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STRATEGIC INVESTMENT UNDER UNCERTAINTY
WITH FIRST- AND SECOND-MOVER ADVANTAGES

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ABSTRACT

We analyze firm entry in a duopoly real-option game. The interaction between first- and second-mover advantages gives rise to a unique Markov subgame-perfect symmetric equilibrium, featuring state-contingent pure and mixed strategies in multiple endogenously-determined regions. In addition to the standard option-value-of-waiting region, a second waiting region arises because of the second-mover advantage. For sufficiently high market demand, waiting preserves the second-mover advantage but forgoes profits. Two disconnected mixed-strategy regions where firms enter probabilistically surface. In one such region, Leader earns monopoly rents while Follower optimally waits. Finally, when the first-mover advantage dominates the second-mover advantage, firms enter using pure strategies.

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1 Introduction

In strategically competitive industries, the first mover is often perceived to have an advantage over followers.¹ However, it is sometimes advantageous to be the second mover. For example, by observing and learning from the first mover’s successes and failures, the second mover can lower its production cost and have a more efficient production technology. The ulcer-relief drug Zantac is a known successful second mover, which enjoyed lower R&D and legal compliance costs than the industry leader (Berndt, Pindyck, and Azoulay, 2003).² Using the oil and gas sector, a widely used example for real-option analyses, Décaire and Wittry (2022) document that “a firm’s incentive to delay its investment grows with the amount of information expected to be released by its peers.” This is because a firm prefers to exercise its growth option as the second mover in order to acquire information revealed by its peers’ investment decisions. Décaire and Wittry (2022) conclude that “In equilibrium, a war-of-attrition regarding the timing of investment decisions among peers ensues, resulting in delays.”

In this paper, we develop a tractable duopoly entry model to parsimoniously capture the equilibrium implications of both the first-mover and second-mover advantages. Our model delivers the following insights. First, when the first-mover advantage dominates, firms play pure entry strategies as in Fudenberg and Tirole (1985). Second, when the second-mover advantage dominates, a firm delays its entry with the hope that its competitor enters as Leader. This is because the entry game is effectively a war of attrition with Leader being the loser of the game, which is consistent with empirical findings in Décaire and Wittry (2022). A key prediction of our model is that firms use mixed strategies to enter probabilistically for some ranges of market demand.³ For empirical support of mixed strategies in dynamic games, see Wang (2009) who analyzes firm price-setting strategies in retail gasoline markets.

An important feature of our model is the nonlinear interaction between the two types of

¹For example, being the first mover may enable a firm to establish strong brand recognition and product loyalty. Also, by entering first, a firm makes it less profitable for its competitor to enter. As a result, the competitor voluntarily waits until the total market demand rises to a higher level before entering the market. In equilibrium, Leader earns monopoly profits until Follower enters.

²For the second-mover advantage analysis using the Zantac case, also see https://insight.kellogg.northwestern.edu/article/the_second_mover_advantage, titled “The Second-Mover Advantage,” based on a marketing research article (Shankar et al., 1998), in Northwestern Kellogg’s Insight.

³Asymmetric pure-strategy equilibria also exist. We characterize these equilibria in Section 7.

advantages, which generates new predictions that would have been absent in a setting with only one of the two advantages. For example, in addition to a pure-strategy entry region, our model also predicts two disconnected waiting regions as well as two disconnected mixed-strategy entry regions. One immediate implication of this five-region prediction is that a firm's market entry propensity is non-monotonic in market demand.

Our model builds on the duopoly model of Dixit and Pindyck (1994) (in Chapter 9), which features the first-mover advantage only.⁴ Two *ex ante* identical firms, a and b , discount cash flows at the rate $r > 0$ and compete to enter a new product market where the demand X_t follows a geometric Brownian motion (with drift μ and volatility σ). Each firm can pay a fixed cost to enter the market at any time. Leader captures the entire demand until the second firm (Follower) enters. Upon entering the market, Follower takes away a half of the total market demand from Leader.⁵ Thus in our model, the first-mover advantage arises from the monopoly rents that Leader enjoys before Follower enters.

We make a single change to the duopoly model of Dixit and Pindyck (1994) by letting Follower's entry cost (K_F) differ from Leader's entry cost (K_L). When the entry-cost ratio $R = K_L/K_F > 1$, firms have a second-mover advantage. Depending on the market demand, the second-mover advantage either dominates or is dominated by the first-mover advantage.

There are three Markov subgame-perfect equilibria: one symmetric equilibrium and two asymmetric equilibria. We characterize solutions in closed form for all three equilibria. In an asymmetric equilibrium, which firm is Leader and which is Follower are known at the beginning. Firms use pure strategies characterized by entry thresholds.⁶ Compared with the asymmetric equilibria, the predictions of the symmetric equilibrium are much richer and the solution is much subtler. For example, probabilistic entry is an essential part of a firm's equilibrium entry strategy when $R > 1$. Moreover, there may exist disjoint mixed-strategy regions and disjoint waiting regions. Pure and mixed entry strategies can coexist.

⁴The duopoly model of Dixit and Pindyck (1994) builds on Smets (1991). Grenadier (1996) develops a real-option duopoly model to study real-estate development cascades and overbuilding. We provide a summary of Grenadier (1996) and compare our model with Grenadier (1996) in Appendix B and Related Literature of Introduction.

⁵To ease exposition, we focus on the case where the two firms equally split the market after Follower's entry. If Leader captures more than half of the duopoly market profits, we then have another source for the first-mover advantage. We can also allow the total market demand to depend on the industry structure (e.g., monopoly or duopoly). The key results of our main model remain valid in these more general settings.

⁶In one asymmetric equilibrium, firm a is Leader with probability one. In the other, firm b is Leader.

Our model has five parameters: the discount rate r , drift μ and volatility σ , and two entry costs (K_L and K_F). For a given set of $(r, \mu, \sigma, K_L, K_F)$, the symmetric equilibrium solution belongs to one of the three cases: **A**, **B**, and **C**.

In **Case A** where the entry-cost ratio R is so large that the second-mover advantage always dominates the first-mover advantage, firms prefer entering as Follower under all circumstances. However, without Leader there would be no Follower. Put differently, while being Follower is always better than being Leader, when market demand is high a firm forgoes a large profit each period it waits. A firm takes both the waiting benefit (to capture the second-mover advantage) and the waiting cost (to forgo a large profit) into account when making its entry decision. In equilibrium, when market demand is high, both firms must *probabilistically* enter at the same rate per unit of time.

But how do we determine the equilibrium entry probability (rate) for the mixed strategy? First, when using a mixed strategy, a firm must be indifferent between the strategies that it chooses to mix and its value is independent of its own entry rate.⁷ In equilibrium, the rate at which a firm probabilistically enters is chosen to make its competitor indifferent between entering as Leader and waiting for another period. In our symmetric equilibrium, which we show is Markov subgame perfect, firms enter at the same rate.

When one firm becomes Leader, the other luckily becomes Follower and its value immediately jumps upward reflecting the equilibrium result that the second-mover advantage dominates in **Case A**. After Leader is determined, Follower solves a single-firm real-option problem by choosing its optimal entry time to obtain its market share. Follower has two entry strategies: It either enters immediately after Leader does or waits until the market demand rises further. Which strategy Follower chooses critically depends on market demand X_t and the degree of the second-mover advantage measured by the entry-cost ratio R .

First, what happens when market demand is very high? In this case, Follower enters immediately after Leader does because Follower's profits are high. This suggests that there exists an endogenous threshold for market demand X_t , $\bar{x}$, above which firms play mixed strategies. That is, the $x \geq \bar{x}$ region is a probabilistic entry region. And once Leader is

⁷Otherwise, the firm should always choose the strategy that yields a higher payoff with probability one, contradicting the result that the equilibrium involves mixed strategies. See Tadelis (2013) for a textbook treatment on the mixed-strategy equilibrium.

determined, Follower immediately enters, leaving no monopoly rents for Leader.

Second, when market demand is low, firms wait for both the option value of waiting (as in standard single-firm real-option models) and the waiting-to-be-Follower (to lower its entry cost) reasons. This suggests that there exists another endogenous threshold for market demand, $\tilde{x}$, below which firms wait. That is, the $x < \tilde{x}$ region is the waiting region.

Third, what happens when market demand X_t further exceeds $\tilde{x}$? In this scenario, the market demand X_t for a single firm as a monopolist is sizeable but it is not large enough for two firms to share. It is thus not in Follower's interest to enter. Instead, Follower preserves its option value of waiting until market demand further rises. Anticipating Follower's entry strategy, both firms adopt the mixed strategy to enter probabilistically when market demand is medium. However, as market demand further rises, Follower's incentive to enter the market becomes stronger, which *reduces* the duration that Leader can earn monopoly rents and hence *discourages* firms to probabilistically enter as Leader.

This suggests another threshold $\underline{x}$, which defines a second mixed-strategy region where $x \in [\tilde{x}, \underline{x}]$. In this region, firms probabilistically enter as Leader. Once Leader is determined, Follower waits and then enters when market demand further rises to a threshold x_F that is higher than $\underline{x}$. A firm's equilibrium entry rate $\lambda(X_t)$ as Leader is generally not monotonic as Follower's incentive to enter increases with X_t , undermining firms' *ex ante* incentives to enter as Leader.

Finally, as market demand further rises, anticipating Follower's immediate entry, both firms wait with probability one until market demand reaches the threshold, $\bar{x}$, above which firms play mixed strategies as we discussed earlier. The $x \in (\underline{x}, \bar{x})$ region is a second waiting region (between the two probabilistic entry regions).

In sum, there are four regions in general for **Case A** where the second-mover advantage dominates: 1.) In the $x < \tilde{x}$ region, firms wait; 2.) In the $x \in [\tilde{x}, \underline{x}]$ region, firms enter probabilistically with Leader earning equilibrium monopoly rents; 3.) In the $x \in (\underline{x}, \bar{x})$ region, firms wait; and 4.) In the $x \geq \bar{x}$ region, firms enter probabilistically with Follower entering immediately after Leader and leaving no monopoly rents to Leader. Note that the two waiting regions and the two probabilistic entry regions alternate.

Next, we turn to **Case C** where there is no second-mover advantage as Leader's entry cost

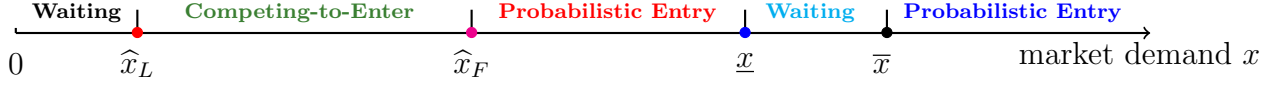


Figure 1: FIVE-REGION EQUILIBRIUM SOLUTION FOR THE GENERAL CASE.

is lower than Follower's ($R \leq 1$).⁸ When market demand is high, firms rush to enter and one firm is randomly selected as Leader in a way that *ex ante* rents that the two firms earn are equalized. There are two equilibrium regions: the waiting (for the standard real-option argument) region where $x < \hat{x}_L$ and the rushing-to-enter region (to capture the *first-mover advantage*) where $x \geq \hat{x}_L$. The endogenous threshold $\hat{x}_L$ is the turning point above which firms rush to become Leader.

Finally, we are ready to analyze **Case B** where R is larger than one but not very large. Because **Case B** is the intermediate case between **Case A** and **Case C**, we expect its solution is even richer, which indeed features five regions. On one hand, similar to **Case A**, the second-mover advantage dominates if market demand is sufficiently high, exceeding an endogenous threshold $\hat{x}_F$. On the other hand, with a weaker second-mover advantage (a lower R than in **Case A**), the first-mover advantage dominates in the $[\hat{x}_L, \hat{x}_F]$ region, which is to the right of the $(0, \hat{x}_L)$ waiting region. Recall that when market demand further rises above $\hat{x}_F$, firms anticipate Follower's stronger incentive to enter, which implies a shorter monopoly duration and lower monopoly rents for Leader, discouraging their *ex ante* incentives to enter as Leader. This explains why two firms compete to enter in the $[\hat{x}_L, \hat{x}_F]$ region but enter probabilistically in the $(\hat{x}_F, \underline{x}]$ region.

In sum, **Case B** in general features five equilibrium regions, as summarized in Figure 1.⁹

Related Literature. Our paper contributes to the strategic real-options literature.¹⁰ This literature has focused on the first-mover advantage. By introducing the entry-cost wedge between Leader and Follower, we generate new predictions that models with only the first-

⁸The duopoly model in Chapter 9 of Dixit and Pindyck (1994) corresponds to the special $R = 1$ case in our **Case C**.

⁹**Case B** has two subcases: **Subcase B₁** and **Subcase B₂**. The difference between the two subcases is the entry-cost ratio R , which is larger in **Subcase B₁**. In **Subcase B₂**, there are four regions in equilibrium as the fifth region (the mixed strategy region where Leader earns monopoly rents) disappears.

¹⁰Fudenberg, Gilbert, Stiglitz, and Tirole (1983) model preemption games (e.g., patent races) in deterministic settings. Smets (1991) studies irreversible investment in a duopoly setting and analyzes an asymmetric leader-follower equilibrium. Murto (2004) studies a duopoly exit game and focuses on pure strategies.

mover advantage cannot capture. For example, we show that a model with just five parameters (discount rate, drift and volatility of market demand, and two entry costs) can generate a five-region solution due to the interaction between the first- and second-mover advantages.

Grenadier (1996) studies real-estate redevelopment by incorporating the following feature into the duopoly real-option model in Chapter 9 of Dixit and Pindyck (1994): When Leader starts its redevelopment, Follower loses a fraction of its rents from its existing businesses. While both our and his models generate a non-monotonic firm entry propensity with respect to market demand, the mechanisms are different in the two models. In our model, the non-monotonicity arises from the strategic interaction induced by the second-mover advantage considerations. In contrast, there is no second-mover advantage in Grenadier (1996). Therefore, different from our model, there are no mixed strategies and no interaction between the first-mover and second-mover advantages in Grenadier (1996). Appendix B provides a detailed comparison between the Grenadier (1996) model and our model.

Grenadier (2002) and Back and Paulsen (2009) analyze continuous-time oligopoly capital-accumulation games. They show that competition causes firms to speed up their investment rates. Different from them, the second-mover advantage plays a crucial role and also firm entry is non-monotonic in market demand in our model. Also the underlying single-firm monopoly problem in their models is a singular-control problem and that is why investment is characterized by a running maximum capital stock.¹¹ In contrast, the underlying single-firm monopoly problem for our model is an optimal stopping-time problem and that is why investment is characterized by an endogenous cutoff threshold. In sum, the economic insights and mathematical analysis of our model are complementary to but quite different from those in Grenadier (2002) and Back and Paulsen (2009).

Whether an equilibrium is subgame perfect or not is a key issue in the real-option games literature. Back and Paulsen (2009) show that the perfectly competitive outcome, produced by closed-loop strategies that are firms' mutually best responses, is Markov subgame perfect in their capital accumulation game model. Similarly, we show that the equilibria in our model are Markov subgame perfect.¹²

¹¹Kogan (2001, 2004) develop production-based equilibrium asset-pricing models, using singular controls to model investment irreversibility.

¹²Firms in Back and Paulsen (2009) have no incentives to deviate from their equilibrium investment strate-

Lambrecht (2001) develops a duopoly *exit* model in a real-option setting with two *ex ante* heterogeneous firms and studies pure-strategy equilibria.¹³ In contrast, we analyze how *ex ante* identical firms become *ex post* heterogeneous via their market *entry* decisions and characterize pure-strategy and mixed-strategy equilibria as well as hybrid-strategy equilibria, which feature state-contingent pure and mixed strategies. In Dai, Jiang, and Wang (2023), we study equilibrium capital structure and firm default by incorporating firm reputation into a duopoly setting, and show that there exists a symmetric mixed strategy equilibrium.

The second-mover advantage part of our model mechanism is related to but different from the classic war-of-attrition argument.¹⁴ First, while classic attrition games are about firm exit, our model is about firm entry and thus has to be solved via backward induction. Second, while firm payoffs in war-of-attrition games are often exogenous (Levin, 2004), they are endogenous in our model. This is because Leader’s payoff depends on Follower’s equilibrium strategy and contains an option-value component. Importantly, our paper focuses on the interaction between the first- and the second-mover advantages, which induces the coexistence of state-dependent mixed and pure strategies in our duopoly model, absent in standard war-of-attrition models. Finally, a firm’s entry propensity is not monotonic in market demand unlike in war-of-attrition models.

Our model is also related to Chamley and Gale (1994) and Grenadier (1999), in which firms delay entry timing in anticipation of information spillover from peers’ decisions. Unlike these social-learning-based models, our model features entry-cost savings for the second mover generating the coexistence of state-contingent first-mover and second-mover advantages.

There is also a growing literature that integrates industrial organization considerations into asset pricing models. Dou, Ji, and Wu (2022) extend the standard Lucas-tree asset pricing model to allow for endogenous strategic competition. Chen, Dou, Guo, and Ji (2022) study how strategic competition and financial distress dynamically interact.

gies, which generate a perfectly competitive outcome with zero profits, as choosing not to invest also yields zero profits. Firms in our model do not deviate from their equilibrium strategies, as entering as Leader and waiting for another period yield the same payoff in the mixed-strategy region.

¹³Firms’ initial capital structures are exogenously given in Lambrecht (2001). This is the source of *ex ante* firm heterogeneity in his model. Lambrecht and Perraudin (2003) introduce incomplete information into an equilibrium real-option exercising model. Anderson, Friedman, and Oprea (2010) generalize Lambrecht and Perraudin (2003) to settings with multiple firms.

¹⁴Section 8.1 of Tirole (1988) and Levin (2004) offer introductions to the war-of-attrition literature.

2 Model

In this section, we set up an entry game in which two *ex ante* identical firms choose their timing to enter a new market.

2.1 Industry Structure and Market Demand

Let τ_L and τ_F denote the stochastic time when Leader and Follower enter the market, respectively. By definition, $\tau_F \geq \tau_L$. Which firm becomes Leader is endogenous and random. The industry structure has three phases. First, before either firm enters ($t < \tau_L$), neither firm receives any cash flow. Second, after Leader enters at τ_L and before Follower enters at τ_F , the industry is a monopoly. In this monopoly stage, Leader captures the total market profit, $\{X_t; t \in [\tau_L, \tau_F)\}$, where $\{X_t\}$ follows a geometric Brownian motion (GBM) as in McDonald and Siegel (1986) and Dixit and Pindyck (1994):

$$dX_t = \mu X_t dt + \sigma X_t dZ_t. \quad (1)$$

The parameters, μ and $\sigma > 0$, are the drift and volatility of the growth rate of X , $\{Z_t; t \geq 0\}$ is a standard Brownian motion, and $X_0 = x_0 > 0$ is the initial known value of X .

Third, after Follower enters at τ_F , the industry becomes a duopoly and the total market profit changes from X_t to DX_t , where the parameter D measures the total market profits for the two firms. If $D < 1$, the total market profit is lower under duopoly than under monopoly. This captures the effect of competition on total industry profits. We split the total industry profits DX_t into $D_L X_t$ for Leader and $D_F X_t$ for Follower, where $D_L + D_F = D$. To ease exposition, in the main body of our paper following Dixit and Pindyck (1994) we assume that the two firms equally split the total industry profits ($D_L = D_F = D/2$) so that each firm receives profits at the rate of $\{DX_t/2\}$ in the duopoly market. Of course, the first mover may have a bigger brand name recognition and thus enjoy a larger market share after Follower enters. We can capture this feature by requiring $D_L > D_F$ (see Appendix C for details.)

Next, we turn to the key assumption of our model on firm entry costs. We assume that Leader incurs a larger entry cost than Follower does as Leader may have to pay additional innovation and marketing costs, learn about a new product market, and work with local governments in the new markets. Follower can save some of the costs by observing Leader's

actions, learning from Leader's experiences and mistakes, and even possibly imitating Leader's success and copying Leader's strategies. We capture the second-mover advantage by letting Follower's entry cost be lower than Leader's. Let $K_L > 0$ and $K_F > 0$ denote the entry cost that Leader and Follower have to pay at their entry time τ_L and τ_F , respectively. Let R denote the entry-cost ratio:

$$R = K_L/K_F. \quad (2)$$

As long as $R > 1$, there is a second-mover advantage.

We can further enrich our modelling of the second-mover advantage by assuming that the longer Follower learns from Leader (i.e., the longer $\tau_F - \tau_L$), the more cost reduction Follower receives, and moreover, the marginal benefit of Follower's cost reduction decreases as Follower's learning time $\tau_F - \tau_L$ increases. In Internet Appendix I.A, we analyze our model with this richer time-varying second-mover-advantage specification.

In sum, two *ex ante* identical firms, firm a (Alice's) and firm b (Bob's), maximize their values by taking how firm entry changes industry structure and market profits as given. Let τ_a and τ_b respectively denote firm a 's and b 's stochastic entry time before Leader is determined. Leader's entry time is then given by

$$\tau_L = \min\{\tau_a, \tau_b\} = \tau_a \wedge \tau_b. \quad (3)$$

Both firms are risk-neutral and discount profits at the constant interest rate r . As in the standard real-option models, we require $r > \mu$ and $r > 0$, which ensure that firm value is finite. Below we summarize these assumptions, which apply throughout our analysis:

$$\text{Assumptions : } r > \mu, \quad r > 0, \quad K_L > 0, \quad K_F > 0, \quad D \in (0, 2). \quad (4)$$

We require $D \in (0, 2)$ so that Follower's entry lowers Leader's profits.

Before we solve our duopoly problem, we first summarize the solution for the classic single firm's real-option problem (McDonald and Siegel, 1986; Dixit and Pindyck, 1994). The monopoly solution will help us better understand the mechanism of our duopoly model.

2.2 Monopoly Solution

A firm with an exclusive market entry opportunity chooses its entry time, τ_M , to solve:

$$M(x) = \max_{\tau_M \geq t} \mathbb{E}_t^x \left[e^{-r(\tau_M - t)} \left(\int_{\tau_M}^{\infty} e^{-r(s - \tau_M)} X_s ds - K_L \right) \right], \quad (5)$$

where $X_t = x > 0$ and $\mathbb{E}_t^x[\cdot] = \mathbb{E}_t[\cdot | X_t = x]$. The monopolist's optimal entry is characterized by a trigger strategy in that $\tau_M^* = \inf\{s \geq t : X_s \geq x_M\}$, where the optimal threshold, x_M , is given by

$$x_M = \frac{\beta}{\beta - 1}(r - \mu)K_L \quad (6)$$

and $\beta > 1$ measures optionality and is given by¹⁵

$$\beta = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}. \quad (7)$$

Let $\Pi(x)$ denote the monopolist's value function after entry:

$$\Pi(x) = \mathbb{E}_t^x \left[\int_t^\infty e^{-r(s-t)} X_s ds \right] = \frac{x}{r - \mu}. \quad (8)$$

In the waiting region where $x < x_M$, the monopolist's value $M(x)$ is

$$M(x) = \underbrace{\left(\frac{x}{x_M} \right)^\beta}_{\text{PV of \$1 paid at } \tau_M^*} \underbrace{(\Pi(x_M) - K_L)}_{\text{NPV at } \tau_M^*}. \quad (9)$$

As the stochastic entry time τ_M^* is characterized by the trigger policy (x_M) , before entry at any time t , the monopolist's value equals the product of (i) the time- t value of a \$1 paid at τ_M^* , given by $(x/x_M)^\beta$, and (ii) the NPV $(\Pi(x_M) - K_L)$ collected at τ_M^* . In the $x \geq x_M$ region, the firm enters the market immediately and therefore

$$M(x) = \Pi(x) - K_L, \quad x \geq x_M. \quad (10)$$

As $\beta > 1$, $M(x)$ is globally increasing and convex in x . Next, we sketch out our solution method for the duopoly model.

2.3 Duopoly Model Solution Procedure

We solve our duopoly model using backward induction as illustrated in Figure 2. After both firms have entered ($t > \tau_F$), they equally split profits and thus are valued at $D\Pi(x)/2$. This is **Step 0** in Figure 2. Next, we calculate Follower's and Leader's values after Leader's entry but before Follower's entry, i.e., for the (τ_L, τ_F) period. This is **Step 1** in Figure 2.

¹⁵That is, β is the larger root of the fundamental quadratic equation, $\sigma^2 z(z - 1)/2 + \mu z - r = 0$, for the GBM X process (1) in standard real option models.

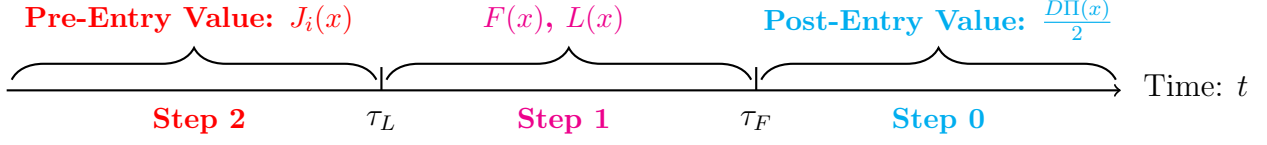


Figure 2: This figure summarizes various value functions for a given pair of entry timing (τ_L, τ_F) in three stochastic periods: $t < \tau_L$ (before Leader's entry); $t \in (\tau_L, \tau_F)$; and $t > \tau_F$ (after Follower's entry). In the $t > \tau_F$ period, firm value equals $D\Pi(x)/2$ where $\Pi(x) = x/(r - \mu)$. In the $t \in (\tau_L, \tau_F)$ period, $F(x)$ and $L(x)$ are Follower's and Leader's value functions. Finally, in the $t < \tau_L$ period (before Leader's entry), $J_i(x)$ is firm i 's value.

Defining Follower's Pre-entry and Leader's Post-entry Values: $F(x)$ and $L(x)$.

Follower's value in the (τ_L, τ_F) period is given by:

$$F(x) = \max_{\tau_F \geq t} \mathbb{E}_t^x \left[\int_{\tau_F}^{\infty} e^{-r(s-t)} \frac{DX_s}{2} ds - e^{-r(\tau_F-t)} K_F \right], \quad (11)$$

where $X_t = x > 0$. Let τ_F^* denote Follower's optimal entry time for (11). Taking τ_F^* as given, we define Leader's post-entry value function, $L(x)$, for $t \in (\tau_L, \tau_F^*)$ as:

$$L(x) = \mathbb{E}_t^x \left[\int_t^{\infty} e^{-r(s-t)} X_s ds - \int_{\tau_F^*}^{\infty} e^{-r(s-t)} \left(1 - \frac{D}{2} \right) X_s ds \right], \quad (12)$$

where the first term in (12) gives time- t value if Leader were to monopolize the market indefinitely and the second term gives the time- t value taken away by Follower from entry time τ_F^* onward.¹⁶

2.3.1 Step 1: Solving $F(x)$, $L(x)$, and Follower's Optimal Entry Time τ_F^* .

Using backward induction, we first jointly solve Follower's optimal entry time τ_F^* and its closed-form value function $F(x)$, and then calculate Leader's post-entry value $L(x)$.

Follower's Optimal Entry Threshold τ_F^* and Pre-entry Value $F(x)$. After paying the cost K_F at its chosen entry time τ_F^* , Follower receives $DX_t/2$ for all $t \geq \tau_F^*$. Therefore, Follower's entry decision boils down to a monopolist's real-option problem analyzed in Subsection 2.2 but with an entry cost of K_F paid at τ_F and a subsequent stochastic flow payoff of $DX_t/2$ for all $t \geq \tau_F$. Follower's value $F(x)$ is thus given by:

$$F(x) = \left(\frac{D\Pi(x_F)}{2} - K_F \right) \left(\frac{x}{x_F} \right)^\beta, \quad x < x_F, \quad (13)$$

¹⁶Note that $F(x)$ includes Follower's entry cost K_F but $L(x)$ does not include Leader's entry cost K_L as $L(x)$ is calculated for $t > \tau_L$.

$$F(x) = \frac{D\Pi(x)}{2} - K_F, \quad x \geq x_F, \quad (14)$$

where the optimal entry threshold, x_F , is given by

$$x_F = \frac{2}{D} \frac{\beta}{\beta - 1} (r - \mu) K_F. \quad (15)$$

Note that the entry threshold x_F is now proportional to Follower's entry cost K_F and the coefficient $2/D$ in (15) reflects the assumption that Follower's post-entry profit is $DX_t/2$. As in standard real-option models, Follower's pre-entry value $F(x)$ is increasing and convex. The higher the volatility σ , the higher the value $F(x)$.

Leader's Post-entry Value $L(x)$. Solving $L(x)$ defined in (12), we obtain

$$L(x) = \Pi(x) - \left(1 - \frac{D}{2}\right) \Pi(x_F) \left(\frac{x}{x_F}\right)^\beta, \quad x < x_F, \quad (16)$$

$$L(x) = \frac{D\Pi(x)}{2}, \quad x \geq x_F. \quad (17)$$

In the $x \geq x_F$ region, both Leader and Follower are in the market and they equally split the market demand and are valued at $D\Pi(x)/2$. In the $x < x_F$ region, Leader's time- t value $L(x)$ equals the difference between $\Pi(x)$ and $(1 - \frac{D}{2}) \Pi(x_F) \left(\frac{x}{x_F}\right)^\beta$, the latter of which equals the value of Leader's lost profits caused by Follower's entry. Note that solving $L(x)$ is a pure valuation problem as there is no decision made by Leader involved. Leader's value $L(x)$ is concave in the $x < x_F$ region but linear in the $x \geq x_F$ region. Therefore, $L(x)$ is not globally concave. This property has important equilibrium implications as we show later.

Next, we turn to **Step 2**, the final and key step of our analysis for the $[0, \tau_L)$ period. In this period, firms formulate their optimal entry strategies into a market with no incumbents.

2.3.2 **Step 2: Determining Leader and Its Entry Time τ_L**

For a pair of entry strategy (τ_a, τ_b) , firm i 's value function $J_i(x)$ at time t is given by

$$\mathbb{E}_t^x \left[e^{-r(\tau_L - t)} \left(\mathbf{1}_{\tau_i < \tau_{-i}} (L(X_{\tau_i}) - K_L) + \mathbf{1}_{\tau_i > \tau_{-i}} F(X_{\tau_{-i}}) + \mathbf{1}_{\tau_i = \tau_{-i}} \frac{L(X_{\tau_i}) - K_L + F(X_{\tau_i})}{2} \right) \right], \quad (18)$$

where $\tau_L = \tau_i \wedge \tau_{-i}$, $X_t = x > 0$, and $\mathbf{1}_A$ is an indicator function that equals one if event A occurs and zero otherwise. The first term in (18) describes the event where firm i becomes Leader, the second term describes the event where firm i becomes Follower, and the last term accounts for the scenario where the two firms enter at the same time.

Firm i chooses its optimal entry time τ_i to maximize its value given in (18) taking into the best response of its competitor, firm $-i$. Next, we characterize our model solution and focus on Markov subgame-perfect equilibria.

3 Characterizing Model Solution via Three Cases

For our remaining analysis, we set $D = 1$ to ease our exposition. Depending on the value of R , the solution falls into one of the three cases, **Case A**, **Case B**, and **Case C**.

