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RUNNING PRIMARY DEFICITS FOREVER IN A DYNAMICALLY EFFICIENT ECONOMY: FEASIBILITY AND OPTIMALITY

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ABSTRACT

Government debt can be rolled over forever without primary surpluses in some stochastic economies, including some economies that are dynamically efficient. In an overlapping-generations model with constant growth rate, g, of labor-augmenting productivity, and with shocks to the durability of capital, we show that along a balanced growth path, the maximum sustainable ratio of bonds to capital is attained when the risk-free interest rate, r[sub]f, equals g. Furthermore, this maximal ratio maximizes utility per capita along a balanced growth path and ensures that the economy is dynamically efficient.

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What is the maximum amount (possibly zero) of bonds that a government can rollover forever without running primary budget surpluses? If the government can rollover a positive amount of bonds forever, what is the optimal amount of bonds to rollover? Blanchard's (2019) presidential address to the American Economic Association spurred renewed scholarly interest in these questions. Only a year later, global events brought these issues into the public realm as governments around the world ran huge deficits to deal with the catastrophic economic consequences of the covid-19 pandemic.

In the absence of uncertainty, the ability of the government to rollover bonds forever is determined by the "r vs g" comparison, as it is colloquially known, where r is the net rate of return on all assets, including government bonds, and g is the growth rate of the capital stock. Specifically, if and only if $r < g$ along a balanced growth path in a competitive economy, government bonds can be rolled over forever and will shrink as a share of the economy over time. Also, if and only if $r < g$, the economy suffers from a dynamically inefficient overaccumulation of capital. That is, government bonds can be rolled over forever if and only if the economy is dynamically inefficient.

In the presence of uncertainty, the link between dynamic inefficiency and the feasibility of rolling over government bonds forever is more nuanced. Put simply, the rate of return on capital is the rate of return relevant for assessing dynamic efficiency, but the riskfree interest rate is the rate of return relevant for assessing whether the government can rollover its bonds forever. Uncertainty breaks the equality of these two rates of return. As we will show, in some dynamically efficient competitive economies with a constant growth rate $g \geq 0$, it is possible for the riskfree interest rate, r_f, to be less than g, which makes permanent rollover of government bonds feasible. This possibility of permanently rolling over debt in a dynamically efficient economy does not exist in deterministic, dynamically efficient, competitive economies.

Our principal findings in this paper result from both positive and normative analyses of sustainable levels of the ratio of government bonds to the capital stock, which we define as levels of this ratio that permit government bonds to be rolled over forever *without any primary surpluses*. In our positive analysis, we find that if $r_f < g$ along a balanced growth path without government bonds (which can be the case in some dynamically efficient economies, and must be the case in all dynamically inefficient economies), the maximum sustainable level of the bond-capital ratio is strictly positive. Starting from zero government bonds, increasing the amount of bonds reduces capital, thereby driving up the marginal product of capital and the constellation of rates of return until, at the maximum sustainable bond-capital ratio, $r_f = g$.

Our normative analysis examines the optimal sustainable level of the bond-capital ratio, specifically the sustainable level of this ratio that maximizes welfare, measured as the utility of consumers in the steady state. We find that the marginal impact on welfare of an increase in this ratio is positive along balanced growth paths with $r_f < g$ and is non-negative along balanced growth paths with $r_f = g$. Therefore, since the bond-capital ratio is not sustainable for $r_f > g$, the optimal sustainable level of the bond-capital ratio equals its maximum sustainable value, which is attained when $r_f = g$. At the optimal level of the bond-capital ratio, government bonds play the dual role of (1) eliminating any overaccumulation of capital that would occur if the bond-less economy were dynamically inefficient and (2) providing safe assets to be held in the portfolios of risk-averse consumers. When the bond-capital ratio is at its optimal level ($r_f = g$), the economy is in the dynamically efficient region. A marginal increase in the bond-capital ratio reduces capital and expected aggregate consumption. Despite this reduction in expected aggregate consumption, the benefit of increased risk sharing from debt issuance is large enough to prevent a reduction in welfare.

The model in this paper is crafted so that along a balanced growth path, the capital stock per unit of effective labor is constant but the rate of return on capital is stochastic. To illustrate the mechanism in its simplest form, we preview the model in the case in which (1) there is no labor-augmenting technical progress, so $g = 0$, and (2) the government wastes any funds collected when it issues new bonds in excess of contemporaneous interest payments on existing government bonds. The model has overlapping generations of a constant number of people who live for two periods, earn labor income only in the first period of life, and save some of their wage income to provide for consumption in the second period of life. Output in each period is produced with labor and capital according to a Cobb-Douglas production function without a productivity shock, which implies that wage income is non-stochastic. Consumers save a constant fraction of their wage income because they earn non-asset income only in the first period and they have Epstein-Zin-Weil (Epstein and Zin (1989) and Weil (1990)) preferences with an intertemporal elasticity of substitution equal to one. Therefore, aggregate saving of the young consumers is non-stochastic, which makes total assets, the sum of capital and government bonds non-stochastic.

The uncertainty in our model economy enters through a stochastic shock to the durability of capital that makes the rate of capital depreciation, and hence the rate of return on capital, stochastic. Bulow and Summers (1984), p. 25, argue that "capital risk," which they associate with the stochastic nature of depreciation, is far larger than "income risk," which they associate with the stochastic nature of the marginal product of capital. The uncertainty about the rate of capital depreciation drives a wedge between the expected rate of return on capital and the riskfree interest rate. However, in this simple form of the model, the evolution of the capital stock, wage income, as well as the consumption and saving of the young generation, are all invariant to the distribution and realizations of the durability shock. Thus, there is a useful dichotomy in the equilibrium of the economy. We can determine the equilibrium values of aggregate saving and the capital stock, without any consideration of financial values and without any consideration of the realizations or the distribution of the durability shocks.[1] Despite the deterministic evolution of the capital stock, we show that the rate of return on capital and hence the consumption of the old generation are risky because the depreciation rate of capital is stochastic. The stochastic nature of consumption when old implies that the pricing kernel is stochastic.

The simplicity of the model in the case described above has several useful features. Because wage income per unit of effective labor, and hence aggregate saving of young consumers per unit of effective labor, are constant along a balanced growth path, there is no chance that adverse shocks will reduce aggregate saving below the amount needed to absorb the equilibrium amount of government bonds. Thus, government bonds are riskfree and the riskfree interest rate is the appropriate market interest rate on these bonds.[2] In the simple

[1]This dichotomy continues to hold if we relax the assumption (1) that $g = 0$, but it does not hold if we relax assumption (2) that the government wastes any funds it receives when it issues new bonds in excess of interest payments on existing government bonds.

[2]Bertocchi (1994) and Binswanger (2005) discuss the unsustainability of government bonds in economies in which the capital stock evolves stochastically. Our model features a non-stochastic capital stock and

case of the model described above, the equilibrium size of the capital stock is invariant to the distribution of durability shocks. In our quantitative analysis, we illustrate that the maximum sustainable bond-capital ratio, which is the optimal value of this ratio, is an increasing function of the variance of the durability shock. We also illustrate that the maximum sustainable amount of government bonds is an increasing function of the coefficient of relative risk aversion.

1 Literature Review

The celebrated Golden Rule of capital accumulation derived by Phelps (1961) characterizes the capital stock per capita that maximizes consumption per capita along a deterministic balanced growth path. If the rate of return on capital, r, equals the growth rate of the economy, g, then consumption per capita is at the highest feasible level in the long run. If the saving rate exceeds the rate consistent with the Golden Rule, then the capital stock per capita exceeds the Golden Rule level so $r < g$ and consumption per capita is less than in the Golden Rule; that is, there is a dynamically inefficient overaccumulation of capital. Diamond (1965) develops and analyzes an overlapping generations economy with optimizing consumers and competitive firms and finds that if the optimal saving of young consumers is sufficiently high, either because consumers are very patient or the wage share of total income is very high, then capital per capita can exceed the Golden Rule level along a balanced growth path and there is a dynamically inefficient overaccumulation of capital. However, if young consumers use some of their saving to hold government bonds, they will hold a smaller amount of capital in their portfolios, driving down the aggregate capital stock, possibly by enough to eliminate any dynamically inefficient overaccumulation of capital.

Cass (1972) provides a complete characterization of dynamic inefficiency that does not depend on consumers' preferences. The Cass criterion simply asks whether it is feasible to increase aggregate consumption at some date without having to reduce aggregate consumption at some other date(s). In a deterministic steady state, the Cass condition is the same as in Phelps and Diamond, that is, an economy is dynamically inefficient if and only if

avoids the unsustainability problem in Bertochhi and Binswanger.

$r < g$. However, the general version of the Cass criterion also applies to economies outside the steady state.

A government bond that is rolled over forever is often regarded as a bubble, which is an asset with zero fundamental value that nevertheless has a positive market value. Tirole (1985) develops a tight link between dynamic inefficiency and the feasibility of bubbles, in his words "the existence of bubbles is conditioned by the efficiency of the bubbleless equilibrium." (Tirole (1985), p. 1076) In our context, a "bubbleless equilibrium" has zero government bonds. Part (a) of Proposition 1 in Tirole (1985) states that if the economy without any government bonds is dynamically efficient, then equilibrium in the economy cannot contain bubbles. Tirole's statement holds in deterministic economies where the rates of return on capital and government bonds are equal. The introduction of government bonds reduces capital, thereby increasing the common rate rates of return on government bonds and capital. If the economy was dynamically efficient without government bonds, then $r > g$ initially and this increase in r induced by government bonds increases the excess of r over g, which implies that government bonds cannot be rolled over forever.

In order for bubbles to be feasible in a deterministic dynamically efficient economy, there must be a wedge between the rates of return on government bonds, r_f, and on capital, r. Specifically, r must exceed r_f and the growth rate g must lie between r_f and r. In deterministic models, Farhi and Tirole (2011) and Martin and Ventura (2012) provide this wedge by introducing a wedge between borrowing and lending rates for firms. Also in a deterministic framework, Ball and Mankiw (2023) introduces monopoly power by firms, which drives a wedge between the marginal product of capital, r, and the user cost of capital, which is based on the interest rate on government bonds, r_f. Aguiar et al. (2021) also examine the role of monopoly power in driving a wedge between the rate of return on capital and the interest rate on government bonds, but their analysis includes idiosyncratic risk. Like Ball and Mankiw (2023), however, in their analysis the aggregate rate of return of capital is not uncertain.[3]

[3] Amol and Luttmer (2022) also examines fiscal policies in economies with idiosyncratic uncertainty but no aggregate uncertainty. The analysis in that paper does not include monopoly but is conducted in a two-sector model with a capital goods sector and a consumption goods sector. As in Ball and Mankiw (2023) and Aguiar et al. (2021), but unlike in our paper, the aggregate return to capital in Amol and Luttmer (2022) is not random.

The early literature on dynamic efficiency (Diamond (1965), Cass (1972)) focussed on deterministic models. In those models, the conditions for dynamic efficiency are the same as the conditions that rule out scope for Pareto improvements. However, the literature has long recognized that the introduction of uncertainty can break the link between dynamic efficiency and the conditions that rule out any scope for Pareto improvements[4] or infinite debt rollover.[5] Roughly speaking, the sufficient condition for dynamic efficiency in Zilcha (1991) is that the expected logarithmic return to capital exceeds the logarithmic growth rate of the economy, whereas the sufficient condition for infinite debt rollover tests whether the yield of a GDP-contingent, pure-discount bond converges to a negative number as maturity approaches infinity.[6] Typically, these conditions are stated in terms of endogenous quantities (returns). The common approach in the literature stops short of fully solving the underlying model and showing that in some dynamically efficient stochastic economies it is possible to roll over debt forever.[7] The model that we develop is designed to present a dynamically efficient stochastic economy in which it is simple to verify that government bonds can be rolled over forever.

Aggregate uncertainty drives a wedge between the riskfree rate, r_f, and the rate of return on capital, r. Abel, Mankiw, Summers, and Zeckhauser (1989), hereafter AMSZ, proves that if the rate of return on capital is greater than the growth rate in all states and at all times, then the economy is dynamically efficient; and since the rate of return on capital is always greater than the growth rate, the riskfree rate is greater than the growth rate and hence it is not feasible to rollover government bonds forever. Alternatively, AMSZ proves that if the rate of return on capital is less than the growth rate in all states and at all times,

[4]See, e.g., Bertocchi (1991), Barbie et al. (2007), Binswanger (2005), Bloise and Reichlin (2023).

[5]See, e.g., Kocherlakota (2023a).

[6]A GDP-contingent, pure discount bond is a bond that makes a single payment at maturity, T, and that payment is proportional to Y_T, which is output at time T. See Kocherlakota (2023a). The sufficient conditions in Bloise and Reichlin (2023) for achieving Pareto improvements are closely related to the condition in Kocherlakota (2023a).

[7]While, in principle, one could try to verify empirically that the conditions for dynamic efficiency and the feasibility of infinite debt rollover are satisfied, in practice this approach faces challenges. The Zilcha condition on dynamic efficiency requires knowledge of the returns to capital and aggregate growth rates (but does not require any assumptions on marginal utilities in different economic states). However, testing the condition for the feasibility of infinite debt rollover requires information about the valuation of a GDP-linked payoff at infinite maturity. Because governments don't issue GDP-linked bonds, and there are no markets for GDP futures, the yields on long-term GDP-linked bonds can neither be observed, nor easily inferred from existing securities without a model that describes marginal utilities.

then the economy is dynamically inefficient; and since the rate of return on capital is always lower than the growth rate of capital, the riskfree rate is less than the growth rate and it is feasible to rollover government bonds forever. Thus, in the situations that can be declared dynamically efficient or dynamically inefficient by the sufficient conditions in AMSZ, the link between dynamic inefficiency and the feasibility of bubbles continues to hold. However, the AMSZ conditions are not applicable in economies where the rate of return on capital is sometimes greater than the growth rate and sometimes less than the growth rate.

Zilcha (1990, 1991) steps into the gap left by AMSZ and derives a characterization of dynamic efficiency in stochastic economies in which the rate of return on capital can sometimes exceed and sometimes falls short of the growth rate, g, of the economy. Zilcha adapts Cass's definition of dynamic efficiency in a natural way to stochastic economies and derives a remarkable condition for dynamic inefficiency when the economy grows at a constant rate, g: $E\{\ln(1+r)\} < \ln(1+g)$. Zilcha presents this condition as necessary and sufficient for dynamic inefficiency. However, Rangazas and Russell (2005) and Barbie and Kaul (2009) point out a subtle but important statistical issue that implies that the Zilcha condition is necessary, but not sufficient, for dynamic inefficiency. Equivalently, $E\{\ln(1+r)\} \geq \ln(1+g)$ is sufficient for dynamic efficiency. Recently, Bloise and Reichlin (2023, p. 640) presents a sufficient condition for dynamic efficiency, which they refer to as "capital efficiency." Their sufficient condition states that if "capital is sufficiently productive with some probability," then capital is not overaccumulated.[8]

Blanchard and Weil (2001) presents four examples that debunk various simplistic views about the relation between dynamic inefficiency and the feasibility of rolling over government bonds forever. In particular, the third and fourth examples in that paper illustrate that government bonds can be rolled over forever in some dynamically efficient economies. Those two examples feature models of stochastic storage so that output is linear in the capital stock. Without concavity in the capital stock, the issue of capital overaccumulation is moot.[9] Barro (2023) uses a model with stochastic depreciation as in our model, but specifies $Y_t = AK_t$,

[8]The condition is $\max_i \min_j \frac{1+g_{ij}}{R_{ij}} < 1$, where g_{ij} and R_{ij} are, respectively, the growth rate of output and the rate of return on capital, when the economy moves from state i to state j in a finite state space.

[9]Footnote 11 in Blanchard and Weil (2001) points out that these models could be extended to incorporate concavity in the capital stock by specifying output to be $Y_t = K_t^\alpha - \delta K_t$, where δ is a random variable, but they do not work out the implications of this model.

so that, as in Blanchard and Weil's specification with simply stochastic storage, there is no concavity in the capital stock.

Blanchard's presidential address (Blanchard (2019)) is a far-ranging analysis of both empirical and theoretical issues related to the rollover of government bonds. It carefully documents that the recent situation with safe interest rates below growth rates is not unusual in historical data. In simulations reminiscent of the eponymous deficit gamble in Ball, Elmendorf, and Mankiw (1998), Blanchard finds that even if government bonds cannot be rolled over forever, it is likely that they can be rolled over for many decades before investors become unable, or unwilling, to buy newly issued government bonds.

Since Blanchard's presidential address, at least three papers have appeared with simple titles that involve comparisons of the rate of return and the growth rate. Cochrane (2021b), simply titled "$r < g$," is a forceful warning against the notion that when the riskfree interest rate is lower than the growth rate, the government can rely on growing itself out of debt. An attempted permanent rollover of bonds is bound to fail eventually, especially if government deficits are large. "The constraint on public debt when $r < g$ but $g < m$" (Reis (2021)) develops a model in which the interest rate on government bonds (r) is less than the growth rate of the economy (g), opening the possibility that government bonds can be rolled over, and yet the marginal product of capital (m) is greater than g so the economy is dynamically efficient. That paper derives the fiscal capacity of the economy, which is a limit on the ratio of government spending to the amount of bonds outstanding. Barro (2023), simply titled "r Minus g," provides data on (arithmetic) averages in each of 14 OECD countries of rates of return on bonds and equities and growth rates of GDP per capita and consumption per capita. However, the Zilcha criterion directly implies that for the purpose of assessing dynamic efficiency, one must use the geometric means of rates of return and growth rates rather than arithmetic means.

Two interesting questions – one positive and one normative – remain unanswered in the papers described above. First, what is the *maximum* amount of government bonds that can be rolled over forever? The fiscal capacity in Reis (2021) is related to this question but, as mentioned above, fiscal capacity is the maximum ratio of government spending to the amount of bonds outstanding, rather than the maximum ratio of the amount of government bonds outstanding to the capital stock that is the focus of our analysis. Second, what is the *optimal* amount of bonds to rollover along a balanced growth path? Blanchard (2019), Falkenheim (2022), Ball and Mankiw (2023), and Kocherlakota (2023b) provide interesting analyses and discussions of the marginal impact of government bonds on welfare, where, as in our paper, welfare is defined to be the level of utility of consumers along a balanced growth path.[10] However, none of these four papers derives analytically the optimal sustainable amount of government bonds in a dynamically efficient economy with aggregate shocks to the return of capital.[11] Angeletos et al. (2023) analyzes optimal debt issuance in a model where the planner maximizes a discounted objective balancing distortionary effects of taxation, seignorage, and improved risk sharing.

