that Seller 1 hires Influencer 1 and use the original pricing strategy. For all consumers, they either stay outside of the reach of Seller 2, or fall within the consumer base of Seller 2 but attracted by Seller 1. In the latter case,

$$\begin{aligned} y_1(1 - \|x - \theta_1\|/I_1) - p_1 &> y_1(1 - \|x - \theta_1\|/I_2) - p_1 \\ &\geq y_2(1 - \|x - \theta_2\|/I_1) - p_2 > y_2(1 - \|x - \theta_2\|/I_2) - p_2 \end{aligned}$$

We used the fact that $I_1 \geq I_2$ in the first and the third inequality, and the second inequality follows from the assumed market dominance. This implies that Seller 1 and Influencer 1 will jointly deviate and form a new group, and thus negative assortative matching is sub-optimal.

Case (iii). First, we compute $\Pi(i, j)$ and start with $\Pi(1, 1)$. Note that $\beta = \pi$ implies that the consumer share is symmetric around θ_j for both influnecers. Fix prices (p_1, p_2), the cutoff type is given by the indifference condition that $y_1(1 - s_1/I_1) - p_1 = y_2(1 - s_2/I_2) - p_2$ where $s_1 = \pi - s_2$ is Seller 1's share on one side, and we can solve it to get $s_1 = \frac{I_1(I_2(p_2-p_1)+I_2(y_1-y_2)+\pi y_2)}{y_1 I_2+y_2 I_1}$. Thus, Group 1's profit is $\Pi(1, 1) = 2s_1 p_1$ and Group 2 gets $\Pi(2, 2) = 2(\pi - s_1)p_2$. By taking derivatives over $\Pi(1, 1)$ and $\Pi(2, 2)$ w.r.t. p_1 and p_2 respectively, we get $\frac{\partial \Pi(1,1)}{\partial p_1} = \frac{2I_1 I_2(y_1-y_2+p_2-2p_1)+2\pi I_1 y_2}{y_1 I_2+y_2 I_1}$ and $\frac{\partial \Pi(2,2)}{\partial p_2} = \frac{2I_1 I_2(y_2-y_1+p_1-2p_2)+2\pi I_2 y_1}{y_1 I_2+y_2 I_1}$. The second-order conditions can be directly checked. We use the first-order conditions to get $p_1^* = \frac{I_1 I_2(y_1-y_2)+\pi(I_2 y_1+2I_1 y_2)}{3I_1 I_2}$ and $p_2^* = \frac{I_1 I_2(y_2-y_1)+\pi(I_1 y_2+2I_2 y_1)}{3I_1 I_2}$, and we can further calculate profits as $\Pi(1, 1) = \frac{a_1}{b_1}$ and $\Pi(2, 2) = \frac{a_2}{b_1}$, where $b_1 = 9I_1 I_2(y_1 I_2 + y_2 I_1)$ and

$$a_1 = 2\big(I_1 I_2(y_1 - y_2) + \pi(y_1 I_2 + 2y_2 I_1)\big)^2, \quad a_2 = 2\big(I_1 I_2(y_2 - y_1) + \pi(y_2 I_1 + 2y_1 I_2)\big)^2.$$

Similarly, we calculate $\Pi(1, 2)$ and $\Pi(2, 1)$ under the alternative matching $k(j) = 2 - j$. Specifically, $\Pi(1, 2) = \frac{a_3}{b_2}$ and $\Pi(2, 1) = \frac{a_4}{b_2}$, where $b_2 = 9I_1 I_2(y_1 I_1 + y_2 I_2)$ and

$$a_3 = 2\big(I_1 I_2(y_1 - y_2) + \pi(y_1 I_1 + 2y_2 I_2)\big)^2, \quad a_4 = 2\big(I_1 I_2(y_2 - y_1) + \pi(y_2 I_2 + 2y_1 I_1)\big)^2.$$

To see that $\Pi(1, 1) \geq \Pi(1, 2)$, it suffices to show that $a_1 b_2 - a_3 b_1 \geq 0$. Denote $I_1 = aI_2$, $y_1 = by_2$ and $I_2 = c\pi$. Note that $a \geq 1, b \geq 1$ and $c \geq 1$. Then, $a_1 b_2 - a_3 b_1 = 9ac^5\pi^7 y_2^3 * (g_0 + g_1(c - 1) + g_2(c^2 - 1))$, where $g_2 = (a - 1)a^2(b - 1)^3$, $g_1 = 2a(a^2 - 1)(b - 1)b$ and $g_0 = (4 - b)b + a^2(-1 + 5b - 2b^2 + b^3) + a(4 - 2b + 5b^2 - b^3)$. Obviously, $g_1 \geq 0$ and $g_2 \geq 0$. To see that $g_0 \geq 0$, note that $g_0 = (a - 1)(ab(b - 1)^2 + b^2) + a(b(2b - 1) + (4b - 1) + 4) + 4b > 0$.

Meanwhile, to see that $\Pi(1, 1) \geq \Pi(2, 1)$, it suffices to show that $a_1 - a_4 \geq 0$ because $b_1 \leq b_2$. Again, note that $(a_1 - a_4) \propto 2\pi(y_1 + y_2)(2I_1 + I_2)(y_1 - y_2)\big(\pi I_2 + 2I_1(I_2 - \pi)\big) \geq 0$ when $I_2 \geq \pi$. The proof concludes. □

A.5 Proof of Proposition 3

Proof. Consider the following equilibrium conjecture: (i) in the labor market, it features assortative matching, that is, $k(j) = j$ for $j \in \{1, 2\}$; and (ii) in the product market, Seller 1, matched with Influencer 1, prices the product at $p_1^* = \frac{y_1}{2}$, and earns a total profit of $\Pi_1 = \frac{y_1 I_1}{2}$. Seller 2 is effectively forced out of the market and sets $p_2^* = 0$.

Now, we verify that this constitutes an equilibrium. First, given $p_1^* = \frac{y_1}{2}$, a consumer, whose type x satisfies $\|x - \theta_2\| \le I_2$, always prefers Seller 1 to Seller 2 even when $p_2 = 0$, as long as the following two conditions hold. One, the consumer with type $x = \theta_2$ satisfies $y_1 * \big(1 - \beta/I_1\big) - p_1^* \ge y_2$. Two, the consumer with type x such that $\|x - \theta_2\| = I_2$ & $\|x - \theta_1\| = \beta + I_2$ satisfies $y_1 * \big(1 - (\beta + I_2)/I_1\big) - p_1^* \ge 0$. Simplifying these two equations yields the condition that $\frac{\beta}{I_1} \le \min\{\frac{1}{2} - \frac{y_2}{y_1}, \frac{1}{2} - \frac{I_2}{I_1}\}$.

Second, given consumers' equilibrium choices and Seller 2's pricing strategy, Seller 1 has no incentive to deviate from the monopolist pricing. Meanwhile, given other participants' equilibrium strategies, Seller 2 chooses $p_2^* = 0$ because she cannot attract any consumer by setting $p_2^* > 0$.

Third, anticipating equilibrium profits, Influencer 1 chooses to match with Seller 1 and gets a payoff of $w_1 = \frac{(1-\gamma)y_1 I_1}{2}$. Instead, if she deviates to Seller 2, Influencer 1 can at most get $\hat{w}_1 = \frac{(1-\gamma)y_2 I_1}{2} < w_1$. Given Influencer 1's equilibrium matching choice, Influencer 2 can only match with Seller 2. The proof concludes. $\square$

A.6 Proof of Example 1

Proof. First, note that the two influencers are homogeneous for sellers, which means that sellers' incentives are trivial under balanced matching. Given this and the matched groups' profits, the influencer from the seller-influencer group with a bigger profit has no incentive to deviate when the bargaining power $(\gamma, 1-\gamma)$ is fixed. Thus, the other influencer also has no incentive to deviate, and we get $k(j) = j$ for $j \in \{1, 2\}$.

