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ABSTRACT

We analyze a duopoly real-option entry game where the second mover has a cost advantage over the first mover. The equilibrium solution features five regions. In addition to the option-value-of-waiting and competing-to-enter (first-mover-advantage) regions (Fudenberg and Tirole, 1985; Grenadier, 1996), three new regions appear due to the second-mover advantage: a waiting-to-be-Follower region and two disconnected probabilistic-entry regions. Only when market demand is very high does Follower immediately enter after Leader does. The second-mover advantage causes firms to use state-contingent mixed strategies, significantly delaying their entry timing. Our model generates new predictions, e.g., entry likelihood is non-monotonic in market demand.

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1 Introduction

In this paper, we study strategic real-option exercising decisions by building on Grenadier (1996), which is the duopoly formulation of the classic single-firm real-option framework (McDonald and Siegel, 1986; Dixit and Pindyck, 1994).[1] One of the most important predictions of standard duopoly entry models, e.g., Fudenberg and Tirole (1985) and Grenadier (1996), is that firms exercise their real options too soon relative to the socially efficient level because firms want to capture the first-mover advantage: the monopoly rents earned by the Leader until the Follower's entry. As we show, this result critically depends on the assumption that firms have the same entry costs.

However, in many real-world settings the second mover pays a lower entry cost and/or has a more efficient production technology than the first mover. The ulcer-relief drug Zantac is a known successful second mover (Berndt, Pindyck and Azoulay, 2003).[2] While the Leader of the industry may enjoy the first-mover advantage, it also pays higher learning, R&D, and other costs than the Follower. By observing and learning from the pioneer's successes and failures, the Follower in effect lowers its own entry cost. Décaire and Wittry (2022) provide evidence in support of the second-mover advantage in a classic real-option setting: the oil and gas sector.

Motivated by the analyses in Berndt, Pindyck and Azoulay (2003) and Décaire and Wittry (2022), we make a single change to Grenadier (1996): allowing the second mover to have a lower (or more broadly different) entry cost than the first mover. Incorporating the second-mover advantage into Grenadier (1996) fundamentally alters the economic predictions in duopoly entry games. For example, when market demand is high, rather than competing to enter as the first mover (using pure strategies), firms prefer to enter as the second mover

[1] Abel, Dixit, Eberly and Pindyck (1996) make the connection between the real options approach and the q theory of investment (Hayashi, 1982; Abel and Eberly, 1994). Grenadier and Malenko (2010) develop a Bayesian real-options approach and Orlov, Skrzypacz and Zryumov (2020) study Bayesian persuasion in a real-options environment. Grenadier and Malenko (2011) analyze real-option signaling games. Real-options models are widely used in Corporate Finance to study mergers (Lambrecht, 2004), takeovers (Morellec and Zhdanov, 2008), and external equity financing (Hugonnier, Malamud and Morellec, 2015), among others.

[2] Also see the piece in Northwestern Kellogg's Insight: https://insight.kellogg.northwestern.edu/article/the_second_mover_advantage, titled "The Second-Mover Advantage," which is based on a marketing research article (Shankar, Carpenter and Krishnamurthi, 1998).

(using mixed strategies). Moreover, the *interaction* between the first-mover and second-mover advantages generates new predictions that would have been absent had we studied either the first-mover or second-mover advantage alone in the classic real-option framework (McDonald and Siegel, 1986).

Next, we sketch out our duopoly real-option model. Two *ex ante* identical firms compete to enter a new product market. To ease exposition, we assume that the market demand X_t is exogenous and follows a geometric Brownian motion. Each firm can enter the market by paying a one-time fixed cost. The first mover captures the entire demand until Follower enters. Upon entering the market, Follower takes one half of the market demand away from Leader as in Fudenberg and Tirole (1985) and Grenadier (1996).[3] Our key new assumption is that Follower's (fixed) entry cost (K_F) is different from Leader's entry cost (K_L). As Follower and Leader are endogenously determined, the entry-cost ratio $R = K_L/K_F$, which measures how large the second-mover advantage is, plays a key role in our model.

In Grenadier (1996), firms balance the option value of waiting against the first-mover advantage. When the former dominates, firms wait. Otherwise, firms rush to enter and Leader is randomly selected using the *rent equalization* principle (Fudenberg and Tirole, 1985).

In contrast, our model solution falls into one of the three cases: A, B, and C (more specifically one of the five subcases: A_1, A_2, B_1, B_2, and C), depending on parameter values.[4] For the subcase with the richest predictions, Subcase B_1, the equilibrium solution features *five* regions. Compared with Grenadier (1996), three new equilibrium regions surface in the domain where the second-mover advantage dominates the first-mover advantage.

Additionally, for all five subcases, we can show that *two measures*, the entry-cost ratio $R = K_L/K_F$ and the real-optionality measure β,[5] are necessary and sufficient to fully charac-

[3] We can generalize our model so that the total market demand depends on the industry structure (e.g., monopoly or duopoly). We can also relax the equal-profit-split assumption by allowing for other profit-split assumptions after both firms enter. Our main results continue to hold.

[4] We show that Case A can be divided into two subcases, Subcase A_1 and Subcase A_2; and Case B can be divided into two subcases, Subcase B_1 and Subcase B_2.

[5] This measure, β, is the positive (larger) root of the classic fundamental quadratic equation in the real-options literature (McDonald and Siegel, 1986; Dixit and Pindyck, 1994).

terize the economics of our duopoly entry model. Intuitively speaking, the ratio R measures the second-mover advantage, β captures the real optionality of firm entry, and these two measures, R and β, jointly pin down the equilibrium tradeoff between the first-mover and second-mover advantages in the real-option framework, fully characterizing the solution.

Next, we discuss the economics of our duopoly entry model incrementally by analyzing the key results for each of the five subcases. We start with Case A where R is so large that the second-mover advantage globally dominates the first-mover advantage.

In Case A, firms strictly prefer being Follower. However, without Leader there would be no Follower. Each period a firm waits, it forgoes a large profit when the market demand is sufficiently high. But there is also a benefit of waiting. A firm's entry cost is lowered to K_F when its competitor becomes Leader. In equilibrium, *probabilistic entry*, the mid-of-the-road strategy between waiting and entering (via pure strategies), is optimal. Put differently, probabilistic entry is a compromise and Leader is randomly selected to enter in equilibrium. We then use closed-form solutions to answer the following questions.

Under what market conditions do firms choose to enter probabilistically versus to wait? And what are firms' equilibrium entry rates in the probabilistic entry regions? What is Follower's equilibrium strategy and how does that influence firms' entry strategies as Leader? Does Leader earn equilibrium monopoly rents? If so, for how long and under what conditions?

The answers to these questions depend on the parameters of the duopoly model. Case A has two subcases: A_1 and A_2. In Subcase A_1, the entry-cost ratio $R = K_L/K_F$ is so large that Follower always enters immediately after Leader does, leaving Leader with no monopoly rents. In equilibrium, firms wait in the $x < \overline{x}$ region and enter probabilistically in the $x \geq \overline{x}$ region, where $\overline{x}$ is the endogenous cutoff value for the total market demand $X_t = x$ at t.[6]

What is the equilibrium entry rate in the probabilistic entry region where $x \geq \overline{x}$? By entering as Leader, a firm in effect collects the stochastic profit $\{X_t/2\}$ indefinitely.[7] However, by entering, the firm forgoes the opportunity to lower its entry cost as Follower. In a

[6]Once one firm enters, the other immediately follows and as a result the two firms split the market equally.

[7]In Subcase A_1, as Leader shares the market with Follower in equilibrium, Leader only collects $X_t/2$. The one-time fixed cost K_L paid upon entry is equivalent to a perpetual payment of rK_L in present value.

mixed-strategy equilibrium, firms are indifferent between entering and waiting for another period. This indifference condition pins down the equilibrium entry rate, related to but different from that in war-of-attrition games.[8]

What if the entry-cost ratio R is lower than in Subcase A$_1$ but the second-mover advantage remains significant? Then we have *four* regions in equilibrium divided by three cutoff values ($\widetilde{x}$, $\underline{x}$, and $\overline{x}$ in ascending order). This is our Subcase A$_2$.

In the $x < \widetilde{x}$ region, firms wait. In the $x \geq \overline{x}$ region, firms play mixed entry strategies.[9] Between the $x < \widetilde{x}$ and $x \geq \overline{x}$ regions are: 1.) a second probabilistic entry region where the market demand is moderate: $x \in [\widetilde{x}, \underline{x}]$ and 2.) a second waiting region (between the two probabilistic entry regions) where $x \in (\underline{x}, \overline{x})$.

The intuition for the second probabilistic entry region is as follows. When market demand is moderate (i.e., for $x \in [\widetilde{x}, \underline{x}]$), half of the market demand is not enough for the second mover to immediately follow Leader's entry. Therefore, Leader collects equilibrium monopoly rents for a stochastic duration, which in turn encourages firms to enter as the first mover.

Why do firms wait in the region where $x \in (\underline{x}, \overline{x})$? This is because waiting yields a higher value than probabilistic entry, the (only) other alternative, when the second-mover advantage dominates. Market demand x in this region is not high enough for a firm to probabilistically enter with no monopoly rents, nor offers a firm monopoly rents for a long enough duration. In sum, waiting is the optimal strategy for $x \in (\underline{x}, \overline{x})$. This waiting motive is very different from the standard irreversibility-induced option-value-of-waiting motive, explaining why we have two disconnected waiting regions, unlike the single firm's real option problem.

Note that because of the second-mover advantage there are two disconnected entry regions, implying that the likelihood of firm entry is non-monotonic with respect to market demand, absent in the first-mover-advantage-based models, e.g., Grenadier (1996).

We next turn to Case C where Leader's entry cost is weakly lower than Follower's ($R \leq 1$) so that there is no second-mover advantage.

[8]Section 8.1 of Tirole (1988) and Levin (2004) provide introductions to the war-of-attrition literature.

[9]Once a firm enters as Leader, the other follows immediately, leaving no monopoly rents for Leader, similar to the probabilistic entry region in Subcase A$_1$.

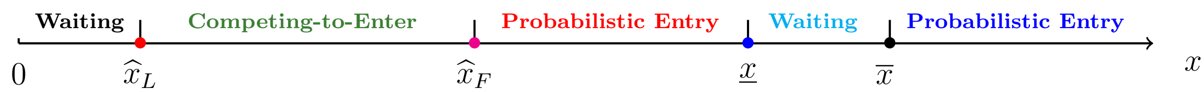

Figure 1: Five-Region Equilibrium Solution for the General Case: Subcase B_1.

Our analysis of Case C extends Grenadier (1996), corresponding to our $R = 1$ special case. When market demand is sufficiently high, firms rush to enter and one firm is randomly selected as Leader in a way that *ex ante* rents are equalized between the two firms (Fudenberg and Tirole, 1985). In equilibrium there are two regions: the waiting (for the standard real-option argument) region where $x < \widehat{x}_L$ and the *first-mover-advantage* region where $x \geq \widehat{x}_L$. The endogenous cutoff value $\widehat{x}_L$ is the turning point above which firms prefer to be Leader.

We now analyze Case B, the intermediate case between Case A and Case C, where the entry-cost ratio R is larger than one but not too large. The first-mover and second-mover advantages co-exist and there are (up to) five regions in equilibrium. Figure 1 demonstrates the five regions in equilibrium for Subcase B_1, which we explain below.[10]

The far left region where $x < \widehat{x}_L$ is the standard option-value of-waiting region. In the second region where $x \in [\widehat{x}_L, \widehat{x}_F]$, the first-mover advantage dominates. As in Case C, the solution for these two regions are fully characterized on their own where $\widehat{x}_L$ and $\widehat{x}_F$ are the two points at which being Leader and Follower yield the same value.

In the $x > \widehat{x}_F$ domain, there are three new regions (absent in Case C) where the second-mover advantage dominates in equilibrium: 1.) a probabilistic entry $x \in (\widehat{x}_F, \underline{x}]$ region where Leader earns monopoly rents; 2.) a second probabilistic entry $x \geq \overline{x}$ region where Leader earns no monopoly rents; 3.) and a (second) waiting $x \in (\underline{x}, \overline{x})$ region between the two probabilistic entry regions. Follower's value at $\widehat{x}_F$ serves as a key boundary condition in the $x > \widehat{x}_F$ domain. We show that the two smooth-pasting conditions at the two endogenous cutoff values ($\underline{x}$ and $\overline{x}$) divide the $x > \widehat{x}_F$ domain into the three regions discussed above.

In terms of technical contributions, we provide an equilibrium definition for the duopoly entry game (featuring both first-mover and second-mover advantages), characterize the equi-

[10] In Subcase B_2 where R is slightly larger than one, there are four regions in equilibrium as the fifth region (the mixed strategy region where Leader earns monopoly rents) disappears.

librium solutions via variational inequalities, and derive closed-form expressions for equilibrium strategies and value functions.[11]

While we have mainly focused on symmetric equilibria, we also analyze asymmetric pure-strategy equilibria. We show that Leader's value in pure-strategy equilibria equals a firm's pre-entry value in the mixed-strategy equilibrium and provide an economic connection between the pure-strategy equilibria and the mixed-strategy symmetric equilibrium.

Finally, we quantify our model's predictions using Subcase A_1 as an example. We characterize the distributions of entry time using partial differential equations with economically intuitive boundary conditions for both pure-strategy and mixed-strategy equilibria. We find large socially inefficient entry delays and substantial option value erosion. Moreover, the mixed-strategy equilibrium is far more inefficient than the pure-strategy equilibria.

Related Literature

Our paper is closely related to Grenadier (1996, 2002), and Back and Paulsen (2009).[12] We provide a unified analysis of real-option duopoly entry game where both the first-mover and second-mover advantages exist. Grenadier (1996) is a special case of our model where $R = K_L/K_F = 1$. As a result, there is no second-mover advantage and hence no mixed strategy in equilibrium in his paper.[13]

Grenadier (2002) and Back and Paulsen (2009) analyze continuous-time oligopoly capital-accumulation games. Their analyses build on an individual firm's optimal *singular control* and show that competition causes firms to speed up investment. In contrast, our duopoly entry game builds on a firm's stopping-time problem and we show that the second-mover advantage can significantly delay entry timing. In sum, the economic insights and mathematical analysis of our model are complementary to but quite different from those in Grenadier

[11] In our game-theoretic setting, equilibria are not characterized by simple cutoff strategies, e.g., as in McDonald and Siegel (1986), but rather multiple cutoff values implied by variational inequalities.

[12] Fudenberg, Gilbert, Stiglitz and Tirole (1983) model preemption games (e.g., patent races) in deterministic settings. Smets (1991) studies irreversible investment in a duopoly setting and analyzes an asymmetric leader-follower equilibrium. Murto (2004) studies a duopoly exit game and focuses on pure strategies.

[13] The model in Fudenberg and Tirole (1985) is a deterministic version of Grenadier (1996). Weeds (2002) integrates a real-options model with strategic interactions by incorporating technological uncertainty into models along the lines of Grenadier (1996).

(2002) and Back and Paulsen (2009).

Lambrecht (2001) develops a duopoly exit model in a real-option setting with two *ex ante* heterogeneous firms and studies pure-strategy equilibria.[14] In contrast, we analyze how *ex ante* identical firms become *ex post* heterogeneous via their market *entry* decisions and characterize pure-strategy and mixed-strategy equilibria as well as hybrid-strategy equilibria, which involve both pure and mixed strategies.

The second-mover advantage in our model gives firms incentives to delay entry. The mechanism in our model is related to but different from that in classic war-of-attrition games.[15] First, while classic attrition games are about firm exit, our model is about firm entry. Second, while firm payoffs in war-of-attrition games are often exogenous (Levin, 2004), they are endogenous in our model. This is because Leader's payoff depends on Follower's equilibrium entry strategy and contains an option-value component. Third, the interaction between the first-mover and the second-mover advantages induces the coexistence of mixed strategies and pure strategies in our duopoly entry model, which is absent in standard war-of-attrition models. Finally, the likelihood of entry is not monotonic in market demand.

Our model is also related to Chamley and Gale (1994) and Grenadier (1999), in which firms delay entry timing in anticipation of information spillover from their peer's decisions. Unlike these social-learning-based models, our model features complete information, entry-cost savings for the second mover, and the coexistence of first-mover and second-mover advantages.

There is also a growing literature that integrates industrial organization considerations into asset pricing models. For example, Dou, Ji and Wu (2021) extend the standard Lucas-tree asset pricing model to allow for endogenous strategic competition. Chen, Dou, Guo and Ji (2022) study how strategic competition and financial distress dynamically interact.

[14]The heterogeneity arises from exogenously given different capital structures for two incumbents. Lambrecht and Perraudin (2003) introduce incomplete information into an equilibrium real-option exercising model. Anderson, Friedman and Oprea (2010) generalize Lambrecht and Perraudin (2003) to settings with multiple firms.

[15]For war-of-attrition-style duopoly exit models, see Ghemawat and Nalebuff (1985, 1990), Fudenberg and Tirole (1986), and Hendricks, Weiss and Wilson (1988), among others. Steg (2015) analyzes mixed strategies in symmetric stochastic war-of-attrition games.

2 Model

In this section, we set up an entry game in which two *ex ante* identical firms choose their optimal timing to enter a new market.

