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INTERTEMPORAL DEPENDENCE, IMPATIENCE, AND DYNAMICS

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ABSTRACT

This paper develops simple geometric methods for analyzing dynamic behavior in models with intertemporally dependent consumer tastes. Since the preferences studied do not assume time-additivity, they allow the marginal utility of consumption on a given date to vary with consumption on other dates. Intertemporal dependence is induced by the presence of a variable individual rate of time preference. The optimal consumption responses to transitory and anticipated changes in incomes and interest rates are easily derived and are similar in important ways to the responses implied by the standard model with constant time preference. Intuitive explanations of the first-order conditions describing optimal paths are provided.

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1. Introduction

This paper develops simple geometric methods for analyzing dynamic models with intertemporally dependent consumer tastes. One appealing feature of such preferences is that they relax the widely used but restrictive assumption of time-additivity, thereby allowing the marginal utility of consumption on a given date to vary with consumption on other dates. Intertemporal dependence is introduced through the presence of a variable individual rate of time preference. This type of preference setup was introduced by Uzawa (1968), and it has recently been extended and clarified by Epstein (1987a).[1]

Mathematical convenience, rather than innate plausibility, has always been the main rationale for assuming time-additive preferences in economic modeling. A growing body of empirical evidence suggests, however, that the assumption of intertemporal independence may be inadequate in practice, even as an approximation. Tests of the stochastic intertemporal Euler equations implied by recursive time-additive preferences have typically produced strong statistical rejections, leading a number of researchers to posit some form of intertemporal dependence in tastes (see, for example, Dunn and Singleton 1986, Hayashi 1985,

[1]In Obstfeld (1982), I applied Uzawa preferences to study a small open economy's response to a permanent, unanticipated terms-of-trade shock. Further applications include Nairay's (1984) proof that an individual optimum exists, and Bergman's (1985) study of capital-asset pricing in a stochastic environment. Uzawa-Epstein preferences fall into the broader category of general recursive preferences, as defined by Koopmans (1986). Lucas and Stokey (1984) and Epstein (1987b) study the stability of steady states under quite general recursive preferences. Epstein and Hynes (1983) examine some well-known dynamic economic models using a particular specification of recursive utility. Judd (1985) uses a preference specification that includes Epstein's (1987a) as a special case, but his focus is on steady-state results. Becker, Boyd, and Sung (1989) provide a general existence theorem for recursive-preference problems of the type explored below.

and Heaton 1988). Intertemporal dependence has been invoked to help explain phenomena as diverse as the high return to U.S. equity investments over the last century (Constantinides 1988) and drug addiction (Becker and Murphy 1988). Finally, certain implications of time-additive preferences seem to clash with the cross-sectional patterns of consumption and income growth actually observed in the world economy (Summers and Carroll 1989).

In a deterministic setting with a fixed number of competitive infinitely-lived households, constant time-preference rates fail to produce a determinate steady state in which all households have positive net worth and consumption (Becker 1980). Translated to a world of small open economies, each inhabited by an immortal representative decision-maker, the result implies either that the long-run global distribution of wealth is indeterminate, or that the economy with the lowest time-preference rate (if one exists) eventually comes to own all the world's outside wealth. Whether these stark and (for some purposes) analytically inconvenient implications are reasonable is essentially an empirical matter. They can be avoided by assuming an overlapping-generations structure with disconnected cohort budget constraints, or by introducing income uncertainty. In at least some applications, however, such modifications may complicate analysis considerably while being economically beside the point.[2]

Despite the strong empirical and theoretical cases for

[2]In a nonmonetary open-economy setting, Buiter (1981) shows that models with finite overlapping lifetimes can produce a unique, nondegenerate long-run distribution of wealth between countries. Weil (1989) observes that Buiter's result survives generalization to a world in which new infinitely-lived households are continually born, provided the new households are not linked altruistically to existing ones. Clarida (1989) examines the stationary distribution of wealth in a stochastic model with possibly different fixed national rates of time preference.

dispensing with constant time-preference rates, convenient analytical methods for studying alternative models have not entered into general use. Important stability results for models with endogenous discounting have been established, but these say little or nothing about the effects on optimal consumption plans of a wide variety of common economic events. Readers of the literature in this area often are left with the impression that the models studied are both analytically intractable and based on restrictive assumptions.[3]

Contrary to this impression, however, the dynamic behavior implied by variable time preference is easy to analyze and interpret in many cases of interest. This paper supports this claim in two ways. First, it offers an intuitive account of the Uzawa-Epstein model, one that stresses the sense in which it generalizes the standard intertemporal preference model. Second, it develops the simple diagrammatic machinery needed to study how such models respond to rather complicated disturbances, such as transitory movements in fiscal variables or interest rates. Models of endogenous time preference are no more difficult to analyze than models of habit-forming consumption. Indeed, endogenous time preference can be viewed as a particular special case of habit formation.

From a formal point of view, the main dynamic complication due to endogenous time preference is the nontrivial influence of an additional costate variable equal, at an optimum, to the maximized lifetime utility function. The extra costate can be eliminated in interesting special cases, however; and even when

[3]Thus, Blanchard and Fischer (1989, p. 75) warn students that endogenous time-preference models of the Uzawa variety are "not recommended for general use."

this is impossible, it may be quite easy to account for its effects. The behavior implied by endogenous time preference is often strikingly similar, at least in qualitative terms, to that implied by the standard model. The main difference seems to be that the sometimes troublesome nonconvergence and hysteresis implied by constant time preference disappear.

The paper is organized as follows. Section 2 analyzes individual optimization in an economy with discrete trading periods, starting out with a two-period model and then generalizing to an infinite horizon. Particular attention is paid to the effect on time preference of current consumption. The intuition gained in section 2 is applied in section 3, which describes a continuous-time formulation more amenable to diagrammatic methods. Section 4, which is the heart of the paper, explores the geometry of the system of dynamic necessary optimality conditions derived in section 3. This geometry is used to derive the individual's responses to exogenous changes in the parameters of the maximization problem. Section 5 presents a final application, to the problem of optimal capital accumulation in a closed economy.

2. A Discrete-Time Formulation

It is useful to begin with a discrete-time formulation of the Uzawa-Epstein approach to modeling intertemporally dependent preferences. The goal of this prologue is primarily expository. A discrete-time exposition clarifies (i) the nature of time preference; (ii) the role of certain regularity conditions in assuring existence and uniqueness of optima; and (iii) the meaning of the first-order necessary conditions for optimality that arise in the continuous-time case.

Until section 5, which studies neoclassical growth, the exposition is phrased in terms of an individual consumer's maximization problem. The analysis applies equally well, however, to a small open economy facing a given world interest rate, or to a closed economy in which the marginal product of capital is a technological constant.

2.1. *A Two-Period Model*

The simplest case arises when a consumer lives for two periods and maximizes the lifetime utility from consumption,

$$(1) \quad U(c_1,c_2) = u(c_1) + u(c_2)exp[-\theta(c_1)].$$

Maximization is carried out over nonnegative consumption levels, subject to the lifetime budget constraint

$$(2) \quad c_1 + c_2 exp(-r) \leq a_1,$$

in which r is the instantaneous real interest rate and a_1 the consumer's total real wealth at the start of period 1.

The consumer discounts second-period felicity by the factor $exp[-\theta(c_1)]$, which is allowed to depend on first-period consumption.[4] In the usual time-additive setup, $\theta(c) = \theta$, a constant, so the discount factor for future felicities u(c) does not depend on consumption choices.

Provided $U(c_1,c_2)$ satisfies regularity conditions, a unique

[4] The discussion follows the convention of Arrow and Kurz (1970) in referring to the subutility functions u(c) appearing in (1) as *felicities*. In contrast, the term "utility" always refers to the consumer's intertemporal objective.

interior solution to the consumer's problem exists and is characterized by the familiar marginal equalities emphasized by Irving Fisher (1930). Consistent with the infinite-horizon perspective adopted later, regularity of the lifetime utility function is achieved through conditions imposed on the building-block functions u(c) and $\theta(c)$.[5]

First, there are some standard assumptions. Both u(c) and $\theta(c)$ are assumed to be twice continuously differentiable. The discount rate $\theta(c)$ is taken to be *strictly* positive [$\theta(c) > 0$]. And u(c) is required to be strictly increasing [$u'(c) > 0$] and strictly concave [$u''(c) < 0$].[6]

In addition, some more specialized conditions--which are obviously much stronger than necessary--suffice to deliver Fisher's intertemporal optimum. One possible set of sufficient conditions consists of the following inequalities:

(3) $\theta''(c) \leq 0;$

(4) $u'(c_1) - \theta'(c_1)u(c_2)exp[-\theta(c_1)] > 0, \quad \forall\ c_1,\ c_2;$

(5) $u(c)u''(c) - u'(c)^2 \geq 0.$

Condition (3) requires $\theta(c)$ to be concave. Condition (4) (as shown in a moment) is the requirement that lifetime utility rise with first-period consumption. Finally, condition (5), which

[5]As will be apparent, some of these conditions are cardinal in nature. All predictions of the model, however, are invariant with respect to affine transformations of the lifetime utility function.

[6]Appropriate Inada conditions can be used to assure interior solutions.

restricts u(c) to be negative,

$$(6) \quad u(c) < 0 ,$$

states that log[-u(c)] is a convex function of consumption.

The role of these assumptions is best appreciated by examining first- and second-order conditions for an optimum. Let λ_1 be the Lagrange multiplier attached to constraint (2). Then the first-order conditions are

$$(7) \quad U_1(c_1,c_2) = u'(c_1) - \theta'(c_1)u(c_2)exp[-\theta(c_1)] = \lambda_1,$$

$$(8) \quad U_2(c_1,c_2) = u'(c_2)exp[-\theta(c_1)] = \lambda_1 exp(-r).$$

Equation (7) equates the marginal utility of initial consumption--a positive number, by (4)--to the shadow price of wealth λ_1. Importantly, the marginal utility of initial consumption is the sum of two components: first, the additional first-period *felicity* from an increase in c_1, and second, the resulting change in the *discounted* value of second-period felicity. (The second component would disappear with a constant discount factor, $\theta'(c) = 0$). Equation (8) has the marginal present utility of consumption falling over time at rate r.