Kocherlakota (2023b) considers an economy where consumers face idiosyncratic "rare disasters." In the production-economy version of the model in Kocherlakota (2023b), the returns to capital and bonds are equal, so that the conditions on infinite debt rollover and dynamic inefficiency are identical. In our paper the return to capital and the return on bonds are distinct because of aggregate uncertainty about the return on capital. This difference in the rates of return allows the possibility that infinite debt rollover is possible in a dynamically efficient economy. We show that if the riskfree interest rate on government bonds is less than the growth rate of the economy, it is optimal to increase the quantity of debt to the maximal sustainable amount. Although reducing capital in a dynamically efficient economy reduces expected aggregate consumption, the additional risk sharing provided by an increased amount of riskfree government bonds implies that, on net, it is optimal to increase the amount of bonds to the maximum sustainable level.

Our result that the government can increase consumer welfare by issuing additional debt

[10] Aoki et al. (2014) show that in situations in which bubbles exist in equilibrium, welfare in the bubbly equilibrium is higher than in the equilibrium without bubbles. However, it does not examine the marginal impact on welfare of an increase in the size of the bubble.

[11] The working paper version (NBER WP 29138, August 2021) of Kocherlakota (2023b) concludes that "as long as there is a public debt bubble (in this class of models), agents are better off in the long run if the government changes its policy choices so as to increase the debt and deficit" (p. 20). Aiyagari and McGrattan (1998) numerically computes the optimal value of government debt in a model with uninsurable idiosyncratic risks, but no aggregate risks. It finds that the welfare loss resulting from the actual level of debt in the US economy (at the time of that paper's writing), rather than the optimal level of debt as computed in their model, is small. Brumm and Hussmann (2023) numerically compute optimal debt in a variety of models, including calibrated OLG models and models in which investors obtain "convenience yields."

whenever $r_f < g$ is consistent with the conditions that permit Pareto improvements in Bertocchi (1991), Barbie et al. (2007) and more recently Bloise and Reichlin (2023), who show that the condition permitting a Pareto improvement can hold in some economies without dynamically inefficient overaccumulation of capital.[12] Bloise and Reichlin provide diagnostics to assess whether a given allocation of consumption is Pareto-optimal and whether that allocation is dynamically inefficient in the sense of having overaccumulation of capital. Their main finding implies that when $r_f < g$, an allocation is not Pareto optimal, as long as the planner is permitted to freely reallocate consumption across time and across people. While this finding suggests that Pareto optimality requires increasing r_f to a value at least as high as g, they do not describe the implementation of such a Pareto improvement using market instruments. By contrast, we focus on a particular market instrument, namely, non-contingent government bonds that can be rolled over forever without primary surpluses, as a means to increase the riskfree rate. In order to conduct this analysis, we have to analyze the general equilibrium effects of an increase in the amount of government bonds on allocations of consumption, portfolio allocations, and asset returns. We also characterize the level of government bonds (relative to the capital stock) that achieves the optimal balanced growth path.

Finally, in addition to the positive and normative questions that share a common answer, our paper also offers fresh insights about the intertemporal government budget constraint. It is typical in discussions of the sustainability of fiscal policy (O'Connell and Zeldes (1988), Wilcox (1989), and Bohn (1995)), the pricing of government bonds (Jiang et al. (2019)), and the fiscal theory of the price level (Cochrane (2021a)) to assume that the value of government debt equals the expected present value of the sum of primary government surpluses over the infinite future, for an appropriate path of discount rates over time. This budget constraint is often described as a transversality condition or, more accurately, as a No Ponzi Game (NPG) condition. In our paper, the NPG condition is violated by design.[13] The NPG condition is

[12]Since in our model the risk free rate is constant along a balanced-growth path, the results in Bloise and Reichlin (2023) imply that if $r_f < g$ then there is scope for a Pareto improvement.

[13]As shown in Santos and Woodford (1997), in an economy in which the present value of the stream of present and future aggregate consumption is infinite, there is room for the NPG condition to fail. Nevertheless, even though the NPG condition fails in our model when $r_f < g$, thereby enabling permanent rollover of government debt, the market value of the existing capital stock is finite, because it is the valuation of the stream of profits to the remaining portion of a depreciating capital stock that approaches zero over time.

often invoked to rule out the possibility of rolling over debt forever. While the NPG arises naturally in some contexts, we examine situations in which permanent rollover of debt is both feasible and optimal, and the NPG condition need not, and does not, hold. Contrary to the literature in which the value of government debt equals the sum of present values of future primary surpluses, in our model, the value of government debt is positive even though all future primary government surpluses are non-positive.

2 A Graphical Preview of the Main Results

The model developed and analyzed in this paper has the convenient property that the capital stock per unit of effective labor is constant along a balanced growth path, which implies that the expected net rate of return on capital, $\bar{r}$, is constant. It is convenient to focus on the expected adjusted gross rate of return on capital, $\overline{R} \equiv \frac{1+\bar{r}}{1+g}$, where g is the constant growth rate of capital along a balanced growth path. For the graphical preview in this section (and only in this preview), the adjusted gross rate of return on capital, R, has a symmetric two-point distribution with mean $\overline{R}$. Specifically, $\Pr\left\{R = R_H \equiv \overline{R} + \sigma\right\} = \frac{1}{2} = \Pr\left\{R = R_L \equiv \overline{R} - \sigma\right\}$.

Figure 1 shows the possible adjusted gross rates of return on capital along a balanced growth path, with R_H on the horizontal axis and R_L on the vertical axis. In the absence of shocks to the rate of return on capital, $R_H = R_L$ so the locus of possible pairs (R_H, R_L) is the 45-degree line labeled "Certainty: $R_H = R_L = R_f$," which passes through the origin. The area above and to the left of the 45-degree line is grayed out because $R_H \geq R_L$ by definition.

At Point F on the certainty line, $\frac{1+r}{1+g} = R_H = R_L = 1$, so that $r = g$, which is the Golden Rule. As shown by Phelps (1961), along balanced growth paths with $r = g$, the economy attains the maximum possible consumption per capita. Along the segment of the certainty line to the southwest of Point F, $r < g$, which implies, as shown by Phelps (1961) and Diamond (1965), that the economy has overaccumulated capital, in the sense that it is possible to increase aggregate consumption today without having to reduce aggregate consumption at any future date. This situation is described as dynamic inefficiency. Alternatively, along

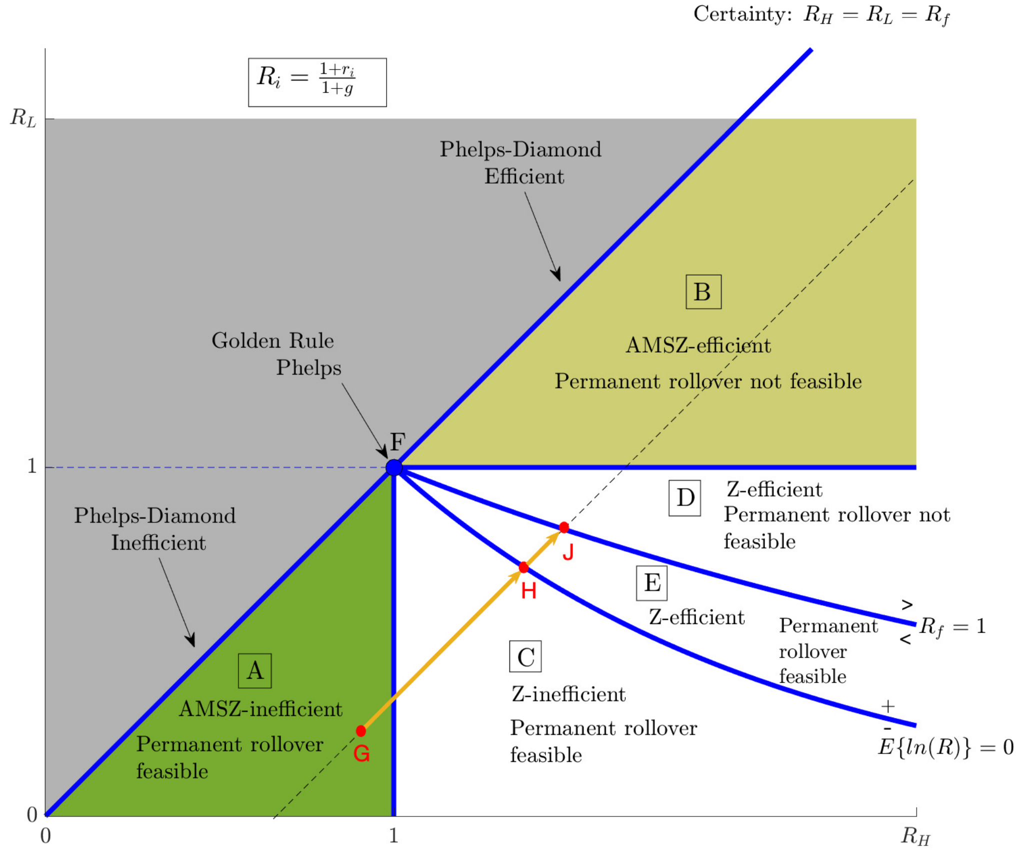

Figure 1: A Graphical Summary of Dynamic Efficiency and Feasibility of Debt Rollover when $\zeta = 0$, $\gamma = 1$, and $\mathcal{B} = 0$.

the segment of the certainty line to the northeast of Point F, $r > g$, which implies that the economy is dynamically efficient in the following sense: it is impossible to increase aggregate consumption at any time without decreasing aggregate consumption at some other time. In a world of certainty, all assets have the same rate of return and, in particular, the riskfree interest rate on bonds, r_f, equals the rate of return on capital, r. When r_f and g are both constant, government bonds can be rolled over forever, if and only if $r_f < g$ so the economy is dynamically inefficient.

Now consider the situation in which σ, the standard deviation of R, is positive so that (R_H, R_L) lies in one of the five regions below the 45-degree line. In two of these regions, the AMSZ criterion that compares r with g gives a decisive assessment of dynamic efficiency. In

Region A, labelled "AMSZ-inefficient," both R_H and R_L are less than one so $r < g$ for both possible values of R. Therefore, the AMSZ criterion implies that the economy is dynamically inefficient. Furthermore, since $R_L < R_H < 1$, the adjusted gross riskfree interest rate, R_f, which satisfies $R_L \leq R_f \leq R_H$, is always less than one. In this case, $r_f < g$, so that a dollar of government debt that is rolled over at the riskfree interest rate will shrink over time toward zero.

In Region B, labelled "AMSZ-efficient," both R_H and R_L are greater than one so the net rate of return on capital is always greater than the growth rate. Therefore, the AMSZ criterion implies that the economy is dynamically efficient. Furthermore, since $1 < R_L < R_H$, the adjusted gross riskfree interest rate, which satisfies $R_L \leq R_f \leq R_H$, is always greater than one. That is, $r_f > g$ and it is impossible to rollover government bonds, without primary surpluses, at the riskfree rate forever.

The AMSZ criteria are silent about Regions C, D, and E, where the adjusted gross rate of return on capital, R, is sometimes greater than one and sometimes smaller than one. Zilcha (1991) states that a balanced growth path is dynamically inefficient if and only if $E\{\ln R\} < 0$, though, as discussed in the introduction, this condition is necessary, but not sufficient, for dynamic inefficiency. In the symmetric two-point distribution in Figure 1, $E\{\ln R\} = \frac{1}{2}(\ln R_H + \ln R_L) = \frac{1}{2}\ln R_H R_L$, so the locus of points for which $E\{\ln R\} = 0$ is simply the rectangular hyperbola $R_L = R_H^{-1}$, which is the downward-sloping curve originating at Point F and labelled $E\{\ln R\} = 0$. In Region C, $R_L R_H < 1$ and hence $E\{\ln R\} < 0$ so we label Region C as "Z-inefficient," to indicate that points in this region satisfy Zilcha's necessary condition for dynamic inefficiency. By contrast, in Regions D and E, $R_L R_H > 1$ and hence $E\{\ln R\} > 0$ so the economy is dynamically efficient according to the Zilcha criterion; thus, we label Regions D and E as "Z-efficient." Finally, we note that along the curve where $E\{\ln R\} = 0$, Zilcha's necessary condition for dynamic inefficiency, $E\{\ln R\} < 0$, is violated, so dynamic efficiency prevails on this curve.

The government can rollover its bonds forever if $r_f \leq g$, equivalently, if $R_f \leq 1$. The higher of the two downward-sloping curves through Point F is the locus of (R_H, R_L) for which $R_f = 1$. For points above this curve (Regions B and D), $R_f > 1$ and permanent rollover of government bonds is not feasible. For points below the $R_f = 1$ curve (Regions

A, C, and E), $R_f < 1$ and permanent rollover of government bonds, without any primary surpluses, is feasible. Region E is of particular interest because, as we show in this paper, it contains dynamically efficient economies (since it lies above the $E\{\ln R\} = 0$ locus) in which government bonds can be rolled over forever without primary surpluses (since it lies below the $R_f = 1$ locus). Such a situation is impossible in deterministic economies.[14]

Now consider the impact of introducing and increasing the amount of riskfree government bonds. Suppose that Point G represents the balanced growth path in an economy without government bonds. Since it is located in Region A, this balanced growth path is dynamically inefficient and government bonds can be rolled over forever. As we will show in the main part of the paper, the introduction of government bonds reduces capital and drives up $\overline{R}$, the expected rate of return on capital, which increases R_H and R_L by the same amount that $\overline{R}$ increases. That is, the introduction of government bonds is represented by movement to the northeast along the dashed line through point G with slope equal to one. Consider the introduction of a quantity of government bonds that moves the balanced growth path to Point H on the curve labeled $E\{\ln R\} = 0$. Since the economy is Z-efficient along (and above) this curve, the movement from Point G to Point H eliminates the Z-inefficient overaccumulation of capital.

Next consider introduction of a greater amount of government bonds that moves the balanced growth path to Point J. Note that an even larger increase in government bonds is not feasible without primary surpluses (which we rule out) because such points to the northeast of Point J have $R_f > 1$, equivalently, $r_f > g$.

Now we address the normative question: which point along the dashed line between Points H and J maximizes utility per capita along a balanced growth path? As we prove in Proposition 6, utility per capita is strictly increasing as we steadily move along the dashed

[14] Points in Region C satisfy Zilcha's necessary condition for dynamic inefficiency. Bloise and Reichlin (2023, p. 640), "argue that capital is not overaccumulated whenever $\widehat{\rho} < 1$, where $\widehat{\rho}$ is $\max_i \min_j \frac{1+g_{ij}}{R_{ij}} < 1$, where g_{ij} and R_{ij} are, respectively, the growth rate of output and the rate of return on capital, when the economy moves from state i to state j in a finite state space. Their example 4.1, stated in the notation of this paper, is $R_L < 1 < R_H$ and $g_{ij} = 0$, which corresponds to the union of Regions C, D, and E in Figure 1. In this case, "$\widehat{\rho} = \min\{1/R_L, 1/R_H\} < 1$, thus ruling out capital overaccumulation." Hence, although Region C is "Z-inefficient," it is characterized by a dynamically efficient accumulation of capital according to the Bloise-Reichlin criterion. Thus, using the Bloise-Reichlin criterion expands the set of dynamically efficient economies (relative to the Zilcha criterion) in which permanent rollover of government bonds is feasible.

line from Point H to Point J. Indeed, when the economy reaches Point J, the impact on utility per capita of additional government bonds remains non-negative. Since any further increase in government bonds is infeasible without budget surpluses, Point J represents the utility-maximizing amount of government bonds. This result implies that to maximize utility per capita along a balanced growth path, the amount of government bonds must exceed the amount of bonds needed to eliminate dynamic inefficiency. Even though Z-inefficiency has been eliminated by moving to point H, there is still scope to increase utility along a balanced growth path by increasing the amount of government bonds and thereby decreasing the capital stock to the point at which R_f is driven to equal one (at point J).

3 Deterministic Capital Stock with Risky Returns

We now introduce the formal model. We begin with production, saving, and investment in this section, and turn to portfolio allocation and asset pricing in Section 4.

Aggregate output at time t, Y_t, is produced by competitive firms using capital, K_t, and effective labor, $G^t N$, where $G \equiv 1 + g \geq 1$ is the gross growth rate of labor-augmenting productivity and N is the constant number of young consumers in each time period. The production function is $Y_t = F\left(K_t, G^t N\right) = \left(G^t N\right)^{1-\alpha} K_t^{\alpha}$, where $0 < \alpha < 1$, and it is convenient to write the production function in intensive form as

$$y_t = k_t^{\alpha}, \tag{1}$$

where $y_t \equiv \frac{Y_t}{G^t N}$ and $k_t \equiv \frac{K_t}{G^t N}$. Capital depreciates at the rate $0 \leq \delta - \varepsilon_t \leq 1$ per period, where the durability shock ε_t is an i.i.d., non-degenerate random variable with mean zero, so $\delta > 0$ is the expected depreciation rate in each period. The durability shock is the only source of uncertainty in the model; the production function, $Y_t = F(K_t, G^t N)$, is deterministic. Online Appendix B shows that our model with a deterministic production function is isomorphic to a particular model that has shocks in the production function.

The economy is populated by people who live for two periods. In period t, N people are born and each of these people inelastically supplies G^t units of effective labor, earns wage income $W_t = (1-\alpha)(G^t)^{1-\alpha} K_t^\alpha N^{-\alpha} = (1-\alpha) G^t k_t^\alpha$ and receives a lump-sum transfer, τ_t.