2.1 Market Demand and Industry Structure

The total market profit is governed by a stochastic process, $\{X_t; t \geq 0\}$. As in McDonald and Siegel (1986), Dixit and Pindyck (1994), and Grenadier (1996), we assume that $\{X_t; t \geq 0\}$ follows a geometric Brownian motion (GBM):

$$dX_t = \mu X_t dt + \sigma X_t d\mathcal{Z}_t \,, \tag{1}$$

where μ is the expected growth rate of X, $\sigma > 0$ is the constant volatility for the growth rate of X, $\{\mathcal{Z}_t; t \geq 0\}$ is a one-dimensional standard Brownian motion, and the initial value of X is known: $X_0 = x_0 > 0$.[16]

Let τ_L and τ_F denote the stochastic time when Leader and Follower enter the market, respectively. By definition, $\tau_F \geq \tau_L$. Let $K_L > 0$ and $K_F > 0$ denote the fixed entry cost that Leader and Follower have to pay at their respective entry time τ_L and τ_F. We interpret K_L and K_F as the present value of all expenses that Leader and Follower incur, respectively.

The key assumption of our model is that Leader incurs a larger entry cost than Follower does as Leader may have to pay additional innovation and marketing costs, learn about a new product market, and work with local governments in the new markets. Follower can save some of the costs by observing Leader's actions, learning from Leader's experiences and mistakes, and even possibly imitating Leader's success and copying Leader's strategies.

The industry structure has three phases. First, before either firm enters ($t < \tau_L$), neither firm receives any cash flow. Which firm becomes Leader is endogenous and random. Second, after Leader enters at τ_L and before Follower enters at τ_F, Leader receives monopoly profits: $\{X_s; s \in [\tau_L, \tau_F)\}$. Third, after Follower enters at τ_F, the economy permanently switches from a monopoly to a duopoly setting in which Follower and Leader equally split the total

[16]Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ denote the probability space. We assume that the process $\{\mathcal{Z}_t; t \geq 0\}$ is progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.

market profit and both receive profits indefinitely: $\{X_s/2; s \geq \tau_F\}$.

In sum, two *ex ante* identical firms, firm a (Alice's) and firm b (Bob's), maximize their values by taking the total market profit $\{X_s; s \geq 0\}$ process and the industry structure described above as given. Let τ_a and τ_b respectively denote firm a's and b's stochastic entry time before Leader is determined. Leader's entry time is then given by

$$\tau_L = \min\{\tau_a, \tau_b\} = \tau_a \wedge \tau_b \,. \tag{2}$$

Both firms are risk-neutral and discount profits at the constant interest rate r. As in the standard real-option models, we require $r > \mu$ and $r > 0$, which ensure that firm value is finite. As we show later, the ratio between Leader's entry cost (K_L) and Follower's (K_F) plays a crucial role in our analysis. Let R denote the entry-cost ratio:

$$R = K_L/K_F \,. \tag{3}$$

As long as $R > 1$, there is a second-mover advantage. Below we summarize these assumptions, which apply throughout our analysis:

$$\textit{Assumptions}: \quad r > \mu, \quad r > 0, \quad K_L > 0, \quad K_F > 0 \,. \tag{4}$$

Before we solve our duopoly problem, we first summarize the solution for the classic single firm's real-option problem (McDonald and Siegel, 1986; Dixit and Pindyck, 1994). The monopoly solution will help us better understand the mechanism of our duopoly model.

2.2 Monopoly Solution

A firm with an exclusive market entry opportunity chooses its entry time, τ_M, to solve:

$$M(x) \;=\; \max_{\tau_M \geq t} \;\; \mathbb{E}_t^x\left[e^{-r(\tau_M - t)}\left(\int_{\tau_M}^{\infty} e^{-r(s-\tau_M)} X_s ds - K_L\right)\right] , \tag{5}$$

where $X_t = x > 0$ and $\mathbb{E}_t^x[\cdot] = \mathbb{E}_t[\cdot | X_t = x]$. The monopolist's optimal entry is characterized by a trigger strategy in that $\tau_M^* = \inf\{s \geq t : X_s \geq x_M\}$, where the optimal threshold, x_M, is given by

$$x_M = \frac{\beta}{\beta - 1}(r - \mu) K_L \tag{6}$$

and $\beta > 1$ measures optionality and is given by[17]

$$\beta = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}. \tag{7}$$

Let $\Pi(x)$ denote the monopolist's value function after entry:

$$\Pi(x) = \mathbb{E}_t^x \left[\int_t^\infty e^{-r(s-t)} X_s ds \right] = \frac{x}{r-\mu}. \tag{8}$$

In the waiting region where $x < x_M$, the monopolist's value $M(x)$ is

$$M(x) = \underbrace{\left(\frac{x}{x_M}\right)^\beta}_{\text{PV of \$1 paid at } \tau_M^*} \underbrace{(\Pi(x_M) - K_L)}_{\text{NPV at } \tau_M^*}. \tag{9}$$

As the stochastic entry time τ_M^* is characterized by the trigger policy (x_M), before entry at any time t, the monopolist's value equals the product of (i) the time-t value of a \$1 paid at τ_M^*, given by $(x/x_M)^\beta$, and (ii) the NPV $(\Pi(x_M) - K_L)$ collected at τ_M^*. In the $x \geq x_M$ region, the firm enters the market immediately and therefore

$$M(x) = \Pi(x) - K_L, \quad x \geq x_M. \tag{10}$$

As $\beta > 1$, $M(x)$ is globally increasing and convex in x. Next, we sketch out our solution method for the duopoly model.

2.3 Duopoly Model Solution Procedure

We solve our duopoly model using backward induction as illustrated in Figure 2. After both firms have entered, i.e., for $t \geq \tau_F$, they equally split profits, valued at $\Pi(x)/2$. This is Step 0 in Figure 2. Next, we calculate Follower's and Leader's value after Leader's entry but before Follower's entry, i.e., for the $[\tau_L, \tau_F)$ period. This is Step 1 in Figure 2.

Defining Follower's Pre-entry and Leader's Post-entry Values: $F(x)$ and $L(x)$. Follower's value in the $[\tau_L, \tau_F)$ period is given by:

$$F(x) = \max_{\tau_F \geq t} \mathbb{E}_t^x \left[\int_{\tau_F}^\infty e^{-r(s-t)} \frac{X_s}{2} ds - e^{-r(\tau_F - t)} K_F \right], \tag{11}$$

where $X_t = x > 0$. Let τ_F^* denote Follower's optimal entry time for (11). Taking τ_F^* and $F(x)$ as given, we define Leader's post-entry value function, $L(x)$, for any $t \in [\tau_L, \tau_F^*)$ as

[17] That is, β is the larger root of the fundamental quadratic equation, $\sigma^2 z(z-1)/2 + \mu z - r = 0$, for the GBM X process (1) in standard real option models.

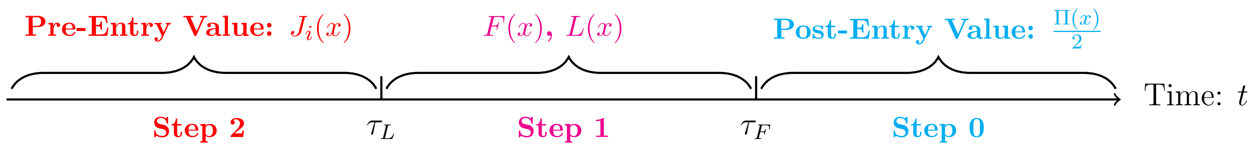

Figure 2: This figure summarizes various value functions for a given pair of entry timing (τ_L, τ_F) in three time periods: $t < \tau_L$ (before Leader's entry); $t \in [\tau_L, \tau_F)$; and $t \geq \tau_F$ (after Follower's entry). $\Pi(X_t) = X_t/(r-\mu)$ is the total market capitalization. $F(x)$ and $L(x)$ are Follower's and Leader's value functions in the $t \in [\tau_L, \tau_F)$ period. $J_i(x)$ is firm i's value before Leader's entry.

follows:

$$L(x) = \mathbb{E}_t^x \left[\int_t^{\infty} e^{-r(s-t)} X_s ds - \int_{\tau_F^*}^{\infty} e^{-r(s-t)} \frac{X_s}{2} ds \right], \tag{12}$$

where the first term in (12) gives time-t value if Leader were to monopolize the market indefinitely and the second term gives the time-t value taken away by Follower from entry time τ_F^* onward.[18]

2.3.1 Step 1: Solving $F(x)$, $L(x)$, and Follower's Optimal Entry Time τ_F^*.

Using backward induction, we first jointly solve Follower's optimal entry time τ_F^* and its closed-form value function $F(x)$, and then calculate Leader's post-entry value $L(x)$.

Follower's Optimal Entry Threshold τ_F^* and Pre-entry Value $F(x)$. At any time t after Leader enters ($t \geq \tau_L$), by paying an entry cost K_F at its chosen entry time τ_F^*, Follower occupies half of the total market. Therefore, Follower's entry decision boils down to a monopolist's real-option problem analyzed in Section 2.2 but with an entry cost of K_F and a stochastic flow payoff of $X_t/2$. Follower's value $F(x)$ is thus given by:

$$F(x) = \left(\frac{\Pi(x_F)}{2} - K_F \right) \left(\frac{x}{x_F} \right)^{\beta}, \quad x < x_F, \tag{13}$$

$$F(x) = \frac{\Pi(x)}{2} - K_F, \quad x \geq x_F, \tag{14}$$

[18]Note that as in Grenadier (1996), $F(x)$ includes Follower's entry cost K_F but $L(x)$ does not include Leader's entry cost K_L as $L(x)$ is calculated for $t \geq \tau_L$.

where the optimal entry threshold, x_F, is given by

$$x_F = \frac{2\beta}{\beta - 1}(r - \mu)K_F\,. \tag{15}$$

Note that the entry threshold x_F is now proportional to Follower's entry cost K_F and the multiple 2 is due to the assumption that Follower's profit is a half of the total industry profits. As in standard real option models, Follower's pre-entry value $F(x)$ is increasing and convex. The higher the volatility σ, the higher the value $F(x)$.

Leader's Post-entry Value $L(x)$. Solving $L(x)$ defined in (12), we obtain

$$L(x) = \Pi(x) - \frac{\Pi(x_F)}{2}\left(\frac{x}{x_F}\right)^{\beta}, \quad x < x_F\,, \tag{16}$$

$$L(x) = \frac{\Pi(x)}{2}, \quad x \geq x_F\,. \tag{17}$$

In the $x \geq x_F$ region, both Leader and Follower are in the market and they equally split the market demand and hence both are valued at $\Pi(x)/2$. In the $x < x_F$ region, Leader's time-t value $L(x)$ equals the difference between the total market capitalization $\Pi(x)$ and $\frac{\Pi(x_F)}{2}\left(\frac{x}{x_F}\right)^{\beta}$, which equals the value of Leader's lost profits caused by Follower's entry. Note that solving $L(x)$ is a pure valuation problem as there is no decision by Leader involved. Leader's value $L(x)$ for the $x < x_F$ region is concave but $L(x)$ for the $x \geq x_F$ region is linear. Therefore, $L(x)$ is not globally concave. This property has important equilibrium implications in our duopoly model as we show later.

Next, we turn to Step 2, the final and key step of our analysis for the $[0, \tau_L)$ period. In this period, firms formulate their optimal entry strategies into a market with no incumbents.

2.3.2 Step 2: Determining Leader and Its Entry Time τ_L

For a pair of entry strategy (τ_a, τ_b), firm i's value function $J_i(x)$ at time t is given by

$$\mathbb{E}_t^x\left[e^{-r(\tau_L - t)}\left(\mathbf{1}_{\tau_i < \tau_{-i}}(L(X_{\tau_i}) - K_L) + \mathbf{1}_{\tau_i > \tau_{-i}}F(X_{\tau_{-i}}) + \mathbf{1}_{\tau_i = \tau_{-i}}\frac{L(X_{\tau_i}) - K_L + F(X_{\tau_i})}{2}\right)\right], \tag{18}$$

where $\tau_L = \tau_i \wedge \tau_{-i}$, $X_t = x > 0$, and $\mathbf{1}_A$ is an indicator function that equals one if event A occurs and zero otherwise. The first term in (18) describes the event where firm i becomes Leader, the second term describes the event where firm i becomes Follower, and the last term accounts for the scenario where the two firms enter at the same time.

Firm i chooses its optimal entry time τ_i to maximize its value given in (18) taking into the best response of its competitor, firm $-i$. Next, we characterize our model solution and focus on Markov perfect equilibria.

3 Characterizing Model Solution via Three Cases

In this section, we show that depending on how large the entry-cost ratio $R = K_L/K_F$ is, our model solution falls into one of the three cases, Case A, Case B, and Case C.

In Case A, the entry-cost ratio R is so large that firms are better off being Follower for all levels of market demand x. We refer to Case A as the second-mover-advantage case.

In Case C where the entry-cost ratio R is weakly less than one, there is no second-mover advantage at all. In equilibrium, firms trade off the first-mover advantage and the standard option value of waiting, as highlighted in Fudenberg and Tirole (1985) and Grenadier (1996).[19] We refer to Case C as the first-mover-advantage case.

Finally, in Case B where the entry-cost ratio R is larger than one but only by a moderate margin, both the first-mover and second-mover advantages coexist in equilibrium.

Next, we formally describe these three cases of our duopoly model solution.

Proposition 1 *Let R_{AB} be given by*[20]

$$R_{AB} = \left(\frac{2^\beta}{\beta+1}\right)^{\frac{1}{\beta-1}} > 1. \tag{19}$$

Depending on how large the entry-cost ratio $R = K_L/K_F$ is, our duopoly model solution falls into one of the following three cases.

Case A. *If $R > R_{AB}$, then the second-mover advantage globally dominates. This is because the following inequality holds for all x:*

$$L(x) - K_L < F(x)\,, \quad x > 0\,. \tag{20}$$

[19] Grenadier (1996) corresponds to the special $R = 1$ case of our duopoly model.

[20] We obtain $R_{AB} = K_{AB}/K_F$, by solving for K_{AB}, which is the unique root of the following equation for $K_L \in (K_F, 2K_F)$: $\frac{L(x_M)-K_L}{x_M^\beta} = \frac{F(x_F)}{x_F^\beta}$.

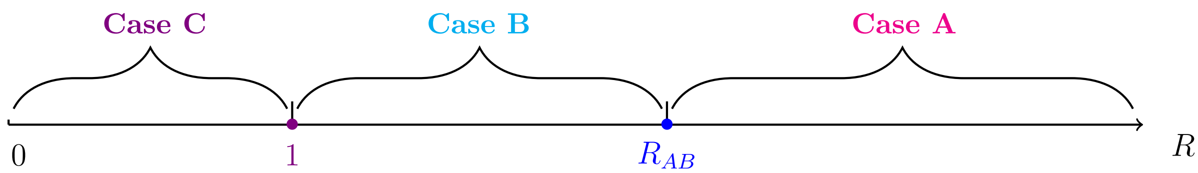

Figure 3: This figure summarizes all the three cases of the duopoly model solution: ($i.$): Case A: $R > R_{AB}$; ($ii.$) Case B: $1 < R \leq R_{AB}$; and ($iii.$) Case C: $R \leq 1$. The cutoff value is $R_{AB} = \left(\frac{2^\beta}{\beta+1}\right)^{\frac{1}{\beta-1}} > 1$ and β is the optionality measure given in (7).

Case B. *If $1 < R < R_{AB}$, then $L(x) - K_L = F(x)$ has two roots, $\widehat{x}_L$ and $\widehat{x}_F$:* [21]

$$L(x) - K_L > F(x)\,, \quad \widehat{x}_L < x < \widehat{x}_F, \tag{21}$$

$$L(x) - K_L < F(x)\,, \quad x < \widehat{x}_L \quad or \quad x > \widehat{x}_F, \tag{22}$$

and $\widehat{x}_L < x_M < \widehat{x}_F < x_F$.[22] *The first-mover advantage dominates in the $\widehat{x}_L < x < \widehat{x}_F$ region and the second-mover advantage dominates in $x > \widehat{x}_F$ region. Both firms wait due to the option-value-of-waiting considerations in the $x < \widehat{x}_L$ region.*

Case C. *If $R \leq 1$, then $L(x) - K_L = F(x)$ has a unique root $\widehat{x}_L$ in the $(0, x_F)$ domain and*[23]

$$L(x) - K_L \geq F(x)\,, \quad x > \widehat{x}_L, \tag{23}$$

$$L(x) - K_L < F(x)\,, \quad x < \widehat{x}_L, \tag{24}$$

and $\widehat{x}_L < x_M$. The first-mover advantage dominates in $x > \widehat{x}_L$ region. Both firms wait due to the option-value-of-waiting considerations in the $x < \widehat{x}_L$ region.

At the core of Proposition 1 is whether $F(x)$ is larger than $L(x) - K_L$ or not.

Next, we analyze Case A, where $F(x) > L(x) - K_L$ holds for all $x > 0$. We focus on symmetric equilibria in Section 4 and analyze asymmetric equilibria in Section 7.

[21]If $R = R_{AB}$, the two roots for the $L(x) - K_L = F(x)$ equation reduce to one root: $\widehat{x}_L = \widehat{x}_F = x_M$, where x_M is the monopolist's entry threshold given in (6). For all $x \neq x_M$, $L(x) - K_L < F(x)$.