Under the assumptions already made, the utility function $U(c_1,c_2)$ is strictly concave:

$$U_{11} = u''(c_1) + [(\theta'(c_1)^2 - \theta''(c_1)]u(c_2)exp[-\theta(c_1)] < 0;$$

$$U_{22} = u''(c_2)exp[-\theta(c_1)] < 0;$$

$$U_{11}U_{22} - U_{12}^2 = u''(c_1)u''(c_2)exp[-\theta(c_1)]$$

$$+ \theta'(c_1)^2[u(c_2)u''(c_2) - u'(c_2)^2]exp(-2\theta(c_1)]$$

$$- \theta''(c_1)u''(c_2)u(c_2)exp[-2\theta(c_1)] > 0.$$

The consumer's choice problem therefore has the unique optimum shown at point A in figure 1, where (2) holds as an equality.

2.2. *The Rate of Time Preference*

So far, no direct stand has been taken on the sign of the derivative $\theta'(c)$, that is, on the question of how initial consumption influences the discount factor applied to future felicity. A closely related question asks how the rate of time preference changes as consumption changes; but the two questions are not the same in general. Figure 1 is useful in illustrating the distinction.

Conditions (7) and (8) together equate the marginal rate of substitution of future for present consumption to the inverse market discount factor:

$$MRS(c_1,c_2) = U_1(c_1,c_2)/U_2(c_1,c_2) = exp(r).$$

The pure subjective rate of time preference can be defined as the natural logarithm of this marginal rate of substitution for a *constant* consumption path, $c_1 = c_2 = c$. In terms of figure 1, rates of time preference depend on the slopes of the indifference curves U and U′ as they cross the 45° line. If the subjective discount rate is a constant θ, the time-preference rate is fixed at θ as well; but generally, $\log\mathrm{MRS}(c,c) = \log\{\exp[\theta(c)] - \theta'(c)u(c)/u'(c)\}$ depends on c.[7] Figure 1 illustrates the utility contours of an individual whose time-preference rate rises with consumption. The interpretation is that the individual's impatience to consume increases as actual consumption rises.

What determines whether time preference rises with consumption, as in the figure, or falls? Define the elasticities

$$(9) \qquad \epsilon_u = -cu'/u > 0, \quad \epsilon_{u'} = -cu''/u' > 0, \quad \epsilon_{\theta'} = -c\theta''/\theta' \gtrless 0,$$

and consider the total derivative

$$(10) \qquad d\mathrm{MRS}(c,c)/dc = \theta'(c)\{\exp[\theta(c)] - 1 + \epsilon_u^{-1}(\epsilon_{u'} - \epsilon_{\theta'})\}.$$

Notice that if $\theta'(c) < 0$, then $\epsilon_{\theta'} < 0$ and the time-preference rate is necessarily decreasing in consumption. But if $\theta'(c) > 0$, the time preference rate can rise or (if $\epsilon_{\theta'}$ exceeds $\epsilon_{u'}$ by a sufficiently large amount) fall as consumption rises. Intuitively, if $\theta'(c)$ falls very rapidly relative to $u'(c)$ as c rises, $U_1(c,c)$ can fall toward $u_1(c,c)$ so quickly that future consumption becomes more valuable, at the margin, relative to present.

There is considerable disagreement over whether impatience to consume should increase or decrease as actual consumption rises. Koopmans (1986, pp. 94-95) suggests that a majority of economists would make an introspective argument in favor of *decreasing* impatience; while Epstein (1987a, pp. 73-74), who surveys the debate on this issue, offers several counter-arguments. Lucas and Stokey (1984) point out that something like increasing

[7]Use (7) and (8) to derive the last expression.

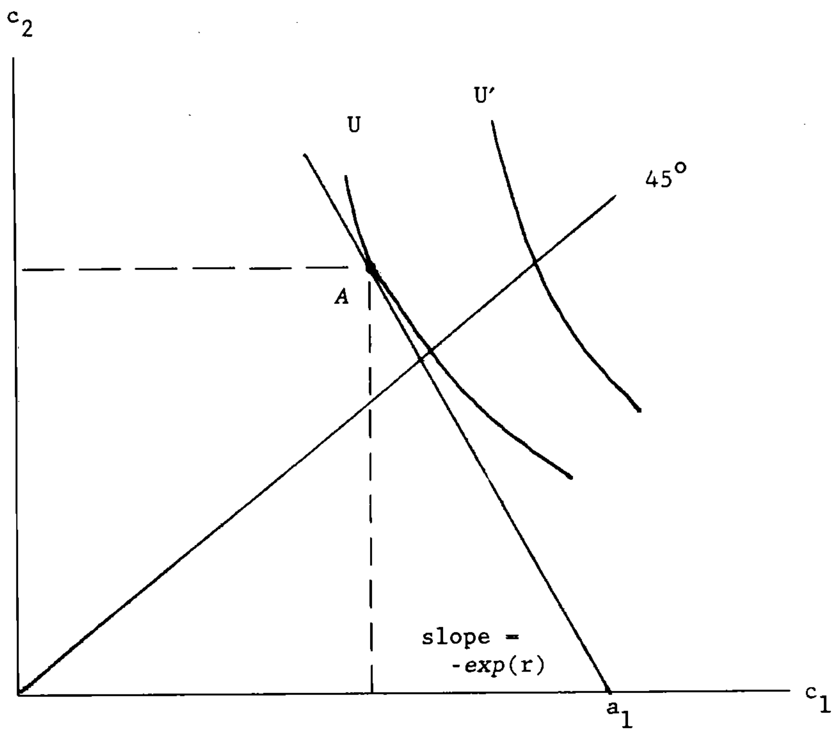

Figure 1: Optimal consumption with a variable time preference rate

impatience--a sort of diminishing *private* returns to saving--is needed to produce unique, stable, nondegenerate steady-state wealth distributions in a deterministic infinite-horizon setting.[8]

In the present two-period setting, none of the results derived above requires assumptions about the slopes of the time-preference function *log*MRS(c,c) or the subjective discount-rate function $\theta(c)$. (It is true, however, that if $\theta'(c) < 0$ and $\theta(c)$ is concave, $\theta(c)$ turns negative at some finite consumption level.)

Matters are different once the consumer's horizon is extended indefinitely (as will become evident in section 4). In that case the assumption $\theta'(c) > 0$ is necessary for stability of the long-run optimal consumption plan (and indeed, it may be necessary for the existence of an individual optimum). While the rate of time preference still need not equal $\theta(c)$, the two rates do coincide in the steady state. Since it is necessary for stability, the condition

$$(11) \qquad \theta'(c) > 0$$

will generally be imposed from now on, although further arguments in favor of this choice will be introduced in context.

Probably the best way of appreciating the meaning of inequality (11) is to look at the cross-partial derivative of the utility function,

$$(12) \qquad U_{12}(c_1,c_2) = -\theta'(c_1)u'(c_2)exp[-\theta(c_1)].$$

[8]Svensson and Razin (1983) recognize a related point in the setting of a small open economy, but they do not regard it as a compelling argument in favor of a time preference rate that rises with consumption.

If $\theta'(c) > 0$, U_{12} is negative and consumptions in the two periods are Edgeworth substitutes: higher consumption in period 1, say, lowers the marginal utility of period-2 consumption by raising the discount rate for second-period felicity. Edgeworth substitutability is behind the infinite-horizon stability result mentioned above. The wealth accumulation process converges when $\theta'(c) > 0$ because, as wealth and consumption rise, the marginal private return to further saving, which depends on the marginal utility of future consumption, falls.

If $\theta'(c) < 0$, however, consumptions in different periods are Edgeworth complements, not substitutes, and a rise in present consumption raises the marginal utility of future consumption. An intertemporal complementarity assumption might well be plausible in a model where goods are habit forming, but it seems rather unlikely applied to consumption in general. This is an additional, introspective argument in favor of the assumption that the subjective discounting of future felicity rises with consumption. As argued in the next subsection, the argument becomes even more compelling in an infinite-horizon setting.

2.3. *The Infinite-Horizon Case*

For a consumer with an infinite planning horizon, the analogue of utility function (1) is

$$(13)\qquad U(C_1) = \sum_{t=1}^{\infty} u(c_t)\mathit{exp}[-\sum_{s=1}^{t-1}\theta(c_s)],$$

which is maximized subject to the constraint

$$(14) \quad \sum_{t=1}^{\infty} c_t exp[-r\cdot(t-1)] \le a_1.$$

In (13), C_1 stands for the consumption path c_1, c_2, c_3, ... To ensure convergence of the sum (13), the conditions that u(c) be bounded below and that $\theta(c)$ be bounded away from zero can be assumed (Epstein 1987a, p. 75).

Introducing again the Lagrange multiplier λ_1, one obtains the first-order necessary conditions

$$(15) \quad u'(c_t)exp[-\sum_{s=1}^{t-1}\theta(c_s)] - \theta'(c_t)\sum_{\nu=t+1}^{\infty} u(c_\nu)exp[-\sum_{s=1}^{\nu-1}\theta(c_s)]$$

$$= \lambda_1 exp[-r\cdot(t-1)], \quad \forall t \ge 1.$$

The conditions in (15) are completely analogous to (7) and (8), and they have the same interpretation. The present marginal utility due to an increase in consumption on date t equals the marginal felicity that results, plus the induced effects on the weights attached to felicities accruing *after* date t. (Naturally, this total is subjectively discounted to date 1.) As before, the marginal utility of consumption is planned to fall at rate r.

