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ABSTRACT

This paper proposes an analytic representation of perturbations in heterogeneous agent economies with aggregate shocks. Treating the underlying distribution as an explicit state variable, a single value function defined on an infinite-dimensional state space provides a fully recursive representation of the economy: the ‘Master Equation’ introduced in the mathematics mean field games literature. I show that analytic local perturbations of the Master Equation around steady-state deliver dramatic simplifications. The First-order Approximation to the Master Equation (FAME) reduces to a standard Bellman equation for the directional derivatives of the value function with respect to the distribution and aggregate shocks. The FAME has six main advantages: (i) finite dimension; (ii) closed-form mapping to steady-state objects; (iii) applicability when many distributional moments or prices enter individuals' decision such as dynamic trade, urban or job ladder settings; (iv) block-recursivity bypassing further fixed points; (v) mapping to analytic sequence-space derivatives; (vi) fast implementation using standard numerical methods. I develop the Second-order Approximation to the Master Equation (SAME) and show that it shares these properties, making the approach amenable to settings such as asset pricing. I apply the method to two economies: an incomplete market model with unemployment and a wage ladder, and a discrete choice spatial model with migration.

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Introduction

A vibrant frontier in macroeconomics incorporates rich cross-sectional heterogeneity in dynamic general equilibrium. Recent numerial advances have remarkably accelerated the computation of impulse response functions up to first-order perturbations in some leading incomplete market models (Ahn et al., 2018, Auclert et al., 2019). Yet, a conceptual framework rationalizing these numerical techniques and expand their applicability has remained elusive. What is the relationship between economic fundamentals and equilibrium outcomes in perturbations of heterogeneous agent models? How handle settings that fall beyond the natural scope of these methods, such as frictional labor markets or dynamic discrete choice? Are higher order perturbations that include nonlinearities and aggregate risk feasible?

In this paper, I propose answers to these questions using a new conceptual framework for perturbations in heterogeneous agent economies. I propose a representation of dynamic economies with heterogeneity that is analytic, low-dimensional, handles flexible general equilibrium feedbacks between the distribution and individual decisions, and applies systematically to perturbations of any order. These results rely on two key ideas. First, I use a state-space approach and treat the distribution of underlying heterogeneity as an explicit state variable in individual decisions. A single value function equation set on the space of distributions summarizes the equilibrium: the 'Master Equation.' Second, I take analytic—instead of numerical—perturbations of the Master Equation in the distribution and aggregate shocks and characterize the directional derivatives of the value function.

Specifically, the first core idea in this paper is to represent dynamic general equilibrium economies in fully recursive form. For concreteness, consider the Krusell and Smith (1998) economy. Households face uninsurable idiosyncratic labor productivity risk, and may borrow and save in a risk-free asset. A representative firm rents capital and hires labor. Abstract from aggregate shocks for now: the economy simply starts out of steady-state. Households' consumption and savings forward-looking decisions are fully determined by the sequence of future interest and wage rates. These prices in turn depend on the underlying distribution of asset holdings and idiosyncratic productivity through the firms' decision and market clearing. The distribution of assets and productivity evolves over time according to the optimal savings decisions of individuals. The classic difficulty in characterizing this economy is that individual decisions are forward-looking in time, while the evolution of the infinite-dimensional distribution is backward-looking in time. Prices are the fixed point of this forward-backward system that clear the capital and labor market.

I include the underlying distribution of heterogeneity as an explicit state variable in households' decision problem. Knowledge of the distribution fully characterizes prices. Households know the law of motion of the distribution. Thus, they forecast the future path of the distribution and hence prices. Households' decision problem then depends on the distribution just as on any other state variable. The only notable difference is that the distribution is an infinite-dimensional object, rather than a finite-dimensional state vector.

The resulting representation of the economy is the Master Equation. It was recently characterized in the mathematics mean field games literature by Cardaliaguet et al. (2019). It consists of a single Bellman equation that describes the entire behavior of a system of interacting agents. In the Krusell and Smith (1998) example, the Master Equation defines a value function that depends on a given households' idiosyncratic states—assets and productivity—as well as the underlying distribution of assets and productivity of all other households of the economy. The Master Equation is a Markovian representation of the economy because it includes as a state variable all the necessary information to forecast the evolution of the economy. It merges the fixed point on decisions, prices and the distribution into a single object.

The logic underpinning the Master Equation representation is more general than the Krusell and Smith (1998) example. At the same, the analysis in Cardaliaguet et al. (2019) imposes restrictions that are at odds with most economic applications of interest. Therefore, I expand the scope of the Master Equation to encompass a large class of continuous-time dynamic general equilibrium economies that nest many applications of interest.[1]

The second core idea is to simplify the Master Equation by focusing on local perturbations around a deterministic steady-state. Consider an impulse in the distribution, that moves the economy away from its steady-state. For instance, suppose the economy simply starts out of steady-state. I explicitly perturb the Master Equation along any such distributional impulse using generalized derivatives in infinite-dimensional spaces—namely, Fréchet derivatives. I preserve the full nonlinearity of individual decisions with respect to idiosyncratic states. Critically, relative to numerical techniques that first discretize and next linearize, I start by taking an internally consistent analytic perturbation first, before any computational discretization is applied. Key to this step is the use of continuous time, that streamlines the mapping between individual decisions and the evolution of the distribution.

The First-order Approximation to the Master Equation (FAME) without aggregate shocks results in a Bellman equation with six key properties. First, it is low-dimensional. Its solution, the 'Impulse Value', consists of the directional derivatives of the value function with respect to the distribution. It encodes how individuals value changes in the distribution. In the FAME, the Impulse Value depends on only twice the number of idiosyncratic states, down from infinity in the fully nonlinear Master Equation. In the Krusell and Smith (1998) example, the Impulse Value has dimension four. This drastic dimension reduction is a feature of the local perturbation. To know their Impulse Value, individuals must know their own idiosyncratic states—for instance assets and productivity. Individuals must also know where the distributional impulse is happening—at another possible pair of assets and productivity. Thus, they must keep track of another set of idiosyncratic states that index which distributional impulse they are contemplating.

Second, the FAME depends in closed form on known and interpretable steady-state objects through

[1] I let individual states follow controlled jump-diffusion processes that are flexible functionals of the underlying distribution to handle job search and dynamic discrete choice models. I let idiosyncratic states face constraints, to handle occasionally binding borrowing constraints. I let mass points develop in the distribution, consistently with borrowing constraints.

a systematic structure. The FAME depends on the steady-state law of motion of idiosyncratic states, such as assets and productivity. It also depends on the direct impact of distributional impulses on individual utility and transition probabilities between states. Finally, the FAME depends on the general equilibrium response of all individuals in the economy through its effect on the law of motion of the distribution. By virtue of the local perturbation, these objects all have explicit expressions that are linked together in equilibrium by the FAME.

Third, the FAME applies equally well to settings in which few or many prices summarize feedback between the general equilibrium and individual decisions. For instance, in dynamic spatial models, households must keep track of several prices per location to decide where to migration. In search and matching models with job-to-job search, the entire distribution of wage offers matters directly for workers. Because the FAME is a perturbation with respect to the entire distribution, it applies directly to those examples. For instance, Bilal and Rossi-Hansberg (2023) use the FAME to evaluate the cost of climate change in a model of the U.S. economy disaggregated into over 3,000 counties.

Fourth, the FAME provides a block-recursive representation of equilibrium and impulse response functions. The FAME inherits the block-recursivity from the Master Equation, in that it merges the value function and the distributional fixed points. Once the Impulse Value is known, the evolution of the distribution over time is obtained without solving any additional fixed point. Leveraging block-recursivity and the closed-form dependence on steady-state, I show that the speed of convergence back to steady-state depends on the dominant eigenvalue of a transition operator encoding general equilibrium effects.

Fifth, the FAME provides a conceptual bedrock for computational perturbation techniques. It is the mathematically exact foundation for the state-space numerical methods in Reiter (2009) and Ahn et al. (2018). I also show that it underpins sequence-space numerical methods as in Auclert et al. (2019). I derive an analytic counterpart to the sequence-space Jacobians in Auclert et al. (2019) and show the equivalence between Impulse Values and sequence-space Jacobians. This equivalence is key for implementation, as state-space impulse values are particularly useful in settings with many prices while sequence-space Jacobians are especially powerful in settings with few prices.

Sixth, the FAME offers a streamlined and efficient implementation using standard finite difference methods. Building on the analytic representation of the the FAME, I show that it displays a specific separability structure because an individual's own idiosyncratic state and the distributional impulse propagate symmetrically but independently. Once discretized, the FAME takes the form of a modified Sylvester matrix equation for which standard routines exist. In the Krusell and Smith (1998) example, computation typically takes a tenth of a second and requires a couple dozen lines of code.

The FAME extends readily to the presence of aggregate shocks. A similar perturbation approach delivers two key insights. To first order, the Impulse Value splits into two distinct components. The first component is simply the Impulse Value from the deterministic FAME, the 'deterministic Impulse Value'. The second component is the 'stochastic Impulse Value'. The latter represents how individuals value an aggregate shock. It satisfies a similar FAME than the distributional Impulse Value and may be solved or computed analogously. The economy remains block-recursive, and the linearized law of motion of the distribution evolution equation now features an additional component that represents the response of individual decisions to aggregate shocks. That same law of motion defines the invariant distribution in the stochastic steady-state. Numerically, it is the solution to a simple linear system.

The Master Equation provides a systematic approach to perturbations of increasing order. Conceptually, the Second-order Approximation to the Master Equation (SAME) is the same as the FAME. I show that the SAME defines a value function that depends again only on steady-state objects in closed form. Its solution now depends on three times the number of idiosyncratic states because pairwise impulses in the distribution matter to second order. The SAME handles asset pricing applications that require second-order perturbations to depart from certainty equivalence.

I illustrate the Master Equation approach with two distinct applications. The first application is purely analytic, and studies a dynamic location choice model that may also be broadly interpreted as an occupation, industry or product choice model. In this setting, I leverage the FAME to connect measurable preference and technology parameters to the speed of convergence back to steady-state—formally the dominant eigenvalue of the transition operator governing general equilibrium forces. As a result, the FAME provides a direct mapping between measurable primitives and the dynamic response to the economy to an initial population shock.

The second application is quantitative. I evaluate the impact of state-dependent Unemployment Insurance (UI) in an environment similar to Krusell and Smith (1998) but for the labor market. I explicitly model unemployment, endogenous vacancy creation, and workers search for jobs on and off the job. Solving such a model falls beyond what existing methods are designed for. In contrast, the FAME is well-equipped to handle this model.

In such an environment, state-dependent UI already leads to a trade-off even without nominal rigidities. Raising UI generosity in downturns provides valuable relief to unemployed, credit-constrained individuals. At the same time, the tax burden associated with this spending falls upon wealthy net savers, thereby crowding out investment and depressing vacancy creation going forward. On net, it is unclear whether state-dependent UI harms households that it intends to help.

I calibrate the model to micro evidence on the incidence of recessions and marginal propensities to consume by income status. After solving for steady-state, computing the Impulse Values takes under 5 seconds on a laptop. Solving for any impulse response function or the invariant distribution in stochastic-steady state subsequently takes less than a second. The entire implementation of the FAME requires less than 200 lines of Matlab code.

I find that a moderate increase in UI generosity delivers large direct welfare gains to credit-constrained unemployed households. In contrast, the crowding out effects on capital are minimal. Together, these observations imply that state-dependent UI is an effective tool. While I abstract from nominal rigidities, they would further amplify the potency of UI.

This paper relates to four strands of literature. First, I build on the mathematics mean field games literature. They were first introduced in sequential form by Lasry and Lions (2006) and Huang et al.

(2006). The Master Equation was discussed in Lions (2011), and formally analyzed in Cardaliaguet et al. (2019).[2] I complement this literature by proposing a flexible formulation of the Master Equation that is amenable to a wide class of economic applications, and by characterizing explicitly its first- and second-order perturbations.

Second, this paper also relates to the set of papers that characterize impulse response functions analytically in specific heterogeneous agent models by studying their spectral properties (Gabaix et al., 2016, Alvarez and Lippi, 2021, Liu and Tsyvinski, 2020). I complement this literature by characterizing the role and determination of the dominant eigenvalue in a wider class of economies.

Third, this paper connects to literature proposing computational methods for first-order perturbations of impulse response functions in heterogeneous agent economies in state space form (Reiter, 2009, Ahn et al., 2018). These methods first discretize, then linearize to first order, an economy with heterogeneity. They treat the resulting finite but high-dimensional system as a standard rational expectations system. By reversing the order—linearizing first, discretizing next—the FAME is the internally consistent foundation for this computational approach. The FAME provides an economic interpretation of numerical output that may be otherwise difficult to see through and simplifies implementation. Crucially, the Master Equation approach also delivers a systematic approach to higher order perturbations such as the SAME.[3] Since this paper was first circulated in 2021, Bhandari et al. (2023) build on its key insights to characterize perturbations in discrete time heterogeneous agent economies.

Fourth, the Master Equation approach relates to numerical linearization techniques that leverage the sequence-space representation of the economy (Boppart et al., 2018, Auclert et al., 2019). These sequence-space approaches are designed for first-order perturbations, in contrast to the Master Equation approach. When the economy admits a simple sequence-space representation, I show that the FAME features a separability property that permits a lower-dimensional implementation through sequence-space Jacobians that I derive in closed form.

Fifth, the Master Equation approach connects to numerical methods to solve heterogeneous agent models globally. Schaab (2021) also uses the Master Equation to propose a global, adaptative sparse grid strategy that builds on Brumm and Scheidegger (2017). This paper instead proposes an analytic perspective on local perturbations of the Master Equation.[4]

The remainder of this paper is organized as follows. Section 1 presents the intuition behind the Master Equation approach in the context of the Krusell and Smith (1998) economy. Section 2 defines

[2] See also Carmona and Delarue (2018a) and Carmona and Delarue (2018b) for the dual, probabilistic approach to mean-field games and the Master Equation.

[3] The analytic nature of perturbations in the FAME and the SAME relate to the analytic perturbation approach in Bhandari et al. (2018). They also propose first and second-order perturbation of heterogeneous agent economies. However, they require that all shocks, both aggregate and idiosyncratic, are small enough. The FAME and the SAME instead preserve full nonlinearity with respect to idiosyncratic uncertainty by only requiring that aggregate shocks are small. Childers (2018) proposes a hybrid approach where somme differentiation is analytic but does not recover smaller-dimensional Bellman equations as in the FAME.

[4] Ahn et al. (2018) mention the Master Equation in their Appendix as a possible justification for numerical state-space approaches but do not establish the connection formally.

the Master Equation for a general economy. Section 3 derives the FAME and discusses connections to existing state-space and sequence-space approaches. Section 4 describes the SAME. Section 5 presents two applications. The last section concludes. Proofs and additional details may be found in the Appendix.

1 Motivating example

This section describes the Krusell and Smith (1998) economy and how to use the Master Equation approach to solve for impulse response functions. I start with a deterministic economy that starts out of steady-state, before introducing aggregate shocks.

1.1 Setup

Time $t \geq 0$ is continuous and runs forever. There are no aggregate shocks for now, but the economy may start out of steady-state. Individuals are endowed with idiosyncratic time-varying productivity y_t every instant. Their productivity endowment follows a stationary stochastic process, and is independent across individuals. The process is defined by its generator $M(y)$. The generator is a functional operator that encodes conditional expectations under the income process, starting from the point y. For instance, if productivity follows a diffusion, $dy_t = \mu(y_t)dt + \sigma(y_t)dW_t$, then the generator is the functional operator $M(y)[V] = \mu(y)V'(y) + \frac{\sigma(y)^2}{2}V''(y)$. Households are endowed with initial asset holdings a_0.

Households solve a standard income fluctuation problem by deciding how much to consume and save every period in a single risk-free asset a. For brevity, I denote by $x = (a, y)$ the pair of households' idiosyncratic states. Households' value function $V_t(x)$ satisfies the Hamilton-Jacobi-Bellman equation:[5]

$$\rho V_t(x) - \frac{\partial V_t}{\partial t}(x) \quad = \quad \max_{c \geq 0} u(c) + L_t(x, c)[V_t] \tag{1}$$

where I define

$$L_t(x, c)[V] = (r_t a + w_t y - c)\frac{\partial V}{\partial a}(x) + M(y)[V] \tag{2}$$

the expectation of future values given that assets a_t evolve according to the budget constraint $da_t = (r_t a_t + w_t y_t - c_t)dt$ and productivity follows its exogenous stochastic process. The functional operator $L_t(x, c)$ is the generator of the stochastic process for households' idiosyncratic state $x = (a, y)$. Denote by $\hat{c}_t(x)$ the optimal consumption decision of households.

A representative firm operates a production technology $Y = \bar{Z}K^\alpha N^{1-\alpha}$ and rents assets from households at the interest rate r_t. The firm transforms assets into productive capital K that does not depreciate. The firm's optimality conditions together with market clearing imply that the real interest

[5] The value function is restricted to have at most linear growth as assets and income approach infinity. This restriction is the recursive analogue to the No-Ponzi condition in the sequential formulation of problem (1).

rate r_t and the wage rate w_t clear the capital and labor market:

$$r_t = \alpha \left(\frac{\iint y g_t(x) dx}{\iint a g_t(x) dx} \right)^{1-\alpha} \equiv \mathcal{R}(g_t) \quad ; \quad w_t = (1-\alpha) \left(\frac{\iint a g_t(x) dx}{\iint y g_t(x) dx} \right)^{\alpha} \equiv \mathcal{W}(g_t), \tag{3}$$

where $g_t(x)$ is the probability distribution function of households over assets and income at calendar time t. The functionals $\mathcal{R}, \mathcal{W}$ capture how the interest and wage rates react to shifts in the distribution of households.

The distribution $g_t(x)$ evolves over time according to its law of motion, the Kolmogorov forward equation:

$$\frac{\partial g_t}{\partial t}(x) \quad = \quad L_t^*(x, \hat{c}_t(x))[g] \tag{4}$$

where

$$L_t^*(x, \hat{c}_t(x))[g] = -\frac{\partial}{\partial a}\Big(s_t(x) g_t(x) \Big) + M^*(y)[g_t] \tag{5}$$

and $s_t(x) = r_t a + w_t y - \hat{c}_t(x)$ denotes the equilibrium savings rate. M^* is the formal adjoint of the operator M, which is the functional equivalent of the matrix transpose. Similarly, the operator L^* is the adjoint of the operator L^*.

1.2 The Master Equation

The Master Equation approach considers the individual decision problem (1) in fully recursive form. This approach is distinct from the common sequential view of (1), that takes the path of interest and wage rates $(r_t, w_t), t \geq 0$ as an input. Making the individual decision problem (1) recursive requires defining the value function on the underlying space of distributions g. While the distribution is infinite-dimensional, it may be viewed as a larger analog of any other finite-dimensional state variable. Just as with any other state variable, including the distribution g as an explicit state variables requires knowing its drift over time (and possibly volatility). Crucially, the Kolmogorov forward equation (4) encodes precisely that law of motion.

I build towards the Master Equation in three steps. The first step is to recognize how the value of a household depends on equilibrium objects. In this example, the value depends only on the interest and wage rates r_t, w_t.

The second step is to express prices and any other equilibrium objects that may enter the value function, as functionals of the underlying primitive distributions that define the relevant aggregate state. In this example, interest and wage rates $r_t = \mathcal{R}(g_t), w_t = \mathcal{W}(g_t)$ depend on the ratio of marginals of the joint distribution of assets and wealth from the capital and labor market clearing conditions (3). Thus, the capital and labor market clearing conditions (3) provides the required map directly through the functionals $\mathcal{R}, \mathcal{W}$. Substituting into the value of households, I obtain

$$\rho V_t(a, y) - \frac{\partial V}{\partial t}(a, y) \quad = \quad \max_c u(c) + L(x, c, g_t)[V_t] \tag{6}$$

where $L(x, c, g_t)[V] = (\mathcal{R}(g_t) a + \mathcal{W}(g_t) y - c) \frac{\partial V}{\partial a}(a, y) + M(y)[V]$. So far, the transformation of the Bellman equation (1) into (6) is mostly notational: I have substituted the time-dependent price sequence with the distribution-dependent price functionals.

The cornerstone of the Master Equation approach lies in the third step. Replace the time dependence of the value function itself by an explicit dependence on the distribution g. This substitution amounts to a change of variables:

$$V_t(x) \equiv V(x, g_t). \tag{7}$$

Using identity (7), the decision problem may be made fully recursive by re-expressing the time derivative in the Bellman equation (6). The first step is to recognize that, by the chain rule:

$$\frac{\partial V_t}{\partial t}(a, y) = \left\langle \frac{\partial V}{\partial g}(x, g_t), \frac{\partial g_t}{\partial t} \right\rangle. \tag{8}$$

The chain rule in (8) is one that applies in infinite-dimensional spaces. It involves slightly more notation than the usual chain rule in finite dimension, but follows the exact same logic.

First, the brackets $\langle \cdot, \cdot \rangle$ denote an inner product in the appropriate functional space. In this application, the inner product turns out to be $\langle \varphi, \psi \rangle = \int \varphi(a, y)\psi(a, y) da dy$ on the space of square integrable functions. This inner product is the natural generalization of the Euclidian inner product $\langle \varphi, \psi \rangle = \sum_{n=1}^{N} \varphi_n \psi_n$ when dealing with functions rather than vectors.

Second, the derivative of the value with respect to the distribution, $\frac{\partial V}{\partial g}$, must be understood in an appropriate space for the distribution g. The relevant notion in most economic applications turns out to be that of Frechet derivative, the natural generalization of derivatives in finite-dimensional spaces to infinite-dimensional Hilbert spaces.

To gain intuition, suppose temporarily that the possible set of assets and wages was discrete and finite, indexed by n. The value function would become a vector $(V_n)_{n=1}^N$, and the derivative $\frac{\partial V}{\partial g}$ would simply represent the gradient of the value vector with respect to the mass at each ones of these points. Namely, one could write $g \equiv (g_n)_{n=1}^N$, and thus $\frac{\partial V}{\partial g} = \left(\frac{\partial V_n}{\partial g_1}, ..., \frac{\partial V_n}{\partial g_N}\right)_{n=1}^N$.

The Frechet derivative simply extends the notion of gradient to the case when the underlying idiosyncratic state space is continuous rather than discrete. In particular, the Frechet derivative $\frac{\partial V}{\partial g}(x, g)$ is itself a function of the direction in which the derivative is taken, x'—just as with a finite dimensional gradient. The notation $\frac{\partial V}{\partial g}(x, g)$ implicitly omits the dependence on x', but I also write explicitly $\frac{\partial V}{\partial g}(x, x', g)$ when needed.