[22]Recall that x_M and x_F are the monopolist's and Follower's entry thresholds given in (6) and (15), respectively.

[23]The inequality (23) is strict when $R < 1$. When $R = 1$, the inequality is strict for $\widehat{x}_L < x < x_F$ and holds with equality for $x \geq x_F$.

4 Solution for Case A: Mixed-Strategy Equilibrium

In Case A, the entry-cost ratio is so large, $R > R_{AB} > 1$, that $F(x) > L(x) - K_L$ holds for all $x > 0$. As a result, between waiting and entering as Leader, a firm strictly prefers waiting which yields a higher payoff. But there is no Follower without Leader. Does this mean there is no equilibrium for Case A? The above reasoning implies that there is no *symmetric* pure-strategy equilibrium for Case A. Importantly and somewhat surprisingly there exists an economically intuitive symmetric *mixed-strategy* equilibrium which we analyze in this section. First, we define the mixed-strategy equilibrium.

4.1 Definition of Markov Mixed-Strategy Equilibrium

Let $\lambda_i(X_t)$ denote the rate at which firm i becomes Leader over a small time interval $[t, t+dt]$ where $t < \tau_L$. Firm i's entry time τ_i is a doubly stochastic process as its associated rate $\{\lambda_i(X_t)\}_{t \geq 0}$ is also stochastic.[24] Next, we define feasible Markov mixed strategies and the Markov perfect mixed-strategy equilibrium.

Definition 1 An entry rate $\lambda_i(x)$ is a measurable function from $\mathbb{R}_+$ to $\mathbb{R}_+$. A pair of Markov strategy $(\lambda_a(\,\cdot\,), \lambda_b(\,\cdot\,))$ is feasible if and only if for any $t > 0$, $\int_0^t \lambda_i(X_s)ds < \infty$ almost surely. Let Φ denote the set of all feasible Markov mixed strategies.

Definition 2 Let $J_i(x; \lambda_a, \lambda_b)$ denote firm i's value at time t defined in (18) for a given $X_t = x > 0$ and a feasible Markov mixed strategy pair (λ_a, λ_b). A feasible strategy pair $(\lambda_a^*, \lambda_b^*)$ is

[24] Stopping time τ is doubly stochastic if the underlying counting process $\{\mathcal{N}_t\}_{t\geq 0}$ whose first jump time τ is doubly stochastic. A counting process $\{\mathcal{N}_t\}_{t\geq 0}$ is doubly stochastic if its associated intensity process $\{\lambda_t\}_{t\geq 0}$ is $\{\mathcal{F}_t\}_{t\geq 0}$-predictable and for all t and $s > t$, conditional on the σ-algebra generated by $\{\mathcal{N}_u\}_{u\in[0,t]}$ and $\mathcal{F}_s$, the random variable $(\mathcal{N}_s - \mathcal{N}_t)$ has a Poisson distribution with parameter $\int_t^s \lambda_u du$. Now we apply these definitions to our model. Let $\{\mathcal{G}_t\}_{t\geq 0}$ be the σ-algebra generated by $\{\mathcal{F}_t\}_{t\geq 0}$ and $\{\mathcal{N}_t^i\}_{t\geq 0}$, where $i = a, b$. For any $t \geq 0$ and $s > t$, conditional on the σ-algebra generated by $\mathcal{G}_t \bigcup \mathcal{F}_s$, the counting processes $\{\mathcal{N}_u^a - \mathcal{N}_t^a\}_{u\in[t,s]}$ and $\{\mathcal{N}_u^b - \mathcal{N}_t^b\}_{u\in[t,s]}$ are independent and the random variable $(\mathcal{N}_s^i - \mathcal{N}_t^i)$ has a Poisson distribution with parameter $\int_t^s \lambda_i(X_u)du$ for $i = a, b$. Firm i's entry time τ_i is thus doubly stochastic with the underlying counting process $\{\mathcal{N}_t^i\}_{t\geq 0}$ and the associated intensity process $\{\lambda_i(X_t)\}_{t\geq 0}$. See Lando (1998) and Duffie (2005) among others for applications of doubly stochastic processes to affine credit-risk models.

a *Markov perfect mixed-strategy equilibrium* if for any $x > 0$, the following conditions hold:

$$J_a(x; \lambda_a^*, \lambda_b^*) \geq J_a(x; \lambda_a, \lambda_b^*), \quad \forall\, (\lambda_a, \lambda_b^*) \in \Phi, \tag{25}$$

$$J_b(x; \lambda_a^*, \lambda_b^*) \geq J_b(x; \lambda_a^*, \lambda_b), \quad \forall\, (\lambda_a^*, \lambda_b) \in \Phi. \tag{26}$$

Let $V_i(x)$ denote firm i's equilibrium value function: $V_i(x) = J_i(x; \lambda_a^*, \lambda_b^*)$.

4.2 Closed-Form Markov Perfect Mixed-Strategy Equilibrium

In this subsection, we first discuss the economic mechanism underlying a firm's entry decision and then we provide a mathematical proof of our equilibrium solution by extending the variational inequality method for a single firm's entry problem to our duopoly setting.

For a given mixed strategy pair $(\lambda_a(x), \lambda_b(x))$, the following HJB equation for firm i's value, $J_i(x) = J_i(x; \lambda_a(x), \lambda_b(x))$, holds:

$$rJ_i(x) = \frac{\sigma^2 x^2}{2} J_i''(x) + \mu x J_i'(x) + \lambda_i(x)[L(x) - K_L - J_i(x)] + \lambda_{-i}(x)[F(x) - J_i(x)]\,, \tag{27}$$

where $L(x)$ is given by (16)-(17) and $F(x)$ is given by (13)-(14). The intuition for the HJB equation (27) is as follows. The first two terms on the right side capture the standard diffusion and drift effects of X on $J_i(x)$. The third term describes the effect of firm i's own mixed (entry) strategy on its value and this term equals zero in equilibrium as a rational firm will only mix with strictly positive probabilities between two strategies that yield the same value.[25] The last term in (27) describes the effect of the competitor's mixed entry strategy on firm i's value. If the competitor enters, firm i becomes Follower and its value function jumps from $J_i(x)$ to $F(x)$. The firm's optimality requires that the sum of these four terms on the right side equals the annualized firm value $rJ_i(x)$ (Duffie, 2001).

It is worth noting although X is continuous, firm value is discontinuous and jumps when its competitor enters the market. This is an example where strategic interactions generate endogenous uncertainty (jump shocks) because firms play mixed strategies in equilibrium.

Next, we turn to the symmetric Markov perfect equilibrium. Let $\lambda^*(x) = \lambda_a^*(x) = \lambda_b^*(x)$ denote the symmetric equilibrium Markov perfect mixed strategy. Equation (18) and

[25]Otherwise, it is always better for the firm to play the pure strategy (waiting or entering) that yields the higher value: $\max\{L(x) - K_L, J_i(x)\}$. Despite in equilibrium this term is zero, we leave it in the HJB equation (27) to better understand the economic mechanism.

inequality (20) together imply the following for $V_i(x)$, firm i's equilibrium value function:

$$L(x) - K_L \le V_i(x) \le F(x)\,, \quad x > 0\,. \tag{28}$$

That is, *ex ante* firm i's value must be weakly larger than $L(x) - K_L$, Leader's net payoff upon entry at τ_L, and weakly lower than Follower's value $F(x)$ because the second-mover advantage globally dominates at all $x > 0$.

There are two scenarios to consider: 1.) $\lambda^*(x) > 0$ and 2.) $\lambda^*(x) = 0$. When $\lambda^*(x) > 0$, the firm must be indifferent between entering the market (becoming Leader) and waiting. That is, the value functions from the two strategies must equal:

$$V_i(x) = L(x) - K_L \quad \text{if} \quad \lambda^*(x) > 0\,, \tag{29}$$

which implies

$$\lambda^*(x) = 0 \quad \text{if} \quad V_i(x) > L(x) - K_L\,. \tag{30}$$

Using (27) and (29), we obtain the following HJB equation for $V_i(x)$:

$$rV_i(x) = \frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) + \lambda^*(x)[F(x) - V_i(x)]\,, \tag{31}$$

which holds for both $\lambda^*(x) > 0$ and $\lambda^*(x) = 0$ cases. Re-arranging (31) yields the following expression for $\lambda^*(x)$ for all $x > 0$:

$$\lambda^*(x) = \frac{rV_i(x) - \left[\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x)\right]}{F(x) - V_i(x)}\,. \tag{32}$$

When $\lambda^*(x) > 0$, substituting $V_i(x) = L(x) - K_L$ given in (29) into (32), we obtain

$$\lambda^*(x) = \frac{rL(x) - \left[\frac{\sigma^2 x^2}{2} L''(x) + \mu x L'(x)\right] - rK_L}{F(x) - (L(x) - K_L)}\,. \tag{33}$$

Since we have closed-form solutions for $L(x)$ as given by (16)-(17) and $F(x)$ as given by (13)-(14), we have an explicit formula for the equilibrium entry rate $\lambda^*(x)$.

The mechanism inducing firms to enter probabilistically in our Case A is related to that causing firms to exit probabilistically in war-of-attrition games (see Levin (2004) for a PhD teaching note on wars of attrition). Unlike standard war-of-attrition exit games, ours is an *entry timing* game with stochastic payoffs. In our Case A, when market demand x is sufficiently high, entering as Leader is profitable but Leader is not the winner but rather the loser of the game in the war-of-attrition sense as Follower's value is higher than Leader's net payoff $F(x) > L(x) - K_L$ at all levels of x. Also, because of irreversible entry and stochastic market demand, the option value of waiting is another key force in our model.

To complete our model solution, we still need to solve $V_i(x)$ in the $\lambda^*(x) = 0$ region and characterize the $\lambda^*(x) > 0$ region. We show that $V_i(x)$ for $x > 0$ is the unique solution for the following variational inequality (See Appendix B):

$$\max\left\{\frac{\sigma^2 x^2}{2}V_*''(x) + \mu x V_*'(x) - rV_*(x), (L(x) - K_L) - V_*(x)\right\} = 0\,, \tag{34}$$

subject to the following boundary conditions: [26]

$$V_*(x) = 0 \quad \text{at} \quad x = 0, \tag{35}$$

$$V_*(x) - (L(x) - K_L) \to 0 \quad \text{as } x \to \infty\,. \tag{36}$$

The variational inequality (34) is analogous to that of a monopolist's real option problem, but their economic implications are different. Mathematically, we generalize the variational-inequality analysis in standard real-option models to our mixed-strategy equilibrium model.

Solving the variational-inequality problem (34)-(36), we obtain firm i's equilibrium value $V_i(x) = V_*(x)$. Using (30), (33), and $V_i(x) = V_*(x)$, we obtain the equilibrium entry rate $\lambda^*(x)$. We summarize these results in the following theorem:

Theorem 1 *For* **Case A** *where $R > R_{AB}$, there exists a symmetric Markov perfect equilibrium. In this equilibrium, $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is the unique solution for the variational-inequality problem* (34)-(36) *in the $x \geq 0$ domain. The equilibrium strategy is given by $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, where $\lambda^*(x) = 0$ in the $V_*(x) > L(x) - K_L$ region and firms enter probabilistically at the strictly positive rate of $\lambda^*(x)$ given in* (33) *in the $V_*(x) = L(x) - K_L$ region.*

While Theorem 1 fully describes the solution for Case A where $R > R_{AB}$, we can further divide Case A into two subcases: Subcase A_1 and Subcase A_2, depending on whether the

[26]Because $X = 0$ is an absorbing state for a geometric Brownian motion process X, firm value must be zero as stated in (35). The condition given in (36) follows from the equilibrium result that firms must enter probabilistically when demand is sufficiently high.

entry-cost ratio $R = K_L/K_F$ is larger or smaller than $R_{A_1A_2}$ given by[27]

$$R_{A_1A_2} = \left(\frac{2^\beta - 1}{\beta}\right)^{\frac{1}{\beta-1}} . \tag{37}$$

It is straightforward to show $R_{A_1A_2} > R_{AB}$, where R_{AB} is given in (19).

In Subcase A$_1$, $R > R_{A_1A_2}$ holds and in Subcase A$_2$, $R_{AB} < R \leq R_{A_1A_2}$ holds. In Subsection 4.3, we obtain explicit solutions for firm's value $V_i(x)$ and $\lambda^*(x)$ for these two subcases. As we will show, even though there is only a second-mover advantage in equilibrium for both subcases, the equilibrium strategies are quite different for the two subcases.

4.3 Two Subcases: Subcase A$_1$ and Subcase A$_2$

First, we solve Subcase A$_1$ where $R > R_{A_1A_2}$ in closed form for $V_i(x)$ and $\lambda^*(x)$.

4.3.1 Subcase A$_1$: $R > R_{A_1A_2}$

Solution. For Subcase A$_1$, there exists a threshold $\overline{x}$ dividing the $x > 0$ real line into two regions: 1.) the waiting region where $x < \overline{x}$ and $V_*(x) > L(x) - K_L$ and 2.) the probabilistic entry region where $x \geq \overline{x}$ and $V_*(x) = L(x) - K_L$. The variational inequality (34) is simplified to the following ordinary differential equation (ODE) in the waiting ($x < \overline{x}$) region:

$$\frac{\sigma^2 x^2}{2} V_*''(x) + \mu x V_*'(x) - r V_*(x) = 0 \ , \tag{38}$$

subject to the following value-matching and smooth-pasting conditions at the threshold $\overline{x}$:

$$V_*(x) \ = \ L(x) - K_L, \tag{39}$$

$$V_*'(x) \ = \ L'(x) \, . \tag{40}$$

While these two boundary conditions resemble the standard value-matching and smooth-pasting conditions for a single firm's optimal threshold in the standard models, the economics underpinning (39)-(40) is different from standard real-option models, which we discuss later.

A key result for Subcase A$_1$ is that Follower enters immediately after Leader does in that $\tau_F^* = \tau_L^*+$. This implies that Leader earns no monopoly rents in equilibrium. Therefore, in

[27] We obtain $R_{A_1A_2} = K_{A_1A_2}/K_F$ by solving for $K_{A_1A_2}$, which is the unique root of the following equation for $K_L \in (K_F, 2K_F)$: $\frac{L(x_M)-K_L}{x_M^\beta} = \frac{L(2x_M)-K_L}{(2x_M)^\beta}$.

the probabilistic entry region, Leader's NPV netting out of its entry cost is given by

$$V_*(x) \;=\; \frac{\Pi(x)}{2} - K_L\,, \quad x \geq \overline{x}\,. \tag{41}$$

Using Leader's linear net payoff function at entry in the $x \geq \overline{x}$ region, (41), and solving the ODE (38) in the $x < \overline{x}$ region subject to the value-matching and smooth-pasting conditions, (39) and (40), we obtain the closed-form expressions for $V_*(x)$:

$$V_*(x) \;=\; \left(\frac{x}{\overline{x}}\right)^{\beta}\left(\frac{\Pi(\overline{x})}{2} - K_L\right)\,, \quad x < \overline{x}\,, \tag{42}$$

where the threshold $\overline{x}$ is given by

$$\overline{x} \;=\; \frac{2\beta}{\beta-1}(r-\mu)K_L = 2x_M\,. \tag{43}$$

Note that $\overline{x}$ is the lower bound for Leader's optimal (probabilistic) entry region. In our model, even when $X_t \geq \overline{x}$, the firm may still be waiting as firms play mixed entry strategies.

Now we verify the equilibrium result that as soon as one firm enters probabilistically, the other also immediately enters. This is because $\overline{x} = 2x_M > x_F$ which follows from a comparison of (43) for $\overline{x}$ and (15) for x_F under the second-mover advantage: $R > R_{A_1A_2} > 1$. Finally, substituting (14) and (17) into (33) gives the equilibrium entry rate:

$$\lambda^*(x) = \frac{x/2 - rK_L}{K_L - K_F} > 0\,, \quad x \geq \overline{x}. \tag{44}$$

The numerator in (44) equals firm i's *net income*, which equals the equilibrium profit for a duopoly, $x/2$, minus rK_L, the interest expense of financing the entry cost K_L. The entry rate $\lambda^*(x)$ increases linearly with x for $x \geq \overline{x}$ and approaches ∞ as $x \to \infty$. Intuitively, the higher the market demand x the more likely a firm enters so as to end the waiting game and collect profits sooner. Next we summarize the solution for Subcase A$_1$ where $R > R_{A_1A_2}$.

Proposition 2 *For* **Subcase A$_1$** *($R > R_{A_1A_2}$), there exists a symmetric Markov perfect mixed-strategy equilibrium where the threshold $\overline{x}$ given in (43) divides the $x > 0$ real line into two solution regions. In the $x < \overline{x}$ region, firms wait ($\lambda^*(x) = 0$) and $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is given in (42). In the $x \geq \overline{x}$, region, firms enter probabilistically at the same rate, $\lambda^*_a(x) = \lambda^*_b(x) = \lambda^*(x)$, where $\lambda^*(x)$ is given in (44), and $V_a(x) = V_b(x) = V_*(x)$, where $V_*(x)$ is given in (41). As soon as one firm enters, the other enters immediately: $\tau^*_F = \tau^*_L+$.*

The *ex ante* probability that either firm becomes Leader is one half.