In **Case A**, the entry-cost ratio R is so large that firms are better off being Follower for all levels of market demand x . This is because the second-mover advantage dominates the first-mover advantage for all levels of x in this case.

In **Case C** where the entry-cost ratio R is weakly less than one, there is no second-mover advantage at all. In equilibrium, firms trade off the first-mover advantage and the standard option value of waiting, as highlighted in Fudenberg and Tirole (1985), Dixit and Pindyck (1994), and Grenadier (1996).¹⁷ We refer to **Case C** as the first-mover-advantage case.

Finally, in **Case B** where the entry-cost ratio R is moderately larger than one (i.e., between **Case A** and **Case C**), whether the first-mover or the second-mover advantage dominates depends on market demand x . We show that the interaction between the two types of advantage generates rich predictions.

Next, we describe these three cases of our model solution. In Appendix E and Internet Appendix I.B, we provide proofs of all propositions, theorems, and supporting lemmas.

Proposition 1 *Let R_{AB} be given by*¹⁸

$$R_{AB} = \left(\frac{2^\beta}{\beta + 1} \right)^{\frac{1}{\beta-1}} > 1. \quad (19)$$

Depending on how large the entry-cost ratio $R = K_L/K_F$ is, our duopoly model solution falls into one of the following three cases.

Case A. *If $R > R_{AB}$, then the second-mover advantage globally dominates. This is because the*

¹⁷Dixit and Pindyck (1994) corresponds to the special $R = 1$ case of our duopoly model.

¹⁸We obtain $R_{AB} = K_{AB}/K_F$, by solving for K_{AB} , which is the unique root of the following equation for $K_L \in (K_F, 2K_F)$: $\frac{L(x_M) - K_L}{x_M^\beta} = \frac{F(x_F)}{x_F^\beta}$.

following inequality holds for all x :

$$L(x) - K_L < F(x), \quad x > 0. \quad (20)$$

Case B. If $1 < R < R_{AB}$, then $L(x) - K_L = F(x)$ has two roots, $\hat{x}_L$ and $\hat{x}_F$.¹⁹

$$L(x) - K_L > F(x), \quad \hat{x}_L < x < \hat{x}_F, \quad (21)$$

$$L(x) - K_L < F(x), \quad x < \hat{x}_L \text{ or } x > \hat{x}_F, \quad (22)$$

and $\hat{x}_L < x_M < \hat{x}_F < x_F$.²⁰ The first-mover advantage dominates in the $\hat{x}_L < x < \hat{x}_F$ region and the second-mover advantage dominates in $x > \hat{x}_F$ region. Both firms wait due to the option-value-of-waiting considerations in the $x < \hat{x}_L$ region.

Case C. If $R \leq 1$, then $L(x) - K_L = F(x)$ has a unique root $\hat{x}_L$ in the $(0, x_F)$ domain and²¹

$$L(x) - K_L \geq F(x), \quad x > \hat{x}_L, \quad (23)$$

$$L(x) - K_L < F(x), \quad x < \hat{x}_L, \quad (24)$$

and $\hat{x}_L < x_M$. There is only a first-mover advantage in the $x > \hat{x}_L$ region. Both firms wait due to the option-value-of-waiting considerations in the $x < \hat{x}_L$ region.

At the core of Proposition 1 is whether $F(x)$ is larger than $L(x) - K_L$ or not.

Next, we analyze **Case A**, where $F(x) > L(x) - K_L$ holds for all $x > 0$. We focus on symmetric equilibria in Section 4 and analyze asymmetric equilibria in Section 7.

4 Solution for **Case A**: Mixed-Strategy Equilibrium

In **Case A**, the entry-cost ratio is so large, $R > R_{AB} > 1$, that $F(x) > L(x) - K_L$ holds for all $x > 0$. As a result, between waiting and entering as Leader, a firm strictly prefers waiting which yields a higher payoff. But there is no Follower without Leader. Does this mean there is no equilibrium for **Case A**? The above reasoning implies that there is no symmetric pure-strategy equilibrium for **Case A**. But there exists an economically intuitive

¹⁹If $R = R_{AB}$, the two roots for the $L(x) - K_L = F(x)$ equation reduce to one root: $\hat{x}_L = \hat{x}_F = x_M$, where x_M is the monopolist's entry threshold given in (6). For all $x \neq x_M$, $L(x) - K_L < F(x)$.

²⁰Recall that x_M and x_F are the monopolist's and Follower's entry thresholds given in (6) and (15), respectively.

²¹The inequality (23) is strict when $R < 1$. When $R = 1$, the inequality is strict for $\hat{x}_L < x < x_F$ and holds with equality for $x \geq x_F$.

symmetric *mixed-strategy* equilibrium which we analyze in this section. First, we define the mixed-strategy equilibrium.

4.1 Definition of Markov Mixed-Strategy Equilibrium

How does a firm enter using a mixed strategy in a continuous-time model? It does so probabilistically with a rate over a small time increment dt . Our mixed-strategy equilibrium is related to the mixed-strategy equilibrium in classic war-of-attrition games (Levin, 2004), which feature constant equilibrium exit rates.²² As market demand X_t is stochastic in our model, the entry rate for our mixed-strategy equilibrium depends on X_t . Let $\lambda_i(X_t)$ denote the rate at which firm i becomes Leader over a small time interval $[t, t + dt]$ where $t < \tau_L$. Firm i 's entry time τ_i is a doubly stochastic process as its associated rate $\{\lambda_i(X_t)\}_{t \geq 0}$ is also stochastic.²³ Next, we define feasible Markov mixed strategies and the Markov perfect mixed-strategy equilibrium.

Definition 1 An entry rate $\lambda_i(x)$ is a measurable function from $\mathbb{R}_+$ to $\mathbb{R}_+$. A pair of Markov strategy $(\lambda_a(\cdot), \lambda_b(\cdot))$ is feasible if and only if for any $t > 0$, $\int_0^t \lambda_i(X_s) ds < \infty$ almost surely. Let Φ denote the set of all feasible Markov mixed strategies.

Definition 2 Let $J_i(x; \lambda_a, \lambda_b)$ denote firm i 's value at time t defined in (18) for a given $X_t = x > 0$ and a feasible Markov mixed strategy pair (λ_a, λ_b) . A feasible strategy pair $(\lambda_a^*, \lambda_b^*)$ is a *Markov perfect mixed-strategy equilibrium* if for any $x > 0$, the following conditions hold:

$$J_a(x; \lambda_a^*, \lambda_b^*) \geq J_a(x; \lambda_a, \lambda_b^*), \quad \forall (\lambda_a, \lambda_b^*) \in \Phi, \quad (25)$$

$$J_b(x; \lambda_a^*, \lambda_b^*) \geq J_b(x; \lambda_a^*, \lambda_b), \quad \forall (\lambda_a^*, \lambda_b) \in \Phi. \quad (26)$$

²²For war-of-attrition models, see Ghemawat and Nalebuff (1985, 1990), Fudenberg and Tirole (1986), and Hendricks, Weiss, and Wilson (1988) with deterministic payoffs and Steg (2015) with stochastic payoffs.

²³Stopping time τ is doubly stochastic if the underlying counting process $\{\mathcal{N}_t\}_{t \geq 0}$ whose first jump time τ is doubly stochastic. A counting process $\{\mathcal{N}_t\}_{t \geq 0}$ is doubly stochastic if its associated intensity process $\{\lambda_t\}_{t \geq 0}$ is $\{\mathcal{F}_t\}_{t \geq 0}$ -predictable and for all t and $s > t$, conditional on the σ -algebra generated by $\{\mathcal{N}_u\}_{u \in [0, t]}$ and $\mathcal{F}_s$, the random variable $(\mathcal{N}_s - \mathcal{N}_t)$ has a Poisson distribution with parameter $\int_t^s \lambda_u du$. Now we apply these definitions to our model. Let $\{\mathcal{G}_t\}_{t \geq 0}$ be the σ -algebra generated by $\{\mathcal{F}_t\}_{t \geq 0}$ and $\{\mathcal{N}_t^i\}_{t \geq 0}$, where $i = a, b$. For any $t \geq 0$ and $s > t$, conditional on the σ -algebra generated by $\mathcal{G}_t \cup \mathcal{F}_s$, the counting processes $\{\mathcal{N}_u^a - \mathcal{N}_t^a\}_{u \in [t, s]}$ and $\{\mathcal{N}_u^b - \mathcal{N}_t^b\}_{u \in [t, s]}$ are independent and the random variable $(\mathcal{N}_s^i - \mathcal{N}_t^i)$ has a Poisson distribution with parameter $\int_t^s \lambda_i(X_u) du$ for $i = a, b$. Firm i 's entry time τ_i is thus doubly stochastic with the underlying counting process $\{\mathcal{N}_t^i\}_{t \geq 0}$ and the associated intensity process $\{\lambda_i(X_t)\}_{t \geq 0}$. See Lando (1998) and Duffie (2005) among others for applications of doubly stochastic processes to affine credit-risk models.

Let $V_i(x)$ denote firm i 's equilibrium value function: $V_i(x) = J_i(x; \lambda_a^*, \lambda_b^*)$.

Note that the mixed-strategy equilibrium in Definition 2 is subgame perfect.²⁴

4.2 Closed-Form Markov Perfect Mixed-Strategy Equilibrium

In this subsection, we first discuss the economic mechanism underlying a firm's entry decision and then we provide a mathematical proof of our equilibrium solution by extending the variational inequality method for a single firm's entry problem to our duopoly setting.

For a given mixed strategy pair $(\lambda_a(x), \lambda_b(x))$, the following HJB equation for firm i 's value, $J_i(x) = J_i(x; \lambda_a(x), \lambda_b(x))$, holds:

$$rJ_i(x) = \frac{\sigma^2 x^2}{2} J_i''(x) + \mu x J_i'(x) + \lambda_i(x)[L(x) - K_L - J_i(x)] + \lambda_{-i}(x)[F(x) - J_i(x)], \quad (27)$$

where $L(x)$ is given by (16)-(17) and $F(x)$ is given by (13)-(14). The intuition for the HJB equation (27) is as follows. The first two terms on the right side capture the standard diffusion and drift effects of X on $J_i(x)$. The third term describes the effect of firm i 's own mixed (entry) strategy on its value and this term equals zero in equilibrium as a rational firm will only mix with strictly positive probabilities between two strategies that yield the same value.²⁵ The last term in (27) describes the effect of the competitor's mixed entry strategy on firm i 's value. If the competitor enters, firm i becomes Follower and its value function jumps from $J_i(x)$ to $F(x)$. The firm's optimality requires that the sum of these four terms on the right side equals the annualized firm value $rJ_i(x)$ (Duffie, 2001).

Rewriting (27) and using $\lambda_i(x)[L(x) - K_L - J_i(x)] = 0$, we obtain:

$$\lambda_{-i}(x) = \frac{rJ_i(x) - \left[\frac{\sigma^2 x^2}{2} J_i''(x) + \mu x J_i'(x) \right]}{F(x) - J_i(x)}. \quad (28)$$

That is, in order to make firm i indifferent between probabilistically entering and waiting, its competitor's entry rate, $\lambda_{-i}(x)$, has to equal the right side of (28).

It is worth pointing out that although X is continuous, firm value is discontinuous and jumps when its competitor enters the market. This is an example where strategic interactions

²⁴We can expand the set of a firm's admissible entry strategy to allow for any adaptive entry time τ_i satisfying $\{\tau_i \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$ and still characterize the equilibrium using $\lambda(\cdot)$ in Definition 1.

²⁵Otherwise, it is always better for the firm to play the pure strategy (waiting or entering) that yields the higher value: $\max\{L(x) - K_L, J_i(x)\}$. Although in equilibrium this term is zero, we leave it in the HJB equation (27) to better understand the economic mechanism.

generate endogenous uncertainty via an equilibrium jump process because firms play mixed strategies in equilibrium.

Next, we turn to the symmetric Markov perfect equilibrium. Let $\lambda^*(x) = \lambda_a^*(x) = \lambda_b^*(x)$ denote the symmetric equilibrium Markov perfect mixed strategy. Equation (18) and inequality (20) together imply the following for $V_i(x)$, firm i 's equilibrium value function:

$$L(x) - K_L \leq V_i(x) \leq F(x), \quad x > 0. \quad (29)$$

That is, *ex ante* firm i 's value must be weakly larger than $L(x) - K_L$, Leader's net payoff upon entry at τ_L , and weakly lower than Follower's value $F(x)$ because the second-mover advantage globally dominates at all $x > 0$.

There are two scenarios to consider: 1.) $\lambda^*(x) > 0$ and 2.) $\lambda^*(x) = 0$. When $\lambda^*(x) > 0$, the firm must be indifferent between entering the market (becoming Leader) and waiting. That is, the value functions from the two strategies must equal:

$$V_i(x) = L(x) - K_L \quad \text{if} \quad \lambda^*(x) > 0, \quad (30)$$

which implies

$$\lambda^*(x) = 0 \quad \text{if} \quad V_i(x) > L(x) - K_L. \quad (31)$$

Using (27) and (30), we obtain the following HJB equation for $V_i(x)$:

$$rV_i(x) = \frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) + \lambda^*(x) [F(x) - V_i(x)], \quad (32)$$

which holds for both $\lambda^*(x) > 0$ and $\lambda^*(x) = 0$ cases. Re-arranging (32) yields the following expression for $\lambda^*(x)$ for all $x > 0$:

$$\lambda^*(x) = \frac{rV_i(x) - \left[\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) \right]}{F(x) - V_i(x)}. \quad (33)$$

When $\lambda^*(x) > 0$, substituting $V_i(x) = L(x) - K_L$ given in (30) into (33), we obtain

$$\lambda^*(x) = \frac{rL(x) - \left[\frac{\sigma^2 x^2}{2} L''(x) + \mu x L'(x) \right] - rK_L}{F(x) - (L(x) - K_L)}. \quad (34)$$

Since we have closed-form solutions for $L(x)$ as given by (16)-(17) and $F(x)$ as given by (13)-(14), we have an explicit formula for the equilibrium entry rate $\lambda^*(x)$.

The mechanism inducing firms to enter probabilistically in our **Case A** is related to that causing firms to exit probabilistically in war-of-attrition games (see Levin (2004) for a PhD teaching note on wars of attrition). Unlike standard war-of-attrition exit games, ours is

an *entry timing* game with stochastic payoffs. In our **Case A**, when market demand x is sufficiently high, entering as Leader is profitable but Leader is not the winner but rather the loser of the game in the war-of-attrition sense as Follower's value is higher than Leader's net payoff $F(x) > L(x) - K_L$ at all levels of x . Also, because of irreversible entry and stochastic market demand, the option value of waiting is another key force in our model.

To complete our model solution, we still need to solve $V_i(x)$ in the $\lambda^*(x) = 0$ region and characterize the $\lambda^*(x) > 0$ region. We show that $V_i(x)$ for $x > 0$ is the unique solution for the following variational inequality (See Appendix D):

$$\max \left\{ \frac{\sigma^2 x^2}{2} V''_*(x) + \mu x V'_*(x) - r V_*(x), (L(x) - K_L) - V_*(x) \right\} = 0, \quad (35)$$

subject to the following boundary conditions:²⁶

$$V_*(x) = 0 \quad \text{at } x = 0, \quad (36)$$

$$V_*(x) - (L(x) - K_L) \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (37)$$

The variational inequality (35) is analogous to that of a monopolist's real option problem, but their economic implications are different. Mathematically, we generalize the variational-inequality analysis in standard real-option models to our mixed-strategy equilibrium model.

Solving the variational-inequality problem (35)-(37), we obtain firm i 's equilibrium value $V_i(x) = V_*(x)$. Using (31), (34), and $V_i(x) = V_*(x)$, we obtain the equilibrium entry rate $\lambda^*(x)$. We summarize these results in the following theorem:

Theorem 1 *For **Case A** where $R > R_{AB}$, there exists a symmetric Markov perfect equilibrium. In this equilibrium, $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is the unique solution for the variational-inequality problem (35)-(37) in the $x \geq 0$ domain. The equilibrium strategy is given by $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, where $\lambda^*(x) = 0$ in the $V_*(x) > L(x) - K_L$ region and firms enter probabilistically at the strictly positive rate of $\lambda^*(x)$ given in (34) in the $V_*(x) = L(x) - K_L$ region.*

While Theorem 1 fully describes the solution for **Case A** where $R > R_{AB}$, we can further divide **Case A** into two subcases: **Subcase A₁** and **Subcase A₂**, depending on whether the

²⁶Because $X = 0$ is an absorbing state for a geometric Brownian motion process X , firm value must be zero as stated in (36). The condition given in (37) follows from the equilibrium result that firms must enter probabilistically when demand is sufficiently high.

entry-cost ratio $R = K_L/K_F$ is larger or smaller than $R_{A_1A_2}$ given by²⁷

$$R_{A_1A_2} = \left(\frac{2^\beta - 1}{\beta} \right)^{\frac{1}{\beta-1}}. \quad (38)$$

It is straightforward to show $R_{A_1A_2} > R_{AB}$, where R_{AB} is given in (19).

In **Subcase A₁**, $R > R_{A_1A_2}$ holds and in **Subcase A₂**, $R_{AB} < R \leq R_{A_1A_2}$ holds. In Subsection 4.3, we obtain explicit solutions for firm's value $V_i(x)$ and $\lambda^*(x)$ for these two subcases. As we will show, even though there is only a second-mover advantage in equilibrium for both subcases, the equilibrium strategies are quite different for the two subcases.

4.3 Two Subcases: **Subcase A₁** and **Subcase A₂**

First, we solve **Subcase A₁** where $R > R_{A_1A_2}$ in closed form for $V_i(x)$ and $\lambda^*(x)$.

4.3.1 **Subcase A₁**: $R > R_{A_1A_2}$

Solution. For **Subcase A₁**, there exists a threshold $\bar{x}$ dividing the $x > 0$ real line into two regions: 1.) the waiting region where $x < \bar{x}$ and $V_*(x) > L(x) - K_L$ and 2.) the probabilistic entry region where $x \geq \bar{x}$ and $V_*(x) = L(x) - K_L$. The variational inequality (35) is simplified to the following ordinary differential equation (ODE) in the waiting ($x < \bar{x}$) region:

$$\frac{\sigma^2 x^2}{2} V_*''(x) + \mu x V_*'(x) - r V_*(x) = 0, \quad (39)$$

subject to the following value-matching and smooth-pasting conditions at the threshold $\bar{x}$:

$$V_*(x) = L(x) - K_L, \quad (40)$$

$$V_*'(x) = L'(x). \quad (41)$$

While these two boundary conditions resemble the standard value-matching and smooth-pasting conditions for a single firm's optimal threshold in the standard models, the economics underpinning (40)-(41) is different from standard real-option models, which we discuss later.

A key result for **Subcase A₁** is that Follower enters immediately after Leader does in that $\tau_F^* = \tau_L^* +$. This implies that Leader earns no monopoly rents in equilibrium. Therefore, in

²⁷We obtain $R_{A_1A_2} = K_{A_1A_2}/K_F$ by solving for $K_{A_1A_2}$, which is the unique root of the following equation for $K_L \in (K_F, 2K_F)$: $\frac{L(x_M) - K_L}{x_M^\beta} = \frac{L(2x_M) - K_L}{(2x_M)^\beta}$.

the probabilistic entry region, Leader's NPV netting out of its entry cost is given by

$$V_*(x) = \frac{\Pi(x)}{2} - K_L, \quad x \geq \bar{x}. \quad (42)$$

Using Leader's linear net payoff function at entry in the $x \geq \bar{x}$ region, (42), and solving the ODE (39) in the $x < \bar{x}$ region subject to the value-matching and smooth-pasting conditions, (40) and (41), we obtain the closed-form expressions for $V_*(x)$:

$$V_*(x) = \left(\frac{x}{\bar{x}}\right)^\beta \left(\frac{\Pi(\bar{x})}{2} - K_L\right), \quad x < \bar{x}, \quad (43)$$

where the threshold $\bar{x}$ is given by

$$\bar{x} = \frac{2\beta}{\beta - 1}(r - \mu)K_L = 2x_M. \quad (44)$$

Note that $\bar{x}$ is the lower bound for Leader's optimal (probabilistic) entry region. In our model, even when $X_t \geq \bar{x}$, the firm may still be waiting as firms play mixed entry strategies.

Now we verify the equilibrium result that as soon as one firm enters probabilistically, the other also immediately enters. This is because $\bar{x} = 2x_M > x_F$ which follows from a comparison of (44) for $\bar{x}$ and (15) for x_F under the second-mover advantage: $R > R_{A_1A_2} > 1$. Finally, substituting (14) and (17) into (34) gives the equilibrium entry rate:

$$\lambda^*(x) = \frac{x/2 - rK_L}{K_L - K_F} > 0, \quad x \geq \bar{x}. \quad (45)$$

The numerator in (45) equals firm i 's *net income*, which equals the equilibrium profit for a duopoly, $x/2$, minus rK_L , the interest expense of financing the entry cost K_L . The entry rate $\lambda^*(x)$ increases linearly with x for $x \geq \bar{x}$ and approaches ∞ as $x \rightarrow \infty$. Intuitively, the higher the market demand x the more likely a firm enters so as to end the waiting game and collect profits sooner. Next we summarize the solution for **Subcase A₁** where $R > R_{A_1A_2}$.

Proposition 2 For **Subcase A₁** ($R > R_{A_1A_2}$), there exists a symmetric Markov perfect mixed-strategy equilibrium where the threshold $\bar{x}$ given in (44) divides the $x > 0$ real line into two solution regions. In the $x < \bar{x}$ region, firms wait ($\lambda^*(x) = 0$) and $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is given in (43). In the $x \geq \bar{x}$, region, firms enter probabilistically at the same rate, $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, where $\lambda^*(x)$ is given in (45), and $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is given in (42). As soon as one firm enters, the other enters immediately: $\tau_F^* = \tau_L^*$.

The *ex ante* probability that either firm becomes Leader is one half.

Throughout our paper, we use figures to supplement our formal analysis to further deepen

our understanding of the economic mechanism. We start with **Subcase A₁**. First, we discuss our choices of parameter values for the figures.

Parameter Choices. Our duopoly model has five parameters in total. First, for the risk-free rate (r), the expected growth rate (drift) of the profit process (μ), and the volatility of the profit growth rate (σ), we choose commonly used values: $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$ per annum, following the standard practice in real-options and contingent-claim literature, e.g., Grenadier (1996) and Leland (1994). The implied optionality measure given in (7) is $\beta = 1.70$. Substituting $\beta = 1.70$ into (38) for $R_{A_1A_2}$ and (19) for R_{AB} , we obtain $R_{A_1A_2} = 1.49$ and $R_{AB} = 1.3$ that split **Case A** into **Subcase A₁** and **Subcase A₂**.

Which subcase our model solution falls into only depends on two measures: (1.) the optionality measure β given in (7), which determines the cutoff values $R_{A_1A_2}$ and R_{AB} , and (2.) the entry-cost ratio $R = K_L/K_F$, which determines how strong the second-mover advantage is. Finally, we set Follower's entry cost to $K_F = 0.5$.

Graphical Illustration. To demonstrate the economics of **Subcase A₁**, it is necessary and sufficient to choose a value of $R = K_L/K_F$ that is larger than $R_{A_1A_2} = 1.49$. We choose $R = 1.6$, which implies Leader's entry cost of $K_L = 0.8$.

In Panel A of Figure 3, we highlight how to graphically pin down firm value before Leader is determined (i.e., $t \leq \tau_L^*$). First, we plot $L(x) - K_L$, which is concave in the $x < x_F$ region and linear in the $x \geq x_F$ region (the dashed red line). Second, we plot Follower's value $F(x)$, which is increasing and convex (the magenta dash-dotted line). Because $L(x) - K_L < F(x)$ holds for all $x > 0$, neither firm wants to be Leader with probability one. Moreover, firm value $V_i(x)$ must satisfy $L(x) - K_L \leq V_i(x) \leq F(x)$ in equilibrium.

Third, we pin down firm value $V_a(x) = V_b(x) = V_*(x)$ by smoothly pasting a convex curve (from the origin) onto the $L(x) - K_L$ payoff line. Doing so determines the endogenous threshold $\bar{x}$ (the solid black dot): To the left of $\bar{x}$ is the increasing convex $V_i(x)$ (the black solid line) and to the right of $\bar{x}$ is the straight net payoff line $L(x) - K_L = \Pi(x)/2 - K_L$ (the blue solid straight line).

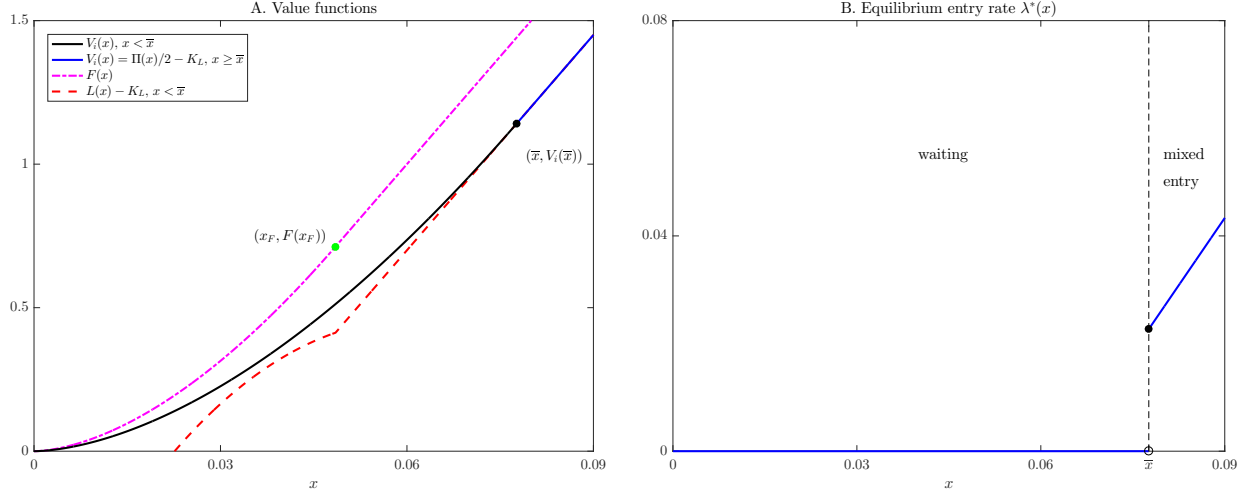


Figure 3: VALUE FUNCTIONS $V_a(x) = V_b(x) = V_*(x)$ AND ENTRY RATES $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$ IN THE SYMMETRIC MARKOV PERFECT MIXED-STRATEGY EQUILIBRIUM IN **SUBCASE A₁**. The threshold value dividing the market demand x into the two regions is: $\bar{x} = 2x_M$, where x_M is the entry threshold for a monopolist with an entry cost of K_L . The two regions are: 1.) the $x < \bar{x}$ waiting region and 2.) the $x \geq \bar{x}$ probabilistic entry region. Panel A plots pre-entry firm value $V_*(x)$ (solid line), Follower's value $F(x)$ (dash dotted line), and Leader's net payoff upon entry $L(x) - K_L$ (dashed line). Panel B plots the equilibrium entry rate: $\lambda^*(x) = \frac{x/2 - rK_L}{K_L - K_F}$ in the $x \geq \bar{x}$ region and $\lambda^*(x) = 0$ in the $x < \bar{x}$ waiting region. Parameter values are $R = 1.6$, $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$, which imply $R_{A_1A_2} = 1.49$, $R_{AB} = 1.3$, $x_F = 0.0485$, and $\bar{x} = 2x_M = 0.0776$.

Panel B of Figure 3 plots the equilibrium entry rate $\lambda^*(x)$ that supports $V_i(x) = V_*(x)$ obtained in Panel A. The vertical dashed line in Panel B divides the solution into two regions. To the left of $\bar{x}$ is the waiting region where $\lambda^*(x) = 0$. To the right of $\bar{x}$ is the probabilistic entry region where $\lambda^*(x) = \frac{x/2 - rK_L}{K_L - K_F}$. The mixed-strategy equilibrium is a compromised outcome between the two firms. As a firm waits for the other to enter, it forgoes the opportunity of collecting profits $x/2 - rK_L$, but preserves the option value of being the second mover and saving $\Delta K = K_L - K_F$. In equilibrium, the linear $\lambda^*(x)$ entry rate makes the firm indifferent between entering and waiting. The higher the value of x , the higher the costs of forgoing one-period profit and thus the more likely it enters to end the game sooner.

Finally, we point out that the lower bound of x that firms are willing to probabilistically enter, $\bar{x}$ given in (44), equals twice the value of x_M , the entry threshold of a monopolist (with an entry cost of K_L). This is because in equilibrium Leader only collects $X_t/2$ and never enjoys monopoly rents. Next, we turn to **Subcase A₂**.

4.3.2 Subcase A₂: $R \in (R_{AB}, R_{A_1A_2}]$

Solution Overview. Compared with Subcase A₁ where $R > R_{A_1A_2}$, the entry-cost ratio R for Subcase A₂ is lower but still larger than one and lies in the region $(R_{AB}, R_{A_1A_2}]$. Although the second-mover advantage for Subcase A₂ is lower than that for Subcase A₁, firms still prefer to be Follower for all $x > 0$. The solution for Subcase A₂ is richer and subtler than for Subcase A₁ and can still be obtained in closed form.

We will show that there are four regions in equilibrium for Subcase A₂: two disconnected waiting regions and two disconnected probabilistic entry regions. There are two waiting regions because there are two distinct waiting motives: one is due to the standard option value and the other is due to the second-mover advantage. Depending on the market demand x , we have two probabilistic entry regions: When x is very high, Follower enters immediately after Leader leaving no monopoly rents for Leader. When x is in the intermediate range, we also have a mixed strategy equilibrium but Follower voluntarily waits so that Leader can make enough monopoly profits for a sufficiently long period.