Second, we construct an equilibrium in which $p_1^C \ge p_2^C \ge 0$ because the first influencer-seller group is stronger in the sense that it offers a better product. In particular, Seller 1 is targeting the most valuable consumers and the cutoff type x^* is pinned down by

$$y_1(1 - \|x^* - \theta\|/I) - p_1 = y_2(1 - \|x^* - \theta\|/I) - p_2$$

Since $y_1 \ge y_2$, all consumers, whose type x satisfies $\|x - \theta\| < \|x^* - \theta\|$, purchase from Seller 1. Furthermore, given this, Seller 2 attracts consumers whose type x satisfies with $\|x - \theta\| \ge \|x^* - \theta\|$ and $\|x - \theta\| \le \|x^{**} - \theta\|$ where $y_2(1 - \|x^{**} - \theta\|/I) - p_2 = 0$. Thus, we can calculate the demand. Specifically, for Seller 1, $q_1 = 2\|x^* - \theta\| = 2I * (1 - \frac{p_1 - p_2}{y_1 - y_2})$, and $q_2 = 2(\|x^{**} - \theta\| - \|x^* - \theta\|) = 2I * (\frac{p_1 - p_2}{y_1 - y_2} - \frac{p_2}{y_2})$ for Seller 2. Then, we can further compute profits as $\Pi_1 = 2I * p_1 * (1 - \frac{p_1 - p_2}{y_1 - y_2})$ and $\Pi_2 = 2I * p_2 * (\frac{p_1 - p_2}{y_1 - y_2} - \frac{p_2}{y_2})$. Taking derivatives over

Π_m with respect to p_m for $m = 1, 2$,

$$\frac{2I(-2p_1^C + p_2^C + y_1 - y_2)}{y_1 - y_2} = 0, \text{ and } \frac{I(-4p_2^C y_1 + 2p_1^C y_2)}{(y_1 - y_2)y_2} = 0$$

Solving these two equations yields the desired solution. Moreover, the second order conditions are trivially satisfied and thus the first-order conditions fully characterize all solutions. By submitting the prices $(p_1^C, p_2^C) = (\frac{2y_1(y_1-y_2)}{4y_1-y_2}, \frac{y_2(y_1-y_2)}{4y_1-y_2})$ into the demand functions and the profits, we obtain all the desired results after simple algebra manipulation. Finally, we impose $I \leq \pi$ so that the consumer size is well-defined (i.e., $3Iy_1/(4y_1 - y_2) \leq \pi$). □

A.7 Proof of Example 2

Proof. The matching part is trivial. Hence, we only consider price competition in the product market. We construct an equilibrium which features $p_1^C \geq p_2^C \geq 0$. This is because, if $p_1 < p_2$, then the second seller is priced out of the market because $\beta = 0$, $y_1 = y_2$ and $I_1 \geq I_2$.

First, we compute consumer share for both sellers. Given that $p_1 \geq p_2$, the consumer whose type x is close to θ will purchase from seller 2 because the utility gap, compared with purchasing from seller 1, equals $(p_1 - p_2) - y * \|x - \theta\| * (I_1 - I_2)/I_1 I_2$. This further implies that there exists a cutoff style type x^* such that all consumers satisfying $\|x - \theta\| < \|x^* - \theta\|$ are served by seller 2, that is, $(p_1 - p_2) = y * \|x - \theta\| * (I_1 - I_2)/I_1 I_2$. Solving it yields $\|x^* - \theta\| = \frac{I_1 I_2(p_1-p_2)}{y(I_1-I_2)}$. Meanwhile, consumers with $\|x - \theta\| > \|x^* - \theta\|$ are attracted by seller 1 as long as it generates a positive utility, which implies a second cutoff type x^{**} given by $y\,(1 - \|x^{**} - \theta\|/I_1) - p_1 = 0$, which yields $\|x^{**} - \theta\| \leq I_1\,(1 - p_1/y)$. Thus, consumers whose type satisfies $\|x^* - \theta\| < \|x - \theta\| \leq \|x^{**} - \theta\|$ purchase from seller 1.

Second, we compute profits as $\Pi_1 = p_1 * \big((1-p_1/y)I_1 - \frac{I_1 I_2(p_1-p_2)}{y(I_1-I_2)}\big)$ and $\Pi_2 = p_2 * \frac{I_1 I_2(p_1-p_2)}{y(I_1-I_2)}$. Obviously, Π_j is concave in p_j for $j = 1, 2$. Thus, the equilibrium is pinned down by first order conditions such that $\frac{I_1}{y(I_1-I_2)}\big(I_2(p_2 - y) + I_1(y - 2p_1)\big) = 0$ and $\frac{I_1 I_2}{y(I_1-I_2)}(p_1 - 2p_2) = 0$. Solving these two equations, we get the desired solution (p_1^C, p_2^C), and by simple algebra, we can obtain the profits for both groups. Furthermore, note that we impose $I_1 \leq \pi$ so that the consumer size are well-defined (i.e., $\|x^{**} - \theta\| \leq \pi$). The proof concludes. □

A.8 Proof of Example 3

Proof. Here, we only construct the equilibrium for price competition. First, we compute sellers' profits Π_j^C for $j = 1, 2$. Fix p_2 and consider seller 1's profit. First of all, if p_1 and p_2 are close and not too small (to be discussed shortly), we expect two sellers share consumers along the short arc between θ_1 and θ_2, which is pinned down by equating the utility from purchasing the two sellers, that is, $p_2 - p_1 = (\|x^* - \theta_1\| - \|x^* - \theta_2\|) * y/I$. This is an indifference condition. Define $s_j = \|x - \theta_j\|$ for $j = 1, 2$. If the two sellers indeed share consumers along the short arc, we further have $\sum_{j=1}^{2} s_j = \beta$. Combined with the indifference condition, we can solve them to get $s_j = \frac{1}{2}(\beta + (p_{3-j} - p_j) * I/y)$. Note that s_j is well-defined only when all consumers are served along the short arc between θ_1 and θ_2, that is, $I * (1 - p_1/y) + I * (1 - p_2/y) \geq \|\theta_1 - \theta_2\|$, which further reduces to $p_1 + p_2 \leq (2 - \beta/I) * y$. We will verify this condition later and argue how to find the equilibrium when it fails. In this case, $\Pi_1^C = p_1 * (s_1 + I(1 - p_1/y))$, where $s_j = \frac{1}{2}(\beta + (p_{3-j} - p_j) * I/y)$.

Furthermore, when p_1 is sufficiently low relative to p_2, then $s_1 > \beta$ might occur, which implies $\|x - \theta_1\| - \|x - \theta_2\| = \beta$. In particular, seller 1 beats seller 2 and grabs the whole market when $p_1 \leq p_2 - y\beta/I$, and seller 1 loses all consumers when $p_1 \geq p_2 + y\beta/I$. In other words, when $p_1 \leq p_2 - \beta * y/I$, $\Pi_1^C = 2p_1 * (1 - p_1/y)I$, and when $p_1 \geq p_2 + \beta * y/I$, we have $\Pi_1^C = 0$. Note that there are two discontinuity points for Π_1^C. Similarly, we can write down the profit function for seller 2 by symmetry.

Second, we solve the price competition equilibrium. Obviously, it is sub-optimal for Seller 1 to set $p_1 \geq p_2 + y\beta/I$, which leads to a zero profit. Then, we start with the case in which p_1 and p_2 are sufficiently close. We can plug in the formula for s_j and derive first-order conditions: $\frac{y\beta + I(p_2 - 6p_1 + 2y)}{2y} = 0$ and $\frac{y\beta + I(p_1 - 6p_2 + 2y)}{2y} = 0$. Solving these two equations yields $p_1^* = p_2^* = \frac{y(2I+\beta)}{5I}$, and we can further plug p_j^* into Π_j^C to get $\Pi_1^C = \frac{3y(2I+\beta)^2}{50I}$, as long as the price range condition that $p_2^* - y\beta/I < p_1^* < p_2^* + y\beta/I$ is satisfied.

However, we also need to check that $p_1^* + p_2^* \leq (2 - \beta/I) * y$ indeed holds, which further reduces to $\beta \leq 6I/7$. When it fails (i.e., $\beta > \frac{6}{7}I$), define $\hat{p}_1^* = \hat{p}_2^* = (1 - \beta/2I) * y$. Fix $p_2 = \hat{p}_2^*$, the term $p_1 * ((1 - p_1/y) * I + s_1)$ is strictly increasing in p_1 for $p_1 \leq p_1^*$. Thus, the first (influencer-seller) group has no incentive to deviate downward. Meanwhile, for $p_1 > \hat{p}_1^*$, the term $p_1 * (1 - p_1/y) * I$ is strictly decreasing for $p_1 > \hat{p}_1^*$ since $\hat{p}_1^* \geq \frac{y}{2}$. This implies that the first group has no incentive to deviate upward. Thus, $(\hat{p}_1^*, \hat{p}_2^*)$ constitutes an equilibrium when $\frac{6}{7}I < \beta < I$, as long as no seller is priced out of the market.