Throughout our paper, we use figures to supplement our formal analysis to further deepen our understanding of the economic mechanism. We start with Subcase A_1. First, we discuss our choices of parameter values for the figures.

Parameter Choices. Our duopoly model has five parameters in total. First, for the risk-free rate (r), the expected growth rate (drift) of the profit process (μ), and the volatility of the profit growth rate (σ), we choose commonly used values: $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$ per annum, following the standard practice in real-options and contingent-claim literature, e.g., Grenadier (1996) and Leland (1994). The implied optionality measure given in (7) is $\beta = 1.70$. Substituting $\beta = 1.70$ into (37) for $R_{A_1A_2}$ and (19) for R_{AB}, we obtain $R_{A_1A_2} = 1.49$ and $R_{AB} = 1.3$ that split Case A into Subcase A_1 and Subcase A_2.

Which subcase our model solution falls into only depends on two measures: (1.) the optionality measure β given in (7), which determines the cutoff values $R_{A_1A_2}$ and R_{AB}, and (2.) the entry-cost ratio $R = K_L/K_F$, which determines how strong the second-mover advantage is. Finally, we set Follower's entry cost to $K_F = 0.5$.

Graphical Illustration. To demonstrate the economics of Subcase A_1, it is necessary and sufficient to choose a value of $R = K_L/K_F$ that is larger than $R_{A_1A_2} = 1.49$. We choose $R = 1.6$, which implies Leader's entry cost of $K_L = 0.8$.

In Panel A of Figure 4, we highlight how to graphically pin down firm value before Leader is determined (i.e., $t \leq \tau_L^*$). First, we plot $L(x) - K_L$, which is concave in the $x < x_F$ region and linear in the $x \geq x_F$ region (the dashed red line). Second, we plot Follower's value $F(x)$, which is increasing and convex (the magenta dash-dotted line). Because $L(x) - K_L < F(x)$ holds for all $x > 0$, neither firm wants to be Leader with probability one. Moreover, firm value $V_i(x)$ must satisfy $L(x) - K_L \leq V_i(x) \leq F(x)$ in equilibrium.

Third, we pin down firm value $V_a(x) = V_b(x) = V_*(x)$ by smoothly pasting a convex curve (from the origin) onto the $L(x) - K_L$ payoff line. Doing so determines the endogenous threshold $\overline{x}$ (the solid black dot): To the left of $\overline{x}$ is the increasing convex $V_i(x)$ (the black solid line) and to the right of $\overline{x}$ is the straight net payoff line $L(x) - K_L = \Pi(x)/2 - K_L$ (the blue solid straight line).

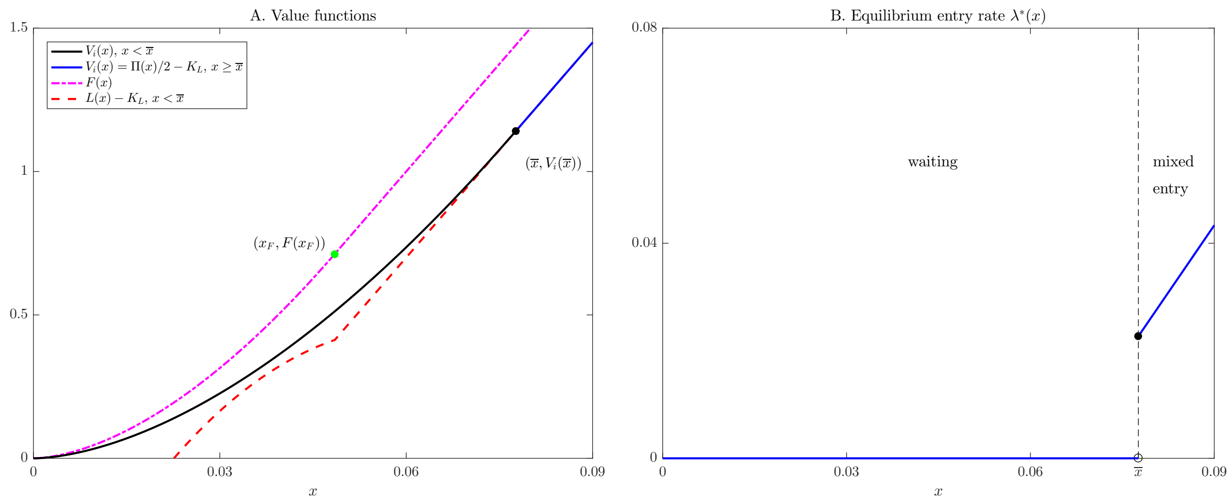

Figure 4: Value functions $V_a(x) = V_b(x) = V_*(x)$ and entry rates $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$ in the symmetric Markov perfect mixed-strategy equilibrium in Subcase A_1. The threshold value dividing the market demand x into the two regions is: $\overline{x} = 2x_M$, where x_M is the entry threshold for a monopolist with an entry cost of K_L. The two regions are: 1.) the $x < \overline{x}$ waiting region and 2.) the $x \geq \overline{x}$ probabilistic entry region. Panel A plots pre-entry firm value $V_*(x)$ (solid line), Follower's value $F(x)$ (dash dotted line), and Leader's net payoff upon entry $L(x) - K_L$ (dashed line). Panel B plots the equilibrium entry rates: $\lambda^*(x) = \frac{x/2 - rK_L}{K_L - K_F}$ in the $x \geq \overline{x}$ region and $\lambda^*(x) = 0$ in the $x < \overline{x}$ waiting region. Parameter values are $R = 1.6$, $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$, which imply $R_{A_1A_2} = 1.49$, $R_{AB} = 1.3$, $x_F = 0.0485$, and $\overline{x} = 2x_M = 0.0776$.

Panel B of Figure 4 plots the equilibrium entry rate $\lambda^*(x)$ that supports $V_i(x) = V_*(x)$ obtained in panel A. The vertical dashed line in panel B divides the solution into two regions. To the left of $\overline{x}$ is the waiting region where $\lambda^*(x) = 0$. To the right of $\overline{x}$ is the probabilistic entry region where $\lambda^*(x) = \frac{x/2 - rK_L}{K_L - K_F}$. The mixed-strategy equilibrium is a compromised outcome between the two firms. As a firm waits for the other to enter, it forgoes the opportunity of collecting profits $x/2 - rK_L$, but preserves the option value of being the second mover and saving $\Delta K = K_L - K_F$. In equilibrium, the linear $\lambda^*(x)$ entry rate makes the firm indifferent between entering and waiting. The higher the value of x, the higher the costs of forgoing one-period profit and thus the more likely it enters to end the game sooner.

Finally, we point out that the lower bound of x that firms are willing to probabilistically enter, $\overline{x}$ given in (43), equals twice the value of x_M, the entry threshold of a monopolist (with an entry cost of K_L). This is because in equilibrium Leader only collects $X_t/2$ and never enjoys monopoly rents. Next, we turn to Subcase A_2.

4.3.2 Subcase A_2: $R \in (R_{AB}, R_{A_1A_2}]$

Solution Overview. Compared with Subcase A_1 where $R > R_{A_1A_2}$, the entry-cost ratio R for Subcase A_2 is lower but still larger than one and lies in the region $(R_{AB}, R_{A_1A_2}]$. Although the second-mover advantage for Subcase A_2 is lower than that for Subcase A_1, firms still prefer to be Follower for all $x > 0$.

The solution for Subcase A_2 is richer and subtler than for Subcase A_1 and can still be obtained in closed form. There are four regions in equilibrium for Subcase A_2: two disconnected waiting regions and two disconnected probabilistic entry regions. There are two waiting regions because there are two distinct waiting motives: one is due to the standard option value and the other is due to the second-mover advantage. Depending on the market demand x, we have two probabilistic entry regions: When x is very high, Follower enters immediately after Leader leaving no monopoly rents for Leader. When x is in the intermediate range, we also have a mixed strategy equilibrium but Follower voluntarily waits so that Leader can make enough monopoly profits for a sufficiently long period.

Next, we obtain the four-region solution for firm value $V_*(x)$ by using smooth-pasting conditions with the net payoff function being $L(x) - K_L$. Doing so, we obtain three cutoff values of market demand x: $\widetilde{x}$, $\underline{x}$, and $\overline{x}$ that define the four regions. Without loss of generality, let $0 < \widetilde{x} < \underline{x} < \overline{x}$. Why do we have four regions (with three smooth-pasting conditions in Subcase A_2) rather than two regions (with only one smooth-pasting condition in Subcase A_1)? Geometrically, this is because the second-mover advantage (R) is smaller in Subcase A_2 than in Subcase A_1, which leaves less wiggle room between $F(x)$ and $L(x) - K_L$, leading $V_*(x)$ to paste onto $L(x) - K_L$ at three points.

Consequently, $[\widetilde{x}, \underline{x}]$ and $[\overline{x}, \infty)$ are the two probabilistic regions where $\lambda^*(x) > 0$, while $(0, \widetilde{x})$ and $(\underline{x}, \overline{x})$ are the two waiting regions where $\lambda^*(x) = 0$. Solving (38) in the $(0, \widetilde{x})$ and $(\underline{x}, \overline{x})$ waiting regions subject to the value-matching and smooth-pasting conditions (39)-(40)

at $x = \widetilde{x}$, $\underline{x}$, and $\overline{x}$, yields the four-region solution, which we discuss in detail below.[28]

Four-Region Solution. First, in the standard waiting region where $x \in [0, \widetilde{x})$, $V_*(x)$ is convex and given by

$$V_*(x) = \left(\frac{x}{\widetilde{x}}\right)^\beta (L(\widetilde{x}) - K_L) \,, \quad x \in [0, \widetilde{x}) \,. \tag{45}$$

Note that firm value equals the product of (a.) the present value of a dollar paid at $\tau := \inf\{s : X_s = \widetilde{x}\}$ and (b.) the net payoff if the firm enters at τ as Leader. We can prove that the threshold below which waiting is a firm's dominant strategy, $\widetilde{x}$, equals x_M, the entry threshold of a monopolist with an entry cost of K_L, given by (6):

$$\widetilde{x} = x_M \,. \tag{46}$$

Second, in the $x \geq \overline{x}$ region, as soon as one firm enters as Leader, the other follows immediately because the market demand is sufficiently high (and formally $x_F < \overline{x}$). This mechanism is the same as the one for the probabilistic entry region in Subcase A_1. Therefore, firm value (in this probabilistic entry region) is also linear, $V_*(x) = \Pi(x)/2 - K_L$, and $\lambda^*(x)$ has the same linear expression as (44) for Subcase A_1. The only difference is the value of $\overline{x}$ in Subcase A_2 is different from $\overline{x} = 2x_M$ given in (43) in Subcase A_1.

Third, in the $x \in [x_M, \underline{x}]$ region, firms enter probabilistically and a firm's pre-entry value is thus given by $V_*(x) = L(x) - K_L$, which is concave. As soon as Leader is determined, the other firm waits and Leader thus collects monopoly rents until $\tau_F^* = \inf\{s : X_s \geq x_F\}$. Technically, this follows from $\underline{x} < x_F$. The equilibrium entry rate function that makes firms indifferent between entering as Leader and waiting in this region is:

$$\lambda^*(x) = \frac{x - rK_L}{F(x) - (L(x) - K_L)}, \quad x \in [x_M, \underline{x}]. \tag{47}$$

The numerator in (47) is the opportunity cost of not collecting by waiting and the denominator in (47) is the net benefit of waiting to be Follower: the value gap between becoming Leader immediately and being Follower.

[28]Mathematically, we can prove that the variational-inequality problem (34)-(36) boils down to the smooth-pasting conditions at $\widetilde{x}$, $\underline{x}$, and $\overline{x}$, which define the four regions. Firm value, $V_*(x)$, is the solution to the variational inequality problem (34)-(36), which satisfies the ordinary differential equation (38) in the two waiting regions $(0, \widetilde{x})$ and $(\underline{x}, \overline{x})$.

Fourth, in the $x \in (\underline{x}, \overline{x})$ region, firms wait because they prefer to enter as Follower to lower their entry costs. Solving the ODE (38) in this region subject to the value-matching and smooth-pasting conditions (39)-(40) at $\underline{x}$ and $\overline{x}$, we obtain the closed-form expression:

$$V_*(x) \quad = \quad \Theta(x; \underline{x}, \overline{x}), \quad x \in (\underline{x}, \overline{x}) \,, \tag{48}$$

where $\Theta(x; a, b)$ for any $a \le x \le b$ is given by

$$\Theta(x; a, b) = \theta_1(a, b)x^{\beta} + \theta_2(a, b)x^{\gamma} \,, \tag{49}$$

$\beta > 1$ is given in (7), and $\gamma < 0$ is given by[29]

$$\gamma = \frac{-(\mu - \frac{1}{2}\sigma^2) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} \,. \tag{50}$$

Finally, in Lemma 2 of Appendix B, we prove $x_M < \underline{x} < x_F < \overline{x}$ and characterize the pair $(\underline{x}, \overline{x})$ defining the waiting (to be Follower) region. Below we summarize the duopoly model solution for Subcase A_2.

Proposition 3 *For* **Subcase A_2** *($R_{AB} < R \le R_{A_1A_2}$),*[30] *firm i's value function is given by $V_i(x) = V_*(x)$ for all $x > 0$. In the $x < x_M$ and $x \in (\underline{x}, \overline{x})$ regions, $V_*(x)$ is given by (45) and (48), respectively. In the $x \in [x_M, \underline{x}]$ and $x \in [\overline{x}, \infty)$ regions, $V_*(x) = L(x) - K_L$. The cutoff values, $\underline{x}$ and $\overline{x}$, are given in (B.8) via the smooth-pasting conditions in Lemma 2. The symmetric Markov perfect equilibrium strategy is given by $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$. In both $x < x_M$ and $x \in (\underline{x}, \overline{x})$ regions, firms wait: $\lambda^*(x) = 0$. In the $x \in [x_M, \underline{x}]$ and $x \ge \overline{x}$ regions, firms enter stochastically at the rate of $\lambda^*(x) > 0$ given in (47) and (44), respectively.*

After Leader is determined at $\tau_L^* = \tau_a^* \wedge \tau_b^*$, the other firm enters at $\tau_F^* = \inf\{s : X_s \ge x_F\}$, where x_F is given in (15). In the $x \in [x_M, \underline{x}]$ region, Leader earns monopoly rents in the (τ_L^*, τ_F^*) period but in the $x \ge \overline{x}$ region, Leader earns no monopoly rents as $\tau_F^* = \tau_L^* +$.

Graphical Illustration. Next, we use Figure 5 to corroborate our analysis of Subcase A_2. We set the entry-cost ratio at $R = 1.4$, which lies in the $(R_{AB}, R_{A_1A_2}] = (1.30, 1.49]$ region. The implied Leader's entry cost is $K_L = 0.7$.

[29] Mathematically, γ is the smaller (and negative) root of the fundamental quadratic equation, $\sigma^2 z(z - 1)/2 + \mu z - r = 0$, for the GBM X process (1). See (B.2) for the expressions of $\theta_1(a, b)$ and $\theta_2(a, b)$.

[30] R_{AB} and $R_{A_1A_2}$ are given in (19) and (37), respectively.

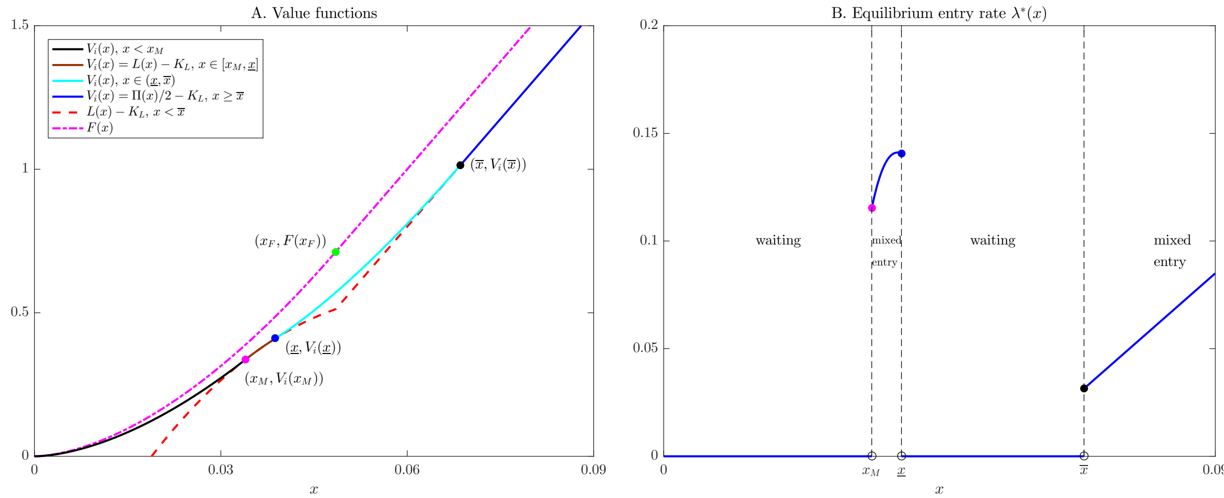

Figure 5: VALUE FUNCTIONS AND ENTRY RATE IN THE SYMMETRIC EQUILIBRIUM IN SUBCASE A_2. Parameter values are $R = 1.4$, $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$, which imply $R_{A_1A_2} = 1.49$, $R_{AB} = 1.3$, $x_M = 0.0340$, $\underline{x} = 0.0388$, $\overline{x} = 0.0686$, and $x_F = 0.0485$.