Next, we obtain the solution for firm value $V_*(x)$ by using the smooth-pasting conditions with the payoff function being $L(x) - K_L$. Importantly, we show that there are *three* cutoff values of market demand x that define the four regions. Without loss of generality, we order the three threshold in an ascending order: $\tilde{x}$, $\underline{x}$, and $\bar{x}$. Then, $[\tilde{x}, \underline{x}]$ and $[\bar{x}, \infty)$ are the two mixed-strategy regions where a firm probabilistically enters as Leader, and $(0, \tilde{x})$ and $(\underline{x}, \bar{x})$ are the two waiting regions.²⁸

Four-Region Solution. First, in the standard waiting region where $x \in [0, \tilde{x})$, $V_*(x)$ is convex and given by

$$V_*(x) = \left(\frac{x}{\tilde{x}}\right)^\beta (L(\tilde{x}) - K_L), \quad x \in [0, \tilde{x}). \quad (46)$$

Note that firm value equals the product of (a.) the present value of a dollar paid at $\tau := \inf\{s : X_s = \tilde{x}\}$ and (b.) the net payoff if the firm enters at τ as Leader. Using the value-matching and smooth-pasting conditions (40)-(41) at $x = \tilde{x}$, we can prove that $\tilde{x}$ equals x_M , the entry

²⁸Mathematically, we can prove that the variational-inequality problem (35)-(37) boils down to the smooth-pasting conditions at $\tilde{x}$, $\underline{x}$, and $\bar{x}$, which define the four regions. Firm value, $V_*(x)$, is the solution to the variational inequality problem (35)-(37), which satisfies the ordinary differential equation (39) in the two waiting regions $(0, \tilde{x})$ and $(\underline{x}, \bar{x})$.

threshold of a monopolist with an entry cost of K_L , given by (6):

$$\tilde{x} = x_M. \quad (47)$$

Second, in the $x \geq \bar{x}$ region, as soon as one firm enters as Leader, the other follows immediately because the market demand is sufficiently high (and formally $x_F < \bar{x}$). This mechanism is the same as the one for the probabilistic entry region in **Subcase A₁**. Therefore, firm value (in this probabilistic entry region) is also linear, $V_*(x) = \Pi(x)/2 - K_L$, and $\lambda^*(x)$ has the same linear expression as (45) for **Subcase A₁**. The only difference is the value of $\bar{x}$ in **Subcase A₂** is different from $\bar{x} = 2x_M$ given in (44) in **Subcase A₁**.

Third, in the $x \in [x_M, \underline{x}]$ region, firms enter probabilistically and a firm's pre-entry value is thus given by $V_*(x) = L(x) - K_L$, which is concave. As soon as Leader is determined, the other firm waits and Leader thus collects monopoly rents until $\tau_F^* = \inf\{s : X_s \geq x_F\}$. Technically, this follows from $\underline{x} < x_F$. The equilibrium entry rate function that makes firms indifferent between entering as Leader and waiting in this region is:

$$\lambda^*(x) = \frac{x - rK_L}{F(x) - (L(x) - K_L)}, \quad x \in [x_M, \underline{x}]. \quad (48)$$

The numerator in (48) is the opportunity cost of not collecting by waiting and the denominator in (48) is the net benefit of waiting to be Follower: the value gap between becoming Leader immediately and being Follower.

Fourth, in the $x \in (\underline{x}, \bar{x})$ region, firms wait because they prefer to enter as Follower to lower their entry costs. Solving the ODE (39) in this region subject to the value-matching and smooth-pasting conditions (40)-(41) at $\underline{x}$ and $\bar{x}$, we obtain the closed-form expression:

$$V_*(x) = \Theta(x; \underline{x}, \bar{x}), \quad x \in (\underline{x}, \bar{x}), \quad (49)$$

where $\Theta(x; a, b)$ for any $a \leq x \leq b$ is given by

$$\Theta(x; a, b) = \theta_1(a, b)x^\beta + \theta_2(a, b)x^\gamma, \quad (50)$$

$\beta > 1$ is given in (7), and $\gamma < 0$ is given by²⁹

$$\gamma = \frac{-(\mu - \frac{1}{2}\sigma^2) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}. \quad (51)$$

Finally, in Lemma 2 of Appendix D, we prove $x_M < \underline{x} < x_F < \bar{x}$ and characterize the pair

²⁹ Mathematically, γ is the smaller (and negative) root of the fundamental quadratic equation, $\sigma^2 z(z - 1)/2 + \mu z - r = 0$, for the GBM X process (1). See (D.2) for the expressions of $\theta_1(a, b)$ and $\theta_2(a, b)$.

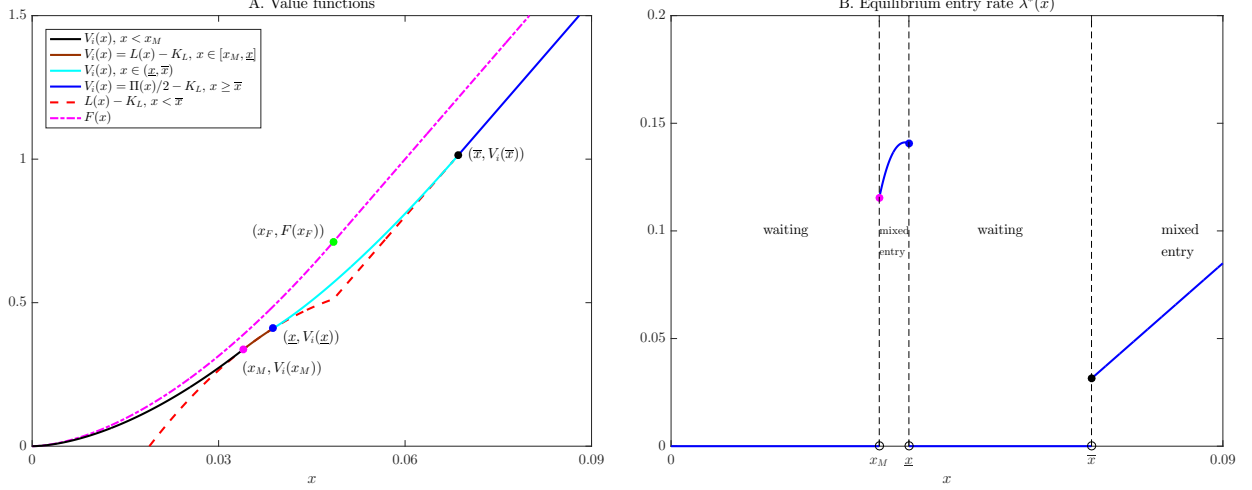


Figure 4: VALUE FUNCTIONS AND ENTRY RATE IN THE SYMMETRIC EQUILIBRIUM IN **SUB-CASE A₂**. Parameter values are $R = 1.4$, $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$, which imply $R_{A_1A_2} = 1.49$, $R_{AB} = 1.3$, $x_M = 0.0340$, $\underline{x} = 0.0388$, $\bar{x} = 0.0686$, and $x_F = 0.0485$.

$(\underline{x}, \bar{x})$ defining the waiting (to be Follower) region. Below we summarize the duopoly model solution for **Subcase A₂**.

Proposition 3 For **Subcase A₂** ($R_{AB} < R \leq R_{A_1A_2}$),³⁰ firm i 's value function is given by $V_i(x) = V_*(x)$ for all $x > 0$. In the $x < x_M$ and $x \in (\underline{x}, \bar{x})$ regions, $V_*(x)$ is given by (46) with $\tilde{x} = x_M$ and (49), respectively. In the $x \in [x_M, \underline{x}]$ and $x \in [\bar{x}, \infty)$ regions, $V_*(x) = L(x) - K_L$. The thresholds, $\underline{x}$ and $\bar{x}$, are given in (D.8) via the smooth-pasting conditions in Lemma 2. The symmetric Markov perfect equilibrium strategy is given by $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$. In both $x < x_M$ and $x \in (\underline{x}, \bar{x})$ regions, firms wait: $\lambda^*(x) = 0$. In the $x \in [x_M, \underline{x}]$ and $x \geq \bar{x}$ regions, firms enter stochastically at the rate of $\lambda^*(x) > 0$ given in (48) and (45), respectively.

After Leader is determined at $\tau_L^* = \tau_a^* \wedge \tau_b^*$, the other firm enters at $\tau_F^* = \inf\{s : X_s \geq x_F\}$, where x_F is given in (15). In the $x \in [x_M, \underline{x}]$ region, Leader earns monopoly rents in the (τ_L^*, τ_F^*) period but in the $x \geq \bar{x}$ region, Leader earns no monopoly rents as $\tau_F^* = \tau_L^*$.

Graphical Illustration. Next, we use Figure 4 to corroborate our analysis of **Subcase A₂**. We set the entry-cost ratio at $R = 1.4$, which lies in the $(R_{AB}, R_{A_1A_2}] = (1.30, 1.49]$ region. The implied Leader's entry cost is $K_L = 0.7$.

³⁰ R_{AB} and $R_{A_1A_2}$ are given in (19) and (38), respectively.

Although K_L is lowered to 0.7 from 0.8, $F(x) > L(x) - K_L$ still holds for all $x > 0$. Therefore, the second-mover advantage, while weakened, still dominates in **Subcase A₂**. As in **Subcase A₁**, a firm's pre-entry value $V_i(x)$ has to satisfy $L(x) - K_L \leq V_i(x) \leq F(x)$ and firm optimality requires that $V_i(x)$ smoothly pastes onto $L(x) - K_L$ somewhere. Geometrically, reducing K_L to 0.7 narrows the wiggle room for $V_i(x)$ to paste onto $L(x) - K_L$. In equilibrium, $V_i(x)$ has to change the sign of its curvature twice by smoothly pasting onto the Leader's net payoff line $L(x) - K_L$ *three* times at x_M , $\underline{x}$, and $\bar{x}$. That is, the shape of $V_i(x)$ changes from convex in the $x \leq x_M$ region to concave in the $x \in (x_M, \underline{x})$ region and then back to convex in the $x \in (\underline{x}, \bar{x})$ region. The corresponding equilibrium strategies are waiting, probabilistic entry, and again waiting, respectively.

In sum, Figure 4 confirms our model's prediction that as x increases from zero to ∞ , a firm finds itself in one of the four mutually exclusive regions: 1.) the first waiting region (to preserve option value); 2.) the first mixed entry strategy (with monopoly profits); 3.) the second waiting region (second-mover advantage); and 4.) the second mixed entry strategy (no monopoly profits).

4.3.3 Summary of **Case A**

In sum, the second-mover advantage dominates for all $x > 0$ in **Case A**, where $R > R_{AB} > 1$ and R_{AB} is given in (19). As a result, we only have two types of regions: waiting and probabilistic entry regions. This is because entry with probability one is a dominated strategy in a symmetric equilibrium. We further divide **Case A** into two subcases. For **Subcase A₁** ($R > R_{A_1A_2}$), there is one waiting region and one probabilistic entry region, where $R_{A_1A_2}$ is given in (38). For **Subcase A₂** ($R \in (R_{AB}, R_{A_1A_2}]$), there are two disconnected waiting regions alternating with two disconnected probabilistic entry regions.

Next, we analyze **Case C** where $R \leq 1$.

5 **Case C: First-Mover Advantage Only**

In **Case C** ($R \leq 1$), there is no second-mover advantage. The equilibrium is determined by firms' tradeoff between the first-mover advantage and the option value of waiting.

Let $\mathcal{E}_i \subset (0, \infty)$ denote a closed set associated with firm i 's entry strategy: firm i enters at t if and only if $X_t \in \mathcal{E}_i$. Let Ψ denote the set of all feasible entry strategies $(\mathcal{E}_a, \mathcal{E}_b)$ and let $J_i(X_t; \mathcal{E}_a, \mathcal{E}_b)$ denote the associated time- t value of firm i defined by (18). Next, we define the pure-strategy equilibrium.

Definition 3 A pair of entry strategy $(\mathcal{E}_a^*, \mathcal{E}_b^*)$ is a *pure-strategy equilibrium* if for any $x > 0$ the following conditions hold:

$$J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq J_a(x; \mathcal{E}_a, \mathcal{E}_b^*), \quad \forall (\mathcal{E}_a, \mathcal{E}_b^*) \in \Psi, \quad (52)$$

$$J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq J_b(x; \mathcal{E}_a^*, \mathcal{E}_b), \quad \forall (\mathcal{E}_a^*, \mathcal{E}_b) \in \Psi. \quad (53)$$

Let $V_i(x)$ denote firm i 's equilibrium value function: $V_i(x) = J_i(x; \mathcal{E}_a^*, \mathcal{E}_b^*)$.

Proposition 1 shows that for **Case C** where $R \leq 1$, $F(x)$ intersects with $L(x) - K_L$ at $\hat{x}_L$ and $L(x) - K_L \geq F(x)$ for any $x \geq \hat{x}_L$. Therefore, in the $x \geq \hat{x}_L$ region, both firms want to enter as Leader but only one firm can be randomly selected (with 50% probability) to be Leader. This is the *rent equalization principle* of Dixit and Pindyck (1994), and Grenadier (1996), which implies that the equilibrium firm value for both firms is:

$$V_i(x) = \frac{L(x) - K_L + F(x)}{2} \quad (54)$$

for $x \geq \hat{x}_L$. In the $x < \hat{x}_L$ region, firms optimally wait and the equilibrium firm value is: $V_i(x) = F(x)$ as in Dixit and Pindyck (1994). Next, we summarize the solution in the following theorem.

Theorem 2 Consider **Case C** where $R \leq 1$. Let $\hat{x}_L$ be the unique root of $L(x) - K_L = F(x)$ in the $(0, x_F)$ region for **Case C** in Proposition 1. Then there exists a pure strategy equilibrium such that firm i 's equilibrium value $V_i(x)$ equals $F(x)$, where $F(x)$ is given in (13) in the $x < \hat{x}_L$ region, and $V_i(x)$ is given by (54) in the $x \geq \hat{x}_L$ region. Both firms wait in the $x < \hat{x}_L$ region. In the $x \in [\hat{x}_L, x_F)$ region, firms compete to enter and one firm is randomly selected to enter immediately as Leader and the other optimally waits until $\tau_F^* = \inf\{s : X_s \geq x_F\}$ to enter as Follower. In the $x \geq x_F$ region, the two firms in effect simultaneously enter with one chosen to be Leader randomly.

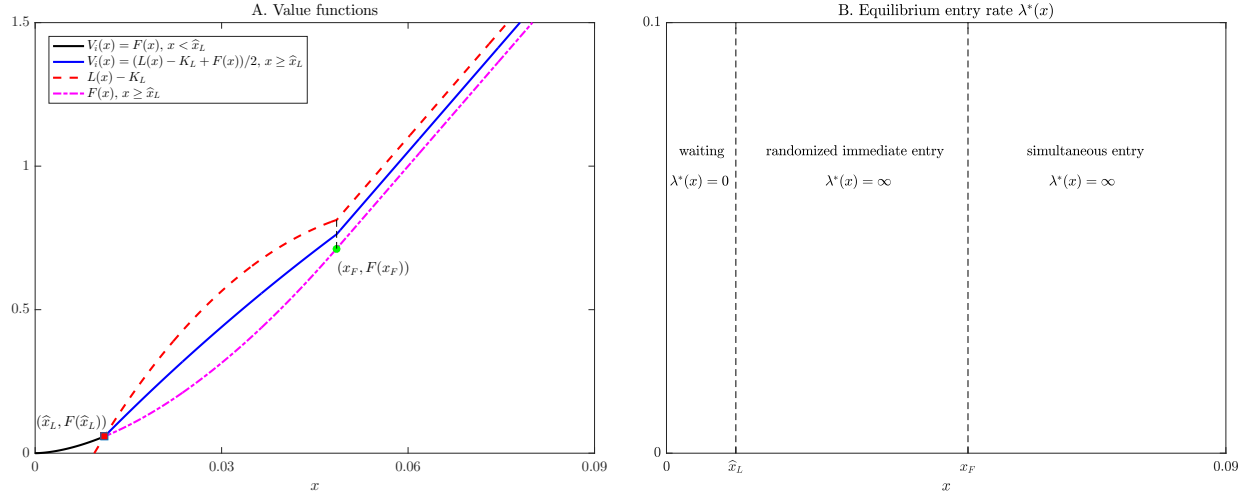


Figure 5: VALUE FUNCTIONS AND ENTRY RATE IN THE SYMMETRIC EQUILIBRIUM IN **CASE C**. Parameter values are $R = 0.8$, $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$. The two cutoff values of the three x regions are $\hat{x}_L = 0.0112$ and $x_F = 0.0485$.

Graphical Illustration. Next, we use Figure 5 to highlight the key results of **Case C**. We set the entry-cost ratio at $R = 0.8 < 1$. The implied Leader’s entry cost is $K_L = 0.4$.

First, we note that $L(x) - K_L > F(x)$ holds in the $x > \hat{x}_L = 0.0112$ region, which implies that the first-mover advantage dominates and both firms want to enter first. To select Leader, we need a randomization device while keeping the *ex ante* rents for the two firms equal. The solid (blue) line depicts the value function $V_i(x)$ given in (54) in the entry region. To the left of the red square is the $x < \hat{x}_L = 0.0112$ region, where both firms wait. Note that $\hat{x}_L = 0.0112 < x_M = 0.0194$. That is, the option value of waiting is eroded as emphasized in Dixit and Pindyck (1994). Mathematically, we prove $\hat{x}_L < x_M$ for $R \in (0, 1]$ in Lemma 1.

Second, in the $x \geq x_F$ subregion (recall that $x_F > \hat{x}_L$ and in our example $x_F = 0.0485 > \hat{x}_L = 0.0112$), Follower immediately enters after Leader is randomly chosen. This is the “simultaneous entry” region in Panel B of Figure 5. Third, in the $x \in [\hat{x}_L, x_F]$ subregion, the (lucky) Leader collects monopoly rents until Follower enters when X_t reaches x_F for the first time. This is the “randomized immediate entry” region in Panel B of Figure 5.

In sum, $V_*(x)$ is convex in the $x < \hat{x}_L$ waiting region, concave in the $x \in [\hat{x}_L, x_F]$ sequential-entry region, and linear in the $x \geq x_F$ simultaneous-entry region. Note that all entry decisions are pure strategies. Mathematically, there is no smooth-pasting condition involved for **Case C**, as there is no second-mover advantage and firms compete to be the first mover as soon as

Leader's net payoff $L(x) - K_L$ exceeds Follower's value $F(x)$.

Notation-wise, for pure entry strategies, although we do not explicitly refer to equilibrium entry rates, we write $\lambda^*(x) = \infty$ in the entry region and $\lambda^*(x) = 0$ in the waiting region.

6 Case B: Both First- and Second-mover Advantages

In this section, we analyze Case B where $1 < R \leq R_{AB}$. Because the range of R for Case B lies between that for Case A and for Case C, we expect that both first-mover and second-mover advantages (the key force behind Case C and for Case A, respectively) influence the equilibrium outcomes. Indeed, our analysis of Case B confirms the key results in Case A and Case C and also generates new insights that depend on the *interaction* between the two types of advantages in our real-option context.

Next, we summarize the solution for Case B. In Appendix A, we define the equilibrium involving both pure and mixed strategies.

6.1 Closed-Form Markov Perfect Equilibria

Theorem 3 *Consider Case B where $1 < R \leq R_{AB}$. Let $\hat{x}_L$ and $\hat{x}_F$ be the two roots of $L(x) - K_L = F(x)$ in the $(0, x_F)$ region for Case B in Proposition 1.³¹ Then there exists a symmetric Markov perfect equilibrium with the following properties:*

1. *In the $x \leq \hat{x}_F$ domain, firms only play pure strategies.*
 - (a) *In the $x < \hat{x}_L$ region, both firms wait and $V_a(x) = V_b(x) = F(x)$.*
 - (b) *In the $x \in [\hat{x}_L, \hat{x}_F]$ region, firms compete to become Leader with one firm being randomly selected as Leader and $V_a(x) = V_b(x) = (L(x) - K_L + F(x))/2$.*
2. *In the $x > \hat{x}_F$ domain, firms play mixed strategies. Firm value is $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is the unique solution to the variational inequality (35) in the $x > \hat{x}_F$ domain subject to the boundary conditions: (37) as $x \rightarrow \infty$ and*

$$V_*(\hat{x}_F) = F(\hat{x}_F). \quad (55)$$

³¹If $R = R_{AB}$, the two roots of $L(x) - K_L = F(x)$ are the same and moreover equal x_M , the entry threshold of a (hypothetical) monopolist with entry cost K_L : $\hat{x}_L = \hat{x}_F = x_M$.

The equilibrium strategy is $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, where $\lambda^*(x) > 0$ is given by (34) in the probabilistic entry region:

$$\mathcal{R}^E := \{x > \hat{x}_F : V_*(x) = L(x) - K_L\} \quad (56)$$

and $\lambda^*(x) = 0$ for any x in the $x > \hat{x}_F$ domain but not in $\mathcal{R}^E$, i.e., $x \in (\hat{x}_F, \infty) \setminus \mathcal{R}^E$.

Intuitively speaking, the cutoff value $\hat{x}_F$ divides the total market demand x into two domains: (1.) the $x \leq \hat{x}_F$ domain where firms play pure strategies in equilibrium as in **Case C** and (2.) the $x > \hat{x}_F$ domain where firms play mixed strategies as in **Case A**. We provide additional discussions of the key results including Follower's strategies in Subsection 6.2 using figures.

As for **Case A**, there are also two subcases for **Case B**. Let $R_{B_1B_2}$ denote the level of the entry-cost ratio R that solves $\hat{x}_F = \underline{x}$, where $\underline{x}$ is given in Lemma 2.³² The two subcases of **Case B** are (i) **Subcase B₁** where $R_{B_1B_2} < R \leq R_{AB}$ and (ii) **Subcase B₂** where $1 < R \leq R_{B_1B_2}$, as shown in Figure 6. The solution for **Subcase B₁** features *five* regions and the solution for **Subcase B₂** features *four* regions. For both subcases, there are two regions to the left of $\hat{x}_F$: the $x < \hat{x}_L$ waiting region and the $x \in [\hat{x}_L, \hat{x}_F]$ entry region where firms compete to be Leader and one firm is luckily selected. Theorem 3 summarizes the solutions in the $x < \hat{x}_L$ and $x \in [\hat{x}_L, \hat{x}_F]$ regions which apply to both subcases. Next, we summarize the solutions in the $x > \hat{x}_F$ domain for the two subcases.

Proposition 4 *The solution in the $x > \hat{x}_F$ domain for **Case B** is as follows.*

1. **Subcase B₂** where $1 < R \leq R_{B_1B_2}$. There are two regions ($x \in (\hat{x}_F, \bar{x})$ and $x \geq \bar{x}$) where the second-mover advantage dominates. In the $x \in (\hat{x}_F, \bar{x})$ region, both firms wait and firm i 's value is $V_i(x) = \Theta(x; \hat{x}_F, \bar{x})$, where $\Theta(x; a, b)$ for any $x \in [a, b]$ is given by (50) and $\bar{x}$ is given in case (ii) in Lemma 4. In the $x \geq \bar{x}$ region, both firms enter probabilistically at the rate of $\lambda^*(x) > 0$ given in (45) and firm i 's value is $V_i(x) = \Pi(x)/2 - K_L$.
2. **Subcase B₁** where $R_{B_1B_2} < R \leq R_{AB}$. There are three regions ($x \in (\hat{x}_F, \underline{x}]$, $x \in (\underline{x}, \bar{x})$, and $x \geq \bar{x}$) where the second-mover advantage dominates. In the $x \in (\hat{x}_F, \underline{x}]$ region, both firms enter probabilistically at the rate of $\lambda^*(x) > 0$ given in (48), and firm i 's value is

³²To be precise, $R_{B_1B_2} = \sup \{R \in (1, R_{AB}) : \underline{x} \leq \hat{x}_F\}$.

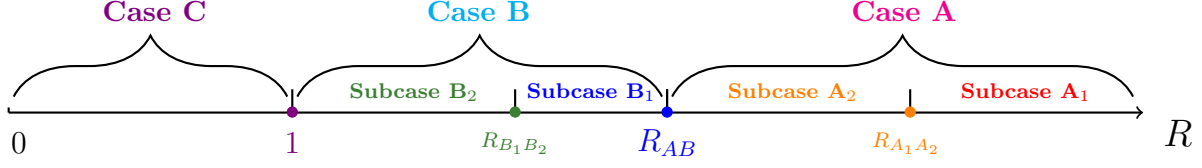


Figure 6: THIS FIGURE SUMMARIZES ALL CASES OF THE DUOPOLY MODEL SOLUTION WITH FOUR ENTRY-COST RATIO ($R = K_L/K_F$) THRESHOLDS, $R_{A_1 A_2} > R_{AB} > R_{B_1 B_2} > 1$: **SUBCASE A₁**: $R > R_{A_1 A_2}$; **SUBCASE A₂**: $R_{AB} < R \leq R_{A_1 A_2}$; **SUBCASE B₁**: $R_{B_1 B_2} < R \leq R_{AB}$; **SUBCASE B₂**: $1 < R \leq R_{B_1 B_2}$; AND **CASE C**: $R \leq 1$.

given by $V_i(x) = L(x) - K_L$. In the $x \in (\underline{x}, \bar{x})$ region, both firms wait and firm i 's value is $V_i(x) = \Theta(x; \underline{x}, \bar{x})$, where $\Theta(x; a, b)$ for any $x \in [a, b]$ is given by (50) and the cutoffs $\underline{x}$ and $\bar{x}$ are given in Lemma 2. In the $x \geq \bar{x}$ region, both firms enter probabilistically at the rate of $\lambda^*(x) > 0$ given in (45) and firm i 's value is $V_i(x) = \Pi(x)/2 - K_L$.

6.2 Comparing Subcase B₁ vs Subcase B₂: Graphical Illustration

For the triplet (r, μ, σ) , we use the same (annualized) parameter values as for **Case A** in Subsection 4.3: $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$. This triplet (r, μ, σ) pins down the optionality measure $\beta = 1.70$, the cutoff value between **Case A** and **Case B**, $R_{AB} = 1.30$, and the cutoff value for the two subcases of **Case B**, $R_{B_1 B_2} = 1.19$. We choose $R = 1.28 \in (R_{B_1 B_2}, R_{AB}) = (1.19, 1.30)$ to illustrate the economics of **Subcase B₁** and $R = 1.18 \in (1, R_{B_1 B_2}) = (1, 1.19)$ to illustrate the economics of **Subcase B₂**.

In Panels A and B of Figure 7, we plot the solution for **Subcase B₂**. First, by intersecting $L(x) - K_L$ with $F(x)$ as in **Case C**, we obtain the two regions on the left: (1.) The $x < \hat{x}_L = 0.021$ region where firms wait to preserve the option value and (2.) the $x \in [\hat{x}_L, \hat{x}_F] = [0.021, 0.042]$ region where firms compete to enter as Leader as discussed earlier. Graphically, we determine the remaining parts of our model solution by smoothly pasting a curve starting from the magenta square at $(\hat{x}_F, F(\hat{x}_F))$ onto Leader's net payoff line $L(x) - K_L$. This convex curve is the equilibrium firm value $V_*(x)$ for $x > \hat{x}_F$ where the second-mover advantage dominates. Moreover, the smooth-pasting condition at $x = \bar{x}$ divides the $x > \hat{x}_F$ domain into two regions: the $x \geq \bar{x}$ region where firms play mixed entry strategies and the $(\hat{x}_F, \bar{x})$ region where firms wait to lower entry costs.

In sum, for **Subcase B₂**, firms have four strategies: 1) waiting for the standard option value

reason (subject to entry competition) in the $x < \hat{x}_L$ region where $\lambda^*(x) = 0$ as shown in Panel B; 2.) competing to enter as Leader due to the first-mover advantage in the $x \in [\hat{x}_L, \hat{x}_F]$ region where $\lambda^*(x) = \infty$; 3.) waiting with the hope of becoming Follower to lower entry costs in the $x \in (\hat{x}_F, \bar{x})$ region where $\lambda^*(x) = 0$; and 4.) entering probabilistically in the $x \geq \bar{x}$ region. The first two regions resemble the solution for **Case C** while the latter two regions resemble the solution in **Subcase A₁**. Note that the threshold, $\bar{x}$, dividing the two regions where the second-mover advantage dominates, depends on the threshold: $\hat{x}_F$. That is, there is a feedback effect from the first-mover advantage to the second-mover advantage.

Panel B of Figure 7 plots the equilibrium entry rates. We emphasize that in the fourth region where $x \geq \bar{x}$, firms probabilistically enter at the rate of $(x/2 - rK_L)/(K_L - K_F)$ as Follower immediately enters after Leader does and therefore Leader enjoys no monopoly rents. This is the same as in the $x \geq \bar{x}$ region of **Subcase A₁**.

Panels C and D of Figure 7 plot the solution for **Subcase B₁**. Compared with **Subcase B₂** (see Panels A and B of Figure 7), there is a new fifth region for **Subcase B₁**: the probabilistic entry region where $x \in (\hat{x}_F, \underline{x}]$. Unlike the probabilistic entry region $x \geq \bar{x}$ where Follower immediately enters after Leader and hence Leader enjoys no monopoly rents, Leader enjoys a (stochastic) period of monopoly rents in the $x \in (\hat{x}_F, \underline{x}]$ region.

Also note that $V_i(x)$ is concave in the $x \in (\hat{x}_F, \underline{x}]$ region, while convex in the $x > \underline{x}$ region. This new $(\hat{x}_F, \underline{x}]$ region arises in **Subcase B₁** as we increase the second-mover advantage measured by R . The intuition is as follows. As R increases for a fixed K_F , the $[\hat{x}_L, \hat{x}_F]$ region becomes narrower,³³ leaving more room for Leader to earn monopoly rents (as $\hat{x}_F$ is further to the left of Follower's entry threshold x_F). As a result, a symmetric equilibrium where firms play mixed entry strategies in the $(\hat{x}_F, \underline{x}]$ region becomes feasible. This explains why we have a new (fifth) region as we move from **Subcase B₂** to **Subcase B₁**.

Panel D plots the equilibrium entry rate $\lambda^*(x)$ in the $(\hat{x}_F, \underline{x}]$ region, which equals the forgone monopoly profit $(x - rK_L)$ divided by the net benefit of being Follower: $F(x) - (L(x) - K_L)$. Because of embedded optionality, $\lambda^*(x)$ given in (48) is highly nonlinear.

³³Lemma 1 shows that $\hat{x}_L$ increases and $\hat{x}_F$ decreases with R for **Case B**.