For future reference, it is helpful to introduce two definitions. Denote the current-value marginal utility of consumption by

$$(16) \quad q_t = \lambda_1 exp[\sum_{s=1}^{t-1}\theta(c_s) - r\cdot(t-1)],$$

and the current value of utility from date t+1 onward by

$$(17) \quad \phi_t = \sum_{\nu=t+1}^{\infty} u(c_\nu) exp[-\sum_{s=t}^{\nu-1} \theta(c_s)].$$

With the help of these definitions, (15) can be rewritten as

$$(18) \quad u'(c_t) - \theta'(c_t)\phi_t = q_t,$$

which expresses (15) in current-value terms. The multiperiod analogue of (4),

$$(19) \quad u'(c_t) - \theta'(c_t)\phi_t \geq 0 \quad \forall\ t,\ C_1,$$

is assumed from now on.

Before going on to pursue dynamic analysis in a continuous-time framework, it is useful to look briefly at the implications of (13) about the complementarity/substitutability of consumption in different periods. The cross-partial derivative of $U(C_1)$ with respect to consumption on two dates t and v is

$$U_{tv}(C_1) = -\theta'(c_t)[u'(c_v) - \theta'(c_v)\phi_v] exp[-\sum_{s=1}^{v-1} \theta(c_s)].$$

Given assumption (19), the condition $\theta'(c) > 0$ is necessary and sufficient for consumptions on different dates to be Edgeworth substitutes. Further, the expression above suggests that as $v - t \to \infty$, the cross-partial's current value goes to zero (at least for consumption paths bounded away from zero).

The utility specification with $\theta'(c) > 0$ therefore has the intuitive implication that consumption on any given date lowers the marginal utility of consumption on successive dates, but with a strength that declines as those dates become more distant. The implication seems most attractive when decision intervals are short, because it is then more likely that a higher level of consumption in one period lowers the marginal utility from consuming in the next.

3. The Model in Continuous Time

A continuous-time formulation of the model greatly eases the study of its dynamic behavior. Differential equations describing first-order necessary conditions are derived below using optimal control theory, and in a way that brings out the analogy between the present model and other models with time-nonseparabilities.

3.1. *First-Order Conditions*

To place the model in a form amenable to solution by optimal-control methods, assume that the consumer maximizes

$$(20) \qquad U[C(0)] = \int_0^\infty u[c(t)]\,exp\{-\int_0^t \theta[c(s)]ds\}dt$$

subject to

$$(21) \qquad \dot{a}(t) = ra(t) - c(t), \qquad a(0) > 0 \text{ given.}^9$$

In (20), C[(0)] is a (piecewise differentiable) consumption path originating at t = 0. Define the cumulated "excess" subjective

[9]It is assumed that the implicit constraint $a(t) \geq 0$ never binds at an optimum.

discount rate[10] by

$$\Theta(t) = \int_0^t \{\theta[c(s)] - r\}ds. \tag{22}$$

Then the consumer's problem can be given the alternative formulation: maximize

$$U[C(0)] = \int_0^\infty u[c(t)]exp[-\Theta(t)]exp(-rt)dt \tag{23}$$

subject to (21) and

$$\dot{\Theta}(t) = \theta[c(t)] - r, \qquad \Theta(0) = 0. \tag{24}$$

Since the felicity function $u(c)exp(-\Theta)$ is strictly concave in c and Θ under the assumptions made in the last section, the above problem is of the same general form as some others in which marginal consumption utilities are intertemporally dependent, for example, the addiction problem analyzed by Becker and Murphy (1988). The essential feature of the latter model is that felicity directly depends on a stock variable whose rate of accumulation depends, in turn, on consumption of some of the available goods. Here the relevant stock is Θ, an indicator of accumulated impatience.

[10]Uzawa (1968) uses a similar transformation to change variables from t to a cumulated time-preference measure in equations (20) and (21); he then analyzes the individual's optimal-control problem as one involving a single state variable, assets. As pointed out by Kompas and Abdel-Razeq (1987), however, this procedure is invalid when the transition equation (21) is nonautonomous. Since I will want to study anticipated disturbances below, and these shift equation (21) over time, a more general solution approach involving the two state variables $a(t)$ and $\Theta(t)$ is used below.

Necessary conditions for an optimum are derived by introducing the costate variables, $\tilde{q}(t)$ and $\tilde{\phi}(t)$, and the current-value Hamiltonian

$$(25) \quad H[c(t),a(t),\theta(t),\tilde{q}(t),\tilde{\phi}(t)] = u[c(t)]exp[-\theta(t)] + \tilde{q}(t)[ra(t) - c(t)] - \tilde{\phi}(t)\{\theta[c(t)] - r\}.$$

The conditions can be expressed as

$$u'[c(t)]exp[-\theta(t)] - \theta'[c(t)]\tilde{\phi}(t) = \tilde{q}(t),$$

$$\dot{\tilde{q}}(t) = 0,$$

$$\dot{\tilde{\phi}}(t) = r\tilde{\phi}(t) - u[c(t)]exp[-\theta(t)].$$

(See Arrow and Kurz 1970, pp. 48-49.) To simplify the dynamic analysis, it is convenient to rescale the costate variables so as to eliminate $\theta(t)$ from the above equations. Let $q(t) \equiv \tilde{q}(t)exp[\theta(t)]$ and $\phi(t) \equiv \tilde{\phi}(t)exp[\theta(t)]$. Then the necessary conditions take the form

$$(26) \quad u'[c(t)] - \theta'[c(t)]\phi(t) = q(t),$$

$$(27) \quad \dot{q}(t) = q(t)\{\theta[c(t)] - r\},$$

$$(28) \quad \dot{\phi}(t) = \theta[c(t)]\phi(t) - u[c(t)].$$

Notice that when $\phi(t)$ converges to some definite long-run value, as will be assumed below, differential-equation (28) has the solution

$$(29) \quad \phi(t) = \int_t^\infty u[c(\nu)]exp(-\int_t^\nu \theta[c(s)]ds)d\nu;$$

in words, $\phi(t)$--just as in equation (17)--is the discounted present value of the future felicity stream from the standpoint of time t. Since the costate transition equation (27) is the continuous-time version of (16), (29) implies that condition (26) above has the same basic interpretation as did (18), to which it is formally identical.

The interpretation is as follows. Consider the term

$$u'[c(t)] - \theta'[c(t)]\int_t^\infty u[c(\nu)]exp(-\int_t^\nu \theta[c(s)]ds)d\nu$$

on the left-hand side of (26). If U[C(t)] is the lifetime utility function maximized at time t, the preceding expression measures the increase in U[C(t)] caused by a small increase in the consumption path at and near its starting point--a generalized notion of the marginal utility of consumption at time t. In technical terms, the relevant notion of derivative here is the *Volterra derivative* of U[C(t)] with respect to c(t),[11] denoted $D_V\{U[C(t)],c(t)\}$. Just as the marginal felicity of consumption rises at rate $\theta - r$ in the standard model, here the Volterra derivative of utility with respect to initial consumption rises at rate $\theta[c(t)] - r$.

[11]See Epstein and Hynes (1983) and Epstein (1987). An early application of Volterra derivatives to intertemporal choice problems is Wan (1970).

3.2. *The Role of the Time-Preference Rate*

The quantity

$$(30) \quad D_V(U[C(t)], c(t)) exp[-\Theta(t) - rt] = \left\{u'[c(t)] - \theta'[c(t)]\phi(t)\right\} exp[-\Theta(t) - rt],$$

which is the t = 0 present value (in utility terms) of the left-hand side of (26), measures the increase in U[C(0)] due to an upward perturbation of the consumption path at time t ≥ 0. It is the Volterra derivative $D_V(U[C(0)], c(t))$.

Expression (30) can be used to calculate the rate of time preference in this continuous-time model. In analogy with the definition adopted in section 2, it is reasonable to associate a time-preference rate with the rate of decrease in the marginal utility of consumption along a *locally constant* consumption path. Clearly, the rate of time preference associated with any locally constant c(t) on a consumption path C(0) generally will vary with c(t) (as before), as well as with subsequent consumption on C(0).

Epstein and Hynes (1983) and Epstein (1987a) suggest this local approach to defining time preference.[12] They consider a consumption path C(0) and a point c(t) on it such that $\dot{c}(t) = 0$. The rate of time preference at that point, denoted ρ, is (minus) the logarithmic time derivative of $D_V(U[C(0)], c(t))$:

$$\rho = -\frac{d}{dt} log D_V(U[C(0)], c(t))\Big|_{\dot{c}(t)=0}.$$

Application of this formula and of (28) to (30) shows that ρ

[12]In contrast, Koopmans' (1986) definition assumes a globally constant consumption path.

depends only on c(t) and $\phi(t)$:[13]

$$(31) \quad \rho = \rho(c,\phi) = \theta(c)\left\{1 + \frac{[\phi - u(c)/\theta(c)]}{[u'(c) - \phi\theta'(c)]}\,\theta'(c)\right\} > 0.$$

In a stationary state with constant consumption $\bar{c}$, $\phi = \bar{\phi} = u(\bar{c})/\theta(\bar{c})$, so (31) implies that $\rho(\bar{c},\bar{\phi}) = \theta(\bar{c})$ in this case. More generally, however, (31) shows that the time-preference rate exceeds the discount rate $\theta(c)$ when consumption is rising toward a stationary state, and is below the discount rate in the opposite case.[14] Formally, equation (30) shows why this is so: a rise in the rate of increase of ϕ, other things the same, causes the marginal utility of consumption to fall more quickly over time. This property of the model captures some of the intuition behind the "introspective" finding that people with lower consumption levels discount the future more heavily.