First, $F(x) > L(x) - K_L$ still holds for all $x > 0$ in Subcase A_2 in that the second-mover advantage dominates globally. Second, we can geometrically fit a smooth curve for $V_i(x)$ at *three* smooth-pasting points: x_M, $\underline{x}$, and $\overline{x}$ with $L(x) - K_L$ being the net payoff line because $K_L = 0.7$ is lower than $K_L = 0.8$ in Subcase A_1. As a result, Figure 5 confirms our model's prediction that as x increases from zero to ∞, a firm finds itself in one of the four mutually exclusive regions: 1.) the first waiting region (to preserve option value); 2.) the first mixed entry strategy (with monopoly profits); 3.) the second waiting region (second-mover advantage); and 4.) the second mixed entry strategy (no monopoly profits).

4.3.3 Summary of Case A

In sum, the second-mover advantage dominates for all $x > 0$ for Case A, where $R > R_{AB} > 1$. As a result, we only have two types of regions: waiting and probabilistic entry regions. This is because entry with probability one is a strictly dominated strategy in a symmetric equilibrium. Depending on whether $R > R_{A_1A_2}$ or not, we have two subcases: Subcase A_1 and Subcase A_2, discussed in detail earlier in this section.

Next, we analyze Case C where $R \leq 1$.

5 Case C: Equilibrium with First-Mover Advantage

In Case C ($R \leq 1$), there is no second-mover advantage. The equilibrium is determined by firms' tradeoff between the first-mover advantage and the option value of waiting.

Let $\mathcal{E}_i \subset (0, \infty)$ denote a closed set associated with firm i's entry strategy: firm i enters at t if and only if $X_t \in \mathcal{E}_i$. Let Φ denote the set of all feasible entry strategies $(\mathcal{E}_a, \mathcal{E}_b)$ and let $J_i(X_t; \mathcal{E}_a, \mathcal{E}_b)$ denote the associated time-t value of firm i defined by (18). Next, we define the pure-strategy equilibrium.

Definition 3 A pair of entry strategy $(\mathcal{E}_a^*, \mathcal{E}_b^*)$ is a *pure-strategy equilibrium* if for any $x > 0$ the following conditions hold:

$$J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq J_a(x; \mathcal{E}_a, \mathcal{E}_b^*), \quad \forall\, (\mathcal{E}_a, \mathcal{E}_b^*) \in \Phi, \tag{51}$$

$$J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq J_b(x; \mathcal{E}_a^*, \mathcal{E}_b), \quad \forall\, (\mathcal{E}_a^*, \mathcal{E}_b) \in \Phi. \tag{52}$$

Let $V_i(x)$ denote firm i's equilibrium value function: $V_i(x) = J_i(x; \mathcal{E}_a^*, \mathcal{E}_b^*)$.

Proposition 1 shows that for Case C where $R \leq 1$, $F(x)$ intersects with $L(x) - K_L$ at $\widehat{x}_L$ and $L(x) - K_L \geq F(x)$ for any $x \geq \widehat{x}_L$. Therefore, in the $x \geq \widehat{x}_L$ region, both firms want to enter as Leader but only one firm can be randomly selected (with 50% probability) to be Leader. This is the *rent equalization principle* of Fudenberg and Tirole (1985) and Grenadier (1996), which implies that the equilibrium firm value for both firms is:

$$V_i(x) = \frac{L(x) - K_L + F(x)}{2} \tag{53}$$

for $x \geq \widehat{x}_L$. In the $x < \widehat{x}_L$ region, firms optimally wait and the equilibrium firm value is: $V_i(x) = F(x)$ as in Grenadier (1996). Next, we summarize the solution in the following theorem.

Theorem 2 *Consider Case C where $R \leq 1$. Let $\widehat{x}_L$ be the unique root of $L(x) - K_L = F(x)$ in the $(0, x_F)$ region for Case C in Proposition 1. Then there exists a pure strategy equilibrium such that firm i's equilibrium value $V_i(x)$ equals $F(x)$, where $F(x)$ is given in (13) in the $x < \widehat{x}_L$ region, and $V_i(x)$ is given by (53) in the $x \geq \widehat{x}_L$ region. Both firms wait in the $x < \widehat{x}_L$ region. In the $x \in [\widehat{x}_L, x_F)$ region, firms compete to enter and one firm is randomly selected*

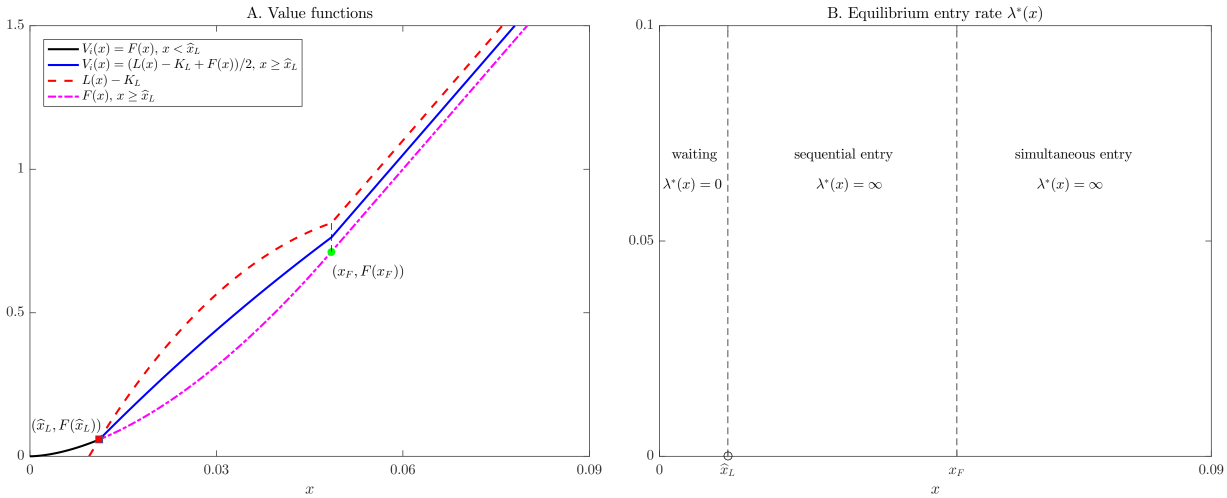

Figure 6: VALUE FUNCTIONS AND ENTRY RATE IN THE SYMMETRIC EQUILIBRIUM IN CASE C. Parameter values are $R = 0.8$, $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$. The two cutoff values of the three x regions are $\widehat{x}_L = 0.0112$ and $x_F = 0.0485$.

to enter immediately as Leader and the other optimally waits until $\tau_F^* = \inf\{s : X_s \geq x_F\}$ to enter as Follower. In the $x \geq x_F$ region, the two firms in effect simultaneously enter with one chosen to be Leader randomly.

Graphical Illustration. Next, we use Figure 6 to highlight the key results of Case C. We set the entry-cost ratio at $R = 0.8 < 1$. The implied Leader's entry cost is $K_L = 0.4$.

First, we note that $L(x) - K_L > F(x)$ holds in the $x > \widehat{x}_L = 0.0112$ region, which implies that the first-mover advantage dominates and both firms want to enter first. To select Leader, we need a randomization device while keeping the *ex ante* rents for the two firms equal (Fudenberg and Tirole, 1985). The solid (blue) line depicts the value function $V_i(x)$ given in (53) in the entry region. To the left of the red square is the $x < \widehat{x}_L = 0.0112$ region, where both firms wait. Note that $\widehat{x}_L = 0.0112 < x_M = 0.0194$. That is, the option value of waiting is eroded as emphasized in Grenadier (1996). Mathematically, we prove $\widehat{x}_L < x_M$ for $R \in (0, 1]$ in Lemma 1.

Second, in the $x \geq x_F$ subregion (recall that $x_F > \widehat{x}_L$ and in our example $x_F = 0.0485 > \widehat{x}_L = 0.0112$), Follower immediately enters after Leader is randomly chosen. This is the "simultaneous entry" region in Panel B of Figure 6. Third, in the $x \in [\widehat{x}_L, x_F)$ subregion, the (lucky) Leader collects monopoly rents until Follower enters when X_t reaches x_F for the first time. This is the "sequential entry" region in Panel B of Figure 6.

In sum, $V_*(x)$ is convex in the $x < \widehat{x}_L$ waiting region, concave in the $x \in [\widehat{x}_L, x_F)$ sequential-entry region, and linear in the $x \geq x_F$ simultaneous-entry region. Note that all entry decisions are pure strategies. Mathematically, there is no smooth-pasting condition involved for Case C, as there is no second-mover advantage and firms compete to be the first mover as soon as Leader's net payoff $L(x) - K_L$ exceeds Follower's value $F(x)$.

Notation-wise, for pure entry strategies, although we do not explicitly refer to equilibrium entry rates, we write $\lambda^*(x) = \infty$ in the entry region and $\lambda^*(x) = 0$ in the waiting region.

6 Case B: First- and Second-mover Advantages

In this section, we analyze Case B where $1 < R \leq R_{AB}$. Because the range of R for Case B lies between that for Case A and for Case C, we expect that both first-mover and second-mover advantages (the key force behind Case A and for Case C, respectively) influence the equilibrium outcomes. Indeed, our analysis of Case B confirms the key results in Case A and Case C and also generates new insights that depend on the *interaction* between the two types of advantages in our real-option context.

Next, we summarize the solution for Case B. In Appendix A, we define the equilibrium involving both pure and mixed strategies.

6.1 Closed-Form Markov Perfect Equilibria

Theorem 3 *Consider Case B where $1 < R \leq R_{AB}$. Let $\widehat{x}_L$ and $\widehat{x}_F$ be the two roots of $L(x) - K_L = F(x)$ in the $(0, x_F)$ region for Case B in Proposition 1.*[31] *Then there exists a symmetric Markov perfect equilibrium with the following properties:*

1. *In the $x \leq \widehat{x}_F$ domain, firms only play pure strategies.*

   (a) *In the $x < \widehat{x}_L$ region, both firms wait and $V_a(x) = V_b(x) = F(x)$.*

[31] If $R = R_{AB}$, the two roots of $L(x) - K_L = F(x)$ are the same and moreover equal x_M, the entry threshold of a (hypothetical) monopolist with entry cost K_L: $\widehat{x}_L = \widehat{x}_F = x_M$.

(b) *In the* $x \in [\widehat{x}_L, \widehat{x}_F]$ *region, firms compete to become Leader with one firm being randomly selected as Leader and* $V_a(x) = V_b(x) = (L(x) - K_L + F(x))/2$.

2. *In the* $x > \widehat{x}_F$ *domain, firms play mixed strategies. Firm value is* $V_a(x) = V_b(x) = V_*(x)$, *where* $V_*(x)$ *is the unique solution to the variational inequality* (34) *in the* $x > \widehat{x}_F$ *domain subject to the boundary conditions:* (36) *as* $x \to \infty$ *and*

$$V_*(\widehat{x}_F) = F(\widehat{x}_F). \tag{54}$$

The equilibrium strategy is $\lambda_a^*(x) = \lambda_b^*(x) = \lambda^*(x)$, *where* $\lambda^*(x) > 0$ *is given by* (33) *in the probabilistic entry region:*

$$\mathcal{R}^E := \{x > \widehat{x}_F : V_*(x) = L(x) - K_L\} \tag{55}$$

and $\lambda^*(x) = 0$ *for any* x *in the* $x > \widehat{x}_F$ *domain but not in* $\mathcal{R}^E$, *i.e.,* $x \in (\widehat{x}_F, \infty) \setminus \mathcal{R}^E$.

Intuitively speaking, the cutoff value $\widehat{x}_F$ divides the total market demand x into two domains: (1.) the $x \leq \widehat{x}_F$ domain where firms play pure strategies in equilibrium as in Case C and (2.) the $x > \widehat{x}_F$ domain where firms play mixed strategies as in Case A. We provide additional discussions of the key results including Follower's strategies in Subsection 6.2 using figures.

As for Case A, there are also two subcases for Case B. Let $R_{B_1B_2}$ denote the level of the entry-cost ratio R that solves $\widehat{x}_F = \underline{x}$, where $\underline{x}$ is given in Lemma 2.[32] The two subcases of Case B are (i) Subcase B_1 where $R_{B_1B_2} < R \leq R_{AB}$ and (ii) Subcase B_2 where $1 < R \leq R_{B_1B_2}$, as shown in Figure 7. The solution for Subcase B_1 features *five* regions and the solution for Subcase B_2 features *four* regions. For both subcases, there are two regions to the left of $\widehat{x}_F$: the $x < \widehat{x}_L$ waiting region and the $x \in [\widehat{x}_L, \widehat{x}_F]$ entry region where firms compete to be Leader and one firm is luckily selected. Theorem 3 summarizes the solutions in the $x < \widehat{x}_L$ and $x \in [\widehat{x}_L, \widehat{x}_F]$ regions which apply to both subcases. Next, we summarize the solutions in the $x > \widehat{x}_F$ domain for the two subcases.

Proposition 4 *The solution in the* $x > \widehat{x}_F$ *domain for* **Case B** *is as follows.*

1. **Subcase B_2** *where* $1 < R \leq R_{B_1B_2}$. *There are two regions (*$x \in (\widehat{x}_F, \overline{x})$ *and* $x \geq \overline{x}$*) where the second-mover advantage dominates. In the* $x \in (\widehat{x}_F, \overline{x})$ *region, both firms*

[32] To be precise, $R_{B_1B_2} = \sup\{R \in (1, R_{AB}) : \underline{x} \leq \widehat{x}_F\}$.

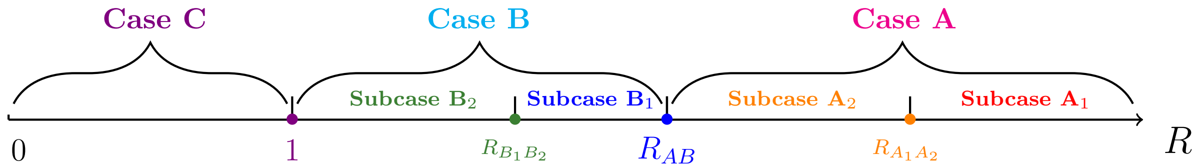

Figure 7: This figure summarizes all cases of the duopoly model solution with four entry-cost ratio ($R = K_L/K_F$) thresholds, $R_{A_1A_2} > R_{AB} > R_{B_1B_2} > 1$: Subcase A$_1$: $R > R_{A_1A_2}$; Subcase A$_2$: $R_{AB} < R \leq R_{A_1A_2}$; Subcase B$_1$: $R_{B_1B_2} < R \leq R_{AB}$; Subcase B$_2$: $1 < R \leq R_{B_1B_2}$; and Case C: $R \leq 1$.

wait and firm i's value is $V_i(x) = \Theta(x; \widehat{x}_F, \overline{x})$, where $\Theta(x; a, b)$ for any $x \in [a, b]$ is given by (49) and $\overline{x}$ is given in case (ii) in Lemma 4. In the $x \geq \overline{x}$ region, both firms enter probabilistically at the rate of $\lambda^*(x) > 0$ given in (44) and firm i's value is $V_i(x) = \Pi(x)/2 - K_L$.

2. **Subcase B$_1$** *where $R_{B_1B_2} < R \leq R_{AB}$. There are three regions ($x \in (\widehat{x}_F, \underline{x}]$, $x \in (\underline{x}, \overline{x})$, and $x \geq \overline{x}$) where the second-mover advantage dominates. In the $x \in (\widehat{x}_F, \underline{x}]$ region, both firms enter probabilistically at the rate of $\lambda^*(x) > 0$ given in (47), and firm i's value is given by $V_i(x) = L(x) - K_L$. In the $x \in (\underline{x}, \overline{x})$ region, both firms wait and firm i's value is $V_i(x) = \Theta(x; \underline{x}, \overline{x})$, where $\Theta(x; a, b)$ for any $x \in [a, b]$ is given by (49) and the cutoffs $\underline{x}$ and $\overline{x}$ are given in Lemma 2. In the $x \geq \overline{x}$ region, both firms enter probabilistically at the rate of $\lambda^*(x) > 0$ given in (44) and firm i's value is $V_i(x) = \Pi(x)/2 - K_L$.*

6.2 Comparing Subcase B$_1$ vs Subcase B$_2$: Graphical Illustration

For the triplet (r, μ, σ), we use the same (annualized) parameter values as for Case A in Subsection 4.3: $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$. This triplet (r, μ, σ) pins down the optionality measure $\beta = 1.70$, the cutoff value between Case A and Case B, $R_{AB} = 1.30$, and the cutoff value for the two subcases of Case B, $R_{B_1B_2} = 1.19$. We choose $R = 1.28 \in (R_{B_1B_2}, R_{AB}) = (1.19, 1.30)$ to illustrate the economics of Subcase B$_1$ and $R = 1.18 \in (1, R_{B_1B_2}) = (1, 1.19)$ to illustrate the economics of Subcase B$_2$.