Third, to finish the equilibrium construction, we need to ensure that the case is also globally optimal in which p_1 and p_2 are close, which means that Seller 1 has no incentive to deviate downward by a big price cut to undercut Seller 2 and force her out of the product market. To this end, Seller 1 only needs to set $p_1 = p_2^* - y\beta/I$, which leads to a profit:

$$\hat{\Pi}_1 = 2I * (p_2^* - y\beta/I) * (1 - (p_2^* - y\beta/I)/y)$$

Note that this is the most profitable deviation since $\Pi_1 = 2I(1 - p_1/y)p_1$ is strictly decreasing for all $p_1 \leq p_2^* - y\beta/I$. To support the equilibrium, it requires that $\hat{\Pi}_1 \leq \Pi_1^C$, which leads to the condition of β_0. The proof concludes. □

A.9 Proof of Proposition 4

Proof. First, by the formula of $\Pi_1^C(\beta)$ (see equation (6)), we can check that $\Pi_1^C(\beta)$ is strictly increasing for all $\beta \in \big(\beta_0, I\big]$, which implies that the term, $G(\beta) := \Pi_1^C(\beta) - \big(\Pi_1^C(I) - f_E\big)$, has at most one solution for all $\beta \in \big(\beta_0, I\big]$. Indeed, note that $G(I) = f_E$ and $G(\beta_0) < 0$ by the assumed condition equation (7). Hence, there exists a unique $\tilde{\beta}$ well defined. Further, by monotonicity, we have $G(\beta) > 0$ for $\beta > \tilde{\beta}$ and $G(\beta) < 0$ for $\beta \in [\beta_0, \tilde{\beta})$.

Second, consider $\beta > \tilde{\beta}$. By symmetry, we only need to consider the incentive for Seller 1. Given that Seller 2 hires Influencer 2, Seller 1 can choose to hire Influencer 1 and get a payoff of $\gamma\Pi_1^C(\beta)$, or pay a fixed cost and select a location such that $\|\alpha_1 - \theta_2\| \geq I$), which yields a payoff of $\gamma(\Pi_1^C(I) - f_E)$. However, note that $G(\beta) > 0$ for $\beta > \tilde{\beta}$, or equivalently, $\gamma\Pi_1^C(\beta) \geq \gamma(\Pi_1^C(I) - f_E)$. Thus, it is optimal for Seller 1 to hire Influencer 1.

Third, consider $\beta \leq \tilde{\beta}$. We show that there exists an equilibrium in which Seller 1 hires Influencer 1 and Seller 2 pays the fixed cost f_E. Seller 2, given that Seller 1 hires Influencer 1, can choose to hire Influencer 2 and obtain a payoff of $\gamma\Pi_1^C(\beta)$, or pay the fixed cost f_E and select a location to avoid competition and gets $\gamma(\Pi_1^C(I) - f_E)$. By the fact $G(\beta) < 0$ for $\beta \leq \tilde{\beta}$, she has an incentive to pay the fixed cost f_E. Furthermore, given that Seller 2 chooses maximum differentiation, Seller 1 has the incentive to hire Influencer 1 and gets a profit of $\gamma\Pi_1^C(I)$. The proof concludes. □

A.10 Proof of Lemma 2

Proof. Note that by bilateral Nash bargaining, we can focus on the joint group profits when considering R&D investments. The proof consists of two parts.

Part (i). Compute $\Pi_{H,H}$, $\Pi_{L,L}$, $\Pi_{L,H}^j$ and $\Pi_{H,L}^j$ for $j = 1, 2$.

First, by Example 3 and condition i), $\Pi_{H,H} = yA(\beta)$ and $\Pi_{L,L} = A(\beta)$ for $j = 1, 2$, where $A(\beta) = \frac{3}{50}(2I + \beta)^2$. Second, we compute profits under asymmetric quality investment, say (H, L), meaning that only Group 1 chooses high quality. By the assumed condition ii), Group 2 can still attract some consumers even Group 1 sets a price at $p_1 = 0$. Now, denote by p_j the price charged by Group $j = 1, 2$. For Group 1, the size of consumers served along the long arc is $y_1(1-\|x-\theta_1\|)-p_1 \geq 0$ or equivalently $(1-p_1/y)$. Meanwhile, along the short arc, the cutoff consumer's type x^* satisfies $y(1 - \|x^* - \theta_1\|) - p_1 = 1 - (\beta - \|x^* - \theta_2\|) - p_2$, which yields $s_1 := \|x^* - \theta_1\| = \frac{p_2 - p_1 + (y-1)+\beta}{y+1}$ and $s_2 = \beta - s_1$. We can further compute group profits as $\Pi_{H,L}^j = (1 - p_j/y_j) * p_j + p_1 s_j$ for $j = 1, 2$. Using the first-order conditions, we get $p_1^* = \frac{2(1+2\beta)y+(8+3\beta)y^2+4y^3}{8+19y+8y^2}$ and $p_2^* = \frac{4+(8+3\beta)y+2(1+2\beta)y^2}{8+19y+8y^2}$. Meanwhile, we need to make sure that the cutoff type x^* gets a non-negative utility, that is, $y(1-\|x^*-\theta\|)-p_1^* \geq 0$, which reduces to $\frac{y\left(2(5-6\beta)(1+y^2)+(22-25\beta)y\right)}{(1+y)(8+19y+8y^2)} \geq 0$, which trivially holds under condition i). Furthermore, we can directly verify that the second order conditions are satisfied. Thus, we can compute $\Pi_{H,L}^1 = \frac{y(1+2y)\left(2+8y+4y^2+\beta(4+3y)\right)^2}{(1+y)(8+19y+8y^2)^2}$ and $\Pi_{H,L}^2 = \frac{y(2+y)\left(4+8y+2y^2+\beta(3y+4y^2)\right)^2}{(1+y)(8+19y+8y^2)^2}$. Finally, note that $\Pi_{H,L}^j = \Pi_{L,H}^i$ for $i \neq j$.

Part (ii). Nash Equilibrium Construction.

We first state two properties as below:

1) Fix y. $V_1(\beta, y) > V_2(\beta, y)$. Here, $V_1(\beta, y) = \Pi_{H,L}^1 - \Pi_{L,L}$ and $V_2(\beta, y) = \Pi_{H,H} - \Pi_{L,H}^1$.

2) Fix y. Both $V_1(\beta, y)$ and $V_2(\beta, y)$ are strictly increasing in β.

We first verify the equilibrium under these two properties and prove them shortly.

First, by condition iii), $V_1(\beta_0, y) < F_V < V_2(5/6, y)$ and property 1), we have

$$V_1(5/6, y) > V_2(5/6, y) > F_V > V_1(\beta_0, y) > V_2(\beta, y)$$

which, together with property 2), the strict monotonicity of $V_1(\beta, y)$ and $V_2(\beta, y)$, implies that there exist $\underline{\beta}, \overline{\beta} \in (\beta_0, 5/6)$ such that $F_V = V_1(\underline{\beta}, y) = V_2(\overline{\beta}, y)$ and $\overline{\beta} > \underline{\beta}$.

To summarize:

(a) For $\beta \geq \overline{\beta}$, $V_2(\beta, y) \geq F_V$, or equivalently, $\Pi_{H,H} \geq F_V + \Pi^1_{L,H}$. Thus, given that Influencer 2 chooses high quality, it is optimal for Influencer 1 to invest. By symmetry, Influencer 2 also chooses to invest, and thus (H, H) is a Nash Equilibrium.

(b) For $\underline{\beta} \leq \beta < \overline{\beta}$, we have both $V_1(\beta, y) \geq F_V$ and $V_2(\beta, y) < F_V$, that is, $\Pi^1_{H,L} - \Pi_{L,L} \geq F_V$ and $\Pi_{H,H} - \Pi^2_{H,L} < F_V$ since $\Pi^2_{H,L} = \Pi^1_{L,H}$. These two conditions read as follows. One, given that Influencer 2 chooses low quality, it is optimal for Influencer 1 to invest. Two, given that Influencer 1 chooses high quality, it is optimal for Influencer 2 to choose low quality. Thus, (H, L) is a Nash Equilibrium, so is (L, H) by symmetry.

(c) For $\beta < \underline{\beta}$, $V_1(\beta, y) < F_V$, or equivalently, $\Pi^1_{H,L} - \Pi_{L,L} < F_V$, which implies that even if Influencer 2 does not invest, it is optimal for Influencer 1 not to invest. By symmetry, (L, L) is a Nash Equilibrium.

Now, it suffices to verify property 1) and 2) on $V_1(\beta, y)$ and $V_2(\beta, y)$. To this end, we write down the formulas and check them one by one.

First, with y fixed, $V_1(\beta, y)$ and $V_2(\beta, y)$ are strictly increasing in β. Note that

$$\begin{aligned} V_1(\beta, y) &= \frac{(y-1)}{50(y+1)(8+19y+8y^2)} \\ &\times \left\{1600y^5 + 32M_1 * y^4 + 4M_2 * y^3 + 25M_3 * y^2 + 8M_4 * y + 192(2+\beta)^2\right\} \\ V_2(\beta, y) &= \frac{(y-1)}{50(y+1)(8+19y+8y^2)} \\ &\times \left\{1600 + 32M_1 * y + 4M_2 * y^2 + 25M_3 * y^3 + 8M_4 * y^4 + 192(2+\beta)^2 y^5\right\} \end{aligned}$$

where $M_1 = 251 + 51\beta - 6\beta^2$, $M_2 = 3704 + 1604\beta - 99\beta^2$, $M_3 = 500 + 340\beta + 3\beta^2$ and $M_4 = 623 + 548\beta + 62\beta^2$. With simple algebra, we can show that all the quadratic terms $M_j (j = 1, 2, 3, 4)$ are positive and strictly increasing for $\beta \in (\beta_0, 5/6)$, which verifies the monotonicity of both $V_1(\beta, y)$ and $V_2(\beta, y)$.