To focus on the impact of government borrowing, we simplify other aspects of fiscal policy. Specifically, we assume that the government does not purchase goods or services and that all taxes and transfers are lump-sum. Let B_t be the amount of one-period government bonds outstanding at the beginning of period t, which are held by the old generation of consumers at time t. These bonds were bought at the end of the preceding period, when the currently-old consumers were young. Government budget accounting implies

$$B_{t+1} = (1 + r_{f,t}) B_t + D_t, \tag{2}$$

where D_t is the primary government budget deficit during period t and $r_{f,t}$ is the riskfree interest rate on government bonds bought at the end of period $t-1$ and maturing in period t. Define $g_{B,t}$ to be the growth rate of government bonds from the end of period $t-1$, when the amount of government bonds equals B_t, to the end of period t, when the amount of government bonds equals B_{t+1}, so $B_{t+1} = (1 + g_{B,t}) B_t$ and equation (2) can be rewritten as

$$(g_{B,t} - r_{f,t}) B_t = D_t. \tag{3}$$

When the primary deficit, D_t, is positive, the government acquires funds to pay for the primary deficit by issuing additional bonds in excess of the interest payments on existing bonds, which is a form of seignorage. The government uses its seignorage to pay for lump-sum transfer payments to young consumers or to pay for wasteful government purchases, or some mix of the two. In period t, total transfers to the N young consumers are $N\tau_t = \zeta D_t$, where $0 \leq \zeta \leq 1$ is the share of the primary deficit that is used to make transfer payments to young consumers and $(1-\zeta) D_t$ is spent wastefully.[15] Therefore, the lump-sum transfer received by each young consumer in period t, is $\tau_t = \zeta \frac{D_t}{N}$, where D_t is given by equation (3),

[15] If, instead of wasting $(1-\zeta) D_t$, the government purchases public goods that enter utility functions additively separably from consumption when young and when old, such purchases would not affect the equilibrium values of consumption, saving, or rates of return. However, any utility associated with public goods would need to be considered when analyzing the impact of government bonds on welfare in Section 6.

so

$$\tau_t = \zeta \left(g_{B,t} - r_{f,t} \right) \frac{B_t}{N}. \tag{4}$$

Young consumers in period t each consume c_t^y and save $s_t = W_t + \tau_t - c_t^y$. The aggregate saving of the young generation, $S_t \equiv N s_t$, is used to purchase assets, $A_{t+1} = K_{t+1} + B_{t+1}$, consisting of capital, K_{t+1}, and one-period riskfree government bonds, B_{t+1}. Thus, the aggregate capital stock in period $t+1$ is

$$K_{t+1} = A_{t+1} - B_{t+1} = S_t - B_{t+1}. \tag{5}$$

The rate of return on capital purchased at the end of period t and used in period $t+1$ is

$$r_{t+1} = \alpha k_{t+1}^{\alpha - 1} - \delta + \varepsilon_{t+1}, \tag{6}$$

which comprises the marginal product of capital in the production function in equation (1), $\alpha \left(G^{t+1} N \right)^{1-\alpha} K_{t+1}^{\alpha-1} = \alpha k_{t+1}^{\alpha-1}$, less the depreciation rate, $\delta - \varepsilon_{t+1}$.

At the end of period t, young consumers hold a fraction, λ_{t+1}, of their portfolios in riskfree government bonds with interest rate, $r_{f,t+1}$, and the remaining fraction, $1 - \lambda_{t+1}$, in risky capital with rate of return r_{t+1}. In the following period, when these consumers are old, they do not work. The generation of old consumers in period $t+1$ uses the gross return on total assets, $\left(1 + r_{a,t+1}\right) A_{t+1}$, to pay for its consumption, $N c_{t+1}^o$, where

$$r_{a,t+1} \equiv \lambda_{t+1} r_{f,t+1} + \left(1 - \lambda_{t+1}\right) r_{t+1} \tag{7}$$

is the rate of return on total assets.

Each person born in period t has an Epstein-Zin-Weil utility function with intertemporal elasticity of substitution (IES) equal to one. We use the specification for a consumer who lives for two periods that is used in the second example in Blanchard and Weil (2001)[16]

$$U_t = (1-\beta)\ln c_t^y + \beta \ln\left(\left[E_t\left\{\left(c_{t+1}^o\right)^{1-\gamma}\right\}\right]^{\frac{1}{1-\gamma}}\right), \text{ where } \gamma > 0. \tag{8}$$

To ensure a non-negative rate of time preference, we assume that $\beta \le \frac{1}{2}$.

To solve the consumption/saving problem, use $s_t = W_t + \tau_t - c_t^y$ and $c_{t+1}^o = (1 + r_{a,t+1})\, s_t$ in equation (8) to write the consumption/saving problem as

$$\max_{s_t} (1-\beta)\ln(W_t + \tau_t - s_t) + \beta \ln s_t + \frac{\beta}{1-\gamma}\ln E_t\left\{(1 + r_{a,t+1})^{1-\gamma}\right\}. \tag{9}$$

The joint impact of $IES = 1$ and the assumption that consumers do not earn wage income or receive transfers in the second period of life is that the optimal value of s_t is independent of $r_{a,t+1}$. The solution to the maximization problem in equation (9) is

$$s_t = \beta (W_t + \tau_t), \tag{10}$$

which implies

$$c_t^y = (1-\beta)(W_t + \tau_t) \tag{11}$$

and

$$c_{t+1}^o = (1 + r_{a,t+1})\,\beta (W_t + \tau_t). \tag{12}$$

Aggregate saving in period t is $S_t = Ns_t = N\beta(W_t + \tau_t) = \beta(NW_t + N\tau_t) = \beta[(1-\alpha)Y_t + \zeta(g_{B,t} - r_{f,t})B_t]$. Therefore, equation (5) implies that the aggregate capital stock in period

[16] If $\gamma = 1$, we treat the utility function in equation (8) as $U_t = (1-\beta)\ln c_t^y + \beta E_t\left\{\ln c_{t+1}^o\right\}$. Brumm et al. (2021) extend the two-period version of Epstein-Zin-Weil preferences used in Blanchard and Weil (2001), Blanchard (2019), and in this paper, so that the planner treats the uncertainty of the consumption of future young consumers in the same way that the consumers themselves treat uncertainty about consumption when old. In our model, the dynamic path of capital, hence wage income, hence consumption of the young, are all deterministic and therefore the planner's preferences used in Brumm et al. (2021) imply the same welfare criterion as we use.

$t+1$ is

$$K_{t+1} = S_t - B_{t+1} = \beta \left[(1-\alpha) Y_t + \zeta \left(g_{B,t} - r_{f,t} \right) B_t \right] - B_{t+1}. \tag{13}$$

Divide both sides of equation (13) by $G^{t+1}N$, define the bond-capital ratio, $\mathcal{B}_t \equiv \frac{B_t}{K_t}$, and rearrange the resulting equation to obtain

$$k_{t+1} = G^{-1}\beta \left[(1-\alpha) k_t^{\alpha} + \zeta \left(g_{B,t} - r_{f,t} \right) \mathcal{B}_t k_t \right] - \mathcal{B}_{t+1} k_{t+1}. \tag{14}$$

From this point onward, we focus on balanced growth paths along which K_t, Y_t, and B_t all grow at rate $g \geq 0$, so that $g_{B,t}$, $k_t \equiv \frac{K_t}{G^t N}$, $y_t \equiv \frac{Y_t}{G^t N}$, $b_t \equiv \frac{B_t}{G^t N}$ and $\mathcal{B}_t \equiv \frac{B_t}{K_t}$ are all constant, with values g, k, y, b, and $\mathcal{B}$, respectively. Also, since the durability shock is i.i.d., the riskfree interest rate is constant and equal to r_f. Throughout, we use the notational convention that variables without time subscripts represent constant values along balanced growth paths.

Equation (14) implies that along a balanced growth path, the marginal product of capital is

$$\alpha k^{\alpha-1} = \frac{\alpha}{(1-\alpha)\beta} \left[(1+\mathcal{B}) G - \beta\zeta \left(g - r_f \right) \mathcal{B} \right]. \tag{15}$$

Remarkably, the ratio of capital to effective labor, k, in equation (15) is constant despite the shocks to the durability of capital. In the case with $\zeta = 0$, the marginal product of capital along a balanced growth path is invariant to the distribution of the durability shock. However, if $\zeta \neq 0$, then the marginal product of capital in equation (15) depends on the riskfree interest rate, which, as we will see below, depends on the distribution of the durability shock.

Despite the fact that k_t is constant along a balanced growth path, the rate of return on capital is stochastic. Use equation (6), which implies that the rate of return on capital along a balanced growth path is $r = \alpha k^{\alpha-1} - \delta + \varepsilon$ (where ε is the random durability shock), and equation (15) to obtain

$$r = \frac{\alpha}{(1-\alpha)\beta}\left[(1+\mathcal{B})G - \beta\zeta(g-r_f)\mathcal{B}\right] - \delta + \varepsilon. \tag{16}$$

It will be convenient to use the ratios of the gross rates of return to the gross growth rate, G. Specifically, along a balanced growth path, $R \equiv \frac{1+r}{G}$ is the "adjusted gross rate of return" on capital, $R_f \equiv \frac{1+r_f}{G}$ is the "adjusted gross riskfree interest rate" and $R_a \equiv \frac{1+r_a}{G} = \lambda R_f + (1-\lambda)R$.

Use $G^{-1}(g-r_f) = G^{-1}(1+g-(1+r_f)) = 1-R_f$ and equation (16) to obtain

$$R \equiv \frac{1+r}{G} = \overline{R} + G^{-1}\varepsilon, \tag{17}$$

where

$$\overline{R} = \overline{R}(\mathcal{B}, R_f) \equiv \frac{\alpha}{(1-\alpha)\beta}\left[1+\mathcal{B} - \beta\zeta(1-R_f)\mathcal{B}\right] + (1-\delta)G^{-1}, \tag{18}$$

is the expected "adjusted gross rate of return on capital" along a balanced growth path. Note that $\frac{\partial\overline{R}(\mathcal{B},R_f)}{\partial\mathcal{B}} = \frac{\alpha}{(1-\alpha)\beta}\left[1-\beta\zeta+\beta\zeta R_f\right] > 0$ because $0 < \beta \leq \frac{1}{2}$ and $0 \leq \zeta \leq 1$; also $\frac{\partial\overline{R}(\mathcal{B},R_f)}{\partial R_f} = \frac{\alpha}{1-\alpha}\zeta\mathcal{B} \geq 0$.

4 Portfolio Allocation and Asset Pricing

Young consumers choose portfolios consisting of riskfree government bonds and risky capital. Formally, the optimal share of riskfree government bonds in a young consumer's portfolio is

$$\lambda_{t+1} = \arg\max_{\lambda_{t+1}} \frac{\beta}{1-\gamma}\ln E_t\left\{(R_{a,t+1})^{1-\gamma}\right\}, \tag{19}$$

where, as discussed earlier, $R_{a,t+1} = \lambda_{t+1}R_{f,t+1} + (1-\lambda_{t+1})R_{t+1}$. The first-order condition associated with this maximization problem along a balanced growth path is

$$E\left\{(\lambda R_f + (1-\lambda)R)^{-\gamma}(R_f - R)\right\} = 0. \tag{20}$$

Viewing R_f and the distribution of R as given, equation (20) determines the optimal value of λ. Alternatively, viewing λ and the distribution of R as given, equation (20) can be viewed as a financial market equilibrium condition that determines R_f as a function of the equilibrium value of λ and the distribution of R. In a financial market equilibrium with a given value of $\mathcal{B} \equiv \frac{B}{K}$, the share of the aggregate portfolio that is held in riskfree government bonds is $\lambda = \frac{B}{K+B} = \frac{\mathcal{B}}{1+\mathcal{B}}$ and the adjusted gross rate of return on total assets is $R_a = R_f + \frac{1}{1+\mathcal{B}}(R - R_f)$.

Lemma 1 *For any distribution of* $R > 0$, $R_f = \frac{E_t\left\{R_a^{1-\gamma}\right\}}{E_t\left\{R_a^{-\gamma}\right\}}$.

Since $R_a = R_f + \frac{1}{1+\mathcal{B}}(R - R_f)$, equations (17), (18), and (20) imply that for a given value of $\mathcal{B}$, the equilibrium value of R_f satisfies

$$f(\mathcal{B}, R_f) \equiv E\left\{\left(R_f + \frac{1}{1+\mathcal{B}}\left[\overline{R}(\mathcal{B}, R_f) + G^{-1}\varepsilon - R_f\right]\right)^{-\gamma}\left(\overline{R}(\mathcal{B}, R_f) + G^{-1}\varepsilon - R_f\right)\right\} = 0. \tag{21}$$

To analyze the general equilibrium relationship between $\mathcal{B}$ and R_f, it is useful to know the signs of the partial derivatives of $f(\mathcal{B}, R_f)$ with respect to $\mathcal{B}$ and R_f. As shown in Appendix A, it is straightforward to prove the following lemma.

Lemma 2 $\frac{\partial f(\mathcal{B}, R_f)}{\partial \mathcal{B}} > 0$.

The sign of $\frac{\partial f(\mathcal{B}, R_f)}{\partial R_f}$ is more nuanced. To analyze the sign of $\frac{\partial f(\mathcal{B}, R_f)}{\partial R_f}$, we define R_{inf}, as the infimum of the adjusted gross risky rate of return on capital, R, over all possible values of $\mathcal{B} \geq 0$ and ε.

Definition 1 $R_{\text{inf}} \equiv \inf_{\mathcal{B} \geq 0, \varepsilon} R$, *which equals* $\frac{\alpha}{(1-\alpha)\beta} + (1 - \delta + \inf \varepsilon) G^{-1} > 0$.

Along a balanced growth path, government bonds can be rolled over forever if and only if $R_f \leq 1$, so we focus our attention only on situations with $R_f \leq 1$. Provided that the distribution of ε is non-degenerate, the absence of arbitrage opportunities implies $R_{\text{inf}} < R_f$.

The following lemma presents a sufficient condition for $\frac{\partial f(\mathcal{B}, R_f)}{\partial R_f} < 0$.

Lemma 3 *If* $R_f \leq 1$ *and* $\gamma < \Lambda \equiv \frac{1-\frac{\zeta\beta}{1-\beta\zeta}(1-R_{\text{inf}})}{1+\frac{\alpha}{1-\alpha}\zeta}\frac{1}{1-R_{\text{inf}}}$, *then* $\frac{\partial f(\mathcal{B},R_f)}{\partial R_f} < 0$.

Five comments are in order. First, the upper bound Λ in this lemma is a function only of the parameters of the model. Second, since $0 \leq \zeta \leq 1$, $\beta \leq \frac{1}{2}$, and $R_{\text{inf}} > 0$, the upper bound, Λ, is positive. Third, the condition on γ in Lemma 3 is a sufficient, but not necessary, condition, so $\frac{\partial f(\mathcal{B},R_f)}{\partial R_f} < 0$ for a larger set of values of γ than specified in this lemma. Fourth, if the conditions in Lemma 3 hold, so that $\frac{\partial f(\mathcal{B},R_f)}{\partial R_f} < 0$ for all $R_f \in (0,1]$, then for any given $\mathcal{B}$, there is at most one value of $R_f \in (0,1]$ for which $f(\mathcal{B},R_f) = 0$; if such a value R_f exists, we denote it as $R_f(\mathcal{B})$. In particular, if $f(0,1) < 0$, then $R_f(0) < 1$; if $f(0,1) = 0$, then $R_f(0) = 1$; and if $f(0,1) > 0$, then $R_f(0) > 1$, in which case there is no value of $\mathcal{B} \geq 0$ for which $R_f(\mathcal{B}) = 1$.[17] Fifth, in the case in which $\zeta = 0$, the upper bound on γ becomes particularly simple: $\Lambda = \frac{1}{1-R_{\text{inf}}}$.

The following proposition is a direct consequence of Lemmas 2 and 3.

Proposition 1 *Whenever* $R_f \leq 1$ *and if* $\gamma < \Lambda$, *then* R_f *is an increasing function of* $\mathcal{B}$, *which we denote as* $R_f(\mathcal{B})$.

With the exception of Proposition 3, all propositions that follow assume that $R_f'(\mathcal{B}) > 0$ whenever $R_f \leq 1$. Proposition 1 provides a sufficient condition for this assumption to be true, namely, $\gamma < \Lambda$. Since this condition is sufficient, but not necessary, for the results of these later propositions, they potentially apply to a larger set of economies than the set of economies for which $\gamma < \Lambda$.

The fiscal authority can directly control the normalized value of government bonds outstanding, $b_t \equiv \frac{B_t}{G^t}$, which is constant along a balanced growth path. However, along a balanced growth path, we express the marginal product of capital in equation (15) and asset returns in equations (18) and Proposition 1 as functions of the bond-capital ratio, $\mathcal{B}$, and we will treat $\mathcal{B}$ as a choice variable of the fiscal authority.[18]

[17] Note that $f(0,0) = E\left\{R^{1-\gamma}\right\} > 0$ and that $f(0,x)$ is continuous in x. Therefore, since $\frac{\partial f(\mathcal{B},R_f)}{\partial R_f} < 0$ for $R_f \in (0,1]$, we can deduce that if $f(0,1) \leq 0$, there is a unique $R_f \in (0,1]$ such that $f(0,R_f) = 0$. Alternatively, since $\frac{\partial f(\mathcal{B},R_f)}{\partial R_f} < 0$ for all $R_f \in (0,1]$, if $f(0,1) > 0$, there is no value of $R_f \in (0,1]$ for which $f(\mathcal{B},R_f) = 0$ for any positive value of $\mathcal{B}$. That is, if $f(0,1) > 0$, then the equilibrium value of R_f must exceed one.

[18] To find the value of b that corresponds to a chosen value of $\mathcal{B}$, first use $f(\mathcal{B},R_f) = 0$ from equation (21)

The following proposition shows that when $R_f'(\mathcal{B}) > 0$, an increase in $\mathcal{B}$ reduces the value of k along a balanced growth path.

Proposition 2 *If $R_f'(\mathcal{B}) > 0$, then along a balanced growth path, $\frac{dk}{d\mathcal{B}} < 0$.*

5 Dynamic Efficiency and the Feasibility of Rollover

The link between dynamic inefficiency and the feasibility of rollover is particularly stark in deterministic economies because R_f, the adjusted gross riskfree interest rate equals R, the adjusted gross rate of return on capital. If $R_f = R < 1$, then the economy is dynamically inefficient and, since $R_f < 1$, government bonds can be rolled over forever; alternatively, if $R_f = R > 1$, then the economy is dynamically efficient and, since $R_f > 1$, government bonds cannot be rolled over forever.[19] However, in stochastic economies, R_f and R generally differ since R is stochastic. In such economies, there are parameter configurations for which $E\{\ln R\} > 0$, indicating that the economy is dynamically efficient and nevertheless, $R_f \leq 1$ so that at least a small amount of government bonds can be rolled over forever.

As we show in Section 6, the situation with $R_f = 1$ is particularly interesting. The following proposition presents a class of economies with dynamically efficient balanced growth paths along which $R_f = 1$ so that government bonds can be rolled over forever.