In Panels A and B of Figure 8, we plot the solution for Subcase B$_2$. First, by intersecting

$L(x) - K_L$ with $F(x)$ as in Case C, we obtain the two regions on the left: (1.) The $x < \widehat{x}_L = 0.021$ region where firms wait to preserve the option value and (2.) the $x \in [\widehat{x}_L, \widehat{x}_F] = [0.021, 0.042]$ region where firms compete to enter as Leader as discussed earlier. Graphically, we determine the remaining parts of our model solution by smoothly pasting a curve starting from the magenta square at $(\widehat{x}_F, F(\widehat{x}_F))$ onto Leader's net payoff line $L(x) - K_L$. This convex curve is the equilibrium firm value $V_*(x)$ for $x > \widehat{x}_F$ where the second-mover advantage dominates. Moreover, the smooth-pasting condition at $x = \overline{x}$ divides the $x > \widehat{x}_F$ domain into two regions: the $x \geq \overline{x}$ region where firms play mixed entry strategies and the $(\widehat{x}_F, \overline{x})$ region where firms wait to lower entry costs.

In sum, for Subcase B_2, firms have four strategies: 1) waiting for the standard option value reason (subject to entry competition) in the $x < \widehat{x}_L$ region where $\lambda^*(x) = 0$ as shown in panel B; 2.) competing to enter as Leader due to the first-mover advantage in the $x \in [\widehat{x}_L, \widehat{x}_F]$ region where $\lambda^*(x) = \infty$; 3.) waiting with the hope of becoming Follower to lower entry costs in the $x \in (\widehat{x}_F, \overline{x})$ region where $\lambda^*(x) = 0$; and 4.) entering probabilistically in the $x \geq \overline{x}$ region. The first two regions resemble the solution for Case C while the latter two regions resemble the solution in Subcase A_1. Note that the threshold, $\overline{x}$, dividing the two regions where the second-mover advantage dominates, depends on the threshold: $\widehat{x}_F$. That is, there is a feedback effect from the first-mover advantage to the second-mover advantage.

Panel B of Figure 8 plots the equilibrium entry rates. We emphasize that in the fourth region where $x \geq \overline{x}$, firms probabilistically enter at the rate of $(x/2 - rK_L)/(K_L - K_F)$ as Follower immediately enters after Leader does and therefore Leader enjoys no monopoly rents. This is the same as in the $x \geq \overline{x}$ region of Subcase A_1.

Panels C and D of Figure 8 plot the solution for Subcase B_1. Compared with Subcase B_2 (see Panels A and B of Figure 8), there is a new fifth region for Subcase B_1: the probabilistic entry region where $x \in (\widehat{x}_F, \underline{x}]$. Unlike the probabilistic entry region $x \geq \underline{x}$ where Follower immediately enters after Leader and hence Leader enjoys no monopoly rents, Leader enjoys a (stochastic) period of monopoly rents in the $x \in (\widehat{x}_F, \underline{x}]$ region.

Also note that $V_i(x)$ is concave in the $x \in (\widehat{x}_F, \underline{x}]$ region, while convex in the $x \geq \overline{x}$ region. This new $(\widehat{x}_F, \underline{x}]$ region arises in Subcase B_1 as we increase the second-mover advantage measured by R. The intuition is as follows. As R increases for a fixed K_F, the $[\widehat{x}_L, \widehat{x}_F]$ region becomes narrower,[33] leaving more room for Leader to earn monopoly rents (as $\widehat{x}_F$ is further to the left of Follower's entry threshold x_F). As a result, a symmetric equilibrium where firms play mixed entry strategies in the $(\widehat{x}_F, \underline{x}]$ region becomes feasible. This explains why we have a new (fifth) region as we move from Subcase B_2 to Subcase B_1.

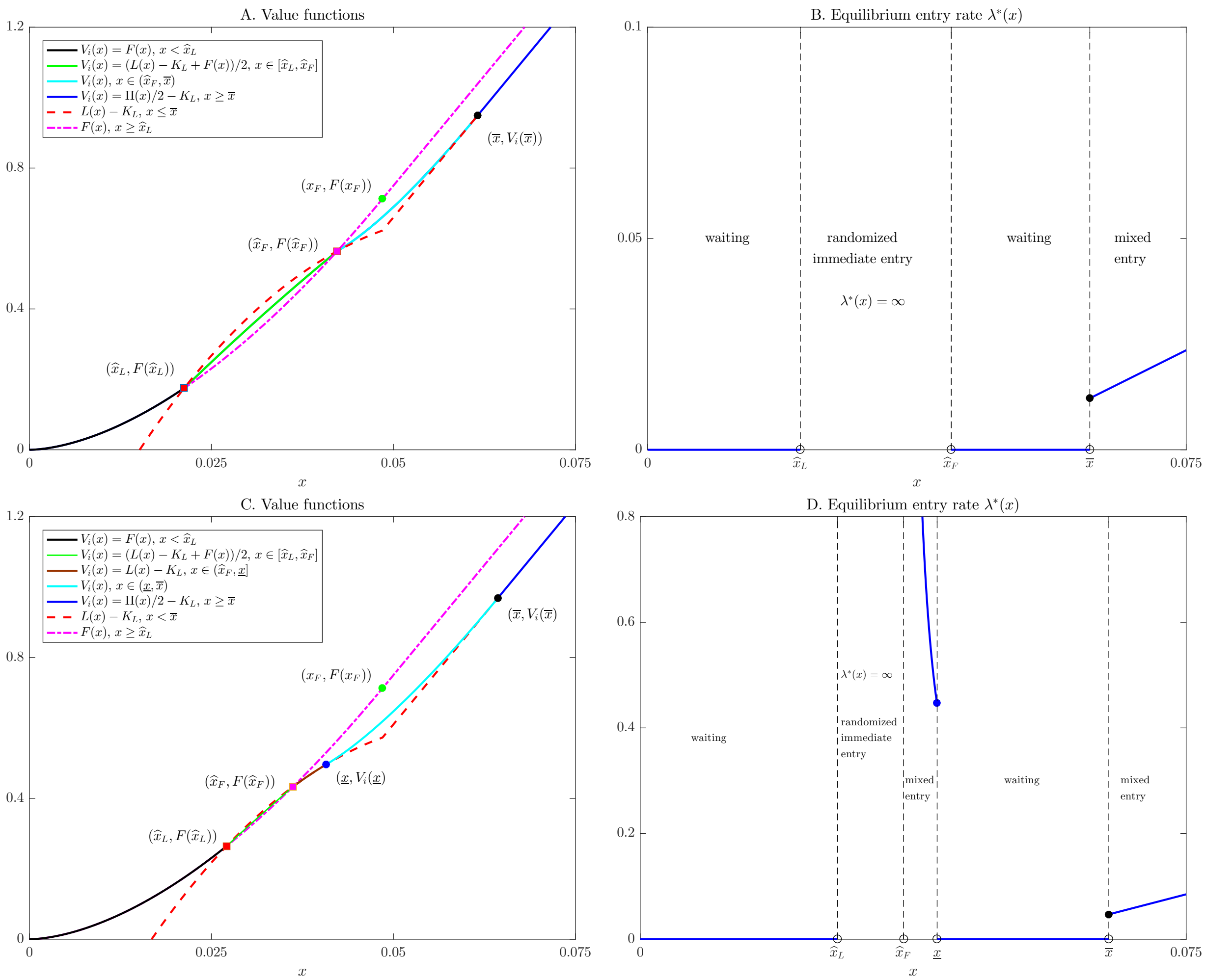

Figure 8: VALUE FUNCTIONS AND ENTRY RATES IN THE SYMMETRIC EQUILIBRIA OF SUBCASE B_2 (PANELS A-B) AND SUBCASE B_1(PANELS C-D). For both subcases, we set $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$, which imply $R_{AB} = 1.3$, $R_{B_1B_2} = 1.19$, and $x_F = 0.0485$. We choose $R = 1.18$ for Subcase B_2 and $R = 1.28$ for Subcase B_1. The three cutoff values defining the four regions in Subcase B_2 are $\widehat{x}_L = 0.021$, $\widehat{x}_F = 0.042$, and $\overline{x} = 0.062$. The four cutoff values defining the five regions in Subcase B_1 are $\widehat{x}_L = 0.027$, $\widehat{x}_F = 0.036$, $\underline{x} = 0.041$, and $\overline{x} = 0.064$.

[33] Lemma 1 shows that $\widehat{x}_L$ increases and $\widehat{x}_F$ decreases with R for Case B.

Panel D plots the equilibrium entry rate $\lambda^*(x)$ in the $(\widehat{x}_F, \underline{x}]$ region, which equals the forgone monopoly profit $(x - rK_L)$ divided by the net benefit of being Follower: $F(x) - (L(x) - K_L)$. Because of embedded optionality, $\lambda^*(x)$ given in (47) is highly nonlinear.

6.3 Summary of Case B: $1 < R \leq R_{AB}$

Case B is the most general case where both the first-mover and second-mover advantages are present. Depending on the value of R, the solution fits into Subcase B_1 or Subcase B_2.

For Subcase B_2 where $1 < R \leq R_{B_1B_2}$, there are four regions: two disconnected waiting regions (one to preserve the option value and the other to lower entry costs), the pure entry strategy region where rents are equalized (Fudenberg and Tirole, 1985), and the probabilistic entry region where Leader enjoys no monopoly rents in equilibrium.

For Subcase B_1 where $R_{B_1B_2} < R \leq R_{AB}$, we have a new (fifth) region in addition to the four regions as in Subcase B_2. This new region appears between the $[\widehat{x}_L, \widehat{x}_F]$ region where the first-mover advantage dominates and the second waiting region $(\underline{x}, \overline{x})$ where firms want to lower their entry costs. In this new region, once a firm becomes Leader, it enjoys monopoly rents for a stochastic duration as the other firm chooses to wait and therefore $V_i(x)$ is concave.[34] Finally, we emphasize that the *interaction* between the two types of advantages in our real-option context fundamentally alters how these five regions are determined and connected. For example, firms may enter in one of three different ways: pure strategy, probabilistic entry with or without monopoly rents (of stochastic duration). Moreover, firm entry is not monotonic as market demand increases. In sum, game-theoretic considerations when both first- and second-mover advantages are present fundamentally enrich the equilibrium real-option exercising decisions and firm valuation.

7 Pure- vs Mixed-strategy Equilibria: A Comparison

In this section, we first analyze the pure-strategy equilibria and then compare them with the mixed-strategy equilibrium. We further analyze and quantify the distribution of time to

[34] The insights for the other four regions are similar to those for the four regions of Subcase B_2.

Leader's entry $\tau_L^* - t$ and value losses for an industry as a whole. We find that the second-mover advantage fundamentally changes both the qualitative and quantitative implications of duopoly competition. For brevity, we focus on Subcase A_1 where $R > R_{A_1 A_2}$.

7.1 Solution for Pure-strategy Equilibria

Consider the pure-strategy equilibrium where firm a is Leader and firm b is Follower.[35] Let $P_L(x)$ denote Leader's value in this equilibrium. Firm a solves the following problem:

$$P_L(x) = \max_{\tau \geq t} \mathbb{E}_t^x \left[e^{-r(\tau - t)} (L(X_\tau) - K_L) \right] . \tag{56}$$

Let x_L denote firm a's optimal entry threshold. First, we show that Leader's value $P_L(x)$ in this pure-strategy equilibrium equals firm value $V_i(x) = V_*(x)$ in the mixed-strategy equilibrium: $P_L(x) = V_*(x)$ where $V_*(x)$ is given in (41)-(42). Second, Leader's optimal entry time is: $\tau_L^* = \inf\{s \geq t : X_s \geq x_L\}$, where Leader's optimal entry threshold x_L equals the cutoff in the mixed-strategy equilibrium $\overline{x}$: $x_L = \overline{x}$ and $\overline{x}$ is given in (43). Third, Follower enters at $\tau_F^* = \inf\{s \geq \tau_L^* : X_s \geq x_F\}$, where x_F is given in (15). Because $\overline{x} = 2x_M > x_F$, Follower enters immediately after Leader does: $\tau_F^* = \tau_L^*+$. Hence, Follower's value, $P_F(x)$, is

$$P_F(x) = \mathbb{E}_t^x \left[e^{-r(\tau_L^* - t)} (\frac{\Pi(X_{\tau_L^*})}{2} - K_F) \right] , \tag{57}$$

where $\tau_L^* = \inf\{s \geq t : X_s \geq \overline{x}\}$. Solving (57), we obtain the following closed-form solutions:

$$P_F(x) = F(x) = \Pi(x)/2 - K_F, \quad x \geq \overline{x}, \tag{58}$$

$$P_F(x) = (x/\overline{x})^\beta F(\overline{x}) = (x/\overline{x})^\beta (\Pi(\overline{x})/2 - K_F), \quad x < \overline{x} . \tag{59}$$

Next, we summarize the key results for the pure-strategy equilibria.

Theorem 4 *There are two asymmetric pure-strategy equilibria for Subcase A_1 where $R > R_{A_1 A_2}$. Leader enters at $\tau_L^* = \inf\{s \geq t : X_s \geq x_L\}$ where $x_L = \overline{x}$ as given in* (43) *and Leader's value is $P_L(x) = V_*(x)$, where $V_*(x)$ is given in* (41)-(42)*. Because Follower's entry threshold x_F is lower than Leader's threshold x_L: $x_L = \overline{x} > x_F$, Follower enters immediately after Leader ($\tau_F^* = \tau_L^*+$) and Follower's value $P_F(x)$ is given by* (58)-(59)*.*

[35] This equilibrium is supported by beliefs that firm a is Leader and firm b is Follower with probability one. Making firm a Follower and firm b Leader, we obtain the other pure-strategy equilibrium.

7.2 Comparing Mixed-strategy with Pure-strategy Equilibria

In Figure 9, we plot Leader's and Follower's value functions, $P_L(x)$ and $P_F(x)$, for the asymmetric pure-strategy equilibria and then compare them with firm i's value function $V_i(x) = V_*(x)$ (where $i = a, b$) for the symmetric mixed-strategy equilibrium.

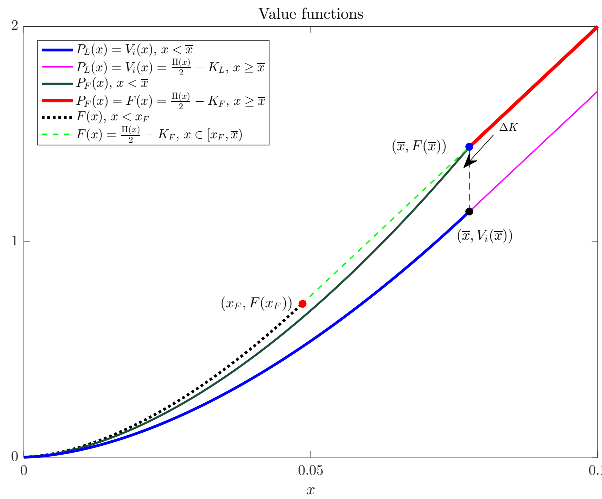

Figure 9: COMPARING VALUE FUNCTIONS FOR PURE-STRATEGY AND MIXED-STRATEGY EQUILIBRIA FOR SUBCASE A_1. Leader's value in a pure-strategy equilibrium, $P_L(x)$, equals firm value in the mixed-strategy equilibrium: $P_L(x) = V_i(x) = V_*(x)$, and Follower's value in a pure-strategy equilibrium, $P_F(x)$, is higher than Leader's value: $P_F(x) > P_L(x)$. Leader's entry threshold in a pure-strategy equilibrium, x_L, equals the cutoff value, $\overline{x}$, between the probabilistic entry region and the waiting region in the mixed-strategy equilibrium: $x_L = \overline{x} = 0.0776$, where $\overline{x}$ is given in (43). Parameter values are $R = 1.6$, $K_F = 0.5$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$, which imply $K_L = 0.8$ and $x_F = 0.0485$.

In a pure-strategy equilibrium, firms are pre-determined to be Leader and Follower due to beliefs. The solid lines depict the equilibrium pre-entry Leader's value $P_L(x)$ where the blue segment is increasing and convex in the waiting region ($x < x_L = \overline{x}$) and the magenta solid line is Leader's net linear payoff function $\Pi(x)/2 - K_L$ in the entry region ($x \geq x_L = \overline{x}$).

Note that $P_L(x)$ in pure-strategy equilibria equals firm value $V_i(x)$ in the mixed-strategy equilibrium. This is because $P_L(x)$ and $V_i(x)$ are both determined by a smooth-pasting condition with the same net payoff functions $\Pi(x)/2 - K_L$. Also, we can show $\overline{x} = 2x_M > x_F$.

Now we turn to Follower's value $P_F(x)$. The solid red line gives $P_F(x)$ in the $x \geq x_L = \overline{x}$ region where both firms are in the market. This is because in equilibrium Follower immediately enters after Leader. Also, Follower's pre-entry value function $P_F(x)$ in the waiting region ($x < \overline{x}$) is increasing and convex (the solid green line.) Because Leader's entry threshold $x_L = \overline{x}$ is higher than Follower's unconstrained entry threshold x_F given in (15), Follower's equilibrium entry threshold thus equals $x_L = \overline{x}$. Follower's value in a pure-strategy equilibrium $P_F(x)$ must be lower than Follower's unconstrained value function $F(x)$, i.e., $P_F(x) < F(x)$ and also the smooth-pasting condition does not hold for $P_F(x)$ at its equilibrium entry threshold $x_L = \overline{x}$.[36] To ease exposition, we use solid lines to draw all the on-the-equilibrium-path value functions.