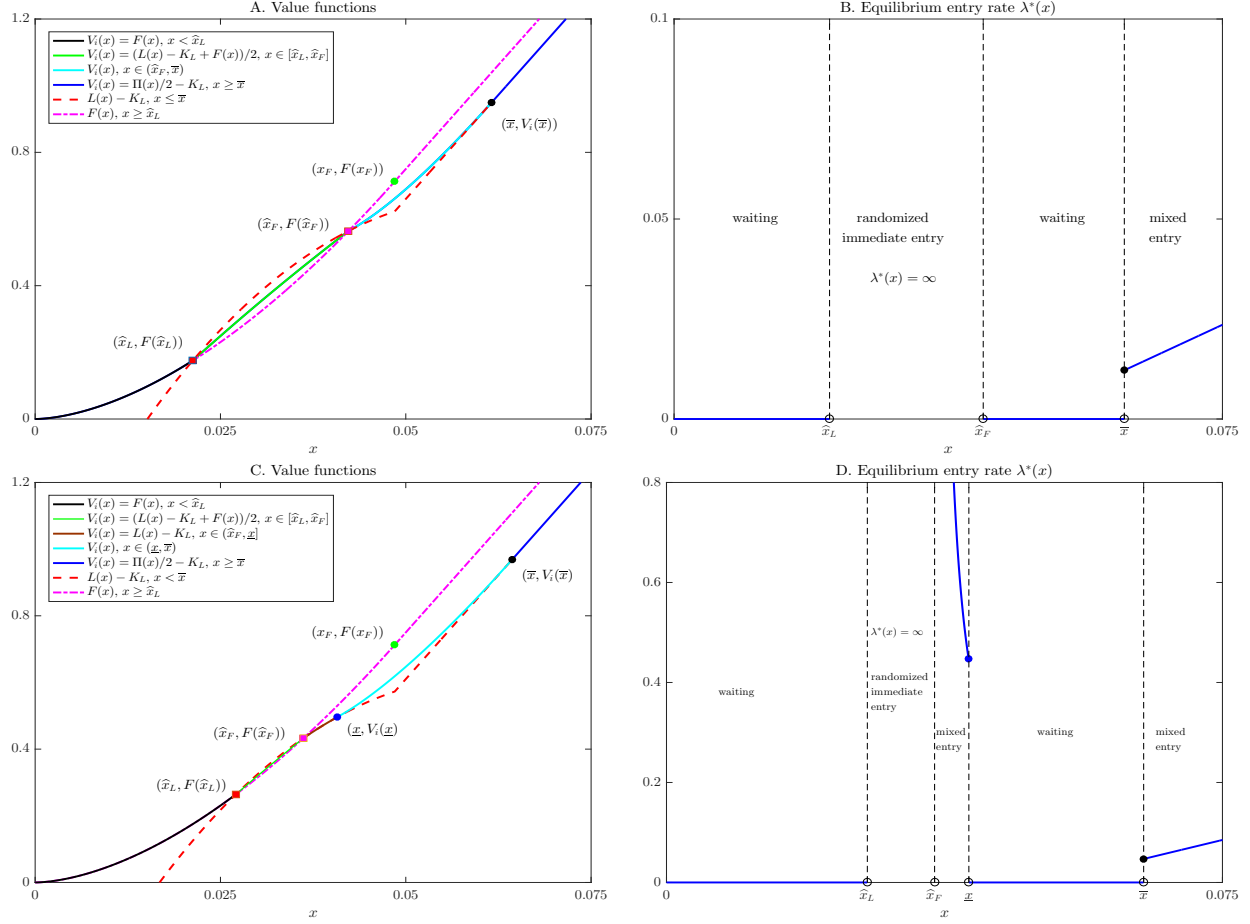


Figure 7: VALUE FUNCTIONS AND ENTRY RATES IN THE SYMMETRIC EQUILIBRIA OF **SUBCASE B₂** (PANELS A-B) AND **SUBCASE B₁** (PANELS C-D). For both subcases, we set $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$, which imply $R_{AB} = 1.3$, $R_{B_1B_2} = 1.19$, and $x_F = 0.0485$. We choose $R = 1.18$ for **Subcase B₂** and $R = 1.28$ for **Subcase B₁**. The three cutoff values defining the four regions in **Subcase B₂** are $\hat{x}_L = 0.021$, $\hat{x}_F = 0.042$, and $\bar{x} = 0.062$. The four cutoff values defining the five regions in **Subcase B₁** are $\hat{x}_L = 0.027$, $\hat{x}_F = 0.036$, $\underline{x} = 0.041$, and $\bar{x} = 0.064$.

6.3 Summary of **Case B**: $1 < R \leq R_{AB}$

Case B is the most general case where both the first-mover and second-mover advantages are present. Depending on the value of R , the solution fits into **Subcase B₁** or **Subcase B₂**.

For **Subcase B₂** where $1 < R \leq R_{B_1B_2}$, there are four regions: two disconnected waiting regions (one to preserve the option value and the other to lower entry costs), the pure entry strategy region where rents are equalized, and the probabilistic entry region where Leader enjoys no monopoly rents in equilibrium.

For **Subcase B₁** where $R_{B_1B_2} < R \leq R_{AB}$, we have a new (fifth) region in addition to the four regions as in **Subcase B₂**. This new region appears between the $[\hat{x}_L, \hat{x}_F]$ region where the first-mover advantage dominates and the second waiting region $(\underline{x}, \bar{x})$ where firms want to lower their entry costs. In this new region, once a firm becomes Leader, it enjoys monopoly rents for a stochastic duration as the other firm chooses to wait and therefore $V_i(x)$ is concave.³⁴ Finally, we emphasize that the *interaction* between the two types of advantages in our real-option context fundamentally alters how these five regions are determined and connected. For example, firms may enter in one of three different ways: pure strategy, probabilistic entry with or without monopoly rents (of stochastic duration). Moreover, firm entry is not monotonic as market demand increases. In sum, game-theoretic considerations when both first- and second-mover advantages are present fundamentally enrich the equilibrium real-option exercising decisions and firm valuation.

7 Comparing Pure- and Mixed-strategy Equilibria

In this section, we first solve for the asymmetric pure-strategy equilibria, and then compare the value functions and the distribution of time to Leader's entry $\tau_L^* - t$ in pure- and mixed-strategy equilibria. We find that the second-mover advantage fundamentally changes both the qualitative and quantitative implications of duopoly competition. For brevity, we focus on **Subcase A₁** where $R > R_{A_1A_2}$ and relegate analyses for other cases to Internet Appendix I.C. In Appendix F, we provide a comparison between the solution of our duopoly model with that of the cooperative benchmark.

7.1 Solution for Pure-strategy Equilibria

Consider the asymmetric pure-strategy equilibrium where firm a is Leader and firm b is Follower.³⁵ In this equilibrium, $\tau_a = \tau_L$ and let $P_L(x)$ denote Leader's value. Firm a 's problem

³⁴The insights for the other four regions are similar to those for the four regions of **Subcase B₂**.

³⁵This equilibrium is supported by beliefs that firm a is Leader and firm b is Follower with probability one. Making firm a Follower and firm b Leader, we obtain the other pure-strategy equilibrium. See Section 5 for our definition of a pure-strategy equilibrium.

given in (18) is simplified to the following problem:

$$P_L(x) = \max_{\tau \geq t} \mathbb{E}_t^x \left[e^{-r(\tau-t)} (L(X_\tau) - K_L) \right]. \quad (57)$$

Let x_L denote firm a 's optimal entry threshold. First, note that Leader's value $P_L(x)$ in this equilibrium equals firm value $V_i(x) = V_*(x)$ in the mixed-strategy equilibrium: $P_L(x) = V_*(x)$ where $V_*(x)$ is given in (42)-(43). This is because the value function $P_L(x)$ for the stopping-time problem (57) is characterized by the same variational inequality (35). Second, Leader's optimal entry time is $\tau_L^* = \inf\{s \geq t : X_s \geq x_L\}$, where $x_L = \bar{x}$ is given in (44).

Now we turn to Follower's problem. Taking Leader's strategy $\tau_L^* = \inf\{s \geq t : X_s \geq \bar{x}\}$ as given, let $P_F(x)$ denote Follower's optimal value for the following problem:

$$P_F(x) = \max_{\tau_F \geq \tau_L^*} \mathbb{E}_t^x \left[e^{-r(\tau_F-t)} \left(\frac{\Pi(X_{\tau_F})}{2} - K_F \right) \right]. \quad (58)$$

Solving (58), we obtain Follower's optimal entry time: $\tau_F^* = \inf\{s \geq \tau_L^* : X_s \geq x_F\}$, where x_F is given in (15). Because Leader's optimal entry threshold is larger than Follower's: $x_L = \bar{x} = 2x_M > x_F$, Follower enters immediately after Leader does: $\tau_F^* = \tau_L^* +$. Therefore, Follower's value, $P_F(x)$, is given by the following closed-form expressions:

$$P_F(x) = F(x) = \Pi(x)/2 - K_F, \quad x \geq \bar{x}, \quad (59)$$

$$P_F(x) = (x/\bar{x})^\beta F(\bar{x}) = (x/\bar{x})^\beta (\Pi(\bar{x})/2 - K_F), \quad x < \bar{x}. \quad (60)$$

Next, we summarize the key results for the pure-strategy equilibria.

Theorem 4 *There are two asymmetric pure-strategy equilibria for **Subcase A₁** where $R > R_{A_1 A_2}$. Leader enters at $\tau_L^* = \inf\{s \geq t : X_s \geq \bar{x}\}$ and Leader's value is $P_L(x) = V_*(x)$, where $\bar{x}$ is given in (44) and $V_*(x)$ is given in (42)-(43). Because Follower's entry threshold x_F is lower than Leader's threshold: $x_L = \bar{x} > x_F$, Follower enters immediately after Leader ($\tau_F^* = \tau_L^* +$) and Follower's value $P_F(x)$ is given by (59)-(60).*

7.2 Value Functions in Pure- and Mixed-strategy Equilibria

In Figure 8, we plot Leader's and Follower's value functions, $P_L(x)$ and $P_F(x)$, for the asymmetric pure-strategy equilibria and then compare them with firm i 's value function $V_i(x) = V_*(x)$ (where $i = a, b$) for the symmetric mixed-strategy equilibrium.

In a pure-strategy equilibrium, firms are pre-determined to be Leader and Follower due to

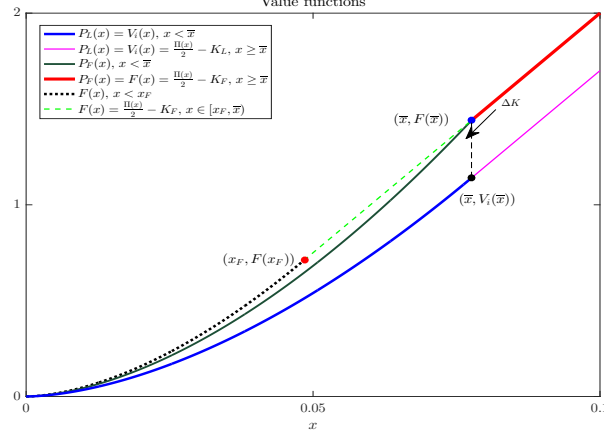


Figure 8: COMPARING VALUE FUNCTIONS FOR PURE-STRATEGY AND MIXED-STRATEGY EQUILIBRIA FOR **SUBCASE A₁**. Leader's value in a pure-strategy equilibrium, $P_L(x)$, equals firm value in the mixed-strategy equilibrium: $P_L(x) = V_i(x) = V_*(x)$, and Follower's value in a pure-strategy equilibrium, $P_F(x)$, is higher than Leader's value: $P_F(x) > P_L(x)$. Leader's entry threshold in a pure-strategy equilibrium, x_L , equals the cutoff value, $\bar{x}$, between the probabilistic entry region and the waiting region in the mixed-strategy equilibrium: $x_L = \bar{x} = 0.0776$, where $\bar{x}$ is given in (44). Parameter values are $R = 1.6$, $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$, which imply $K_L = 0.8$ and $x_F = 0.0485$.

beliefs. The solid lines depict the equilibrium pre-entry Leader's value $P_L(x)$ where the blue segment is increasing and convex in the waiting region ($x < x_L = \bar{x}$) and the magenta solid line is Leader's net linear payoff function $\Pi(x)/2 - K_L$ in the entry region ($x \geq x_L = \bar{x}$).

Note that $P_L(x)$ in pure-strategy equilibria equals firm value $V_i(x)$ in the mixed-strategy equilibrium. This is because $P_L(x)$ and $V_i(x)$ are both determined by a smooth-pasting condition with the same net payoff functions $\Pi(x)/2 - K_L$. Also, we can show $\bar{x} = 2x_M > x_F$.

Now consider Follower's value $P_F(x)$. Because Leader's entry threshold $x_L = \bar{x}$ is higher than Follower's entry threshold x_F given in (15), Follower immediately enters after Leader and $P_F(x)$ in the waiting region ($x < \bar{x}$) is increasing and convex (the solid green line.) Note that $P_F(x) < F(x)$ and also the smooth-pasting condition does not hold for $P_F(x)$ at its equilibrium entry threshold $x_L = \bar{x}$.³⁶ As $K_L > K_F$, Follower's value in the pure-strategy equilibria is larger than in the mixed-strategy equilibrium: $P_F(x) > V_i(x)$. The industry's total market capitalization in a pure-strategy equilibrium is thus larger than in the mixed-strategy equilibrium for all $x > 0$.³⁷

³⁶The black dotted and green dashed line segments for $F(x)$ in Figure 8 aid our understanding of the model's mechanism and solutions but are off-the-equilibrium path.

³⁷This follows from $P_L(x) + P_F(x) - [V_a(x) + V_b(x)] = P_F(x) - V_i(x) = P_F(x) - P_L(x) > 0$, as $P_L(x) =$

Note that in our pure-strategy equilibria, Leader still exercises its entry option too late compared with the socially optimal level. This is because Leader anticipates no monopoly rents in equilibrium. This result differs from those in simple war-of-attrition games, where the pure-strategy equilibria are socially efficient as one firm immediately drops out (Levin, 2004). Why are our pure-strategy equilibria socially inefficient? This is because Leader (the loser in the attrition game) also has a real option in our model. This result highlights the rich predictions generated by the interaction between the real-option value and the second-mover advantage in our stochastic entry game.

A key feature of our model is that Leader enters probabilistically in the mixed-strategy equilibrium even when market demand is high. We next show that probabilistic entry substantially lengthens the time it takes for a firm to become Leader: $\tau_L^* - t$. We demonstrate the economic significance of this result by comparing the distribution of $\tau_L^* - t$ in the mixed-strategy equilibrium with that in pure-strategy equilibria.

7.3 Time to Entry $\tau_L^* - t$ in Pure- and Mixed-strategy Equilibria

Definitions. Fix a calendar date T and let $X_t = x$ at $t \leq T$. Let $G^{\text{mixed}}(t, x; T)$ denote the time- t cumulative distribution function (CDF) that a firm enters as Leader before T in the mixed-strategy equilibrium. Similarly, let $G^{\text{pure}}(t, x; T)$ denote the time- t CDF for the same event in the pure-strategy equilibria. Mathematically, for any $x > 0$ and time $t \in [0, T]$:

$$G^{\text{mixed}}(t, x) = \mathbb{P}_t^x(\tau_L^{\text{mixed}} - t \leq T - t) \quad \text{and} \quad G^{\text{pure}}(t, x) = \mathbb{P}_t^x(\tau_L^{\text{pure}} - t \leq T - t), \quad (61)$$

where superscripts, mixed and pure, indicate that Leader's entry time τ_L^* in the mixed- and pure-strategy equilibria (characterized in Proposition 2 and Theorem 4), respectively.

For *every* sample path, Leader enters sooner in a pure-strategy equilibrium than in the mixed-strategy equilibrium. In both types of equilibria, firm entry is characterized by trigger strategies and the entry threshold is the same, which implies that entry is only possible in the $x \geq \bar{x}$ region. But the economic forces underpinning the entry strategies in the $x \geq \bar{x}$ region are different: In a pure-strategy equilibrium, Leader enters with probability one but in contrast both firms enter probabilistically at the rate of $\lambda^*(x) = (x/2 - rK_L)/(K_L - K_F)$ in $\overline{V_a(x) = V_b(x)}$ and $P_F(x) > P_L(x)$ (implied by the second-mover advantage).

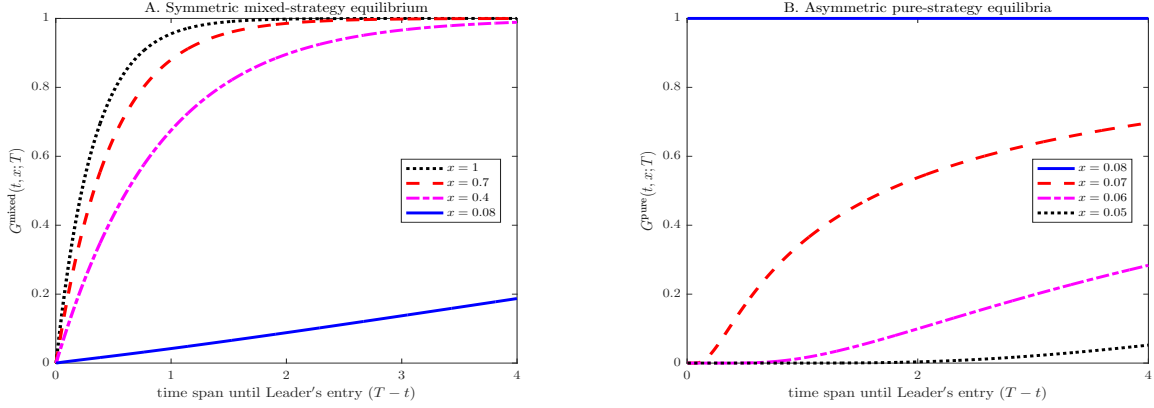


Figure 9: CDFs OF TIME TO ENTRY $\tau_L^* - t$ IN PURE-STRATEGY AND MIXED-STRATEGY EQUILIBRIA. Panel A plots the CDF of $\tau_L^* - t$ in the mixed-strategy equilibrium for four levels of market demand: $x = 0.08, 0.4, 0.7$, and 1. Panel B plots the CDF of $\tau_L^* - t$ in the pure-strategy equilibria for four levels of market demand: $x = 0.05, 0.06, 0.07$, and 0.08. Parameter values are $K_F = 0.5$, $K_L = 0.8$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$.

the mixed-strategy equilibrium. As a result, entry can be much delayed in the mixed-strategy equilibrium than in the pure-strategy equilibria. Below we present quantitative results for entry-timing distributions, relegating the analytical results to Appendix G.

Panel A of Figure 9 plots the CDFs $G^{\text{mixed}}(t, x; T)$ of $\tau_L^* - t$ in the mixed-strategy equilibrium for four levels of x : 0.08, 0.4, 0.7, 1. When $X_t = x = 0.08$, firms enter within one year with a small probability (4.21%). Even within four years, firms only enter with 18.7% probability. In contrast, in a pure-strategy equilibrium, as $X_t = x = 0.08 > \bar{x} = 0.0776$, entry occurs with probability one. This comparison of CDFs for the mixed-strategy and pure-strategy equilibria shows that quantitative predictions of the model are very different depending on which equilibrium we choose.

In the mixed-strategy equilibrium, entry can take significantly much longer time. For example, even when market demand is very high, e.g., $X_t = x = 1$, (recall this is a flow variable and Follower's one-time lumpy entry cost is only $K_F = 0.5$, half of one year's profit X_t), there is still $4.5\% = 1 - G^{\text{mixed}}(t, 1; t + 1)$ probability that firms have not entered within one year. This is in sharp contrast with the prediction in a pure-strategy equilibrium where entry is immediate provided that $x \geq \bar{x} = 0.0776$ as we discussed earlier.

The key takeaway from our analysis of distribution of $\tau_L^* - t$ is that the mixed-strategy equilibrium can be much more inefficient and entry is significantly delayed than the pure-

strategy equilibria, which are also inefficient.

8 Conclusion

We analyze how a firm optimally takes into account both the first-mover and the second-mover advantages when deciding its entry decision in a classic duopoly real-option framework (McDonald and Siegel, 1986; Dixit and Pindyck, 1994). Our general model solution ([Subcase B₁](#)) features five regions, defined by four endogenous cutoff values $(\hat{x}_L, \hat{x}_F, \underline{x}, \bar{x})$ in ascending order. In addition to the option-value of waiting region where $x \in (0, \hat{x}_L)$ and the competing-to-enter region where $x \in [\hat{x}_L, \hat{x}_F]$ (due to the first-move advantage) as in Dixit and Pindyck (1994), there are three new regions in the $x > \hat{x}_F$ domain where the second-mover advantage dominates: 1.) in the $x \in (\hat{x}_F, \underline{x}]$ region, firms enter probabilistically and Leader earns monopoly rents until Follower enters; 2.) in the $x \geq \bar{x}$ region, firms enter probabilistically with no equilibrium monopoly rents for Leader; and 3.) in the $x \in (\underline{x}, \bar{x})$ region between the two probabilistic entry regions, firms wait.

Our model generates new quantitative and testable predictions. For example, firm entry is non-monotonic with respect to market demand and can occur either in clusters or sequentially. Also in contrast to the classic real-option model's prediction, a firm's (pre-entry) option value can be concave in market demand x and decrease with market volatility in the probabilistic entry region (with monopoly rents) due to the interactive effect between imperfect competition and the second-mover advantage.

To sharpen the key mechanism of duopoly entry games, we have purposefully chosen a minimalistic setting. A firm has complete information about its competitor's cost structure and type. One important extension that we plan to pursue is to incorporate the effects of reputation as in Kreps and Wilson (1982), Milgrom and Roberts (1982), and Abreu and Gul (2000) into our duopoly entry game. Another interesting extension of our model is to grant the first mover with monopoly rents for some periods to capture industrial policies, e.g., patent protection for newly developed drugs. Indeed, a key reason for patent protection is to encourage firm innovation and entry, consistent with our duopoly model's excessive entry delay prediction caused by the second-mover advantage.

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Appendices

A Definition of Equilibrium for the General Case

Definition 4 A Markov entry strategy for firm $i \in \{a, b\}$ is a pair: $\varphi_i = (\mathcal{E}_i, \lambda_i(x))$, where $\mathcal{E}_i \subseteq \mathbb{R}_+$ is a closed set and the entry rate $\lambda_i(x)$ is a measurable function from $\mathbb{R}_+ \setminus \mathcal{E}_i$ to $\mathbb{R}_+$. Firm i enters the market for sure when $X_t \in \mathcal{E}_i$ and randomly at an intensity rate of $\lambda_i(X_t)$ when $X_t \notin \mathcal{E}_i$. A Markov strategy pair $(\varphi_a, \varphi_b) = \{(\mathcal{E}_a, \lambda_a), (\mathcal{E}_b, \lambda_b)\}$ is feasible if and only if $\int_0^t \lambda_i(X_s) ds < \infty$ almost surely for any $t < \inf\{s \geq 0 : X_s \in \mathcal{E}_a \cup \mathcal{E}_b\}$. Let Φ denote the set of all feasible entry strategies.

Given $X_0 = x_0 > 0$ and a feasible Markov strategy pair $(\varphi_a, \varphi_b) = \{(\mathcal{E}_a, \lambda_a), (\mathcal{E}_b, \lambda_b)\}$, the entry time pair (τ_a, τ_b) is determined by the joint distribution:

$$\begin{aligned} \mathbb{P}^{x_0}(\tau_a \leq t_a, \tau_b \leq t_b) &= \mathbb{E}^{x_0} \left[\int_{s \in [0, t_b]} \int_{t \in [0, t_a]} dG_a(t) dG_b(s) \right] \\ &= \mathbb{E}^{x_0} [(G_a(t_a) - G_a(0))(G_b(t_b) - G_b(0))], \quad t_a \geq 0, t_b \geq 0, \end{aligned} \quad (\text{A.1})$$

where $G_i(t)$ is the conditional distribution of firm i 's entry time τ_i given $\{X_s; s \geq 0\}$:

$$G_i(t) = 1 - \left(1 - \mathbf{1}_{t \geq \inf\{s \geq 0 : X_s \in \mathcal{E}_i\}}\right) e^{-\int_0^t \lambda_i(X_u) du}. \quad (\text{A.2})$$

Definition 5 Let $J_i(x; \varphi_a, \varphi_b)$ denote firm i 's value at time t defined in (18) for a given $X_t = x > 0$ and a feasible Markov strategy pair $(\varphi_a, \varphi_b) = \{(\mathcal{E}_a, \lambda_a), (\mathcal{E}_b, \lambda_b)\}$. A feasible entry strategy pair $\{\varphi_a^*, \varphi_b^*\}$ forms a *Markov perfect equilibrium* if for any $x > 0$, we have

$$J_a(x; \varphi_a^*, \varphi_b^*) \geq J_a(x; \varphi_a, \varphi_b^*), \quad \forall \varphi_a = (\mathcal{E}_a, \lambda_a) \text{ s.t. } \{\varphi_a, \varphi_b^*\} \in \Phi, \quad (\text{A.3})$$

$$J_b(x; \varphi_a^*, \varphi_b^*) \geq J_b(x; \varphi_a^*, \varphi_b), \quad \forall \varphi_b = (\mathcal{E}_b, \lambda_b) \text{ s.t. } \{\varphi_a^*, \varphi_b\} \in \Phi. \quad (\text{A.4})$$

B Comparing Grenadier (1996) with Our Model

In this appendix, we highlight how the key mechanism in our model fundamentally differs from that in Grenadier (1996). To do so, we first summarize the solution of a version of Grenadier (1996) that preserves his model's key mechanism.³⁸

³⁸To facilitate comparisons between his model and ours, we only provide technical details that are essential for readers to understand the key differences between the two models.

Grenadier (1996) is in effect an extension of the model of Chapter 9 in Dixit and Pindyck (1994) with the following two features. First, each firm receives a constant (rental) profit Q per period before a firm exercises its real-estate redevelopment option as Leader. Second, Leader's redevelopment lowers its competitor's (rental) profit from Q to zero.³⁹

In sum, firm profits in Grenadier (1996) are as follows: 1.) for $t \leq \tau_L$, both firms receive Q per period; 2.) for $t \in (\tau_L, \tau_F)$, Leader receives the entire market demand $\{X_t\}$ from its new development and Follower receives zero profits; 3.) for $t \geq \tau_F$, both firms receive $\{X_t/2\}$ equally splitting the market demand.

Given the entry game's structure, firm i solves the following problem in Grenadier (1996):

$$\mathbb{E}_t^x \left[e^{-r(\tau_L-t)} \left(\mathbf{1}_{\tau_i < \tau_{-i}} (L(X_{\tau_i}) - K_L) + \mathbf{1}_{\tau_i > \tau_{-i}} F(X_{\tau_{-i}}) + \mathbf{1}_{\tau_i = \tau_{-i}} \frac{L(X_{\tau_i}) - K_L + F(X_{\tau_i})}{2} \right) + \int_t^{\tau_L} e^{-r(s-t)} Q ds \right], \quad (\text{B.1})$$

where $L(x)$ is Leader's value given by (16)-(17) and $F(x)$ is Follower's value given by (13)-(14). Compared with the model in Chapter 9 of Dixit and Pindyck (1994), the new term in Grenadier (1996) is the integral on the second line in (B.1), which captures the value of constant rental profits Q that firm i receives before Leader enters at τ_L . Let $V_i(x)$ denote firm i 's equilibrium value function for (B.1).

Next, we summarize the solution of (B.1), which features two cases:

- (i) If the profit flow Q is large, there are four equilibrium regions.
- (ii) If Q is small, there are three equilibrium regions as in our [Case C](#).

Because case (ii) in Grenadier (1996) effectively generates the same results as the model in Chapter 9 of Dixit and Pindyck (1994) (which is [Case C](#) of our model), we do not discuss the details but simply plot the solution in Panel B of Figure 10 for reference.

For our remaining analysis, we focus on case (i) where Q is large, which is the economically more interesting case analyzed in Grenadier (1996). There are four equilibrium regions in total:

³⁹In Grenadier (1996), redevelopment options have other features. First, it takes time for a firm to complete its redevelopment (time to build). Second, Leader's redevelopment lowers Follower's profits from Q to ρQ until Follower exercises its redevelopment option. While realistic, these two features are not necessary to highlight the key differences between our model and Grenadier (1996). To ease comparisons between his model and ours, we set both ρ and time to build to zero.

two waiting regions alternate with the two pure-strategy entry regions where firms rush to enter as Leader.

First, when market demand is low: $x \in (0, \hat{x}_L)$, firms wait to preserve their option values. Second, when market demand is high ($x \geq x_J$), both firms rush to enter regardless, as the value of taking half of the market share is so large that entering simultaneously with the other firm is sufficiently profitable to justify entry. Third, when market demand is medium (i.e., $\hat{x}_L \leq x \leq x_F$), which is to the immediate right of the first region where $x < \hat{x}_L$, both firms rush to enter as Leader. As one firm is selected randomly as Leader, the other firm optimally chooses to wait to preserve its option value. This is the case where the total demand is high enough for one firm to be sufficiently profitable but not enough for Follower to enter (once taking its option value of waiting into account.)

Finally, when market demand is medium high, i.e., between the two pure-strategy entry regions: $x_F < x < x_J$, both firms decide to wait. This is because had one firm entered, the other would also enter (because the market demand is higher than x_F , the threshold above which Follower optimally enters). Anticipating Follower's optimal response along this path, both firms decide to wait as equally splitting the market demand in the $x_F < x < x_J$ region is not profitable for firms. Intuitively, Follower's immediate entry credibly deters firms from entering first as Leader in this region.

We summarize the model's solution for case (i) in Panel A of Figure 10. In the four regions, firms alternate between waiting and entering as Leader with probability one, implying a zero-one binary form of non-monotonic entry strategy in Grenadier (1996). The non-monotonic entry likelihood in our model arises from the interaction between the first and second mover advantages ($R = K_L/K_F > 1$).

Because of the second-mover advantage, our model solution features mixed strategies in the probabilistic entry region and thus the entry likelihood takes a value between zero and one in that region. In contrast, because Leader and Follower have the same entry cost ($R = K_L/K_F = 1$) in Grenadier (1996) as in Chapter 9 of Dixit and Pindyck (1994), the second-mover advantage is absent in his model. As a result, firms do not use mixed entry strategies, which implies that firm entry is a binary decision, and moreover, there is no interaction/feedback effect between the first-mover and second-mover advantages in Grenadier

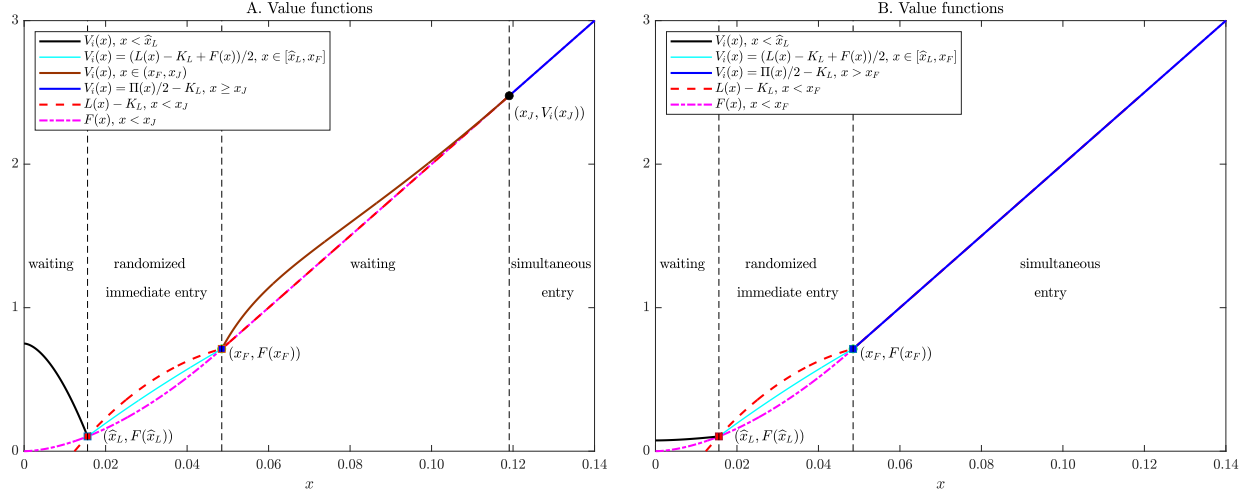


Figure 10: VALUE FUNCTIONS AND FIRM ENTRY STRATEGIES IN GRENADIER (1996). Parameter values are $Q = 0.03$ in Panel A and $Q = 0.003$ in Panel B, and $K_L = K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$ in both panels. The equilibrium cutoff values are: $x_J = 0.1191$ in Panel A and $(\hat{x}_L = 0.0156, x_F = 0.0485)$ in both panels.