Second, with y fixed, $V_1(\beta, y) > V_2(\beta, y)$ for all $\beta \in (\beta_0, 5/6)$.

$$V_1(\beta, y) - V_2(\beta, y) = \frac{(y-1)^2}{50(y+1)(8+19y+8y^2)} \times \left\{64 * M_5 + 40 * M_6 y + M_7 * y^2 + 40 * M_6 y^3 + 64 * M_5 y^4\right\}$$

where $M_5 = 13 - 12\beta - 3\beta^2$, $M_6 = 97 - 88\beta - 22\beta^2$, and $M_7 = 6196 - 5604\beta - 1351\beta^2$.

To see that $M_5 > 0$, note that there are two solutions $\beta_1 \approx -4.89$ and $\beta_2 \approx 0.89$. Hence, $M_5 > 0$ for all $\beta \in (-4.89, 0.89)$, and thus $M_5 > 0$ for all $\beta \in (\beta_0, 5/6)$. We can prove $M_6 > 0$ and $M_7 > 0$ similarly. The proof concludes. □

A.11 Proof of Proposition 6

Proof. We use superscript $l \in \{a, b\}$ to denote a term before and after power acquisition. For example, w_j^b and w_j^a correspond to Influencer j's wage before and after power acquisition. Define SW to be the total welfare when all potential consumers are served by the best-matched influencer. Define $\Delta w_j = w_j^a - w_j^b$, and $\Delta SW = SW^a - SW^b$. Obviously, Influencer j will invest iff $C_P < \Delta w_j$. Furthermore,

(i) there exists under-investment iff: $\Delta SW > |\{j : C_P > \Delta w_j\}| * C_P$;

(ii) there exists over-investment iff: $\Delta SW < |\{j : C_P > \Delta w_j\}| * C_P$.

Case (i). The condition that $kI < \frac{\pi}{2}$ guarantees the feasibility of monopolist pricing (i.e., $p_1^* = p_2^* = \frac{y}{2}$), and thus $\Pi_j^b = \frac{yI}{2}$ and $\Pi_j^a = \frac{kyI}{2}$. Note that $w_j^l = (1-\gamma)\Pi_j^l$ for $j \in \{1, 2\}$ and $l \in \{a, b\}$. Hence, influencer j will invest iff $C_P \leq w_j^a - w_j^b = \frac{(1-\gamma)(k-1)yI}{2}$. Meanwhile, note that the total welfare, before power acquisition, is $SW^b = 2\int_{y(1-\|x-\theta_1\|/I)\geq 0} y(1-\|x-\theta_1\|)dx = 2yI$ and similarly $SW^a = 2kyI$, and thus $\Delta SW = SW^a - SW^b = 2(k-1)yI$. Thus, there exists under-investment if $\frac{(1-\gamma)(k-1)yI}{2} < C_P < 2(k-1)yI$.

Case (ii). First, note that both sellers can get positive profits by attracting consumers sufficiently loyal to their own influencers. Meanwhile, since $I \geq \frac{3}{2}\pi \geq \pi$, all consumers are served. The condition that $I \geq \frac{3}{2}\pi$ ensures that the cutoff consumer indifferent between purchasing from the first seller and the second one will get a positive utility. Since some influencer(s) might acquire additional power, we use I_1 and I_2 in the equilibrium construction.

Second, given prices (p_1, p_2), the cutoff consumer's type x^* is determined by the equation $y(1 - s_1/I_1) - p_1 = y(1 - s_2/I_2) - p_2$, where $s_j := \|x^* - \theta_j\|$ is the size of consumers attracted by seller j on one side. We can solve it to get $s_j = \frac{\pi y + I_1 I_2(p_{3-j} - p_j)}{y(I_1+I_2)}$ and thus seller j's profit is given by $\Pi_j = 2s_j p_j$. By taking derivatives w.r.t. p_j over Π_j, we get $\frac{\partial \Pi_j}{\partial p_j} = \frac{2I_1I_2(p_{3-j}-2p_j)+2I_j\pi y}{y(I_1+I_2)}$. We can solve it to get $p_1 = \frac{\beta y(2I_1+I_2)}{3I_1I_2}$ and $p_2 = \frac{\beta y(I_1+2I_2)}{3I_1I_2}$. The profits are given by

$$\Pi_j = \frac{2\pi^2 y(2I_j + I_{3-j})^2}{9I_1I_2(I_1 + I_2)} \tag{A.2}$$

and the cutoff type consumer gets a utility of $u(x^*) = \frac{y\left(3I_1I_2(I_1+I_2)-2\pi(I_1^2+I_2^2)-5\pi I_1I_2\right)}{3I_1I_2(I_1+I_2)}$. We can verify that the cutoff type consumer indeed gets a positive utility. To see it, when $I_1 = I_2 = I$, then it reduces to $y - \frac{3\pi y}{2I} \geq 0$, which holds if and only if $I \geq \frac{3}{2}\pi$. When $I_1 = I_2 = kI$, it reduces to $kI \geq \frac{3}{2}\pi$. When $I_1 = kI$ and $I_2 = I$, it reduces to $u(x^*) = y - \frac{\pi y(2+5k+2k^2)}{3Ik(1+k)}$. Note that $\frac{du(x^*)}{dk} = \frac{\pi y(2+4k+3k^2)}{3k^2(1+k)^2} > 0$, and thus $u(x^*) \geq 0$ for any $k \geq 1$ as long as $I \geq \frac{3}{2}\pi$.

Third, we analyze the incentive for power acquisition. Obviously, before power acquisition, $w_1^b = w_2^b = \frac{(1-\gamma)\pi^2 y}{I}$. Next, we consider the incentive for power acquisition when the opponent influencer, say Influencer 2, does not acquire additional power. This means $I_1 = kI$ and $I_2 = I$. After power acquisition by influencer 1, by the profit formula, equation (A.2), we have $w_1^a = \frac{2\pi^2 y(1-\gamma)(1+2k)^2}{9Ik(1+k)}$ and $w_2^a = \frac{2\pi^2 y(1-\gamma)(2+k)^2}{9Ik(1+k)}$. This further implies $\Delta w_1 = \frac{\pi^2 y(1-\gamma)(2+k)(1-k)}{9Ik(1+k)} < 0$ and thus influencer 1 has no incentive to acquire additional power. Finally, we consider the incentive for power acquisition when the opponent influencer acquires power. By symmetry, we consider Influencer 1. If he chooses not to acquire power, he gets $\hat{w}_1^b = \frac{2\pi^2 y(1-\gamma)(2+k)^2}{9Ik(1+k)}$. If he acquires power, he gets $\hat{w}_1^a = \frac{(1-\gamma)\pi^2 y}{kI}$. Thus, $\Delta \hat{w}_1 = \frac{\pi^2 y(1-\gamma)(1-k)(1+2k)}{9Ik(1+k)} < 0$.

Fourth, we compute the cost range of C_P where power acquisition improves social welfare. Since internal transfer from consumers to sellers through product price does not impact total welfare, we can measure social welfare as $SW = \sum_{j=1}^{2} \int_{\|x-\theta_j\| \le s_j} y * (1 - \|x - \theta_j\|/I_j)dx$. Then we can compute the social welfare SW_0 when no influencer acquires power, SW_1 when only one influencer acquires power, and SW_2 when both influencers acquire power as follows: $SW_0 = \frac{\pi(4I-\pi)y}{2I}$, $SW_1 = \frac{1}{9}\pi y \left(18 - \frac{\pi\left(k^2+7k+1\right)}{I(k^2+k)}\right)$, and $SW_2 = \frac{\pi(4kI-\pi)y}{2kI}$. It is easy to show that $SW_2 > SW_1 > SW_0$ and $2(SW_1 - SW_0) > SW_2 - SW_0$, and thus power acquisition improves social welfare as long as $C_P < \min\{SW_1 - SW_0, \frac{1}{2}(SW_2 - SW_0)\} = \frac{\pi^2 y(7k+2)(k-1)}{18Ik(k+1)}$. The proof concludes. □

A.12 Proof of Proposition 7

Proof. Consider the following equilibrium strategies: (i) On-equilibrium path, that is, $\beta \ge \frac{I_1+I_2}{2}$. We specify that $k(j) = j$, and that payoffs and wages for sellers and influencers are given by $U_j = \gamma \Pi_j$ and $w_j = (1-\gamma)\Pi_j$ where $\Pi_j = \frac{y_j I_j}{2}$. Note that by Proposition 2, we only need to verify that $\beta \ge \frac{I_1+I_2}{2}$. (ii) Off-equilibrium path, that is, $\beta < \frac{I_1+I_2}{2}$. We specify that Seller 1 always rejects Influencer 2, and Influencer 1 always rejects Seller 2. Furthermore, given the matching outcome, the two seller-influencer groups play a price competition game and if an equilibrium exists, payoff are specified by Nash bargaining. Also note that when pure strategy equilibria fail to exist, payoffs for both seller-influencer groups are strictly bounded away from their monopolist profits $\Pi_j^m = \frac{y_j I_j}{2}$.