By combining equations (26)-(28), we obtain a dynamic equation for consumption that clarifies the role of the time-preference rate defined above. Time-differentiation of (26) shows that $[u''(c) - \theta''(c)\phi]\dot{c} - \theta'(c)\dot{\phi} = \dot{q}$. Use (28) to eliminate $\dot{\phi}$ and (27) to eliminate $\dot{q}$; then apply (26) to eliminate q. After some manipulation, the solution for $\dot{c}$ that emerges is

$$(32) \quad \dot{c} = \frac{[u'(c) - \theta'(c)\phi]}{[u''(c) - \theta''(c)\phi]}\left[\rho(c,\phi) - r\right],$$

which states that consumption is optimally falling when the

[13]Explicit time-indexation is dropped from now on except when there is a risk of confusion.

[14]Notice that although (31) has been calculated along a path with locally constant consumption, the formula can be applied to *any* path. The function $\rho(c,\phi)$ simply measures the rate at which marginal utility *would* fall with time, given c and ϕ, if c were to remain momentarily fixed.

time-preference rate exceeds the interest rate, rising in the opposite case, but stationary when $c = \bar{c}$ and

$$\rho(\bar{c},\bar{\phi}) = r = \theta(\bar{c}).$$

Of course, when $\theta(c)$ is a constant θ, (32) reduces to the familiar $\dot{c} = [u'(c)/u''(c)](\theta - r)$.

One final characterization of an optimal program is noted. Denote by $V[a(0)]$ the maximized value of (23) (under the constraints). If the optimal costate $\phi(0)$ has the convergent representation given in (29), then the value function $V[a(0)]$ and $\phi(0)$ are the equal. Furthermore, by analogy with standard results, we would expect that $V[a(0)]$ is differentiable and that its derivative satisfies $V'[a(0)] = D_V\{U[C(0)],c(0)\} = u'[c(0)] - V[a(0)]\theta'[c(0)]$ for the optimal consumption path. Since the foregoing relationships will continue to hold for $t > 0$, (26) and (28) imply that

$$V'[a(t)]\dot{a}(t) = q(t)\dot{a}(t) = \theta[c(t)]\left\{ V[a(t)] - \frac{u[c(t)]}{\theta[c(t)]} \right\}.$$

An analogous expression holds in the constant time-preference model. It implies that the value of saving (in utility terms) equals the annuity flow of the discrepancy between maximized utility and the utility obtained by maintaining the current consumption level permanently. The "interest rate" used to calculate this flow is the subjective discount rate.[15]

[15]A key assumption used in deriving this relation is that *all* exogenous factors impinging on the system are constant over time.

3.3. *Relationship to Uzawa's (1968) Formulation*

It is easily shown that intertemporal preferences of the form suggested by Uzawa (1968) are within the class studied by Epstein (1987a). The basic argument follows quite closely one that Nairay (1984) used to prove that the Uzawa problem has a solution.

Uzawa assumed that utility is given by the functional

$$(33)\quad U[C(0)] = \int_0^{\infty} u[c(t)]\,exp\left(-\int_0^t \hat{\theta}[c(s)]ds\right)dt,$$

where $\hat{\theta}(c) = \delta(u(c))$ for some function $\delta:\mathbb{R} \rightarrow \mathbb{R}$. The functions u(c) and $\delta(u)$ are assumed to satisfy the restrictions

$$(34)\quad u(0) = 0, \quad u(c) > 0, \quad u'(c) > 0, \quad u''(c) < 0,$$

$$(35)\quad \delta(u) > 0, \quad \delta'(u) > 0, \quad \delta''(u) > 0, \quad \delta(u) - u\delta'(u) > 0.$$

Notice that the discount rate is increasing in consumption, because $\hat{\theta}'(c) = \delta'[u(c)]u'(c) > 0$ [see (11)]. To obtain Epstein's formulation, one must add to Uzawa's conditions (34) and (35) the requirement that $\hat{\theta}(c)$ be concave, $\hat{\theta}''(c) = \delta'[u(c)]u''(c) + \delta''[u(c)]u'(c)^2 \le 0$.

Under the preceding specification, it is easy to see that $u(c)/\hat{\theta}(c) = u(c)/\delta[u(c)]$ is strictly increasing in c but remains bounded as $c \rightarrow \infty$, even if u(c) doesn't.[16] Define $\gamma \equiv \sup_{c>0}\{u(c)/\hat{\theta}(c)\}$ and use this definition to create a new felicity function, $\hat{u}(c) \equiv u(c) - \gamma\hat{\theta}(c)$.

[16]Under (35), one can always find a linear function a + bu, $a \ge 0$, $b > 0$, such that $\delta(u) \ge a + bu$ for $u \ge 0$. Then $u/\delta(u) \le u/(a + bu)$, so even if $\lim_{c\to\infty} u(c) = \infty$, $\lim_{c\to\infty} u/(a + bu) = 1/b \ge \lim_{c\to\infty} u/\delta(u)$.

The point of defining $\hat{u}(c)$ is that

$$\hat{U}[C(0)] = \int_0^\infty \hat{u}[c(t)]exp\{-\int_0^t \hat{\theta}[c(s)]ds\}dt = U[C(0)] - \gamma,$$

so maximization of $\hat{U}[C(0)]$ leads to the same optimal consumption path as maximization of $U[C(0)]$. However, $\hat{u}(c)$ not only is negative [see (6)], it also is increasing and strictly concave in c.[17] Thus $\hat{u}(c)$ satisfies conditions sufficient to yield Epstein's formulation, with the possible exception of log-convexity [see (5)]. Log-convexity is not, however, an essential property for most purposes.

4. The Geometry of Dynamic Adjustment

The dynamics implied by an optimal plan can be described in terms of three variables. Two of these, consumption and wealth, are present in the standard time-additive setup. In the time-nonadditive case, however, it is also necessary to keep track of the present discounted value of future utility, since that variable influences the marginal utility of current consumption.

This section analyzes the properties of the resulting three-variable dynamic system. It is useful to analyze first a two-variable subsystem of the full system within which the full system's essential properties are apparent. The extension to three dimensions is then straightforward. Concluding the section are

[17]Proof: $\hat{u}'(c) = u'(c) - \gamma\theta'(c) = u'(c)\{1 - \gamma\delta'[u(c)]\}$ and $\hat{u}''(c) = u''(c) - \gamma\theta''(c) = -\gamma\delta''[u(c)]u'(c)^2 + u''(c)\{1 - \gamma\delta'[u(c)]\}$. Therefore, by (34) and (35), $\hat{u}'(c) > 0$ and $\hat{u}''(c) < 0$ if $1 - \gamma\delta'(u) > 0$ $\forall u$. To establish this last inequality, observe that by (35), $1 > [u/\delta(u)]\delta'(u)$ $\forall u$; but since $\delta''(u) > 0$, $1 > [\lim_{u\to\infty} u/\delta(u)]\delta'(u) = \gamma\delta'(u)$.

some applications of the model to anticipated disturbances.

4.1 *Dynamics of Lifetime Utility and Consumption*

The dynamic behavior of optimal plans is described by three equations, (21), (28), and (32). They are reproduced below for easy reference:

(21) $\dot{a} = ra - c$

(28) $\dot{\phi} = \theta(c)\phi - u(c)$

(32) $\dot{c} = \frac{[u'(c) - \theta'(c)\phi]}{[u''(c) - \theta''(c)\phi]} \left[\rho(c,\phi) - r\right].$

Three-variable systems typically are difficult to analyze. That this will not be the case here is due to the block-recursive nature of the equations of motion. Assets a enter the system only through equation (21); (28) and (32) form a separate subsystem in ϕ and c. Thus, while consumption drives asset accumulation, the level of assets affects lifetime utility and consumption only by determining the initial optimal choices of those variables, Once these initial choices are made, ϕ and c evolve autonomously.

Figures 2 and 3 show two possible configurations of the phase diagram described by equations (28) and (32). Along the $\dot{\phi} = 0$ schedule, future lifetime utility equals the current felicity level divided by the current subjective discount rate: $\phi = u(c)/\theta(c)$. Along the $\dot{c} = 0$ schedule, the time preference and interest rates coincide: $\rho(c,\phi) = r$. The shape of the latter schedule is derived by observing that