(1996). In sum, while both our model and Grenadier (1996) predict that firm entry is non-monotonic in market demand, the predicted entry dynamics and the underlying mechanisms in the two models are quite different, as we have discussed above.

C Different Market Shares for Leader and Follower

In this appendix, we generalize the model in Section 2 by allowing Follower and Leader to have different market shares in the duopoly stage. Specifically, Leader and Follower receive profits $\{D_L X_s; s \geq \tau_F\}$ and $\{D_F X_s; s \geq \tau_F\}$, respectively, where $D_L > 0$ and $D_F > 0$ are constants. The total market demand under duopoly $(D_L + D_F)X_t$ can differ from X_t under monopoly. Moreover, Leader's market share $D_L/(D_L + D_F)$ can differ from Follower's market share $D_F/(D_L + D_F)$.

Follower's value in the (τ_L, τ_F) period is given by:

$$F(x) = \max_{\tau_F \geq t} \mathbb{E}_t^x \left[\int_{\tau_F}^{\infty} e^{-r(s-t)} D_F X_s ds - e^{-r(\tau_F-t)} K_F \right]. \quad (\text{C.1})$$

Let τ_F^* denote Follower's optimal entry time for (C.1). Taking τ_F^* as given, we write Leader's post-entry value function, $L(x)$, for $t \in (\tau_L, \tau_F^*)$ as:

$$L(x) = \mathbb{E}_t^x \left[\int_t^{\tau_F^*} e^{-r(s-t)} X_s ds + \int_{\tau_F^*}^{\infty} e^{-r(s-t)} D_L X_s ds \right]. \quad (\text{C.2})$$

Using effectively the same procedure as in Section 2.3, we obtain Follower's value:

$$F(x) = (D_F \Pi(x_F) - K_F) \left(\frac{x}{x_F} \right)^\beta, \quad x < x_F, \quad (\text{C.3})$$

$$F(x) = D_F \Pi(x) - K_F, \quad x \geq x_F, \quad (\text{C.4})$$

where Follower's optimal entry threshold, x_F , is given by

$$x_F = \frac{1}{D_F} \frac{\beta}{\beta - 1} (r - \mu) K_F. \quad (\text{C.5})$$

Solving $L(x)$ defined in (C.2), we obtain

$$L(x) = \Pi(x) - (1 - D_L) \Pi(x_F) \left(\frac{x}{x_F} \right)^\beta, \quad x < x_F, \quad (\text{C.6})$$

$$L(x) = D_L \Pi(x), \quad x \geq x_F. \quad (\text{C.7})$$

For our remaining analysis in this appendix, we further make the following assumption:

Assumption 1 *Market shares satisfy $D_L/D_F > 1$ with $D_L \in (0, 1)$ and $R = K_L/K_F$ is sufficiently large that $L(x) - K_L < F(x)$ for $x \in (0, x_F]$, where x_F is given in (C.5).*

Because Leader has a larger market share than Follower: $D_L/D_F > 1$, the first-mover advantage must dominate for sufficiently large market demand X_t . On the other hand, as $R = K_L/K_F$ is sufficiently large, the second-mover advantage dominates for medium market demand (when $x \leq x_F$). Under Assumption 1, we deliver both first-mover and second-mover advantages on the equilibrium path in our model without getting overly involved in non-essential detailed technical analyses.

Mathematically, there exists a unique root, $\hat{x}$, for the equation $L(x) - K_L = F(x)$ in the $x > 0$ region, which is a firm's indifference condition between entering as Leader with probability one and being Follower. Solving $L(x) - K_L = F(x)$, we obtain:

$$\hat{x} = \frac{K_L - K_F}{D_L - D_F} (r - \mu). \quad (\text{C.8})$$

We can verify that the following inequalities hold:

$$L(x) - K_L > F(x), \quad x > \hat{x}, \quad (\text{C.9})$$

$$L(x) - K_L < F(x), \quad 0 < x < \hat{x}. \quad (\text{C.10})$$

That is, a firm is strictly better off by entering first as Leader if and only if $x > \hat{x}$. As we expect, in equilibrium, the first-mover advantage dominates when market demand is sufficiently high

($x > \hat{x}$) and the second-mover advantage dominates if otherwise ($0 < x < \hat{x}$). Additionally, we can show that $\hat{x} > x_F$, where x_F is given by (C.5), under Assumption 1.

We summarize the symmetric equilibrium strategy in the following theorem.

Theorem 5 *Under Assumption 1, there exists a symmetric Markov perfect equilibrium with the following properties:*

1. *In the $x \geq \hat{x}$ region where $\hat{x}$ is given in (C.8), two firms simultaneously enter. One firm is randomly chosen to be Leader and the other firm immediately enters. Pre-entry firm value functions are thus equalized: $V_a(x) = V_b(x) = (L(x) - K_L + F(x))/2$.*
2. *In the $x < \hat{x}$ region, firms wait or play mixed entry strategies. Pre-entry firm value functions are equal: $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is the unique solution to the variational inequality (35) in the $x < \hat{x}$ domain subject to the boundary conditions:*

$$V_*(0) = 0 \quad \text{and} \quad V_*(\hat{x}) = F(\hat{x}). \quad (\text{C.11})$$

The equilibrium entry rates are $\lambda_a^(x) = \lambda_b^*(x) = \lambda^*(x)$, where $\lambda^*(x) > 0$ is given by (34) in the probabilistic entry region: $\mathcal{R}^E := \{x < \hat{x} : V_*(x) = L(x) - K_L\}$, and $\lambda^*(x) = 0$ for all x in the $x < \hat{x}$ domain but not in $\mathcal{R}^E$, i.e., $x \in (0, \hat{x}) \setminus \mathcal{R}^E$.*

In Figure 11, we plot the value functions in Panel A and the equilibrium entry rate $\lambda^*(x)$ in Panel B, using $D_L = 0.58$, $D_F = 0.45$, $K_L = 1.4$, and $K_F = 0.5$. The equilibrium solution for this simple case features three regions. To the left of $\bar{x} = \frac{1}{D_L} \frac{\beta}{\beta-1} (r - \mu) K_L = 0.117$ is the waiting region where $\lambda^*(x) = 0$ and $V_a(x) = V_b(x) = (x/\bar{x})^\beta (L(\bar{x}) - K_L)$. In the middle region where $x \in [\bar{x}, \hat{x})$, firms enter probabilistically and a firm's pre-entry value is thus equal $L(x) - K_L$, the net payoff upon immediate entry: $V_a(x) = V_b(x) = L(x) - K_L$. Using (34) and (C.7), we can pin down the equilibrium entry rate $\lambda^*(x) = \frac{D_L x - r K_L}{F(x) - (L(x) - K_L)}$ in this middle region as $\bar{x} > x_F$. Finally, to the right of $\hat{x}$ is the simultaneous entry region where the first-mover advantage dominates and firms rush to enter. Moreover, Follower immediately enters after Leader is randomly chosen, as the market demand is sufficiently high.

In sum, we have purposefully chosen parameter values to obtain a three-region solution for a symmetric equilibrium: (1) a waiting region where $x < \bar{x}$ as in the standard single-firm

real-option problem, (2) a probabilistic-entry region where $x \in [\bar{x}, \hat{x})$, and (3) a pure-strategy entry region where $x \geq \hat{x}$ and the first-mover advantage dominates.

In an even richer setting (by changing parameter values), we expect an even richer solution. For example, we expect that a six-region case, which combines the five regions analyzed in [Subcase B₁](#) with the region for sufficiently large market demand X_t where the first-mover advantage dominates due to $D_L/D_F > 1$, can naturally emerge in a setting with parameter values close to [Subcase B₁](#).

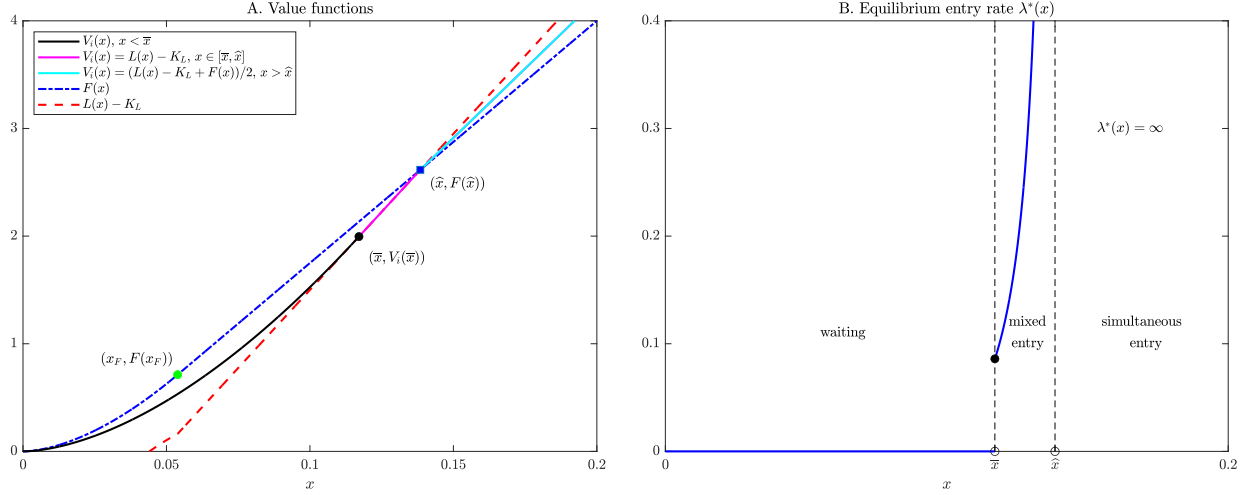


Figure 11: VALUE FUNCTIONS $V_a(x) = V_b(x) = V_*(x)$ AND EQUILIBRIUM PROBABILISTIC ENTRY RATES $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$ WHEN $D_L/D_F > 1$. The threshold values dividing the market demand x into the three regions are: $\bar{x} = \frac{1}{D_L} \frac{\beta}{\beta-1} (r - \mu) K_L$ and $\hat{x} = \frac{K_L - K_F}{D_L - D_F} (r - \mu)$. The three regions are: 1.) the $x < \bar{x}$ waiting region, 2.) the $x \in [\bar{x}, \hat{x})$ probabilistic entry (mixed-strategy) region, and 3.) the $x \geq \hat{x}$ simultaneous entry region. Parameter values are $D_L = 0.58$, $D_F = 0.45$, $K_L = 1.4$, $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$. The equilibrium thresholds are: $x_F = 0.054$, $\bar{x} = 0.117$, and $\hat{x} = 0.139$.

D Some Technical Results

Deriving the Variational Inequality (35) for [Case A](#). The HJB equation (32) and the inequality $V_i(x) \leq F(x)$ given in (29) together imply

$$\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) - r V_i(x) \leq 0.$$

Substituting (31) into (32), we obtain

$$\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) - r V_i(x) = 0 \quad \text{if} \quad L(x) - K_L < V_i(x).$$

Combining the above with $L(x) - K_L \leq V_i(x)$ given in (29), we obtain the variational inequality (35).

Formulas of $\theta_1(a, b)$ and $\theta_2(a, b)$. Let $\tau_{ab} = \inf\{s \geq t : X_s \leq a \text{ or } X_s \geq b\}$ for a given pair (a, b) satisfying $0 < a < b$ and let $\Theta(x; a, b)$ denote the following present value:

$$\Theta(x; a, b) = \mathbb{E}_t^x[e^{-r(\tau_{ab}-t)}(L(X_{\tau_{ab}}) - K_L)], \quad x \in [a, b]. \quad (\text{D.1})$$

We can show that $\Theta(x; a, b)$ is given by (50), where $\theta_1(a, b)$ and $\theta_2(a, b)$ solve:

$$\theta_1 a^\beta + \theta_2 a^\gamma = L(a) - K_L \quad \text{and} \quad \theta_1 b^\beta + \theta_2 b^\gamma = L(b) - K_L.$$

Solving the above equations yields

$$\theta_1(a, b) = \frac{(L(a) - K_L)b^\gamma - (L(b) - K_L)a^\gamma}{a^\beta b^\gamma - a^\gamma b^\beta} \quad \text{and} \quad \theta_2(a, b) = \frac{(L(b) - K_L)a^\beta - (L(a) - K_L)b^\beta}{a^\beta b^\gamma - a^\gamma b^\beta}. \quad (\text{D.2})$$

Next, we present two lemmas used several times in the main body of the paper.

Lemma 1 *We characterize the cutoff values, $\hat{x}_L$ and $\hat{x}_F$, as follows.*

(i) *For $R \in (1, R_{AB}]$ (Case B), $\hat{x}_L$ and $\hat{x}_F$, the two roots of $L(x) - K_L = F(x)$ in Proposition 1, have the following closed-form expressions:*

$$\hat{x}_F = \bar{\eta}(R)(r - \mu)K_F \quad \text{and} \quad \hat{x}_L = \underline{\eta}(R)(r - \mu)K_F, \quad (\text{D.3})$$

where $\bar{\eta}(R)$ and $\underline{\eta}(R)$ are given by

$$\bar{\eta}(R) := \sup\{y > 0 : y - \frac{\beta+1}{\beta-1} \left(\frac{\beta-1}{2\beta}\right)^\beta y^\beta = R\}, \quad (\text{D.4})$$

$$\underline{\eta}(R) := \inf\{y > 0 : y - \frac{\beta+1}{\beta-1} \left(\frac{\beta-1}{2\beta}\right)^\beta y^\beta = R\}. \quad (\text{D.5})$$

We can further show: $x_F > \hat{x}_F > x_M > \hat{x}_L$ for $R \in (1, R_{AB})$. For the special case $R = R_{AB}$, $x_F > \hat{x}_F = x_M = \hat{x}_L$. Finally, $\lim_{R \rightarrow 1+} \hat{x}_F = x_F$.

(ii) *For $R \in (0, 1]$ (Case C), the equation $L(x) - K_L = F(x)$ has only one root in the $x < x_F$ region: $\hat{x}_L$ given in (D.3). Finally, $\hat{x}_L < x_M$.*

(iii) *The root $\hat{x}_L$ increases in $R \in (0, R_{AB}]$ and the root $\hat{x}_F$ decreases in $R \in (1, R_{AB}]$.*

Lemma 2 For any $R \in [1, R_{A_1 A_2})$, there exists a unique pair of thresholds $(\underline{x}, \bar{x})$ in the domain $(x_M, x_F) \times (2x_M, \infty)$ satisfying⁴⁰

$$\Theta(\underline{x}; \underline{x}, \bar{x}) = L(\underline{x}) - K_L, \quad \Theta_x(\underline{x}; \underline{x}, \bar{x}) = L'(\underline{x}), \quad (\text{D.6})$$

$$\Theta(\bar{x}; \underline{x}, \bar{x}) = \frac{\Pi(\bar{x})}{2} - K_L, \quad \Theta_x(\bar{x}; \underline{x}, \bar{x}) = \frac{\Pi'(\bar{x})}{2}, \quad (\text{D.7})$$

where $\Theta(x; a, b)$, x_M , and x_F are given by (50), (6), and (15), respectively. Also, $\underline{x}$ and $\bar{x}$ are continuously differentiable as functions of R in $R \in [1, R_{A_1 A_2})$. Moreover, $\underline{x}$ and $\bar{x}$ are given by

$$\underline{x} = (1 + u) \frac{\beta}{\beta - 1} (r - \mu) K_L \quad \text{and} \quad \bar{x} = (1 + U) \frac{2\beta}{\beta - 1} (r - \mu) K_L, \quad (\text{D.8})$$

where (u, U) is the unique solution pair to the following system of equations in the domain $(0, 2/R - 1) \times (0, \infty)$:

$$U(1 + U)^{-\gamma} = 2^\gamma u(1 + u)^{-\gamma}, \quad (\text{D.9})$$

$$H(U) = 2^\beta H(u) - \frac{\beta}{\beta - 1} R^{\beta-1}. \quad (\text{D.10})$$

The function $H(\cdot)$ in (D.10) is defined as follows:

$$H(z) = \frac{(1 - \gamma) \frac{\beta}{\beta-1} (1 + z) + \gamma}{(\beta - \gamma)(1 + z)^\beta}. \quad (\text{D.11})$$

When $R = R_{A_1 A_2}$, we can show that $\underline{x} = x_M$, $\bar{x} = 2x_M$, and for any $x \in [\underline{x}, \bar{x}]$, $\Theta(x; \underline{x}, \bar{x}) = V_*(x)$ where $V_*(x)$ is given by (42)-(43).

We relegate our proofs of Lemmas 1 and 2 to Internet Appendix I.B.

Finally, we determine $R_{B_1 B_2}$, the cutoff value of R for [Subcase B₁](#) and [Subcase B₂](#).

Determining $R_{B_1 B_2}$. First, Lemma 2 implies that $\underline{x}$ is continuous in $R \in [1, R_{AB}]$ and satisfies $x_M < \underline{x} < x_F$ for any $R \in [1, R_{AB}]$. Second, Lemma 1 implies that $\hat{x}_F$ is continuous in $R \in (1, R_{AB}]$ and satisfies $\hat{x}_F \rightarrow x_F$ as $R \rightarrow 1+$ and $\hat{x}_F = x_M$ for $R = R_{AB}$. Combining these two results, we conclude: $\underline{x} > \hat{x}_F$ for $R = R_{AB}$, $\underline{x} < \hat{x}_F$ as $R \rightarrow 1+$, $R_{B_1 B_2} = \sup \{ R \in (1, R_{AB}) : \underline{x} \leq \hat{x}_F \}$ is well-defined, $R_{B_1 B_2} \in (1, R_{AB})$, and

$$\underline{x} > \hat{x}_F, \quad \text{if } R \in (R_{B_1 B_2}, R_{AB}]. \quad (\text{D.12})$$

⁴⁰Recall that in this case, we have $x_F \leq 2x_M$, which implies $\bar{x} > x_F$.

E Proofs

We now prove the theorems and propositions in the main text. Let $\mathcal{A}V(x)$ denote the infinitesimal generator operating on a function $V(x)$:

$$\mathcal{A}V(x) = \frac{\sigma^2}{2}x^2V''(x) + \mu xV'(x) - rV(x). \quad (\text{E.1})$$

Proof of Proposition 1: Let $K_{AB} = R_{AB}K_F$. Using (16)-(17) for $L(x)$ and (13)-(14) for $F(x)$, we can verify $K_{AB} = \max_{x>0} [L(x) - F(x)]$. Then, for any $x > 0$, we have

$$L(x) - K_L - F(x) \leq \max_{x>0} [L(x) - F(x)] - K_L = K_{AB} - K_L = K_F(R_{AB} - R) < 0$$

for $R > R_{AB}$. We thus have proven **Case A** in Proposition 1. We relegate our proofs for **Case B** and **Case C** in Proposition 1 to Internet Appendix I.B. $\square$

Before proving Theorem 1, we introduce the following lemma for the variational-inequality equation (35) for **Case A**.

Lemma 3 *Let $V_*(x)$ be the solution to the variational-inequality problem (35)-(37) in the $x \geq 0$ region. Let $\mathcal{R}^E$ denote the probabilistic entry region:*

$$\mathcal{R}^E := \{x > 0 : V_*(x) = L(x) - K_L\}. \quad (\text{E.2})$$

(i) For **Subcase A₁** where $R > R_{A_1A_2}$, $V_*(x)$ is given by (42)-(43) and $\mathcal{R}^E = [\bar{x}, \infty)$, where $\bar{x} = 2x_M$.

(ii) For **Subcase A₂** where $R_{AB} < R \leq R_{A_1A_2}$, $V_*(x)$ is given below:

$$V_*(x) = \left(\frac{x}{x_M}\right)^\beta (L(x_M) - K_L), \quad x \in [0, x_M], \quad (\text{E.3})$$

$$V_*(x) = L(x) - K_L, \quad x \in [x_M, \underline{x}], \quad (\text{E.4})$$

$$V_*(x) = \Theta(x; \underline{x}, \bar{x}), \quad x \in (\underline{x}, \bar{x}), \quad (\text{E.5})$$

$$V_*(x) = L(x) - K_L = \frac{\Pi(x)}{2} - K_L, \quad x \geq \bar{x}, \quad (\text{E.6})$$

where $\Theta(x; a, b)$ is given by (50) and the thresholds, $\underline{x}$ and $\bar{x}$, are given in Lemma 2. Finally, $\mathcal{R}^E = [x_M, \underline{x}] \cup [\bar{x}, \infty)$.

We relegate our proof of Lemma 3 to Internet Appendix I.B.

Proof of Theorem 1: Let $g(x) := \mathcal{A}V_*(x)$ for $x \geq 0$, where $\mathcal{A}V$ is the infinitesimal generator given in (E.1). Substituting the closed-form expressions for $V_*(x)$ for both **Subcase A₁** and **Subcase A₂** into (E.1)⁴¹, we obtain

$$g(x) = \mathcal{A}V_*(x) = \lambda^*(x)[L(x) - K_L - F(x)], \quad x > 0, \quad (\text{E.7})$$

where $\lambda^*(x)$ is given by (34) for any $x \in \mathcal{R}^E$ and $\lambda^*(x) = 0$ for any $x \in (0, \infty) \setminus \mathcal{R}^E$. Using the expression for $\lambda^*(x)$ given in (34) and $L(x)$ given in (16)-(17), we obtain:

$$g(x) = \mathbf{1}_{x \in \mathcal{R}^E} [(rK_L - x)\mathbf{1}_{x < x_F} + (rK_L - x/2)\mathbf{1}_{x > x_F}]$$

for any $x > 0$.

For **Subcase A₁**, using $\lambda^*(x)$ given by (34) for any $x \in \mathcal{R}^E$, where $\mathcal{R}^E = [2x_M, \infty)$, we obtain $\lambda^*(x) = \frac{x/2 - rK_L}{K_L - K_F}$. Similarly, for **Subcase A₂**, in the $[\bar{x}, \infty)$ region, we also obtain $\lambda^*(x) = \frac{x/2 - rK_L}{K_L - K_F}$. Therefore, there exist a positive value x' and a positive constant λ' , such that $\lambda^*(x) \geq \lambda' > 0$ for all $x > x'$, which further implies:

$$e^{-\int_t^\infty \lambda^*(X_s)ds} = 0, \quad \text{almost surely.} \quad (\text{E.8})$$

Next, we complete our proof in two steps. First, we show that it is suboptimal for firm a to deviate from its equilibrium strategy if firm b does not (Step 1).

Step 1: We prove $V_(x) \geq J_a(x; \lambda_a, \lambda^*)$ where $(\lambda_a, \lambda^*) \in \Phi$.*

Let τ_a and τ_b be firm a 's and b 's stochastic entry time associated with the strategy pair $(\lambda_a, \lambda_b) = (\lambda_a, \lambda^*)$, where $\lambda_a \neq \lambda^*$. Let $\tau := \min\{\tau_a, \tau_b\}$.

For both **Subcase A₁** and **Subcase A₂**, $V_*(x)$ is twice continuously differentiable except at finite points and is globally continuously differentiable (see Lemma 3.) Applying Itô's Lemma to $e^{-rs}V_*(X_s)$ for $s \in [t, \tau]$ and taking expectations at time t , we obtain the following expression for $V_*(x)$:

$$V_*(x) = \mathbb{E}_t^x[e^{-r(\tau-t)}V_*(X_\tau)] - \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right]. \quad (\text{E.9})$$

Recall that $V_*(x)$ satisfies the variational inequality (35), we have $V_*(x) \geq L(x) - K_L, \forall x > 0$. Substituting it into the right side of (E.9), we obtain the following inequality:

$$V_*(x) \geq \mathbb{E}_t^x[e^{-r(\tau-t)}(L(X_\tau) - K_L)] - \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right]. \quad (\text{E.10})$$

⁴¹According to Lemma 3, the variational-inequality problem (35)-(37) admits a unique solution, $V_*(x)$, given by (42)-(43) for **Subcase A₁** and (E.3)-(E.6) for **Subcase A₂**, respectively.

Note that

$$\begin{aligned} J_a(x; \lambda_a, \lambda^*) &= \mathbb{E}_t^x \left[e^{-r(\tau-t)} [\mathbf{1}_{\tau_a < \tau_b} (L(X_\tau) - K_L) + \mathbf{1}_{\tau_a > \tau_b} F(X_\tau)] \right] \\ &= \mathbb{E}_t^x [e^{-r(\tau-t)} (L(X_\tau) - K_L)] - \mathbb{E}_t^x [\mathbf{1}_{\tau_a > \tau_b} e^{-r(\tau-t)} (L(X_\tau) - K_L - F(X_\tau))], \end{aligned} \quad (\text{E.11})$$

where the second equality follows from the property: $\mathbf{1}_{\tau_a = \tau_b} = 0$ almost surely. Using (E.11) and (E.10), we obtain

$$J_a(x; \lambda_a, \lambda^*) \leq V_*(x) + \mathbb{E}_t^x \left[\int_t^\tau e^{-r(s-t)} \mathcal{A}V_*(X_s) ds - \mathbf{1}_{\tau_a > \tau_b} e^{-r(\tau-t)} (L(X_\tau) - K_L - F(X_\tau)) \right]. \quad (\text{E.12})$$

We can simplify the first term on the right side of (E.12) as follows:

$$\begin{aligned} \mathbb{E}_t^x \left[\int_t^\tau e^{-r(s-t)} \mathcal{A}V_*(X_s) ds \right] &= \mathbb{E}_t^x \left[\int_t^\tau e^{-r(s-t)} g(X_s) ds \right] \\ &= \mathbb{E}_t^x \left[\int_t^\infty \int_t^{\tau_a \wedge z} e^{-r(s-t)} g(X_s) \lambda^*(X_z) e^{-\int_t^z \lambda^*(X_u) du} ds dz \right] \\ &= \mathbb{E}_t^x \left[\int_t^{\tau_a} \int_s^\infty \lambda^*(X_z) e^{-\int_t^z \lambda^*(X_u) du} dz e^{-r(s-t)} g(X_s) ds \right] \\ &= \mathbb{E}_t^x \left[\int_t^{\tau_a} e^{-\int_t^s (r + \lambda^*(X_u)) du} g(X_s) ds \right] \\ &= \mathbb{E}_t^x \left[\int_t^{\tau_a} e^{-\int_t^s (r + \lambda^*(X_u)) du} \lambda^*(X_s) [L(X_s) - K_L - F(X_s)] ds \right] \\ &= \mathbb{E}_t^x [\mathbf{1}_{\tau_a > \tau_b} e^{-r(\tau_b-t)} [L(X_{\tau_b}) - K_L - F(X_{\tau_b})]] \end{aligned} \quad (\text{E.13})$$

using (E.7), Tonelli's Theorem (to interchange the integration order in the third equality as $g(x) \leq 0$ and $\lambda^*(x) \geq 0$ for any $x > 0$), integration by parts, and (E.8). Combining (E.12) and (E.13) yields $J_a(x; \lambda_a, \lambda^*) \leq V_*(x)$.

Step 2: We prove $V_(x) = J_a(x; \lambda^*, \lambda^*)$.*

Recall that τ_a^* and τ_b^* are firms a 's and b 's stochastic entry time, respectively, associated with strategy $(\lambda_a(x), \lambda_b(x)) = (\lambda^*(x), \lambda^*(x))$ and $\tau^* := \min\{\tau_a^*, \tau_b^*\}$. Because $\lambda^*(x) = 0$ for any $x \in (0, \infty) \setminus \mathcal{R}^E$, we have $X_{\tau^*} \in \mathcal{R}^E$, which implies $V_*(X_{\tau^*}) = L(X_{\tau^*}) - K_L$. Therefore, we can see that (E.10)-(E.12) hold with equality if λ_a , τ_a , τ_b and τ therein are set to λ^* , τ_a^* , τ_b^* and τ^* , respectively. We have thus shown $V_*(x) = J_a(x; \lambda^*, \lambda^*)$.

In sum, combining our analyses in Steps 1 and 2, we obtain $J_a(x; \lambda^*, \lambda^*) \geq J_a(x; \lambda_a, \lambda^*)$. By symmetry, we also have $J_b(x; \lambda^*, \lambda^*) \geq J_b(x; \lambda^*, \lambda_b)$ for $(\lambda^*, \lambda_b) \in \Phi$. $\square$

Proof of Proposition 2: This is implied by Theorem 1 and part (i) in Lemma 3. $\square$

Proof of Proposition 3: This is implied by Theorem 1 and part (ii) in Lemma 3. $\square$

Proof of Theorem 2: Let $\tau_a^* = \tau_b^* = \hat{\tau} := \inf\{s \geq t : X_s \geq \hat{x}_L\}$. We prove that (τ_a^*, τ_b^*) is the equilibrium strategy pair in three steps.

First, because $L(x) - K_L \geq F(x)$ holds in $x \geq \hat{x}_L$ region, it is optimal for firms to compete to enter as Leader in this region. Leader is selected randomly via the rent-equalization principle of Dixit and Pindyck (1994) and Grenadier (1996), which implies $V_i(x) = (L(x) - K_L + F(x))/2$.

Second, we analyze the solution in the $x \in (0, \hat{x}_L)$ region. As both firms wait in the $(0, \hat{x}_L)$ region and compete to enter only when $\{X_s; s \geq 0\}$ exceeds $\hat{x}_L$, firm i 's value is given by

$$V_i(x) = \mathbb{E}_t^x \left[e^{-r(\hat{\tau}-t)} \frac{L(X_{\hat{\tau}}) - K_L + F(X_{\hat{\tau}})}{2} \right] = \mathbb{E}_t^x [e^{-r(\hat{\tau}-t)} F(X_{\hat{\tau}})] = F(x), \quad (\text{E.14})$$

where the first equality is due to definition (18), the second equality follows from $L(\hat{x}_L) - K_L = F(\hat{x}_L)$ (see Proposition 1) and $X_{\hat{\tau}} = \hat{x}_L$, and the last equality follows from the property that Follower's present value is a martingale in its waiting region.

Third, we show that firms have no incentives to deviate from the strategy pair (τ_a^*, τ_b^*) . Suppose firm a chooses its entry time τ_a , deviating from τ_a^* , and firm b chooses $\tau_b = \tau_b^*$. Let $\tau := \min\{\tau_a, \tau_b^*\}$. Using the definition of $J_i(x)$ given in (18), we obtain

$$J_a(x) \leq \mathbb{E}_t^x [e^{-r(\tau-t)} F(X_{\tau})] = F(x) = V_a(x), \quad (\text{E.15})$$

where the inequality in (E.15) follows from 1.) $\tau_b = \hat{\tau}$ and 2.) the property that $X(s) \leq \hat{x}_L$ and $L(X_s) - K_L \leq F(X_s)$ hold for any $s \in [t, \tau]$ (see Proposition 1), the first equality follows from the property that $\{F(X_s); s \geq 0\}$ is a martingale in the pre-entry region $X_s \leq x_F$ and the second equality follows from (E.14). Therefore, firm a has no incentives to deviate from τ_a^* . The same analysis holds for firm b . We thus have proven (τ_a^*, τ_b^*) is the equilibrium strategy pair. $\square$

Before proving Theorem 3, we introduce the following lemma for the variational-inequality equation (35) for [Case B](#).