The only thing we need to prove is that $\beta \ge \frac{I_1+I_2}{2}$. Consider Influencer 2's incentive and the argument for Influencer 1 is identical. On equilibrium path, he gets a wage of $w_2 = \frac{(1-\gamma)y_2 I_2}{2}$. Instead, if he chooses θ_2 such that $\|\theta_1^* - \theta_2\| < \frac{I_1+I_2}{2}$. Since seller 1 is still matched with Influencer 1 on the off-equilibrium path, his wage $\tilde{w}_2$ satisfies that $\tilde{w}_2 \le w_2$ because the joint profit $\tilde{\Pi}_2$ satisfies that $\tilde{\Pi}_2 \le p_2(1-p_2/y_2)I_2 \le p_2^m(1-p_2^m/y)I_2 = \Pi_2^m = \frac{y_2 I_2}{2}$. The first inequality says that Influencer 2 can at most attract all consumers within his influence reach, and the second one says that it is weakly dominated by the monopoly price. Hence, he has an incentive to choose $\|\theta_1^* - \theta_2^*\| \ge \frac{I_1+I_2}{2}$. □

A.13 Proof of Lemma 3

Proof. The proof consists of two parts.

Part (i). We check three cases one by one. First, consider heterogeneous product quality (i.e., $I_1 = I_2 = I, \beta = 0$ and $y_1 \geq y_2$). Example 1 characterizes equilibrium wages w_j for $j = 1, 2$ under balanced matching. Now, consider unbalanced matching. If Seller 1 hires both influencers, $\hat{\Pi}_{\{1,2\}} = \frac{y_1 I}{2} = \hat{\Pi}_{\{1\}} = \hat{\Pi}_{\{2\}}$, which implies that $\hat{w}_1 = \hat{w}_2 = 0$ by bilateral Nash bargaining and thus Influencer 2 rejects being hired by Seller 1 together with Influencer 1 since $w_2 = \frac{(1-\gamma)\Pi_2^C}{2} > \hat{w}_2$. Similarly, we can show that Influencer 2 also rejects being hired together with Influencer 1 by Seller 2.

Second, consider heterogeneous influence power (i.e., $\beta = 0$, $y_1 = y_2$ and $I_1 \geq I_2$). Similarly, Example 2 characterizes equilibrium wages w_j for $j = 1, 2$. Now, consider unbalanced matching. If Seller 1 hires both influencers, then the optimal pricing strategy is $p_1^* = p_2^* = \frac{y_1}{2}$. Note that $\hat{\Pi}_{\{1,2\}} = \hat{\Pi}_{\{1\}} = \frac{y_1 I_1}{2}$, which implies $\hat{w}_2 = 0$ by bilateral Nash bargaining. Since $w_2 = \frac{(1-\gamma)\Pi_2^C}{2} > \hat{w}_2$, Influencer 2 rejects being hired together with Influencer 1 by Seller 1. Similarly, Influencer 2 rejects the offer that Seller 2 hires both influencers.

Third, consider style dispersion (i.e., $y_1 = y_2 = y$, $I_1 = I_2 = I$ and $\beta > 0$). Here, we assume $\beta \geq \beta_0$ so that by Example 3, $w_j = (1-\gamma)\Pi_1^C(\beta)$, where $\Pi_1^C(\beta)$ is given by equation (6). Now, consider unbalanced matching. By Lemma A.2, $\hat{w}_1 = \hat{w}_2 = (1-\gamma)(\Pi_{\{1,2\}} - \Pi_1) = (1-\gamma)\left(\Pi_{\{1,2\}} - \frac{yI}{2}\right)$, where $\Pi_{\{1,2\}}$ is defined in Lemma A.2. Then, it reduces to check that $\Pi_1^C \geq \Pi_{\{1,2\}} - \frac{yI}{2}$. Note that it suffices to show it for the special case that $y = 1$ and $I = 1$. We can check it case by case. For instance, when $\beta \in [\beta_0, \frac{2I}{3}]$, it reduces to $\frac{3}{50}(2+\beta)^2 \geq \frac{1}{8}(2+\beta)^2 - \frac{1}{2}$. Because $\frac{3}{50} < \frac{1}{8}$, the inequality is most restrictive when $\beta = \frac{2}{3}$. It is easy to check that it indeed holds for $\beta = \frac{2}{3}$ for all $\beta \in [0, \frac{2}{3}]$, so it is true for all $\beta \in [\beta_0, \frac{2}{3}]$. Similarly, we can check it for when $\beta \in [2/3, 6/7]$, $\beta \in (6/7, 1]$ and $\beta > 1$. To summarize, $\hat{w}_1 < w_1$, and thus Influencer 1 rejects being hired together Influencer 2 by Seller 1. By symmetry, Seller 2 cannot hire both influencers in equilibrium.

Part (ii). First, consider balanced matching. Then, by Proposition 2, case i), $k(j) = j$ for $j = 1, 2$, and payoffs are given by: $U_1 = \frac{\gamma y_1 I_1}{2}$, $U_2 = \frac{\gamma y_2 I_2}{2}$, $w_1 = \frac{(1-\gamma)y_1 I_1}{2}$ and $w_2 = \frac{(1-\gamma)y_2 I_2}{2}$. Second, consider unbalanced matching. If Seller 1 hires both influencers, then $\Pi_{1,2} = \Pi_1 + \Pi_2$, $\Pi_1 = \frac{y_1 I_1}{2}$ and $\Pi_2 = \frac{y_1 I_2}{2}$, and thus $U_1 = \frac{\gamma y_1(I_1+I_2)}{2}$, $U_2 = 0$, $w_1 = \frac{(1-\gamma)y_1 I_1}{2}$ and $w_2 = \frac{(1-\gamma)y_1 I_2}{2}$. Meanwhile, if Seller 2 hires both influencers, then $\Pi_{1,2} = \Pi_1 + \Pi_2$, $\Pi_1 = \frac{y_2 I_1}{2}$ and $\Pi_2 = \frac{y_2 I_2}{2}$, and thus $U_1 = 0$, $U_2 = \frac{\gamma y_2(I_1+I_2)}{2}$, $w_1 = \frac{(1-\gamma)y_2 I_1}{2}$ and $w_2 = \frac{(1-\gamma)y_2 I_2}{2}$. By comparing influencers' wages, we can see that unbalanced matching with Seller 1 hiring both influencers is incentive compatible for both influencers. The proof concludes. □

A.14 Proof of Proposition 8

Proof. Note that when products are homogeneous (i.e., $y_1 = y_2$), since transfers between influencers, consumers and sellers do not impact total welfare, efficiency only depends on the size of consumers served, which further depends on equilibrium prices. The proof consists of two parts.

Part (i). Denote W_U and W_B to be the total welfare under unbalanced matching and balanced matching respectively. Now, it suffices to show that $W_U \leq W_B$ and we have three cases:

Case (1). Heterogeneous influencer power (i.e., $y_1 = y_2 = y$, $\beta = 0$ and $I_1 \geq I_2$). Under unbalanced matching, $p_1^* = p_2^* = \frac{y}{2}$, and the total size of consumer served is just I_1 (i.e., $\{x \in \mathbf{S}^1 : y\left(1 - \frac{1}{I_1}\|x-\theta\|\right) - p_1^* \geq 0\}$). In contrast, under balanced matching, the marginal consumer faces an equilibrium price given by $p_1^C = \frac{2y(I_1-I_2)}{4I_1-I_2}$ (see Example 2), which implies that the total size of consumer base is bigger than I_1 because $\frac{2y(I_1-I_2)}{4I_1-I_2} \leq \frac{y}{2}$. Thus, $W_U \leq W_B$.