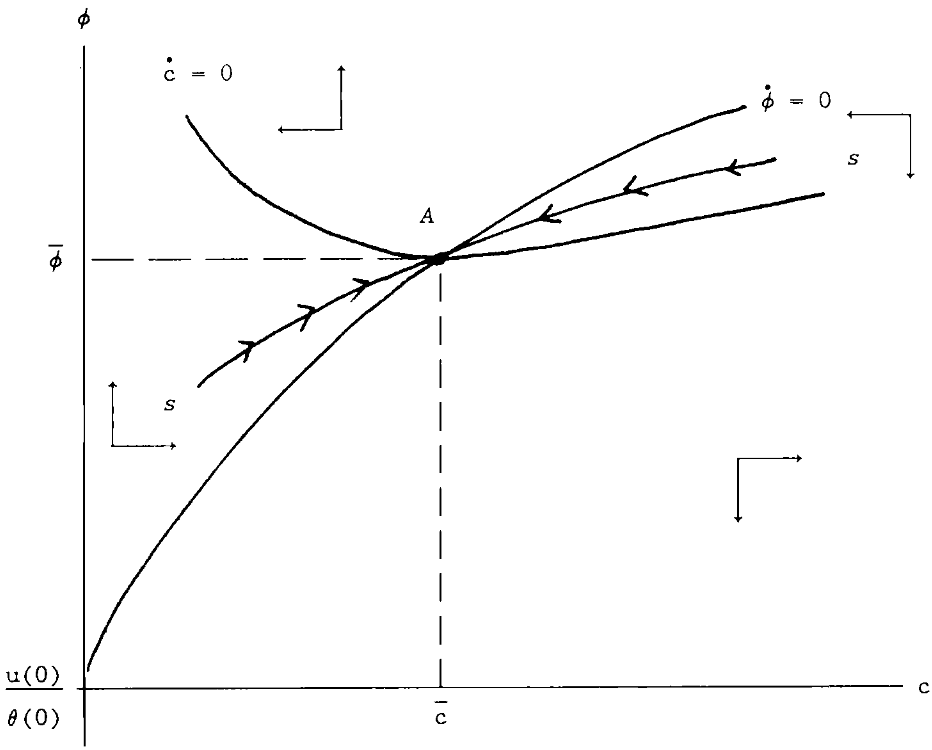

Figure 2: Adjustment when $\epsilon_{u'} > \epsilon_{\theta'}$

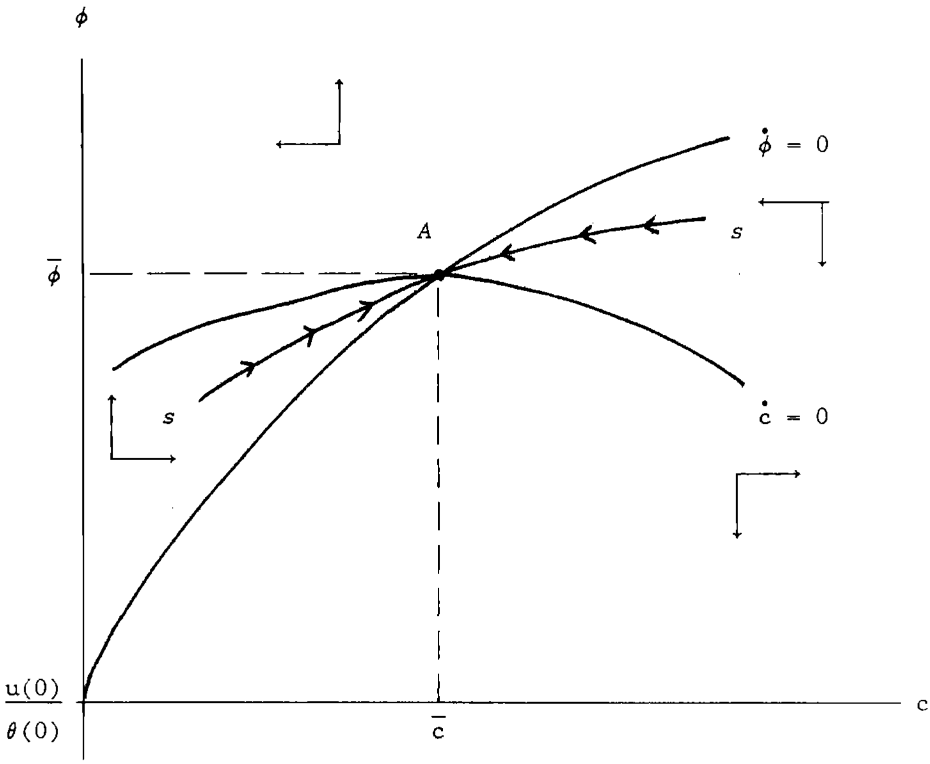

Figure 3: Adjustment when $\epsilon_{u'} < \epsilon_{\theta'}$

$$(36)\quad \left.\frac{d\phi}{dc}\right|_{\rho=r} = -\rho_1/\rho_2 = \left[\frac{\epsilon_u}{c(\epsilon_u + \epsilon_\theta)}\right]\left[\phi - \frac{u(c)}{\theta(c)}\right]\left(\epsilon_{\theta'} - \epsilon_{u'}\right),$$

where $\epsilon_\theta \equiv c\theta'(c)/\theta(c)$ and the other elasticities are defined in (9) above. Since $\bar{\phi} = u(\bar{c})/\theta(\bar{c})$ at the stationary state A, the $\dot{c} = 0$ schedule has a zero slope there. Where $\phi > u(c)/\theta(c)$, the schedule has a positive slope if $\epsilon_{\theta'} - \epsilon_{u'} > 0$ and a negative slope in the opposite case; where $\phi < u(c)/\theta(c)$, this dependence is reversed.

The two possible patterns of dynamics in the figures reflect the ambiguity in the sign of $\rho_1 = \partial\rho/\partial c$. As in the two-period model of section 2, the sign depends on that of $\epsilon_{u'} - \epsilon_{\theta'}$. Specifically, *sign*$(\partial\rho/\partial c)$ = *sign*$(\phi - u/\theta)(\epsilon_{u'} - \epsilon_{\theta'})$.[18] This ambiguity does not arise if the subjective discount rate is of the Uzawa form, $\hat{\theta}(c) = \delta[u(c)]$, because $\epsilon_{\hat{\theta}'} = \epsilon_{u'} - c\delta''(u')^2/\delta'u' < \epsilon_{u'}$. Thus, Epstein's formulation does enrich the class of behaviors consistent with the endogenous time-preference model.

In either case, the stationary state A is a saddlepoint. Lifetime utility and consumption rise or fall together along the unique path *ss* converging to this position. The rate of time preference $\rho(c,\phi)$ falls toward the discount rate $\theta(c)$ as c increases along *ss*, and rises toward it in the opposite case; both rates equal the interest rate r in the long run. Notice that the

[18]It is straightforward to establish that the $\dot{\phi} = 0$ locus is strictly concave, as drawn. Notice also that in the case shown in figure 2, the $\dot{c} = 0$ locus can never intersect the $\dot{\phi} = 0$ locus from below, a situation that would produce a second stationary point. The reason is that the $\dot{\phi} = 0$ locus has a strictly positive slope--equal to $[u'(c) - \theta'(c)\phi]/\theta(c)$--while the $\dot{c} = 0$ locus has a slope of zero whenever $\dot{\phi} = 0$ [see equation (36)].

convergence pattern in the figures implies a positive correlation between anticipated consumption growth and anticipated income growth, contrary to the simplest permanent-income model of consumption. Intertemporally dependent preferences therefore have the potential to rationalize the patterns of consumption and income comovement noted by Summers and Carroll (1989).

How should one interpret figures 2 and 3, which in themselves provide no way of determining the initial choices $\phi(0)$ and $c(0)$? Remember that the dynamics of ϕ and c are embedded in a larger system involving the stock of assets, a, which is a predetermined state variable. It is the initial value of assets, $a(0)$, relative to the long-run value $\bar{a} = \bar{c}/\rho$, that determines the initial position. As the next subsection shows figures like 2 and 3 can be thought of as projections of the full, three-variable system onto the two-dimensional (c,ϕ) plane. The stable path in that plane is just the projection of a unique stable path converging to $(\bar{c},\bar{\phi},\bar{a})$ in the full three-dimensional system.

The critical implication of the full system's block-recursive structure is that, given the parameters of the differential equations governing ϕ and c, system dynamics depend *only* on the current values of ϕ and c. As a result, figures 2 and 3 represent utility and consumption dynamics in any situation where the $\dot{\phi}$ and $\dot{c}$ equations are expected to remain constant over time. Importantly, such situations include those in which certain changes are anticipated to occur in the future, provided these changes do not affect the parameters appearing in (28) or (32).

Before discussing this feature of the model in detail, I note the relation between the saddlepoint-stability of the model and the assumption that the discount rate increases with current consumption. Linear approximation of the two-variable system around point A yields two *positive* characteristic roots when $\theta'(c) < 0$. Since the addition of equation (21) entails one more positive root, r, the full system can have the two-positive, one-negative root pattern needed for saddlepoint stability only if $\theta'(c) > 0$.

4.2. *The System in Three Dimensions*

A complete, three-dimensional view of the system requires the inclusion of the asset accumulation equation, (21). Figure 4 illustrates the dynamics of convergence to this long-run equilibrium, which is labeled A and corresponds to the point A in figures 2 and 3. The (c,ϕ) plane in figure 4 reproduces the subsystem dynamics shown in those figures. The $\dot{a} = 0$ locus in the (c,a) plane is the line along which $c = ra$; assets rise to the right of this line and fall to its left, at rates independent of the value of ϕ. An optimal path that converges to A from an initial asset stock of $a(0) < \bar{a}$, say, must entail an initial consumption level $c(0) < ra(0)$, so that assets are increasing, as well as an initial lifetime utility level $\phi(0) < \bar{\phi}$. The subsystem saddlepath *ss* is the projection onto the (c,ϕ) plane of the full-system saddlepath *SS*.

To see how at least one convergent path *SS* can be constructed, let $c[t;c(0),\bar{c}]$ be a consumption path satisfying the subsystem consisting of (28) and (32), as well as the boundary conditions $c[0;c(0),\bar{c}] = c(0)$, $\lim_{t\to\infty} c[t;c(0),\bar{c}] = \bar{c}$. In words, $c[t;c(0),\bar{c}]$ is just the portion of the stable path in figure 2 or 3 originating at c(0). Then if c(0) in figure 4 is chosen so that the intertemporal budget constraint $\int_0^\infty c[t;c(0),\bar{c}]\,exp(-rt)dt = a_0$ holds, the path starting at $[c(0),\phi(0),a(0)]$ converges to point A.