Proposition 3 *Assume that the distribution of ε is non-degenerate and that $(1-\delta+\varepsilon_{\inf})\,G^{-1} \geq 0$. Also, assume that $\zeta = 0$, $\gamma = 1$, and $\frac{1}{2} < \frac{\alpha}{(1-\alpha)\beta} < 1-(1-\delta)\,G^{-1}$. Then for some $\mathcal{B} > 0$ there is a balanced growth with $R_f = 1$ and $E\{\ln R\} > 0$ so that the economy is dynamically efficient.*

The purpose of Proposition 3 is to provide a completely specified example of a dynamically efficient economy in which $R_f = 1$ so that government bonds can be rolled over forever without primary surpluses. The assumptions that $\gamma = 1$ and $\zeta = 0$ are not necessary for

to express R_f as a function of $\mathcal{B}$, as in Proposition 1. Then use the values of $\mathcal{B}$ and R_f along with equation (15) to determine the value of k associated with the given value of $\mathcal{B}$. Finally, use the values of k and $\mathcal{B}$ to compute the normalized amount of bonds outstanding along a balanced growth path as $b = \mathcal{B}k$.

[19]In the borderline case in which $R_f = R = 1$, the economy is dynamically efficient and yet government bonds can be rolled over forever. This departure from the association between dynamic inefficiency and the feasibility of rolling over government bonds forever in deterministic economies is a knife-edge case.

this finding. The quantitative application in Section 7 presents examples, for various values of γ different than 1, and values of ζ equal to 0 and 1, of dynamically efficient balanced growth paths in which government bonds can be rolled over forever.

6 Maximum and Optimal Amounts of Sustainable Government Debt

In this section, we analyze two questions about sustainable levels of government debt. First we address a positive question: what is the maximum sustainable value of $\mathcal{B}$ along a balanced growth path? Then we address a normative question: what is the sustainable value of $\mathcal{B}$ that maximizes utility along a balanced growth path? Remarkably, we find that utility along a balanced growth path is maximized by the maximum sustainable value of $\mathcal{B}$.

Definition 2 *A constant value of $\mathcal{B}$ along a balanced growth path is sustainable if government debt can be rolled over forever at the riskfree interest rate without any primary budget surpluses in the future. Define $\mathcal{B}_{\max}$ as the maximum sustainable value of $\mathcal{B}$.*

Remark 1 *Along a balanced growth path with constant R_f, a constant value of $\mathcal{B}$ is sustainable if and only if $R_f \leq 1$.*

Proposition 4 *If $R_f'(\mathcal{B}) > 0$ whenever $R_f(\mathcal{B}) \leq 1$, then*

1. *if $R_f(0) \geq 1$, then $\mathcal{B}_{\max} = 0$.*
2. *if $R_f(0) < 1$, then*
   (a) *$\mathcal{B}_{\max}$ is the unique root of $R_f(\mathcal{B}) = 1$; equivalently, $f(\mathcal{B}_{\max}, 1) = 0$*
   (b) *$\mathcal{B}_{\max} \leq \widehat{\mathcal{B}} \equiv \max\left\{0, \frac{(1-\alpha)\beta}{\alpha} - 1\right\}$ is finite*
   (c) *$\mathcal{B}$ is sustainable if and only if $0 \leq \mathcal{B} \leq \mathcal{B}_{\max}$.*

Corollary 1 *If $R_f'(\mathcal{B}) > 0$ whenever $R_f(\mathcal{B}) \leq 1$, then $\mathcal{B}_{\max}$ is invariant to ζ.*

The parameter ζ appears only in the transfers to young consumers. Along a balanced growth path the normalized value of transfers is $\tau = \frac{\tau_t}{G^t} = \zeta\,(g - r_f)\,\frac{B}{G^t N} = \zeta\,(g - r_f)\,\mathcal{B}k$. Proposition 4 implies that either $\mathcal{B}_{\max} = 0$ or $g - r_f = 0$. In either case, transfers are zero, and hence ζ is irrelevant when $\mathcal{B} = \mathcal{B}_{\max}$.[20]

Proposition 4 shows that if $\mathcal{B}_{\max} > 0$, then $R_f = 1$ when $\mathcal{B} = \mathcal{B}_{\max}$. The following proposition states that if none of the seignorage revenue is wasted ($\zeta = 1$), then a marginal increase in $\mathcal{B}$ reduces expected aggregate consumption.

Proposition 5 *Assume that $\zeta = 1$ and suppose that $R_f'(\mathcal{B}) > 0$ whenever $R_f(\mathcal{B}) \leq 1$. Along a balanced growth path with $R_f = 1$, we have $\frac{dE\{c\}}{dB} < 0$, where $E\{c\}$ is the expected value of normalized aggregate consumption per effective unit of labor, $(C_t^y + C_t^o)/G^t$.*

Proposition 5 implies that in the left neighborhood of $\mathcal{B} = \mathcal{B}_{\max}$, a marginal increase in $\mathcal{B}$ reduces expected aggregate consumption (in the absence of waste). Nevertheless, as we show in the remainder of this section, an increase in $\mathcal{B}$ increases welfare whenever $\mathcal{B} < \mathcal{B}_{\max}$.

As in Blanchard (2019), Falkenheim (2022), Ball and Mankiw (2023) and Kocherlakota (2023b), our measure of welfare is the utility of consumers along a balanced growth path.

Definition 3 *Define $u_t \equiv U_t - t\ln G$, which is constant along a balanced growth path. The optimal sustainable value of $\mathcal{B}$ along any balanced growth path is* $\arg\max_{\mathcal{B}\in[0,\mathcal{B}_{max}]} u(\mathcal{B})$.

To evaluate utility along a balanced growth path for a given value of $\mathcal{B}$, define $w \equiv \frac{W_t}{G^t} = (1-\alpha)\,k^\alpha$ and use the expression for U_t in equation (8) along a balanced growth path to obtain[21]

$$u(\mathcal{B}) = \ln w + \ln\left(1 + \frac{\tau_t}{W_t}\right) + \frac{\beta}{1-\gamma}\ln E\left\{R_a^{1-\gamma}\right\} + constant. \tag{22}$$

The following definition will be useful in analyzing the impact of a change in the bond-capital ratio, $\mathcal{B}$, on $u(\mathcal{B})$.

[20] Note that $f(\mathcal{B}, R_f)$ in equation (21) depends on ζ only through $\overline{R}$ in equation (18). When $R_f = 1$, $\overline{R}(\mathcal{B}, R_f)$ is invariant to ζ and hence $f(\mathcal{B}, R_f)$ is invariant to ζ.

[21] Substitute $c_t^y = (1-\beta)(W_t + \tau_t)$ and $c_{t+1}^o = (1 + r_{a,t+1})\,\beta\,(W_t + \tau_t)$ into equation (8) to obtain $U_t = \ln W_t + \ln\left(1 + \frac{\tau_t}{W_t}\right) + \frac{\beta}{1-\gamma}\ln E_t\left\{(1 + r_{a,t+1})^{1-\gamma}\right\} + constant$. Subtract $\ln G^t$ from both sides of the resulting equation and use $u = U_t - \ln G^t$ and $\ln w = \ln W_t - \ln G^t$ to obtain equation (22).

Definition 4 $\Omega \equiv [1 + \mathcal{B} + \beta\zeta (R_f - 1) \mathcal{B}] G$.

To interpret Ω, note that equation (15) can be rewritten[22] as $\alpha k^{\alpha-1} = \frac{\alpha}{(1-\alpha)\beta}\Omega$, which implies that Ω is proportional to the marginal product of capital, $\alpha k^{\alpha-1}$.

Proposition 6 *Assume that $R_f'(\mathcal{B}) > 0$ whenever $R_f \leq 1$. If $R_f \leq 1$, then*

$$u'(\mathcal{B}) = \underset{(+)}{\frac{1}{\Omega}} \underset{(+)}{\frac{1}{R_f}} \left[\begin{array}{c} (1 - R_f) \underbrace{\left(1 - \beta\zeta \frac{\mathcal{B}}{1+\mathcal{B}}\right)}_{(+)} \left(\underbrace{\frac{\alpha}{1-\alpha} \frac{d\Omega}{d\mathcal{B}}}_{(+)} + \mathcal{B}\, G \underbrace{\beta \frac{dR_f}{d\mathcal{B}}}_{+} \right) \\ + \underbrace{\frac{1}{1+\mathcal{B}}}_{(+)} \beta\zeta G R_f (1 - R_f) + (1 - \zeta) \mathcal{B} \underbrace{R_f G \beta \frac{dR_f}{d\mathcal{B}}}_{(+)} \end{array} \right].$$

Therefore, if $R_f \leq 1$, then $u'(\mathcal{B}) \geq 0$ with strict inequality unless $R_f = 1$ and $(1 - \zeta) \mathcal{B} = 0$.

As shown in Lemma 8 in Appendix A, an increase in $\mathcal{B}$ has opposing effects on $\ln w$ and $\frac{\beta}{1-\gamma} \ln E_t \{R_a^{1-\gamma}\}$. An increase in $\mathcal{B}$ decreases the capital stock, thereby reducing $\ln w$ but increasing the marginal product of capital, which increases rates of return and hence increases $\frac{\beta}{1-\gamma} \ln E_t \{R_a^{1-\gamma}\}$. Despite these opposing effects, whenever $R_f \leq 1$, the impact of an increase in $\mathcal{B}$ on utility is non-negative, as stated in Proposition 6.

Corollary 2 below, which follows from Proposition 6, provides a simple expression for $u'(B)$ in the case with $\zeta = 0$. This expression allows us to show how the opposing effects of a decrease in the wage and an increase in the rate of return on capital are related through the factor price frontier in a way that the utility-increasing effect of the increase in rates of return dominates the utility-decreasing effect of the reduction in the wage.

Corollary 2 *Assume that $R_f'(\mathcal{B}) > 0$ whenever $R_f \leq 1$. If $\zeta = 0$ and $R_f \leq 1$, then*

$$u'(\mathcal{B}) = \frac{\beta}{1+\mathcal{B}} \frac{1}{R_f} \left[(1 - R_f) \frac{\alpha}{(1-\alpha)\beta} + \mathcal{B} \frac{dR_f}{d\mathcal{B}} \right] \geq 0,$$

with strict inequality unless $R_f = 1$ and $\mathcal{B} = 0$.

[22] Use the fact that $g - r_f = (1 + g) - (1 + r_f) = G - GR_f = G (1 - R_f)$.

Corollary 2 is proved formally in Appendix A. Here we provide a heuristic derivation of $u'(\mathcal{B})$ to illustrate how the welfare-decreasing impact of the decrease in wage income is dominated by the welfare-increasing impact of the increase in rates of return on capital and bonds, when $\zeta = 0$ and $R_f \leq 1$. To demonstrate that the main features of $u'(\mathcal{B})$ do not depend on Epstein-Zin-Weil utility with IES equal to one, here we consider an additively separable two-period utility function $u = u^y(c^y) + E\{u^o(c^o)\}$, where, for $i = y, o$, the utility function $u^i(c^i)$ is strictly increasing and strictly concave with $\lim_{c^i \to 0} u^{i\prime}(c^i) = \infty$ and $\lim_{c^i \to \infty} u^{i\prime}(c^i) = 0$. For simplicity, we also assume for the purposes of this heuristic derivation that $G = 1$.

The heuristic derivation proceeds in the three steps. First, for an individual consumer who has optimally chosen to consume c^y when young and c^o when old, the envelope theorem implies that the impact on utility of a change in $\mathcal{B}$ equals the change in utility if the consumer reduces c^y by the amount of lost wage income in the first period and increases c^o by the additional return on the $\frac{K}{N}$ units of capital and $\frac{B}{N}$ units of bonds held in the second period. That is,

$$u'(\mathcal{B}) = u^{y\prime}(c^y)\frac{dW}{d\mathcal{B}} + E\{u^{o\prime}(c^o)\}\left(\frac{K}{N}\frac{dR}{d\mathcal{B}} + \frac{B}{N}\frac{dR_f}{d\mathcal{B}}\right), \tag{23}$$

where, in addition to the envelope theorem, we have used the fact that $\frac{K}{N}\frac{dR}{d\mathcal{B}} + \frac{B}{N}\frac{dR_f}{d\mathcal{B}}$ is non-stochastic in our model.

Second, use the factor-price frontier, $N\frac{dF_N}{dK} + K\frac{dF_K}{dK} = 0$, which implies $\frac{dW}{d\mathcal{B}} = -\frac{K}{N}\frac{dR}{d\mathcal{B}}$, to obtain[23]

$$u'(\mathcal{B}) = -u^{y\prime}(c^y)\frac{K}{N}\frac{dR}{d\mathcal{B}} + E\{u^{o\prime}(c^o)\}\frac{K}{N}\frac{dR}{d\mathcal{B}} + E\{u^{o\prime}(c^o)\}\frac{B}{N}\frac{dR_f}{d\mathcal{B}}. \tag{24}$$

Interestingly, the first and second terms on the right hand side of equation (24) are both proportional to $\frac{K}{N}\frac{dR}{d\mathcal{B}}$, though the first term is multiplied by $-u^{y\prime}(c^y) < 0$ while the second term is multiplied by $E\{u^{o\prime}(c^o)\} > 0$.

[23] Since $G = 1$ and $F(K, N)$ is homogeneous of degree one in K and N, Euler's theorem implies $NF_N + KF_K = F$. Differentiating this equation with respect to K yields $N\frac{dF_N}{dK} + K\frac{dF_K}{dK} = 0$. Use $F_N = W$, $R = F_K + 1 - \delta + \varepsilon$, and hence $\frac{dR}{dK} = \frac{dF_K}{dK}$ to obtain $\frac{dW}{dK} = -\frac{K}{N}\frac{dR}{dK}$, and finally the chain rule to obtain $\frac{dW}{d\mathcal{B}} = \frac{dW}{dK}\frac{dK}{d\mathcal{B}} = -\frac{K}{N}\frac{dR}{dK}\frac{dK}{d\mathcal{B}} = -\frac{K}{N}\frac{dR}{d\mathcal{B}}$.

Third, to see which of the first two terms in equation (24) is dominant, use the first-order condition for the optimal intertemporal allocation of consumption, along with the fact that R_f is not stochastic, to obtain $u^{y\prime}(c^y) = R_f E\{u^{o\prime}(c^o)\}$, which implies

$$u'(\mathcal{B}) = u^{y\prime}(c^y)\left[\left(\frac{1}{R_f} - 1\right)\frac{K}{N}\frac{dR}{d\mathcal{B}} + \frac{1}{R_f}\frac{B}{N}\frac{dR_f}{d\mathcal{B}}\right]. \tag{25}$$

In situations in which government bonds can be rolled over forever, that is, when $R_f \leq 1$, we have $\left(\frac{1}{R_f} - 1\right)\frac{K}{N}\frac{dR}{d\mathcal{B}} \geq 0$, because $E\{u^{o\prime}(c^o)\} \geq u^{y\prime}(c^y)$ implies that the change in wage income in the first period, $\frac{dW}{d\mathcal{B}} = -\frac{K}{N}\frac{dR}{d\mathcal{B}}$, reduces u by no more than the equal-sized increase in income accruing to initial capital in the second period, $\frac{K}{N}\frac{dR}{d\mathcal{B}}$, increases u.

To express equation (25) in a form similar to the expression in Corollary 2, use (1) $G = 1$, which implies $k = \frac{K}{N}$, and (2) $\frac{dR}{d\mathcal{B}} = \frac{\alpha}{(1-\alpha)\beta}$ (from equations (17) and (18) with $\zeta = 0$), to obtain[24]

$$u'(\mathcal{B}) = u^{y\prime}(c^y)\, k\frac{1}{R_f}\left[(1 - R_f)\frac{\alpha}{(1-\alpha)\beta} + \mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]. \tag{26}$$

If the time separable utility function is logarithmic, specifically $u^y(c^y) = (1-\beta)\ln c^y$ and $u^o(c^o) = \beta \ln c^o$, so that $IES = 1$, the expression for $u'(\mathcal{B})$ in equation (26) is identical to the expression in Corollary 2, which was derived for Epstein-Zin-Weil preferences with $IES = 1$.[25]

Proposition 7 *If $R_f'(\mathcal{B}) > 0$ whenever $R_f(\mathcal{B}) \leq 1$, then* $\arg\max_{\mathcal{B}\in[0,\mathcal{B}_{max}]} u(\mathcal{B})$ *equals* $\mathcal{B}_{\max}$*, that is, utility per effective unit of labor along a balanced growth path is maximized by the maximum sustainable value of* $\mathcal{B}$*.*

[24] Equation (26) states that the sign of $u'(\mathcal{B})$ depends on R_f, but not on R. In contrast, Blanchard (2019) shows that the sign of the impact of an increase in debt on welfare depends on both the riskfree rate and the risky rate of return on capital. This difference arises because in our model, the effect of a change in the capital stock on the marginal product of capital (the second derivative of the production function with respect to K) is non-stochastic. In addition to making the sign of $u'(\mathcal{B})$ depend only on the sign of $1 - R_f$, this feature allows us to derive exact expressions for the marginal impact on welfare of an increase in $\mathcal{B}$, without relying on approximations.

[25] To prove this statement, it suffices to prove that $\frac{\beta}{1+\mathcal{B}} = u^{y\prime}(c^y)\, k = \frac{1-\beta}{c^y}k$. Since $\zeta = 0$ and $G = 1$, $c^y = (1-\beta)\, w$ and hence $\frac{1-\beta}{c^y} = \frac{1}{w}$; also $\beta N w = K + B = (1+\mathcal{B})\, K$, which implies $\beta w = (1+\mathcal{B})\, k$ and hence $k = \frac{\beta w}{1+\mathcal{B}}$. Therefore, $\frac{1-\beta}{c^y}k = \frac{1}{w}\frac{\beta w}{1+\mathcal{B}} = \frac{\beta}{1+\mathcal{B}}$, which, as stated at the beginning of this footnote, suffices to prove the statement.

Since the proof of Proposition 7 is both simple and instructive, we present it here. If $R_f(0) \geq 1$, then $\mathcal{B}_{\max} = 0$ and the closed interval $[0, \mathcal{B}_{\max}]$ is a singleton so $\arg\max_{[0,\mathcal{B}_{\max}]} u(\mathcal{B})$ equals $\mathcal{B}_{\max} = 0$. Alternatively, if $R_f(0) < 1$, then $\mathcal{B}_{\max} > 0$ and Statement 2c of Proposition 4 implies all $\mathcal{B}$ in $[0, \mathcal{B}_{\max}]$ are sustainable. Proposition 6 implies that $u'(\mathcal{B}) > 0$ for all $\mathcal{B} \in [0, \mathcal{B}_{\max})$, so $\arg\max_{\mathcal{B}\in[0,\mathcal{B}_{\max}]} u(\mathcal{B})$ is a corner solution with $\mathcal{B} = \mathcal{B}_{\max}$.