The second step to remove the time derivative is to recognize that the change in the distribution, $\frac{\partial g_t}{\partial t}$ is precisely given by the law of motion of the distribution (4). Following the same logic whereby all equilibrium objects—prices and distributions—were expressed as functionals of g in the Bellman equation (6), the evolution equation (4) may be expressed as a function of g only. Write the savings rate $s_t(x) \equiv s(x, g_t) = \mathcal{R}(g_t)a + w - \hat{c}(x, g_t)$. As in the change of variables in the Bellman equation (6), these observations may be summarized by changing variables in the functional operator that encodes the evolution of the distribution $L_t^*(x, \hat{c}_t(x)) \equiv L^*(x, g_t)$. The dependence on the distribution g is both explicit through the interest rate, and implicit through the optimal consumption decision. I express the dependence of the coefficients of the operator L^* on the distribution g_t through the parenthesis $g_t \mapsto L^*(x, g_t)[\cdot]$. I express the action of the operator holding the coefficients fixed through the square brackets $g_t \mapsto L^*(x, \cdot)[g_t]$. Again, conditional on the aggregate state g_t, the coefficients of the operator L^* do not depend on time, leading to

$$\frac{\partial g_t}{\partial t}(x) = L^*(x, g_t)[g_t]. \tag{9}$$

Combining the chain rule (8) with the law of motion of the distribution (9), I finally obtain

$$\frac{\partial V_t}{\partial t}(x) = \int \frac{\partial V}{\partial g}(x, x', g_t) L^*(x', g_t)[g_t] dx'. \tag{10}$$

Identity (10) expresses the time derivative of the value function through the distributional derivative of the value function $\frac{\partial V}{\partial g}$, as well as the operator that encodes the time evolution of the distribution, L^*. Time dependence now only runs through the distribution g_t.

Combining the previous observations, I rewrite the Bellman equation (6) as

$$\rho V(x, g) \quad = \quad \max_c u(c) + L(x, c, g_t)[V] + \int \frac{\partial V}{\partial g}(x, x', g_t) L^*(x', g_t)[g_t] dx'. \tag{11}$$

Equation (11) is the Master Equation. Arriving at the representation (11) has required many definitions, but the payoff is substantial: the Master Equation (11) is a fully recursive—or Markovian—representation of the household's problem.

Inspection of the Master Equation (11) reveals that there is no need to keep track of a separate law of motion for the distribution. This law of motion has precisely been incorporated into the value function through its last term $\int \frac{\partial V}{\partial g}(x, x', g_t) L^*(x', g_t)[g_t] dx'$. As a result, the Master Equation (11) is the only equation that needs be solved to characterize the equilibrium. It is this property has lead it to be called the 'Master Equation' in the mathematics mean field games literature. In practice, the representation of the equilibrium is now block-recursive: the evolution of the distribution along any particular equilibrium realization follows ex-post, once the solution to the Master Equation is known.

The recursive nature of the Master Equation allows to leverage standard recursive methods to characterize and compute the solution to (11). However, the fully nonlinear Master Equation (11) is defined on a infinite-dimensional state space that includes the distribution g. Therefore, it remains difficult to compute nonlinearly in practice.

To overcome this practical difficulty, I combine the Master Equation (11) with local perturbation methods. It turns out that this combination provides a powerful closed form characterization of the linearized Master Equation, and drastically reduces the dimensionality of the problem.

1.3 The FAME

I start from a locally isolated steady-state of the economy. It is given by a steady-state value function $V^{SS}(x)$ and a steady-state distribution $g^{SS}(x)$. I then consider the First-order Approximation to the Master Equation (FAME) (11) around the distribution g^{SS}. Specifically, I only require that deviations in the distribution—that I denote by $h = g - g^{SS}$ and call distributional impulses—are small in the mean squared error norm. I do not require that idiosyncratic uncertainty is small. Instead, the present approximation preserves the full nonlinearity of decisions with respect to individual states.[6] The main innovation in this paper is to take the perturbation analytically instead of numerically.

The critical observation in the FAME is that the value function $V(x, g)$ then becomes, to a first order, a linear functional of the distributional impulse h. Namely, to first order:

$$V(x, g) \approx V^{SS}(x) + \int v(x, x')h(x')dx'. \tag{12}$$

The function $v(x, x')$ encodes how the value function evaluated at the point $x = (a, y)$ responds to small impulses in the distribution around the steady-state. To first order, only the effect of the impulse direction by direction need be considered, and the expansion is additive in the impulse h. The pairwise effects of the impulses is second order and thus drop out to first order.

I call the function $v(x, x')$ the 'determinstic Impulse Value Function' or simply the Impulse Value. This terminology is motivated by the observation that $v(x, x')$ exactly encodes how the value function responds to a small impulse h in the underlying distribution relative to steady-state. It can be interpreted as the general equilibrium effect of adding one household at x' on the value of a household at x.

By construction of the Impulse Value, it coincides with the directional derivative evaluated at steady-state:

$$v(x, x') = \frac{\partial V}{\partial g}(x, x', g^{SS}).$$

To build intuition, the analogue of equation (12) with a finite state space would simply be $V_n(g) \approx V_n^{SS} + \sum_{k=1}^{N} v_{nk}h_k$ with $v_{nk} = \frac{\partial V_n}{\partial g_k}(g^{SS})$. The integral in equation (12) merely generalizes this notation to settings with a continuous state space.

The goal of the FAME is to derive restrictions that determine the Impulse Value. To that end, I follow a similar strategy to perturbation methods in representative agent economies such as the Real Business Cycle (RBC) model. I substitute the definition of the Impulse Value (12) into the nonlinear Master Equation (11). I then take a first-order approximation in the distributional impulse h. Since the Master Equation must hold for all h, the final step is simply to use the method of undetermined coefficients. To build intuition, recall that when linearizing the RBC model, there is a finite number of coefficients to identify, for instance one coefficient for how the value function depends on impulses in the aggregate capital stock. With heterogeneity, the only difference is that the 'coefficients' are themselves functions, such as the Impulse Value $v(x, x')$ itself.

[6] Namely, I do not approximate the equilibrium around a degenerate distribution as in Preston and Roca (2007), Mertens and Judd (2017), or Bhandari et al. (2018). My approach is similar in spirit to Reiter (2009), Ahn et al. (2018) or Auclert et al. (2019)

The calculation described above leads to the FAME:

$$\begin{aligned}\rho v(x,x') \;=\;& \overbrace{u'(c^{SS}(x))D(x,x')}^{\text{Direct price impact}} + \overbrace{\mathcal{L}(x)[v(\cdot,x')]}^{\substack{\text{Partial equilibrium:}\\ \text{continuation value}\\ \text{from shocks to } x}} + \overbrace{\mathcal{L}(x')[v(x,\cdot)]}^{\substack{\text{General equilibrium:}\\ \text{continuation value from}\\ \text{propagation of impulse at } x'}} \\ &- \underbrace{\int v(x,x'')\frac{\partial}{\partial a''}\left(g^{SS}(x'')\Big(D(x'',x') - \frac{1}{u''(c^{SS}(x''))}\frac{\partial v}{\partial a''}(x'',x')\Big)\right)dx''}_{\substack{\text{General equilibrium: weighted average of changes in savings rates}\\ \text{of other households } x'' \text{ in response to impulse at } x'}},\end{aligned} \tag{13}$$

where

$$\begin{aligned}D(x,x') &= \big(\mathcal{R}_0 a' + \mathcal{R}_1 y'\big)a + \big(\mathcal{W}_0 a' + \mathcal{W}_1 y'\big)y \\ \mathcal{L}(x) &= L(x, c^{SS}(x), g^{SS})\end{aligned}$$

and where $\mathcal{R}_0, \mathcal{R}_1, \mathcal{W}_0, \mathcal{W}_1$ are constants that depend only the steady-state distribution and are given in Appendix A.1.

Equation (13) is the FAME. Its right-hand-side has four component. Each one encodes a particular force that affects how the value of household $x = (a, y)$ changes in equilibrium when an additional household enters the economy at $x' = (a', y')$.

The first component in the FAME is the direct price impact. When the distribution changes, prices also change. The movement in prices affects households' disposable income as encoded in the price impact function $D(x, x')$. The price impact function D depends linearly on the household's state: a household with more assets benefits more from a rise in the interest rate. The price impact function D also depends linearly in the point x' in the state space where the distributional impulse is occurring—where household x is contemplating an excess mass of other households x'. A given distributional impulse at x' affects the interest rate more if a' is high or if y' is high. Therefore, the function D is larger at larger a' and y'. A similar logic underlies the impact of the distributional impulse through the wage rate. The impact of the distributional impulse on households' consumption drops out due to the envelope condition—the first-order optimality condition always holds in continuous time. By virtue of the local perturbation, households then convert changes in disposable income into utils using their steady-state marginal utility of consumption.

The second component in the FAME encodes a partial equilibrium force, similar to households' continuation value in (1). Even out of steady-state, households form expectations about their own assets and labor market productivity. Crucially, by virtue of the first-order perturbation, household need only evaluate those expectations using steady-state prices and consumption policy functions. Thus, they use the steady-state continuation value operator $\mathcal{L}$ that involves only steady-state transition probabilities. This operator acts on the first argument x of the Impulse Value, that represents the dependence of their value on their own own state variable.

The third component in the FAME represents a first general equilibrium force. When contemplating the effect of an additional household at x' on the economy, households at x expect this additional household at x' to behave just as any other household. Household x' consumes, saves and received labor market shocks. Thus, the additional household at x' travels through the state space. Keeping track of where they go matters to project the economy forward in time and evaluate what tomorrow's distribution will be. The FAME shows that this expectation is summarized by the steady-state expectation operator $\mathcal{L}$. Once more, because of the local perturbation, only steady-state transition probabilities matter to first order. Crucially, the steady-state operator $\mathcal{L}$ acts on the second argument x' of the Impulse Value value, that represents the effect of an additional household at x' on the value of household x.

The fourth component in the FAME encodes a second general equilibrium force. It represents how household x values the change in the law of motion of the distribution that arises because of an additional household at x'. Why would the law of motion change? An additional household at x' affects prices. Because prices change, all households in the economy—represented by the integral over x''—change their savings behavior. This change in savings behavior affects the law of motion of the distribution to first order, and thus affects any given household x after weighting by the steady-state distrbution g^{SS} and converting to utils using the Impulse Value $v(x, x'')$. The change in savings rates of any other household x'' is then given by the innermost bracket. It involves the price impact function D net of the first-order change in consumption $\frac{\partial_{a''} v(x'', x')}{u''(c^{SS}(x''))}$. This expression for the consumption response follows from linearizing the first-order condition for consumption. It is a 'distributional Marginal Propensity to Consume' (dMPC): it represents how consumption changes in response to a distributional impulse.

Despite its notational complexity, the FAME is in fact remarkably simple. It has four striking features, which Section 3 shows hold much more broadly than in the present example.

(i) From infinite to finite dimension. The FAME (13) is a standard Bellman equation in finite dimension. The dimensionality of the Impulse Value is simply twice that of the original problem, instead of being infinite-dimensional like the nonlinear Master Equation (11). This drastic simplification stems from the local perturbation. Households located at x in the state space need only consider isolated impulses at any other possible x' in the distribution, since any pairwise impulses would lead to a second-order deviation in the value function.

(ii) Closed form steady-state dependence. Second, by virtue of the analytic nature of the perturbation, all the objects entering in the FAME are steady-state objects with closed form expressions. This observation implies that, once the nonlinear steady-state of the model is known, no additional calculation is needed to write down and solve the FAME.

(iii) Block-recursivity. The FAME inherits the block-recursivity from the Master Equation. The FAME is the only fixed point that must be solved to know individual behavior along any impulse response. There is no additional price or distributional fixed point to solve because such fixed points have already been embedded into the Master Equation. The FAME uncovers that this joint fixed point has a simple structure that may be solved efficiently.

Once the Impulse Value is known, it is straightforward to apply a similar perturbation argument to the law of motion of the distribution (4).[7] To first order,

$$\frac{\partial h_t}{\partial t}(x) \quad = \quad \mathcal{L}^*(x)[h_t] + \mathcal{K}(x)[h], \tag{14}$$

where the operator $\mathcal{K}$ is defined by the integral formula

$$\mathcal{K}(x)[h] = \int K(x,x')h(x')dx' \ , \ K(x,x') = -\frac{\partial}{\partial a}\left(g^{SS}(x)\left(D(x,x') - \frac{1}{u''(c^{SS}(x))}\frac{\partial v}{\partial a}(x,x')\right)\right). \tag{15}$$

Equation (14) encodes the time evolution of any impulse h in the distribution over time. Its two components represent distinct forces that mirror those that the FAME (13) uncovered.

The first term $\mathcal{L}^*[h]$ represents the partial equilibrium evolution of the impulse h. This partial equilibrium evolution is computed holding transition probabilities fixed at their steady-state values.

The second term $\mathcal{K}[h]$ represents the general equilibrium response of the economy. It encodes changes savings behavior of all households in response to the distributional impulse. These changes in behavior stem are embedded in the price impact and dMPCs that enter the definition of the kernel K. The kernel K coincides with the expression in the general equilibrium component of the FAME (13) because both represent changes in savings rates.

(iv) Efficient numerical implementation. Due to its standard Bellman equation structure, the FAME can be solved efficiently using standard finite dimension schemes. Denote by L the steady-state transition matrix that discretizes the operator L. It encodes transitions within a 2-dimensional state space, assets and productivity. With 100 grid points in each dimension, such a matrix has 100 million (10^8) entries. Its sparse nature allows standard software to easily handle its size. However, in this example, the Impulse Value is a 4-dimensional object. Naively constructing the associated transition matrix would lead it to have 10 million billion (10^{16}) entries. Even with a sparse structure, it may prove difficult for standard computational software.

A closer look at the FAME reveals that its particular structure bypasses this difficulty. The propagation of the relevant state (x, x') is encoded in the second and third components, $\mathcal{L}(x)[v(\cdot, x')] + \mathcal{L}(x')[v(x, \cdot)]$. These expectational terms exhibit a specific separability and symmetry structure because household x and household x' evolve independently from each other. Section 3 shows that this property holds much more broadly.

The implication for numerical implementation is that the Impuse Value may be efficiently represented as a square matrix, rather than a column vector. Let v denote its discretization. Then the discretization of $\mathcal{L}(x)[v(\cdot, x')] + \mathcal{L}(x')[v(x, \cdot)]$ is simply the sum of two matrix products, $\mathrm{Lv} + \mathrm{vL}^T$, where the ^{T} superscript denotes the matrix transpose. Crucially, this computation never requires to construct a 4-by-4-dimensional matrix, only the 2-by-2-dimensional steady-state matrix. Section 3 leverages this observation to construct an algorithm to solve the FAME numerically with only steady-state dimensionality

[7] In fact, equation (14) is already known. It follows immediately from the linearization of the last component in (11).

With the Impulse Value v at hand, the time evolution of the distribution obtains immediately by iterating forward on the law of motion of the distribution (14) without solving any additional fixed point problem: the equilibrium is block-recursive. With the distributional impulse h_t along any equilibrium path, impulse response may then be immediately computed.

1.4 Aggregate shocks with the FAME

So far, there were no aggregate shocks in order to focus on the role of the distribution. I now introduce aggregate shocks in the economy. Aggregate productivity Z is no longer constant, but fluctuates over time: $Z_t = \bar{Z}e^{\varepsilon z_t}$. For the sake of simplicity, this section assumes that log productivity follows a continuous-time AR(1) process: $dz_t = -\mu z_t dt + dW_t$ with associated generator $\mathcal{A}(z) = -\mu z\partial_z + \frac{1}{2}\partial_{zz}$. The parameter $\varepsilon > 0$ represents the overall scale of aggregate shocks.

The Master Equation (11) enriched with aggregate shocks becomes

$$\rho V(x,\varepsilon,z,g) \quad = \quad \max_c u(c) + L(x,c,\varepsilon,z,g)[V] + \int \frac{\partial V}{\partial g}(x,x',\varepsilon,z,g)L^*(x',\varepsilon,z,g)[g]dx' + \mathcal{A}(z)[V] \tag{16}$$

Relative to the Master Equation without aggregate shocks (11), the Master Equation with aggregate shocks (16) now depends on aggregate productivity εz_t. This dependence is explicit in the operator L through the price functionals that now depend on aggregate productivity. This dependence in turn implies that the value function depends both on z and ε. In addition, households expect aggregate productivity to fluctuate over time. These expectations result in the additional continuation value $\mathcal{A}(z)[V]$. It is also possible to consider deterministic transitional dynamics in response to a one-time shock. In that case, one needs simply to keep track of time t instead of the aggregate productivity state z.[8]

The law of motion of the distribution (4) also extends with aggregate shocks:

$$dg_t(x) = L^*(x,\varepsilon,z_t,g_t)[g_t]\ dt. \tag{17}$$

The law of motion in (17) is now a stochastic version of the Kolmogorov forward equation (4). The coefficients of the operator L^* change stochastically following shocks to aggregate productivity z_t, and thus define a stochastic partial differential equation. Despite its apparent complexity, the logic and intuition underlying the law of motion (17) is entirely similar to that of finite-dimensional evolution equations. For instance, the law of motion for assets for employed individuals is a unidimensional stochastic differential equation $da_t = s_t(x_t)dt$. The law of motion (17) is a mere infinite-dimensional analog thereof. The only addition is that interactions between the entries of the infinite-dimensional vector $g_t(x)$ that are relevant for its time dynamics are picked up by cross-sectional derivatives.[9]

[8] In that case, the Master Equation approach simply removes the contribution to the time derivative in the individual decision problem coming from the distribution g_t, but keeps the time derivative coming from the aggregate shock.

[9] The Master Equation (16) is non-stochastic, while the SPDE (17) is, because the Master Equation (16) conditions on the current value of the aggregate shock z, while the SPDE (17) takes the sequence of realized aggregate shocks z_t as given.

With the Master Equation (16) at hand, it is then straightforward to extend the perturbation argument in the deterministic FAME to aggregate shocks. There are now two objects that are small. The first one is, as before, the deviation in the distribution $g - g^{SS}$. The second object is the scale of aggregate shocks ε. The natural benchmark is that $\varepsilon h \equiv g - g^{SS}$ has scale ε.[10]

When ε is small, the first-order approximation to the value function is:

$$V(x, z, \varepsilon, g^{SS} + \varepsilon h) \approx V^{SS}(x) + \varepsilon \left\{ \int v(x, x')h(x')dx' + \omega(x, z) \right\}. \tag{18}$$

ω is the 'stochastic Impulse Value': the direct effect of aggregate shocks on the value function.[11]

Just as in any Taylor expansion, the first-order approximation is additive in the response to a distributional impulse h and the aggregate shock z. Any pairwise perturbation involving an impulse in the aggregate shock together with an impulse in the distribution h is of scale ε^2, and thus second-order.

The separability in equation (18) is critical and represents certainty equivalence. It implies that the deterministic Impulse Value $v(x, x')$ with respect to the distributional impulse h satisfies the deterministic FAME (13) evaluated at $Z = \bar{Z}$. In particular, the deterministic Impulse Value x can be solved for independently from aggregate shocks. This observation mirrors the linearization of representative agent models such as the RBC model. There too, the deterministic component in the value function is independent from the stochastic component.

Mirroring the derivation of deterministic FAME (13), I identify coefficients on the aggregate shock after substituting the first-order expansion (18) into the nonlinear Master Equation (11). The stochastic Impulse Value $\omega(x, z)$ then satisfies the stochastic FAME:

$$\begin{aligned} \rho\omega(x,z) \quad = \quad & \overbrace{z\Omega(x)u'(c^{SS}(x))}^{\text{Direct price impact}} + \overbrace{\mathcal{L}(x)[\omega(\cdot,z)]}^{\substack{\text{Partial equilibrium:}\\ \text{continuation value}\\ \text{from shocks to } x}} + \overbrace{\mathcal{A}(z)[\omega(x,\cdot)]}^{\substack{\text{Continuation value from}\\ \text{aggregate shocks } z}} \\ - \quad & \underbrace{\int v(x,x'')\frac{\partial}{\partial a'}\left(g^{SS}(x'')\left(z\Omega(x'') - \frac{1}{u''(c^{SS}(x''))}\frac{\partial\omega}{\partial a''}(x'',z)\right)\right)dx''}_{\substack{\text{General equilibrium: weighted average of changes in savings rates}\\ \text{of other households } x'' \text{ in response to aggregate shock } z}} \end{aligned} \tag{19}$$

where $\Omega(x) \equiv \mathcal{R}_2 a + \mathcal{W}_2 y$, and $\mathcal{R}_2, \mathcal{W}_2$ are constant that depend only on g^{SS} and are given in Appendix A.1.

The structure of the stochastic FAME (19) mirrors that of the deterministic FAME (13). The main difference is simply the expression for the direct price impact of aggregate shocks Ω. As with

[10] Consider the stochastic steady-state. Suppose the economy starts at its deterministic steady-state $g_0 = g^{SS}$, and is then hit by a sequence of aggregate shocks. Provided the deterministic steady-state is stable, g_t will remain in a neighborhood of g^{SS} of typical size ε because the typical size of aggregate shocks is ε. Thus, in the stochastic steady-state, the deviation in the distribution $g_t - g^{SS}$ should be expected to be of comparable scale to that of aggregate shocks.

[11] Depending on the stochastic process for aggregate productivity z, the stochastic Impulse Value ω needs not be linear in z. In this section's example, it turns out that it can be easily proven because the productivity process is an unrestricted diffusion process. However, when there are reflecting boundaries or non-symmetric jump terms in the productivity process, ω is no longer linear in general. By contrast, the process for the distribution is always an unrestricted diffusion process, and thus the distribution always enters as a linear functional.

the deterministic FAME, all the objects that enter into the stochastic FAME (19) are evaluated at the deterministic steady-state. They are thus immediately known given given the steady-state.

As with the deterministic FAME, the stochastic FAME (19) is a standard Bellman equation that may again be solved with standard methods. Given that the deterministic Impulse Value v is known by the time one solves the stochastic FAME, inspection of (19) reveals that the stochastic Impulse Value ω solves a linear equation. By contrast, the deterministic FAME solves a quadratic equation (13) in v. This observation mirrors the linearization of the RBC model for instance, in which the deterministic component solves a quadratic scalar equation, while the stochastic component solves a linear equation given the solution to the deterministic component.

With both Impulse Values v, ω at hand, I turn back to the law of motion of the distribution and obtain the evolution of the impulse h_t in the distribution up to a first order:

$$dh_t(x) \quad = \quad \Big\{\mathcal{L}^*(x)[h_t] + \mathcal{K}(x)[h_t] + S(x, z_t)\Big\}\, dt, \tag{20}$$

where $S(x, z)$ represents the change in savings rates of households at x in response to aggregate shocks, and has a similar expression to K:

$$S(x, z) \quad = \quad \frac{\partial}{\partial a}\left(g^{SS}(x)\left(\Omega(x)z - \frac{1}{u''(c^{SS}(x))}\frac{\partial \omega}{\partial a}(x, z)\right)\right) \tag{21}$$

Equation (20) is the linearized version of the SPDE in (17). Iterating forward on (20) for a given sequence of aggregate shocks z_t then delivers any desired impulse response function.

This section developed the main ideas and benefits of the FAME in the context of the Krusell and Smith (1998) example. In Sections 2 and 3, I generalize the approach to nest many possible economic models that fall beyond the scope of traditional methods, and introduce the necessary formalism to make the statements in Section 1 rigorous.