Case (2). Heterogeneous influencers' style type (i.e., $y_1 = y_2$, $I_1 = I_2$ and $\beta > 0$). Here, we only focus on the case in which a pure strategy equilibrium exists (i.e., $\beta \geq \beta_0$) and the equilibrium prices are p_j^C in Example 3. In contrast, under unbalanced matching, the equilibrium prices are given by p_j^* in Lemma A.2. We can directly verify that $p_1^C = p_2^C \leq p_1^* = p_2^*$ for $\beta \geq \beta_0$. Thus, total welfare is higher under regulated matching, because a consumer who purchases the product under unmatched matching is also willing to buy it under regulated matching. Hence, $W_U \leq W_B$.

Case (3). Heterogeneous product quality (i.e., $I_1 = I_2 = I$, $\beta = 0$ and $y_1 \geq y_2$). Under unbalanced matching, seller 1 hires both influencers and set a price at $p_j^* = \frac{y_j}{2}$. Influencer 2 is not active in the market. Hence, $W_U = \int_{\left\{x \in \mathbf{S}^1 : \|x-\theta\| \leq \frac{1}{2}I\right\}} y(1 - \|x-\theta\|/I)dx = \frac{3}{4}y_1 I$.

In contrast, under balanced matching, the equilibrium outcome is given by Example 1, and thus $W_B = \int_{R_1} y(1-\|x-\theta\|/I_1)dx + \int_{R_2} y(1-\|x-\theta\|/I_2)dx$, where $R_1 = \{x \in \mathbf{S}^1 : \|x-\theta\| \leq \frac{2Iy_1}{4y_1-y_2}\}$ and $R_2 = \{x \in \mathbf{S}^1 : \frac{2Iy_1}{4y_1-y_2} < \|x-\theta\| \leq \frac{3Iy_1}{4y_1-y_2}\}$. We can further compute $W_B = \frac{Iy_1(14y_1^2-4y_1y_2-y_2^2)}{(4y_1-y_2)^2} = Iy_1 * f(x)$ where $f(x) = (14x^2-x-1)/(4x-1)^2$ and $x := y_1/y_2$.[20] Note that $f'(x) = -\frac{12(x-1)}{(4x-1)^3} < 0$, and thus $f(x)$ is strictly decreasing for $x > 1$. Moreover, since $\lim_{x\to 1} f(x) = 1$ and $\lim_{x\to\infty} f(x) = \frac{7}{8}$, we have $W_B \geq \frac{7}{8}Iy_1 > \frac{3}{4}Iy_1 = W_U$.

Part (ii). First, note that when $\beta \geq \frac{I_1+I_2}{2}$, each influencer charges a monopolist price $p_j^* = \frac{y}{2}$ when hired by a seller with product quality y, and serves a sub-population such that $x \in R_j$ where $R_j := \left\{x \in \mathbf{S}^1 : \|x-\theta_j\| \leq \frac{I}{2}\right\}$. Hence, there is zero over-lapping among consumers served by the two influencers.

Under unbalanced matching, both influencers are hired by Seller 1. Furthermore, similar to that of unbalanced matching in the heterogeneous product quality case, we can compute that $W_U = \frac{3}{4}y_1(I_1+I_2)$. Similarly, under balanced matching, $W_B = \frac{3}{4}y_1I_1 + \frac{3}{4}y_2I_2$. Obviously, $W_U > W_R$ as long as $y_1 > y_2$. All the proofs conclude. □

[20]The skipped algebra is available upon request.

A.15 Proof of Lemma A.1

Proof. First, consider the case that $I_1 + I_2 \le 2\pi$. The monopolist seller sets prices $p_1^* = p_2^* = \frac{y}{2}$ and Influencer j attracts consumers with $x \in \mathbf{S}^1$ such that $\|x - \theta_j\| \le \frac{I_j}{2}$ for $j = 1, 2$. Note that there exists no overlapping in consumers served since $\frac{I_1}{2} + \frac{I_2}{2} \le \pi$. Thus, $\Pi_{\{1,2\}} = \Pi_{\{1\}} + \Pi_{\{2\}} = \frac{y(I_1+I_2)}{2}$.

Second, consider the case that $I_1 + I_2 > 2\pi$. Note that both influencers attract a positive share of consumers. If not, then suppose that Influencer 2 is serving no consumers (and the other case is identical). For $p_1 \ge 0$, we can always charge a slightly higher price for all consumers sufficiently close to θ_2. Meanwhile, suppose Influencer 1 serves consumers $x \in \mathbf{S}^1$ such that $\|x - \theta_1\| \le s_1$ with $s_1 \in [0, \pi)$. Prices p_1 and p_2 are set such that the indifferent consumer is marginal (i.e., the type x such that $\|x - \theta_1\| = s_1$ gets a zero payoff), otherwise we can increase p_j at least for one influencer. Hence, $p_1 = y\,(1 - s_1/I_1)$ and $p_2 = y\,(1 - (\pi - s_1)/I_2)$. Then, the total revenue can be written as a function of s_1 as $\Pi_{\{1,2\}} = 2p_1 * s_1 + 2p_2 * (\pi - s_1)$. Maximizing the joint revenue function, it yields $p_1^* = p_2^* = y * (1 - \pi/(I_1 + I_2))$, $s_1 = \pi I_1/(I_1 + I_2)$ and $s_2 = \pi I_2/(I_1 + I_2)$, which implies $\Pi_{\{1,2\}} = 2\pi y \left(1 - \frac{\pi}{I_1+I_2}\right)$ when $I_1 + I_2 > 2\pi$. The proof concludes. □

A.16 Proof of Lemma A.2

Proof. First of all, when $\beta \ge I$ holds, the optimal pricing strategy is given by $p_j^* = p_2^* = \frac{y}{2}$ and each influencer serves consumers such that $\|x - \theta_j\| \le \frac{I}{2}$ for $j = 1, 2$. Since there is no overlapping in the consumer base, it yields $\Pi_{\{1,2\}} = 2\Pi_{\{1\}} = 2 * \frac{1}{2} yI = yI$.

Next, consider the case $\beta < I$. We show that in equilibrium, we have: (i) consumers along the short arc between θ_1 and θ_2 are all served; (ii) both influencers get a positive share along the short arc; and (iii) $p_1 = p_2$.

First, to see (i), note that if a positive measure of consumers is not served, then $\|x - \theta_j\| < \frac{I}{2}$ holds for some $j \in \{1, 2\}$ because $\beta < I$, which implies that $p_j > I/2$. However, since $\Pi_j = p_j(1 - p_j/y)I$, it is profitable to set $\hat{p}_j = p_j - \varepsilon$ for a sufficiently small $\varepsilon > 0$.

Second, to see (ii), note that if Influencer 2 is not active along the short arc, it must be that $y(1 - \beta/I) - p_1 \ge y - p_2$, which further implies that Influencer 2 is not actively serving any consumer, because for any consumer with $\|x - \theta_1\| = \|x - \theta_2\| + \beta$, we also have $y(1 - \beta/I - \|x - \theta_2\|/I) - p_1 \ge y(1 - \|x - \theta_2\|/I) - p_2$. However, this is sub-optimal because the seller's profit cannot exceed Π_1 in this case and it is dominated by the profit function proposed in the lemma.

Third, to see (iii), suppose not, assume that $p_1 < p_2$. The symmetric case that $p_1 > p_2$ is proved similarly. Consider $(\hat{p}_1, \hat{p}_2) = (p_1 + \varepsilon, p_2 - \varepsilon)$ with $\varepsilon > 0$ sufficiently small, and we show that $(\hat{p}_1, \hat{p}_2)$ dominates (p_1, p_2). By i) and ii), all consumers between θ_1 and θ_2 on the short arc are served and denote by x^* the cutoff consumer type, and define $s_j = \|x^* - \theta_j\|$ for $j \in \{1, 2\}$. Obviously, we have two conditions: $y(1 - s_1/I) - p_1 = y(1 - s_2/I) - p_2$ and $s_1 + s_2 = \beta$.