Lemma 4 *Let $V_*(x)$ be the unique solution to the variational-inequality problem (35) in the*

$x > \hat{x}_F$ region subject to the boundary conditions (37) and (55). Let $\Theta(x; a, b)$ for any $x \in [a, b]$ be given by (50) and let $\mathcal{R}^E$ denote the probabilistic entry domain defined in (56).

(i) For **Subcase B₁** where $R_{B_1 B_2} < R \leq R_{AB}$, we have

$$V_*(x) = L(x) - K_L, \quad x \in [\hat{x}_F, \underline{x}], \quad (\text{E.16})$$

$$V_*(x) = \Theta(x; \underline{x}, \bar{x}), \quad x \in (\underline{x}, \bar{x}), \quad (\text{E.17})$$

$$V_*(x) = \frac{\Pi(x)}{2} - K_L, \quad x \geq \bar{x}, \quad (\text{E.18})$$

where the cutoff, $\hat{x}_F$, is given in Lemma 1 and the cutoffs, $\underline{x}$ and $\bar{x}$, are given in Lemma 2. The $\mathcal{R}^E$ domain is the union of two disconnected regions: $\mathcal{R}^E = (\hat{x}_F, \underline{x}] \cup [\bar{x}, \infty)$.

(ii) For **Subcase B₂** where $1 < R \leq R_{B_1 B_2}$, we have

$$V_*(x) = \Theta(x; \hat{x}_F, \bar{x}), \quad x \in [\hat{x}_F, \bar{x}], \quad (\text{E.19})$$

$$V_*(x) = \frac{\Pi(x)}{2} - K_L, \quad x \geq \bar{x}, \quad (\text{E.20})$$

where $\hat{x}_F$ is given in Lemma 1 and $\bar{x}$ is the unique solution of the following equation:

$$\Gamma(\hat{x}_F, y) = F(\hat{x}_F) \quad (\text{E.21})$$

in the $y > 2x_M$ region. The function $\Gamma(x, y)$ in (E.21) is defined as follows:

$$\Gamma(x, y) = \frac{\frac{1}{2}(1 - \gamma)\Pi(y) + \gamma K_L}{\beta - \gamma} \left(\frac{x}{y}\right)^\beta + \frac{\frac{1}{2}(\beta - 1)\Pi(y) - \beta K_L}{\beta - \gamma} \left(\frac{x}{y}\right)^\gamma. \quad (\text{E.22})$$

Finally, the probabilistic entry region is given by $\mathcal{R}^E = [\bar{x}, \infty)$.

We relegate our proof of Lemma 4 to Internet Appendix I.B.

Proof of Theorem 3: Using Lemma 4 and similar arguments as in Theorems 1 and 2, we can prove Theorem 3. We relegate our proof of Theorem 3 to Internet Appendix I.B. $\square$

Proof of Proposition 4: This is implied by Theorem 3 and parts (i)-(ii) of Lemma 4. $\square$

Proof of Theorem 4: First, we prove $P_L(x) = V_*(x)$. Applying Itô's Lemma to $e^{-rs}V_*(X_s)$ for $s \in [t, \tau]$ where $\tau \geq t$ is a stopping time, we obtain

$$V_*(x) = \mathbb{E}_t^x[e^{-r(\tau-t)}V_*(X_\tau)] - \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right] \geq \mathbb{E}_t^x[e^{-r(\tau-t)}(L(X_\tau) - K_L)], \quad (\text{E.23})$$

where the inequality follows from $\mathcal{A}V_*(x) \leq 0$ and $V_*(x) \geq L(x) - K_L$ (Lemma 3-(i)). Also note that when $\tau = \tau_L^*$, (E.23) holds with equality. This is because $x_L = \bar{x}$, $\mathcal{A}V_*(x) = 0$ for

$x < \bar{x}$, and $V_*(x) = L(x) - K_L$ for $x \geq \bar{x}$. Therefore, we have shown

$$V_*(x) = \max_{\tau \geq t} \mathbb{E}_t^x [e^{-r(\tau-t)}(L(X_\tau) - K_L)] = P_L(x). \quad (\text{E.24})$$

Next, we prove that $\mathcal{E}_a^* = [x_L, \infty)$ and $\mathcal{E}_b^* = \emptyset$ form an asymmetric pure-strategy entry equilibrium in two steps.

Step 1: We show that Leader has no incentives to deviate its strategy from $\mathcal{E}_a^ = [x_L, \infty)$ to an alternative strategy $\mathcal{E}_a$.*

Let $\tau_a = \inf\{s \geq t : X_s \in \mathcal{E}_a\}$. Using

$$\mathbb{E}_t^x [e^{-r(\tau_a^*-t)}(L(X_{\tau_a^*}) - K_L)] = V_*(x) = \max_{\tau \geq t} \mathbb{E}_t^x [e^{-r(\tau-t)}(L(X_\tau) - K_L)], \quad (\text{E.25})$$

we obtain

$$\mathbb{E}_t^x [e^{-r(\tau_a^*-t)}(L(X_{\tau_a^*}) - K_L)] \geq \mathbb{E}_t^x [e^{-r(\tau_a-t)}(L(X_{\tau_a}) - K_L)]. \quad (\text{E.26})$$

Inequality (E.26) together with $\mathcal{E}_b^* = \emptyset$ implies that Leader has no incentive to deviate.

Step 2: We show that Follower has no incentives to deviate its strategy from $\mathcal{E}_b^ = \emptyset$ to another strategy $\mathcal{E}_b$.*

Using the definition of $P_F(x)$ given in (58), we obtain $J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = P_F(x)$. Let $\tau_a^* := \inf\{s \geq t : X_s \geq \bar{x}\}$, $\tau_b := \inf\{s \geq t : X_s \in \mathcal{E}_b\}$, and $\tau := \min\{\tau_a^*, \tau_b\}$. For any $x \geq \bar{x}$, we conclude from the properties that $F(x) \geq L(x) - K_L$ and $e^{-rs}F(X_s)$ is a supermartingale that

$$J_b(x; \mathcal{E}_a^*, \mathcal{E}_b) \leq \mathbb{E}_t^x [e^{-r(\tau-t)}F(X_\tau)] \leq F(x) = P_F(x).$$

For any $x \in (0, \bar{x})$, applying Itô's Lemma to $e^{-rs}P_F(X_s)$ where $s \in [t, \tau]$, we obtain

$$\begin{aligned} P_F(x) &= \mathbb{E}_t^x [e^{-r(\tau-t)}P_F(X_\tau)] = \mathbb{E}_t^x \left[e^{-r(\tau-t)} \left(F(X_{\tau_a^*}) \mathbf{1}_{\tau_a^* \leq \tau_b} + P_F(X_\tau) \mathbf{1}_{\tau_a^* > \tau_b} \right) \right] \\ &\geq \mathbb{E}_t^x \left[e^{-r(\tau-t)} \left(F(X_{\tau_a^*}) \mathbf{1}_{\tau_a^* \leq \tau_b} + (L(X_\tau) - K_L) \mathbf{1}_{\tau_a^* > \tau_b} \right) \right] \geq J_b(x; \mathcal{E}_a, \mathcal{E}_b^*), \end{aligned}$$

where the second equality uses $X_{\tau_a^*} \geq \bar{x}$ and $P_F(x) = F(x)$ for all $x \geq \bar{x}$, the first inequality follows from $X_\tau \leq \bar{x}$ and $P_F(x) \geq L(x) - K_L$ for any $x \leq \bar{x}$, and the last inequality uses the result: $F(x) \geq L(x) - K_L$ for any $x > 0$. $\square$

F Cooperative Benchmark and Value Erosion

In this appendix, we study the effect of competition on welfare by comparing our duopoly competition model solution to the cooperative benchmark solution.

F.1 Cooperative Benchmark

In the cooperative benchmark model, we choose a pair of entry timing (τ_L, τ_F) to solve:

$$W(x) = \max_{\tau_F \geq \tau_L \geq t} \mathbb{E}_t^x \left[\int_{\tau_L}^{\infty} e^{-r(s-t)} X_s ds - K_L e^{-r(\tau_L-t)} - K_F e^{-r(\tau_F-t)} \right]. \quad (\text{F.1})$$

Note that we have set $D = 1$. The solution for (F.1) features: (1.) Follower never enters and (2.) Leader adopts the monopolist's strategy, by setting $\tau_F = \infty$ and $\tau_L = \tau_M^*$, where τ_M^* is given in Section 2.2. Why $\tau_F = \infty$? This is because Follower grabs a fraction of market share from Leader but does not increase market demand for the industry. It is thus optimal from the industry's perspective to keep one firm out forever. Why is it optimal for the other firm to behave as a monopolist? This follows from the previous $\tau_F = \infty$ result and the standard real-option analysis. That is, using $\tau_F = \infty$ and solving (5), we obtain the monopolist's solution in that the total value in the cooperative benchmark is given by $W(x) = M(x)$, where $M(x)$ is given in (9) for $x < x_M$ and in (10) for $x \geq x_M$, respectively.

F.2 Option Value Erosion in Pure- and Mixed-strategy Equilibria

As in Weeds (2002) and others, we measure inefficiency by dividing the total market capitalization of the duopoly industry, $V_a(x) + V_b(x)$, by the total market capitalization of the industry in the cooperative benchmark, $W(x)$, and subtracting this ratio from one. Let $\Delta(x)$ denote this inefficiency measure:

$$\Delta(x) = 1 - \frac{V_a(x) + V_b(x)}{W(x)}, \quad (\text{F.2})$$

where $W(x) = M(x)$ and $M(x)$ is the monopolist's market value given in (9) for $x < x_M$ and given in (10) for $x \geq x_M$.

In Panels A and B of Figure 12, we plot $\Delta(x)$ for mixed-strategy and pure-strategy equilibria, respectively. First, the industry value loss $\Delta(x)$ decreases with x for both types of equilibria, which is consistent with our intuition. Second, in the mixed-strategy equilibrium (Panel A), $\Delta(x)$ for empirically plausible levels of market demand is very large. For example, $\Delta(x) = 39\%$ in the $x \leq x_M$ region. Also at the threshold above which firms play mixed entry strategy, $x = \bar{x} = 2x_M$, $\Delta(\bar{x}) = (\beta - 1)/(\beta + 1) = 26\%$, independent of Leader's entry cost K_L . (see the three black dots on the dashed black line.)

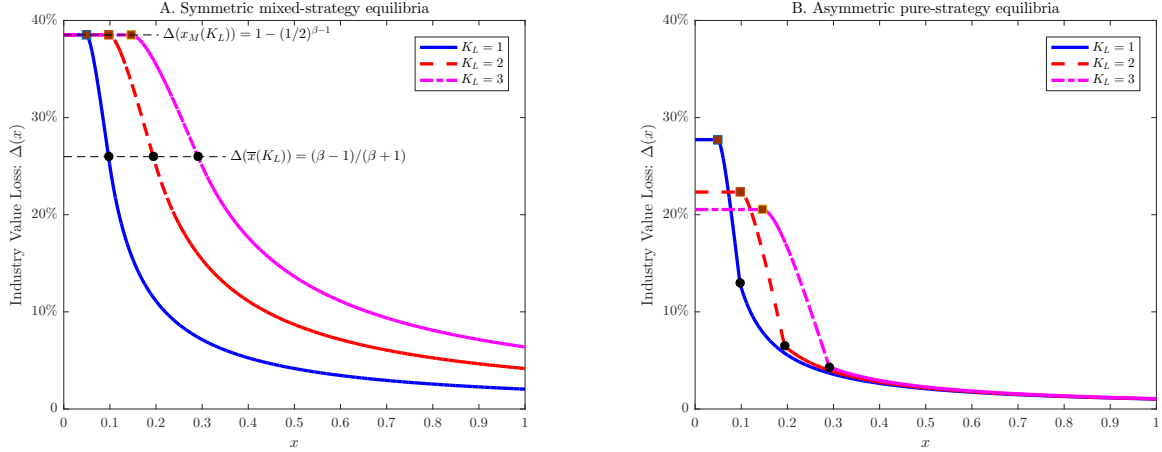


Figure 12: INDUSTRY VALUE LOSS $\Delta(x)$. Panels A and B plot $\Delta(x)$ for the mixed- strategy equilibrium and the pure-strategy equilibria, respectively. Both types of equilibria are socially inefficient. Quantitatively, the mixed-strategy equilibrium is significantly more inefficient than the pure-strategy equilibria. Parameter values are $K_F = 0.5$, $K_L = 1, 2$ and 3 , $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$.

Third, in the pure-strategy equilibria (Panel B), competition also significantly erodes firm option value. For example, $\Delta(x) = 27.7\%$ in the $x \leq x_M = 0.0485$ region for the case $K_L = 1$. Note that even for pure-strategy equilibria, entry is inefficiently and significantly delayed (where the optimal entry threshold increases from x_M to $\bar{x} = 2x_M$).

Comparing the two panels in Figure 12 makes it clear that the mixed-strategy equilibrium is much more inefficient than the pure-strategy equilibria. The intuition is as follows. In the mixed-strategy equilibrium firms play a war-of-attrition game even when market demand is very high and are only willing to enter probabilistically (with the hope that the other firm becomes Leader). In contrast, in a pure-strategy equilibrium, there is no uncertainty which firm is Leader. For brevity, we do not analyze value losses for other cases, which have even richer economics due to the interaction between the first-mover and second-mover advantages in a standard real-option framework.

In sum, when the second-mover advantage is the dominating force, e.g., **Subcase A₁** of our model, the value loss for the industry as a whole is very large for both types of equilibria, especially in the mixed-strategy equilibrium due to excessively delayed entry. This is in sharp contrast with the predictions in Dixit and Pindyck (1994) where firms in equilibrium make preemptive moves and enter sooner than a monopolist. Of course, in our general case

encompassing both first- and second-mover advantages, firms exercise their entry option either too soon or too late depending on x in a nonlinear and non-monotonic way. We leave these analyses out due to space considerations.

G Distribution Functions for Entry Time $\tau_L^* - t$

In this appendix, we derive analytical results for cumulative distribution functions (CDFs) for the symmetric equilibrium, asymmetric equilibria, and the cooperative benchmark.

CDF for the Mixed-strategy Equilibrium: $G^{\text{mixed}}(t, x; T)$. The CDF for time to entry $\tau_L^* - t$ satisfies the following partial differential equation (PDE) for $t < T$ and all $x > 0$:

$$G_t^{\text{mixed}}(t, x) + \mu x G_x^{\text{mixed}}(t, x) + \frac{1}{2} \sigma^2 x^2 G_{xx}^{\text{mixed}}(t, x) + 2\lambda^*(x)(1 - G^{\text{mixed}}(t, x)) = 0, \quad (\text{G.1})$$

subject to the boundary conditions: $G^{\text{mixed}}(t, 0) = 0$, $\lim_{x \rightarrow \infty} G^{\text{mixed}}(t, x) = 1$ for $t \in [0, T)$, and $G^{\text{mixed}}(T, x) = 0$ for $x \in (0, \infty)$. The last term in PDE (G.1) captures the effect of mixed strategies on the CDF. As either firm can become Leader, Leader is determined at the rate of $2\lambda^*(x)$ and the CDF jumps from $G^{\text{mixed}}(t, x)$ to one at Leader's entry time $\tau_L^* = \tau_a^* \wedge \tau_b^*$. The first three terms in the PDE (G.1) are the standard terms describing the calendar time effect, the drift effect of x , and the volatility effect of x on the CDF.

CDF for the Pure-strategy Equilibria: $G^{\text{pure}}(t, x; T)$. The CDF for $\tau_L^* - t$ in the pure-strategy equilibria, $G^{\text{pure}}(t, x)$, satisfies the following PDE for $t < T$ and $x \in [0, \bar{x})$:

$$G_t^{\text{pure}}(t, x) + \mu x G_x^{\text{pure}}(t, x) + \frac{1}{2} \sigma^2 x^2 G_{xx}^{\text{pure}}(t, x) = 0, \quad x \in [0, \bar{x}), \quad (\text{G.2})$$

subject to the boundary conditions: $G^{\text{pure}}(t, \bar{x}) = 1$, $G^{\text{pure}}(t, 0) = 0$ for $t \in [0, T)$, and $G^{\text{pure}}(T, x) = 0$ for $x \in [0, \bar{x})$. Solving (G.2), we obtain the following closed-form solution for the CDF:

$$G^{\text{pure}}(t, x) = N(d_2) + (x/\bar{x})^{(1-2\mu/\sigma^2)} N(d_1), \quad (\text{G.3})$$

where $N(\cdot)$ is the CDF for the standard normal distribution and

$$d_1 = d_2 - (2\mu/\sigma^2 - 1) \sigma \sqrt{T - t}, \quad (\text{G.4})$$

$$d_2 = \frac{\ln(x/\bar{x}) + (\mu - \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}}. \quad (\text{G.5})$$

The first term $N(d_2)$ in (G.3) is the time- t probability for the $X_T \geq \bar{x}$ event.⁴² The second term is the probability for all the events where $X_T < \bar{x}$ but $\{X_s; s \in (t, T)\}$ exceeds $\bar{x}$ at least once at some $s \in (t, T)$.

CDF for the Cooperative Benchmark: $G^{\text{bench}}(t, x; T)$. In Appendix F.1, we have shown that Leader's entry time in the cooperative benchmark is the same as the monopolist's entry time: $\tau_L^* = \tau_M^*$, where τ_M^* is given in Section 2.2. Therefore, the CDF for $\tau_L^* - t$ is given by (G.3) with $\bar{x}$ being replaced by x_M . Recall that $\bar{x} = 2x_M$ (see (44) in Subsection 4.3.1), we have $G^{\text{bench}}(t, x) = G^{\text{pure}}(t, 2x)$.

⁴²The first term is analogous to the conditional (risk-neutral) probability that the option holder receives a strictly positive payoff at the option maturity date in the Black-Scholes option pricing formula.

Internet Appendix to
“Strategic Investment under Uncertainty with First- and
Second-mover Advantages”

I.A Time-varying Second-mover Advantage

In this appendix, we allow the second mover’s advantage to depend on $\tau_L - \tau_F$. Intuitively, the longer that Follower waits after Leader enters, the more the former learns from the latter. Moreover, the marginal second-mover advantage, e.g., the marginal benefit of Follower’s cost reduction, decreases as Follower’s waiting time $\tau_F - \tau_L$ increases. For example, Follower’s marginal benefit of learning decreases as it learns from Leader over time. Let $K_F(\tau_F - \tau_L)$ denotes Follower’s entry cost at the moment it enters at τ_F . We capture these above ideas by using the following entry cost specification for Leader:

$$K_F(\tau_F - \tau_L) = K_L e^{-\zeta(\tau_F - \tau_L)}, \quad (\text{IA.1})$$

where $\zeta > 0$ is a parameter that captures time-varying second-mover advantage. Equation (IA.1) implies that there is no second-mover advantage if Follower enters immediately after Leader does in that $K_F(0) = K_L$. The specification in (IA.1) is more realistic than the one in the main body of our paper, removing firms’ incentives to wait just a split second solely for the purpose of enjoying an instant entry-cost reduction: $\Delta K = K_L - K_F$. By making $K_F(\cdot)$ decreasing and convex in the inter-entry time $\tau_F - \tau_L$, we capture the idea that the second mover advantage increases but the marginal second mover advantage decreases as Follower continues to learn from Leader by waiting to enter.

Taking Follower’s optimal entry time for (IA.3), τ_F^* , as given, Leader’s value at τ_L is the same as in Section 2.3 and is given by

$$L(X_{\tau_L}) = \mathbb{E}_{\tau_L} \left[\int_{\tau_L}^{\infty} e^{-r(s-\tau_L)} X_s ds - \int_{\tau_F^*}^{\infty} e^{-r(s-\tau_L)} \frac{X_s}{2} ds \right]. \quad (\text{IA.2})$$

Follower’s value at $t = \tau_L$ is given by

$$F(X_{\tau_L}) = \max_{\tau_F \geq \tau_L} \mathbb{E}_{\tau_L} \left[\int_{\tau_F}^{\infty} e^{-r(s-\tau_L)} \frac{X_s}{2} ds - e^{-r(\tau_F - \tau_L)} K_F(\tau_F - \tau_L) \right]. \quad (\text{IA.3})$$

For notational convenience, let $Y_t = e^{\zeta(t-\tau_L)} X_t$, $t \geq \tau_L$. We can show that Y_t is a GBM with

drift $\mu + \zeta$ and volatility σ :

$$dY_t = (\mu + \zeta)X_t dt + \sigma Y_t dZ_t \quad t \geq \tau_L, \quad Y_{\tau_L} = X_{\tau_L}. \quad (\text{IA.4})$$

Using Y_t and (IA.1), we can write Follower's value at τ_L as follows:

$$\begin{aligned} F(X_{\tau_L}) &= \max_{\tau_F \geq \tau_L} \mathbb{E}_{\tau_L} \left[e^{-r(\tau_F - \tau_L)} \left(\frac{\Pi(X_{\tau_F})}{2} - K_L e^{-\zeta(\tau_F - \tau_L)} \right) \right] \\ &= \max_{\tau_F \geq \tau_L} \mathbb{E}_{\tau_L} \left[e^{-(r+\zeta)(\tau_F - \tau_L)} \left(\frac{\Pi(Y_{\tau_F})}{2} - K_L \right) \right], \end{aligned} \quad (\text{IA.5})$$

where $\Pi(x)$ is given in (8). In effect, after Leader enters, Follower solves an optimal entry problem with the same entry cost K_L as Leader, but the payoff being Y rather than X and a discount rate of $r + \zeta$ rather than r .

Follower's optimal entry time is given by $\tau_F^* = \inf\{s \geq \tau_L : Y_s \geq y_F\}$, where y_F is a constant and given by

$$y_F = 2 \frac{\alpha}{\alpha - 1} (r - \mu) K_L. \quad (\text{IA.6})$$

In (IA.6), $\alpha \in (1, \beta)$ measures optionality for the Y_t process with a discount rate $r + \zeta$:

$$\alpha = \frac{-(\mu + \zeta - \frac{1}{2}\sigma^2) + \sqrt{(\mu + \zeta - \frac{1}{2}\sigma^2)^2 + 2(r + \zeta)\sigma^2}}{\sigma^2}. \quad (\text{IA.7})$$

Follower's value $F(X_{\tau_L})$ is thus given by:

$$F(X_{\tau_L}) = \left(\frac{\Pi(y_F)}{2} - K_L \right) \left(\frac{X_{\tau_L}}{y_F} \right)^\alpha, \quad X_{\tau_L} < y_F, \quad (\text{IA.8})$$

$$F(X_{\tau_L}) = \frac{\Pi(X_{\tau_L})}{2} - K_L, \quad X_{\tau_L} \geq y_F. \quad (\text{IA.9})$$

Solving $L(X_{\tau_L})$ defined in (IA.2), we obtain

$$L(X_{\tau_L}) = \Pi(X_{\tau_L}) - \frac{\Pi(y_F)}{2} \left(\frac{X_{\tau_L}}{y_F} \right)^\alpha, \quad X_{\tau_L} < y_F, \quad (\text{IA.10})$$

$$L(X_{\tau_L}) = \frac{\Pi(X_{\tau_L})}{2}, \quad X_{\tau_L} \geq y_F. \quad (\text{IA.11})$$

The equation $L(x) - K_L = F(x)$ has a unique root $\hat{x}$ in the $(0, y_F)$ domain and

$$L(x) - K_L = F(x), \quad x \geq y_F, \quad (\text{IA.12})$$

$$L(x) - K_L > F(x), \quad x \in (\hat{x}, y_F), \quad (\text{IA.13})$$

$$L(x) - K_L < F(x), \quad x < \hat{x}. \quad (\text{IA.14})$$

We summarize the symmetric equilibrium strategy in the following theorem.

Theorem A.1 *There exists a symmetric Markov perfect equilibrium with the following properties:*

1. *In the $x \geq y_F$ region where y_F is given in (IA.6), two firms simultaneously enter and $V_a(x) = V_b(x) = \Pi(x)/2 - K_L$.*
2. *In the $x \in [\hat{x}, y_F)$ region, firms compete to become Leader with one firm being randomly selected as Leader and $V_a(x) = V_b(x) = (L(x) - K_L + F(x))/2$.*
3. *In the $x < \hat{x}$ region, firms wait or play mixed entry strategies. Pre-entry firm values are equal: $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is the unique solution to the variational inequality (35) in the $x < \hat{x}$ domain subject to the boundary conditions:*

$$V_*(0) = 0 \quad \text{and} \quad V_*(\hat{x}) = F(\hat{x}). \quad (\text{IA.15})$$

The equilibrium entry rates are $\lambda_a^(x) = \lambda_b^*(x) = \lambda^*(x)$, where $\lambda^*(x) > 0$ is given by (34) in the probabilistic entry region: $\mathcal{R}^E := \{x < \hat{x} : V_*(x) = L(x) - K_L\}$, and $\lambda^*(x) = 0$ for all x in the $x < \hat{x}$ domain but not in $\mathcal{R}^E$, i.e., $x \in (0, \hat{x}) \setminus \mathcal{R}^E$.*

In Figure I-1, we plot the value functions in Panel A using the solid line and the equilibrium entry rate $\lambda^*(x)$ in Panel B. The equilibrium solution for this case features four regions. To the left of $\underline{x}$ is the waiting region where $\lambda^*(x) = 0$ and $V_a(x) = V_b(x) = (x/\underline{x})^\beta (L(\underline{x}) - K_L)$.¹ In the mixed entry region where $x \in [\underline{x}, \hat{x})$, firms enter probabilistically and firm values are equal: $V_a(x) = V_b(x) = L(x) - K_L$. Using (34) and (IA.10), we can pin down the equilibrium entry rate $\lambda^*(x) = \frac{x - \frac{(\alpha-1)\zeta}{2}\Pi(y_F)\left(\frac{x}{y_F}\right)^\alpha - rK_L}{F(x) - (L(x) - K_L)}$ in this region. In the region where $x \in [\hat{x}, y_F)$, firms compete to become Leader with one being randomly selected as Leader and $V_a(x) = V_b(x) = (L(x) - K_L + F(x))/2$. Finally, to the right of y_F is the simultaneous entry region where both firms enter instantly as the market demand is high even if two firms split.

¹The threshold $\underline{x}$ is determined by the smooth-pasting condition: $V_i'(\underline{x}) = L'(\underline{x})$.

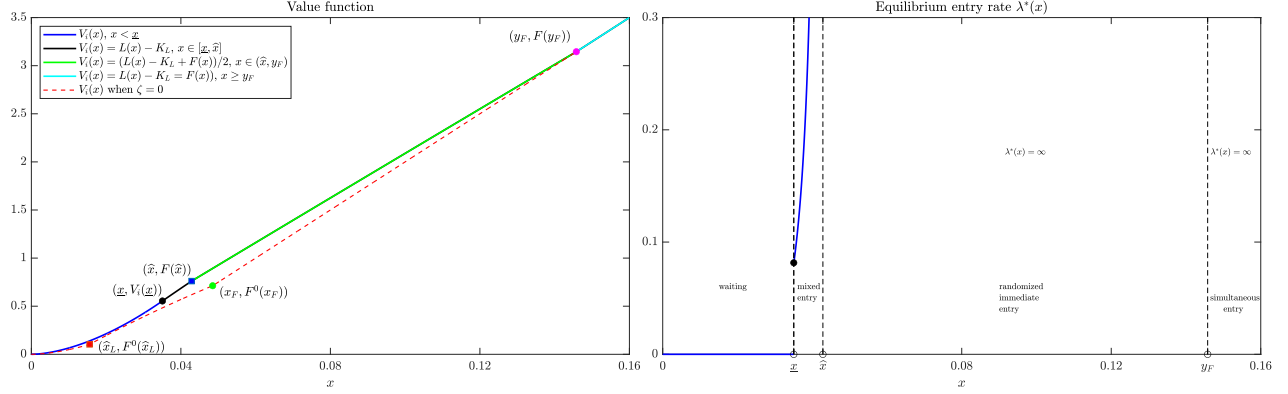


Figure I-1: VALUE FUNCTIONS $V_a(x) = V_b(x) = V_*(x)$ AND EQUILIBRIUM PROBABILISTIC ENTRY RATES $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$ WITH A TIME-VARYING SECOND-MOVER ADVANTAGE. The threshold values dividing the total market demand x into the four regions are: $\underline{x}$, $\hat{x}$, and y_F . The four regions are: 1.) the $x < \underline{x}$ waiting region, 2.) the $x \in [\underline{x}, \hat{x}]$ probabilistic entry (mixed-strategy) region, 3.) the $x \in [\hat{x}, y_F]$ randomized immediate entry (with Follower voluntarily waiting) and 4.) the $x \geq y_F$ simultaneous entry region. Parameter values are $\zeta = 0.1$, $K_L = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$. The equilibrium thresholds are: $\underline{x} = 0.0351$, $\hat{x} = 0.0429$, and $y_F = 0.1458$. Additionally, we have $\hat{x}_L = 0.0156$ and $x_F = 0.0485$.

As a comparison, we also plot the equilibrium value when $\zeta = 0$ using the red dash line in Panel A of Figure I-1. $F^0(x)$ denotes the Follower's value when $\zeta = 0$ and is given by (13)-(14). The $\zeta = 0$ result corresponds to **Case C** with $R = 1$ in the main body of the paper (see Theorem 2 for its solution characterization).

A key takeaway is that the equilibrium value $V_i(x)$ is higher when $\zeta > 0$ than when $\zeta = 0$. This is because Follower's stronger incentive to wait when $\zeta > 0$ ($y_F > x_F$) allows Leader to earn more monopoly rents (as the waiting region ($x \leq y_F$) when $\zeta > 0$ is wider than the waiting region ($x \leq x_F$) when $\zeta = 0$). In sum, introducing a time-varying second-mover advantage into the setting where $R = 1$, analyzed in Chapter 9 of Dixit and Pindyck (1994), makes not only Follower but also Leader better off.

I.B Additional Proofs

In this Internet Appendix, we provide proofs for **Case B** and **Case C** of Proposition 1, Theorem 3, and all the lemmas in Appendices D-E.

Proof of Lemma 1: Denote $\psi(y) := y - \frac{\beta+1}{\beta-1} \left(\frac{\beta-1}{2\beta} \right)^\beta y^\beta$ where $y > 0$. We can verify that $L(x) - F(x) = \psi\left(\frac{x}{(r-\mu)K_F}\right) K_F$ for $x \leq x_F$ and $L(x) - F(x) = K_F = \psi\left(\frac{2\beta}{\beta-1}\right) K_F$ for $x \geq x_F$.

Therefore, we can write $L(x) - K_L - F(x)$ as:

$$L(x) - K_L - F(x) = K_F \left[\psi \left(\frac{x}{(r - \mu)K_F} \right) - R \right], \quad x \leq x_F, \quad (\text{IB.1})$$

$$L(x) - K_L - F(x) = K_F \left[\psi \left(\frac{2\beta}{\beta - 1} \right) - R \right], \quad x \geq x_F, \quad (\text{IB.2})$$

where $R = K_L/K_F$.