2 The Master equation

This section builds on the insights from Section 1 and develops a general formulation of the Master Equation in the deterministic case. The case with aggregate shocks is postponed to Section 3.3.

2.1 Notation and setup

This section sets up the notation for the remainder of the paper, most of which is to handle mass points in the distribution symmetrically to a smooth density.

Time $t \geq 0$ is continuous and runs forever. The economy is populated by a unit measure of agents. Agents are characterized by their individual state vector $x \in \bar{X} \subset \mathbb{R}^{D_X}$, where $X = (\underline{x}_1, \overline{x}_1) \times \ldots \times (\underline{x}_{D_X}, \overline{x}_{D_X})$ is a D_X-dimensional hypercube.[12] Let $\bar{X}$ denote its closure in the Euclidian norm. $\bar{X}$ is

[12]Working with a hypercube is not strictly necessary for most of the results below. However, it makes the notation much lighter—in a relative sense—to handle mass points. Without mass points, virtually all the results below go through for a general open domain X without additional notation.

endowed with the Borel $\sigma-$algebra, and a base measure η. Individuals may choose a control variable $c \in \bar{\Gamma} \subset \mathbb{R}^{D_C}$, where Γ is open.[13]

The base measure η plays a key role in the sequel. It encodes a priori information about where the distribution is absolutely continuous, and where it may develop mass points. If the example of Section 1 was enriched with an occasionally binding borrowing constraint $a \geq \underline{a}$, one would define $d\eta(a, y) = (da + \delta_{\{\underline{a}\}}(da)) \otimes dy$, where $\delta_{\{\underline{a}\}}$ denotes the Dirac measure at $\underline{a}$, and $\otimes$ denotes the tensor product of measures. This definition then allows for the possibility of a mass point at the borrowing constraint $\underline{a}$.

The base measure allows to handle only densities with respect to that base measure, and thus treats mass points and smooth densities symmetrically.[14] In the sequel, I always impose that the base measure is a product measure of marginal measures. The marginal measure along dimension i in turn consists of the Lebesgue measure together with a countable set of possible mass points. These possible mass points are located at $x_{in} \in [\underline{x}_i, \overline{x}_i]$, for a countable set $\mathcal{N}_i \subset \mathbb{N}$ of indices. Specifically, I henceforth impose the following assumption.

Assumption 1. *(Base measure)*
$d\eta(x) = d\eta_1(x_1) \otimes ... \otimes d\eta_{D_X}(x_{D_X})$, where, for all $i = 1...D_X$ and all Borel-measurable subset $Z \subset [\underline{x}_i, \overline{x}_i]$, $\eta_i(Z) = \ell(Z_i) + \sum_{n \in \mathcal{N}_i} \mathbb{1}\{x_{in} \in Z\}$, where $\ell(Z)$ denotes the Lebesgue measure of Z.

The domain of the state x is the union of the open domain X together with the set of possible mass points when they lie at the boundary. Specifically , let $I_i = (\underline{x}_i, \overline{x}_i) \cup \{\inf_n x_{in}\} \cup \{\sup_n x_{in}\}$, and $\widehat{X} = I_1 \times ... \times I_{D_X}$. Denote by ∂X the boundary of X, and by $\widehat{\partial X}$ the boundary of X net of the the edges where there are mass points. Denote $m_i = \{x_{in}, n \in \mathcal{N}_i\}$ the set of possible mass points along direction i. Let $M_i = \{x \in \widehat{X} : x_i \in m_i\}$ be the set of states x that have their i^{th} coordinate on a possible mass point, and $M = \cup_i M_i$ the set of states where any one coordinate falls upon a possible mass point.

Agents' state x_t evolves over time according to a controlled jump-diffusion process

$$dx_t = b_{0t}(x_t, c_t)dt + \sigma_{0t}(x_t, c_t) \cdot dW_t + \iint (y - x_t) f_{0t}(x_t, c_t, y, \nu) d\eta(y) N(dt) d\nu. \quad (22)$$

b_{0t} is the $\mathbb{R}^{D_X}$-valued drift of the process. It may depend directly on calendar time t, the current state x_t, as well as the control c_t. Similarly, σ_0 is the $\mathbb{R}^{D_X \times D_W}$-valued function volatility of the process. It governs how individual states respond to the $\mathbb{R}^{D_W}$-valued Brownian motion W. Importantly, W is independent across agents. The Poisson jump measure N encodes the frequency of jump increments. The density f_0 captures the density of increments and their frequency over the base measures η, N.

[13] To keep the exposition as concise as possible, discussion of filtrations and adaptedness are omitted. See Carmona and Delarue (2018a) for an in-depth exposition.

[14] In principle, it is possible to work without a base measure. In that case, the law of motion of the distribution is set in the space of measures. Consequently, one needs to develop the formalism of derivatives with respect to general measures in the Wassertstein space. See Cardaliaguet et al. (2019) for details. Introducing this formalism is beyond the scope of this paper, and is also irrelevant for many economic applications of interest in which a priori knowledge of where mass points may develop is often available.

In the notation is implicit that jumps are also independent across agents. Finally, the variable ν represents idiosyncratic shocks that may occur upon a jump. This last degree of freedom lets the framework handle taste shocks common in the dynamic discrete choice literature.

The state x_t may include discrete indicators for different types of agents, for instance employed workers and unemployed workers, workers and firms, regions or countries. The process x_t is assumed to remain within $\bar{X}$, either through reflection at the boundary of X, or through an appropriate combination of drift and volatility at the boundary. Without further restrictions, equation (22) encompasses a large class of stochastic processes in continuous time, and nests the vast majority of those commonly used in macroeconomic applications.

I use the notion of weak derivatives to handle mass points in the distribution. Let f be a η-measurable function. Its weak derivative $\frac{\partial f}{\partial x_i}$ is, when it exists, defined by duality. Suppose that there exists a η-measurable function w_i such that

$$\int f(x)\frac{\partial \varphi}{\partial x_i}(x)d\eta(x) = -\int w_i(x)\varphi(x)d\eta(x)$$

for all continuously differentiable functions φ that vanish on $\widehat{\partial X}$. In that case, define its weak derivative $\frac{\partial f}{\partial x_i}$ to be w_i. See Online Appendix E for details.

When f is continuously differentiable, the weak derivative coincides with the classical derivative. When f has a jump in direction i at some $x_0 \in X$, then the weak derivative in the sense of generalized functions is a measure: it is a Dirac mass point multiplied by the size of the jump: $\frac{\partial f}{\partial x_i}(x_0)dx \equiv \Big(\lim_{\varepsilon\downarrow 0} f(x_0+\varepsilon\tau_i) - \lim_{\varepsilon\downarrow 0} f(x_0-\varepsilon\tau_i)\Big)\delta_{x_0}(dx)$, where τ_i is the unit vector pointing in direction i. In that case, f admits a weak derivative only if $x_0 \in M_i$. Then, the weak derivative as a η-measurable function simply the size of the jump at x_0: $\lim_{\varepsilon\downarrow 0} f(x_0+\varepsilon\tau_i) - \lim_{\varepsilon\downarrow 0} f(x_0-\varepsilon\tau_i)$.

Two functional spaces are useful in the sequel. The first is the space of square-integrable functions with respect to the base measure η. The second is the second Sobolev space:

$$L^2 \equiv \left\{ f : \bar{X} \to \mathbb{R} \text{ is } \eta\text{-measurable} \;\Big|\; \int f(x)^2 d\eta(x) < \infty \right\}, \qquad H^2 \equiv \left\{ f \in L^2 \;\Big|\; \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2} \in L^2 \right\}. \quad (23)$$

The notation L^2 should not be confused with the notation for the generator of the process, L. The second Sobolev space H^2 consists of all square-integrable functions that have square-integrable first and second weak derivatives.

I now define two functional operators related to the state process x_t. The first operator L_{0t} is the generator L_{0t} of the state process and encodes conditional expectations of functions of x_t. For any

$V \in H^2$, define:[15]

$$L_{0t}(x,c)[V] = \sum_{i=1}^{D_X} b_{0,i,t}(x,c)\frac{\partial V}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{D_X}\Sigma_{0,ij}(x,c)\frac{\partial^2 V}{\partial x_i \partial x_j}(x) + \iint f_{0t}(x,c,y,\nu)\big(V(y)-V(x)\big)d\eta(y)d\nu \quad (24)$$

where $\Sigma_{0t}(x,c) = \sigma_{0t}(x,c)\sigma_{0t}(x,c)^T$, and recall that the ^{T} superscript denotes the matrix transpose.

The second operator is the formal adjoint operator L^*_{0t}.[16] It encodes how the cross-sectional probability distribution of the process x_t evolves over time. For any $g \in H^2$, define:

$$L^*_{0t}(x,c)[g] = -\sum_{i=1}^{D_X}\frac{\partial}{\partial x_i}\Big(g(x)b_{0,i,t}(x,c)\Big) + \frac{1}{2}\sum_{i,j=1}^{D_X}\frac{\partial^2}{\partial x_i \partial x_j}\Big(\Sigma_{0,ij}(x,c)g(x)\Big) + \iint f_{0t}(y,c,x,\nu)g(y)d\eta(y)d\nu - g(x)\iint f_{0t}(x,c,y,\nu)d\eta(y)d\nu \quad (25)$$

2.2 Individual optimization problem and distribution

Armed with the notation above, I define agents' decision problem. They solve the following time-dependent Bellman equation with possible constraints on the state variable:

$$\rho V_t(x) - \frac{\partial V_t}{\partial t}(x) = \max_{c \in \bar{C}} u_{0t}(x,c,V_t) + L_{0t}(x,c)[V_t]$$
$$\text{s.t.} \quad C_{0t}\left(x, V_t(x), \frac{\partial V_t}{\partial x}(x)\right) \geq 0, \; x \in B \quad (26)$$

In equation (26), the flow payoff u_{0t} may depend on time, the current state, but also the value function V directly. This formulation embeds recursive preferences such as Epstein-Zin, as well as bargaining models of the labor market.[17]

The function C_{0t} captures constraints on the state x_t.[18] For instance, if there was a credit constraint $a_t \geq \underline{a}$ in the economy of Section 1, consumption c must be such that $r\underline{a} + w - c \geq 0$, i.e. $c \leq r\underline{a} + w$. This constraint on the control may equivalently be re-stated on the value function by $\frac{\partial V_t}{\partial a} \geq u'(r_t\underline{a} + w)$. Therefore, $C_{0t}\left(a, y, V_t, \frac{\partial V_t}{\partial a}\right) = \frac{\partial V_t}{\partial a} - u'(r_t a + w)$ for $a = \underline{a}$. The state constraint inequality holds for all x in a set B that I assume without loss of generality to be included in the set of possible mass points as well as the boundary of the domain: $B \subset M \cap \partial X$.

It is useful to define the evolution of the distribution in terms of the density $g_t(x)$ with respect to the base measure η in order to accommodate the presence of possible mass points. g_t satisfies the law

[15] For points on the boundary of $\bar{X}$, $x \in \partial X$, it is understood that derivatives are taken with respect to the interior directions to X, and all functions are extended by 0 outside of $\widehat{X}$.

[16] It is the adjoint of L_{0t} on $H^2 \cap \{f \text{ continuous and } f(x) = 0 \text{ for all } x \in \widehat{\partial X}\}$.

[17] For Epstein-Zin preferences, set $u_t(x,c,V) = \rho V(x) + \rho\frac{1-\gamma}{1-\psi}V(x)\left[\left(c/((1-\gamma)V(x))^{\frac{1}{1-\gamma}}\right)^{1-\frac{1}{\psi}} - 1\right]$ and set $\rho = 0$ in the left-hand-side of (26).

[18] See Fleming and Soner (2006) for more details.

of motion:

$$\frac{\partial g_t}{\partial t}(x) = L^*_{0t}(x, \hat{c}_t(x))[g_t]. \tag{27}$$

The notion of weak derivative with respect to the base measure η is key to systematically handle mass points in the evolution of the distribution. As long as mass points develop only where η allows them to, the weak derivative is well-defined in the space H^2: for all $g \in H^2$, $L^*_{0t}(x, \hat{c}_t(x))[g] \in L^2$.

When the economy is stationary, and provided u_0 and L_0 are time-independent, the Bellman equation (26) becomes

$$\rho V(x) \quad = \quad \max_{c \in \bar{C}} u_0(x, c, V) + L_0(x, c)[V] \ \text{ s.t. } \ \mathcal{C}_0\left(x, V(x), \frac{\partial V}{\partial x}(x)\right) \geq 0, \ x \in B \tag{28}$$

Similarly, the law of motion of the distribution (27) becomes

$$0 = L^*_0(x, \hat{c}(x))[g] \tag{29}$$

2.3 General equilibrium

I now specify how the flow payoff u_t as well as the process for the productivity process L_{0t} depend on the underlying distribution g_t.[19]

Assumption 2. *(Distribution dependence)*
There exist functionals u, b, σ, f, C such that u_0 and the coefficients of L_0 satisfy

$$u_{0t}(x, c, V) = u(x, c, V, g_t), \qquad b_{0t}(x, c) = b(x, c, g_t), \qquad \sigma_{0t}(x, c) = \sigma(x, c, g_t),$$
$$f_{0t}(x, c, y) = f(x, c, y, g_t), \qquad C_{0t}\left(x, V_t, \frac{\partial V_t}{\partial x}\right) = C\left(x, V, \frac{\partial V_t}{\partial x}, g_t\right) \tag{30}$$

In addition, $u, b, \sigma, f, \mathcal{C}$ are continuously L^2-Frechet-differentiable in V and g.

I impose Assumption 2 henceforth. It captures how prices and other general equilibrium forces feed back into individual decisions. Assumption 2 is typically mediated through market clearing conditions. It is widely satisfied in applications, in which the dependence on calendar time is a shorthand for dependence on the underlying aggregate state of the economy. The aggregate state of the economy in turn consists of the distribution of individual states. In the example of Section 1, the interest and wage rates are simple moments of the distribution. Thus, Assumption 2 is satisfied.

Importantly, Assumption 2 allows for a much more flexible dependence of the individual decision problem than only a few prices. There can be any arbitrary number of prices that matter, such as in a dynamic migration model. The distribution can also matter directly for individual decisions, as in a search-and-matching model with job-to-job search. I now define an equilibrium of the economy.

[19] To keep the exposition minimal, I assume that time dependence runs only through the distribution g_t as in Section 1. It is not difficult to let time affects the economy deterministically. In that case, one needs only treat time as another state variable.

Definition 1. *(Equilibrium in sequential form)*
An equilibrium in sequential form of the economy consists of a path of distributions $(g_t)_{t\geq 0} \in H^2$, a path of values $(V_t)_{t\geq 0} \in H^2$ such that (26) and (27) holds for all times $t \geq 0$, and g_0 is given.

There is no need to require any market to clear, since market clearing is embedded in Assumption 2. Similarly, a steady-state equilibrium is defined as follows.

Definition 2. *(Steady-state equilibrium)*
A steady-state equilibrium of the economy consists of a distribution $g^{SS} \in H^2$, a value $V^{SS} \in H^2$ such that (28) and (29) hold.

2.4 From the time-dependent problem to the Master Equation

Now turn to the Master Equation. Under Assumption 2, the economy may be represented fully recursively as a value function defined on idiosyncratic states as well as the space of distributions $\widehat{X} \times H^2$. As in Section 1, the key step is to change variables from calendar time t to the distribution g_t by writing $V_t(x) \equiv V(x, g_t)$.

Since L^2 is a Hilbert space when equipped with the inner product $\langle f, g\rangle = \int f(x)g(x)d\eta(x)$, Frechet derivatives are well-defined in L^2. Suppose for now that $g \mapsto V(x, g)$ admits a Frechet derivative for η-almost all x. It consists of a linear bounded operator from L^2 onto itself. Using the Riesz representation theorem, it may in turn be represented by an L^2 function. Denote this L^2 function by $y \mapsto \frac{\partial V}{\partial g}(x, g, y)$.

The same change of variables as in Section 1, using the chain rule for Frechet derivatives and the law of motion of the distribution (27), delivers:

$$\frac{\partial V_t}{\partial t}(x) = \left\langle \frac{\partial V}{\partial g}(x, g, \cdot), \frac{\partial g_t}{\partial t}(\cdot) \right\rangle = \left\langle \frac{\partial V}{\partial g}(x, g, \cdot), L^*(\cdot, \hat{c}(\cdot), g)[g] \right\rangle \tag{31}$$

Substituting identity (31) into the Bellman equation (26), I obtain the Master Equation.

Definition 3. *(Master Equation)*
The Master Equation is defined by

$$\begin{aligned} \rho V(x, g) &= \max_{c\in\bar{\Gamma}} u(x, c, V, g) + L(x, c, g)[V] + \int \frac{\partial V}{\partial g}(x, x', g) L^*(x', \hat{c}(x'), g)\,[g]\, d\eta(x') \\ & \quad s.t. \quad C\left(x, V(x, g), \frac{\partial V}{\partial x}(x, g), g\right) \geq 0 \end{aligned} \tag{32}$$

for functions $\widehat{X} \times H^2 \ni (x, g) \mapsto V(x, g)$ that are L^2-Frechet-differentiable in g η-a.e. in x.

The Master Equation (32) delivers a natural definition of a recursive equilibrium.

Definition 4. *(Equilibrium in recursive form)*
An equilibrium in recursive form consists of a solution V to the Master Equation (32).

Definition 4 emphasizes that a value function that solves the Master Equation is the only object that is needed to describe the equilibrium. Through the integral that enters the right-hand-side of (32), the endogenous evolution of the distribution is fully taken into account by agents. By construction, both notions of equilibrium coincide whenever defining a solution to the Master Equation is possible.

Proposition 1. *(Coincidence of recursive and sequential competitive equilibrium)*
Suppose that there exists an equilibrium in recursive form given by a solution to the Master Equation (32) $V(x,g)$. Define $V_t(x) \equiv V(x,g_t)$, and let g_t solve (27). Then the pair (V_t, g_t) defines an equilibrium in sequential form.

A natural and important question is of course when does a solution to the Master Equation exist at all. Several set of assumptions have been proposed recently in the mathematics mean field games literature. Since this literature is still growing, the set of assumptions is still limited at the time of writing. In particular, typical assumptions exclude several key economics application, such as the presence of a credit constraint or multiplicative interactions between prices and states in the savings rate. Therefore, I do not list those assumptions in this paper, and merely point to existing results.[20] These assumptions also guarantee that the value function $V(x,g)$ is L^2-Frechet-differentiable in the distribution g up to second order. In addition, these assumptions are typically the same as those required for existence and uniqueness of an equilibrium in sequential form.

It is not the purpose of this paper to attempt expanding the set of assumptions leading to existence and uniqueness results for the fully non-linear Master Equation—though one may hope that such results will eventually become available for most setups of economics interest. Instead, this paper is concerned with the more practical question of local approximations to the Master Equation conditional on the existence of at least on isolated steady-state equilibrium. Therefore, I shall merely impose the following assumption in the sequel.

Assumption 3. *(Existence, local uniqueness and regularity)*
There exists at least one isolated steady-state equilibrium V^{SS}, g^{SS}. There exists a solution $V(x,g)$ to the Master Equation locally around g^{SS}. This solution is continuously L^2-Frechet-differentiable in g in a neighborhood of g^{SS}, η-almost everywhere in x. V is continuous in x. The $D_X - 1$-dimensional boundary of the set of points such that the state constraint holds with equality is continuously Frechet-differentiable in g around $g = g^{SS}$, η-almost everywhere in x.

I impose Assumption 3 henceforth. It opens the door to local perturbations of the the Master Equation, which is the subject of the next section.

3 The FAME

This section derives the FAME, highlights its properties and derives its implications for the local stability of the steady-state. Finally, this section proposes an efficient numerical implementation to compute the solution to the FAME.

[20] See for instance Theorem 2.4.2. p. 39 in Cardaliaguet et al. (2019) or Theorem 4.2.1. p. 295 in Carmona and Delarue (2018b).

3.1 The deterministic FAME

Start by fixing a locally isolated steady-state equilibrium g^{SS}, V^{SS}, which I will be refer to as "the" steady-state for brevity in the sequel. Consider a small perturbation of the distribution around the steady-state, $g = g^{SS} + h$.

Under Assumption 3, the value function $V(x, g^{SS} + h)$ is a linear functional of the impulse in the distribution, $h = g - g^{SS}$ to first order:

$$V(x,g) = V^{SS}(x) + \int v(x,x')h(x')d\eta(x') + \mathcal{O}_x(||h||^2_{H^2}) \tag{33}$$

where the notation $\mathcal{O}_x$ is the "large O" notation, indexed by x to highlight that it may not be uniform. As in Section 1, $v(x,x')$ is the deterministic Impulse Value. It is the directional derivative of the value function with respect to the distribution g, evaluated at steady-state g^{SS}: $v(x,x') = \frac{\partial V}{\partial g}(x,x',g^{SS})$.

In light of Section 1, it is natural to expect the Impulse Value to satisfy a Bellman equation. To define that Bellman equation as concisely as possible, it is useful to introduce additional notation for steady-state objects. Denote objects evaluated at steady-state functions by a script letter. For instance, denote $\mathcal{L}(x) \equiv L(x, c^{SS}(x), V^{SS}, g^{SS})$. Similarly, denote partial usual or Frechet derivatives by the corresponding subscript. For instance, denote $u_g(x,x') \equiv \frac{\partial u}{\partial g}(x,x',V^{SS},g^{SS})$. For the Frechet derivatives of the adjoint operator $\mathcal{L}^*$ with respect to values V and controls c, I introduce an additional argument. For instance, for a small perturbation in controls $dc(x)$, I write $\mathcal{L}^*_c(x, dc(x))[g]$ to highlight that the derivatives embedded in the operator $\mathcal{L}^*_c$ also apply to $dc(x)$. The functional $dc \mapsto \mathcal{L}^*_c(x,dc)[g]$ is linear. See Appendix A.2 for the full list of notation.

Building on the linearization of the value function in (33), it is possible to linearize how individuals' optimal control depends on the distributional impulse through the first-order optimality condition. It is natural to expect that, to a first order,

$$\hat{c}(x, g^{SS} + h, V) \approx c^{SS}(x) + \int \mathcal{M}(x,x',v)h(x')d\eta(x'). \tag{34}$$

The kernel $\mathcal{M}(x,y,v)$ is the distributional Marginal Propensity to Control (dMPC). It encodes how individual controls respond to a small distributional impulse. It is a function of the point at which consumption is evaluated, x, and the direction x' in which the impulse occurs. It is also a function of the Impulse Value v, since the Impulse Value determines individuals' valuations of the impulse h. I need the following regularity condition, satisfied in most applications of interest, e.g. when the individual decision problem is strictly concave.

Assumption 4. *(Control regularity)*
The optimal control $\hat{c}(x,V,g)$ *is interior for* $||g - g^{SS}||_{L^2}$ *small enough.* $\mathcal{U}(x) \equiv u^{SS}_{cc}(x) + \mathcal{L}_{cc}(x)[V^{SS}]$ *is an invertible matrix for* η*-a.a.* x.