At the corner solution, where $\mathcal{B} = \mathcal{B}_{\max}$, we have $R_f = 1$ and therefore when $\zeta = 1$, Proposition 5 implies $\frac{dE\{c\}}{d\mathcal{B}} < 0$. This observation highlights an important difference between deterministic economies and stochastic economies. In deterministic economies, government bonds can be rolled over forever without primary surpluses only if the economy is dynamically inefficient, so the reduction of capital induced by an increase in government bonds actually increases aggregate consumption. However, in stochastic economies, government bonds can be rolled over forever without primary surpluses in some dynamically efficient economies. In those cases, the decrease in capital reduces expected aggregate consumption, which would decrease welfare. But the additional risk sharing enabled by an increase in government bonds increases welfare. Remarkably, the increase in welfare resulting from increased risk sharing offsets the decrease in welfare from reduced expected aggregate consumption when $\mathcal{B} = \mathcal{B}_{\max}$. Therefore, the marginal impact on welfare of an increase in government bonds is zero if $\zeta = 1$; however, as discussed immediately below, if $0 \leq \zeta < 1$, so that the government wastes some of its seignorage revenue, the marginal impact on welfare is positive.

Corollary 1, which states that $\mathcal{B}_{\max}$ is invariant to ζ, implies that the main result of the paper, Proposition 7, does not depend on whether the government uses its seignorage to make transfers to young consumers or simply wastes these resources. However, the value of ζ determines whether $u'(\mathcal{B}_{\max})$ is positive or zero. Specifically, Proposition 6 implies that if $\mathcal{B}_{\max} > 0$, then $u'(\mathcal{B}_{\max}) = \frac{1}{\Omega}(1-\zeta)\mathcal{B}_{\max} G\beta \frac{dR_f}{d\mathcal{B}}$, which is positive if $0 \leq \zeta < 1$ and is zero if $\zeta = 1$. Recall that in period t the amount of seignorage is $(g - r_f) B_t$ and that a fraction $1 - \zeta$ of this seignorage is wasted by the government. Therefore[26], $\kappa(\mathcal{B}) \equiv (1-\zeta)[1 - R_f(\mathcal{B})]\mathcal{B}Gk$ is the amount of seignorage wasted per unit of effective labor along a balanced growth path. Differentiating $\kappa(\mathcal{B})$ with respect to $\mathcal{B}$ and evalu-

[26]Dividing the wasted resources $(1-\zeta)(g - r_f) B_t = (1-\zeta) G (1 - R_f) B_t$ by the effective units of labor $G^t N$ and using $\mathcal{B} = \frac{B_t}{K_t}$, the amount wasted resources per unit of effective labor along a balanced growth path is $\kappa(\mathcal{B}) \equiv (1-\zeta)[1 - R_f(\mathcal{B})]\mathcal{B}Gk$.

ating this derivative at $\mathcal{B} = \mathcal{B}_{\max}$ (where $R_f = 1$) yields $\kappa'(\mathcal{B}) = -(1-\zeta)\mathcal{B}_{\max}Gk\frac{dR_f}{d\mathcal{B}}$, so $u'(\mathcal{B}_{\max}) = \frac{\beta}{\Omega k}[-\kappa'(\mathcal{B})]$. Therefore, for $\mathcal{B}$ in the neighborhood just below $\mathcal{B}_{\max}$, the (approximate) marginal utility associated with an increase in $\mathcal{B}$, $u'(\mathcal{B}_{\max})$, is proportional to the reduction in waste associated with an increase in $\mathcal{B}_{\max}$, $\frac{\beta}{\Omega k}[-\kappa(\mathcal{B})]$. Provided that $0 \leq \zeta < 1$, both the marginal utility, $u'(\mathcal{B}_{\max})$, and the reduction in waste, $-\kappa'(\mathcal{B})$, are positive. However, if $\zeta = 1$, there is no waste of resources and both $u'(\mathcal{B}_{\max})$ and $\kappa'(\mathcal{B}_{\max})$ equal zero. Regardless of the presence of waste ($0 \leq \zeta < 1$) or the absence of waste ($\zeta = 1$), the optimal sustainable value of $\mathcal{B}$ is $\mathcal{B}_{\max}$.

7 Quantitative Application

In this section, we provide a quantitative illustration of (a) $\mathcal{B}_{\max}$, which is the maximum sustainable value of $\mathcal{B}$ as well as the welfare-maximizing value of $\mathcal{B}$, and (b) the value of $\mathcal{B}$, denoted $\mathcal{B}^*$, for which $E\{\ln R\} = 0$. Any value of $\mathcal{B}$ greater than or equal to $\mathcal{B}^*$ will lead to a dynamically efficient balanced growth path.

We set the capital share α equal to 0.33. We interpret a "period" as 30 years and set $\beta = 0.353$, so that the annualized discount factor, $\left(\frac{\beta}{1-\beta}\right)^{1/30}$, is 0.98 per year, which implies an annual discount rate of 2% per year. We assume that labor-augmenting productivity grows at the rate of 1% per year, so $G = 1.35$. For risk aversion we consider $\gamma = 1, 3, 8$, and 10. Finally, we model the durability shock, ε, as a lognormal variable minus a constant, and choose the parameters so that when the ratio of government bonds to the capital stock, $\mathcal{B}$, equals 0.5, $E\{(1+r)\} = E\{GR\}$ matches a target value of $(1+m)^{30}$, where $m = 0.03$ is an annualized rate of return on unlevered equity and $sd\{(1+r)\} = sd\{GR\}$ matches a target value of $s\sqrt{30}$, where s is an annualized standard deviation of the rate of return on unlevered equity. Further details of the calibration, including a discussion of the implied mean and standard deviation of the rate of return on levered equity, are contained in Online Appendix C.

Tables 1 and 2 report for $\zeta = 0$ and $\zeta = 1$, respectively, $\mathcal{B}_{max}$ and $\mathcal{B}^*$. In both tables, some of the cells for low values of s are blank. In the section of each table that reports $\mathcal{B}_{max}$, a blank indicates that $R_f > 1$ for all nonnegative values of $\mathcal{B}$; thus, there is no positive value

$\zeta = 0$								
	$\mathcal{B}_{\max}$				$\mathcal{B}^*$			
	$\gamma = 1$	$\gamma = 3$	$\gamma = 8$	$\gamma = 10$	$\gamma = 1$	$\gamma = 3$	$\gamma = 8$	$\gamma = 10$
$s = 0.02$								
$s = 0.04$			0.042	0.062				
$s = 0.06$		0.026	0.122	0.150				
$s = 0.08$		0.075	0.193	0.224				
$s = 0.10$	0.017	0.120	0.252	0.285				
$s = 0.12$	0.043	0.162	0.301	0.334				
$s = 0.14$	0.068	0.199	0.342	0.374	0.013	0.013	0.013	0.013
$s = 0.16$	0.091	0.232	0.376	0.408	0.031	0.031	0.031	0.031
$s = 0.18$	0.113	0.261	0.405	0.435	0.049	0.049	0.049	0.049
$s = 0.20$	0.133	0.286	0.429	0.459	0.066	0.066	0.066	0.066
$s = 0.22$	0.152	0.309	0.450	0.478	0.083	0.083	0.083	0.083

Table 1: $\mathcal{B}_{\max}$ is the maximum sustainable, as well as optimal, value of $\mathcal{B}$. $\mathcal{B}^*$ is the value of $\mathcal{B}$ for which $E\{\ln R\} = 0$. γ denotes risk aversion, and s is the annualized standard deviation of the return on capital in an economy with $\mathcal{B} = 0.5$. ζ is set to zero.

$\zeta = 1$								
	$\mathcal{B}_{\max}$				$\mathcal{B}^*$			
	$\gamma = 1$	$\gamma = 3$	$\gamma = 8$	$\gamma = 10$	$\gamma = 1$	$\gamma = 3$	$\gamma = 8$	$\gamma = 10$
$s = 0.02$								
$s = 0.04$			0.042	0.062				
$s = 0.06$		0.026	0.122	0.150				
$s = 0.08$		0.075	0.193	0.224				
$s = 0.10$	0.017	0.120	0.252	0.285				
$s = 0.12$	0.043	0.162	0.301	0.334				
$s = 0.14$	0.068	0.199	0.342	0.374	0.013	0.015	0.018	0.018
$s = 0.16$	0.091	0.232	0.376	0.408	0.032	0.037	0.043	0.044
$s = 0.18$	0.113	0.261	0.405	0.435	0.051	0.059	0.068	0.069
$s = 0.20$	0.133	0.286	0.429	0.459	0.069	0.080	0.091	0.092
$s = 0.22$	0.152	0.309	0.450	0.478	0.087	0.100	0.112	0.114

Table 2: $\mathcal{B}_{\max}$ is the maximum sustainable, as well as optimal, value of $\mathcal{B}$. $\mathcal{B}^*$ is the value of $\mathcal{B}$ for which $E\{\ln R\} = 0$. γ denotes risk aversion, and s is the annualized standard deviation of the return on capital in an economy with $\mathcal{B} = 0.5$. ζ is set to one.

of $\mathcal{B}$ that can be rolled over forever. In the section of each table that reports $\mathcal{B}^*$, a blank indicates that $E\{\ln R\} > 0$ so the economy is dynamically efficient for all non-negative values of $\mathcal{B}$. In all cases in which $\mathcal{B}^* > 0, \mathcal{B}_{\max} > \mathcal{B}^*$. For values of $\mathcal{B}$ in the interval $[\mathcal{B}^*, \mathcal{B}_{\max}]$, the economy is dynamically efficient and $R_f \leq 1$ so that government bonds can be rolled over forever. Both tables show that as risk aversion increases, there is a significant gap between $\mathcal{B}_{max}$ and $\mathcal{B}^*$, so there is a non-trivial set of parameter values for which government bonds can be rolled over forever in dynamically efficient economies. For instance, in Table 1 when $\gamma = 10$ and $s = 0.22$, $\mathcal{B}^* = 0.083$, while $\mathcal{B}_{max} = 0.478$. Therefore, for any value of $\mathcal{B}$ between 0.083 and 0.478, the balanced growth path is dynamically efficient (because $\mathcal{B}$ exceeds 0.083) and $R_f < 1$ (because $\mathcal{B}$ is less than 0.478) so government bonds can be rolled over forever.

To interpret the magnitude of the values of $\mathcal{B}$ in Tables 1 and 2, recall that empirically the level of government debt is often expressed as a multiple of GDP, while the values of $\mathcal{B}$ are expressed as multiples of the capital stock. For an economy in which the capital-output ratio is 2, the debt-GDP ratio is twice as high as the debt-capital ratio, $\mathcal{B}$. In such an economy, the values of $\mathcal{B}$ in Tables 1 and 2, which range from 0 to 0.478, correspond to debt-GDP ratios ranging from 0 to 0.956.

Comparison of Tables 1 and 2 shows the impact of ζ. Overall, the tables show that the impact of ζ on $\mathcal{B}^*$ is quantitatively small and, as stated in Corollary 1, ζ is completely irrelevant for the determination of $\mathcal{B}_{max}$, which is the welfare-maximizing sustainable value of $\mathcal{B}$.

Now we look at the tables in more detail. First consider $\mathcal{B}_{max}$, which is both the maximum and the optimal sustainable value of $\mathcal{B}$. Since $R_f = 1$ when $\mathcal{B} = \mathcal{B}_{max}$, the transfer to young consumers, $\zeta (g - r_f) \mathcal{B} k = \zeta (1 - R_f) G \mathcal{B} k$, equals zero and hence the expected adjusted gross rate of return on capital in equation (18) becomes $\overline{R} = \frac{\alpha}{(1-\alpha)\beta} (1 + \mathcal{B}) + (1 - \delta) G^{-1}$, which is independent of risk aversion, γ, and the volatility parameter, s. An increase in γ or an increase in s increases the risk premium on capital relative to the riskfree rate. With an unchanged $\overline{R}$, the increased risk premium implies that R_f falls. To maintain $R_f = 1$, the value of $\mathcal{B}$ must increase to reduce capital, thereby increasing $\overline{R}$ and R_f. Therefore, moving rightward in each row in the $\mathcal{B}_{max}$ section of each table, γ increases and hence the maximum sustainable $\mathcal{B}$ increases; similarly, moving down each column in this section of these tables, s increases and the maximum sustainable $\mathcal{B}$ increases. As implied by Corollary 1, the sections of Tables 1 and 2 that present $\mathcal{B}_{max}$ are identical to each other.

Now, consider $\mathcal{B}^*$, which is characterized by $E\left\{\ln(\overline{R}+\varepsilon)\right\}=0$. In Table 1, $\zeta=0$, so that the transfer to young consumers, $\zeta\left(g-r_f\right)\mathcal{B}k$, is zero. Hence, as discussed above, equation (18) implies that $\overline{R}=\frac{\alpha}{(1-\alpha)\beta}\left(1+\mathcal{B}\right)+\left(1-\delta\right)G^{-1}$, which is independent of γ and s. Therefore, $E\left\{\ln(\overline{R}+\varepsilon)\right\}$ is independent of γ, so $\mathcal{B}^*$ is independent of γ for given s. By contrast, although an increase in s also has no effect on $\overline{R}$, it reduces $E\left\{\ln(\overline{R}+\varepsilon)\right\}$ because $\ln\left(\cdot\right)$ is a strictly concave function. In order to restore $E\left\{\ln(\overline{R}+\varepsilon)\right\}=0$ when s is increased, $\overline{R}$ must increase, so $\mathcal{B}$ must increase to reduce capital and increase the marginal product of capital. This effect is illustrated in Table 1 by the increasing values of $\mathcal{B}^*$ as one goes down each column in the section of the table devoted to $\mathcal{B}^*$.

In Table 2, where $\zeta=1$, equation (18) implies that $\overline{R}=\frac{\alpha}{(1-\alpha)\beta}\left[1+\mathcal{B}+\beta\left(R_f-1\right)\mathcal{B}\right]+\left(1-\delta\right)G^{-1}$. Thus, with $\zeta=1$, $\overline{R}$ is an increasing function of R_f, which is a decreasing function of γ and s, since both a higher coefficient of relative risk aversion, γ, and higher volatility, s, lead consumers to seek safety in riskfree government bonds, thereby driving R_f downward. Thus, for a given value of s, an increase in γ reduces R_f and hence reduces $\overline{R}$, so to maintain $E\left\{\ln(\overline{R}+\varepsilon)\right\}=0$, $\mathcal{B}$ must increase to increase $\overline{R}$. Alternatively, for given values of γ, an increase in s reduces $E\left\{\ln(\overline{R}+\varepsilon)\right\}$ through two channels. First, the concavity of $\ln\left(\cdot\right)$ implies that for given $\overline{R}$, an increase in s reduces $E\left\{\ln(\overline{R}+\varepsilon)\right\}$. Second, as discussed above, an increase in s reduces R_f and hence reduces $\overline{R}$. To maintain $E\left\{\ln(\overline{R}+\varepsilon)\right\}=0$, $\mathcal{B}$ must increase to increase $\overline{R}$. Finally, note that the nonzero entries for $\mathcal{B}^*$ in Table 2 are higher than in Table 1 because the positive transfers to consumers when $\zeta=1$ in Table 2 increase saving, thereby increasing the capacity of saving to absorb bonds without driving the capital stock low enough to increase $E\left\{\ln(\overline{R}+\varepsilon)\right\}$ above zero.

8 Concluding Remarks

In this paper, we develop an overlapping-generations model to analyze sustainable levels of the ratio of government bonds to the capital stock that can be maintained forever without any future primary government surpluses. To make the analysis easily tractable, the model confines exogenous shocks to the depreciation rate of capital, which is additively separable from the production function so the labor income of young consumers is non-stochastic (though Online Appendix B shows that this model is isomorphic to a model with shocks to the production function). In addition, since (1) consumers have Epstein-Zin-Weil utility functions over their two-period lifetimes with the intertemporal elasticity of substitution set equal to one, and (2) consumers earn labor income (and possibly receive transfers) only in the first period of life, aggregate saving of young consumers is a constant fraction of their income in the first period of life, and hence the evolution of aggregate asset holdings, comprising capital and government bonds, is non-stochastic. Nevertheless, the rate of return on capital is stochastic because it includes the stochastic depreciation rate. Along a balanced growth path, aggregate wage income, the aggregate capital stock, and the aggregate amount of government bonds outstanding all grow at rate g, which is the constant rate of labor-augmenting productivity growth. Therefore, the balanced growth path features constant values of aggregate capital per unit of effective labor, the bond-capital ratio, the riskfree interest rate, and the expected rate of return on capital. Provided that the net riskfree interest rate, r_f, is less than or equal to g along a balanced growth path, the bond-capital ratio is sustainable.

Our model is designed so that both r_f and g are constant along a balanced growth path. Because we are interested in sustainable bond-capital ratios along a balanced growth path, we confine attention to balanced growth paths that feature $r_f \leq g$. Along such paths, the government can roll over its bonds forever without primary budget surpluses and without the bond-capital ratio increasing. There is no chance that young consumers will be unwilling or unable to purchase the bonds that the government issues to rollover its debt, so there is no chance of default on government bonds. Therefore, the market interest rate on government bonds equals the riskfree interest rate. In addition, when $r_f \leq g$, the value of government bonds at a given point in time is <u>not</u> the expected present value of future primary surpluses; along balanced growth paths with $r_f \leq g$, the value of outstanding government bonds is positive and yet all future primary deficits, which equal $(g - r_f)B_t$ at time t, are non-negative and hence all future primary surpluses are non-positive.

This paper has two major findings—one positive and one normative. We focus on levels of the bond-capital ratio that can be sustained forever without any future primary surpluses. The positive finding is that the maximum sustainable bond-capital ratio along a balanced growth path is attained when $r_f = g$. Given that both r_f and g are constant along balanced growth paths, this finding is not surprising. However, the normative finding is surprising (to us, at least). The sustainable bond-capital ratio that maximizes utility along a balanced growth path is the maximum sustainable value of this ratio, that is, the ratio that attains $r_f = g$. Briefly, an increase in the amount of bonds outstanding reduces capital, which reduces aggregate wage income and increases the marginal product of capital and hence increases the rate of return on capital. The reduction in wage income reduces welfare and the increase in the rate of return on capital increases welfare. It follows from the factor-price frontier that the reduction in wage income equals the increase in income accruing to initial capital. Whenever $r_f \leq g$, the welfare-increasing impact of the increased rate of return on capital, which occurs in the second period of life, dominates the welfare-decreasing impact of the reduction in wage income, which occurs during the first period of life. Thus, starting from a balanced growth path with $r_f < g$, an increase in the bond-capital ratio leads to a different balanced growth path with a higher level of welfare. Focusing on long-run welfare along balanced growth paths, it is optimal to increase the bond-capital ratio, and thereby increase r_f, until $r_f = g$, at which point the bond-capital ratio equals its maximal sustainable level.