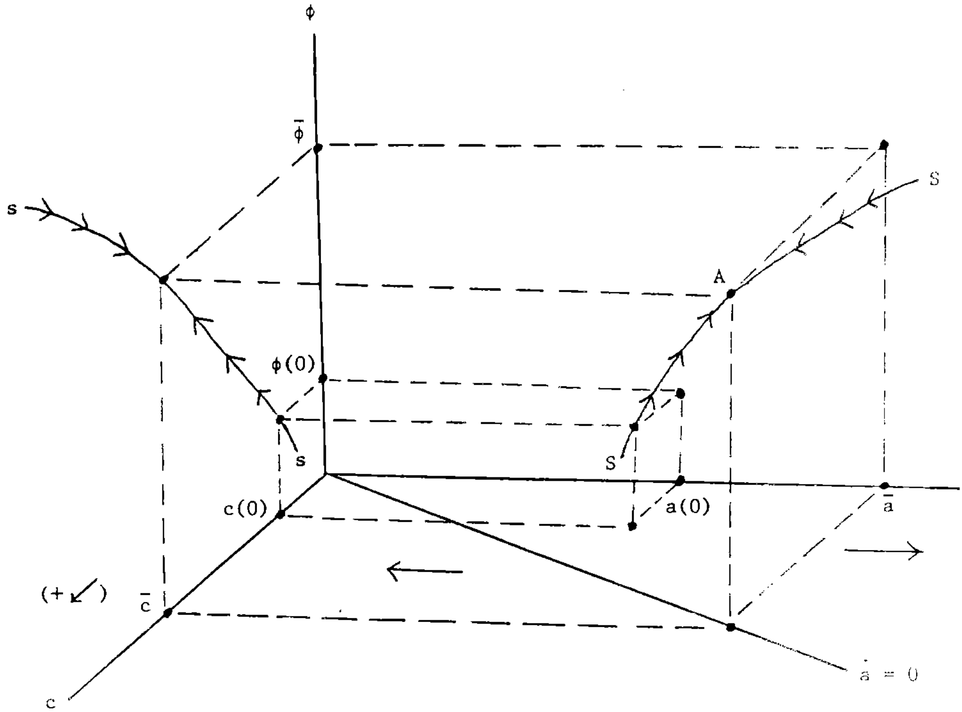

Figure 4: Adjustment in the full system

That *SS* describes optimal plans, given initial asset stocks, follows from the sufficiency theorem for optimal controls (Arrow and Kurz 1970, p. 49). Under assumption (5), the strict concavity of u(c) ensures that the maximand $u(c)exp(-\theta)$ in (23) is strictly concave in c and θ; at the same time, (21) is linear while the right-hand side of (24) is strictly concave in c. It follows that the maximized Hamiltonian (25) is strictly concave in a and θ. Optimality now follows from the conditions $\lim_{t\to\infty}\tilde{q}(t)exp(-rt) \geq 0$, $\lim_{t\to\infty}-\tilde{\phi}(t)exp(-rt) \geq 0$, and $\lim_{t\to\infty}\tilde{q}(t)a(t)exp(-rt) = \lim_{t\to\infty}-\tilde{\phi}(t)\theta(t)exp(-rt) = 0$, which evidently are valid along *SS*.[19]

Figures 2 and 3 make it clear that corresponding to any initial a(0) is a *unique* $[c(0),\phi(0)]$ pair placing the economy on a convergent path to A. If there were another such, $[c(0)',\phi(0)']$ say, we would have to have $\phi(0) = \phi(0)'$, since there can be at most one maximized level of utility corresponding to a given initial asset stock; and this equality implies that c(0) must equal c(0)′ if the consumer's program converges.

In summary, it has been argued that convergent solutions (i) exist, (ii) are optimal, and (iii) are unique. It is assumed from now on that only convergent solutions are optimal solutions.

4.3. *Disturbances that Do Not Affect the* $\dot{c}$ *or* $\dot{\phi}$ *Equations*

Disturbances that do not affect equations (28) or (32) are the easiest to analyze. Such disturbances--whether permanent, anticipated, or transitory in nature--obviously do not alter figures 2 or 3 at all, and in particular do not affect the

[19]As mentioned in footnote 1, the existence of an optimum for variable time-preference problems is established by Becker, Boyd, and Sung (1989) in a general setting. Their results show that the regularity conditions that I imposed above are considerably stronger than necessary.

long-run values $\bar{c}$ and $\bar{\phi}$. Because these types of disturbances must operate through the accumulation equation (21), the long-run asset level $\bar{a}$ does change, as does the relation between asset holdings and optimal consumption levels.

Dynamics are comparatively easy to analyze when (28) and (32) are autonomous (i.e., time-independent) because in this case, the system's motion is confined to a two-dimensional submanifold of Euclidean three-space, R^3, and therefore is easy to visualize. These two-dimensional dynamics follow from the block-recursivity of the full system. Changes in c and ϕ depend only on their current levels, and not on the current asset stock a. If the equations of motion for c and ϕ are autonomous, however, then c and ϕ must always evolve along *ss* in equilibrium, independently of the current value of a. But the set of points (c,ϕ,a) such that $(c,\phi) \in ss$ is a two-dimensional submanifold of R^3. The upper shaded area in figure 5 shows a section of this submanifold, M. [20]

While $c \to \bar{c}$ and $\phi \to \bar{\phi}$ along all trajectories lying on M--and, conversely, all such paths lie on M--paths other than *SS* imply divergent asset stocks. Indeed, the perpendicular joining $(\bar{c},\bar{\phi},\bar{a}) \in SS$ and $(\bar{c},\bar{\phi}) \in ss$ describes an unstable eigenvector of the dynamic system "trapped" on M.

The laws of motion governing paths on M are simple to describe algebraically. Denote the functional relationship between c and ϕ along *ss* by

(37) $\phi = \psi(c), \quad \psi'(c) > 0.$

[20]More formally, let $P_{(c,\phi)}: R^3 \to R^2$ be the projection map $P_{(c,\phi)}[(c,\phi,a)] = (c,\phi)$. Then $M = \{x \in R^3 | \; P_{(c,\phi)}(x) \in ss\}$.

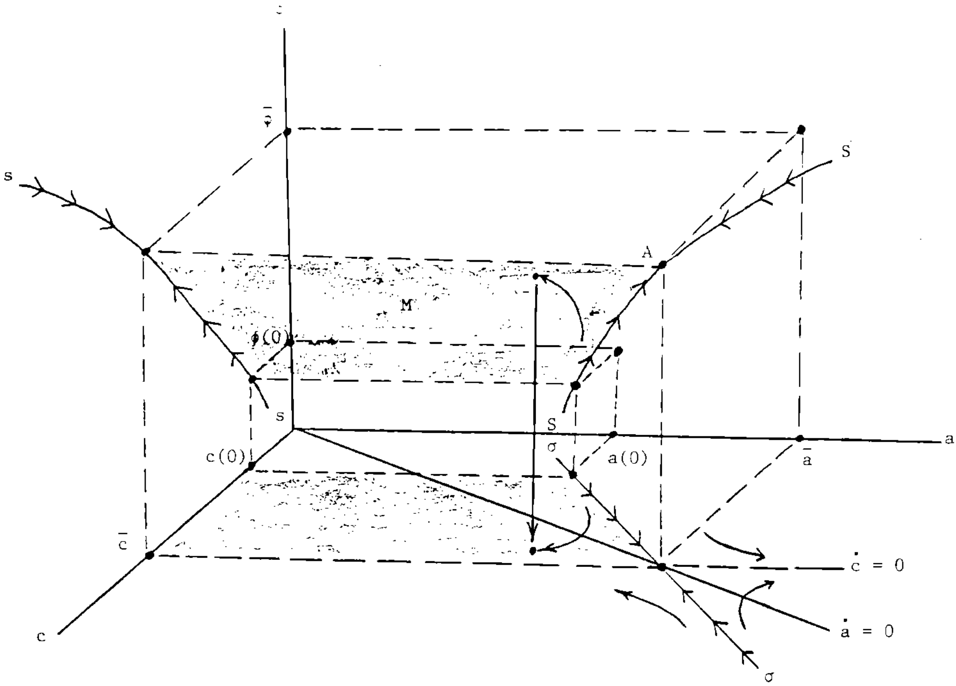

Figure 5: Dynamics of consumption and assets

Then the joint motion of c and a on M described by equation (21) and the equation that results from combining (32) and (37),

$$(38)\qquad \dot{c} = \frac{[u'(c) - \theta'(c)\psi(c)]}{[u''(c) - \theta''(c)\psi(c)]}\left\{\rho[c,\psi(c)] - r\right\}$$

The phase portrait for system described by (21) and (38) is obtained by projecting M onto the (c,a) plane, as figure 5 shows. This projection can be used to analyze any disturbance that shifts the asset-accumulation equation alone.[21] The image of SS, denoted $\sigma\sigma$, is the convergent path of the simplified system.

As an illustration, let $\tau(t)$ be a lump-sum tax levied by the government, so that asset accumulation is given by

$$(39)\qquad \dot{a}(t) = ra(t) - c(t) - \tau(t).$$

Suppose that initially the tax is set at $\tau = 0$, but that it is unexpectedly and permanently raised to $\tau' > 0$. Figure 6, which shows dynamics in the (c,a) plane of figure 5, indicates the adjustment process from an initial long-run equilibrium (point A).

The rise in taxes causes the $\dot{a} = 0$ locus to shift rightward, but it does not affect equation (38), which defines the

[21]Disturbances that shift (28) or (32) would generally invalidate the optimality of the relationship $\phi = \psi(c)$, and therefore would invalidate (38) above. An alternative way of reducing the system consisting of (21), (28), and (32) to one in c and a only is to use the value function defined in section 3.2. Since $\phi = V(a)$ along a convergent path [equation (29)], this function of a may be substituted for ϕ in (32) to yield [with (21)] a system in c and a. This method is, however, even less versatile than the one described in the text. The reason is that in many interesting cases, the value function depends on factors other than a. For example, an anticipated future income increase clearly makes calendar time an additional argument of the value function. In contrast, such a shock does not alter the relation between c and ϕ described by ss.