I impose Assumption 4 henceforth. To gain intuition, it is sometimes useful to introduce additional structure that is satisfied in applications. Denote by L^{diff} the diffusion part of the generator L, and L^{int} the integral part.

Assumption 5. *(Payoff and generator structure 1)*
There are two sets of indices E_1, E_2 *that partition* $\{1, ..., D_X\}$ *such that for* $i \in E_1$, $b_i(x, g, c, V) = b_i(x, g, V) - c_i$, *and for* $i \in E_2$, $L_i^{int}(x, c, g, V) = c_i L^{int}(x, g, V)$. *In addition,* $u(x, c, g, V) = u_0(x, g, V) + \sum_{i \in E_1} u_{1i}(c_i) - \sum_{i \in E_2} u_{2i}(c_i)$. u_{ki} *are strictly concave and satisfy Inada conditions, and* $\bar{\Gamma}_i = [0, +\infty)$.

Assumption 5 ensures that Assumption 4 holds, and that

$$\forall i \in E_1, \quad \hat{c}_i = (u'_{1i})^{-1}\left(\frac{\partial V}{\partial x_i}(x)\right), \qquad \forall i \in E_2, \quad \hat{c}_i = (u'_{2i})^{-1}\left(L_i^{int}(x, g, V)[V]\right).$$

Denote $U_{1i}(x) = \frac{1}{u''_{1i}(\partial_{x_i} V^{SS}(x))}$ and $U_{2i}(x) = \frac{1}{u''_{2i}(L_i^{int}(x, g^{SS}, V^{SS}[V^{SS}])}$. Armed with this notation, I obtain the first-order perturbation in individuals' optimal control in response to a small distributional impulse.

Proposition 2. *(Optimal control)*

$$\begin{aligned} \mathcal{M}(x, x', v) &= -\mathcal{U}(x)^{-1}\left(m_0(x, x') + \mathcal{L}_c(x)[v(\cdot, x')] + \int m_1(x, y) v(y, x') d\eta(y)\right) \\ m_0(x, x') &\equiv u_{cg}(x, x') + \mathcal{L}_{cg}(x, x')[V^{SS}] \quad , \quad m_1(x, x') \equiv u_{cV}(x, x') + \mathcal{L}_{cV}(x, x')[V^{SS}]. \end{aligned}$$

When Assumption 5 holds, then

$$\mathcal{M}_i(x, x', v) = \begin{cases} U_{1i}(x) \partial_{x_i} v(x, x'), & i \in E_1 \\ U_{2i}(x)\left[\mathcal{L}_i^{int}(x)[v(\cdot, x')] + \mathcal{L}_{g,i}^{int}(x, x')[V^{SS}] + \int \mathcal{L}_{V,i}^{int}(x, y)[V^{SS}]\ v(y, x') d\eta(y)\right], & i \in E_2. \end{cases}$$

Proof. See Appendix B.1. □

Proposition 2 characterizes how individuals' controls respond to a distributional impulse h. The dMPC depend on the concavity of the utility function as well as the concavity of the generator L. The dMPC also depends on the corresponding cross-derivatives. To gain intuition, consider only the case with a controlled drift under Assumption 5 ($i \in E_1$). In that case, the dMPC extends directly the nonlinear first-order condition, and is simply the marginal Impulse Value, weighted by the inverse of the second derivative of utility that governs the precautionary motive. Continuous time is key because it ensures that the first-order condition always holds with equality.

With Proposition 2 at hand, I may now proceed to a full characterization of the Impulse Value following the same logic as in Section 1.

Theorem 1. *(FAME)*

$$\begin{aligned} \rho v(x, x') = {} & \overbrace{u_g(x, x') + \int u_V(x, y) v(y, x') d\eta(y) + \mathcal{L}_g(x, x')[V^{SS}] + \int \mathcal{L}_V(x, y)[V^{SS}] v(y, x') d\eta(y)}^{\text{Direct impact}} \\ & + \underbrace{\mathcal{L}(x)\big[v(\cdot, x')\big]}_{\substack{\text{Partial equilibrium:} \\ \text{continuation value from} \\ \text{shocks to } x}} + \underbrace{\mathcal{L}(x')\big[v(x, \cdot)\big]}_{\substack{\text{General equilibrium:} \\ \text{continuation value from} \\ \text{propagation of impulse at } x'}} \\ & + \underbrace{\int v(x, x'')\left\{\mathcal{L}_g^*(x'', x')[g^{SS}] + \int v(y, x') \mathcal{L}_V^*(x'', y)[g^{SS}]\ d\eta(y) + \mathcal{L}_c^*\big(x'', \mathcal{M}(x'', x', v)\big)[g^{SS}]\right\} d\eta(x'')}_{\text{General equilibrium: weighted average of changes in decisions of other agents } x'' \text{ in response to impulse at } x'} \end{aligned} \tag{35}$$

subject to

$$\forall x \text{ such that } \mathcal{C}(x) = 0 \ : \ 0 = \mathcal{C}_g(x, x') + \mathcal{C}_V(x) v(x, x') + \mathcal{C}_p(x) \cdot \frac{\partial v}{\partial x}(x, x') \ \ \forall x'. \tag{36}$$

Proof. See Appendix B.2. □

The structure of the FAME (35) in Theorem 1 is the natural generalization of equation (13) in Section 1. The main addition is the linearization of the state constraint (36). The state constraint binds in the first-order approximation exactly at points where it binds in steady-state. It is natural to expect the state constraint to bind out of steady-state in a neighborhood of the points where it binds in steady-state. Perhaps surprisingly, these set of points turn out to coincide exactly. This conclusion arises because changes in where the state constraint binds in response to an impulse h result in a second-order contribution when interacted with the state constraint itself.

3.2 Deterministic Impulse Response Functions

Once the Impulse Value v is known, the evolution of the distribution is straightforward to compute.

Theorem 2. *(Evolution of the distribution)*

- *The impulse in the distribution $h \in H^2$ follows, to a first order,*

$$\frac{\partial h_t(x)}{\partial t} = \mathcal{L}^*(x)[h_t] + \mathcal{K}(x)[h_t]$$

where the kernel operator $\mathcal{K}$ is defined by $\mathcal{K}(x)[h] \equiv \int K(x, x') h(x') d\eta(x')$, and

$$K(x, x') = \mathcal{L}^*_c\big(x, \mathcal{M}(x, x', v)\big)[g^{SS}] + \mathcal{L}^*_g(x, x')[g^{SS}] + \int \mathcal{L}^*_V(x, y)[g^{SS}] v(y, x') d\eta(x')$$

- *The spectrum of the operator $\mathcal{L}^*$ in H^2 has weakly negative real parts. $\mathcal{L}^*$ has a spectral gap, and the speed of convergence to steady-state is governed by K, to a first order: $\frac{d}{dt}||h_t||_{H^2} \leq \langle h_t, \mathcal{K} h_t \rangle_{H^2}$. If, in addition, K is bounded in L^2, then $||h_t||_{L^2} \leq e^{\lambda^{sup}(\bar{K}) \cdot t} ||h_0||_{L^2}$, where $\lambda^{sup}(\bar{\mathcal{K}})$ is the supremum of non-zero eigenvalues of $\bar{\mathcal{K}} \equiv (\mathcal{K} + \mathcal{K}^*)/2$.*

- *If K is bounded in L^2, the steady-state is asymptotically stable if $\lambda^{sup}(\bar{\mathcal{K}}) < 0$.*

Proof. See Appendix B.3. □

Theorem 2 has two parts. In the first part, I show that the first-order dynamics of the distribution follow a simple linear law of motion. As expected, the evolution depends on the steady-state transition probabilities embedded in the generator $\mathcal{L}^*(x)$. This contribution to the law of motion represents the partial equilibrium response of aggregate dynamics to an impulse h in the distribution.

The evolution of the distribution also depends on the general equilibrium feedback of the economy, as highlighted by the action integral operator $\mathcal{K}(x)$. The action of this operator on the distribution embeds the first-order response of individual controls $\hat{c}$ to an impulse in the distribution, as captured by the first term that may be interpreted as the effect of the dMPCs. The last two terms in the expression for the kernel K reflect the direct impact of a distributional impulse on transition probabilities.

The second part of Theorem 2 leverages the expression for the law of motion to characterize the transitional dynamics more precisely. The operator $\mathcal{L}^*$ is a stochastic operator by construction: it transforms probability distributions into probability distributions. Since it is assumed to have at least one invariant distribution, standard results guarantee that its largest eigenvalue is zero and that it is separated from the second largest by a strictly positive number. The immediate conclusion is that the transitional dynamics generated by $\mathcal{L}^*$ are stable. As a result, whether convergence back to steady-state occurs depends critically on the general equilibrium effects embedded in the operator $\mathcal{K}$.

The final results characterize convergence as a function of the eigenvalues of the operator $\mathcal{K}$. When K is bounded, the associated operator is Hilbert-Schmidt and a spectral theorem applies: its non-zero eigenvalues are real and countable, with an associated countable orthonormal basis of H^2. When the largest eigenvalue is strictly negative, the economy always converges back to steady-state. I obtain a sufficient condition on the eigenvalues of $\mathcal{K}$ rather than a necessary and sufficient condition because the partial equilibrium dynamics could interact with the general equilibrium dynamics.

3.3 Aggregate shocks

I introduce aggregate shocks into the economy of Section 2. They take the form of a Markov stochastic process Z_t with values in $\mathbb{R}^{D_Z}$ that now enters as an argument of u, L and L^*.[21] The stochastic process for Z_t is assumed to depend on a scalar parameter $\varepsilon \geq 0$, that governs how dispersed shocks are.

This section focuses on the case in which aggregate shocks are small. The limit of small aggregate shocks is captured by a local perturbation of the economy when $\varepsilon \to 0$. To make this idea precise, denote by $A(Z, \varepsilon)$ the infinitesimal generator of the stochastic process for Z. Taking the limit of small aggregate shocks requires that the stochastic process remains well-behaved when $\varepsilon \to 0$. I capture this feature with the following regularity condition.

Assumption 6. *(Scalable aggregate shocks)*
There exists a constant $Q_0 > 0$ such that $\mathbb{P}[|z_t| \leq \varepsilon Q_0] = 1$. In addition, for any twice continuously differentiable function φ, $A(\varepsilon z, \varepsilon)[\varphi_\varepsilon] \to \mathcal{A}(z)[\varphi_\varepsilon]$ as $\varepsilon \to 0$, where $\varphi_\varepsilon(z) = \varphi(\varepsilon z)$, and where $\mathcal{A}$ is the infinitesimal generator of a stationary stochastic process with bounded domain.

Recall the particular stochastic process for aggregate shocks of Section 1. The process was a continuous-time AR(1). In that case, the process is scalable by construction because $A(\varepsilon z, \varepsilon)[\varphi_\varepsilon] = \mathcal{A}(z)[\varphi_\varepsilon]$ for all ε.[22] For the FAME to hold also with aggregate shocks, I do not need the relationship

[21] That only the level of aggregate shocks Z_t enters as an argument is not entirely without loss of generality. In particular, it implies that aggregate shocks do not generate direct volatility in the law of motion of individual states. This feature lets me focus attention on the 'first-order Master Equation' in the terminology of Cardaliaguet et al. (2019). The terminology 'first-order' there is independent of the order of local perturbations. When the aggregate shock volatility enters directly the law of motion of individual states as in a portfolio choice problems with aggregate risk, additional terms appear and one obtains the 'second-order Master Equation.' While it is not difficult to generalize the perturbation formulae in this paper to the second-order Master Equation, I leave it for future research.

[22] With the additional assumption that the process is reflected to ensure a bounded support.

between the original aggregate shock process and the rescaled one to hold exactly, but only in the limit. It is then straightforward to write the Master Equation with aggregate shocks, in terms of the rescaled shock $z_t = \frac{Z_t}{\varepsilon}$.

Appendix B.4 provides the Master Equation with aggregate shocks as well as additional details such as regularity conditions similar to those in Section 3.1. As in Section 1, I seek a first-order solution to the nonlinear Master Equation in the form:

$$V(x, z, g^{SS} + \varepsilon h, \varepsilon) \approx V^{SS}(x) + \varepsilon \left\{ \int v(x, x')h(x')d\eta(x') + \omega(x, z) \right\}.$$

v is now called the distributional Impulse Value, and ω is called the stochastic Impulse Value.

Theorem 3. *(FAME with Aggregate Shocks)*
The deterministic Impulse Value v satisfies the FAME of Theorem 1. The stochastic Impulse Value ω satisfies the Bellman equation

$$\begin{aligned} \rho\omega(x, z) &= z \cdot \left\{ u_Z(x) + \mathcal{L}_Z(x)[V^{SS}] \right\} \\ &+ \mathcal{L}(x)[\omega(\cdot, z)] + \mathcal{A}(z)[\omega(x, \cdot)] \\ &+ \int v(x, x') \left\{ \mathcal{L}_c^*\big(y, \overline{\mathcal{M}}(y, z, \omega)\big)[g^{SS}]\Big) + \int \mathcal{L}_V^*\big(x', y, \omega(y, z)\big)[g^{SS}]d\eta(y) \right\} d\eta(x') \end{aligned} \tag{37}$$

subject to

$$\forall x \ s.t. \ \mathcal{C}(x) = 0 \ ; \qquad 0 = \mathcal{C}_V(x)\omega(x, z) + \mathcal{C}_p(x) \cdot \frac{\partial \omega}{\partial x}(x, z) \ \ \forall z, \tag{38}$$

and where the stochastic MPC (sMPC) $\overline{\mathcal{M}}(x, \omega)$ is given by

$$\begin{aligned} \overline{\mathcal{M}}(x, z, \omega) &= -\mathcal{U}(x)^{-1} \left(\overline{m_0}(x) \cdot z + \mathcal{L}_c(x)[\omega(\cdot, z)] + \int \overline{m_1}(x, y)\omega(y, z)d\eta(y) \right) \\ \overline{m_0}(x) &\equiv u_{cZ}(x) + \mathcal{L}_{cZ}(x)[V^{SS}] \\ \overline{m_1}(x, y) &\equiv u_{cV}(x, y) + \mathcal{L}_{cV}(x, y)[V^{SS}]. \end{aligned}$$

Proof. See Appendix B.4. ☐

The first key insight from Theorem 3 is that, with aggregate shocks, the deterministic Impulse Value satisfies the same equation as in the deterministic case. Thus, as in Section 1, the economy is block-recursive. One solves first for the deterministic Impulse Value v independently from the stochastic Impulse Value ω. Only in a second step does one solve for the stochastic Impulse Value ω which satisfies a standard Bellman equation. Theorem 2 extends as follows.

Theorem 4. *(Evolution of the distribution with aggregate shocks)*

- *To first order, the impulse in the distribution h_t follows the SPDE*

$$dh_t(x, z) = \left\{ \mathcal{L}^*(x)[h_t] + \mathcal{K}(x)[h_t] + S(x, z_t) \right\} dt \tag{39}$$

where the forcing term S is given by

$$S(x,z) \quad = \quad \mathcal{L}_c^*\big(x, \overline{\mathcal{M}}(x,z,\omega)\big)[g^{SS}] + z \cdot \mathcal{L}_Z^*(x)[g^{SS}] + \int \mathcal{L}_V^*\big(x, x', \omega(x',z)\big)[g^{SS}] d\eta(x')$$

- *If, in addition, K is bounded in L^2 and $\lambda^{sup}(\bar{\mathcal{K}}) < 0$, then the steady-state equilibrium (g^{SS}, V^{SS}) is stochastically stable: $||h_t||_{L^2} \leq c||h_0||_{L^2}$ for a positive constant $c > 0$. The stochastic steady-state is characterized by a joint invariant distribution $\tilde{h}(x,z)$ that satisfies*

$$0 = \mathcal{L}^*(x)[\tilde{h}(\cdot, z)] + \mathcal{K}(x)[\tilde{h}(\cdot, z)] + S(x,z) + \mathcal{A}^*(z)[\tilde{h}(x, \cdot)] \tag{40}$$

- *If, in addition, K is bounded in L^2, $\lambda^{sup}(\bar{\mathcal{K}}) < 0$, and z_t follows a symmetric process, i.e. $\mathcal{A}(z)[\varphi] = \mathcal{A}(-z)[\varphi(-\ \cdot)]$, then $\omega(x,0) = 0$ η-a.e.. In addition, $h_t \to 0$ when $z_t = 0$ for all $t \geq 0$.*

Proof. See Appendix B.5. □

Theorem 4 extends Theorem 2 to the case with aggregate shocks. The first modification takes the form of the forcing term $S(x,z)$ in the evolution equation (39) for the distribution. It has a similar structure to the general equilibrium operator $\mathcal{K}$. This similarity is natural because both terms represent how shifts in either the distributional impulse or in the aggregate shock affect individual transition probabilities and MPCs.

The second modification is that Theorem 3 now characterizes stability as the presence of a stochastic steady-state rather than convergence back to steady-state. Due to the continual arrival of aggregate shocks, the economy is almost never at its deterministic steady-state, but rather fluctuates around it. Theorem 3 shows how to compute the joint invariant distribution over idiosyncratic states x and aggregate states z with operators that only depend on steady-state values and can be easily implemented numerically. The evolution equation (39) represents one particular trajectory within the continual movement captured by the joint invariant distribution solving (40).

The last result in Theorem 3 provides conditions under which the economy converges back to its deterministic steady-state following a long sequence of realized zero aggregate shocks. This property holds when the stochastic process for aggregate shocks is symmetric, and therefore the aggregate shock Impulse Value is zero for a zero realization of the aggregate shock. When the process for aggregate shocks is asymmetric, one would expect the economy to converge back to steady-state only if the aggregate shock takes a long sequence of identical values that are non-zero, and equal to some value that accounts for the asymmetry in individuals' expectations of future realizations. When the process for aggregate shocks is discrete, finding such a value may not always be possible.

3.4 Numerical implementation

Three key properties of the FAME dramatically simplify the computation of impulse responses. The first property is block-recursivity: first solve the deterministic FAME from Theorem 1, then solve the stochastic FAME of Theorem 3, and finally simulate an Impulse Response using Theorem 4.

The second property is that the FAMEs (35)-(37) have the structure of a standard jump-diffusion Bellman equation in finite dimension. In particular, the general equilibrium effects enters the FAMEs (35)-(37) just as a standard jump term would. Therefore, readily available and highly efficient discretization schemes apply.

The third property is that the FAME provides a closed-form mapping between steady-state objects and all the elements of the Bellman equation to solve. These observations lead to the following numerical scheme, described in pseudo-code below. I focus on the case $u_V, L_V = 0$ and no state constraint for simplicity, but it is straightforward to extend the scheme when these partial derivatives are not zero or the state constraint binds.

Corollary 1. *(Numerical implementation)*
Define grids $\{x_i\}_{i=1}^{I}$*,* $\{z_k\}_{k=1}^{K}$*,* $\{\tau_t\}_{t=1}^{T}$ *and time steps* $\Delta_t = \tau_t - \tau_{t-1}$*. Define the matrices* $\mathrm{v}_{ij} = v(x_i, x_j) \in \mathbb{R}^{I\times I}$ *and* $\omega_{ik} = \omega(x_i, z_k) \in \mathbb{R}^{I\times K}$*. Then:*

1. ***Deterministic FAME.*** *Let* $\mathrm{L}, \mathrm{u}_g, \mathrm{M}, \mathrm{N}$ *and* Pv *discretize* $\mathcal{L}, u_g, \mathcal{L}_g(x,x')[g^{SS}], \mathcal{L}^*(x,x')[g^{SS}]$ *and* $\mathcal{L}_c^*(x, \mathcal{M}(x'', x', v)[g^{SS}]$ *respectively. Then:*
   - *Guess* v^0
   - *Given* v^n*, update* v^{n+1} *by solving the standard Sylvester equation:*

   $$\Big(\rho \mathrm{Id} - \mathrm{L}\Big)\mathrm{v}^{n+1} - \mathrm{v}^{n+1}\Big(\mathrm{L}^T + \mathrm{M} + \mathrm{N}\mathrm{v}^n\Big) \quad = \quad \mathrm{u}_g + \mathrm{N}. \tag{41}$$

   - *Stop when* v^{n+1} *and* v^n *are close enough.*

2. ***Stochastic FAME.*** *Let* $\mathrm{u}_Z, \mathrm{Q}, A$ *discretize* $z u_Z(x), z\mathcal{L}_Z(c)[V^{SS}], \mathcal{A}$ *respectively. Then* ω *solves the Sylvester equation*

   $$\Big(\rho \mathrm{Id} - \mathrm{L} - \mathrm{vP}\Big)\omega - \omega \mathrm{A}^T = \mathrm{u}_Z + \mathrm{Q} \tag{42}$$

3. ***Impulse response functions.*** *Let* K *discretize the kernel* K*. Let* S_t *discretize* $S(\cdot, \tau_t)$ *for any* t*. Given* h_0*, the discretized distributional impulse* h_t *solves the recursion*

   $$\mathrm{h}_{t+1} = \mathrm{h}_t + \Delta_t \Big\{\mathrm{L}^T \mathrm{h}_t + \mathrm{K}\mathrm{h}_t + \mathrm{S}_t\Big\}. \tag{43}$$

Corollary 1 provides a simple way of computing Impulse Values and impulse response functions to first order. The discretization of steady-state operators into matrices follows standard finite difference rules as described in Achdou et al. (2021).

At the heart of Corollary 1 lies the specific structure of the FAME (41). Once discretized, the deterministic FAME becomes a modified Sylvester matrix equation. A standard Sylvester matrix equation is a linear system with a specific structure, so that it may be written as a function of a matrix unknown Y that satisfies $AY + YB = C$ for known matrices A, B, C. Of course, it is always possible to stack this linear system and solve it without exploiting the Sylvester structure. However, doing so would abstract from useful information about the structure of the linear system. Instead, standard routines such as Matlab's `sylvester.m` function solve the standard Sylvester equation much more efficiently than the stacked system.

The deterministic FAME however leads to a Sylvester equation with a quadratic term. Thus, Corollary 1 proposes an iterative scheme leveraging a sequence of standard Sylvester equations. A key observation is to treat the quadratic component vNv consistently with implicit schemes. The first Impulse Value v in the quadratic component represents household's x own Impulse Value. Thus, it is natural to treat it as implicit—solve for it endogenously at every iterative step. The second Impulse Value v in the quadratic component represents the change in control of all other households x''. Thus, it is natural to treat it as explicit—exogenous from the perspective of a given iterative step.[23]

Once the solution to the deterministic Impulse Value is known, the stochastic Impulse Value satisfies a standard Sylvester equation (42). With both Impulse Values at hand, iterating forward on the linearized law of motion (43) is straightforward.

3.5 Connection with existing numerical methods

In this subsection I discuss how the FAME relates to existing numerical approaches that build either on a state-space or a sequence-space representation.