We designed the model to have both constant r_f and g along balanced growth paths so that the assessment of the sustainability of a given bond-capital ratio would be as straightforward as possible. But what if, for instance, the growth rate of labor-augmenting productivity, g, were random. In particular, what if $R_f \equiv \frac{1+r_f}{1+g}$ were random, sometimes greater than one and sometimes less than one? Then there might be some realization paths with R_f persistently greater than one, so that the stock of government bonds eventually exceeds the amount that young households would or could purchase. In such a framework, the notion of sustainability is more nuanced (which is why we designed the model to include constant r_f and g). We leave it as an open question how to characterize sustainability in that framework, and, in particular, how to characterize an appropriate notion of maximal borrowing. If there is some suitable notion of maximal borrowing, does the normative result, that the optimal government borrowing policy is the same as the maximal borrowing policy, generalize beyond the model in this paper? This generalization of the primary normative result would likely hold if, as in the current paper, whenever $r_f < g$, the welfare-increasing effect of an increased rate of return to capital exceeds the welfare-reducing effect of a reduced wage income for any sustainable borrowing policy.

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A Proofs

Proof of Lemma 1. The first-order condition for λ in equation (20) implies $0 = E_t\left\{\frac{(1-\lambda)(R-R_f)}{(\lambda R_f+(1-\lambda)R)^\gamma}\right\} = E_t\left\{\frac{\lambda R_f+(1-\lambda)R-R_f}{(\lambda R_f+(1-\lambda)R)^\gamma}\right\} = E_t\left\{\frac{R_a-R_f}{R_a^\gamma}\right\} = E_t\left\{R_a^{1-\gamma}\right\} - R_f E_t\left\{R_a^{-\gamma}\right\}$, which implies $R_f = \frac{E_t\left\{R_a^{1-\gamma}\right\}}{E_t\left\{R_a^{-\gamma}\right\}}$. ∎

To prove Lemmas 2 and 3, which are used to prove Proposition 1, we present the following two new Lemmas and a new corollary.

Lemma 4 *If $R_f \leq 1$, then $\mathcal{B}\left(R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}} - 1\right) \leq 1 - R_{\inf}$.*

Proof of Lemma 4. Assume that $R_f \leq 1$. Define $\mathcal{A} \equiv \frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}} = \left[\frac{E\left\{R_a^{-\gamma}\right\}}{E\left\{R_a^{-\gamma-1}\right\}}\right]^{-1} = \left[E\left\{\frac{R_a^{-\gamma-1}}{E\left\{R_a^{-\gamma-1}\right\}}R_a\right\}\right]^{-1}$, so $\mathcal{A}^{-1} = E\left\{\frac{R_a^{-\gamma-1}}{E\left\{R_a^{-\gamma-1}\right\}}R_a\right\}$ is a weighted average of R_a and hence exceeds $\inf_\varepsilon R_a = \frac{\mathcal{B}}{1+\mathcal{B}}R_f + \frac{1}{1+\mathcal{B}}R_{\inf}$. Therefore, $\mathcal{A} < \frac{1}{\frac{\mathcal{B}}{1+\mathcal{B}}R_f+\frac{1}{1+\mathcal{B}}R_{\inf}}$ and $R_f\mathcal{A} < \frac{R_f}{\frac{\mathcal{B}}{1+\mathcal{B}}R_f+\frac{1}{1+\mathcal{B}}R_{\inf}}$, which is increasing in R_f. Since $R_f \leq 1$, we have $R_f\mathcal{A} < \frac{1}{\frac{\mathcal{B}}{1+\mathcal{B}}+\frac{1}{1+\mathcal{B}}R_{\inf}} = \frac{1+\mathcal{B}}{\mathcal{B}+R_{\inf}}$. Therefore, $\mathcal{B}\left(R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}} - 1\right) = \mathcal{B}\left(R_f\mathcal{A}-1\right) \leq \mathcal{B}\left(\frac{1+\mathcal{B}}{\mathcal{B}+R_{\inf}} - 1\right) = \mathcal{B}\left(\frac{1-R_{\inf}}{\mathcal{B}+R_{\inf}}\right) = \frac{\mathcal{B}}{\mathcal{B}+R_{\inf}}\left(1-R_{\inf}\right) \leq 1 - R_{\inf}$. ∎

Lemma 5 *Suppose $x > 0$ is a non-degenerate random variable with finite moments. Define $Z \equiv \left[E\left\{x^{-\gamma}\right\}\right]^2 - E\left\{x^{-\gamma-1}\right\}E\left\{x^{1-\gamma}\right\}$. Then $Z < 0$.*

Proof of Lemma 5. Rewrite Z as $Z = E\left\{x^{-\gamma}\right\}E\left\{x^{-\gamma-1}\right\}\left(\frac{E\left\{x^{-\gamma}\right\}}{E\{x^{-\gamma-1}\}} - \frac{E\left\{x^{1-\gamma}\right\}}{E\{x^{-\gamma}\}}\right)$ Observe that Z has the same sign as $\frac{E\left\{x^{-\gamma-1}x\right\}}{E\{x^{-\gamma-1}\}} - \frac{E\left\{x^{-\gamma}x\right\}}{E\{x^{-\gamma}\}}$, which we write as $E\left\{g(x)\,x\right\} - E\left\{h(x)\,x\right\}$,

where $g(x) \equiv \frac{x^{-\gamma-1}}{E\{x^{-\gamma-1}\}}$ and $h(x) \equiv \frac{x^{-\gamma}}{E\{x^{-\gamma}\}}$. Observe that $E\{g(x)\} = 1 = E\{h(x)\}$ and $\frac{g(x)}{h(x)} = \frac{E\left\{x^{-\gamma}\right\}}{E\{x^{-\gamma-1}\}}\frac{1}{x} = A\frac{1}{x}$, so $g(x) = A\frac{1}{x}h(x)$ where $A \equiv \frac{E\left\{x^{-\gamma}\right\}}{E\{x^{-\gamma-1}\}} > 0$, which implies $g(x) \gtreqless h(x)$ as $x \lesseqgtr A$. Therefore, $E\{g(x)x\} - E\{h(x)x\} = \int_0^\infty [g(x) - h(x)]\, x dF(x) = \int_0^\infty [g(x) - h(x)](x - A)\, dF(x) = \int_0^A [g(x) - h(x)](x - A)\, dF(x) +$
$\int_A^\infty [g(x) - h(x)](x - A)\, dF(x) < 0.$ ■

We present, without proof, the following corollary to Lemma 1, which follows immediately from Lemma 5.

Corollary 3 $E\{R_a^{-\gamma}\} - R_f E\{R_a^{-\gamma-1}\} = \frac{1}{E\left\{R_a^{-\gamma}\right\}}\left[(E\{R_a^{-\gamma}\})^2 - E\{R_a^{1-\gamma}\}\{R_a^{-\gamma-1}\}\right] < 0.$

Proof of Lemma 2. Partially differentiate

$$f(\mathcal{B}, R_f) \equiv E\left\{\left(R_f + \frac{1}{1+\mathcal{B}}\left[\overline{R}(\mathcal{B}, R_f) + G^{-1}\varepsilon - R_f\right]\right)^{-\gamma}\left(\overline{R}(\mathcal{B}, R_f) + G^{-1}\varepsilon - R_f\right)\right\}$$

in equation (21) with respect to $\mathcal{B}$, and use $R = \overline{R}(\mathcal{B}, R_f) + G^{-1}\varepsilon$ and $R_a = R_f + \frac{1}{1+\mathcal{B}}\left[\overline{R}(\mathcal{B}, R_f) + G^{-1}\varepsilon - R_f\right]$ to obtain

$$\frac{\partial f(\mathcal{B}, R_f)}{\partial \mathcal{B}} = -\gamma\frac{-1}{(1+\mathcal{B})^2}E\left\{R_a^{-\gamma-1}(R - R_f)^2\right\} - \gamma\frac{1}{1+\mathcal{B}}E\left\{R_a^{-\gamma-1}\left(\frac{\partial\overline{R}(\mathcal{B}, R_f)}{\partial\mathcal{B}}\right)(R - R_f)\right\}$$
$$+ E\left\{R_a^{-\gamma}\frac{\partial\overline{R}(\mathcal{B}, R_f)}{\partial\mathcal{B}}\right\}.$$

Now use the fact that $\frac{\partial\overline{R}(\mathcal{B},R_f)}{\partial\mathcal{B}} = \frac{\alpha}{(1-\alpha)\beta}[1 - \beta\zeta + \beta\zeta R_f]$ is non-stochastic to obtain $\frac{\partial f(\mathcal{B},R_f)}{\partial\mathcal{B}}$ $= \gamma\frac{1}{(1+\mathcal{B})^2}E\left\{R_a^{-\gamma-1}(R - R_f)^2\right\} + \frac{\partial\overline{R}(\mathcal{B},R_f)}{\partial\mathcal{B}}\left[-\gamma E\left\{R_a^{-\gamma-1}\frac{1}{1+\mathcal{B}}(R - R_f)\right\} + E\{R_a^{-\gamma}\}\right]$ and use $R_a - R_f = \frac{1}{1+\mathcal{B}}(R - R_f)$ to obtain $\frac{\partial f(\mathcal{B},R_f)}{\partial\mathcal{B}} = \frac{\gamma}{(1+\mathcal{B})^2}E\left\{R_a^{-\gamma-1}(R - R_f)^2\right\}$
$+\frac{\partial\overline{R}(\mathcal{B},R_f)}{\partial\mathcal{B}}\left[-\gamma E\{R_a^{-\gamma-1}(R_a - R_f)\} + E\{R_a^{-\gamma}\}\right] =$
$\frac{\gamma}{(1+\mathcal{B})^2}E\left\{R_a^{-\gamma-1}(R - R_f)^2\right\} + \frac{\partial\overline{R}(\mathcal{B},R_f)}{\partial\mathcal{B}}\left[-\gamma\left[E\{R_a^{-\gamma}\} - R_f E\{R_a^{-\gamma-1}\}\right] + E\{R_a^{-\gamma}\}\right] > 0,$
where the inequality follows from $\frac{\partial\overline{R}(\mathcal{B},R_f)}{\partial\mathcal{B}} = \frac{\alpha}{(1-\alpha)\beta}[1 - \beta\zeta + \beta\zeta R_f] > 0$ and Corollary 3. ■

Proof of Lemma 3. Partially differentiate
$f(\mathcal{B}, R_f) \equiv E\left\{\left(R_f + \frac{1}{1+\mathcal{B}}\left[\overline{R}(\mathcal{B}, R_f) + G^{-1}\varepsilon - R_f\right]\right)^{-\gamma}\left(\overline{R}(\mathcal{B}, R_f) + G^{-1}\varepsilon - R_f\right)\right\}$ in equation (21) with respect to R_f, and use $R = \overline{R}(\mathcal{B}, R_f) + G^{-1}\varepsilon$ and $R_a = R_f + \frac{1}{1+\mathcal{B}}\left[\overline{R}(\mathcal{B}, R_f) + G^{-1}\varepsilon - R_f\right]$ to obtain $\frac{\partial f(\mathcal{B},R_f)}{\partial R_f} = -\gamma E\left\{R_a^{-\gamma-1}\left(1 + \frac{1}{1+\mathcal{B}}\left[\frac{\partial\overline{R}(\mathcal{B},R_f)}{\partial R_f} - 1\right]\right)(R - R_f)\right\} + E\left\{R_a^{-\gamma}\left(\frac{\partial\overline{R}(\mathcal{B},R_f)}{\partial R_f} - 1\right)\right\}.$
Now use $\frac{\partial\overline{R}(\mathcal{B},R_f)}{\partial R_f} = \frac{\alpha}{1-\alpha}\zeta\mathcal{B}$ to obtain $\frac{\partial f(\mathcal{B},R_f)}{\partial R_f} = -\gamma E\left\{R_a^{-\gamma-1}\left(1 + \frac{1}{1+\mathcal{B}}\left[\frac{\alpha}{1-\alpha}\zeta\mathcal{B} - 1\right]\right)(R - R_f)\right\} +$

$E\left\{R_a^{-\gamma}\right\}\left(\frac{\alpha}{1-\alpha}\zeta\mathcal{B}-1\right) = -\gamma\frac{\mathcal{B}}{1+\mathcal{B}}\left(1+\frac{\alpha}{1-\alpha}\zeta\right)E\left\{R_a^{-\gamma-1}\left(R-R_f\right)\right\}+E\left\{R_a^{-\gamma}\right\}\left(\frac{\alpha}{1-\alpha}\zeta\mathcal{B}-1\right)$. Now use the fact $R_a = R_f + \frac{1}{1+\mathcal{B}}\left(R-R_f\right)$ to replace $\frac{1}{1+\mathcal{B}}\left(R-R_f\right)$ with $R_a - R_f$ to obtain $\frac{\partial f\left(\mathcal{B},R_f\right)}{\partial R_f}$ $= -\gamma\mathcal{B}\left(1+\frac{\alpha}{1-\alpha}\zeta\right)E\left\{R_a^{-\gamma-1}\left(R_a-R_f\right)\right\}+E\left\{R_a^{-\gamma}\right\}\left(\frac{\alpha}{1-\alpha}\zeta\mathcal{B}-1\right)$

$= -\gamma\mathcal{B}\left(1+\frac{\alpha}{1-\alpha}\zeta\right)\left[E\left\{R_a^{-\gamma}\right\}-R_fE\left\{R_a^{-\gamma-1}\right\}\right]+E\left\{R_a^{-\gamma}\right\}\left(\frac{\alpha}{1-\alpha}\zeta\mathcal{B}-1\right)$. To prove that $\frac{\partial f\left(\mathcal{B},R_f\right)}{\partial R_f}<0$, it suffices to prove that $-\gamma\mathcal{B}\left(1+\frac{\alpha}{1-\alpha}\zeta\right)\left[E\left\{R_a^{-\gamma}\right\}-R_fE\left\{R_a^{-\gamma-1}\right\}\right]$

$+ E\left\{R_a^{-\gamma}\right\}\left(\frac{\alpha}{1-\alpha}\zeta\mathcal{B}-1\right)<0$, or, equivalently, $\gamma\mathcal{B}\left(1+\frac{\alpha}{1-\alpha}\zeta\right)\left[\frac{R_fE\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}}-1\right]<1-\frac{\alpha}{1-\alpha}\zeta\mathcal{B}$. Corollary 3 implies $R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}}-1>0$, which, together with Lemma 4, implies $0\leq \mathcal{B}\left(R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}}-1\right)\leq 1-R_{\inf}$ since $\mathcal{B}\geq 0$. Therefore, $\gamma\left(1+\frac{\alpha}{1-\alpha}\zeta\right)\mathcal{B}\left(R_f\frac{E\left\{R_a^{-\gamma-1}\right\}}{E\left\{R_a^{-\gamma}\right\}}-1\right)\leq \gamma\left(1+\frac{\alpha}{1-\alpha}\zeta\right)\left(1-R_{\inf}\right)$, so it suffices to the prove $\gamma\left(1+\frac{\alpha}{1-\alpha}\zeta\right)\left(1-R_{\inf}\right)\leq 1-\frac{\alpha}{1-\alpha}\zeta\mathcal{B}$, or, equivalently,

$$\gamma < \frac{1-\frac{\alpha}{1-\alpha}\zeta\mathcal{B}}{1+\frac{\alpha}{1-\alpha}\zeta}\frac{1}{1-R_{\inf}}. \tag{A.1}$$

The absence of arbitrage opportunities implies $R_f \geq \frac{\alpha}{(1-\alpha)\beta}\left[1+\mathcal{B}-\beta\zeta\left(1-R_f\right)\mathcal{B}\right]+\left(1-\delta+\varepsilon_{\min}\right)G^{-1}$ $= R_{\inf}+\frac{\alpha}{(1-\alpha)\beta}\left[1-\beta\zeta\left(1-R_f\right)\right]\mathcal{B}\geq R_{\inf}+\frac{\alpha}{(1-\alpha)\beta}\left(1-\beta\zeta\right)\mathcal{B}$, so we know that if $R_f\leq 1$, then $R_{\inf}+\frac{\alpha}{(1-\alpha)\beta}\left(1-\beta\zeta\right)\mathcal{B}\leq 1$, which implies $\frac{\alpha}{(1-\alpha)\beta}\left(1-\beta\zeta\right)\mathcal{B}\leq 1-R_{\inf}$, or, equivalently, since $\beta\zeta<1$, $\frac{\alpha}{1-\alpha}\zeta\mathcal{B}\leq\frac{\beta\zeta}{1-\beta\zeta}\left(1-R_{\inf}\right)$. Therefore, $1-\frac{\alpha}{1-\alpha}\zeta\mathcal{B}\geq 1-\frac{\beta\zeta}{1-\beta\zeta}\left(1-R_{\inf}\right)$, so the condition in equation (A.1) will be satisfied if

$$\gamma < \frac{1-\frac{\beta\zeta}{1-\beta\zeta}\left(1-R_{\inf}\right)}{1+\frac{\alpha}{1-\alpha}\zeta}\frac{1}{1-R_{\inf}}. \tag{A.2}$$

■

Proof of Proposition 2. Use the definition of $\overline{R}\left(\mathcal{B},R_f\right)$ in equation (18) and the statements immediately following equation (18) that $\frac{\partial\overline{R}\left(\mathcal{B},R_f\right)}{\partial\mathcal{B}}>0$ and $\frac{\partial\overline{R}\left(\mathcal{B},R_f\right)}{\partial R_f}=\frac{\alpha}{(1-\alpha)\beta}\beta\zeta\mathcal{B}\geq 0$ to obtain $\frac{d\overline{R}\left(\mathcal{B},R_f\right)}{d\mathcal{B}}=\frac{\partial\overline{R}\left(\mathcal{B},R_f\right)}{\partial\mathcal{B}}+\frac{\partial\overline{R}\left(\mathcal{B},R_f\right)}{\partial R_f}R_f'\left(\mathcal{B}\right)$, which is positive if $R_f'\left(\mathcal{B}\right)>0$. Since $\overline{R}\left(\mathcal{B},R_f\right)=\frac{1+r}{G}$ and $r=\alpha k^{\alpha-1}-\delta+\varepsilon$, we have that k is decreasing in $\overline{R}$ and hence is decreasing in $\mathcal{B}$. ■

To prepare for the proof of Proposition 3, we introduce and prove the following two lemmas.