As $K_L > K_F$, Follower's value in the pure-strategy equilibria is larger than in the mixed-strategy equilibrium: $P_F(x) > V_i(x)$. The industry's total market capitalization in a pure-strategy equilibrium is thus larger than in the mixed-strategy equilibrium for all $x > 0$.[37]

Note that in our pure-strategy equilibria, Leader still exercises its entry option too late compared with the socially optimal level. This is because Leader anticipates no monopoly rents in equilibrium. This result differs from those in simple war-of-attrition games, where the pure-strategy equilibria are socially efficient as one firm immediately drops out Levin (2004). Why are our pure-strategy equilibria socially inefficient? This is because Leader (the loser in the attrition game) also has a real option. This result highlights the rich predictions generated by the interaction between the real-option value and the second-mover advantage in our stochastic entry game.

A key feature of our model is that Leader enters probabilistically in the mixed-strategy equilibrium even when market demand is very high. We next show that probabilistic entry substantially lengthens the time it takes for a firm to become Leader: $\tau_L^* - t$. We demonstrate the economic significance of this result by comparing the distribution of $\tau_L^* - t$ in the mixed-strategy equilibrium with that in pure-strategy equilibria.

[36] The black dotted and green dashed line segments for $F(x)$ in Figure 9 aid our understanding of the model's mechanism and solutions but are off-the-equilibrium path.

[37] This follows from $P_L(x) + P_F(x) - [V_a(x) + V_b(x)] = P_F(x) - V_i(x) = P_F(x) - P_L(x) > 0$, as $P_L(x) = V_a(x) = V_b(x)$ and $P_F(x) > P_L(x)$ (implied by the second-mover advantage).

7.3 Time to Entry $\tau_L^* - t$ in Pure- and Mixed-strategy Equilibria

Definitions. Fix a calendar date T and let $X_t = x$ at $t \le T$. Let $G^{\text{mixed}}(t, x; T)$ denote the time-t cumulative distribution function (CDF) that a firm enters as Leader before T in the mixed-strategy equilibrium. Similarly, let $G^{\text{pure}}(t, x; T)$ denote the time-t CDF for the same event in the pure-strategy equilibria. Mathematically, for any $x > 0$ and time $t \in [0, T]$:

$$G^{\text{mixed}}(t, x) = \mathbb{P}_t^x(\tau_L^{\text{mixed}} - t \le T - t) \quad \text{and} \quad G^{\text{pure}}(t, x) = \mathbb{P}_t^x(\tau_L^{\text{pure}} - t \le T - t)\,. \tag{60}$$

In (60), we use superscripts, mixed and pure, to indicate that Leader's entry time τ_L^* in the mixed- and pure-strategy equilibria (characterized in Proposition 2 and Theorem 4), respectively.

For *every* sample path, Leader enters sooner in a pure-strategy equilibrium than in the mixed-strategy equilibrium. In both types of equilibria, firm entry is characterized by trigger strategies and the entry threshold is the same, which implies that entry is only possible in the $x \ge \overline{x}$ region. But the economic forces underpinning the entry strategies in the $x \ge \overline{x}$ region are different: In a pure-strategy equilibrium, Leader enters with probability one but in contrast both firms enter probabilistically at the rate of $\lambda^*(x) = (x/2 - rK_L)/\Delta K$ in the mixed-strategy equilibrium. As a result, entry can be much delayed in the mixed-strategy equilibrium than in the pure-strategy equilibria. Next, we characterize the distribution of entry timing.

CDF for the Mixed-strategy Equilibrium: $G^{\text{mixed}}(t, x; T)$. The CDF for time to entry $\tau_L^* - t$ satisfies the following partial differential equation (PDE) for $t < T$ and all $x > 0$:

$$G_t^{\text{mixed}}(t, x) + \mu x G_x^{\text{mixed}}(t, x) + \frac{1}{2}\sigma^2 x^2 G_{xx}^{\text{mixed}}(t, x) + 2\lambda^*(x)(1 - G^{\text{mixed}}(t, x)) = 0\,, \tag{61}$$

subject to the boundary conditions: $G^{\text{mixed}}(t, 0) = 0$, $\lim_{x\to\infty} G^{\text{mixed}}(t, x) = 1$ for $t \in [0, T)$, and $G^{\text{mixed}}(T, x) = 0$ for $x \in (0, \infty)$. The last term in PDE (61) captures the effect of mixed strategies on the CDF. As either firm can become Leader, Leader is determined at the rate of $2\lambda^*(x)$ and the CDF jumps from $G^{\text{mixed}}(t, x)$ to one at Leader's entry time $\tau_L^* = \tau_a^* \wedge \tau_b^*$. The first three terms in the PDE (61) are the standard terms describing the calendar time effect, the drift effect of x, and the volatility effect of x on the CDF.

CDF for the Pure-strategy Equilibria: $G^{\mathbf{pure}}(t,x;T)$. The CDF for $\tau_L^* - t$ in the pure-strategy equilibria, $G^{\text{pure}}(t,x)$, satisfies the following PDE for $t < T$ and $x \in [0, \overline{x})$:

$$G_t^{\text{pure}}(t,x) + \mu x G_x^{\text{pure}}(t,x) + \frac{1}{2}\sigma^2 x^2 G_{xx}^{\text{pure}}(t,x) = 0\,, \quad x \in [0,\overline{x})\,, \tag{62}$$

subject to the boundary conditions: $G^{\text{pure}}(t,\overline{x}) = 1$, $G^{\text{pure}}(t,0) = 0$ for $t \in [0,T)$, and $G^{\text{pure}}(T,x) = 0$ for $x \in [0,\overline{x})$. Solving (62), we obtain the following closed-form solution for the CDF:

$$G^{\text{pure}}(t,x) = \Phi(d_2) + (x/\overline{x})^{(1-2\mu/\sigma^2)}\,\Phi(d_1), \tag{63}$$

where $\Phi(\,\cdot\,)$ is the CDF for the standard normal distribution and

$$d_1 \;=\; d_2 - \left(2\mu/\sigma^2 - 1\right)\sigma\sqrt{T-t}\,, \tag{64}$$

$$d_2 \;=\; \frac{\ln(x/\overline{x}) + (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,. \tag{65}$$

The first term $\Phi(d_2)$ in (63) is the time-t probability for the $X_T \geq \overline{x}$ event.[38] The second term is the probability for all the events where $X_T < \overline{x}$ but $\{X_s; s \in (t,T)\}$ exceeds $\overline{x}$ at least once at some $s \in (t,T)$. Next, we show that the CDFs of time to entry $\tau_L^* - t$ for the two types of equilibria are not only different qualitatively but also quantitatively.

Comparing CDFs for Mixed-strategy and Pure-strategy Equilibria. Panel A of Figure 10 plots the CDFs $G^{\text{mixed}}(t,x;T)$ of $\tau_L^* - t$ in the mixed-strategy equilibrium for four levels of x: $0.08, 0.4, 0.7, 1$. When $X_t = x = 0.08$, firms enter within one year with a small probability (4.21%). Even within four years, firms only enter with 18.7% probability. In contrast, in a pure-strategy equilibrium, as $X_t = x = 0.08 > \overline{x} = 0.0776$, entry occurs with probability one. This comparison of CDFs for the mixed-strategy and pure-strategy equilibria shows that quantitative predictions of the model are very different depending on which equilibrium we choose. To us, the mixed-strategy equilibrium is more natural and robust as it is symmetric between the two firms.

In the mixed-strategy equilibrium, entry can take significantly much longer time. For example, even when market demand is very high, e.g., $X_t = x = 1$, (recall this is a flow

[38]The first term is analogous to the conditional (risk-neutral) probability that the option holder receives a strictly positive payoff at the option maturity date in the Black-Scholes option pricing formula.

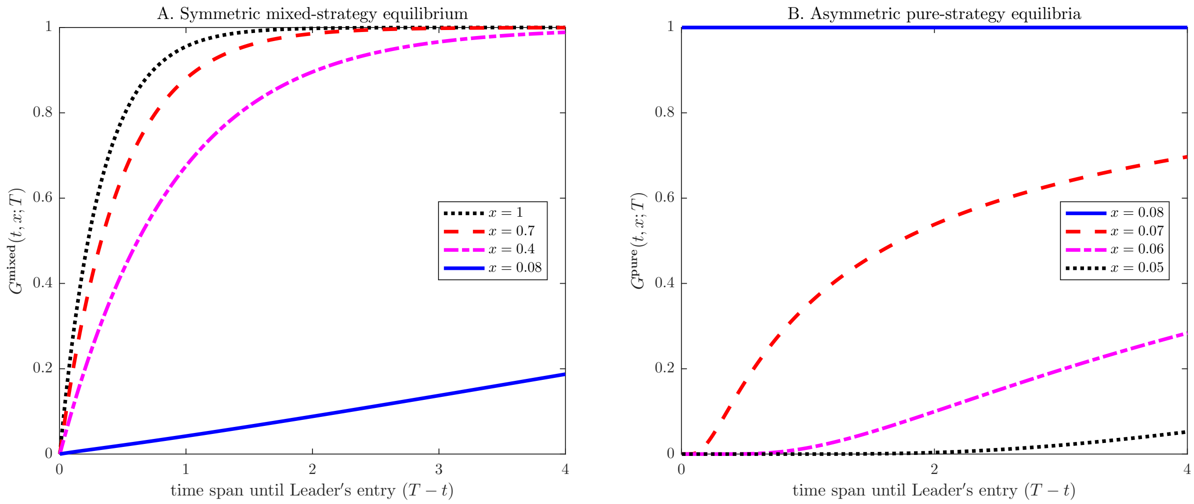

Figure 10: CDFs of time to entry $\tau_L^* - t$ in pure-strategy and mixed-strategy equilibria. Panel A plots the CDF of $\tau_L^* - t$ in the mixed-strategy equilibrium for four levels of market demand: $x = 0.08, 0.4, 0.7,$ and 1. Panel B plots the CDF of $\tau_L^* - t$ in the pure-strategy equilibria for four levels of market demand: $x = 0.05, 0.06, 0.07,$ and 0.08. Parameter values are $K_F = 0.5$, $K_L = 0.8$, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$.

variable and Follower's one-time lumpy entry cost is only $K_F = 0.5$. half of one year's profit X_t), there is still $4.5\% = 1 - G^{\text{mixed}}(t, 1; t+1)$ probability that firms have not entered within one year. This is in sharp contrast with the prediction in a pure-strategy equilibrium where entry is immediate provided that $x \geq \overline{x} = 0.0776$ as we discussed earlier.

The key takeaway from our analysis of distribution of $\tau_L^* - t$ is that the mixed-strategy equilibrium can be much more inefficient and entry is significantly delayed than the pure-strategy equilibria, which are also inefficient.

Next we study the effect of competition on welfare by comparing our duopoly competition model solution to the cooperative duopoly solution.

7.4 Option Value Erosion in Pure- and Mixed-strategy Equilibria

We measure inefficiency by dividing the total market capitalization of the duopoly industry, $V_a(x) + V_b(x)$, by the total market capitalization of the industry in a cooperative duopoly setting, $W(x)$, and subtracting this ratio from one.[39] Let $\Delta(x)$ denote this ineffi-

[39] Mathematically, $W(x) = \max_{\tau_F \geq \tau_L \geq t} \mathbb{E}_t^x \left[\int_{\tau_L}^{\infty} e^{-r(s-t)} X_s ds - K_L e^{-r(\tau_L - t)} - K_F e^{-r(\tau_F - t)}\right]$.

ciency measure:

$$\Delta(x) = 1 - \frac{V_a(x) + V_b(x)}{W(x)}, \tag{66}$$

where $W(x) = M(x)$ and $M(x)$ is the monopolist's market value given in (9) for $x < x_M$ and given in (10) for $x \geq x_M$.

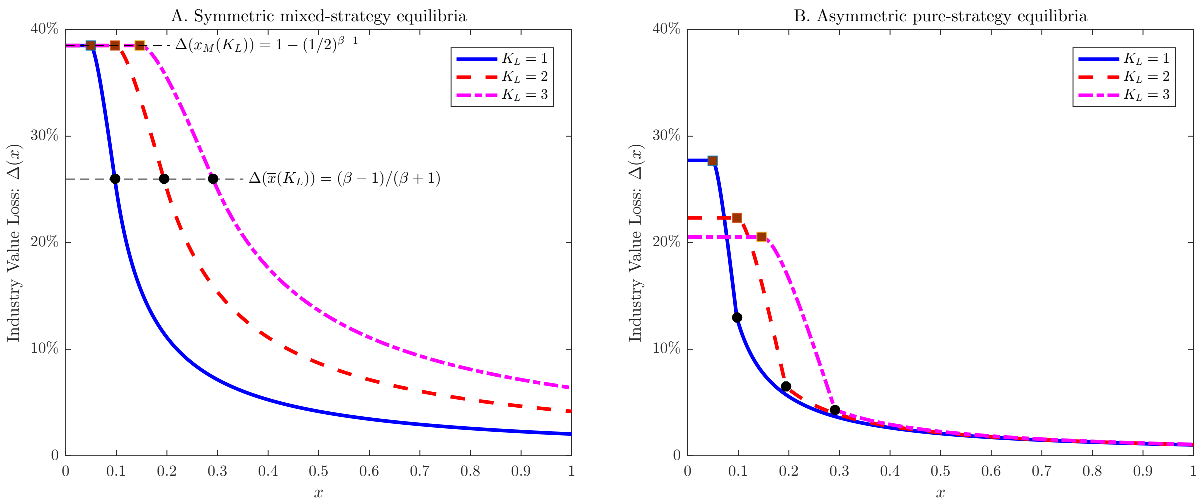

Figure 11: Industry Value Loss $\Delta(x)$. Panels A and B plot $\Delta(x)$ for the mixed-strategy equilibrium and the pure-strategy equilibria, respectively. Both types of equilibria are socially inefficient. Quantitatively, the mixed-strategy equilibrium is significantly more inefficient than the pure-strategy equilibria. Parameter values are $K_F = 0.5$, $K_L = 1, 2$ and 3, $r = 4\%$, $\mu = 2\%$, and $\sigma = 10\%$.

In Panels A and B of Figure 11, we plot $\Delta(x)$ for mixed-strategy and pure-strategy equilibria, respectively. First, the industry value loss $\Delta(x)$ decreases with x for both types of equilibria, which is consistent with our intuition. Second, in the mixed-strategy equilibrium (Panel A), $\Delta(x)$ for empirically plausible levels of market demand is very large. For example, $\Delta(x) = 39\%$ in the $x \leq x_M$ region. Also at the threshold above which firms play mixed entry strategy, $x = \overline{x} = 2x_M$, $\Delta(\overline{x}) = (\beta - 1)/(\beta + 1) = 26\%$, independent of Leader's entry cost K_L. (see the three black dots on the dashed black line.)

Third, in the pure-strategy equilibria (Panel B), competition also significantly erodes firm option value. For example, $\Delta(x) = 27.7\%$ in the $x \leq x_M = 0.0485$ region for the case $K_L = 1$. Note that even for pure-strategy equilibria, entry is inefficiently and significantly delayed (where the optimal entry threshold increases from x_M to $\overline{x} = 2x_M$).

Comparing the two panels in Figure 11 makes it clear that the mixed-strategy equilibrium is much more inefficient than the pure-strategy equilibria. The intuition is as follows. In the mixed-strategy equilibrium firms play a war-of-attrition game even when market demand is very high and are only willing to enter probabilistically (with the hope that the other firm becomes Leader). In contrast, in a pure-strategy equilibrium, there is no uncertainty which firm is Leader. For brevity, we do not analyze value losses for other cases, which have even richer economics due to the interaction between the first-mover and second-mover advantages in a standard real-option framework.

In sum, when the second-mover advantage is the dominating force, e.g., Subcase A_1 of our model, the value loss for the industry as a whole is very large for both types of equilibria, especially in the mixed-strategy equilibrium due to excessively delayed entry. This is in sharp contrast with the predictions in Grenadier (1996) where firms in equilibrium make preemptive moves and enter sooner than a monopolist. Of course, in our general case encompassing both first- and second-mover advantages, firms exercise their entry option either too soon or too late depending on x in a nonlinear and non-monotonic way. We leave these analyses out due to space considerations.

8 Conclusion

In some industries, the second mover faces a lower entry cost and/or has a more efficient production technology than the first mover. We incorporate the second-mover advantage into the duopoly entry game model of Grenadier (1996), where firms trade off the first-mover advantage against the classic option value of waiting (McDonald and Siegel, 1986; Dixit and Pindyck, 1994). Our model solution critically depends on two measures: the optionality measure (β) as in the classic real-option models and the entry-cost ratio ($R = K_L/K_F$), which measures the second-mover advantage. Depending on the values of these two measures (β and R), our closed-form solution fits into one of the five subcases.