State-space approach. The FAME is by nature a state-space approach. It provides the foundation for the computational state-space approach in Ahn et al. (2018) and Reiter (2009).[24] Ahn et al. (2018) rely on automatic differentiation of the nonlinear discretized Bellman equation and law of motion of the distribution to obtain a linear rational expectations system that stacks the linearized Bellman equation and the linearized law of motion of the distribution. They then perform a Blanchard and Kahn (1980) stable root-finding procedure to extract the relevant matrices from the associated Schur decomposition.[25]

In contrast, the FAME emphasizes that linearizing the Bellman equation and the law of motion of the distribution analytically is not only feasible, but also uncovers a systematic structure that remains hidden in Ahn et al. (2018) or Reiter (2009). As a result, the Impulse Values satisfy standard Bellman equations that may be solved using standard and fast recursive methods instead of relying on high-dimensional stable root-finding.

Sequence-space approach. By construction, the FAME also relates to the computational, sequence-space approach in Auclert et al. (2019). To make the connection clear, I propose a closed-form sequence-space linearization and connect it to the FAME.

For concreteness, consider the Krusell and Smith (1998) economy of Section 1 without aggregate shocks. However, it is staighforward to extend the following results to a general economy as in Section

[23] A formal proof of convergence is beyond the scope of this paper. Yet, this implicit-explicit structure leads to robust convergence in practice. By contrast, reversing which Impulse Value is treated as implicit or explicit leads to systematic divergence of the scheme.

[24] Specifically, the distributional Impulse Value v is the analytic counterpart to the matrix $\mathbf{D}_{vg}$ in their notation. The aggregate shock Impulse Value ω is the analytic counterpart to the matrix $\mathbf{D}_{vZ}$ in their notation.

[25] In such a high-dimensional setting, there is in addition no *a priori* guarantee that the stable hyperplane coincides with the true value function.

3 as long as the general equilibrium enters through a finite number of prices.

Consider a small deviation in the sequence of prices $\hat{r}_t, \hat{w}_t$ around stead-state. The first-order perturbation of the value function in sequence space is determined by the functions φ^r, φ^w such that:

$$V_t(x) \approx V^{SS}(x) + \int_t^\infty e^{-\rho(\tau-t)} \varphi^r_{\tau-t}(x) \hat{r}_\tau d\tau + \int_t^\infty e^{-\rho(\tau-t)} \varphi^w_{\tau-t}(x) \hat{w}_\tau d\tau \tag{44}$$

Auclert et al. (2019) propose an algorithm to compute φ^r, φ^w. I show here that using an analytical perturbation that is internally consistent with the model once more uncovers a systematic structure that underpins the algorithm in Auclert et al. (2019). This structure is useful for economic interpretation but also for the design of efficient and simple algorithms. I describe this structure before making the connection with the FAME.

Theorem 5. *(Sequence-space Jacobians of value function)*
φ^r, φ^w satisfy the Bellman equations

$$\begin{aligned} \frac{\partial \varphi^r_t}{\partial t}(x) &= \mathcal{L}(x)[\varphi^r_t], \quad \varphi^r_0(x) = au'(c^{SS}(x)) \\ \frac{\partial \varphi^w_t}{\partial t}(x) &= \mathcal{L}(x)[\varphi^w_t], \quad \varphi^r_0(x) = yu'(c^{SS}(x)). \end{aligned}$$

Proof. See Appendix B.6. □

Theorem 5 reveals that a standard, time-dependent Bellman equations determines the first-order response of the value function to price changes. Similarly to the FAME, this Bellman equation depends only on the steady-state transition probabilities encoded in the steady-state expectation operator $\mathcal{L}$. Through initial conditions, the Bellman equation also depends on the direct price impact.

Theorem 5 also delivers an expression for the first-order change in consumption by linearizing the first-order-condition:

$$\hat{c}_t(x) = \frac{1}{u''(c^{SS}(x))} \left\{ \int_t^\infty e^{-\rho(\tau-t)} \varphi^r_{\tau-t}(x) \hat{r}_\tau d\tau + \int_t^\infty e^{-\rho(\tau-t)} \varphi^w_{\tau-t}(x) \hat{w}_\tau d\tau \right\}. \tag{45}$$

Savings rates are $\hat{s}_t(a, y) = \hat{r}_t a + \hat{w}_t y - \hat{c}_t(a, y)$. Using the linearized savings rate, the law of motion of the distribution in sequence space becomes to first order

$$\frac{\partial h_t}{\partial t}(x) = \mathcal{L}^*(x)[h_t] - \frac{\partial}{\partial a}\Big(\hat{s}_t(x) g^{SS}(x)\Big). \tag{46}$$

Combining the linearized decision rules (45) with the law of motion (46), I relate the distribution at a given point in time to the initial distribution as well as expectations of future prices.

Theorem 6. *(Distribution in sequence space)*

$$h_t(x) \;=\; h^{PE}_t(x) + \int_0^t J^{D,r}_{t-\tau}(x) \hat{r}_\tau d\tau + \int_0^t J^{D,w}_{t-\tau}(x) \hat{w}_\tau d\tau + \int_0^\infty J^r_{t,\tau}(x) \hat{r}_\tau d\tau + \int_0^\infty J^w_{t,\tau}(x) \hat{w}_\tau d\tau$$

where all functions $J \in \{h^{PE}, J^{D,r}, J^{D,w}\}$ satisfy

$$\frac{\partial J_t}{\partial t}(x) = \mathcal{L}^*(x)[J_t]$$

and differ in initial conditions $h_0^{PE}(x) = h_0(x)$*,* $J_0^{D,r}(x) = -\partial_a\big(ag^{SS}(x)\big)$*,* $J_0^{D,w}(x) = -\partial_a\big(yg^{SS}(x)\big)$*. Similarly,* $J^i \in \{J^r_{t,\theta}, J^w_{t,\theta}\}$ *satisfy*

$$\frac{\partial J^i_{t,\theta}}{\partial t}(x) = \mathcal{L}^*(x)[J^i_{t,\theta}] + \mathbb{1}_{\{\theta-t>0\}}\frac{\partial}{\partial a}\Big(\frac{e^{-\rho(\theta-t)}\varphi^i_{\theta-t}(x)}{u''(c^{SS}(x))}g^{SS}(x)\Big)$$

with initial conditions $J_{0,\theta}(x) = 0$*.*

Proof. See Appendix B.7. □

Theorem 6 characterizes the path of the distribution as a function of the path of prices and the sequence-space Jacobians of the distribution J. The Jacobians satisfy simple PDEs that are akin to laws of motion. The first component h^{PE} is simply the partial equilibrium evolution of the initial impulse h_0 under the steady-state law of motion $\mathcal{L}^*$. The next two terms summarize the direct impact of price changes on savings rates through disposable income, cumulated over past times from 0 to t.

The last two terms represent the impact of changing consumption along the transition. These two terms summarize the forward-backward nature of the equilibrium. Consumption being a forward-looking variable, anticipated price changes affect consumption and the integral runs into times $\tau > t$. From the perspective of time $\tau < t$, all anticipated price shocks between times τ and t matter. As savings rate deviate from steady-state at time $\tau < t$, the impact on the distribution then propagates over time up to time t. Thus, the integral also runs between time 0 and time t. The structure of the evolution equation for the corresponding Jacobians $J_{t,\theta}$ mirrors the discrete-time version in Auclert et al. (2019). Here, this structure emerges naturally as a consequence of the analytic linearization.

Theorem 6 is useful to determine the equilibrium sequence of prices through market clearing:

$$\hat{r}_t = \int \widehat{\mathcal{R}}(x)h_t(x)dx \quad , \quad \hat{w}_t = \int \widehat{\mathcal{W}}(x)h_t(x)dx. \tag{47}$$

with $\widehat{\mathcal{R}}(x) = \mathcal{R}_0 a + \mathcal{R}_1 y$ and $\widehat{\mathcal{W}}(x)$ defined similarly. Substituting the results from Theorem 6 into market clearing (47), I obtain that the equilibrium sequence of prices satisfies

$$\hat{r}_t \;=\; r_t^{PE} + \int_0^t \widehat{J^{D,r}}_{t-\tau}\hat{r}_\tau d\tau + \int_0^t \widehat{J^{D,w}}_{t-\tau}\hat{w}_\tau d\tau + \int_0^\infty \widehat{J^r_{t,\tau}}\hat{r}_\tau d\tau + \int_0^\infty \widehat{J^w_{t,\tau}}\hat{w}_\tau d\tau \tag{48}$$

where $\hat{J}(x) \equiv \int \hat{\mathcal{R}}(x)J(x)dx$ for all Jacobians, and a similar equation holds for $\hat{w}_t$. The only term where the initial distribution enters is r_t^{PE}. From Theorem 6, it is a linear functional of the initial distribution h_0.

Equation (48) is a linear, integral equation in the sequence of prices. If it has a solution, this observation implies that the sequence of prices writes $\hat{r}_t = \int R_t(x')h_0(x')dx'$ for a function R_t that represents the inverse of the linear system. Similarly, $\hat{w}_t = \int W_t(x')h_0(x')dx'$. Discretized, the integral equation becomes a linear system that can be readily solved numerically with the Jacobians $\hat{J}$ at hand, and R_t becomes a rectangular matrix.

I can now connect the sequence-space approach to the state-space approach. Contrasting the linearization of the value function in FAME (12) with the value function in the sequence-space approach

(44), I obtain the following result.

Corollary 2. *(FAME and the sequence-space)*

$$v(x,x') = \int_0^\infty e^{-\rho t}\Big\{\varphi_t^r(x)R_t(x') + \varphi_t^w(x)W_t(x')\Big\}dt.$$

Corollary 2 reveals that the Impulse Value is a time average of sequence-space Jacobians. It involves the Jacobians of the value function with respect to prices, since the effect of the distribution only affect individuals through prices. The Impulse Value also relates to the general equilibrium structure of the economy through the solution to the forward-backward price fixed point R_t, W_t.

Of course, it is usually simpler to solve directly for the Impulse Value directly using the FAME (13) than construct the sequence-space Jacobians and back out the Impulse Value. In practice however, when the idiosyncratic state-space is large, it can sometimes be preferable to compute the sequence-space Jacobians and recover the Impulse Value from Corollary 2.

Besides providing a conceptual foundation for the first-order numerical approaches discussed in this subsection, the Master Equation is in addition uniquely suited to obtain higher-order perturbations.

4 The SAME

This section develops the Second-order Approximation to the Master Equation (SAME). The combination of the recursive and state-space structures of the Master Equation approach makes it particularly well-suited to studying second-order perturbations.[26]

To obtain the SAME, the strategy is the same as for the FAME. When ε is small enough, the second-order approximation to the value function is

$$\begin{aligned} V(x,\varepsilon,z,g) \approx\ & \overbrace{V^{SS}(x)}^{\text{Steady-state}} + \overbrace{\varepsilon\Big\{\int v(x,x')h(x')d\eta(x') + \omega(x,z)\Big\}}^{\text{First order}} \\ & + \underbrace{\frac{\varepsilon^2}{2}\Big\{\iint \underbrace{\mathcal{V}(x,x',x'')}_{\substack{\text{2nd-order effect}\\ \text{of distribution}}} h(x')h(x'')d\eta(x')d\eta(x'') + 2\int \underbrace{\Gamma(x,x',z)}_{\substack{\text{Cross effect}\\ \text{of ag. shock.}\\ \text{\& distrib.}}} h(x')d\eta(x') + \underbrace{\Omega(x,z)}_{\substack{\text{2nd-order}\\ \text{effect of}\\ \text{ag. shock}}}\Big\}}_{\text{Second order}}. \end{aligned} \tag{49}$$

The structure of the second-order approximation to the value function mirrors that of the first order approximation. The second-order deterministic Impulse Value $\mathcal{V}(x,x'x,'')$ encodes how deviations in the distribution affect values up to second-order. In contrast to the first order, pairwise deviations at x' and x'' now matter. Formally, $\mathcal{V}(x,x',x'') = \frac{\partial^2 V}{\partial^2 g}(x,x',x'',0,0,g^{SS})$ is the directional Hessian of the value function with respect to the distribution, understood as Frechet derivatives.

[26]In contrast, state-space approaches that rely on linear rational expectation techniques such as Reiter (2009) or Ahn et al. (2018) fall short of handling second-order perturbations. Similarly, sequence-space approaches such as Auclert et al. (2019) also cannot deal with second-order perturbations when there are aggregate shocks because they heavily rely on certainty equivalence.

Aggregate shocks matter up to second order as well, as encoded in the second-order stochastic Impulse Value $\Omega(x, z)$. Up to second order, the cross-effect between deviations in the distribution and aggregate shocks also enters in the cross component $\Gamma(x, x', z)$.

I follows the same strategy as in the FAME to characterize the unknown derivatives $\mathcal{V}, \Gamma, \Omega$. I substitute the second-order approximation (49) into the nonlinear Master Equation (57) in Appendix B.4, and identify 'coefficients' that are again functions. To keep the exposition simple, I focus on the case of drift control only ($E_2 = \emptyset$) under Assumption 5 and no state constraint. It is straightforward to incorporate these additional features. It is useful to define $k(x) = \frac{1}{u''(c^{SS}(x))}$ and $k_p(x) = \frac{u'''(c^{SS}(x))}{u''(c^{SS}(x))^2}$.

The strategy described above leads to the SAME.

Theorem 7. *(SAME)*

$$\begin{aligned}\rho\mathcal{V}(x,x',x'') &= \underbrace{T(x,x',x'')}_{\substack{\text{Direct}\\ \text{2}^{\text{nd}}\text{-order impact}}} + \underbrace{\mathcal{L}(x)[\mathcal{V}(\cdot,x',x'')]}_{\substack{\text{Continuation value from}\\ \text{changes to own state } x}} + \underbrace{\mathcal{L}(x')[\mathcal{V}(x,\cdot,x'')] + \mathcal{L}(x'')[\mathcal{V}(x,x',\cdot)]}_{\substack{\text{Continuation value from propagation in}\\ \text{pair of impulses at } x' \text{ and } x''}} \\ &+ \underbrace{\int \Big(\mathcal{V}(x,y,x'')\sigma(y,x') + \mathcal{V}(x,x',y)\sigma(y,x'')\Big)dy}_{\substack{\text{General equilibrium: 2}^{\text{nd}}\text{-order valuation of}\\ \text{1}^{\text{st}}\text{-order changes in other HHs' savings}}} + \underbrace{\int \mathcal{V}(y,x',x'')\tau(x,y)dy}_{\substack{\text{General equilibrium: 1}^{\text{st}}\text{-order}\\ \text{valuation of 2}^{\text{nd}}\text{-order changes}\\ \text{in other HHs' savings}}}\end{aligned}$$

where $T(x, x, x')$ is given in equation (59) in Appendix C.1 and

$$\sigma(x,y) = -\partial_x\left[g^{SS}(x)(b_g(x,y) - \mathcal{M}(x,y,v))\right] \quad , \quad \tau(x,y) = \partial_y\Big(k(y)v_y(x,y)g^{SS}(y)\Big).$$

The Bellman equations for Γ and Ω are given in equations (63) and (65) in Appendix C.1.

Proof. See Appendix C.1. □

The structure of the SAME for the deterministic Impulse Value in Theorem 7 is analogous to the one found in the deterministic FAME of Theorem 1. In the absence of aggregate shocks, the law of motion of distribution is given by the following result.

Theorem 8. *(Law of motion to second order)*

$$\begin{aligned}\frac{\partial h_t}{\partial t}(x) &= \mathcal{L}^*(x)[h_t] + \mathcal{K}(x)[h_t] \\ &+ \varepsilon \times \left\{\partial_x\left(h_t(x)\int K_{21}(x,x')h_t(x')d\eta(x')\right) + \iint K_{22}(x,x',x'')h_t(x')h_t(x'')d\eta(x')d\eta(x'')\right\},\end{aligned}$$

where the kernels K_{21}, K_{22} are given in Appendix C.2. See Appendix C.2 for the law of motion in the presence of aggregate shocks.

Proof. See Appendix C.2. □

The closed-form expressions in Theorem 7 also deliver a straightforward algorithm to compute the solution to the SAME. Denote by $\mathrm{V}_{ijk} \equiv \mathcal{V}(x_i, x_j, x_k)$ the discretization of the second-order deterministic Impulse Value $\mathcal{V}$ on a grid and into a tensor (a three-dimensional array). Inspection of the SAME reveals that V solves a standard linear Sylvester tensor equation:

$$\mathrm{V} \otimes_1 \mathrm{A}_1 + \mathrm{V} \otimes_2 \mathrm{A}_2 + \mathrm{V} \otimes_3 \mathrm{A}_3 = \mathrm{B}. \tag{50}$$

In equation (50), $\mathrm{A}_i, i \in \{1,2,3\}$ denote standard square matrices. B denotes a tensor. The A_i and B can be mapped to steady-state and first-order objects in closed form using Theorem 7. $\otimes_i$ denotes the standard tensor product along dimension i, which is a direct generalization of matrix products. For instance, $\left(\mathrm{V} \otimes_2 \mathrm{A}_2\right)_{ijk} = \sum_\ell \mathrm{V}_{i\ell k} A_{2,\ell j}$.

It turns out that solving a Sylvester tensor equation is no harder than solving a standard Sylvester matrix equation. It suffices to slice the tensor equation along one dimension. This slicing results in a sequence of standard Sylvester matrix equations that are related to one another. They are related in the same way as rows of the standard Sylvester matrix equation are related to one another after slicing it by row. As a result, the same algorithm that is used to solve the standard Sylvester matrix equation can be used as an outer loop to solve the tensor equation (see Chen and Lu, 2012 for more details).

Together, Theorems 7 and 8 as well as equation (50) demonstrate that taking second-order perturbations of the Master Equation is conceptually no more difficult than taking first-order perturbations—up to slightly lenghtier algebra. This property of the Master Equation approach makes it particularly well-suited to studying settings with aggregate shocks in which nonlinearities matter, in particular environments with asset pricing. In the interest of conciseness, this paper only lays out the groundwork with the SAME and leaves for future research a quantitative application to asset pricing.

5 Applications

This section illustrates the FAME with two distinct economies.

5.1 A location choice model

In this section, I illustrate the FAME in dynamic discrete choice settings. For concreteness, I interpret the framework as a dynamic location choice setting with migration, but the framework may be more broadly interpreted as an industry, occupation or product choice problem.

Consider a unit mass of individuals who choose in which location $i \in \{1, ..., I\}$ to live. From the perspective of the notation in Section 2, the base measure is only a collection of Dirac measures at locations i.

Individuals have Cobb-Douglas flow preferences for a freely traded final good, used as the numeraire, and for housing with share γ. They discount the future at rate ρ. Individuals work for a representative firm that produces the final good using labor and subject to decreasing returns to scale. Finally, individuals are allowed to move at rate δ, in which case they draw extreme-value distributed idiosyncratic preference shocks for potential destinations, with dispersion parameter ν. If they move, they pay a bilateral moving cost τ_{ij}. Locations are endowed with a local productivity A_i, and a fixed supply of housing H_i whose rents are paid to absentee landlords.

Maximizing out the housing and optimal location choices, and clearing labor and housing markets, individuals solve

$$\rho V_{it} - \frac{\partial V_{it}}{\partial t} = U_i(N_{it}) + L_i[V], \quad L_i[V] \equiv \delta \left\{ \frac{1}{\nu} \log \left(\sum_j e^{\nu(V_j - \tau_{ij})} \right) - V_i \right\}. \tag{51}$$

The flow payoff is $U_i(N_i) \equiv A_i H_i^\gamma N_i^{-(1-\gamma)\alpha-\gamma}$.[27] The expectation operator $L_i[V]$ is nonlinear because of the presence of idiosyncratic taste shocks.[28] Location decisions are given by the conditional choice probabilities

$$\pi_{ijt}(V) = \frac{e^{\nu V_{it}}}{\sum_j e^{\nu(V_{jt} - \tau_{ij})}}. \tag{52}$$

The population distribution evolves according to the law of motion:

$$\frac{\partial N_{it}}{\partial t} = \delta \left(\sum_k \pi_{kit}(V) N_{kt} - N_{it} \right) \equiv \mathcal{L}_i^*(V)[N_t]. \tag{53}$$

The notation in equations (51)-(53) highlights that the framework may apply to any dynamic discrete choice framework with bilateral costs in which the payoff from choosing option i depends, in equilibrium, negatively on the number of individuals choosing that option. In the discrete choice setting, the operator $\mathcal{L}_i^*(V)$ is simply the matrix $\pi_t(V)^T - I$. The Master Equation writes

$$\rho V_i(N) = U_i(N_i) + L_i[V] + \mathcal{L}_i^*(V)[N_t].$$

The FAME then takes the form of a nonlinear matrix equation in $v = (v_{ij})_{i,j}$:

$$\rho v = \bar{u} - \nu v \pi^T \bar{N}(\pi - I)v + \delta\big((\pi - I)v + v(\pi^T - I)\big) \tag{54}$$

In the FAME (54), $u_i = \frac{\partial U_i}{\partial L_i} = -\frac{U_i^{SS}}{N_i^{SS}}((1-\gamma)\alpha + \gamma)$, and $\bar{u} = \text{diag}(u)$ and $\bar{N} = \text{diag}(N^{SS})$. $\pi \equiv \pi_{ij}^{SS}$ denotes the matrix of steady-state migration shares. Finally, the linearized law of motion of the distribution is then

$$\frac{\partial h_t}{\partial t} = \delta\Big[(\pi^T - I) + \nu \pi^T \bar{N}(I - \pi)v\Big] h_t \tag{55}$$

Having laid out the economy, I leverage the structure of the FAME to connect primitives of the economy to the speed of a transition to steady-state in Proposition 3 below.

Proposition 3. *(Convergence to steady-state)*
Suppose that $\tau_{ij} = \mathbb{1}_{i \neq j} \tau_0$, and that τ_0 is large relative to other parameters. Then, to a first order in

[27] To understand how equation (51) arises, note that the wage w_i in location i satisfies the firm's first-order condition $w_i = A_i^{\frac{1}{1-\gamma}} N_i^{-\alpha}$, where L_i is the population in location i. r_i is the local rental rate of housing, and clears the local land market $\gamma w_i N_i = r_i H_i$. Therefore, the flow payoff of locating in i is equal to $\gamma^\gamma \frac{w_i}{r_i^\gamma} = U_i(N_i)$.

[28] Crucially, its action on a small perturbation of the value dV still coincides with the adjoint (tranpose) of the operator $\mathcal{L}_i^*$: $L_i[V^{SS} + dV] = L_i[V^{SS}] + \mathcal{L}_i(V^{SS})[dV]$.

$1/\tau_0$*, the population distribution converges to steady according to* $||h_t||_2 \leq e^{-\Lambda t}||h_0||_2$*, where*

$$\Lambda = \frac{\delta\nu}{\rho}\inf_i (1-\pi_{ii}^{SS})N_i^{SS}|u_i|$$

Proposition 3 proposes an explicit convergence condition in this discrete choice setting. Convergence back to steady-state is slower when the migration elasticity ν is low. In that case, idiosyncratic preferences are highly dispersed, and individuals tend to move more randomly. As a result, convergence is sluggish. The rate of convergence depends on payoff elasticities u_i and own migration shares. When local payoff are highly elastic to population changes relative to their overall payoff share in the economy, convergence is fast since initial population deviations result in large payoff differentials across locations. When the own migration share is low, individuals are more mobile and convergence is faster.