Lemma 6 *Assume that the distribution of* ε *is non-degenerate and that* $\left(1-\delta+\varepsilon_{\inf}\right)G^{-1}\geq 0$. *If* $R_f=1$, *then*

1. $\widehat{\mathcal{B}} \equiv \max\left\{0, \frac{(1-\alpha)\beta}{\alpha}(1-R_{\text{inf}})\right\} \leq \max\left\{0, \frac{(1-\alpha)\beta}{\alpha} - 1\right\}$

2. $R_{\text{inf}} + \frac{\alpha}{(1-\alpha)\beta}\widehat{\mathcal{B}} = \max\{R_{\text{inf}}, 1\} \geq 1$

3. $\mathcal{B} \leq \widehat{\mathcal{B}}$.

Proof of Lemma 6. Recall that $R_{\text{inf}} \equiv \inf_{\mathcal{B}\geq 0,\varepsilon} R = \frac{\alpha}{(1-\alpha)\beta} + (1-\delta+\varepsilon_{\text{inf}})\, G^{-1}$, so

$$R_{\text{inf}} \geq \frac{\alpha}{(1-\alpha)\,\beta}. \tag{A.3}$$

Since $R_f = 1$, equations (17) and (18) imply that $R = \frac{\alpha}{(1-\alpha)\beta}(1+\mathcal{B}) + (1-\delta+\varepsilon)\, G^{-1} \geq \frac{\alpha}{(1-\alpha)\beta}(1+\mathcal{B}) + (1-\delta+\varepsilon_{\text{inf}})\, G^{-1} = R_{\text{inf}} + \frac{\alpha}{(1-\alpha)\beta}\mathcal{B}$, with strict inequality if $\varepsilon > \varepsilon_{\text{inf}}$, which we summarize as

$$R \geq R_{\text{inf}} + \frac{\alpha}{(1-\alpha)\,\beta}\mathcal{B}, \text{ with strict inequality of } \varepsilon > \varepsilon_{\text{inf}}. \tag{A.4}$$

Since $R_f = 1$ and the distribution of ε is non-degenerate, the absence of arbitrage opportunities implies that

$$R_{\text{inf}} < 1. \tag{A.5}$$

Statement 1: Equation (A.3) implies $\widehat{B} \equiv \max\left\{0, \frac{(1-\alpha)\beta}{\alpha}(1-R_{\text{inf}})\right\} \leq \max\left\{0, \frac{(1-\alpha)\beta}{\alpha}\left(1 - \frac{\alpha}{(1-\alpha)\beta}\right)\right\} = \max\left\{0, \frac{(1-\alpha)\beta}{\alpha} - 1\right\}$. Statement 2: $R_{\text{inf}} + \frac{\alpha}{(1-\alpha)\beta}\widehat{\mathcal{B}} = R_{\text{inf}} + \frac{\alpha}{(1-\alpha)\beta}\max\left\{0, \frac{(1-\alpha)\beta}{\alpha}(1-R_{\text{inf}})\right\} = R_{\text{inf}} + \max\{0, 1-R_{\text{inf}}\} = \max\{R_{\text{inf}}, 1\} \geq 1$. Statement 3: Assume, contrary to what is to be proved, that $\mathcal{B} > \widehat{\mathcal{B}}$. Then equation (A.4) implies $R > R_{\text{inf}} + \frac{\alpha}{(1-\alpha)\beta}\widehat{\mathcal{B}}$ so Statement 2 implies $R > 1$, which contradicts $R_{\text{inf}} < 1$ in equation (A.5). ∎

Lemma 7 *Assume that the distribution of ε is non-degenerate and that $(1-\delta+\varepsilon_{\text{inf}})\, G^{-1} \geq 0$. If $R_f = 1$ and $\frac{\alpha}{(1-\alpha)\beta} \geq \frac{1}{2}$ then*

1. *$\mathcal{B} \leq R$ for all ε, with strict inequality for $\varepsilon > \varepsilon_{\text{inf}}$*

2. *Let $x \equiv \ln R$. Then $f(x) \equiv \frac{1+\mathcal{B}}{\mathcal{B}+\exp(x)}$ is convex in x, with strict convexity when $\varepsilon > \varepsilon_{\text{inf}}$.*

Proof of Lemma 7. Statement 1: Equation (A.4) in the proof of Lemma 6 implies $\mathcal{B} - R \leq \mathcal{B} - \left(R_{\text{inf}} + \frac{\alpha}{(1-\alpha)\beta}\mathcal{B}\right) = \left(1 - \frac{\alpha}{(1-\alpha)\beta}\right)\mathcal{B} - R_{\text{inf}}$, where the inequality is strict if $\varepsilon > \varepsilon_{\text{inf}}$. It suffices to prove that $\left(1-\frac{\alpha}{(1-\alpha)\beta}\right)\mathcal{B}-R_{\text{inf}}\leq 0$. Equations (A.3) and (A.5) in the proof of Lemma 6 imply that $\frac{\alpha}{(1-\alpha)\beta}<1$, so Statement 3 of Lemma 6 implies $\left(1-\frac{\alpha}{(1-\alpha)\beta}\right)\mathcal{B}-R_{\text{inf}}\leq \left(1-\frac{\alpha}{(1-\alpha)\beta}\right)\widehat{\mathcal{B}}-R_{\text{inf}}=\widehat{\mathcal{B}}-\left(\frac{\alpha}{(1-\alpha)\beta}\widehat{\mathcal{B}}+R_{\text{inf}}\right)\leq\widehat{\mathcal{B}}-1\leq 0$, where the final inequality follows from Statement 2 of Lemma 6. It suffices to prove that $\widehat{\mathcal{B}}\leq 1$, which follows from Statement 1 of Lemma 6 and the assumption $\frac{\alpha}{(1-\alpha)\beta}\geq\frac{1}{2}$, so that $\max\left\{0,\frac{(1-\alpha)\beta}{\alpha}-1\right\}\leq 1$.

Statement 2: Differentiate $f(x)\equiv\frac{1+\mathcal{B}}{\mathcal{B}+\exp(x)}$ with respect to x twice to obtain $f'(x)=-\frac{1+\mathcal{B}}{[\mathcal{B}+\exp(x)]^2}\exp(x)$ and $f''(x)=2\frac{1+\mathcal{B}}{[\mathcal{B}+\exp(x)]^3}[\exp(x)]^2-\frac{1+\mathcal{B}}{[\mathcal{B}+\exp(x)]^2}\exp(x)=(2\exp(x)-[\mathcal{B}+\exp(x)])\times\frac{1+\mathcal{B}}{[\mathcal{B}+\exp(x)]^3}\exp(x)$. Therefore, since $x\equiv\ln R$, $f''(x)$ has the same sign as $\exp(x)-\mathcal{B}=R-\mathcal{B}\geq 0$, where the inequality follows from Statement 1 and is strict for $\varepsilon>\varepsilon_{\text{inf}}$. Therefore, $f''(x)\geq 0$, with strict inequality if $\varepsilon>\varepsilon_{\text{inf}}$. ■

Proof of Proposition 3. First, we prove that there is some value of $\mathcal{B}$ between 0 and $\mathcal{B}^*$ for which $R_f=1$.

If $\mathcal{B}=0$, equation (18) when $\zeta=0$ implies that $\overline{R}=\frac{\alpha}{(1-\alpha)\beta}+(1-\delta)G^{-1}<1$, where the inequality follows from the assumption that $\frac{\alpha}{(1-\alpha)\beta}<1-(1-\delta)G^{-1}$. Since consumers are risk averse and capital has a non-degenerate risk, $R_f<\overline{R}<1$ when $\mathcal{B}=0$. In general, for any non-negative value of $\mathcal{B}$, $R=\overline{R}+G^{-1}\varepsilon=\frac{\alpha}{(1-\alpha)\beta}(1+\mathcal{B})+(1-\delta)G^{-1}+G^{-1}\varepsilon\geq\frac{\alpha}{(1-\alpha)\beta}(1+\mathcal{B})$ for all realizations of ε. Therefore, if $\mathcal{B}=\mathcal{B}^*$, where $\mathcal{B}^*$ is such that $\frac{\alpha}{(1-\alpha)\beta}(1+\mathcal{B}^*)=1$, then $R\geq 1$ for all realizations of R and hence no-arbitrage implies $R_f\geq 1$. Therefore, there is some value of $\mathcal{B}$ between 0 and $\mathcal{B}^*$ for which $R_f=1$.

Now suppose that we have the value of $\mathcal{B}$ for which $R_f=1$. Since $R_f=1$ and $\gamma=1$, Lemma 1 directly implies that $E\{R_a^{-1}\}=1$, which, using $R_a=\frac{1}{1+\mathcal{B}}(\mathcal{B}R_f+R)$ along with $R_f=1$, implies $E\left\{\frac{1+\mathcal{B}}{\mathcal{B}+R}\right\}=1$. Using $f(x)\equiv\frac{1+\mathcal{B}}{\mathcal{B}+\exp(x)}$ defined in Statement 2 of Lemma 7 where $x\equiv\ln R$, we have $E\{f(x)\}=1$. Since the distribution of ε is non-degenerate and $f(x)$ is convex, with strict convexity for $\varepsilon>\varepsilon_{\text{inf}}$, we have $1=E\{f(x)\}\geq f(E(x))=\frac{1+\mathcal{B}}{\mathcal{B}+\exp(E\{x\})}=\frac{1+\mathcal{B}}{\mathcal{B}+\exp(E\{\ln R\})}$. Therefore, $\exp(E\{\ln R\})>1$ and hence $E\{\ln R\}>0$. ■

Proof of Proposition 4. Let $\mathcal{B}_1$ be an arbitrary non-negative value of $\mathcal{B}$ for which $R_f(\mathcal{B})>1$ and let $\mathcal{B}_2$ be the smallest value of $\mathcal{B}$ greater than $\mathcal{B}_1$ for which $R_f(\mathcal{B})=1$. Therefore, $R_f'(\mathcal{B}_2)\leq 0$ and $R_f(\mathcal{B}_2)\leq 1$, which contradicts the assumption that $R_f'(\mathcal{B})>0$ whenever $R_f(\mathcal{B})\leq 1$. Therefore, $R_f(\mathcal{B})>1$ for all $\mathcal{B}\geq\mathcal{B}_1$. We will use this result to prove the statements in this proposition. Statement 1: First, if $R_f(0)>1$, then $R_f(\mathcal{B})>1$ for all positive $\mathcal{B}$, so all positive values of $\mathcal{B}$ are unsustainable. If $R_f(0)=1$, then $R_f(\varepsilon)>1$ in a positive neighborhood of $\varepsilon=0$, and therefore, $R_f>1$ for all positive $\mathcal{B}$, so all positive values of $\mathcal{B}$ are unsustainable. Therefore, if $R_f(0)\geq 1$, then $\mathcal{B}_{\max}=0$. Statement 2a:

Assume that $R_f(0) < 1$. Let $\mathcal{B}_0 > 0$ be the smallest positive $\mathcal{B}$ for which $R_f(\mathcal{B}) = 1$, so $\mathcal{B}_0$ is sustainable. Using the result above in this proof for all values of $\mathcal{B} > \mathcal{B}_0$, $R_f(\mathcal{B}) > 1$, and hence these values of $\mathcal{B}$ are unsustainable. Therefore, $\mathcal{B}_{\max}$ is the unique root of $R_f(\mathcal{B}) = 1$. Statement 2b: Statements 1 and 3 of Lemma 6 imply $\mathcal{B}_{\max} \leq \widehat{\mathcal{B}} \leq \max\left\{0, \frac{(1-\alpha)\beta}{\alpha} - 1\right\}$. Statement 2c: Since there is a unique value of $\mathcal{B}$ for which $R_f(\mathcal{B}) = 1$, we have $R_f(\mathcal{B}) < 1$ for $0 \leq \mathcal{B} \leq \mathcal{B}_{\max}$ and all of these values of $\mathcal{B}$ are sustainable; $R_f(\mathcal{B}) > 1$ for all $\mathcal{B} > \mathcal{B}_{\max}$ and these values are unsustainable, so $\mathcal{B}$ is sustainable if and only if $0 \leq \mathcal{B} \leq \mathcal{B}_{\max}$. ■

Proof of Proposition 5. Along a balanced growth path, expected gross investment is $(g + E\{\delta_t\}) K_t$. With a linearly homogeneous production function, $Y_t = F(K_t, G^t N) = G^t N F_N + K_t F_K$, so expected aggregate consumption is $E\{C_t\} = G^t N F_N + K_t F_K - (g + E\{\delta_t\}) K_t$ (there are no purchases of goods by the government). Therefore, the impact of a marginal change in K_t on $E\{C_t\}$ is $\frac{dE\{C_t\}}{dK} = G^t N \frac{dF_N}{dK} + K_t \frac{dF_K}{dK} + F_K - (g + E\{\delta_t\})$. The factor price frontier (see footnote 23) implies $G^t N \frac{dF_N}{dK} + K_t \frac{dF_K}{dK} = 0$. Therefore, $\frac{dE\{C_t\}}{dK} = F_K - E\{\delta_t\} - g = E\{r_t\} - g > r_f - g = 0$, where $r_t \equiv F_K - \delta_t$ is the rate of return on capital, risk aversion implies $E\{r_t\} > r_f$, and along the optimal balanced growth path with positive government debt, $\mathcal{B} = \mathcal{B}_{\max}$ and $r_f = g$. ■

Lemma 8 *Under the assumptions of Proposition 6,*

1. $\Omega > 0$ *and* $\frac{d\Omega}{d\mathcal{B}} = G\left[1 + \beta\zeta(R_f - 1) + \beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]$
2. $\frac{d\ln w}{d\mathcal{B}} = A_1 \equiv -\frac{\alpha}{1-\alpha}\frac{1}{\Omega}\frac{d\Omega}{d\mathcal{B}} < 0$
3. $\frac{d}{d\mathcal{B}}\left(\frac{\beta}{1-\gamma}\ln E\left\{R_a^{1-\gamma}\right\}\right) = A_2 + A_3$*, where* $A_2 \equiv \beta\frac{\mathcal{B}}{1+\mathcal{B}}\frac{1}{R_f}\frac{dR_f}{d\mathcal{B}} > 0$ *and* $A_3 \equiv \frac{1}{1+\mathcal{B}}\frac{1}{R_f}\frac{\alpha}{1-\alpha}G^{-1}\frac{d\Omega}{d\mathcal{B}} > 0$
4. $\frac{d}{d\mathcal{B}}\ln\left(1 + \frac{\tau_t}{w_t}\right) = A_4 + A_5$*, where* $A_4 \equiv \frac{\beta\zeta G}{\Omega}\frac{1}{1+\mathcal{B}}(1 - R_f)$ *and* $A_5 \equiv -\frac{\beta\zeta G}{\Omega}\mathcal{B}\frac{dR_f}{d\mathcal{B}} < 0$

Proof of Lemma 8. Proof of Statement 1: Since $R_f \geq 0$, $\mathcal{B} \geq 0$, $\beta < 1$ and $\zeta \leq 1$, we have $\Omega \equiv [1 + \mathcal{B} + \beta\zeta(R_f - 1)\mathcal{B}]G \geq [1 + (1 - \beta\zeta)\mathcal{B}]G \geq G > 0$. Differentiating Ω with respect to $\mathcal{B}$ yields $\frac{d\Omega}{d\mathcal{B}} = G\left[1 + \beta\zeta(R_f - 1) + \beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right] \geq (1 - \beta\zeta)G + \beta\zeta\mathcal{B}G\frac{dR_f}{d\mathcal{B}} > 0$.

Proof of Statement 2: (1) Since $w = (1-\alpha)k^\alpha$, we have $\frac{dw}{dk} = \alpha\frac{w}{k}$ and hence $\frac{1}{w}\frac{dw}{dk} = \alpha\frac{1}{k}$; (2) equation (15) implies $k^{a-1} = \frac{1}{(1-\alpha)\beta}[(1+\mathcal{B})G - \beta\zeta(g - r_f)\mathcal{B}] = \frac{1}{(1-\alpha)\beta}\Omega$, so $k = \left(\frac{1}{(1-\alpha)\beta}\Omega\right)^{\frac{1}{\alpha-1}}$ and hence $\frac{dk}{d\mathcal{B}} = -\frac{1}{1-\alpha}k\frac{1}{\Omega}\frac{d\Omega}{d\mathcal{B}}$. Therefore, $\frac{d\ln w}{d\mathcal{B}} = \frac{1}{w}\frac{dw}{dk}\frac{dk}{d\mathcal{B}} = -\frac{\alpha}{1-\alpha}\frac{1}{\Omega}\frac{d\Omega}{d\mathcal{B}}$.

Proof of Statement 3: Use $R_a \equiv \lambda R_f + (1-\lambda)R$ to obtain $\frac{d}{d\mathcal{B}}\left(\frac{\beta}{1-\gamma}\ln E\left\{R_a^{1-\gamma}\right\}\right) = \frac{\beta}{1-\gamma}\frac{1}{E_t\left\{R_a^{1-\gamma}\right\}}(1-\gamma)E\left\{R_a^{-\gamma}\left(\lambda\frac{dR_f}{d\mathcal{B}} + (1-\lambda)\frac{dR}{d\mathcal{B}}\right)\right\}$, where we have used the envelope theorem to ignore $E\left\{R_a^{-\gamma}\left(\frac{d\lambda}{d\mathcal{B}}R_f - \frac{d\lambda}{d\mathcal{B}}R\right)\right\} = 0$. Use the following: (1) $\frac{dR_f}{d\mathcal{B}}$ is non-random;

(2) equations (17) and (18) imply $\frac{dR}{d\mathcal{B}} = \frac{\alpha}{(1-\alpha)\beta}\left(1+\beta\zeta\left(R_f-1\right)+\beta\zeta\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right) = \frac{\alpha}{(1-\alpha)\beta}G^{-1}\frac{d\Omega}{d\mathcal{B}}$, which is non-random; (3) Lemma 1, which implies $\frac{E\left\{R_a^{-\gamma}\right\}}{E\left\{R_a^{1-\gamma}\right\}} = \frac{1}{R_f}$; and (4) $\lambda = \frac{\mathcal{B}}{1+\mathcal{B}}$ to obtain $\frac{d}{d\mathcal{B}}\left(\frac{\beta}{1-\gamma}\ln E\left\{R_a^{1-\gamma}\right\}\right) = \beta\frac{1}{R_f}\left(\frac{\mathcal{B}}{1+\mathcal{B}}\frac{dR_f}{d\mathcal{B}}+\frac{1}{1+\mathcal{B}}\frac{dR}{d\mathcal{B}}\right) = \beta\frac{\mathcal{B}}{1+\mathcal{B}}\frac{1}{R_f}\frac{dR_f}{d\mathcal{B}}+\frac{1}{1+\mathcal{B}}\frac{1}{R_f}\frac{\alpha}{1-\alpha}G^{-1}\frac{d\Omega}{d\mathcal{B}}$.