Solving them yields $s_j = \frac{1}{2}(\beta + I * (p_{3-j} - p_j)/y)$. Now, we can compare the profit under (p_1, p_2) and that under $(\hat{p}_1, \hat{p}_2)$. Specifically, $\Pi = p_1 s_1 + p_2 s_2 + p_1(1 - p_1/y)I + p_2(1 - p_2/y)I$ and $\hat{\Pi} = \hat{p}_1\hat{s}_1 + \hat{p}_2\hat{s}_2 + \hat{p}_1(1 - \hat{p}_1/y)I + \hat{p}_2(1 - \hat{p}_2/y)I$. Now, we can directly check $\hat{\Pi} - \Pi > 0$ as follows. Note that

$$\hat{p}_1(1 - \hat{p}_1/y)I + \hat{p}_2(1 - \hat{p}_2/y)I = I * \left(p_1 + p_2 - (p_1 + \varepsilon)^2/y - (p_2 - \varepsilon)^2/y\right)$$
$$= p_1 + p_2 - p_1^2/y - p_2^2/y + 2(p_2 - p_1)\varepsilon I/y + O(\varepsilon^2) > p_1(1 - p_1/y)I + p_2(1 - p_2/y)I$$

Similarly, $\hat{p}_1\hat{s}_1 - p_1 s_1 = (p_1 + \varepsilon)\left(s_1 - \frac{I\varepsilon}{y}\right) - p_1 s_1 = \varepsilon s_1 - \frac{I\varepsilon}{y} p_1$ and $\hat{p}_2\hat{s}_2 - p_2 s_2 = (p_2 - \varepsilon)\left(s_2 + \frac{I\varepsilon}{y}\right) - p_2 s_2 = -\varepsilon s_2 + \frac{I\varepsilon}{y} p_2$, which implies that

$$\hat{p}_1\hat{s}_1 + \hat{p}_2\hat{s}_2 - (p_1 s_1 + p_2 s_2) = (s_1 - s_2)\varepsilon + (p_2 - p_1)I\varepsilon/y > 0$$

Thus, $(\hat{p}_1, \hat{p}_2)$ is strictly dominant, and the contradiction implies that $p_1 = p_2$.

Finally, we find the optimal pricing strategy and the profit function $\Pi_{\{1,2\}}$. By property (iii), we denote $p_1 = p_2 = p$ and the profit function is $\Pi = p * \beta + 2p(1 - p/y)I$ as long as the indifferent type $\hat{x} = (\theta_1 + \theta_2)/2$ get a non-negative utility, that is , $y(1 - \|x - \theta_1\|/I) - p \geq 0$. The unconstrained optimizer to Π is given by $p^\star = y\beta/(4I) + y/2$. Verification of the non-negative utility for type $\hat{x}$ yields $\beta \leq 2I/3$ and we get $\Pi_{\{1,2\}} = \frac{(2I+\beta)^2 y}{8I}$.

When $\beta \in (2I/3, I)$, $p^\star$ violates the non-negative utility condition for type $\hat{x}$. However, note that $\Pi = p * \beta + 2p(1 - p/y)I$ is strictly increasing for all $p \in (0, p^\star]$. Thus, the optimal price is given by the IR condition for the cutoff type $\hat{x}$, which yields $p^* = y * (1 - \beta/(2I))$ and $\Pi_{\{1,2\}} = \frac{\beta*(2I-\beta)*y}{I}$. The proof concludes. □

Online Appendix

B Additional Results and Proofs

B.1 General Characterizations with Multiple Influencers

Here, we consider a setting with one seller and many influencers and study how influencer competition affects the monopolist seller's payoff, influencers' wages, and total welfare.

Residual multilateral bargaining protocol. We propose a residual bargaining protocol to handle the multilateral bargaining problem. Specifically, given the bargaining parameter $\gamma \in [0,1]$, when a group of influencers are hired $\mathcal{J} \in \{1, \cdots, J\}$, the seller only bargains with influencer $j \in \mathcal{J}$ for the "residual profit gap" $\Pi_{\mathcal{J}} - \Pi_{\mathcal{J}/\{j\}}$, which leads to

$$w_j = (1-\gamma)(\Pi_{\mathcal{J}} - \Pi_{\mathcal{J}/\{j\}})_+, \quad \text{and} \quad U_1 = \Pi_{\mathcal{J}} - \sum_{j \in \mathcal{J}} w_j. \tag{B.1}$$

where $\Pi_{\mathcal{J}}$ is the monopolist profit when the seller hires the influencer group $\mathcal{J}$.[21]

The residual bargaining protocol has several desirable properties: (i) the allocation is unique and feasible because $\sum_{j\in\mathcal{J}} w_j + U_1 \leq \Pi_{\mathcal{J}}$; (ii) it is efficient because $\sum_{j\in\mathcal{J}} w_j + U_1 = \Pi_{\mathcal{J}}$; (iii) it cannot be blocked by a coalition of $S \subseteq \mathcal{J}$ for any $\gamma \in [0,1]$ because $\sum_{j\in S} w_j + U_1 \geq \Pi_S$ and $\sum_{j\in S} w_j \geq 0$; and (iv) it is consistent with the bilateral Nash bargaining in the baseline model.[22] In essence, residual bargaining protocol is an equilibrium refinement. Note that property (iii) is applicable to all reasonable refinements because otherwise the allocation is blocked by excluding influencer j from the group $\mathcal{J}$, which further implies that $w_j \leq \Pi_{\mathcal{J}} - \Pi_{\mathcal{J}/\{j\}}$. In this sense, the residual bargaining protocol is the natural candidate satisfying both property (iii) and property (iv).

For simplicity, all influencers are also assumed to have identical power. First, since there is no fixed searching cost, all influencers are hired by the monopolist seller. Second, we show below that the joint profit is maximized when all $J \geq 1$ influencers are equally distanced. Denote $\theta_{J+1} = \theta_1$, when J influencers are equally distanced.

Lemma B.1 (Equally distant influencers). *If the seller is restricted to hiring J influencers and can freely choose style locations for all influencers, then $\bar{\Pi}_J$ is achieved when all influencers are hired and equally distant (i.e., $\|\theta_j, \theta_{j+1}\| = \frac{2\pi}{J}$ for all $j \in \{1, \cdots, J\}$).*

[21] The seller only shares the residual profit because of a credible threat that if influencer j rejects the offer, the seller switch to hiring the remaining group $\mathcal{J}/\{j\}$ and divide the profit $\Pi_{\mathcal{J}/\{j\}}$.

[22] It can also be generated from coalitional Nash bargaining (Compte and Jehiel, 2010)

With the aid of Lemma B.1,

$$\bar{\Pi}_J = \begin{cases} \frac{yIJ}{2}, & \text{if} \quad J \leq \lfloor \frac{2\pi}{I} \rfloor \\ 2\pi y \left(1 - \frac{\pi}{IJ}\right), & \text{if} \quad J > \lfloor \frac{2\pi}{I} \rfloor \end{cases} \tag{B.2}$$

Equation (B.2) follows from that all influencers are equally distanced and that the cut-off consumer indifferent between two neighboring influencers always receives a zero utility. By residual multilateral bargaining protocol, influencer j bargains with the seller over the residual profit $\left(\bar{\Pi}_J - \bar{\Pi}_{J-1}\right)$, and thus

$$w_j = (1-\gamma)\left(\bar{\Pi}_J - \bar{\Pi}_{J-1}\right) \text{ and } U_1 = \bar{\Pi}_J - J * w_1.$$

For a sufficiently large $J > \lfloor \frac{2\pi}{I} \rfloor$,

$$\frac{\partial\left(\sum_{j=1}^{J} w_j\right)}{\partial J} = -\frac{2\pi^2(1-\gamma)y}{I(J-1)^2} < 0$$

and

$$\frac{\partial U_1}{\partial J} = \frac{2\pi^2 y(J^2(1-\gamma) + (J-1)^2)}{IJ^2(J-1)^2} > 0$$

We can summarize these results as below.

Corollary B.1. *As the number of influencers increases (i.e., $J \uparrow$) and is sufficiently large (i.e., $J > \lfloor \frac{2\pi}{I} \rfloor$), total wages for all influencers is strictly decreasing, the payoff for the seller and total welfare are strictly increasing.*

B.1.1 Proof of Lemma B.1

Proof. Suppose not. Then, there exist three neighboring influencers hired such that $\|\theta_{j-1} - \theta_j\| \neq \|\theta_j - \theta_{j+1}\|$. For simplicity, assume that $\|\theta_{j-1} - \theta_j\| < \|\theta_j - \theta_{j+1}\|$. There are two cases to consider.

Case (i). There exists some consumers not served along the arc between θ_{j-1} and θ_{j+1}. Without loss of generality, we assume there exists consumers not served between θ_j and θ_{j+1}. Then, we can keep (p_{j-1}, p_j, p_{j+1}) unchanged, and shift θ_j to $\theta_j + \delta$ for a sufficiently small $\delta > 0$. This weakly increases the total revenue Π_J, because it weakly increases the size of consumers served by Influencer $(j-1)$.

Case (ii). All consumers are served along the arc between θ_{j-1} and θ_{j+1}. For ease of notation, define $x_{i,i+1} \in \mathbf{S}^1$ the cutoff type indifferent between purchasing from Influencer i and from Influencer $(i+1)$.

Lemma B.2. *The cutoff consumer type indifferent between following two neighboring influencers receive a zero utility.*

Proof. If not, suppose the cutoff type consumer $x_{j-1,j}$ receives a positive utility. Note that it cannot be the case that the consumer $x_{j,j+1}$ also receives a positive utility. Otherwise, we can increase p_j by a small amount without losing any consumers, which leads to a large total revenue. This implies that consumer $x_{j,j+1}$ receives a zero utility.