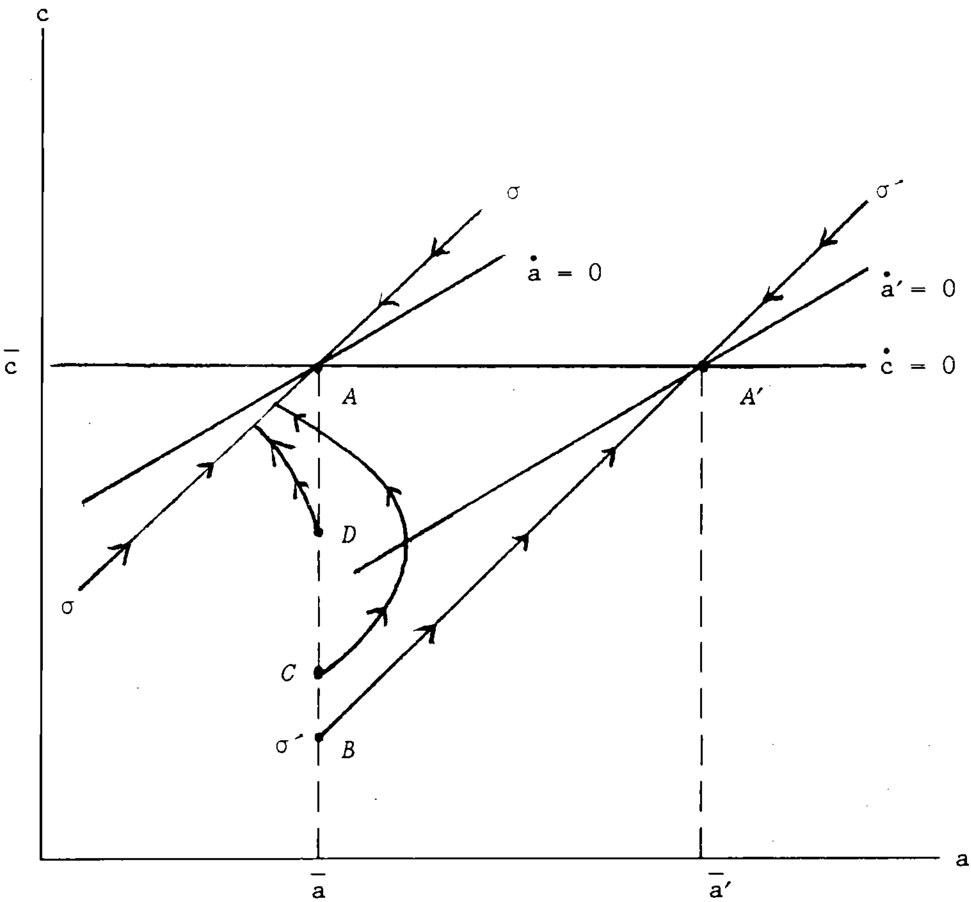

Figure 6: Permanent and transitory tax increases

$\dot{c} = 0$ locus. Long-run consumption is still determined by $\theta(\bar{c}) = r$; however, the long-run asset stock consistent with $\bar{c}$ after the shock is $\bar{a}' = (\bar{c} + \tau')/r > \bar{a}$. This long-run position can be reached only if saving rises immediately. Accordingly, figure 6 shows that consumption drops to the level given by point B on the new saddlepath $\sigma'\sigma'$ converging to A'. Subsequently c rises back to its original level, a result in sharp contrast to the once-and-for-all, permanent drop in consumption predicted by the constant time-preference model with $\theta = r$.

It is simple to extend this analysis to the case of a temporary tax increase. Key to this extension is the observation that consumption will not take a discontinuous jump when the anticipated fall in taxes is implemented. Points C and D in figure 6 show the initial reactions to temporary tax increases of different duration, with point C corresponding to the longer-lived increase.

In both cases, there is an immediate drop in consumption, a reflection of the fall in lifetime income; and in both cases, consumption subsequently rises as the consumer returns to point A. The disturbance pushing the individual to point D leads him to run down his assets while the higher tax is on, as in models with constant time preference. The move to point C, however, is caused by a reduction in lifetime income so large that initially the consumer's saving *rises*. This makes intuitive sense: the longer the duration of the shock, the closer it is to being permanent.

Clearly, the degree of consumption smoothing predicted by the endogenous time preference model is lower than that predicted by traditional permanent-income type models. Other things equal, the consumers described above prefer stronger initial reactions to income shocks, a response pattern that allows them to return eventually to habitual consumption levels.

4.4. *Interest-Rate Disturbances*

The interest rate enters directly into the dynamic equations for consumption and lifetime utility, so the mode of analysis used so far does not automatically apply to all types of interest-rate disturbance. Permanent, unanticipated interest-rate changes remain easy to analyze, however, using the preceding framework.

The long-run relationship between assets and the interest rate is determined by $\theta(r\bar{a}) = r$. While long-run consumption rises if r does, long-run assets can rise or fall. If the initial position is one of long-run equilibrium, a small increase in r raises long-run assets if the elasticity ϵ_{θ} exceeds 1, and lowers them in the opposite case. Figure 7 shows how the economy adjusts in a case where $\epsilon_{\theta} < 1$ at the initial equilibrium. In the case pictured, consumption falls initially to produce the required rise in saving (point B); but it is possible, for different parameter choices, that consumption rises when the disturbance occurs.

When $\epsilon_{\theta} > 1$, long-run assets must fall, and consumption must therefore rise on impact. Consumption overshoots its eventual level in the short run, a result impossible when the subjective discount rate is relatively inelastic with respect to consumption. Since the assumption $\epsilon_{\theta} < 1$ leads to preferences more comparable with time-additive preferences ($\epsilon_{\theta} = 0$), I focus on the inelastic case in discussing the effects of transitory interest-rate movements.

Transitory changes in interest rates are most easily studied in a diagram like figure 8. The figure is drawn on the assumption