The results in Proposition 3 provide a sufficient statistic for the speed of adjustment in terms of estimable elasticities and measurable statistics. For instance, the migration elasticity ν may be estimated using a standard migration gravity estimating equation. The own migration share π_{ii}^{SS} and population N_i^{SS} may be directly read off the data. The payoff elasticity u_i may be estimated using standard instrumental variable methods.

5.2 A consumption-savings model with unemployment and a wage ladder

In this section, I show how to leverage the FAME to efficiently compute impulse responses in an economy with incomplete credit markets and a frictional labor market. I then study how countercyclical UI trade off social insurance against crowding out investment.

Relative to the Krusell and Smith (1998) economy of Section 1, I add four key elements. First, I enrich the economy with an occasional binding borrowing constraint $a \geq \underline{a}$. Second, a second set of agents—absentee owners—own a fixed amount B of claims to capital. Together, these two elements let me obtain high MPCs and retain a one-asset structure. Third, I micro-found the earnings process through a frictional labor market similar to Burdett and Mortensen (1998) and Bilal and Lhuillier (2021). Workers search on and off the job for wage offers. Wages are posted by firms. Firms also post vacancies in order to recruit workers. The labor market clears through a matching function and the equilibrium adjustment of labor market tightness that determines job-finding rates. Fourth, a government taxes employed workers to finance unemployment insurance.

I relegate most of the description of the equilibrium to Appendix D.2. To illustrate how the FAME handles the general equilibrium in this environment, I simply report the decision problem of employed workers, setting profit rebates and transfers to 0:

$$\begin{aligned}\rho V_t(a,y) - \partial_t V_t(a,y) &= \max_c u(c) + (r_t a + w - c)\partial_a V_t(a,y) \\ &+ \lambda_t^E \int_w^\infty \Big(V_t(a,x) - V_t(a,y)\Big)\tilde{f}_t(x)dx + \delta\Big(U_t(a,y) - V_t(a,y)\Big). \qquad (56)\end{aligned}$$

$\tilde{f}_t(x)$ is the equilibrium p.d.f. of new wage offers, and λ_t^E is the equilibrium job-finding rate out of employment. Critically, households need to keep track of the infinite-dimensional distribution $\tilde{f}_t(x)$

of wage offers at all times to make decisions. This distribution is determined by the vacancy posting decisions of firms, that in turn depend on the current distribution of workers across wages $g_t(x)$.

The FAME is uniquely equipped to handle such a setting because it linearizes with respect to the distribution g directly. I compute the deterministic steady-state using standard global methods. Using the finite difference scheme from Corollary 1 on a laptop, computing the distributional Impulse Value takes 5 seconds, and the aggregate shock Impulse Value takes 0.3 seconds. Impulse responses and the invariant distribution in the stochastic steady-state are computed subsequently, in less than a second as well. Together, given the deterministic steady-state, implementing the FAME requires less than 200 lines of Matlab code.

I calibrate the model at the monthly frequency taking most parameter values from the literature as detailed in Appendix D.2. Given my focus on the insurance value of state-dependent UI, several key parameters deserve special attention. First, I chose absentee capital B to match a liquid wealth-to-GDP ratio of 0.5 together with a capital-to-GDP ratio of 5. This target then delivers an empirically plausible distribution of MPCs by income for workers.

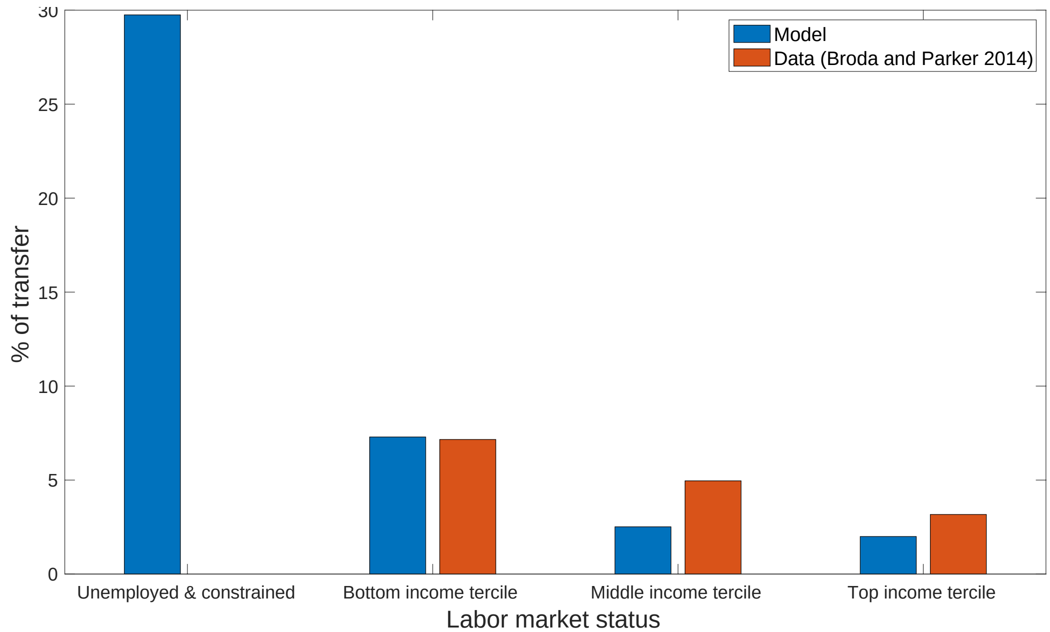
Figure 1(a) displays the monthly MPC of households by income state, in the model and in data as reported by Broda and Parker (2014). Consistently with the data, the model predicts that the MPC is highest for low-income individuals and declines monotonically with income. I also report the MPC of constrained and unemployed households to highlight their particularly large MPC, suggesting potent insurance effects of state-dependent UI.

A second key target to evaluate the insurance effects of state-dependent UI is not only the consumption response conditional on a shock, but also the incidence of shocks. I calibrate the productivity distribution to match the elasticity of individual income to aggregate output by labor market status.

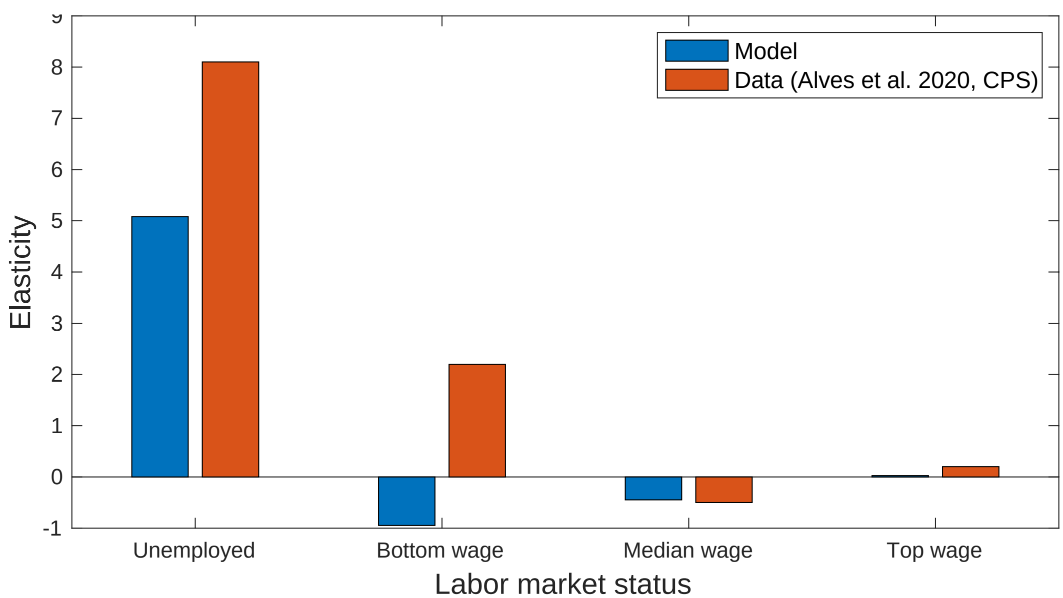
Figure 1(b) displays the correlation between individual income and aggregate output in the model and in the data as reported in Alves et al. (2020). The model broadly replicates the main patterns in the data: unemployed workers are most exposed to business cycle variations, while most employed workers are not. Quantitatively, the model falls short of generating the full elasticity for unemployed workers because the model does not generate exactly enough unemployment volatility compared to the data.[29]

[29] The sign of the elasticity for employed workers follows from composition search externalities that are not crucial for the results of the paper.

(a) Monthly marginal propensity to consume by income, model vs. data.



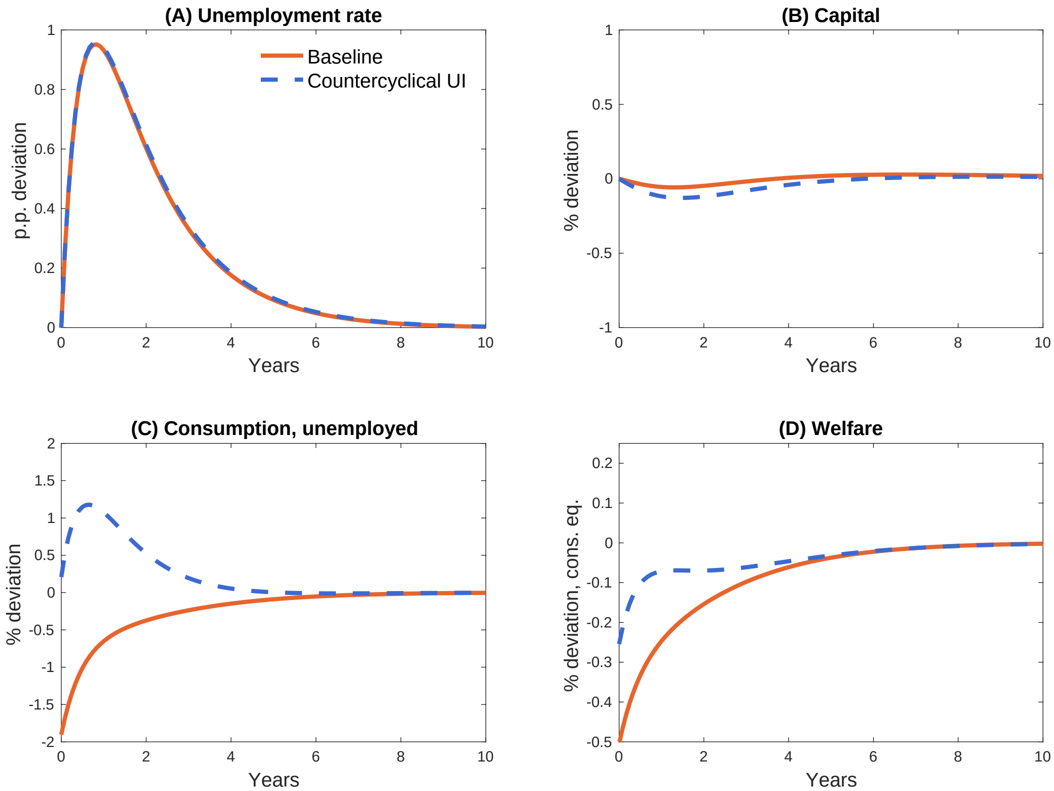
(b) Incidence of business cycles, model vs. data.



Having matched both MPCs and the incidence of business cycles in the cross-section of workers, I ask whether state-dependent UI can help unemployed workers smooth consumption in downturns without crowding out investment so much that it slows down the recovery. I consider two experiments. First, I feed in a mean-reverting sequence of aggregate productivity shocks that generates a one percentage point increase in the unemployment rate, holding UI at its steady-state value. Second, I hit the economy with the same sequence of shocks, but under a rule that increases UI generosity by 5% for every percentage point increase in the unemployment rate. Figure 2 displays the results.

Panel (B) shows that investment is indeed crowded out by the increase in UI, but with a very

Figure 2: Impulse response functions with and without countercyclical UI.



limited impact on the aggregate capital stock. This limited impact is merely explained by the small size of the UI increase: a 5% rise in benefits that only 10% of the population receives (the target for non-employment in the calibration).

Panel (C) shows that with constant UI, unemployed workers have to cut consumption by about 2% on average in the downturn. In welfare terms, it represents a 0.5% decline as shown by panel (D). The increase in UI generosity is effective in reducing the consumption losses of unemployed individuals. In fact, their consumption increases above steady-state on average. However, within unemployed workers, low-liquidity workers still cut consumption despite the transfer, a consequence of both the unequal incidence shown in Figure 1(b) and the high MPCs displayed in Figure 1(a). As a result, welfare does not increase above steady-state on average, but is strongly stabilized as a result of the increase in UI generosity. Together, these results indicate that countercyclical UI can be a powerful tool to help unemployed workers smooth losses in downturns while having only minimal effects on capital accumulation.

Conclusion

This paper proposed a new representation of dynamic general equilibrium economies with cross-sectional heterogeneity. By treating the underlying distribution as an explicit state variable in decision makers' problem, the economy becomes fully recursive and is characterized by the Master Equation. I show that local perturbations in aggregates of the Master Equation deliver interpretable, block-recursive and easily computable representations of equilibrium: the FAME and the SAME.

I highlighted the versality of the FAME in a dynamic discrete choice setting, and an economy with frictional credit and labor markets. The FAME and the SAME apply to many more economic settings. The SAME could be used to investigate asset pricing with cross-sectional heterogeneity. The FAME could be used to introduce aggregate shocks into quantitative spatial and trade models, as highlighted in Bilal and Rossi-Hansberg (2023). The FAME could be used to study the cyclical variability of the unemployment scar, a question that has so far resisted structural analysis.

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Appendix

A Notation

A.1 Additional notation for the motivating example

Define

$$\begin{aligned}
\mathcal{R}_0 &= -\alpha(1-\alpha)\bar{Z}\left(\frac{\iint yg^{SS}(a,y)dady}{\iint ag^{SS}(a,w)}\right)^{1-\alpha}\left(\iint ag^{SS}(a,y)dady\right)^{-1} \\
\mathcal{R}_1 &= \alpha(1-\alpha)\bar{Z}\left(\frac{\iint yg^{SS}(a,y)dady}{\iint ag^{SS}(a,w)}\right)^{1-\alpha}\left(\iint yg^{SS}(a,y)dady\right)^{-1} \\
\mathcal{W}_0 &= \alpha(1-\alpha)\bar{Z}\left(\frac{\iint ag^{SS}(a,y)dady}{\iint yg^{SS}(a,w)}\right)^{\alpha}\left(\iint ag^{SS}(a,y)dady\right)^{-1} \\
\mathcal{W}_1 &= -\alpha(1-\alpha)\bar{Z}\left(\frac{\iint ag^{SS}(a,y)dady}{\iint yg^{SS}(a,w)}\right)^{\alpha}\left(\iint yg^{SS}(a,y)dady\right)^{-1}
\end{aligned}$$

Define also

$$\begin{aligned}
\mathcal{R}_2 &= \alpha\left(\iint ag^{SS}(a,y)dady\right)^{\alpha-1}\left(\iint ag^{SS}(a,y)dady\right)^{1-\alpha} \\
\mathcal{W}_2 &= (1-\alpha)\left(\iint ag^{SS}(a,y)dady\right)^{\alpha}\left(\iint ag^{SS}(a,y)dady\right)^{-\alpha}
\end{aligned}$$

A.2 Additional notation for the FAME

Denote

$$\begin{aligned}
\mathcal{L}(x) &= L(x, c^{SS}(x), V^{SS}, g^{SS}), & \mathcal{C}(x) &= \mathcal{C}\left(x, V^{SS}, \frac{\partial V^{SS}}{\partial x}, g^{SS}\right)) \\
u_g(x,y) &= \frac{\partial u}{\partial g}(x,y,V^{SS},g^{SS}), & u_V(x,y) &= \frac{\partial u}{\partial V}(x,y,V^{SS},g^{SS}) \\
\mathcal{L}_g(x,y) &= \frac{\partial L}{\partial g}(x,y,c^{SS}(x),V^{SS},g^{SS}), & \mathcal{L}_V(x,y) &= \frac{\partial L}{\partial V}(x,y,c^{SS}(x),V^{SS},g^{SS}) \\
\mathcal{C}_g(x,y) &= \frac{\partial C}{\partial g}(x,y,V^{SS},\frac{\partial V^{SS}}{\partial x},g^{SS}), & \mathcal{C}_V(x,y) &= \frac{\partial C}{\partial V}(x,y,V^{SS},\frac{\partial V^{SS}}{\partial x},g^{SS}) \\
\mathcal{C}_p(x,y) &= \frac{\partial C}{\partial p}(x,y,V^{SS},\frac{\partial V^{SS}}{\partial x},g^{SS})
\end{aligned}$$

the Frechet differentials of the flow payoff u, the generator L and the state constraint $\mathcal{C}$ with respect to either the distribution or the value. $\partial/\partial p$ denotes the derivative of $\mathcal{C}$ with respect to $\partial V/\partial x$.

B Proofs for Section 3

B.1 Proof of Proposition 2

The linearized FOCs are

$$\begin{aligned} \forall i \in E_1, \quad dc_i(x) &= \frac{1}{u''_{1i}(c_i^{SS}(x))}\partial_{x_i} dv(x) \\ \forall i \in E_2, \quad dc_i(x) &= \frac{1}{u''_{2i}(c_i^{SS}(x))}\left(\int h(x')d\eta(x')\partial_g \mathcal{L}_i^{int}(x,x') + \int dv(y)d\eta(y)\partial_V \mathcal{L}_i^{int}(x,y)\right) \end{aligned}$$

where now c_i denote steady-state controls. Substituting out dv yields the result.

B.2 Proof of Theorem 1

Since none of the forcing terms in u, L depend directly on the derivatives of g, it is enough to look for a Frechet derivative of V that only loads on h. To ease notation, denote partial derivatives by $\partial_g, \partial_c, \partial_V$, or directly with the corresponding subscript. Also denote $X \cdot Y \equiv \langle X, Y\rangle$. Since c is always interior, taking the FOC and substituting back into the Canonical Master Equation (32), I obtain

$$\begin{aligned} (ME0) \quad \rho V(x,\nu) &= u(\hat{c}(x,V,g),x,g,V) && (ME1) \\ &+ L(x,\hat{c}(x,V,g),g,V)[V] && (ME2) \\ &+ \int \partial_g V(x,x'',g)L^*(x'',\hat{c}(x'',V,g),g,V)[g]d\eta(x'') && (ME3) \end{aligned}$$

I will sometimes omit arguments of function to ease notation. In that case, they are evaluated at $x, g, V, \hat{c}(x,V,g)$. I sometimes denote by $dv(x) = \int v(x,x')h(x')d\eta(x')$ and $dc(x)$ the first-order change in controls. I now expand each component of the Master Equation up to first order.

Left-hand-side (ME0). The first-order contribution of an impulse in h to the left-hand-side is

$$ME0 = \rho \int v(x,x')h(x')d\eta(x')$$

Flow payoff (ME1). The flow gain is, up to first order,

$$\begin{aligned} ME1 &= \int u_g(x,x')h(x')d\eta(x') + u_c(x)\cdot dc(x) + \int u_V(x,y)dv(y)d\eta(y) \\ &= \int u_g(x,x')h(x')d\eta(x') + u_c(x)\cdot dc(x) + \int \left(\int u_V(x,y)v(y,x')d\eta(y)\right)h(x')d\eta(x') \end{aligned}$$

Now note that u depends on V partly through c. So the FOC implies, for all g, V,

$$0 = u_c(x) + L_c(x, dc(x))[V]$$

Generator (ME2). The contribution of the generator term is

$$ME2 = L_c(x,dc(x))[V] + \int L_g(x,x')[V]h(x')d\eta(x') + \int L_V(x,y)[V]dv(y)d\eta(y) + L(x)[dv]$$

Now, passing the operator $\mathcal{L}(x)$ inside the integral for the last term,

$$L(x)[dv] = \int L(x)[v(\cdot, x')]h(x')d\eta(x')$$

Similarly,

$$\int L_V(x,y)[V]dv(y)d\eta(y) \quad = \quad \int \left(\int L_V(x,y)[V]v(y,x')d\eta(y) \right) h(x')d\eta(x')$$

Flow payoff plus generator term (ME1+ME2). An envelope argument obtains where the contributions of ∂_c cancel out:

$$\begin{aligned} ME1 + ME2 \quad &= \quad \int \Big\{ u_g(x,x')h(x') + \int u_V(x,y)v(y,x')d\eta(y) + L_g(x,x')[V] \\ &\quad + \int L_V(x,y)[V]v(y,x')d\eta(y) + L(x)[v(\cdot,x')] \Big\} h(x')d\eta(x') \end{aligned}$$

Optimal control. To expand (ME3), we need to derive the first-order response of the optimal control. The FOC writes

$$0 \quad = \quad u_c(x, \hat{c}(x,g,V), g, V) + L_c(x, \hat{c}(x,g,V), g, V)[V]$$

Denote by by $\widetilde{G}(x,y) \equiv -\Big(u_{cc}(x) + L_{cc}(x)[V^{SS}]\Big)^{-1} G(x,y)$ for any function G. Totally differentiate the FOC to obtain

$$\begin{aligned} dc(x) \quad &= \quad \int \Big\{ m_0(x,x')h(x')d\eta(x') + \int m_1(x,y)v(y,x')d\eta(y) + \widetilde{L_c}(x) \left[v(\cdot,x') \right] \Big\} h(x')d\eta(x') \\ &\equiv \quad \int \mathcal{M}(x,x',v)h(x')d\eta(x') \end{aligned}$$

where

$$m_0(x,x') = \widetilde{u_{cg}}(x,x') + \widetilde{L_{cg}}(x,x')[V] \quad , \quad m_1(x,x') = \widetilde{u_{cV}}(x,y) + \int \widetilde{L_{cV}}(x,y)[V].$$

General equilibrium term (ME3). I obtain, to first order

$$\begin{aligned} ME3 \quad = \quad \int \Big(v(x,x'') + \mathcal{O}_x(||h||) \Big) \Bigg(& L^*(x'')[g] + \int L_g^*(x'',x')[g]h(x')d\eta(x') \\ & + \iint L_V^*(x'',y)[g]v(y,x')h(x')d\eta(y)d\eta(x') \\ & + L_c^* \left(x'', \int \mathcal{M}(x,x',v)h(x')d\eta(x') \right) [g] + L^*(x'')[h] \Bigg) d\eta(x'') \end{aligned}$$

So far, I have considered a local perturbation around any point. To make progress, I now make use of the steady-state property. In steady-state, $L^*(x'')[g] = 0$ for all x'' by definition. Thus, I can negelect the $\mathcal{O}_x(||h||)$ term to first order.