Now, we construct a hiring plan and a price scheme which generates more revenue. Consider the case that $p_{j-1} \geq p_j$ and the other case $p_{j-1} < p_j$ can be proved similarly. We shift θ_j to $\hat{\theta}_j = \theta_j + \delta$ and increase p_j to $\hat{p}_j = p_j + \frac{y}{I}\delta$ where $\delta > 0$ is small. Under the new hiring and pricing scheme, the cutoff consumer type $x_{j,j+1}$ remains unchanged, and the cutoff type $x_{j-1,j}$ shifts to $\hat{x}_{j-1,j} = x_{j-1,j} + \delta$. We can choose a sufficiently small $\delta > 0$ to ensure the consumer $\hat{x}_{j-1,j}$ still receives a positive utility. Now, all consumers between $x_{j-1,j}$ and $x_{j,j+1}$ either pay $p_{j-1} \geq p_j$ or $\hat{p}_j > p_j$. □

Lemma B.2, combined with the condition that $\|\theta_{j-1} - \theta_j\| < \|\theta_j - \theta_{j+1}\|$, implies that $p_{j-1} > p_{j+1}$ because influencer $(j+1)$ needs to serve more consumers than influencer $(j-1)$ to ensure both consumers $x_{j-1,j}$ and $x_{j,j+1}$ receive a zero utility. Denote by a_{j-1} and a_{j+1} the size of consumers served by influencer $(j-1)$ and $(j+1)$, respectively. Since $p_{j-1} > p_{j+1}$, $a_{j-1} < a_{j+1}$ by Lemma B.2.

However, the fact that $p_{j-1} > p_{j+1}$ implies there exists a price scheme more profitable. To see it, consider the new price scheme

$$(\hat{p}_{j-1}, \hat{p}_j, \hat{p}_{j+1}) = \left(p_{j-1} - \frac{y\delta}{I}, p_j, p_{j+1} + \frac{y\delta}{I}\right)$$

and

$$(\hat{\theta}_{j-1}, \hat{\theta}_j, \hat{\theta}_{j+1}) = \left(\theta_{j-1} + \frac{\delta}{2}, \theta_j + \delta, \theta_{j+1} - \frac{\delta}{2}\right)$$

Note that $\hat{a}_{j-1} = a_{j-1} + \delta$ and $\hat{a}_{j+1} = a_{j+1} - \delta$, and the size of consumers served by other influencers remains unchanged, including influencer j. The total revenue change is given by

$$\begin{aligned}
\Delta &= \hat{p}_{j-1}\hat{a}_{j-1} + \hat{p}_{j+1}\hat{b}_{j+1} - p_{j-1}a_{j-1} - p_{j+1}b_{j+1} \\
&= \left(p_{j-1} - \frac{y\delta}{I}\right) * (a_{j-1} + \delta) + \left(p_{j+1} + \frac{y\delta}{I}\right) * (b_{j+1} - \delta) - p_{j-1}a_{j-1} - p_{j+1}b_{j+1} \\
&= \frac{y\delta}{I}(b_{j+1} - a_{j+1}) + (p_{j-1} - p_{j+1})\delta + O(\delta^2)
\end{aligned}$$

By the fact that $p_{j-1} > p_{j+1}$ and $a_{j-1} < a_{j+1}$, this generates a higher total revenue for a sufficiently small $\delta > 0$. This is a contradiction! Hence, it cannot be the case that all consumers are served but influencers are not equally distanced. The proof concludes. □

B.2 Inefficient Power Acquisition under Monopolist Pricing

Proposition B.1 (Inefficient Power Acquisition)**.** *Fix $\beta = \pi$. Suppose $2\pi \leq I$ and $k > 2$. If $\frac{\pi^2 y}{I}\frac{(k-1)}{2(k+1)} < C_P < \frac{\pi^2 y}{I}\frac{(1-\gamma)(k-1)}{(k+1)}$, there exists over-investment in influence power. Instead, if $\frac{\pi^2 y}{I}\frac{(1-\gamma)(k-1)}{k(k+1)} < C_P < \frac{\pi^2 y}{I}\frac{(k-1)}{2k(k+1)}$, there exists under-investment in influence power.*

Proof. First, note that we impose that $I \geq 2\pi$ to simplify calculations. Here, we construct an equilibrium in which only one influencer, say Influencer 1, acquires power and increases his power from I to kI. Under the conjectured equilibrium, for Influencer 1, before power acquisition, by equation (2), we have $\Pi_2^b = 2\pi y(1-\pi/I)$. Meanwhile, by Lemma A.1, $\Pi_{\{1,2\}}^b = 2\pi y(1-\pi/2I)$ and by bilateral Nash bargaining, $w_1^b = (1-\gamma)(\Pi_{\{1,2\}}^b - \Pi_2^b) = (1-\gamma)\pi^2 y/I$.

Second, after power acquisition, again by Lemma A.1, $\Pi_{\{1,2\}}^a = 2\pi y(1-\frac{\pi}{(1+k)I})$, and thus

$$w_1^a = (1-\gamma)(\Pi_{\{1,2\}}^a - \Pi_2^a) = (1-\gamma)\left(2\pi y(1-\frac{\pi}{(1+k)I}) - 2\pi y(1-\frac{\pi}{I})\right) = \frac{2\pi^2 y}{I}\frac{(1-\gamma)k}{k+1}.$$

This further implies that $\Delta w_1 = w_1^a - w_1^b = \frac{\pi^2 y}{I}\frac{(1-\gamma)(k-1)}{(k+1)}$.

Third, given that $I_1 = kI$, for Influencer 2, before power acquisition,

$$w_2^b = (1-\gamma)\left(2\pi y(1-\pi/(I+kI)) - 2\pi y(1-\pi/(kI))\right) = \frac{2\pi^2 y}{I}\frac{(1-\gamma)}{k(k+1)}$$

Instead, given that Influencer 1 acquires power, if Influencer 2 acquires additional power, then $\hat{\Pi}_{\{1,2\}} = 2\pi y(1-\frac{\pi}{2kI})$, $\hat{\Pi}_j = 2\pi y(1-\frac{\pi}{kI})$, which implies

$$w_2^a = (1-\gamma)(\hat{\Pi}_{\{1,2\}} - \hat{\Pi}_1) = (1-\gamma)2\pi y\left(\frac{\pi}{kI} - \frac{\pi}{2kI}\right) = \frac{\pi^2 y}{I}\frac{(1-\gamma)}{k}.$$

and thus $\Delta w_2 = w_2^a - w_2^b = \frac{\pi^2 y}{I}\frac{(1-\gamma)}{k} - \frac{2\pi^2 y}{I}\frac{(1-\gamma)}{k(k+1)} = \frac{\pi^2 y}{I}\frac{(1-\gamma)(k-1)}{k(k+1)}$.

In short, in equilibrium, if $\Delta w_2 < C_P < \Delta w_1$, only Influencer 1 acquires additional influence power. Instead, if $C_P \leq \Delta w_2$, both influencers invest in power acquisition.

Fourth, we can compute total welfare change when only Influencer 1 acquires additional power. By Lemma A.1, $p_1^* = p_2^* = y(1-\frac{\pi}{2I})$, $s_1 = s_2 = \frac{\pi}{2}$, which implies that $SW^b = 4 \times \int_0^{\frac{\pi}{2}} y(1-s/I)dx = 2 * \pi/2 * (y + y(1-\pi/(2I))) = \pi y(2-\pi/(2I))$.

When Influencer 1 acquires additional power, $p_1^* = p_2^* = y(1-\frac{\pi}{(1+k)I}$, $s_1 = \pi - s_2 = \frac{k\pi}{(1+k)}$, and thus $SW^a = 2\int_0^{\frac{k\pi}{(1+k)}} y(1-s/(kI))dx + 2\int_0^{\frac{\pi}{(1+k)}} y(1-s/(I))dx = \pi y(2-\pi/(I+kI))$, which implies that $\Delta SW = SW^a - SW^b = \frac{\pi^2 y}{I}\frac{(k-1)}{2(k+1)}$. Similarly, we can calculate total welfare change by comparing the case that only one influencer acquires power with that in which both influencers do so, which yields $\widehat{\Delta SW} = \frac{\pi^2 y}{I}\frac{(k-1)}{2k(k+1)}$.

Finally, note that when $k > 2$, $\Delta w_2 < \Delta SW$. Thus, if $\Delta SW < C_P < \Delta w_1$, it is sub-optimal for Influencer 1 to acquire power in total welfare, but only Influencer 1 to invest is indeed an equilibrium. Similarly, if $\Delta w_2 < C_P < \widehat{\Delta SW}$, it is socially optimal for both influencers to invest, but only Influencer 1 acquires additional power. □