We can verify that $\psi(y)$ is increasing ($\psi'(y) > 0$) for $y \in (0, y^*)$ and decreasing ($\psi'(y) < 0$) for $y > y^*$, where

$$y^* = \frac{2\beta}{\beta - 1} \left(\frac{2}{\beta + 1} \right)^{1/(\beta - 1)}. \quad (\text{IB.3})$$

We can show $y^* < \frac{2\beta}{\beta - 1}$, $\psi(0) = 0$, $\psi(y^*) = R_{AB}$, and $\psi(\frac{2\beta}{\beta - 1}) = 1$. Using these properties, we can show that for any fixed $R \in (0, R_{AB})$, the $\psi(y) = R$ equation has two positive roots: $\hat{y}_L$ and $\hat{y}_F$, satisfying $0 < \hat{y}_L < y^* < \hat{y}_F$ and the $\psi(y) = R_{AB}$ equation has one positive root: $y = y^* > 0$, where y^* is given in (IB.3). Therefore, (D.4) and (D.5) are well defined, where $\bar{\eta}(R) = \hat{y}_F$, $\underline{\eta}(R) = \hat{y}_L$, $\bar{\eta}(R)$ is decreasing in $R \in (0, R_{AB}]$, and $\underline{\eta}(R)$ is increasing in $R \in (0, R_{AB}]$. Since $\psi(y^*) = R_{AB}$, we have $\bar{\eta}(R_{AB}) = \underline{\eta}(R_{AB}) = y^* = \frac{\beta}{\beta - 1} R_{AB}$. Together with the closed-form expressions of $\hat{x}_L$ and $\hat{x}_F$ in (D.3), we obtain $\hat{x}_F = x_M = \hat{x}_L$ for $R = R_{AB}$. Since $\psi(\frac{2\beta}{\beta - 1}) = 1$, we have $\bar{\eta}(1) = \underline{\eta}(1) = \frac{2\beta}{\beta - 1}$. Together with the closed-form expression of $\hat{x}_F$ given in (D.3), we obtain $\lim_{R \rightarrow 1+} \hat{x}_F = x_F$.

Note that $\psi(y)$ is increasing in $y \in [0, y^*]$ and decreasing in $y \geq y^*$, and for any $R \in (1, R_{AB})$, we have $\psi(\bar{\eta}(R)) = \psi(\underline{\eta}(R)) = R$, $0 < \underline{\eta}(R) < y^* < \bar{\eta}(R) \leq \frac{2\beta}{\beta - 1}$. We then infer

$$\psi(y) < R, \quad y \in (0, \underline{\eta}(R)) \cup (\bar{\eta}(R), +\infty), \quad (\text{IB.4})$$

$$\psi(y) > R, \quad y \in (\underline{\eta}(R), \bar{\eta}(R)). \quad (\text{IB.5})$$

Using (IB.1)-(IB.2), (IB.4)-(IB.5), and (D.3), for any $R \in (1, R_{AB})$ we obtain

$$\begin{aligned} L(x) - K_L &< F(x), & x < \hat{x}_L, \\ L(x) - K_L &= F(x), & x = \hat{x}_L, \\ L(x) - K_L &> F(x), & \hat{x}_L < x < \hat{x}_F, \\ L(x) - K_L &= F(x), & x = \hat{x}_F, \\ L(x) - K_L &< F(x), & x > \hat{x}_F. \end{aligned} \quad (\text{IB.6})$$

When $R \in (1, R_{AB})$, we have

$$x_M = \frac{\beta}{\beta-1}(r-\mu)K_L < R_{AB}\frac{\beta}{\beta-1}(r-\mu)K_F = y^*(r-\mu)K_F < \bar{\eta}(R)(r-\mu)K_F = \hat{x}_F.$$

Note that $\bar{\eta}(R)$ is decreasing in $R \in [1, R_{AB}]$ and $\bar{\eta}(1) = \frac{2\beta}{\beta-1}$. Then, when $R \in (1, R_{AB})$, we obtain $\bar{\eta}(R) < \frac{2\beta}{\beta-1}$, which in turn implies $x_F > \hat{x}_F$ using (D.3).

Next, we consider the $R \in (0, 1)$ case.² Note that $\psi(y)$ is increasing in $y \in [0, y^*]$ and decreasing in $y \geq y^*$. Also for any $R \in (0, 1)$, we have $\psi(\bar{\eta}(R)) = \psi(\underline{\eta}(R)) = R$ and $0 < \underline{\eta}(R) < y^* < \frac{2\beta}{\beta-1} < \bar{\eta}(R)$. We then obtain that $\psi(y) < R$ for any $y \in (0, \underline{\eta}(R))$ and $f(y) > R$ for any $y \in (\underline{\eta}(R), \frac{2\beta}{\beta-1}]$. Using these results and (IB.1)-(IB.2), for $R \in (0, 1)$ we obtain

$$\begin{aligned} L(x) - K_L &< F(x), & x < \hat{x}_L, \\ L(x) - K_L &= F(x), & x = \hat{x}_L, \\ L(x) - K_L &> F(x), & x > \hat{x}_L. \end{aligned} \tag{IB.7}$$

Next, we show that $\hat{x}_L < x_M$ for any $R \in (0, R_{AB})$. For any $R \in (0, R_{AB})$,

$$\psi\left(\frac{\beta}{\beta-1}R\right) = R \left[\frac{\beta}{\beta-1} - \frac{\beta+1}{\beta-1} \frac{1}{2^\beta} R^{\beta-1} \right] > R \left[\frac{\beta}{\beta-1} - \frac{\beta+1}{\beta-1} \frac{1}{2^\beta} R_{AB}^{\beta-1} \right] = R,$$

where the last equality follows from $R_{AB} = \left(\frac{2\beta}{\beta+1}\right)^{\frac{1}{\beta-1}}$. Since $\psi(y)$ is increasing in $y \in [0, y^*]$ and decreasing in $y \geq y^*$ and $\psi(\underline{\eta}(R)) = \psi(\bar{\eta}(R)) = R$, we have $\underline{\eta}(R) < \frac{\beta}{\beta-1}R$. We then derive $\hat{x}_L < x_M$ from (D.3). $\square$

Proof of Proposition 1: The results in [Case B](#) in Proposition 1 are the same as those given in (IB.6) and the results in [Case C](#) in Proposition 1 are the same as those given in (IB.7). We have completed our proofs for [Case B](#) and [Case C](#) of Proposition 1. $\square$

Proof of Lemma 2: We first consider the case where $R = R_{A_1 A_2}$. We can verify that $V_*(x)$ given in (42)-(43) satisfies:

$$V_*(2x_M) = \frac{\Pi(2x_M)}{2} - K_L, \quad V'_*(2x_M) = \frac{\Pi'(2x_M)}{2}, \tag{IB.8}$$

²The $R = 1$ case can be analyzed in the same way.

and

$$\frac{L(x_M) - K_L}{x_M^\beta} = \frac{L(2x_M) - K_L}{(2x_M)^\beta}. \quad (\text{IB.9})$$

Therefore for any $x \leq 2x_M$, we have

$$V_*(x) = \left(\frac{x}{2x_M}\right)^\beta (L(2x_M) - K_L) = \left(\frac{x}{x_M}\right)^\beta (L(x_M) - K_L), \quad (\text{IB.10})$$

where the second equality follows from (IB.9). We can then verify the following value-matching and smooth-pasting conditions:

$$V_*(x_M) = L(x_M) - K_L \quad \text{and} \quad V'_*(x_M) = L'(x_M). \quad (\text{IB.11})$$

Using (IB.9) and formulas for $\theta_1(a, b)$ and $\theta_2(a, b)$ given in (D.2), we obtain:

$$\theta_1(x_M, 2x_M) = \left(\frac{1}{x_M}\right)^\beta (L(x_M) - K_L)$$

and $\theta_2(x_M, 2x_M) = 0$, which together with (IB.10) imply $\Theta(x; x_M, 2x_M) = V_*(x)$ for any $x \in [x_M, 2x_M]$. Equations (IB.8) and (IB.11) imply $\underline{x} = x_M$ and $\bar{x} = 2x_M$ in (D.6)-(D.7).

Next, we turn to the case where $R \in [1, R_{A_1 A_2})$. First, we prove the existence of the pair $(\underline{x}, \bar{x})$ satisfying (D.6)-(D.7).

Proof of the existence of the pair $(\underline{x}, \bar{x})$ for the $R \in [1, R_{A_1 A_2})$ case. First, we can verify that for any $y > 0$, $\Gamma(x, y)$ defined in (E.22) has the following properties:

$$\Gamma(y, y) = \frac{\Pi(y)}{2} - K_L \quad \text{and} \quad \Gamma_x(y, y) = \frac{\Pi'(y)}{2}. \quad (\text{IB.12})$$

Let $\widehat{L}(x) := \Pi(x) - \frac{\Pi(x_F)}{2} \left(\frac{x}{x_F}\right)^\beta - K_L$ where $x > 0$. Next, we show that there exists a pair $(\underline{x}, \bar{x})$ satisfying $\underline{x} > x_M$, $\bar{x} > 2x_M$,

$$\Gamma(\underline{x}, \bar{x}) = \widehat{L}(\underline{x}) \quad \text{and} \quad \Gamma_x(\underline{x}, \bar{x}) = \widehat{L}'(\underline{x}). \quad (\text{IB.13})$$

Let h_1 and h_2 denote the following two constants:

$$h_1 := \frac{1}{\bar{x}^\beta} \frac{\frac{1}{2}(1 - \gamma)\Pi(\bar{x}) + \gamma K_L}{\beta - \gamma}, \quad (\text{IB.14})$$

$$h_2 := \frac{1}{\bar{x}^\gamma} \frac{\frac{1}{2}(\beta - 1)\Pi(\bar{x}) - \beta K_L}{\beta - \gamma}. \quad (\text{IB.15})$$

Then we can show that (IB.13) are equivalent to

$$h_1 \underline{x}^\beta + h_2 \underline{x}^\gamma = \Pi(\underline{x}) - \frac{\Pi(x_F)}{2} \left(\frac{\underline{x}}{x_F} \right)^\beta - K_L, \quad (\text{IB.16})$$

$$h_1 \beta \underline{x}^\beta + h_2 \gamma \underline{x}^\gamma = \Pi(\underline{x}) - \beta \frac{\Pi(x_F)}{2} \left(\frac{\underline{x}}{x_F} \right)^\beta. \quad (\text{IB.17})$$

Equations (IB.16)-(IB.17) hold if and only if h_1 and h_2 are respectively given by

$$h_1 = \frac{1}{\underline{x}^\beta} \left[\frac{(1-\gamma)\Pi(\underline{x}) + \gamma K_L}{\beta - \gamma} - \frac{\Pi(x_F)}{2} \left(\frac{\underline{x}}{x_F} \right)^\beta \right], \quad (\text{IB.18})$$

$$h_2 = \frac{1}{\underline{x}^\gamma} \frac{(\beta - 1)\Pi(\underline{x}) - \beta K_L}{\beta - \gamma}. \quad (\text{IB.19})$$

Using (IB.14) and (IB.15), we can see that (IB.18) and (IB.19) are equivalent to

$$\frac{\frac{1}{2}(1-\gamma)\Pi(\bar{x}) + \gamma K_L}{\beta - \gamma} = \left(\frac{\bar{x}}{\underline{x}} \right)^\beta \left[\frac{(1-\gamma)\Pi(\underline{x}) + \gamma K_L}{\beta - \gamma} - \frac{\Pi(x_F)}{2} \left(\frac{\underline{x}}{x_F} \right)^\beta \right], \quad (\text{IB.20})$$

$$\left[\frac{1}{2}(\beta - 1)\Pi(\bar{x}) - \beta K_L \right] = \left(\frac{\bar{x}}{\underline{x}} \right)^\gamma [(\beta - 1)\Pi(\underline{x}) - \beta K_L], \quad (\text{IB.21})$$

respectively. We introduce the following change of variables:

$$U := \frac{\Pi(\bar{x})(\beta - 1)}{2\beta K_L} - 1, \quad (\text{IB.22})$$

$$u := \frac{\Pi(\underline{x})(\beta - 1)}{\beta K_L} - 1. \quad (\text{IB.23})$$

Then, we can show that $\frac{1}{2}(\beta - 1)\Pi(\bar{x}) = \beta K_L(1 + U)$, $(\beta - 1)\Pi(\underline{x}) = \beta K_L(1 + u)$, $\bar{x}/\underline{x} = 2^{\frac{1+U}{1+u}}$, and $\underline{x}/x_F = \Pi(\underline{x})/\Pi(x_F) = (1+u)R/2$. Therefore, (IB.21) is equivalent to (D.9) and (IB.20) is equivalent to (D.10). Using these calculations, we conclude that a pair $(\underline{x}, \bar{x})$ satisfies $\underline{x} > x_M$, $\bar{x} > 2x_M$ and (IB.13) if and only if (D.8)-(D.10) hold for some $u > 0$ and $U > 0$.

Next, we show that there exist a pair $(u > 0, U > 0)$ such that (D.9)-(D.10) hold. For the $H(\cdot)$ function defined in (D.11), $H(0) = 1/(\beta - 1) > 0$, $\lim_{z \rightarrow \infty} H(z) = 0$, and its derivative is

$$H'(z) = -\frac{\beta[(1-\gamma)(1+z) + \gamma]}{(\beta - \gamma)(1+z)^{\beta+1}} \leq -\frac{\beta}{(\beta - \gamma)(1+z)^{\beta+1}} < 0, \quad z \geq 0. \quad (\text{IB.24})$$

Since $1 \leq R < \left(\frac{2^\beta - 1}{\beta} \right)^{\frac{1}{\beta-1}} = R_{A_1 A_2}$, we have $2^\beta H(0) - \frac{\beta}{\beta-1} R^{\beta-1} > \frac{2^\beta}{\beta-1} - \frac{\beta}{\beta-1} \frac{2^\beta - 1}{\beta} = \frac{1}{\beta-1}$. Then, the continuity and monotonicity of $H(z)$ imply that there exist a pair (u_1, u_2) satisfying

$0 < u_1 < u_2 < +\infty$, such that

$$2^\beta H(u_1) - \frac{\beta}{\beta-1} R^{\beta-1} = \frac{1}{\beta-1} \quad \text{and} \quad 2^\beta H(u_2) - \frac{\beta}{\beta-1} R^{\beta-1} = 0. \quad (\text{IB.25})$$

Additionally, $2^\beta H(z) - \frac{\beta}{\beta-1} R^{\beta-1} \in (0, \frac{1}{\beta-1}]$ for any $z \in [u_1, u_2)$. Since $H(z)$ is smooth and decreases from $\frac{1}{\beta-1}$ to 0 as z increases from 0 to ∞ , we can define a smooth and increasing function $g(z)$ for $z \in [u_1, u_2)$ as follows:

$$g(z) := H^{-1} \left(2^\beta H(z) - \frac{\beta}{\beta-1} R^{\beta-1} \right), \quad z \in [u_1, u_2), \quad (\text{IB.26})$$

where $H^{-1}(\cdot)$ is the inverse function of $H(\cdot)$. We have $g(u_1) = 0$ and $\lim_{u \uparrow u_2} g(u) = \infty$.

Let $\tilde{H}(z) := z(1+z)^{-\gamma}$ for $z \geq 0$, which is smooth and increasing in $z \geq 0$ with the properties of $\tilde{H}(0) = 0$ and $\lim_{z \rightarrow \infty} \tilde{H}(z) = \infty$. Then we can define another smooth and increasing function, $\tilde{g}(z)$ for $z \geq 0$ as follows:

$$\tilde{g}(z) := \tilde{H}^{-1} \left(2^\gamma \tilde{H}(z) \right), \quad z \geq 0, \quad (\text{IB.27})$$

where $\tilde{H}^{-1}(\cdot)$ is the inverse function of $\tilde{H}(\cdot)$. Note that $\tilde{g}(0) = 0$, $\tilde{g}(z) < \tilde{H}^{-1}(\tilde{H}(z)) = z$ for any $z > 0$ as $\gamma < 0$, $\tilde{g}(u_1) - g(u_1) = \tilde{g}(u_1) > \tilde{g}(0) = 0$, and $\tilde{g}(u_2 - \delta) - g(u_2 - \delta) \leq u_2 - g_1(u_2 - \delta) < 0$ for sufficiently small $\delta > 0$ as $\lim_{z \uparrow u_2} g(z) = \infty$. Because $g(z)$ and $\tilde{g}(z)$ are continuous in $z \in [u_1, u_2)$, there exists a value $u \in (u_1, u_2)$ such that $\tilde{g}(u) = g(u)$. By setting $U = \tilde{g}(u) = g(u) > 0$ and using (IB.26)-(IB.27), we conclude that the pair (u, U) satisfy (D.9)-(D.10). Using (IB.22)-(IB.23), $u > 0$ and $U > 0$, we obtain $\underline{x}$ and $\bar{x}$ as given in (D.8), and they satisfy $\bar{x} > 2x_M$ and $\underline{x} > x_M$. We further obtain (IB.13), using $u > 0$, $U > 0$ and (D.8)-(D.10).

Next, we prove $\underline{x} < x_F$ by contradiction. If $\underline{x} \geq x_F$ were to hold, then

$$\Gamma(\underline{x}, \bar{x}) = \hat{L}(\underline{x}) = \Pi(\underline{x}) - \frac{\Pi(x_F)}{2} \left(\frac{\underline{x}}{x_F} \right)^\beta - K_L \leq \frac{\Pi(\underline{x})}{2} - K_L, \quad (\text{IB.28})$$

where the first equality follows from (IB.13) and the inequality follows from $\underline{x} \geq x_F$. For any $y \geq 2x_M$, we can show $\frac{1}{2}(\beta-1)\Pi(y) - \beta K_L \geq 0$ and

$$\frac{1}{2}(1-\gamma)\Pi(y) + \gamma K_L \geq K_L \left[\frac{\beta}{\beta-1}(1-\gamma) + \gamma \right] = K_L \frac{\beta-\gamma}{\beta-1} > 0.$$

Therefore, for any $y \geq 2x_M > x > 0$, we have

$$\begin{aligned} & \Gamma_{xx}(x, y) \\ &= \frac{\frac{1}{2}(1-\gamma)\Pi(y) + \gamma K_L}{(\beta-\gamma)x^2} \beta(\beta-1) \left(\frac{x}{y}\right)^\beta + \frac{\frac{1}{2}(\beta-1)\Pi(y) - \beta K_L}{(\beta-\gamma)x^2} \gamma(\gamma-1) \left(\frac{x}{y}\right)^\gamma > 0. \end{aligned} \quad (\text{IB.29})$$

Using (IB.29) and evaluating (IB.12) at $y = \bar{x}$, we obtain:

$$\Gamma(x, \bar{x}) > \frac{\Pi(x)}{2} - K_L, \quad x \in (0, \bar{x}) \cup (\bar{x}, +\infty). \quad (\text{IB.30})$$

Because the inequality in (IB.28) is strict if $\underline{x} > x_F$, we obtain $\underline{x} = \bar{x} = x_F$ using (IB.30), which contradicts $\bar{x} > 2x_M \geq x_F$. Therefore, $\underline{x} < x_F$ has to hold.

Finally, we prove that (D.6)-(D.7) hold for the pair $(\underline{x}, \bar{x})$ that we have just obtained. Using (IB.13), $\underline{x} < x_F$, and $L(x) = \hat{L}(x)$ in the $x < x_F$ region, we obtain:

$$\Gamma(\underline{x}, \bar{x}) = L(\underline{x}) - K_L \quad \text{and} \quad \Gamma_x(\underline{x}, \bar{x}) = L'(\underline{x}). \quad (\text{IB.31})$$

Using (IB.14)-(IB.15), we obtain $\Gamma(x, \bar{x}) = h_1 x^\beta + h_2 x^\gamma$. Using (IB.31) and $\underline{x} < x_F \leq 2x_M < \bar{x}$, we obtain

$$h_1 \underline{x}^\beta + h_2 \underline{x}^\gamma = L(\underline{x}) - K_L, \quad (\text{IB.32})$$

$$h_1 \bar{x}^\beta + h_2 \bar{x}^\gamma = L(\bar{x}) - K_L. \quad (\text{IB.33})$$

Using (IB.32)-(IB.33), $\theta_1(\underline{x}, \bar{x})\underline{x}^\beta + \theta_2(\underline{x}, \bar{x})\underline{x}^\gamma = L(\underline{x}) - K_L$, and $\theta_1(\underline{x}, \bar{x})\bar{x}^\beta + \theta_2(\underline{x}, \bar{x})\bar{x}^\gamma = L(\bar{x}) - K_L$, we obtain $h_1 = \theta_1(\underline{x}, \bar{x})$, $h_2 = \theta_2(\underline{x}, \bar{x})$, and

$$\Gamma(x, \bar{x}) = \Theta(x; \underline{x}, \bar{x}), \quad x \in [\underline{x}, \bar{x}]. \quad (\text{IB.34})$$

Using (IB.12) and (IB.31), we conclude that (D.6) and (D.7) hold.

Proof of the uniqueness of the pair $(\underline{x}, \bar{x})$ for the $R \in [1, R_{A_1 A_2})$ case. Using $h_1 = \theta_1(\underline{x}, \bar{x})$, $h_2 = \theta_2(\underline{x}, \bar{x})$, (IB.14)-(IB.15) and (IB.18)-(IB.19), we obtain:

$$\theta_1(\underline{x}, \bar{x}) = \frac{1}{\bar{x}^\beta} \frac{\frac{1}{2}(1-\gamma)\Pi(\bar{x}) + \gamma K_L}{\beta-\gamma} = \frac{1}{\underline{x}^\beta} \left[\frac{(1-\gamma)\Pi(\underline{x}) + \gamma K_L}{\beta-\gamma} - \frac{\Pi(x_F)}{2} \left(\frac{\underline{x}}{x_F}\right)^\beta \right], \quad (\text{IB.35})$$

$$\theta_2(\underline{x}, \bar{x}) = \frac{1}{\bar{x}^\gamma} \frac{\frac{1}{2}(\beta-1)\Pi(\bar{x}) - \beta K_L}{\beta-\gamma} = \frac{1}{\underline{x}^\gamma} \frac{(\beta-1)\Pi(\underline{x}) - \beta K_L}{\beta-\gamma}. \quad (\text{IB.36})$$

Let (a, b) denote a pair in the domain $(x_M, x_F) \times (2x_M, \infty)$ satisfying (D.6) and (D.7):

$$\Theta(a; a, b) = L(a) - K_L, \quad \Theta_x(a; a, b) = L'(a), \quad (\text{IB.37})$$

$$\Theta(b; a, b) = \frac{\Pi(b)}{2} - K_L, \quad \Theta_x(b; a, b) = \frac{\Pi'(b)}{2}. \quad (\text{IB.38})$$

We show that (a, b) has to be the same as the $(\underline{x}, \bar{x})$ that we have just characterized.

Using the function $\Gamma(\cdot, \cdot)$ given by (E.22), we obtain

$$\Gamma(b, b) = \frac{\Pi(b)}{2} - K_L \quad \text{and} \quad \Gamma_x(b, b) = \frac{\Pi'(b)}{2}. \quad (\text{IB.39})$$

Using $\Theta(x; a, b) = \theta_1(a, b)x^\beta + \theta_2(a, b)x^\gamma$, (IB.38)-(IB.39), and $\Gamma(x, b) = h_1x^\beta + h_2x^\gamma$, where $h_1 = \frac{1}{b^\beta} \frac{\frac{1}{2}(1-\gamma)\Pi(b) + \gamma K_L}{\beta - \gamma}$ and $h_2 = \frac{1}{b^\gamma} \frac{\frac{1}{2}(\beta-1)\Pi(b) - \beta K_L}{\beta - \gamma}$, we obtain $h_1 = \theta_1(a, b)$ and $h_2 = \theta_2(a, b)$ and thus $\Theta(x; a, b) = \Gamma(x, b)$. Then, using (IB.37), we obtain

$$\Gamma(a, b) = L(a) - K_L \quad \text{and} \quad \Gamma_x(a, b) = L'(a). \quad (\text{IB.40})$$

Using the function $\Gamma(\cdot, \cdot)$ given by (E.22), we obtain

$$\Gamma_y(x, y) = \frac{(1 - \beta)(1 - \gamma)\frac{1}{2}\Pi(y) - \beta\gamma K_L}{(\beta - \gamma)y} \left[\left(\frac{x}{y}\right)^\beta - \left(\frac{x}{y}\right)^\gamma \right]. \quad (\text{IB.41})$$

For any $y \geq 2x_M$, we can show

$$(1 - \beta)(1 - \gamma)\frac{1}{2}\Pi(y) - \beta\gamma K_L \leq (1 - \beta)(1 - \gamma)\Pi(x_M) - \beta\gamma K_L = -\beta K_L < 0$$

and then derive from (IB.41) that

$$\Gamma_y(x, y) > 0, \quad 0 < x < y, \quad y \geq 2x_M. \quad (\text{IB.42})$$

Using (IB.39)-(IB.40) and $\Gamma(x, b) = \theta_1(a, b)x^\beta + \theta_2(a, b)x^\gamma$, we obtain

$$\theta_2(a, b) = \frac{1}{b^\gamma} \frac{\frac{1}{2}(\beta - 1)\Pi(b) - \beta K_L}{\beta - \gamma} = \frac{1}{a^\gamma} \frac{(\beta - 1)\Pi(a) - \beta K_L}{\beta - \gamma}. \quad (\text{IB.43})$$

Since $\left[\frac{1}{2}(\beta - 1)\Pi(x) - \beta K_L\right]x^{-\gamma}$ is increasing in $x \geq 2x_M$, using (IB.36) and (IB.43), we conclude $b = \bar{x}$ if $a = \underline{x}$.

Therefore, we only need to prove that $a = \underline{x}$ holds. Again, we proceed with a proof by contradiction. Suppose $a \neq \underline{x}$.

Using (IB.29), (IB.31), and the concavity of $L(x)$ in the $x \in (0, x_F)$ region ($L''(x) < 0$), we

can show that $\Gamma(x, \bar{x}) > L(x) - K_L$ holds for all $x \in (0, \underline{x}) \cup (\underline{x}, x_F]$. Since $a \neq \underline{x}$ and $a < x_F$, the following inequality has to hold:

$$\Gamma(a, \bar{x}) > L(a) - K_L = \Gamma(a, b). \quad (\text{IB.44})$$

Using $\bar{x} > 2x_M$, $b > 2x_M$, $a < x_F \leq 2x_M$, (IB.42), and (IB.44), we can see that $\bar{x} > b$.

Equations (IB.36) and (IB.43) imply that

$$\begin{aligned} [(\beta - 1)\Pi(a) - \beta K_L] a^{-\gamma} &= \left[\frac{1}{2}(\beta - 1)\Pi(b) - \beta K_L \right] b^{-\gamma} \\ &< \left[\frac{1}{2}(\beta - 1)\Pi(\bar{x}) - \beta K_L \right] \bar{x}^{-\gamma} = [(\beta - 1)\Pi(\underline{x}) - \beta K_L] \underline{x}^{-\gamma}, \end{aligned} \quad (\text{IB.45})$$

where the inequality follows from $\bar{x} > b \geq 2x_M$ and the property that $\left[\frac{1}{2}(\beta - 1)\Pi(x) - \beta K_L \right] x^{-\gamma}$ increases with x in the $x \geq 2x_M$ region. Using $a > x_M$, $\underline{x} > x_M$, (IB.45), and the property that $[(\beta - 1)\Pi(x) - \beta K_L] x^{-\gamma}$ increases with x in the $x \geq x_M$ region, we obtain $a < \underline{x}$.

Therefore, we have

$$\Gamma(\underline{x}, b) < \Gamma(\underline{x}, \bar{x}) = L(\underline{x}) - K_L \leq \Gamma(\underline{x}, b), \quad (\text{IB.46})$$

where the first inequality follows from (IB.42) and $\underline{x} < x_F \leq 2x_M < b < \bar{x}$ and the second inequality follows from $a < \underline{x} < x_F$, $L''(x) < 0$ in the $x \in (0, x_F)$ region, (IB.29), and (IB.40). We thus obtain a contradiction and complete our proof for the uniqueness of the pair $(\underline{x}, \bar{x})$ in the domain $(x_M, x_F) \times (2x_M, \infty)$ satisfying (D.6)-(D.7).

The uniqueness of the pair $(\underline{x}, \bar{x})$ implies that the system of equations (D.9)-(D.10) admits a unique solution pair (u, U) in the domain $(0, 2/R - 1) \times (0, \infty)$. As we have shown, if $u \in [u_1, u_2]$ is a root of $g(u) - \tilde{g}(u) = 0$, then the pair $(u, U) = (u, g(u))$ is one solution to the system of equations (D.9)-(D.10). Because the solution of the system (D.9)-(D.10) is unique in the domain $(0, 2/R - 1) \times (0, \infty)$, the equation $g(u) - \tilde{g}(u) = 0$ must have a unique root in the domain $[u_1, u_2] \cap (0, 2/R - 1)$. Since $g(\cdot)$ is smooth in R and $\tilde{g}(\cdot)$ is independent of R , u being the unique root of $g(u) - \tilde{g}(u) = 0$ is continuously differentiable in R . Therefore, using (D.8), we have shown that both $\underline{x}$ and $\bar{x}$ are continuously differentiable in $R \in [1, R_{A_1 A_2})$.

□

Proof of Lemma 3 : We first prove this lemma for **Subcase A₁**.

Subcase A₁: $R > R_{A_1 A_2}$. For any $x > 0$, let

$$\psi(x, R) := \left(\frac{x}{2x_M(R)} \right)^\beta \left[\frac{\Pi(2x_M(R))}{2} - K_F R \right] - (L(x) - K_F R), \quad (\text{IB.47})$$

where $x_M(R) = \frac{\beta}{\beta-1}(r - \mu)K_F R$.