Proof of Statement 4: Along a balanced growth path, $\frac{\tau_t}{w_t} = \frac{\beta N\tau_t}{\beta N w_t} = \frac{\beta\zeta\left(g-r_f\right)B_t}{\beta N w_t} = \frac{G\beta\zeta\left(1-R_f\right)B_t}{\beta N w_t} = \frac{G\beta\zeta\left(1-R_f\right)B_t}{\beta N(w_t+\tau_t)}\frac{\beta N(w_t+\tau_t)}{\beta N w_t} = \frac{\beta\zeta\left(1-R_f\right)B_{t+1}}{K_{t+1}+B_{t+1}}\left(1+\frac{\tau_t}{w_t}\right) = \beta\zeta\left(1-R_f\right)\frac{\mathcal{B}}{1+\mathcal{B}}\left(1+\frac{\tau_t}{w_t}\right)$. Define $X_t \equiv 1+\frac{\tau_t}{w_t}$, so along a balanced growth path we have $X-1 = \beta\zeta\left(1-R_f\right)\frac{\mathcal{B}}{1+\mathcal{B}}X$, which implies $X = \left[1-\beta\zeta\left(1-R_f\right)\frac{\mathcal{B}}{1+\mathcal{B}}\right]^{-1} = \left(\frac{1}{1+\mathcal{B}}\right)^{-1}\left[1+\mathcal{B}-\beta\zeta\left(1-R_f\right)\mathcal{B}\right]^{-1} = \left(1+\mathcal{B}\right)\left(G^{-1}\Omega\right)^{-1}$ $= G\left(1+\mathcal{B}\right)\Omega^{-1}$. Therefore, $\ln X = \ln G + \ln\left(1+\mathcal{B}\right) - \ln\Omega$. Now differentiate $\ln X$ with respect to $\mathcal{B}$ to obtain $\frac{d\ln X}{d\mathcal{B}} = \frac{1}{1+\mathcal{B}} - \frac{1}{\Omega}\frac{d\Omega}{d\mathcal{B}} = \frac{1}{\Omega}\left(\frac{1}{1+\mathcal{B}}\Omega - \frac{d\Omega}{d\mathcal{B}}\right) = \frac{\beta\zeta G}{\Omega}\left(\left(R_f-1\right)\frac{\mathcal{B}}{1+\mathcal{B}} - \left(R_f-1\right) - \mathcal{B}\frac{dR_f}{d\mathcal{B}}\right)$ $= \frac{\beta\zeta G}{\Omega}\left(\left(1-R_f\right)\frac{1}{1+\mathcal{B}} - \mathcal{B}\frac{dR_f}{d\mathcal{B}}\right)$. ■

Proof of Proposition 6. Differentiate the expression for $u\left(\mathcal{B}\right)$ in equation (22) to obtain $u'(\mathcal{B}) = \frac{d}{d\mathcal{B}}\ln w + \frac{d}{d\mathcal{B}}\ln\left(1+\frac{\tau_t}{W_t}\right) + \frac{\beta}{1-\gamma}\frac{d}{d\mathcal{B}}\ln E\left\{R_a^{1-\gamma}\right\}$, which, using the definitions of A_i for $i \in \{1,2,3,4,5\}$ in Lemma 8 can be written as $\sum_{i=1}^{5} A_i$. However, it is convenient to group the A_i in three sets as follows $u'(\mathcal{B}) = (A_1+A_3)+(A_2+A_5)+A_4$.

$A_1+A_3 = -\frac{\alpha}{1-\alpha}\frac{1}{\Omega}\frac{d\Omega}{d\mathcal{B}} + \frac{1}{1+\mathcal{B}}\frac{1}{R_f}\frac{\alpha}{1-\alpha}G^{-1}\frac{d\Omega}{d\mathcal{B}} = \frac{1}{\Omega}\frac{1}{R_f}\left(-R_f+\frac{1}{1+\mathcal{B}}\Omega G^{-1}\right)\frac{\alpha}{1-\alpha}\frac{d\Omega}{d\mathcal{B}}$
$= \frac{1}{\Omega}\frac{1}{R_f}\left(-R_f+\frac{1}{1+\mathcal{B}}\left[1+\mathcal{B}+\beta\zeta\left(R_f-1\right)\mathcal{B}\right]\right)\frac{\alpha}{1-\alpha}\frac{d\Omega}{d\mathcal{B}} = \frac{1}{\Omega}\frac{1}{R_f}\left(-R_f+1+\beta\zeta\left(R_f-1\right)\frac{\mathcal{B}}{1+\mathcal{B}}\right)\frac{\alpha}{1-\alpha}\frac{d\Omega}{d\mathcal{B}}$
$= \frac{1}{\Omega}\frac{1}{R_f}\left(1-R_f\right)\left(1-\beta\zeta\frac{\mathcal{B}}{1+\mathcal{B}}\right)\frac{\alpha}{1-\alpha}\frac{d\Omega}{d\mathcal{B}}$.

$A_2+A_5 = \beta\frac{\mathcal{B}}{1+\mathcal{B}}\frac{1}{R_f}\frac{dR_f}{d\mathcal{B}} - \frac{\beta\zeta G}{\Omega}\mathcal{B}\frac{dR_f}{d\mathcal{B}} = \frac{1}{\Omega}\frac{1}{R_f}\left(\frac{1}{1+\mathcal{B}}G^{-1}\Omega - \zeta R_f\right)\beta G\mathcal{B}\frac{dR_f}{d\mathcal{B}}$
$= \frac{1}{\Omega}\frac{1}{R_f}\left(1+\left(R_f-1\right)\beta\zeta\frac{\mathcal{B}}{1+\mathcal{B}} - \zeta R_f\right)\beta G\mathcal{B}\frac{dR_f}{d\mathcal{B}} = \frac{1}{\Omega}\frac{1}{R_f}\left[\left(1-R_f\right)\left(1-\beta\zeta\frac{\mathcal{B}}{1+\mathcal{B}}\right)+\left(1-\zeta\right)R_f\right]\beta G\mathcal{B}\frac{dR_f}{d\mathcal{B}}$.

$A_1+A_3+A_2+A_5 = \frac{1}{\Omega}\frac{1}{R_f}\left[\left(1-R_f\right)\left(1-\beta\zeta\frac{\mathcal{B}}{1+\mathcal{B}}\right)\frac{\alpha}{1-\alpha}\frac{d\Omega}{d\mathcal{B}}\right] + \frac{1}{\Omega}\frac{1}{R_f}\left[\left(1-R_f\right)\left(1-\beta\zeta\frac{\mathcal{B}}{1+\mathcal{B}}\right)\beta G\mathcal{B}\frac{dR_f}{d\mathcal{B}} + \left(1-\zeta\right)R_f\beta G\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right] = \frac{1}{\Omega}\frac{1}{R_f}\left[\left(1-R_f\right)\left(1-\beta\zeta\frac{\mathcal{B}}{1+\mathcal{B}}\right)\left(\frac{\alpha}{1-\alpha}\frac{d\Omega}{d\mathcal{B}}+\beta G\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right) + \left(1-\zeta\right)R_f\beta G\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]$.

Therefore,

$u'(\mathcal{B}) = (A_1+A_3)+(A_2+A_5)+A_4 = \frac{1}{\Omega}\frac{1}{R_f}\left[\left(1-R_f\right)\left(1-\beta\zeta\frac{\mathcal{B}}{1+\mathcal{B}}\right)\left(\frac{\alpha}{1-\alpha}\frac{d\Omega}{d\mathcal{B}}+\beta G\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right)\right.$
$\left. + \left(1-\zeta\right)R_f\beta G\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right] + \frac{\beta\zeta G}{\Omega}\frac{1}{R_f}\frac{1}{1+\mathcal{B}}R_f\left(1-R_f\right) =$
$\frac{1}{\Omega}\frac{1}{R_f}\left\{\left(1-R_f\right)\left[\left(1-\beta\zeta\frac{\mathcal{B}}{1+\mathcal{B}}\right)\left(\frac{\alpha}{1-\alpha}\frac{d\Omega}{d\mathcal{B}}+\beta G\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right)+\frac{1}{1+\mathcal{B}}\beta\zeta G R_f\right] + \left(1-\zeta\right)R_f\beta G\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right\}$. ■

Proof of Corollary 2. If $\zeta = 0$, then $\Omega = \left(1+\mathcal{B}\right)G$, so $\frac{d\Omega}{d\mathcal{B}} = G$ and the expression for $u'\left(B\right)$ in Proposition 6 becomes $u'(\mathcal{B}) = \frac{1}{(1+\mathcal{B})G}\frac{1}{R_f}\left[\left(1-R_f\right)\left(\frac{\alpha}{1-\alpha}G+\mathcal{B}G\beta\frac{dR_f}{d\mathcal{B}}\right)+\mathcal{B}R_f G\beta\frac{dR_f}{d\mathcal{B}}\right] = \frac{\beta}{1+\mathcal{B}}\frac{1}{R_f}\left[\left(1-R_f\right)\frac{\alpha}{(1-\alpha)\beta}+\mathcal{B}\frac{dR_f}{d\mathcal{B}}\right]$. ■

Proof of Proposition 7. See the discussion in the text. ■

Online Appendix

B Isomorphic Formulation with Stochastic Output

The model used throughout the main text of the paper–which we will call the *baseline model*–was designed so that the aggregate capital stock, K_t, evolves deterministically and grows at a constant rate, g, along any balanced growth path. As we show in Section 4, the riskfree interest rate, r_f, is also constant along any balanced growth path, so the feasibility of rolling over government debt depends only on the sign of $r_f - g$. In addition, aggregate output, Y_t, evolves non-stochastically in the baseline model. In this appendix, we develop a class of models that are isomorphic to the baseline model in the sense that the evolution of K_t, as well as the evolution of rates of return, including the riskfree interest rate, are identical to those in the baseline model. Nevertheless, except for the baseline model, Y_t evolves stochastically for all models in this isomorphic class. We choose to focus on the baseline model throughout the main text of the paper for expositional simplicity, recognizing that the non-stochastic evolution of Y_t is not at all essential to our findings.

Consider a class of models in which the production function is

$$Y_t = \left(G^t N\right)^{1-\alpha} K_t^\alpha + \eta_{1,t} K_t = \left(k_t^{\alpha-1} + \eta_{1,t}\right) K_t, \tag{B.1}$$

where $\eta_{1,t}$ is an i.i.d. random productivity shock with a mean that can be positive, zero, or negative. Because aggregate wage income, $(1-\alpha)\left(G^t N\right)^{1-\alpha} K_t^\alpha$, is deterministic, aggregate saving of the young and the evolution of K_t are deterministic and identical to those in the baseline model.

In this class of models, the depreciation rate of capital is $\delta - \eta_{2,t}$, where $\eta_{2,t}$ is an i.i.d random variable with arbitrary correlation with $\eta_{1,t}$, and $0 \leq \delta - \eta_{2,t} \leq 1$. In addition, assume that $E\left\{\eta_{1,t} + \eta_{2,t}\right\} = 0$. The (net) rate of return on capital is the marginal product of capital, $\alpha k_t^{\alpha-1} + \eta_{1,t}$, minus the depreciation rate, $\delta - \eta_{2,t}$,

$$r_t = \alpha k_t^{\alpha-1} - \delta + \left(\eta_{1,t} + \eta_{2,t}\right). \tag{B.2}$$

This class of isomorphic models is defined by $\eta_{1,t} + \eta_{2,t} = \varepsilon_t$. Therefore, equation (B.2) can be rewritten as

$$r_t = \alpha k_t^{\alpha-1} - \delta + \varepsilon_t, \tag{B.3}$$

which is identical to equation (6) with $E\left\{\varepsilon_t\right\}=0$. As a consequence, the riskfree rate, $r_{f,t}$, is identical to that in the baseline model. Thus, all of the models in this class of models are isomorphic to the baseline model in the sense that K_t and all rates of return in all periods are identical to their values in the baseline model. Therefore, our major findings about rolling over government debt do not depend on output being deterministic.

C Calibration

Our focus is on the quantitative value of $\mathcal{B}_{\max}$. Corollary 1 states that $\mathcal{B}_{\max}$ is invariant to ζ, so, without loss of generality, we set $\zeta=0$.

Assume that a period equals 30 years. Since $R\equiv\frac{1+r}{G}$ is the adjusted gross rate of return on capital, $1+r=G\times R$ is the gross rate of return on capital (not adjusted for growth). Calibrate the distribution of the durability shock so that along a balanced growth path $E\left\{1+r\right\}=G\times E\left\{R\right\}=\left(1+m\right)^{30}$ where m is the target annual net rate of return on capital; the standard deviation of r, which is equal to the standard deviation of $G\times R$, is set equal to its target $s\sqrt{30}$, where s is the target annual standard deviation of the rate of return on capital.

We specify the distribution of the durability shock so that

$$G^{-1}\varepsilon=\exp\left(z\right)-\overline{R}_0 \tag{C.1}$$

where z is $N\left(\mu,\sigma^2\right)$, and $\overline{R}_0$ is obtained by setting $\mathcal{B}=0$ in equation (18). From equations (17) and (18), the risky rate of return on capital along a balanced growth path is $R=\frac{\alpha}{(1-\alpha)\beta}\left(1+\mathcal{B}\right)+\left(1-\delta\right)G^{-1}+G^{-1}\varepsilon=\left[\overline{R}_0+\frac{\alpha}{(1-\alpha)\beta}\mathcal{B}\right]+\left[\exp\left(z\right)-\overline{R}_0\right]$, so

$$R=\frac{\alpha}{\left(1-\alpha\right)\beta}\mathcal{B}+\exp\left(z\right). \tag{C.2}$$

We choose the parameters μ and σ^2 so that the mean of the risky rate equals the target rate, which is m on an annual basis, so

$$E\left\{1+r\right\} = G\times E\left\{R\right\} =G\times E\left\{\frac{\alpha}{\left(1-\alpha\right)\beta}\mathcal{B}+\exp\left(z\right)\right\}=\left(1+m\right)^{30}. \tag{C.3}$$

Setting the standard deviation of the risky rate, $1+r$, which is $G\sqrt{Var\left\{\exp\left(z\right)\right\}}$, equal to the target standard deviation, $s\sqrt{30}$, implies

$$G\sqrt{Var\left\{\exp\left(z\right)\right\}} = s\sqrt{30}. \tag{C.4}$$

Use $Var\left\{(\exp(z)\right\} = (\exp(\sigma^2)-1)\exp(2\mu+\sigma^2) = (\exp(\sigma^2)-1)\left[\exp(\mu+\frac{1}{2}\sigma^2)\right]^2 = (\exp(\sigma^2)-1)\left[E\left\{\exp\left(z\right)\right\}\right]^2$ to rewrite equation (C.4) as

$$G \times E\left\{\exp\left(z\right)\right\}\sqrt{(\exp(\sigma^2)-1)} = s\sqrt{30}. \tag{C.5}$$

From equation (C.3)

$$G \times E\left\{\exp\left(z\right)\right\} = (1+m)^{30} - G\frac{\alpha}{(1-\alpha)\beta}\mathcal{B}, \tag{C.6}$$

so equation (C.5) implies

$$\sqrt{(\exp(\sigma^2)-1)} = \frac{s\sqrt{30}}{(1+m)^{30} - G\frac{\alpha}{(1-\alpha)\beta}\mathcal{B}}, \tag{C.7}$$

which can be rewritten as

$$\exp(\frac{1}{2}\sigma^2) = \sqrt{1+30\left(\frac{s}{(1+m)^{30} - G\frac{\alpha}{(1-\alpha)\beta}\mathcal{B}}\right)^2}. \tag{C.8}$$

Substitute $\exp\left(\mu\right)\times\exp(\frac{1}{2}\sigma^2)$ for $E\left\{\exp\left(z\right)\right\}$ in equation (C.6) to obtain

$$G \times \exp\left(\mu\right) \times \exp(\frac{1}{2}\sigma^2) = (1+m)^{30} - G\frac{\alpha}{(1-\alpha)\beta}\mathcal{B}, \tag{C.9}$$

so

$$\exp\left(\mu\right) = \frac{G^{-1}(1+m)^{30} - \frac{\alpha}{(1-\alpha)\beta}\mathcal{B}}{\exp(\frac{1}{2}\sigma^2)} = \frac{G^{-1}(1+m)^{30} - \frac{\alpha}{(1-\alpha)\beta}\mathcal{B}}{\sqrt{1+30\left(\frac{s}{(1+m)^{30} - G\frac{\alpha}{(1-\alpha)\beta}\mathcal{B}}\right)^2}}. \tag{C.10}$$

The mean, m, and standard deviation, s, of the rate of return on capital are expressed on an annual basis. To compare these values to familiar values for the mean and standard deviation of annual stock returns, we must take account of the fact that m and s are moments of unlevered rates of return, and the moments of stock returns are levered returns. Let r^A, r_L^A, and r_f^A be rates of return on unlevered equity, levered equity and riskfree assets, all expressed at annual rates (hence the superscript A). They are related to each other by $r^A = \frac{D}{D+E} r_f^A + \frac{E}{D+E} r_L^A = \frac{D/E}{1+D/E} r_f^A + \frac{1}{1+D/E} r_L^A$, where D is the debt owed by the private owners of capital and E is the equity of these owners. Assume that D equals 45% of $D + E$. Therefore, $r^A = 0.45 r_f^A + 0.55 r_L^A$. Hence, if $E\left\{r^A\right\} = 3\%$ per year and $r_f^A = 0.6\%$ per year, then $E\left\{r_L^A\right\} = 4.96\%$ per year. Also, $sd\left\{r^A\right\} = 0.55 sd\left\{r_L^A\right\}$, $sd\left\{r_L^A\right\} = \frac{1}{0.55} sd\left\{r^A\right\}$, so if $sd\left\{r^A\right\} = 0.12$, then $sd\left\{r_L^A\right\} = 0.218$.

Remark 2 *If the value of μ in equation (C.10) positive, then the economy with $\mathcal{B} = 0$ is dynamically efficient and hence the economy is dynamically efficient for any positive $\mathcal{B}$ also.*

To prove this remark, note that equation (C.2) implies that $R_0 = \exp(z)$. Since R is an increasing function of $\mathcal{B}$, we have $R \geq R_0 = \exp(z)$ and hence $E\{\ln R\} \geq E\{\ln R_0\} = E\{z\} = \mu > 0$ for any $\mathcal{B} \geq 0$.