I also make use of the adjoint property between L and L^*: $\int L(x)[\phi]\psi(x)d\eta(x) = \int \phi(x)L^*(x)[\psi]d\eta(x)$. Thus, I can express the last term as an integral over h. Together, these observations imply that to first order around a steady-state:

$$\begin{aligned}ME3 \quad = \quad & \int \Bigg\{ \mathcal{L}(x')[v(x,\cdot)] + \int v(x,x'') \Bigg(\mathcal{L}_g^*(x'',x')[g^{SS}] + \int \mathcal{L}_V^*(x'',y)[g^{SS}] v(y,x') d\eta(y) \\ & + \mathcal{L}_c^*(x'', \mathcal{M}(x'',x',v))[g^{SS}] \Bigg) d\eta(x'') \Bigg\} h(x') d\eta(x')\end{aligned}$$

Equate ME0 = ME1 + ME2 + ME3 around steady-state, for all functions h, conditional on x. The associated vector that integrates against $h(x')$ must be zero in L^2 ("identifying coefficients"). Thus, I obtain the FAME in Theorem 1.

State constraint Expanding the state constraint for $x \in B$, I obtain for all h:

$$\mathcal{C}(x) + \int \Gamma(x,x')h(x')d\eta(x') \geq 0 \quad , \quad \Gamma(x,x') = \mathcal{C}_g(x,x') + \mathcal{C}_V(x)v(x,x') + \mathcal{C}_p(x,x')\frac{\partial v}{\partial x}(y,x').$$

If $\mathcal{C}(x) = 0$, then $\Gamma(x,x') = 0$ for the state constraint inequality to hold for all h. When $\mathcal{C}(x) > 0$, the state constraint inequality holds as long as $||h||$ is small enough. Thus, the set of x where the state constraint holds with equality may in principle depend on h.

Consider a point x a point on the boundary of $\{x \in B : \mathcal{C}(x) > 0\}$ when seen as a manifold in $\mathbb{R}^{D_X - 1}$. Under Assumption 3, I can parametrize the boundary as the set of points $X(h) = x + \int \psi(x,x')h(x')\eta(x')$ for a function $\psi(x,x') \in \mathbb{R}^{D_X}$. Recall that x is on the boundary of the domain. Thus,$\psi_j(x,x') = 0$ for some direction j that depends on which boundary x is located on. Without loss of generality, assume that this coordinate is $j = 1$. Evaluate the state constraint at x and expand in h to a first order:

$$\int \left(\sum_{i=2}^{D_X} \mathcal{C}_{x_i}(x)\psi_i(x,x') + \Gamma(x,x') \right) h(x')d\eta(x') \geq 0.$$

Therefore, the boundary of the constrained set changes as per

$$\Gamma(x,x') + \partial_x \mathcal{C}(x) \cdot \psi(x,x') = 0$$

where $\cdot$ denotes here the inner product in $\mathbb{R}^{D_X}$. Because x is on the border of the constrained set $\{y : \mathcal{C}(y) = 0\}$, by continuity $\mathcal{C}_x(x) = 0$ and $\Gamma(x,x') = 0$. Thus, the first-order approximation does not place additional restrictions on ψ nor Γ, and thus v.

B.3 Proof of Theorem 2

Linearized law of motion. The linearized law of motion follows directly from the first-order expansion in the proof of Theorem 1, component ME3.

Convergence properties. Associating a sesquilinear form with $\mathcal{L}^*$, standard arguments ensure that the associated semigroup on H^2 is positive irreducible, and has compact resolvent. In addition, it is a stochastic semigroup. I assumed that it admits a steady-state distribution. Therefore, by the Krein-Rutman theorem, its spectral bound is 0, and it has a strictly positive spectral gap. Its growth bound coincides with its spectral bound, and so $\forall h, \langle h, \mathcal{L}^* h\rangle_{L^2} \leq 0$. Hence, the growth bound of $\mathcal{L}^* + \mathcal{K}$ is bounded above by the growth bound of $\mathcal{K}$.

When $\mathcal{K}$ is bounded, it defines a compact operator since $\mathcal{K}$ is a Hilbert-Schmidt operator. So is $\bar{\mathcal{K}} = (\mathcal{K} + \mathcal{K}^*)/2$. Thus, the spectrum of $\bar{\mathcal{K}}$ is a sequence of real numbers, and the operator is diagonalizable on a countable orthonormal basis. It is straightforward to see that $\langle h, \mathcal{K}h\rangle = \langle h, \bar{\mathcal{K}}h\rangle$. Thus, the evolution $||h_t||$ is controlled by the supremum of the eigenvalues $\bar{\mathcal{K}}$ (see Kowalski 2009, Chapter 2).

B.4 FAME with aggregate shocks

Definition 5. *(Master Equation with Aggregate Shocks)*
The Master Equation is now defined by

$$\begin{aligned} \rho V^\varepsilon(x,z,g) &= \max_{c \in \bar{C}} u(x, \varepsilon z, c, V, g) + L(x, \varepsilon z, c, g)[V^\varepsilon] + \int \frac{\partial V^\varepsilon}{\partial g}(x,z,y,g,\varepsilon) L^*(y, \varepsilon z, \hat{c}(y), g)\,[g]\, d\eta(y) \\ &+ A(\varepsilon z, \varepsilon)[V^\varepsilon] \\ s.t. \; & C\left(x, \varepsilon z, V^\varepsilon(x,z,g), \frac{\partial V^\varepsilon}{\partial x}(x,z,g), g\right) \geq 0 \end{aligned} \tag{57}$$

for functions $\widehat{X} \times \widehat{z} \times H^2 \times \mathbb{R}_+ \ni (x,z,g,\varepsilon) \mapsto V(x,z,g,\varepsilon)$ that are Frechet-differentiable in g η-a.e. in x.

I assume that u, L, C evaluated at $Z = 0$ coincide with u, L, C in Section 2, and that u, L, C are continuously differentiable in Z. To generalize the results from Section 3, I need a slight variant of the regularity Assumption 3, that also encompasses regularity with respect to the scale of aggregate shocks ε.

Assumption 7. *(Existence, local uniqueness and regularity)*
There exists one isolated steady-state equilibrium V^{SS}, g^{SS}. There exists a solution $V(x,z,g,\varepsilon)$ to the Master Equation in a neighborhood of $(g^{SS}, 0)$ that is continuously Frechet-differentiable in $g \in H^2$ around $g = g^{SS}$, and continuously differentiable in ε around $\varepsilon = 0$, η-almost everywhere in x. V is continuous in x. The $D_X - 1$-dimensional boundary of the set of points such that the state constraint holds with equality is continuously Frechet-differentiable in g around $g = g^{SS}$, and differentiable in ε around $\varepsilon = 0$, η-almost everywhere in x.

Proof of Theorem 3. The proof follows closely that of Theorem 1 and is omitted for brevity.

B.5 Proof of Theorem 4

Kolmogorov forward equation. The derivation mirrors the one in Theorem 2 and is omitted for brevity.

Stochastic stability. Let P_t denote the semigroup associated with $\mathcal{L}^* + \mathcal{K}$. Then

$$h_t = P_t h_0 + \int_0^t P_{t-s} S(\cdot, z_s) ds$$

and therefore

$$||h_t|| \leq |||P_t||| \; ||h_0|| + \int_0^t |||P_{t-s}||| \; ||S(\cdot, z_s)||ds \leq e^{\lambda^{sup}(K)t} \; ||h_0|| + \int_0^t e^{\lambda^{sup}(K)(t-s)} \; ||S(\cdot, z_s)||ds$$

which delivers the stochastic stability result.

Convergence back to steady-state. When the aggregate shock process is symmetric, it is immediate to guess and verify that $\tilde{\omega}(x, z) = \omega(x, z) + \omega(x, -z)$ satisfies the Bellman equation $\rho\tilde{\omega} = 0 + \mathcal{L}(x)[\tilde{\omega}(\cdot, z)] + \mathcal{A}(z)[\tilde{\omega}(x, \cdot)$. Therefore, $\tilde{\omega}(x, z) = 0$, and so $\omega(x, -z) = -\omega(x, z)$ and $\omega(x, 0) = 0$. Then, $S(x, 0) = 0$, and the result follows from Theorem 2.

B.6 Proof of Theorem 5

In general, the perturbation of the value function writes:

$$V_t(x) \approx V^{SS}(x) + \int_t^\infty e^{-\rho(\tau-t)} \varphi^r_{t,\tau}(a, y)\hat{r}_\tau d\tau + \int_t^\infty e^{-\rho(\tau-t)} \varphi^w_{t,\tau}(a, y)\hat{w}_\tau d\tau,$$

where *a priori* the functions φ depend on both calendar time t and the time of the shock τ. Then,

$$\begin{aligned}
\frac{\partial V_t}{\partial t}(x) &= -\varphi^r_{t,t}(x)\hat{r}_t - \varphi^w_{t,t}(x)\hat{w}_t + \rho \int_t^\infty e^{-\rho(\tau-t)} \varphi^r_{t,\tau}(x)\hat{r}_\tau d\tau + \rho \int_t^\infty e^{-\rho(\tau-t)} \varphi^w_{t,\tau}(x)\hat{w}_\tau d\tau \\
&+ \int_t^\infty e^{-\rho(\tau-t)} \frac{\partial \varphi^r_{t,\tau}}{\partial t}(x)\hat{r}_\tau d\tau + \int_t^\infty e^{-\rho(\tau-t)} \frac{\partial \varphi^w_{t,\tau}}{\partial t}(x)\hat{w}_\tau d\tau
\end{aligned}$$

Substituting into the Bellman equation and using the envelope theorem, I obtain, to a first order,

$$\begin{aligned}
&\varphi^r_{t,t}(x)\hat{r}_t + \varphi^w_{t,t}(x)\hat{w}_t - \int_t^\infty e^{-\rho(\tau-t)} \frac{\partial \varphi^r_{t,\tau}}{\partial t}(x)\hat{r}_\tau d\tau - \int_t^\infty e^{-\rho(\tau-t)} \frac{\partial \varphi^w_{t,\tau}}{\partial t}(x)\hat{w}_\tau d\tau \\
= \; &a\hat{r}_t u'(c^{SS}(x)) + y\hat{w}_t u'(c^{SS}(x)) + \int_t^\infty e^{-\rho(\tau-t)} \mathcal{L}(x)[\varphi^r_{t,\tau}]\hat{r}_\tau d\tau + \int_t^\infty e^{-\rho(\tau-t)} \mathcal{L}(x)[\varphi^w_{t,\tau}]\hat{w}_\tau d\tau
\end{aligned}$$

This equation holds for all sequences $\hat{r}_\tau, \hat{w}_\tau$. Thus, identifying coefficients,

$$\varphi^r_{t,t}(a, y) = au'(c^{SS}(x)) \quad , \quad -\frac{\partial \varphi^r_{t,\tau}}{\partial t}(a, y) = \mathcal{L}(x)[\varphi^r_{t,\tau}], \quad \tau \geq t$$

and a similar expression holds for φ^w. I now change variables $\varphi^r_{t,\tau} \equiv \varphi^r_{\tau-t}$, and similarly for φ^w. This change of variables then implies

$$\varphi^r_0(x) = au'(c^{SS}(x)) \quad , \quad \frac{\partial \varphi^r_s}{\partial s}(a, y) = \mathcal{L}(x)[\omega^r_s], \quad s \geq 0.$$

B.7 Proof of Theorem 6

Let $P_t(x, x_0)$ be the semigroup associated with $\mathcal{L}^*$, i.e. the solution to

$$\frac{\partial P_t}{\partial t}(x, x_0) = \mathcal{L}^*(x)[P_t(\cdot, x_0)] \quad , \quad P_0(x) = \boldsymbol{\delta}(x - x_0)$$

where $\boldsymbol{\delta}$ denotes the Dirac measure. Since the PDE $\partial_t \phi = \mathcal{L}^* \phi$ is linear, I may recover the solution to that PDE given any initial condition ϕ_0 as $\phi_t(x) = \int P_t(x, x_0)\phi_0(x_0)dx_0$. Integrating (46), I obtain

$$\begin{aligned}
h_t(x) &= \int P_t(x,x_0)h_0(x_0)dx_0 - \int_0^t \int P_{t-\tau}(x,x_0)\frac{\partial}{\partial a_0}\Big(\hat{s}_\tau(x_0)g^{SS}(x_0)\Big)dx_0 d\tau \\
&= h_t^{PE}(x) - \int_0^t \int P_{t-\tau}(x,x_0)\frac{\partial}{\partial a_0}\left(\Big[\hat{r}_\tau a_0 + \hat{w}_\tau y_0 - \hat{c}_\tau(x_0)\Big]g^{SS}(x_0)\right)dx_0 d\tau \\
&= h_t^{PE}(x) + \int_0^t h_{t-\tau}^{D,r}(x)\hat{r}_\tau d\tau + \int_0^t h_{t-\tau}^{D,w}(x)\hat{w}_\tau d\tau + \underbrace{\int_0^t \int P_{t-\tau}(x,x_0)\frac{\partial}{\partial a_0}\left(\hat{c}_\tau(x_0)g^{SS}(x_0)\right)dx_0 d\tau}_{\equiv C}
\end{aligned}$$

Substituting in the linearized consumption decision (45), the last term C becomes

$$C = \iint d\tau d\theta \mathbb{1}\{\tau \le t\}\mathbb{1}\{\theta \ge \tau\}\int P_{t-\tau}(x,x_0)\left(\gamma_{\theta-\tau}^r(x_0)\hat{r}_\theta + \gamma_{\theta-\tau}^w(x_0)\hat{w}_\theta\right)dx_0$$

where I denoted $\gamma_{\theta-\tau}^r(x_0) = e^{-\rho(\theta-\tau)}\frac{\partial}{\partial a_0}\left(\frac{\varphi_{\theta-\tau}^r(x_0)}{u''(c^{SS}(x_0))}g^{SS}(x_0)\right)$ and $\gamma_{\theta-\tau}^w(x_0) = e^{-\rho(\theta-\tau)}\frac{\partial}{\partial a_0}\left(\frac{\varphi_{\theta-\tau}^w(x_0)}{u''(c^{SS}(x_0))}g^{SS}(x_0)\right)$. Focusing on the contribution from interest changes only C^r, I obtain

$$C^r = \int_0^\infty d\theta \hat{r}_\theta \underbrace{\int_0^{\min\{t,\theta\}} d\tau \int P_{t-\tau}(x,x_0)\gamma_{\theta-\tau}^r(x_0)dx_0}_{\equiv J_{t,\theta}^r(x)}$$

Using the definition of P, I then obtain

$$\frac{\partial J_{t,\theta}^r}{\partial t}(x) = \mathcal{L}^*(x)[J_{t,\theta}^r] + \begin{cases} 0 & \text{if } t > \theta \\ \gamma_{\theta-t}^r(x) & \text{if } t < \theta \end{cases}, \quad J_{0,\theta}^r(x) = 0.$$

C Proofs for Section 4

C.1 Proof of Theorem 7

To shorten notation, I denote $h(dx) \equiv h(x)d\eta(x)$ in the sequel. Recall that to first order,

$$ME1 + ME2 = \left(\int b_g(x,x',g,z)h(dx') + b_z(x,g,z)z\right)V_x(x) + L(x,\hat{c},g,z)[dV]$$

So, to second order and evaluating at steady-state,

$$\begin{aligned}
d[ME1 + ME2] &= V_x^{SS}(x)\left(\iint b_{gg}(x,x',x'')\ h(dx')h(dx'') + 2\int b_{gz}(x,x')zh(dx') + b_{zz}(x)z^2\right) \\
&+ \ 2\left(\int b_g(x,x',g,z)h(dx') + b_z(x,g,z)z\right)dV_x(x) - dc(x)dV_x(x) + \mathcal{L}(x)[d^2V]
\end{aligned}$$

To leading order,

$$\begin{aligned}
dV(x) &= \int v(x,x'')h(dx'')' + \omega(x,z) \\
d^2V(x) &= \iint \mathcal{V}(x,x',x'')h(dx')h(dx'') + 2\int \Gamma(x,x',z)h(dx') + \Omega(x,z)
\end{aligned}$$

Therefore, to second order,

$$\begin{aligned}
d[ME1+ME2] &= V_x^{SS}(x)\left(\iint b_{gg}(x,x',x'')\ h(dx')h(dx'') + 2\int b_{gz}(x,x')zh(dx') + b_{zz}(x)z^2\right) \\
&+ 2\left(\int b_g(x,x')h(dx') + b_z(x)z\right)\left(\int v_x(x,x'')h(dx'') + \omega_x(x,z)\right) \\
&- \left(\int \mathcal{M}(x,x',v)h(dx') + \overline{\mathcal{M}}(x,z,\omega)\right)\left(\int v_x(x,x'')h(dx'') + \omega_x(x,z)\right) \\
&+ \iint \mathcal{L}(x)[\mathcal{V}(\cdot,x',x'')]h(dx')h(dx'') + 2\int \mathcal{L}(x)[\Gamma(\cdot,x',z)]h(dx') + \mathcal{L}(x)[\Omega(\cdot,z)]
\end{aligned}$$

Re-arranging and changing integration indices to symmetrize the second-order terms in h only, I obtain

$$\begin{aligned}
d[ME1+ME2] &= V_x^{SS}(x)\left(\iint b_{gg}(x,x',x'')\ h(dx')h(dx'') + 2\int b_{gz}(x,x')z\ h(dx') + b_{zz}(x)z^2\right) \\
&+ \int \Big(b_g(x,x')v_x(x,x'') + b_g(x,x'')v_x(x,x')\Big)\ h(dx')h(dx'') + 2b_z(x)z\omega_x(x,z) \\
&+ 2\int \Big(b_g(x,x')\omega_x(x,z) + b_z(x)zv_x(x,x')\Big)\ h(dx') \\
&- u''(c^{SS}(x))\Bigg\{\iint \mathcal{M}(x,x',v)\mathcal{M}(x,x'',v)\ h(dx')h(dx'') \\
&\qquad + \int \overline{\mathcal{M}}(x,z,\omega)\mathcal{M}(x,x',v)\ h(dx'') + \overline{\mathcal{M}}(x,z,\omega)^2\Bigg\} \\
&+ \iint \mathcal{L}(x)[\mathcal{V}(\cdot,x',x'')]\ h(dx')h(dx'') + 2\int \mathcal{L}(x)[\Gamma(\cdot,x',z)]\ h(dx') + \mathcal{L}(x)[\Omega(\cdot,z)]
\end{aligned}$$

Now turn to $ME3$. Recall that to first order,

$$ME3 = \int \Big(v(x,x'') + dV_g(x,x'')\Big)\Bigg(L^*(x'')[g] - \underbrace{\partial_{x''}\left(\left(\left(\int b_g(x'',x')h(dx') + zb_z(x'') - dc(x'')\right)g(x'')\right) + L^*(x'')[h]\right)}_{\equiv d\left(L^*(x'')[g]\right)}\Bigg)d\eta(x''$$

To second order and evaluating at steady-state,

$$\begin{aligned}
d^2\big(L^*(x'')[g]\big) &= -2\partial_{x''}\left(\left(\int b_g(x'',x')h(dx') + zb_z(x'') - dc(x)\right)h(x'')\right) \\
&- \partial_{x''}\left(\left(\iint b_{gg}(x'',x',y)h(dx')h(dy) + 2\int zb_{gz}(x'',x')h(dx') + z^2b_{zz}(x'') - d^2c(x)\right)g^{SS}(x'')\right)
\end{aligned}$$

Using that $u'''(c^{SS}(x))(dc(x))^2 + u''(c^{SS}(x))d^2c(x) = d^2V_x(x)$, I obtain

$$
\begin{aligned}
d^2\big(L^*(x'')[g]\big) &= -2\partial_{x''}\left(\left(\int \big(b_g(x'',x') - \mathcal{M}(x'',x',v)\big)h(dx') + zb_z(x'') - \overline{\mathcal{M}}(x'',z,\omega)\right) h(x'')\right) \\
&- \partial_{x''}\left(\left(\iint b_{gg}(x'',x',y)h(dx')h(dy) + 2\int zb_{gz}(x'',x')h(dx') + z^2 b_{zz}(x'')\right) g^{SS}(x'')\right) \\
&+ \partial_{x''}\left(\frac{g^{SS}(x'')}{u''(c^{SS}(x''))}\left(\iint \mathcal{V}_{x''}(x'',x',y)\ h(dx')h(dy) + 2\int \Gamma_{x''}(x'',x',z)\ h(dx') + \Omega_{x''}(x'',z)\right.\right. \\
&\qquad -\frac{u'''(c^{SS}(x''))}{\big(u''(c^{SS}(x''))\big)^2}\left[\iint v_{x''}(x'',x')v_{x''}(x'',y)h(dx')h(dy)\right. \\
&\qquad\qquad \left.\left.\left. +2\int v_{x''}(x'',x')\omega_x(x'',z)h(dx') + \omega_x(x'',z)^2\right]\right)\right)
\end{aligned}
\tag{58}
$$

Finally, to leading order,

$$
dV_g(x,x'') = \int \Big(\mathcal{V}(x,x',x'') + \mathcal{V}(x,x'',x')\Big)h(dx') + 2\Gamma(x,x'',z)
$$

Now, to a leading order, $d[ME3] = \int v(x,x'')\times d^2\big(L^*(x'')[g]\big)\ d\eta(x'') + \int dV_g(x,x'')\times d\big(L^*(x'')[g]\big)\ d\eta(x'')$. Starting with the first component,

$$
\begin{aligned}
& \int v(x,x'') \times d^2\big(L^*(x'')[g]\big)\ d\eta(x'') \\
= \quad & 2\iint v_{x''}(x,x'')\big(b_g(x'',x') - \mathcal{M}(x'',x',v)\big)\ h(dx')h(dx'') + 2\int v_{x'}(x,x')\big(zb_z(x'') - \overline{\mathcal{M}}(x',z,\omega)\big)\ h(dx') \\
+ \quad & \iint \left(\int v_y(x,y)b_{gg}(y,x',x'')g^{SS}(dy)\right) h(dx')h(dx'') + \int \left(2\int v_y(x,y)zb_{gz}(y,x')g^{SS}(dy)\right) h(dx') + \int v_y(x,y)z^2 b_{gzz}(y,x')g^{SS}(dy \\
- \quad & \iint \left(\int v_y(x,y)g^{SS}(dy)k(y)\Big[\mathcal{V}_y(y,x',x'') - k_p(y)v_y(y,x')v_y(y,x'')\Big]\right)\ h(dx')h(dx'') \\
- \quad & 2\int \left(\int v_y(x,y)g^{SS}(dy)k(y)\Big[\Gamma_y(y,x',z) - k_p(y)v_y(y,x')\omega_y(y,z)\Big]\right)\ h(dx') \\
- \quad & \int v_y(x,y)g^{SS}(dy)k(y)\Big[\Omega_y(y,z) - k_p(y)\omega_y(y,z)^2\Big]
\end{aligned}
$$

where I denoted

$$
k(y) = \frac{1}{u''(c^{SS}(y))} \quad , \quad k_p(y) = \frac{u'''(c^{SS}(y))}{\big(u''(c^{SS}(y))\big)^2}.
$$

The second component is, to leading order,

$$
\begin{aligned}
& \int dV_g(x,x'') \times d\big(L^*(x'')[g]\big)\ d\eta(x'') \\
= \ & \int \Big(\int \mathcal{V}(x,x',x'') h(dx') + 2\Gamma(x,x'',z) \Big) \\
& \qquad \times \Big\{ -\partial_{x''} \Big(\Big(\Big(\int \big(b_g(x'',x') - \mathcal{M}(x'',x',v)\big) h(dx') + z b_z(x'') - \overline{\mathcal{M}}(x'',z,\omega) \Big) g^{SS}(x'') \Big) + \mathcal{L}^*(x'')[h] \Big\}\ d\eta(x'') \\
= \ & \iint \Big(\mathcal{L}(x'')[\mathcal{V}(x,x',\cdot)] + \mathcal{L}(x')[\mathcal{V}(x,\cdot,x'')] + \int \mathcal{V}_y(x,y,x'') \big(b_g(y,x') - \mathcal{M}(y,x',v)\big) g^{SS}(dy) \Big) h(dx')h(dx'') \\
+ \ & \int \Big\{ 2\mathcal{L}(x')[\Gamma(x,\cdot,z)] + 2 \int \Big[\Gamma_y(x,y,z)\big(b_g(y,x') - \mathcal{M}(y,x',v)\big) + \big(z b_z(y) - \overline{\mathcal{M}}(y,z,\omega)\big) \Big(\mathcal{V}_y(x,x',y) + \mathcal{V}_y(x,y,x') \Big) \Big] g^{SS}(dy) \Big\} h(d \\
+ \ & 2 \int \Gamma_y(x,y,z) \big(z b_z(y) - \overline{\mathcal{M}}(y,z,\omega)\big) g^{SS}(dy)
\end{aligned}
$$

Putting these equations together and identifying coefficients,[30] I obtain the SAMEs.