First, we can show that $\psi_{xx}(x, R) > 0$ for any $x \in (0, x_F) \cup (x_F, \infty)$ by using $\frac{\Pi(2x_M(R))}{2} - K_F R = K_F R/(\beta - 1) > 0$, $L''(x) < 0$ for $x < x_F$ and $L(x) = \Pi(x)/2$ for $x > x_F$. Using $2x_M(R) > x_F$, $\psi(2x_M(R), R) = 0$, $\psi_x(x, R)|_{x=2x_M(R)} = 0$, and $\psi_{xx}(x, R) > 0 \forall x > x_F$, we obtain

$$\psi(x, R) > 0, \quad \text{for } x \in [x_F, 2x_M(R)), \quad R > 1. \quad (\text{IB.48})$$

Let $\hat{x}_M := x_M(R_{A_1 A_2}) = \frac{\beta}{\beta-1}(r - \mu)K_F R_{A_1 A_2}$. Using (IB.9), we obtain

$$\psi(x, R_{A_1 A_2}) = \left(\frac{x}{\hat{x}_M} \right)^\beta [L(\hat{x}_M) - K_{A_1 A_2}] - (L(x) - K_{A_1 A_2}), \quad (\text{IB.49})$$

where $K_{A_1 A_2} = K_F R_{A_1 A_2}$. Using $\psi(\hat{x}_M, R_{A_1 A_2}) = 0$, $\psi_x(\hat{x}_M, R_{A_1 A_2}) = 0$ and $\psi_{xx}(x, R_{A_1 A_2}) > 0 \forall x \in (0, x_F)$, we obtain

$$\psi(x, R_{A_1 A_2}) \geq 0, \quad x \in (0, x_F), \quad (\text{IB.50})$$

where the equality holds if and only if $x = \hat{x}_M$. Using $\psi_R(x, R) = K_F \left[1 - \left(\frac{x}{2x_M(R)} \right)^\beta \right] > 0$ for any $x < 2x_M(R)$ and using $x_F < 2x_M(R)$ for $R > R_{A_1 A_2} > 1$, we obtain

$$\psi(x, R) > \psi(x, R_{A_1 A_2}) \geq 0, \quad x \in (0, x_F], \quad R > R_{A_1 A_2}. \quad (\text{IB.51})$$

Using $V_*(x)$ given in (42)-(43) where $\bar{x} = 2x_M(R)$, (IB.51), and (IB.48), we obtain

$$V_*(x) - (L(x) - K_L) = \psi(x, R) > 0, \quad x < \bar{x}. \quad (\text{IB.52})$$

Finally, we conclude our proof that $V_*(x)$ given in (42)-(43) solves problem (35)-(37) in the $x \geq 0$ region by using (IB.52) and the following two properties: 1.) for any $x < \bar{x}$, $\mathcal{A}V_*(x) = 0$, where the generator $\mathcal{A}$ is given in (E.1) and 2.) for any $x \geq \bar{x}$, $V_*(x) = \Pi(x)/2 - K_L = L(x) - K_L$ and

$$\mathcal{A}(\Pi(x)/2 - K_L) = rK_L - \frac{x}{2} \leq rK_L - \frac{\bar{x}}{2} = rK_L \left(1 - \frac{\beta}{\beta-1} \frac{r-\mu}{r} \right) < 0, \quad (\text{IB.53})$$

where the last inequality follows from $\frac{\beta}{\beta-1} \frac{r-\mu}{r} > 1$.³

Subcase A₂: $R_{AB} < R \leq R_{A_1A_2}$. Below we provide a proof for the case where $R = K_L/K_F$ satisfies $R_{AB} < R < R_{A_1A_2}$.⁴ First, we can show that $\mathcal{A}V_*(x) = \mathcal{A}\Theta(x; \underline{x}, \bar{x}) = 0$ holds in the $x \in (\underline{x}, \bar{x})$ region, where $\mathcal{A}$ is given in (E.1).

Second, using (IB.31), (IB.29), and $L''(x) < 0$, we can show that $\Gamma(x, \bar{x}) > L(x) - K_L$ for any $x \in (\underline{x}, x_F]$. Combining this result with $\bar{x} > x_F$, (IB.30), and $\Pi(x)/2 - K_L = L(x) - K_L$ for any $x \geq x_F$, we obtain

$$V_*(x) = \Theta(x; \underline{x}, \bar{x}) = \Gamma(x, \bar{x}) > L(x) - K_L, \quad x \in (\underline{x}, \bar{x}), \quad (\text{IB.54})$$

where the second equality follows from (IB.34).

Third, for any $x \geq \bar{x}$, $V_*(x) = \Pi(x)/2 - K_L = L(x) - K_L$ and (IB.53) hold. For any $x \in [x_M, \underline{x}]$, we have $V_*(x) = L(x) - K_L$ and

$$\mathcal{A}(L(x) - K_L) = rK_L - x \leq rK_L - x_M = rK_L \left(1 - \frac{\beta}{\beta-1} \frac{r-\mu}{r}\right) < 0. \quad (\text{IB.55})$$

Finally, for any $x \in (0, x_M)$, using $V_*(x_M) = L(x_M) - K_L$, $V'_*(x_M) = L'(x_M)$, and $V''_*(x) > 0 > L''(x)$, we obtain $V_*(x) > L(x) - K_L$. In addition, we have $\mathcal{A}V_*(x) = 0$ for any $x \in (0, x_M)$.

In sum, we have proven that $V_*(x)$ given in (E.3)-(E.6) solves (35)-(37) for $x \geq 0$. $\square$

Proof of Lemma 4: We first prove this lemma for **Subcase B₁**.

Subcase B₁: $R_{B_1B_2} < R \leq R_{AB}$. Using (D.12), we can follow essentially the same proof procedure as that for **Subcase A₂** to show that $V_*(x)$ satisfies (35) and (37) for any $x \geq \underline{x}$. For any $x \in [\hat{x}_F, \underline{x}]$, we have $V_*(x) = L(x) - K_L$ and

$$\mathcal{A}(L(x) - K_L) = rK_L - x \leq rK_L - x_M = rK_L \left(1 - \frac{\beta}{\beta-1} \frac{r-\mu}{r}\right) < 0 \quad (\text{IB.56})$$

where the first inequality follows from $\hat{x}_F > x_M$ (see Lemma 1). As $F(x) = L(x) - K_L$ for $x = \hat{x}_F$ (see Proposition 1), the boundary condition (55) holds.

³Recall that β is the positive root of the (fundamental) quadratic equation: $g(z) = 0$ where $g(z) = \frac{1}{2}\sigma^2 z(z-1) + \mu z - r$. Using $g(1) = \mu - r < 0$ and $g(z) \rightarrow \infty$ as $z \rightarrow \infty$, we obtain $\beta > 1$. The inequality $\frac{\beta}{\beta-1} \frac{r-\mu}{r} > 1$ is the same as $r > \beta\mu$. This result apparently holds when $\mu \leq 0$. When $\mu > 0$, $g(r/\mu) = \frac{1}{2}\sigma^2(r/\mu)(r/\mu - 1) > 0$ and $g(\beta) = 0$ imply $r > \beta\mu$.

⁴For the special case where $R = R_{A_1A_2}$, the proof is as follows. Using (IB.9) and Lemma 2 and simplifying (E.3)-(E.6), we obtain $\underline{x} = x_M$, $\bar{x} = 2x_M$, and $V_*(x)$ given in (42)-(43).

Subcase B₂: $1 < R \leq R_{B_1 B_2}$. First, note that

$$\begin{aligned} & \Gamma(\widehat{x}_F, 2x_M) - F(\widehat{x}_F) \\ &= \frac{K_L}{\beta - 1} \left(\frac{\widehat{x}_F}{2x_M} \right)^\beta - \frac{K_F}{\beta - 1} \left(\frac{\widehat{x}_F}{x_F} \right)^\beta = \frac{K_L}{\beta - 1} \left(\frac{\widehat{x}_F}{2x_M} \right)^\beta \left[1 - \left(\frac{K_L}{K_F} \right)^{\beta-1} \right] < 0, \end{aligned}$$

using (E.22), $\frac{(1-\gamma)\frac{\Pi(2x_M)}{2} + \gamma K_L}{\beta - \gamma} = \frac{K_L}{\beta - 1}$ and $F(\widehat{x}_F) = \frac{K_F}{\beta - 1} \left(\frac{\widehat{x}_F}{x_F} \right)^\beta$. Combining the above result with $\lim_{y \rightarrow +\infty} \Gamma(\widehat{x}_F, y) = \infty$ and (IB.42), we can show that equation (E.21) admits a unique solution in the $y > 2x_M$ region. Let $\bar{y}$ denote the unique solution of equation (E.21) in the $y > 2x_M$ region.

Note that both $\Gamma(x, \bar{y})$ and $\Theta(x; \widehat{x}_F, \bar{y})$ functions take the form of $\theta_1 x^\beta + \theta_2 x^\gamma$ in the region $x \in [\widehat{x}_F, \bar{y}]$, where θ_1 and θ_2 are constant parameters to be determined by $\Gamma(x, \bar{y}) = L(x) - K_L = \Theta(x; \widehat{x}_F, \bar{y})$ at $x = \widehat{x}_F$ and $x = \widehat{x}_F = \bar{y}$. We thus conclude that $\Gamma(x, \bar{y}) = \Theta(x; \widehat{x}_F, \bar{y})$ holds for any $x \in [\widehat{x}_F, \bar{y}]$. Using $\Gamma(\widehat{x}_F, \bar{y}) = F(\widehat{x}_F)$, $V_*(x)$ given in (E.19)-(E.20) satisfies the boundary condition (55).

Using Lemma 1, we obtain $\widehat{x}_F < x_F$ when $R \in (1, R_{B_1 B_2}]$. Recall $(\underline{x}, \bar{x})$ as given in Lemma 2. Using Lemmas 1-2, one can show that $\widehat{x}_F \geq \underline{x}$. Using (IB.29), (IB.31), and $L''(x) < 0$ for any $x \in (0, x_F)$, we obtain $\Gamma(x, \bar{x}) \geq L(x) - K_L$ for any $x \in (0, x_F)$. Combining this inequality result with $\underline{x} \leq \widehat{x}_F < x_F$, we obtain

$$\Gamma(\widehat{x}_F, \bar{x}) \geq L(\widehat{x}_F) - K_L = F(\widehat{x}_F) = \Gamma(\widehat{x}_F, \bar{y}), \quad (\text{IB.57})$$

where the first equality follows from Proposition 1 and the second equality follows from (E.21). Using $\bar{x} > 2x_M$, $\bar{y} > 2x_M$, $\widehat{x}_F \leq x_F < 2x_M$, (IB.42), and (IB.57), we obtain $\bar{x} \geq \bar{y}$. Differentiating (IB.41) with respect to x , we obtain

$$\Gamma_{xy}(x, y) = \frac{(1 - \beta)(1 - \gamma)\frac{1}{2}\Pi(y) - \beta\gamma K_L}{\beta - \gamma} \left(\beta \frac{x^{\beta-1}}{y^{\beta+1}} - \gamma \frac{x^{\gamma-1}}{y^{\gamma+1}} \right) < 0, \quad 0 < x < y, \quad y \geq 2x_M, \quad (\text{IB.58})$$

where the inequality uses

$$(1 - \beta)(1 - \gamma)\frac{\Pi(y)}{2} - \beta\gamma K_L \leq (1 - \beta)(1 - \gamma)\Pi(x_M) - \beta\gamma K_L = -\beta K_L < 0.$$

We thus have obtained:

$$L'(\widehat{x}_F) \leq L'(\underline{x}) = \Gamma_x(\underline{x}, \bar{x}) \leq \Gamma_x(\widehat{x}_F, \bar{x}) \leq \Gamma_x(\widehat{x}_F, \bar{y}), \quad (\text{IB.59})$$

where the first inequality use $L''(x) < 0$ for $x < x_F$ and $\underline{x} \leq \hat{x}_F < x_F$, the second inequality follows from (IB.29) and $\underline{x} \leq \hat{x}_F < 2x_M$, and the last inequality uses (IB.58), $\hat{x}_F < 2x_M$, and $\bar{x} \geq \bar{y} > 2x_M$. By using (IB.29), $L''(x) < 0$ for any $x \in (0, x_F)$, (IB.59), and $\Gamma(\hat{x}_F, \bar{y}) = F(\hat{x}_F) = L(\hat{x}_F) - K_L$, we can show that

$$\Gamma(x, \bar{y}) > L(x) - K_L, \quad x \in (\hat{x}_F, x_F]. \quad (\text{IB.60})$$

Using (IB.12) and (IB.29), we obtain $\Gamma(x, \bar{y}) > \Pi(x)/2 - K_L = L(x) - K_L$ for any $x \in (x_F, \bar{y})$. Combining this result with (IB.60), we obtain

$$V_*(x) = \Theta(x; \hat{x}_F, \bar{y}) = \Gamma(x, \bar{y}) > L(x) - K_L, \quad x \in (\hat{x}_F, \bar{y}). \quad (\text{IB.61})$$

Additionally, $\mathcal{AV}_*(x) = 0$ holds for any $x \in (\hat{x}_F, \bar{y})$.

For any $x \geq \bar{y} > 2x_M \geq x_F$, $V_*(x) = \Pi(x)/2 - K_L = L(x) - K_L$ holds and

$$\mathcal{A}(\Pi(x)/2 - K_L) = rK_L - \frac{x}{2} \leq rK_L - \frac{2x_M}{2} = rK_L \left(1 - \frac{\beta}{\beta-1} \frac{r-\mu}{r}\right) < 0, \quad (\text{IB.62})$$

where the first inequality follows from $x \geq \bar{y} > 2x_M$. $\square$

Proof of Theorem 3: We prove Theorem 3 for three mutually exclusive regions as follows: 1.) for the $(0, \hat{x}_L)$ region, the proof of Theorem 3 is essentially the same as that of Theorem 2 for **Case C**; 2.) for the $[\hat{x}_L, \hat{x}_F]$ region, firms optimally compete to become Leader because $L(x) - K_L \geq F(x)$; and 3.) for the $x > \hat{x}_F$ region, we prove that the inequalities (A.3) and (A.4) hold.

To ease our exposition of the proof for the $x > \hat{x}_F$ region, recall that $\varphi_i^* = (\mathcal{E}^*, \lambda^*)$, $\mathcal{E}^* = [\hat{x}_L, \hat{x}_F]$, $\lambda^*(x) = 0$ for any $x < \hat{x}_L$ and $x \in (\hat{x}_F, \infty) \setminus \mathcal{R}^E$, and $\lambda^*(x) > 0$ given by (34) for any x in the probabilistic entry region $\mathcal{R}^E$ defined in (56).

Also note that our proof below covers two subcases: 1.) for **Subcase B₁**, the mixed-strategy entry region is given by $\mathcal{R}^E = (\hat{x}_F, \underline{x}] \cup [\bar{x}, \infty)$, where $\underline{x}$ and $\bar{x}$ are given in Lemma 2; 2.) for **Subcase B₂**, the mixed-strategy entry region is given by $\mathcal{R}^E = [\bar{x}, \infty)$, where $\bar{x}$ is the unique solution for (E.21) in the $y > 2x_M$ region.⁵

Now we are ready to prove that the inequality (A.3) hold for any $x > \hat{x}_F$ in the following two steps.

Step 1: Prove $V_(x) \geq J_a(x; \varphi_a, \varphi_b^*)$ in the $x > \hat{x}_F$ region where $(\varphi_a, \varphi_b^*) \in \Phi$.*

Let τ_a and τ_b denote firm a 's and b 's stochastic entry time associated with φ_a and φ_b^* ,

⁵The variational inequality in the $x > \hat{x}_F$ region subject to the boundary conditions (37) and (55) admits a unique solution $V_*(x)$, which is given by (E.16)-(E.18) for **Subcase B₁** and (E.19)-(E.20) for **Subcase B₂**, respectively (see Lemma 4).

respectively, and let $\tau := \min\{\tau_a, \tau_b\}$. Because $\mathcal{E}^* = [\hat{x}_L, \hat{x}_F]$ and $X_t = x > \hat{x}_F$, we have $X_s > \hat{x}_F$ for any $s \in [t, \tau_b]$. For both [Subcase B₁](#) and [Subcase B₂](#), $V_*(x)$ is twice continuously differentiable except at finite points and is globally continuously differentiable. Applying Itô's Lemma to $e^{-rs}V_*(X_s)$ for $s \in [t, \tau]$, we obtain (E.9). Since $V_*(x)$ satisfies (35) in the $x > \hat{x}_F$ region, we obtain $V_*(x) \geq L(x) - K_L$ for all $x \geq \hat{x}_F$. Using this result, (E.9), and $X_s > \hat{x}_F$ for all $s \in [t, \tau]$, we obtain (E.10). Note that

$$\begin{aligned} & J_a(x; \varphi_a, \varphi_b^*) \\ &= \mathbb{E}_t^x \left[e^{-r(\tau-t)} \left(\mathbf{1}_{\tau_a < \tau_b} (L(X_\tau) - K_L) + \mathbf{1}_{\tau_a > \tau_b} F(X_\tau) + \mathbf{1}_{\tau_a = \tau_b} \frac{L(X_\tau) - K_L + F(X_\tau)}{2} \right) \right] \\ &= \mathbb{E}_t^x [e^{-r(\tau-t)} (L(X_\tau) - K_L)] - \mathbb{E}_t^x \left[\mathbf{1}_{\tau_a > \tau_b} e^{-r(\tau-t)} (L(X_\tau) - K_L - F(X_\tau)) \right], \end{aligned} \quad (\text{IB.63})$$

where the second equality follows from $L(\hat{x}_F) - K_L = F(\hat{x}_F)$ (see Proposition 1) and the result that with probability 1, $\tau_a = \tau_b$ only when $X_\tau = \hat{x}_F$. Using (IB.63) and (E.10), we obtain

$$\begin{aligned} & J_a(x; \varphi_a, \varphi_b^*) \\ & \leq V_*(x) + \mathbb{E}_t^x \left[\int_t^\tau e^{-r(s-t)} \mathcal{A}V_*(X_s) ds - \mathbf{1}_{\tau_a > \tau_b} e^{-r(\tau-t)} (L(X_\tau) - K_L - F(X_\tau)) \right]. \end{aligned} \quad (\text{IB.64})$$

Let $\hat{\tau} := \inf\{s \geq t : X_s \in [\hat{x}_L, \hat{x}_F]\}$ and $G(s) := 1 - (1 - \mathbf{1}_{s \geq \hat{\tau}})e^{-\int_t^s \lambda^*(X_u) du}$. We obtain:

$$\begin{aligned} \mathbb{E}_t^x \left[\int_t^\tau e^{-r(s-t)} \mathcal{A}V_*(X_s) ds \right] &= \mathbb{E}_t^x \left[\int_t^{\hat{\tau}} \int_t^{\tau_a \wedge u} e^{-r(s-t)} \mathcal{A}V_*(X_s) ds dG(u) \right] \\ &= \mathbb{E}_t^x \left[\int_t^{\tau_a \wedge \hat{\tau}} \int_s^{\hat{\tau}} dG(u) e^{-r(s-t)} \mathcal{A}V_*(X_s) ds \right] \\ &= \mathbb{E}_t^x \left[\int_t^{\tau_a \wedge \hat{\tau}} e^{-\int_t^s (r + \lambda^*(X_u)) du} \mathcal{A}V_*(X_s) ds \right], \end{aligned} \quad (\text{IB.65})$$

using Tonelli's Theorem, $\mathcal{A}V_*(x) \leq 0$, $G(\hat{\tau}) = 1$, and integration by parts.

Because $\mathcal{A}V_*(x) = \mathbf{1}_{x \in \mathcal{R}^E} [(rK_L - x)\mathbf{1}_{x < x_F} + (rK_L - x/2)\mathbf{1}_{x > x_F}]$ is locally bounded in the $x \geq \hat{x}_F$ region, we can rewrite (IB.65) as follows:

$$\begin{aligned} & \mathbb{E}_t^x \left[\int_{s \in [t, \tau_a \wedge \hat{\tau}]} e^{-\int_t^s (r + \lambda^*(X_u)) du} \mathcal{A}V_*(X_s) ds \right] \\ &= \mathbb{E}_t^x \left[\int_{s \in [t, \tau_a \wedge \hat{\tau}]} e^{-\int_t^s (r + \lambda^*(X_u)) du} \lambda^*(X_s) [L(X_s) - K_L - F(X_s)] ds \right] \\ &= \mathbb{E}_t^x \left[\mathbf{1}_{\tau_a \wedge \hat{\tau} > \tau_b} e^{-r(\tau_b-t)} [L(X_{\tau_b}) - K_L - F(X_{\tau_b})] \right] \\ &= \mathbb{E}_t^x \left[\mathbf{1}_{\tau_a > \tau_b} e^{-r(\tau_b-t)} [L(X_{\tau_b}) - K_L - F(X_{\tau_b})] \right], \end{aligned} \quad (\text{IB.66})$$

where the first equality uses $\mathcal{A}V_*(x) = \lambda^*(x)[L(x) - K_L - F(x)]$ and the last equality follows from $\tau_b \leq \hat{\tau}$, $X_{\hat{\tau}} = \hat{x}_F$, and $L(\hat{x}_F) - K_L = F(\hat{x}_F)$.

Combining (IB.64), (IB.65), and (IB.66), we obtain $V_*(x) \geq J_a(x; \varphi_a, \varphi_b^*)$.

Step 2: Prove $V_(x) = J_a(x; \varphi_a^*, \varphi_b^*)$ in the $x > \hat{x}_F$ region.*

Let τ_a^* and τ_b^* be firm a 's and b 's stochastic entry time, respectively, associated with strategy $(\varphi_a^*, \varphi_b^*)$, and let $\tau^* := \min\{\tau_a^*, \tau_b^*\}$. Because $\lambda^*(x) = 0$ for any $x \in (\hat{x}_F, \infty) \setminus \mathcal{R}^E$, we have $X_{\tau^*} \in \mathcal{R}^E \cup \{\hat{x}_F\}$, which implies $V_*(X_{\tau^*}) = L(X_{\tau^*}) - K_L$. Therefore, (IB.64) holds with equality if $\tau_a = \tau_a^*$, $\tau_b = \tau_b^*$, and $\tau = \tau^*$. We thus have shown $V_*(x) = J_a(x; \varphi_a^*, \varphi_b^*)$.

In sum, we have proven (A.3) via Steps 1 and 2. By symmetry, we can also prove (A.4).

□

I.C Asymmetric Pure-strategy Equilibria

In this appendix, we characterize the asymmetric pure-strategy equilibria for **Subcase A₂**, **Subcase B₁**, and **Subcase B₂** (see Section 7.1 for **Subcase A₁**). For **Case C**, there is only one symmetric pure-strategy equilibrium characterized in Section 5.

Asymmetric Pure-strategy Equilibria for Subcase A₂. As for **Subcase A₁** in Section 7.1, we characterize the asymmetric pure-strategy equilibrium where firm a is Leader and firm b is Follower. Firm a solves problem (57) and its value function $P_L(x)$ is given by $P_L(x) = V_*(x)$, where $V_*(x)$ is given in Proposition 3. Leader's optimal entry time is given by $\tau_L^* = \inf\{s \geq t : X_s \in [x_M, \underline{x}] \cup [\bar{x}, \infty)\}$, where x_M , $\underline{x}$, and $\bar{x}$ are given in Proposition 3.

Taking τ_L^* that we have just characterized above as given, Follower chooses its entry time $\tau_F \geq \tau_L^*$ to maximize (58). Solving (58), Follower chooses its optimal entry time using the threshold strategy: $\tau_F^* = \inf\{s \geq \tau_L^* : X_s \geq x_F\}$. Follower's value, $P_F(x)$, is given by

$$P_F(x) = F(x), \quad x \leq \underline{x}, \quad (\text{IC.1})$$

$$P_F(x) = \frac{F(\underline{x})\bar{x}^\gamma - F(\bar{x})\underline{x}^\gamma}{\underline{x}^\beta\bar{x}^\gamma - \underline{x}^\gamma\bar{x}^\beta}x^\beta + \frac{F(\bar{x})\underline{x}^\beta - F(\underline{x})\bar{x}^\beta}{\underline{x}^\beta\bar{x}^\gamma - \underline{x}^\gamma\bar{x}^\beta}x^\gamma, \quad x \in (\underline{x}, \bar{x}), \quad (\text{IC.2})$$

$$P_F(x) = F(x) = \Pi(x)/2 - K_F, \quad x \geq \bar{x}. \quad (\text{IC.3})$$

Next, we summarize the solution for the pure-strategy equilibria for **Subcase A₂**.

Proposition 5 *There are two asymmetric pure-strategy equilibria for Subcase A₂ where $R_{AB} < R \leq R_{A_1A_2}$. Leader enters at $\tau_L^* = \inf\{s \geq t : X_s \in [x_M, \underline{x}] \cup [\bar{x}, \infty)\}$, where x_M , $\underline{x}$, and $\bar{x}$ are given in Proposition 3. Leader's value is given by $P_L(x) = V_*(x)$, where $V_*(x)$ is given in*

Proposition 3. Follower enters at $\tau_F^* = \inf\{s \geq \tau_L^* : X_s \geq x_F\}$ and Follower's value $P_F(x)$ is given by (IC.1)-(IC.3).

Proof of Proposition 5: Using similar arguments as those for the proofs of Proposition 3, Theorems 1 and 4, we can prove that $\mathcal{E}_a^* = [x_M, \underline{x}] \cup [\bar{x}, \infty)$ and $\mathcal{E}_b^* = \emptyset$ form an asymmetric pure-strategy equilibrium.

Next, we show that Follower's value $P_F(x)$ is given by (IC.1)-(IC.3). Using (58), we obtain

$$\begin{aligned} P_F(x) &= \mathbb{E}_t^x \left[e^{-r(\tau_L^* - t)} \max_{\tau_F \geq \tau_L^*} \mathbb{E}_{\tau_L^*} \left[e^{-r(\tau_F - \tau_L^*)} \left(\frac{\Pi(X_{\tau_F})}{2} - K_F \right) \right] \right] \\ &= \mathbb{E}_t^x \left[e^{-r(\tau_L^* - t)} F(X_{\tau_L^*}) \right], \end{aligned} \quad (\text{IC.4})$$

where the second equality follows from the definition of $F(x)$ given in (11).

Finally, we derive explicit solutions for the three regions. First, for $x \leq \underline{x}$, we have $\tau_L^* = \inf\{s \geq t : X_s \in [x_M, \underline{x}]\}$ and $X_s \leq \underline{x} < x_F$ for any $s \in [t, \tau_L^*]$. Then, using (IC.4) and $\mathcal{A}F(x) = 0$ for $x < x_F$, we conclude that (IC.1) holds, where the generator $\mathcal{A}$ is defined in (E.1). Second, for $x \geq \bar{x}$, Leader enters immediately so that $\tau_L^* = t$. Using this result, (IC.4), and (14), we obtain (IC.3). Third, for $x \in (\underline{x}, \bar{x})$, $\tau_L^* = \inf\{s \geq t : X_s = \underline{x} \text{ or } \bar{x}\}$. Using this result and (IC.4), we obtain (IC.2). $\square$

Asymmetric Pure-strategy Equilibria for Case B. Let $\hat{x}_L$ and $\hat{x}_F$ denote the two roots of $L(x) - K_L = F(x)$ in the $(0, x_F)$ region for Case B in Proposition 1. Firms wait if $x < \hat{x}_L$ and rush to enter as Leader if $x \in [\hat{x}_L, \hat{x}_F]$, where the first-mover advantage dominates. Therefore, in the $x \leq \hat{x}_F$ domain, the pure-strategy equilibrium and value function are the same as those in Part 1 of Theorem 3.

In the $x > \hat{x}_F$ domain, without loss of generality, we analyze the pure-strategy equilibrium where firm a is Leader and firm b is Follower. Firm a solves the following problem:

$$P_L(x) = \max_{\tau \in [t, \hat{\tau}]} \mathbb{E}_t^x \left[e^{-r(\tau - t)} (L(X_\tau) - K_L) \right], \quad (\text{IC.5})$$

where $\hat{\tau} = \inf\{s \geq t : X_s \leq \hat{x}_F\}$. Leader's optimal entry time τ_L^* solves the problem defined in (IC.5). Now we turn to Follower's problem. Taking Leader's optimal entry time τ_L^* as given, Follower chooses $\tau_F \geq \tau_L^*$ to maximize its value $P_F(X)$ by solving (58).

Next, we summarize the asymmetric pure-strategy equilibria in the $x > \hat{x}_F$ domain for the two subcases of Case B: Subcase B₂ and Subcase B₁.

Proposition 6 *The asymmetric pure-strategy equilibria in the $x > \hat{x}_F$ domain for Case B are as follows:*

1. **Subcase B₂** where $1 < R \leq R_{B_1 B_2}$: Leader can enter at two disjoint regions as its optimal entry time is given by $\tau_L^* = \inf\{s \geq t : X_s \geq \bar{x} \text{ or } X_s \leq \hat{x}_F\}$. Leader's value is $P_L(x) = V_i(x)$, where $\bar{x}$ and $V_i(x)$ are given in Part 1 of Proposition 4. Follower either enters immediately or waits for a while depending on $X_{\tau_L^*}$. Specifically, Follower enters immediately after Leader ($\tau_F^* = \tau_L^* +$) if $X_{\tau_L^*} \geq \bar{x}$ and enters at $\tau_F^* = \inf\{s \geq \tau_L^* : X_s \geq x_F\} > \tau_L^*$ if $X_{\tau_L^*} \leq \hat{x}_F$. Follower's value $P_F(x)$ is given by

$$P_F(x) = \frac{F(\hat{x}_F)\bar{x}^\gamma - F(\bar{x})\hat{x}_F^\gamma}{\hat{x}_F^\beta \bar{x}^\gamma - \hat{x}_F^\gamma \bar{x}^\beta} x^\beta + \frac{F(\bar{x})\hat{x}_F^\beta - F(\hat{x}_F)\bar{x}^\beta}{\hat{x}_F^\beta \bar{x}^\gamma - \hat{x}_F^\gamma \bar{x}^\beta} x^\gamma, \quad x \in (\hat{x}_F, \bar{x}), \quad (\text{IC.6})$$

$$P_F(x) = F(x) = \Pi(x)/2 - K_F, \quad x \geq \bar{x}. \quad (\text{IC.7})$$

2. **Subcase B₁** where $R_{B_1 B_2} < R \leq R_{AB}$: Leader can enter at two disjoint regions as its optimal entry time is given by $\tau_L^* = \inf\{s \geq t : X_s \geq \bar{x} \text{ or } X_s \leq \underline{x}\}$. Leader's value is $P_L(x) = V_i(x)$, where $\underline{x}$, $\bar{x}$, and $V_i(x)$ are given in Part 2 of Proposition 4. Follower either enters immediately or waits for a while depending on $X_{\tau_L^*}$. Specifically, Follower enters immediately after Leader's entry ($\tau_F^* = \tau_L^* +$) if $X_{\tau_L^*} \geq \bar{x}$ and enters at $\tau_F^* = \inf\{s \geq \tau_L^* : X_s \geq x_F\} > \tau_L^*$ if $X_{\tau_L^*} \leq \underline{x}$. Follower's value $P_F(x)$ is given by

$$P_F(x) = F(x), \quad x \in (\hat{x}_F, \underline{x}]. \quad (\text{IC.8})$$

$$P_F(x) = \frac{F(\underline{x})\bar{x}^\gamma - F(\bar{x})\underline{x}^\gamma}{\underline{x}^\beta \bar{x}^\gamma - \underline{x}^\gamma \bar{x}^\beta} x^\beta + \frac{F(\bar{x})\underline{x}^\beta - F(\underline{x})\bar{x}^\beta}{\underline{x}^\beta \bar{x}^\gamma - \underline{x}^\gamma \bar{x}^\beta} x^\gamma, \quad x \in (\underline{x}, \bar{x}), \quad (\text{IC.9})$$

$$P_F(x) = F(x) = \Pi(x)/2 - K_F, \quad x \geq \bar{x}. \quad (\text{IC.10})$$

Proof of Proposition 6: Our proof uses similar arguments as those for the proofs of Theorems 3-4 and Proposition 5. $\square$