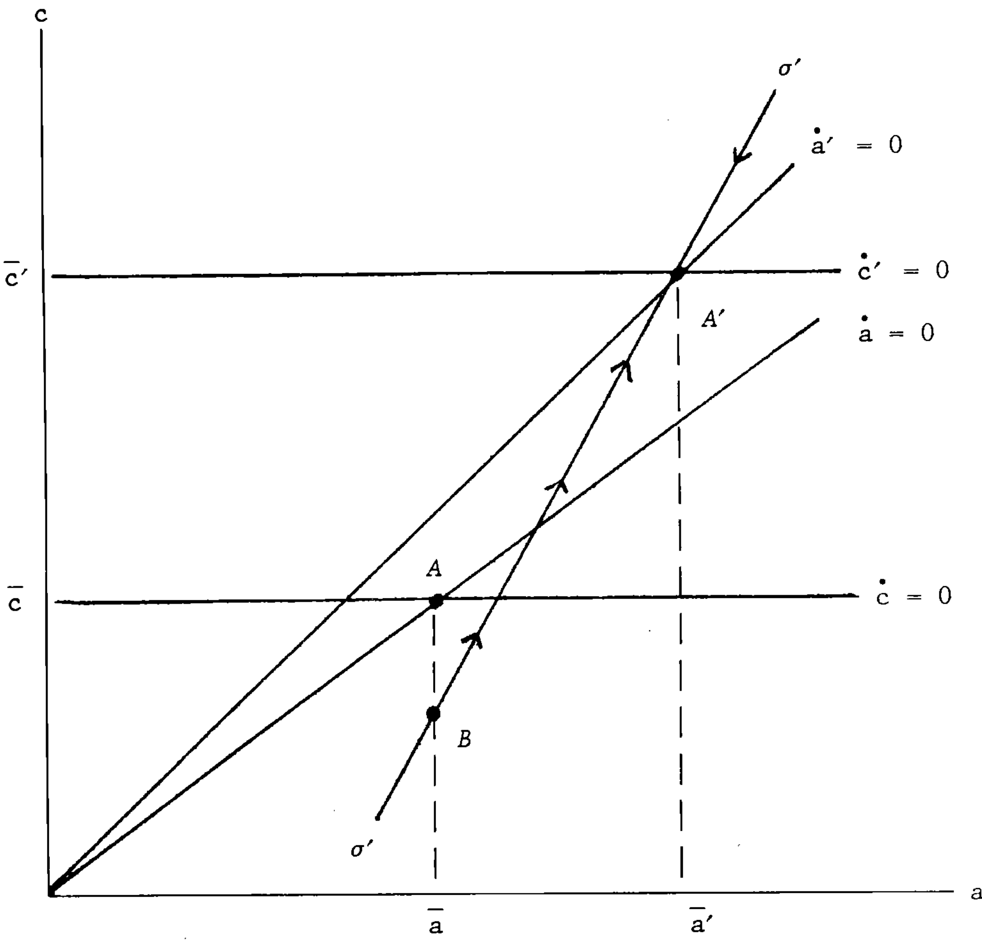

Figure 7: A permanent rise in the interest rate

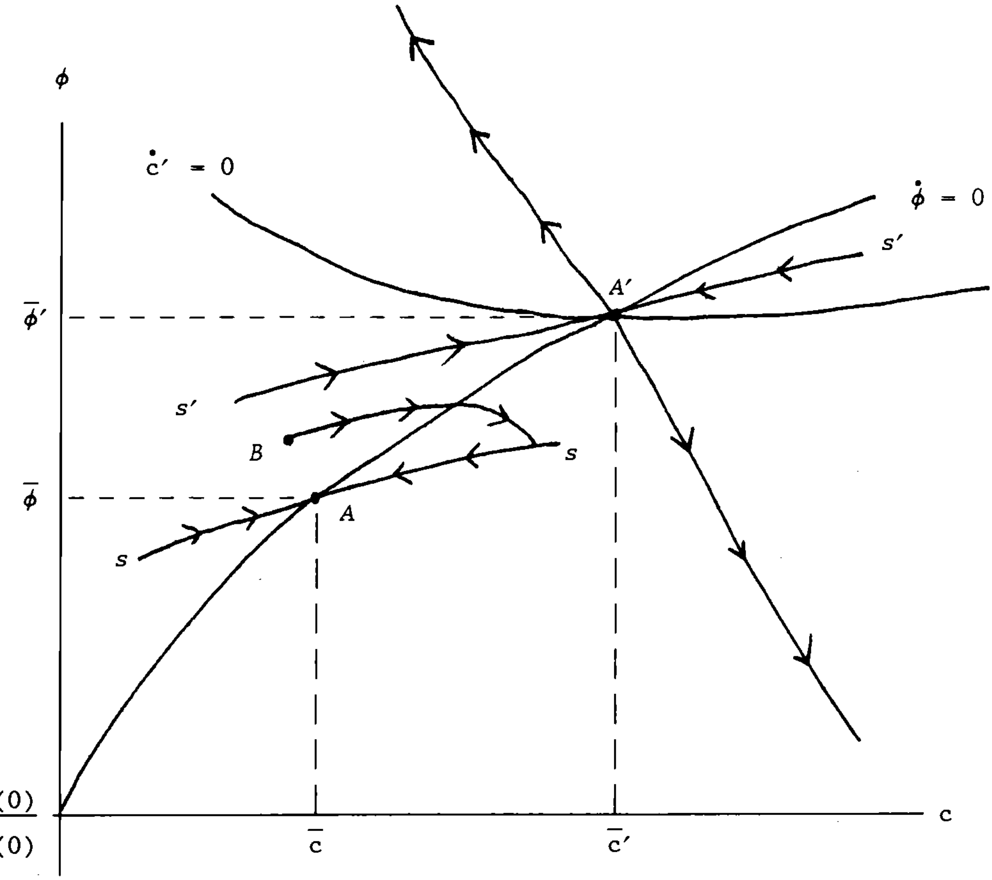

Figure 8: A temporary rise in the interest rate

that consumption initially falls when the interest rises temporarily to r' from r. A *permanent* rise in r would shift *ss* upward to *s's'*, producing a new long-run equilibrium (point A') with higher consumption and higher lifetime utility than at the original long-run equilibrium (point A). An optimal consumption plan necessarily follows the Euler equations of the high-interest-rate system while the interest rate is high; after the rate falls, the optimal plan follows *ss*. Neither consumption nor future lifetime utility can jump at the moment of transition. These conditions lead to the path shown.

Why does figure 8 show ϕ initially rising when the economy moves from point A to point B? If $\bar{a}$ is the initial asset level, it is *feasible* for the consumer to raise consumption immediately to $r'\bar{a}$ and lower it back to $r\bar{a}$ when the interest rate falls. This plan raises lifetime utility above $\bar{\phi}$, and an *optimal* plan necessarily does even better.[22]

The diagram makes clear why the analysis applied to previous disturbances, which assumed a time-invariant consumption-utility relationship of the form $\phi = \psi(c)$, is now inapplicable. During the period before the interest rate reverts to its initial level, consumption must evolve along an *unstable* path of the system in the (c,ϕ) plane. In terms of figure 5, this means that the optimal program involves points off the submanifold *M*. It is therefore no longer legitimate to analyze the comovement of consumption and assets using the two-dimensional projection of *M*.

The optimal response to the disturbance calls for an initial

[22]Let T be the length of the time interval over which the interest rate is at $r' > r$. The feasible plan described (which merely holds saving steady at zero) has a utility value of $exp[-\theta(r'\bar{a})T]\bar{\phi}$ + $\{1 - exp[-\theta(r'\bar{a})T]\}\,\dfrac{u(r'\bar{a})}{\theta(r'\bar{a})} > \bar{\phi} = \dfrac{u(r\bar{a})}{\theta(r\bar{a})}$.

rise in saving. Saving then declines over time, turning negative by the time of the anticipated fall in the interest rate. Notice that over part of the transitional period, ϕ is falling even though consumption is rising, contrary to the positive correlation of these variables along a saddlepath. Lifetime utility is falling despite rising current consumption because the drop in the interest rate is approaching, and consumption is expected to follow a declining path after that event.

It is instructive to compare the above scenario with the response that would occur were the time-preference rate constant at $\theta = r$. Once again, consumption can rise or fall when the interest rate temporarily rises to r'; in the case of logarithmic utility, for example, consumption would remain constant at $\theta\bar{a}$ on impact. After the initial instant, however, consumption rises according to the equation $\dot{c} = (-u'/u'')(r' - \theta)$. This rising transitional consumption path agrees with the result shown in figure 8 for variable time preference. There are two main differences. With a constant θ, assets remain at a new, higher level once r' falls back to r, rather than returning to their original level. (That is, the shock has hysteretic effects.) In addition, the constant-θ case does not admit the possibility of initial consumption overshooting or initial dissaving (which occurs above when $\epsilon_\theta > 1$).

5. Neoclassical Growth in a Closed Economy

A final example applies the above framework to the neoclassical growth problem of Ramsey (1928), Cass (1965), and Koopmans (1965). For this purpose, the maximization problem is interpreted as a planning problem, and net output is assumed to be a function f(k) of the capital stock k. As usual, f(k) is assumed to be twice continuously differentiable, with $f'(k) \geq 0$ over an initial range of capital stocks, and $f''(k) < 0$ everywhere.

Proceeding as before, one derives the necessary conditions

$$(40) \qquad \dot{c} = \frac{[u'(c) - \theta'(c)\phi]}{[u''(c) - \theta''(c)\phi]} \left[\rho(c,\phi) - f'(k)\right],$$

$$(41) \qquad \dot{k} = f(k) - c,$$

and (28). Notice that, provided no parameters are expected to change, the dynamic system described by these equations is fully autonomous. Since the value function V(k) is therefore independent of time, it is legitimate to eliminate (28) by imposing the equality $\phi = V(k)$ valid along convergent paths. Substitution transforms (40) into an equation involving c and k only,

$$(42) \qquad \dot{c} = \frac{[u'(c) - \theta'(c)V(k)]}{[u''(c) - \theta''(c)V(k)]} \left\{\rho[c,V(k)] - f'(k)\right\}.$$

Figure 9 shows a possible configuration of the system described by (41) and (42). Given the convergence assumption made to obtain (42), only the convergent path $\sigma\sigma$ has economic significance. The steady-state values of consumption and capital are uniquely determined by the conditions

$$(43) \qquad \bar{c} = f(\bar{k}),$$

$$(44) \qquad \theta(\bar{c}) = f'(\bar{k}).$$

When $\theta'(c) > 0$, optimal growth has the same essential characteristics as in the standard model.[23] Consumption and capital

[23]If $\theta'(c)$ could be negative, the possibility of multiple steady states would arise

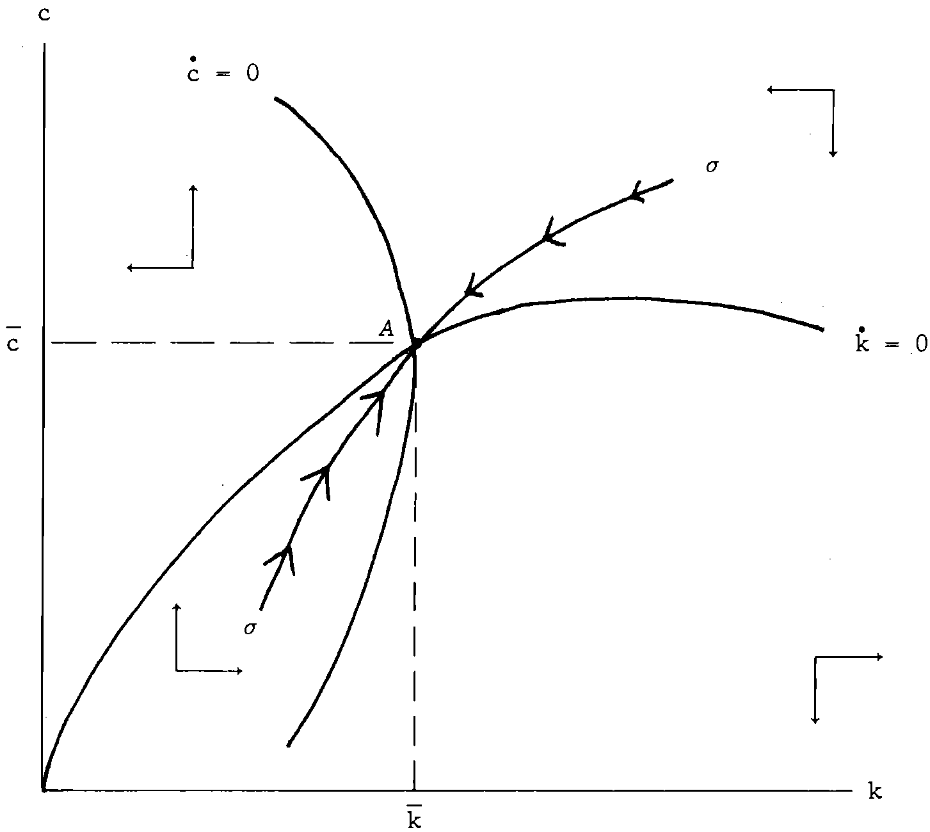

Figure 9: Neoclassical growth with variable time preference

rise or fall together to a steady state A described by a modified golden rule.

6. Concluding Remarks

This exploration of endogenous time-preference models has illustrated the straightforward methods available for understanding the consumption behavior they predict. Models with variable time-preference rates typically yield the more intuitive economic implications of constant-rate models, for example, that consumption should rise during a period of temporarily high interest rates. The main difference is that variable time-preference models imply a well-defined long-run target wealth level. This determinacy is not only empirically plausible, it is also an analytical advantage in many cases.

The assumption of constant time-preference rates has been useful in some recent models of sustained endogenous growth in per capita consumption (see Romer 1989 for a survey). It should be noted, therefore, that the formulation of endogenous time preference described above is fully consistent with those models. An assumption that the concave, increasing discount-rate function $\theta(c)$ is bounded from above would be enough in principle to open the door to unbounded individual consumption growth.

The stability of this paper's models is closely linked to the hypothesis that subjective discount rates are increasing functions of consumption. This hypothesis does not command widespread agreement. It appears, however, to yield more reasonable economic predictions than the alternative view, that people become more willing to defer consumption as consumption possibilities rise.

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