Deterministic SAME.

$$
\begin{aligned}
\rho \mathcal{V}(x,x',x'') \ = \ & u'(c^{SS}(x) b_{gg}(x,x',x'') + b_g(x,x') v_x(x,x'') + b_g(x,x'') v_x(x,x') \\
- \ & u''(c^{SS}(x)) \mathcal{M}(x,x',v) \mathcal{M}(x,x'',v) + \mathcal{L}(x)[\mathcal{V}(\cdot,x',x'')] \\
+ \ & v_{x''}(x,x'') \big(b_g(x'',x') - \mathcal{M}(x'',x',v)\big) + v_{x'}(x,x') \big(b_g(x',x'') - \mathcal{M}(x',x'',v)\big) \\
+ \ & \int v_y(x,y) \Big\{ b_{gg}(y,x',x'') - k(y) \Big[\mathcal{V}_y(y,x',x'') - k_p(y) v_y(y,x') v_y(y,x'') \Big] \Big\} g^{SS}(dy) \\
+ \ & \mathcal{L}(x'')[\mathcal{V}(x,x',\cdot)] + \mathcal{L}(x')[\mathcal{V}(x,\cdot,x'')] \\
+ \ & \int \Big[\mathcal{V}_y(x,y,x'') \big(b_g(y,x') - \mathcal{M}(y,x',v)\big) + \mathcal{V}_y(x,x',y) \big(b_g(y,x'') - \mathcal{M}(y,x'',v)\big) \Big] g^{SS}(dy)
\end{aligned}
$$

Define

$$
\begin{aligned}
T(x,x',x'') \ = \ & \underbrace{b_{gg}(x,x',x'') u'(c^{SS}(x))}_{\text{Direct price}} + \underbrace{b_g(x,x') v_x(x,x'') + b_g(x,x'') v_x(x,x')}_{\text{Cross price-continuation value}} + \underbrace{u''(c^{SS}(x)) \mathcal{M}(x,x',v) \mathcal{M}(x,x'',v)}_{\text{Cross consumption-continuation value}} \\
+ \ & \underbrace{\Big[v_{x'}(x,x') \big(b_g(x',x'') - \mathcal{M}(x',x'',v)\big) + v_{x''}(x,x'') \big(b_g(x'',x') - \mathcal{M}(x'',x',v)\big) \Big]}_{\text{GE: cross: others' savings-impulse} \equiv \text{change in propagation of impulse due to change in savings}} \\
+ \ & \underbrace{\int v_y(x,y) g^{SS}(y) \Big[b_{gg}(y,x',x'') + k(y) k_p(y) v_y(y,x') v_y(y,x'') \Big] dy}_{\text{GE: } 1^{\text{st}}\text{-order valuation of } 2^{\text{nd}}\text{-order changes in others' savings}} \qquad (59)
\end{aligned}
$$

and

$$
\sigma_D(y,x) = -\partial_y \Big(\big(b_g(y,x) - \mathcal{M}(y,x,v)\big) g^{SS}(y) \Big) \quad , \quad \tau(x,y) = +\partial_y \Big(v_y(x,y) k(y) g^{SS}(y) \Big) \qquad (60)
$$

[30] In the case of the second-order expansion, 'identifying coefficients' corresponds to the results stating that if a quadratic form defined by a symmetric operator is equal to another quadratic form defined by a symmetric operator, then both operators must be equal. When either one of the operators is not symmetric, then only their symmetric parts are equal.

so that the deterministic SAME re-writes

$$\begin{aligned}\rho\mathcal{V}(x,x',x'') &= T(x,x',x'') + \mathcal{L}(x)[\mathcal{V}(\cdot,x',x'')] + \mathcal{L}(x')[\mathcal{V}(x,\cdot,x'')] + \mathcal{L}(x'')[\mathcal{V}(x,x',\cdot)] \\ &+ \int \Big(\mathcal{V}(x,y,x'')\sigma_D(y,x') + \mathcal{V}(x,x',y)\sigma_D(y,x'')\Big)dy + \int \mathcal{V}(y,x',x'')\tau(x,y)dy\end{aligned} \tag{61}$$

Cross SAME. The cross SAME writes

$$\begin{aligned}\rho\Gamma(x,x',z) &= u'(c^{SS}(x))b_{gz}(x,x')z + b_g(x,x')\omega_x(x,z) + b_z(x)zv_x(x,x') - u''(c^{SS}(x))\overline{\mathcal{M}}(x,z,\omega)\mathcal{M}(x,x',v) \\ &+ v_{x'}(x,x')\big(zb_z(x') - \overline{\mathcal{M}}(x',z,\omega)\big) + \int v_y(x,y)zb_{gz}(y,x')g^{SS}(dy) \\ &+ \mathcal{L}(x)[\Gamma(\cdot,x',z)] + \mathcal{L}(x')[\Gamma(x,\cdot,z)] + \mathcal{A}(z)[\Gamma(x,x',\cdot)] \\ &- \int v_y(x,y)g^{SS}(dy)k(y)\Big[\Gamma_y(y,x',z) - k_p(y)v_y(y,x')\omega_y(y,z)\Big] \\ &+ \int \Big[\Gamma(x,y,z)\sigma_D(y,x') + \mathcal{V}(x,x',y)\sigma_S(y,z)\Big]dy \\ &+ \int \Gamma(x,y,z)\sigma_S(y,z)dy\end{aligned}$$

where

$$\sigma_S(y,z) = -\partial_y\Big(g^{SS}(y)\left(zb_z(y) - \overline{\mathcal{M}}(y,z,\omega)\right)\Big)$$

Thus, define

$$\begin{aligned}T_C(x,x',z) &= u'(c^{SS}(x))b_{gz}(x,x')z + b_g(x,x')\omega_x(x,z) + b_z(x)zv_x(x,x') - u''(c^{SS}(x))\overline{\mathcal{M}}(x,z,\omega)\mathcal{M}(x,x',v) \\ &+ v_{x'}(x,x')\big(zb_z(x') - \overline{\mathcal{M}}(x',z,\omega)\big) + \int g^{SS}(y)v_y(x,y)\big[zb_{gz}(y,x') + k(y)k_p(y)v_y(y,x')\omega_y(y,z)\big] \\ &+ \int \mathcal{V}(x,x',y)\sigma_S(y,z)dy\end{aligned} \tag{62}$$

The cross SAME then becomes

$$\begin{aligned}\rho\Gamma(x,x',z) &= T_C(x,x',z) + \mathcal{L}(x)[\Gamma(\cdot,x',z)] + \mathcal{L}(x')[\Gamma(x,\cdot,z)] + \mathcal{A}(z)[\Gamma(x,x',\cdot)] \\ &+ \int \Big(\tau(x,y)\Gamma(y,x',z) + \Gamma(x,y,z)\big(\sigma_D(y,x') + \sigma_S(y,z)\big)\Big)dy\end{aligned} \tag{63}$$

Stochastic SAME. The stochastic SAME writes

$$\begin{aligned}\rho\Omega(x,z) &= u'(c^{SS}(x))b_{zz}(x)z^2 + 2b_z(x)z\omega_x(x,z) - u''(c^{SS}(x))\overline{\mathcal{M}}(x,z,\omega)^2 \\ &+ \mathcal{L}(x)[\Omega(\cdot,z)] + \mathcal{A}(z)[\Omega(x,\cdot)] \\ &+ 2\int v_y(x,y)z^2b_{gzz}(y,x')g^{SS}(dy) \\ &- 2\int v_y(x,y)g^{SS}(dy)k(y)\left[\Omega_y(y,z) - k_p(y)\omega_y(y,z)^2\right] \\ &+ 2\int \Gamma_y(x,y,z)\left(zb_z(y) - \overline{\mathcal{M}}(y,z,\omega)\right)g^{SS}(dy)\end{aligned}$$

Define

$$
\begin{aligned}
T_S(x,z) &= u'(c^{SS}(x))b_{zz}(x)z^2 + 2b_z(x)z\omega_x(x,z) - u''(c^{SS}(x))\overline{\mathcal{M}}(x,z,\omega)^2 \\
&+ \int v_y(x,y)z^2 b_{gzz}(y,x')g^{SS}(dy) + 2\int \Gamma(x,y,z)\sigma_S(y,z) \\
&+ \int v_y(x,y)g^{SS}(y)k(y)k_p(y)\omega_y(y,z)^2 dy \qquad (64)
\end{aligned}
$$

The stochastic SAME becomes

$$
\rho\Omega(x,z) = T_S(x,z) + \mathcal{L}(x)[\Omega(\cdot,z)] + \mathcal{A}(z)[\Omega(x,\cdot)] + 2\int \tau(x,y)\Omega(y,z)dy \qquad (65)
$$

C.2 Proof of Theorem 8

I may read off the perturbation of the law of motion of the distribution from the derivation of the SAME, in particular equation (58). I need only introduce the notation:

$$
\begin{aligned}
K_{21}(x,x') &= -2\big(b_g(x,x') - \mathcal{M}(x,x',v)\big) \\
K_{22}(x,x',x'') &= -\partial_x\left(g^{SS}(x)b_{gg}(x,x',x'') - k_p(x)v_y(x,x')v_{y'}(x,x'') - k(x)\mathcal{V}_x(x,x',x'')\right)
\end{aligned}
$$

Below I report the law of motion of the distribution up to second order un the presence of aggregate shocks:

$$
\begin{aligned}
\frac{dh_t(x)}{dt} &= \mathcal{L}^*(x)[h_t] + \mathcal{K}(x)[h_t] + S(x,z_t) \qquad (66) \\
&+ \varepsilon \times \left\{\partial_x\left(h_t(x)\int K_{21}(x,x')h_t(x')d\eta(x')\right) + \iint K_{22}(x,x',x'')h_t(x')h_t(x'')d\eta(x')d\eta(x'')\right\} \\
&- \varepsilon \times \partial_x\Bigg(\Big(z_t b_z(x) - \overline{\mathcal{M}}(x,z_t,\omega)\Big)h_t(x) \\
&\qquad + 2g^{SS}(x)\int\Big[zb_{zg}(x,x') - k(x)\Gamma_x(x,x',z_t) + k(x)k_p(x)v_x(x,x')\omega_x(x,z_t)\Big]h_t(x')d\eta(x')\Bigg) \\
&- \varepsilon \times \partial_x\left(z_t^2 b_{zz}(x)g^{SS}(x) - k(x)\Omega_x(x,z_t) + k(x)k_p(x)\omega_x(x,z_t)^2\right)
\end{aligned}
$$

D Details and proofs for applications

D.1 Proof of Proposition 3

First, to obtain (54), note that the FAME is

$$
\rho v_{ij} = \mathbb{1}_{i=j}u_i + \sum_{n,m} v_{in}\frac{\partial \mathcal{L}_n^*}{\partial V_m}(V^{SS})[\ell^{SS}]v_{mj} + \mathcal{L}_i[v_{\cdot,j}] + \mathcal{L}_j[v_{i,\cdot}]
$$

Then, the operator $\mathcal{L}_i$ coincides with the matrix multiplication by $\delta(\pi - I)$, and π denotes steady-state transition probabilities. $\frac{\partial \mathcal{L}_n^*}{\partial V_m}(V^{SS})[N^{SS}] = -\nu\sum_k \pi_{kn}\pi_{km}N_k^{SS}$ when $n \neq m$, and $\nu\sum_k \pi_{kn}(1-\pi_{kn})N_k$ when $n = m$. The matrix $\left(\frac{\partial \mathcal{L}_n^*}{\partial V_m}(V^{SS})[\ell^{SS}]\right)_{n,m}$ may be thus represented as $\nu\pi^T\bar{\ell}(I-\pi)$.

To prove Proposition 3, denote $\varepsilon = e^{-\tau_0}$ and $\theta_j = \theta_0^{-1} e^{\nu V_j^{SS}|_{\tau_0=+\infty}} = \theta_0^{-1} e^{\rho^{-1}\nu U_j^{SS}}$ for some scaling constant θ_0 to be chosen. I have, to first order in ε:

$$\forall i \neq j,\ \pi_{ij} = \varepsilon \frac{\theta_j}{\theta_i} \quad ; \quad \forall i,\ \pi_{ii} = 1 - \varepsilon \frac{\sum_{j\neq i}\theta_j}{\theta_i}$$

Chose $\theta_0 = \sum_j e^{\rho^{-1}\nu U_j^{SS}}$ so that $\sum_i \theta_i = 1$. Denote

$$\Theta_{ij} = \begin{cases} \frac{\theta_j}{\theta_i} & \text{if} \quad i \neq j \\ -\frac{1-\theta_i}{\theta_i} & \text{if} \quad i = j \end{cases}$$

so that $\pi - I = \varepsilon\Theta$ to first order in ε.

To leading order in ε, the FAME (54) implies that $v = \rho^{-1}\bar{u} + \mathcal{O}(\varepsilon)$. Using Theorem 2, I need only focus on the GE component of the linearized law of motion (53). Thus, I seek to solve for the dominant eigenvalue of the matrix $p = \rho^{-1}\delta\nu\bar{N}\Theta\bar{u}$ within the eigenspace in which $\sum x_i = 0$. I look for solutions (λ, x) to

$$\lambda x_i \quad = \quad \sum_j N_i \Theta_{ij} \tilde{u}_j x_j = \frac{N_i}{\theta_i}\left[\sum_j \theta_j \bar{u}_j x_j - \bar{u}_i x_i\right]$$

Re-arranging, I obtain that

$$\left(\lambda + \frac{N_i\bar{u}_i}{\theta_i}\right) x_i = \frac{N_i}{\theta_i} Q \quad , \quad Q = \sum_j \theta_j \bar{u}_j x_j$$

I can normalize $Q = 1$. Then $x_i = \frac{N_i}{N_i\bar{u}_i + \lambda\theta_i}$. Substituting back into the normalization, I obtain that λ satisfies

$$1 = \sum_i \frac{1}{\frac{1}{\theta_i} + \frac{\lambda}{\bar{u}_i N_i}} \quad \iff \quad 0 = \sum_i \frac{\frac{\theta_i^2}{N_i\bar{u}_i}}{\frac{1}{\lambda} + \frac{\theta_i}{\bar{u}_i N_i}}$$

As expected, 0 is the eigenvalue that corresponds to $\sum_i x_i \neq 0$. We thus look for the largest of other eigenvalues. For those, $\lambda < 0$, otherwise the sum is strictly positive. Denote $\mu = -1/\lambda$. Then $\inf_i \frac{\theta_i}{N_i\bar{u}_i} < \mu < \sup_i \frac{\theta_i}{N_i\bar{u}_i}$, which delivers the result.

D.2 Details for the economy in Section 5.2

Households. Employed households solve, for $a \geq \underline{a}$,

$$\begin{aligned} \rho V(a,y) - \dot{V}(a,y) &= \max_c u(c) + (ra + w + \pi^e - T - c)V_a(a,y) \\ &+ \lambda^E \int_w^\infty \Big(V(a,x) - V(a,y)\Big)\tilde{f}(x)dx + \delta\Big(U(a,y) - V(a,y)\Big). \end{aligned} \tag{67}$$

$\tilde{f}(x)$ is the p.d.f. of new wage offers. π are profits from firms. w denotes real wages. r is the real interest rate. T denote lump-sum taxes on the employed.

Unemployed households solve U solves

$$\rho U(a) - \dot{U}(a) = \max_c u(c) + (ra + b + \pi^u - c)U_a(a) + \lambda^U \int_{\underline{w}(a)}^{\infty} \Big(V(a,x) - U(a)\Big)\tilde{f}(x)dx \quad (68)$$

where $\underline{w}$ is the reservation wage. The reservation wage solves $V(a, \underline{w}(a)) = U(a)$.

Firms. There is an exogenous distribution of new jobs (or firms, with a constant returns production) every period, that differ in productivity z, with p.m.f. $h_0(z)$. They post real wages to solve

$$\max_{w,v} -\frac{c_0 v^{1+\gamma}}{1+\gamma} + q\Big(\phi + (1-\phi)\tilde{G}(w)\Big)vJ(z,w) \quad (69)$$

where $J(z,w)$ is the value of a filled job of productivity z at wage w. It solves

$$\rho J(z,w) - \dot{J}(z,w) = \max_k z^{1-\alpha}k^{\alpha} - w - (r+d)k - \Big[\delta + \lambda^E(1-\tilde{F}(w))\Big] J(z,w) \quad (70)$$

where k is the amount of capital they rent on a competitive market, and d its depreciation rate. I assume that firms fully commit to wages. Wages are not renegotiated, and do not depend on aggregate productivity. Equation (69) reveals that the new wage offer distribution is concentrated on the locus of the optimal wage offer, $w(z)$. Finally, the vacancy decision is given by

$$c_0 v(z)^{1/\gamma} = qJ(z, w(z))$$

and the p.m.f. of new wage offers, $f(w) = V\tilde{f}(w)$, satisfies

$$f(w(z))w'(z) = h_0(z)v(z).$$

Distributions. Let $g^e(a,y,z)$ be the p.m.f. of employed workers, and $g^u(a)$ the one for unemployed workers. It solves, for $a > \underline{a}$,

$$\begin{aligned} \dot{g}^e(a,y,z) &= -\partial_a\Big(s^e(a,y)g^e(a,y,z)\Big) \\ &\quad - \Big[\delta + \lambda^E(1-\tilde{F}(w))\Big] g^e(a,y,z) + \tilde{f}(w(z))w'(z)\boldsymbol{\delta}_{w(z)}(w)\Big[\lambda^U g^u(a) + \lambda^E \tilde{G}(w)\Big] \end{aligned} \quad (71)$$

where s^e is the savings rate of employed workers. The mass of employed workers at $\underline{a}$ follows

$$\begin{aligned} \dot{m}^e(w,z) &= -\Big(s^e(a,y)g^e(a,y,z)\Big)\Big|_{a=\underline{a}} \\ &\quad - \Big[\delta + \lambda^E(1-\tilde{F}(w))\Big] m^e(w,z) + \tilde{f}(w(z))w'(z)\boldsymbol{\delta}_{w(z)}(w)\Big[\lambda^U m^u + \lambda^E \tilde{G}(w)\Big] \end{aligned} \quad (72)$$

Similarly, for unemployed workers,

$$\begin{aligned} \dot{g}^u(a) &= -\partial_a\Big(s^u(a)g^u(a)\Big) - \lambda^U g^u(a) + \delta g^e(a) \\ \dot{m}^u &= -\Big(s^u(a)g^u(a)\Big)\Big|_{a=\underline{a}} - \lambda^U m^u + \delta m^e \end{aligned} \quad (73)$$

Government. A government runs the unemployment insurance program and balances its budget every period:

$$\tau \int w(g^e(w) + m^e(w))dw = b\left(m^u + \int g^u(a)da\right) + rB$$

where B denotes the fixed government debt.

Market clearing. The interest rate clears the capital market:

$$(1-u)\left(\frac{\alpha}{r+d}\right)^{\frac{1}{1-\alpha}} Z = \iint a\Big(g^e(a,w) + g^u(a)\Big)dadw + B$$

Labor market tightness θ satisfies

$$\theta = \frac{V}{u + \xi(1-u)}, \quad V = \int v(w)h_0(w)dw, \quad u = \int g^u(a)da$$

and

$$\lambda^U = m_0\theta^{1-\alpha}, \; \lambda^E = \xi\lambda^U$$

with a matching function $M = m_0 S^\nu V^{1-\nu}$.

Aggregate shocks. Aggregate productivity Z_t follows a continuous-time AR(1) process as in Section 1.

Online Appendix

E Weak derivatives and duality

Unidimensional case. Consider a domain $(-1, 1)$ with a possible mass point at 0. Compute, for a smooth function φ such that $\varphi(-1) = \varphi(1) = 0$,

$$\langle f, \varphi' \rangle = \int_{-1}^{0^-} f(x)\varphi'(x)dx + f_0\varphi_0' + \int_{0^+}^{1} f(x)\varphi'(x)dx$$

where I denote evaluation at the mass point by subscripts to emphasize the role of (non-)smoothness at the possible mass points. Assuming f is differentiable on $(-1, 0)$ and $(0, 1)$, obtain $\langle f, \varphi' \rangle = f(0^-)\varphi(0^-) - \int_{-1}^{0^-} f'(x)\varphi(x)dx + f_0\varphi_0' - f(0^+)\varphi(0^+) - \int_{0^+}^{1} f'(x)\varphi(x)dx$. Finally, make $\langle f', \varphi \rangle$ appear: $\langle f, \varphi' \rangle = f(0^-)\varphi(0^-) + f_0'\varphi_0 + f_0\varphi_0' - f(0^+)\varphi(0^+) - \langle f', \varphi \rangle \equiv J_0 - \langle f', \varphi \rangle$. The key object of interest is therefore $J_0 \equiv f(0^-)\varphi(0^-) + f_0'\varphi_0 + f_0\varphi_0' - f(0^+)\varphi(0^+)$. The duality property requires that the sum of terms around 0, J_0, is equal to zero.

Smooth φ. When φ is continuously differentiable on $(-1, 1)$, and in particular is smooth around 0, then $J_0 = \Big[f_0' - \big(f(0^+) - f(0^-)\big)\Big]\varphi(0) + f_0\varphi_0'$. So clearly, for J_0 to be zero and f to have a weak derivative, one needs

$$f_0' \equiv f(0^+) - f(0^-) \quad , \quad \varphi_0' \equiv 0 = \varphi(0^+) - \varphi(0^-).$$

Thus, the definition of the weak derivative w.r.t. the base measure η imposes that the value of the derivative at possible mass points is equal to the jump there. In particular, if a function is continuous at a possible mass point, then the derivative there that enters into the inner product (but not around) is endogenously 0. It needs not be a requirement.

Multidimensional case. This argument generalizes straightforwardly to multiple dimensions.