Our general model solution (Subcase B_1) features five regions, defined by four endogenous cutoff values ($\widehat{x}_L, \widehat{x}_F, \underline{x}, \overline{x}$) in ascending order. In addition to the option-value of waiting region where $x \in (0, \widehat{x}_L)$ and the competing-to-enter region where $x \in [\widehat{x}_L, \widehat{x}_F]$ (due to the first-move advantage) as in Grenadier (1996), there are three new regions in the $x > \widehat{x}_F$ domain where the second-mover advantage dominates: 1.) in the $x \in (\widehat{x}_F, \underline{x}]$ region, firms enter probabilistically and Leader earns monopoly rents until Follower enters; 2.) in the $x \geq \overline{x}$ region, firms enter probabilistically with no equilibrium monopoly rents for Leader; and 3.) in the $x \in (\underline{x}, \overline{x})$ region between the two probabilistic entry regions, firms wait.

Our model generates new quantitative and testable predictions. For example, firm entry is non-monotonic with respect to market demand and can occur either in clusters or sequentially. Also in contrast to the classic real-option model's prediction, a firm's (pre-entry) option value can be concave in market demand x and decrease with market volatility in the probabilistic entry region (with monopoly rents) due to the interactive effect between imperfect competition and the second-mover advantage. Quantitatively, we find that (a.) the second-mover advantage significantly erodes the industry's market capitalization and (b.) firms significantly delay their entry decisions even when market demand is high, as it is optimal for firms to play mixed entry strategies, engaging in a war-of-attrition game.

To sharpen the key mechanism of duopoly entry games, we have purposefully chosen a minimalistic setting. A firm has complete information about its competitor's cost structure and type. One important extension that we plan to pursue is to incorporate the effects of reputation as in Kreps and Wilson (1982), Milgrom and Roberts (1982), and Abreu and Gul (2000) into our duopoly entry game. Another interesting extension of our model is to grant the first mover with monopoly rents for some periods to capture industrial policies, e.g., patent protection for newly developed drugs. Indeed, a key reason for patent protection is to encourage firm innovation and entry, consistent with our duopoly model's excessive entry delay prediction caused by the second-mover advantage.

Finally, we can generalize our entry game model along several directions, e.g., to allow for a richer cost structure (with both fixed and flow operating costs) and/or a more flexible profit-sharing scheme between Leader and Follower, or to introduce risk premia via a stochastic discount factor to study the asset pricing applications of competition (Duffie, 2001). Importantly, the second-mover advantage that induces firms to play mixed strategies and significantly delay firm entry remains a key force in these extensions.

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Appendices

A Definition of Equilibrium for the General Case

Definition 4 A Markov entry strategy for firm $i \in \{a, b\}$ is a pair: $\varphi_i = (\mathcal{E}_i, \lambda_i(x))$, where $\mathcal{E}_i \subseteq \mathbb{R}_+$ is a closed set and the entry rate $\lambda_i(x)$ is a measurable function from $\mathbb{R}_+ \backslash \mathcal{E}_i$ to $\mathbb{R}_+$. Firm i enters the market for sure when $X_t \in \mathcal{E}_i$ and randomly at an intensity rate of $\lambda_i(X_t)$ when $X_t \notin \mathcal{E}_i$. A Markov strategy pair $(\varphi_a, \varphi_b) = \{(\mathcal{E}_a, \lambda_a), (\mathcal{E}_b, \lambda_b)\}$ is feasible if and only if $\int_0^t \lambda_i(X_s)ds < \infty$ almost surely for any $t < \inf\{s \geq 0 : X_s \in \mathcal{E}_a \cup \mathcal{E}_b\}$. Let Φ denote the set of all feasible entry strategies.

Given $X_0 = x_0 > 0$ and a feasible Markov strategy pair $(\varphi_a, \varphi_b) = \{(\mathcal{E}_a, \lambda_a), (\mathcal{E}_b, \lambda_b)\}$, the entry time pair (τ_a, τ_b) is determined by the joint distribution:

$$\begin{aligned}\mathbb{P}^{x_0}(\tau_a \leq t_a, \tau_b \leq t_b) =& \mathbb{E}^{x_0}\left[\int_{s\in[0,t_b]}\int_{t\in[0,t_a]} dG_a(t)dG_b(s)\right] \\ =& \mathbb{E}^{x_0}\left[(G_a(t_a) - G_a(0))(G_b(t_b) - G_b(0))\right], \quad t_a \geq 0, t_b \geq 0, \end{aligned} \tag{A.1}$$

where $G_i(t)$ is the conditional distribution of firm i's entry time τ_i given $\{X_s; s \geq 0\}$:

$$G_i(t) = 1 - \left(1 - \mathbf{1}_{t \geq \inf\{s\geq 0: X_s \in \mathcal{E}_i\}}\right) e^{-\int_0^t \lambda_i(X_u)du}. \tag{A.2}$$

Definition 5 Let $J_i(x; \varphi_a, \varphi_b)$ denote firm i's value at time t defined in (18) for a given $X_t = x > 0$ and a feasible Markov strategy pair $(\varphi_a, \varphi_b) = \{(\mathcal{E}_a, \lambda_a), (\mathcal{E}_b, \lambda_b)\}$. A feasible entry strategy pair $\{\varphi_a^*, \varphi_b^*\}$ forms a *Markov perfect equilibrium* if for any $x > 0$, we have

$$J_a(x; \varphi_a^*, \varphi_b^*) \geq J_a(x; \varphi_a, \varphi_b^*), \quad \forall\, \varphi_a = (\mathcal{E}_a, \lambda_a)\ s.t.\ \{\varphi_a, \varphi_b^*\} \in \Phi, \tag{A.3}$$

$$J_b(x; \varphi_a^*, \varphi_b^*) \geq J_b(x; \varphi_a^*, \varphi_b), \quad \forall\, \varphi_b = (\mathcal{E}_b, \lambda_b)\ s.t.\ \{\varphi_a^*, \varphi_b\} \in \Phi. \tag{A.4}$$

B Some Technical Results

Deriving the Variational Inequality (34) for Case A. The HJB equation (31) and the inequality $V_i(x) \leq F(x)$ given in (28) together imply

$$\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) - rV_i(x) \leq 0\,.$$

Substituting (30) into (31), we obtain

$$\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) - r V_i(x) = 0 \quad \text{if} \quad L(x) - K_L < V_i(x).$$

Combining the above with $L(x) - K_L \le V_i(x)$ given in (28), we obtain the variational inequality (34).

The Expressions of $\theta_1(a,b)$ and $\theta_2(a,b)$. Let $\tau_{ab} = \inf\{s \ge t : X_s \le a \ \text{ or } \ X_s \ge b\}$ for a given pair (a,b) satisfying $0 < a < b$ and let $\Theta(x;a,b)$ denote the following present value:

$$\Theta(x;a,b) = \mathbb{E}_t^x[e^{-r(\tau_{ab}-t)}(L(X_{\tau_{ab}}) - K_L)], \quad x \in [a,b]. \tag{B.1}$$

We can show that $\Theta(x;a,b)$ is given by (49), where $\theta_1(a,b)$ and $\theta_2(a,b)$ solve:

$$\theta_1 a^\beta + \theta_2 a^\gamma = L(a) - K_L \quad \text{and} \quad \theta_1 b^\beta + \theta_2 b^\gamma = L(b) - K_L.$$

Solving the above equations yields

$$\theta_1(a,b) = \frac{(L(a)-K_L)b^\gamma - (L(b)-K_L)a^\gamma}{a^\beta b^\gamma - a^\gamma b^\beta} \text{ and } \theta_2(a,b) = \frac{(L(b)-K_L)a^\beta - (L(a)-K_L)b^\beta}{a^\beta b^\gamma - a^\gamma b^\beta}. \tag{B.2}$$

Next, we present two lemmas used several times in the main body of the paper.

Lemma 1 *We characterize the cutoff values, $\widehat{x}_L$ and $\widehat{x}_F$, as follows.*

(i) For $R \in (1, R_{AB}]$ (Case B), $\widehat{x}_L$ and $\widehat{x}_F$, the two roots of $L(x) - K_L = F(x)$ in Proposition 1, have the following closed-form expressions:

$$\widehat{x}_F = \overline{\eta}(R)(r-\mu)K_F \quad \text{and} \quad \widehat{x}_L = \underline{\eta}(R)(r-\mu)K_F, \tag{B.3}$$

where $\overline{\eta}(R)$ and $\underline{\eta}(R)$ are given by

$$\overline{\eta}(R) := \sup\{y > 0 : y - \frac{\beta+1}{\beta-1}\left(\frac{\beta-1}{2\beta}\right)^\beta y^\beta = R\}, \tag{B.4}$$

$$\underline{\eta}(R) := \inf\{y > 0 : y - \frac{\beta+1}{\beta-1}\left(\frac{\beta-1}{2\beta}\right)^\beta y^\beta = R\}. \tag{B.5}$$

We can further show: $x_F > \widehat{x}_F > x_M > \widehat{x}_L$ for $R \in (1, R_{AB})$. For the special case $R = R_{AB}$, $x_F > \widehat{x}_F = x_M = \widehat{x}_L$. Finally, $\lim_{R \to 1+} \widehat{x}_F = x_F$.

(ii) For $R \in (0,1]$ (Case C), the equation $L(x) - K_L = F(x)$ has only one root in the $x < x_F$ region: $\widehat{x}_L$ given in (B.3). Finally, $\widehat{x}_L < x_M$.

(iii) The root $\widehat{x}_L$ increases in $R \in (0, R_{AB}]$ and the root $\widehat{x}_F$ decreases in $R \in (1, R_{AB}]$.

Lemma 2 *For any $R \in [1, R_{A_1A_2})$, there exists a unique pair of thresholds $(\underline{x}, \overline{x})$ in the domain $(x_M, x_F) \times (2x_M, \infty)$ satisfying*[40]

$$\Theta(\underline{x}; \underline{x}, \overline{x}) = L(\underline{x}) - K_L, \qquad \Theta_x(\underline{x}; \underline{x}, \overline{x}) = L'(\underline{x}), \tag{B.6}$$

$$\Theta(\overline{x}; \underline{x}, \overline{x}) = \frac{\Pi(\overline{x})}{2} - K_L, \qquad \Theta_x(\overline{x}; \underline{x}, \overline{x}) = \frac{\Pi'(\overline{x})}{2}, \tag{B.7}$$

where $\Theta(x; a, b)$, x_M, and x_F are given by (49), (6), and (15), respectively. Also, $\underline{x}$ and $\overline{x}$ as functions of R are continuously differentiable in $R \in [1, R_{A_1A_2})$. Moreover, $\underline{x}$ and $\overline{x}$ are given by

$$\underline{x} = (1+u)\frac{\beta}{\beta-1}(r-\mu)K_L \quad \text{and} \quad \overline{x} = (1+U)\frac{2\beta}{\beta-1}(r-\mu)K_L, \tag{B.8}$$

where (u, U) is the unique solution pair to the following system of equations in the domain $(0, 2/R - 1) \times (0, \infty)$:

$$U(1+U)^{-\gamma} = 2^\gamma u(1+u)^{-\gamma}, \tag{B.9}$$

$$H(U) = 2^\beta H(u) - \frac{\beta}{\beta-1}R^{\beta-1}, \tag{B.10}$$

with $H(z) = \frac{(1-\gamma)\frac{\beta}{\beta-1}(1+z)+\gamma}{(\beta-\gamma)(1+z)^\beta}$. When $R = R_{A_1A_2}$, we have $\underline{x} = x_M$, $\overline{x} = 2x_M$, and for any $x \in [\underline{x}, \overline{x}]$, $\Theta(x; \underline{x}, \overline{x}) = V_*(x)$ where $V_*(x)$ is given by (41)-(42).

The proofs of Lemmas 1 and 2 are available upon request.

Finally, we determine $R_{B_1B_2}$, the cutoff value of R for the Subcase B_1 and Subcase B_2.

Determining $R_{B_1B_2}$. First, Lemma 2 implies that $\underline{x}$ is continuous in $R \in [1, R_{AB}]$ and satisfies $x_M < \underline{x} < x_F$ for any $R \in [1, R_{AB}]$. Second, Lemma 1 implies that $\widehat{x}_F$ is continuous in $R \in (1, R_{AB}]$ and satisfies $\widehat{x}_F \to x_F$ as $R \to 1+$ and $\widehat{x}_F = x_M$ for $R = R_{AB}$. Combining these two results, we conclude: $\underline{x} > \widehat{x}_F$ for $R = R_{AB}$, $\underline{x} < \widehat{x}_F$ as $R \to 1+$, $R_{B_1B_2} = \sup\{\, R \in (1, R_{AB}) : \underline{x} \leq \widehat{x}_F\}$ is well-defined, $R_{B_1B_2} \in (1, R_{AB})$, and

$$\underline{x} > \widehat{x}_F, \qquad \text{if } R \in (R_{B_1B_2}, R_{AB}]. \tag{B.11}$$

[40]Recall that in this case, we have $x_F \leq 2x_M$, which implies $\overline{x} > x_F$.

C Proofs

We now prove the theorems and propositions in the main text. Let $\mathcal{A}V(x)$ denote the infinitesimal generator operating on a function $V(x)$:

$$\mathcal{A}V(x) = \frac{\sigma^2}{2}x^2V''(x) + \mu xV'(x) - rV(x)\,. \tag{C.1}$$

Proof of Proposition 1: Let $K_{AB} = R_{AB}K_F$. Using (16)-(17) for $L(x)$ and (13)-(14) for $F(x)$, we can verify $K_{AB} = \max_{x>0}\,[L(x) - F(x)]$. Then, for any $x > 0$, we have

$$L(x) - K_L - F(x) \leq \max_{x>0}\,[L(x) - F(x)] - K_L = K_{AB} - K_L = K_F(R_{AB} - R) < 0$$

for $R > R_{AB}$. We thus have proven Case A of Proposition 1. The proofs of results for Case B and Case C in Proposition 1 are available upon request. □

Before proving Theorem 1, we introduce the following lemma for the variational-inequality equation (34) for Case A.

Lemma 3 *Let $V_*(x)$ be the solution to the variational-inequality problem (34)-(36) in the $x \geq 0$ region. Let $\mathcal{R}^E$ denote the probabilistic entry region:*

$$\mathcal{R}^E := \{x > 0 : V_*(x) = L(x) - K_L\}. \tag{C.2}$$

*(i) For **Subcase A_1** where $R > R_{A_1A_2}$, $V_*(x)$ is given by (41)-(42) and $\mathcal{R}^E = [\overline{x}, \infty)$, where $\overline{x} = 2x_M$.*

*(ii) For **Subcase A_2** where $R_{AB} < R \leq R_{A_1A_2}$, $V_*(x)$ is given below:*

$$V_*(x) = \left(\frac{x}{x_M}\right)^{\beta}(L(x_M) - K_L)\,, \quad x \in [0, x_M)\,, \tag{C.3}$$

$$V_*(x) = L(x) - K_L, \quad x \in [x_M, \underline{x}]\,, \tag{C.4}$$

$$V_*(x) = \Theta(x; \underline{x}, \overline{x}), \quad x \in (\underline{x}, \overline{x})\,, \tag{C.5}$$

$$V_*(x) = L(x) - K_L = \frac{\Pi(x)}{2} - K_L, \quad x \geq \overline{x}\,, \tag{C.6}$$

where $\Theta(x; a, b)$ is given by (49) and the thresholds, $\underline{x}$ and $\overline{x}$, are given in Lemma 2. Finally, $\mathcal{R}^E = [x_M, \underline{x}] \cup [\overline{x}, \infty)$.

The proof of Lemma 3 is available upon request.

Proof of Theorem 1: Let $f(x) := \mathcal{A}V_*(x)$ for $x \geq 0$, where $\mathcal{A}V$ is the infinitesimal generator given in (C.1). Substituting the closed-form expressions for $V_*(x)$ for both Subcase A_1 and Subcase A_2 into (C.1)[41], we obtain

$$f(x) = \mathcal{A}V_*(x) = \lambda^*(x)[L(x) - K_L - F(x)], \quad x > 0, \tag{C.7}$$

where $\lambda^*(x)$ is given by (33) for any $x \in \mathcal{R}^E$ and $\lambda^*(x) = 0$ for any $x \in (0, \infty) \setminus \mathcal{R}^E$. Using the expression for $\lambda^*(x)$ given in (33) and $L(x)$ given in (16)-(17), we obtain:

$$f(x) = \mathbf{1}_{x \in \mathcal{R}^E} \left[(rK_L - x)\mathbf{1}_{x < x_F} + (rK_L - x/2)\mathbf{1}_{x > x_F} \right]$$

for any $x > 0$.

For Subcase A_1, using $\lambda^*(x)$ given by (33) for any $x \in \mathcal{R}^E$, where $\mathcal{R}^E = [2x_M, \infty)$, we obtain $\lambda^*(x) = \frac{x/2 - rK_L}{K_L - K_F}$. Similarly, for Subcase A_2, in the $[\overline{x}, \infty)$ region, we also obtain $\lambda^*(x) = \frac{x/2 - rK_L}{K_L - K_F}$. Therefore, there exist an positive value x' and a positive constant λ', such that $\lambda^*(x) \geq \lambda' > 0$ for all $x > x'$, which further implies:

$$e^{-\int_t^\infty \lambda^*(X_s)ds} = 0, \quad \text{almost surely.} \tag{C.8}$$

Next, we complete our proof in two steps. First, we show that it is suboptimal for firm a to deviate from its equilibrium strategy if firm b does not (Step 1).

Step 1: We prove $V_*(x) \geq J_a(x; \lambda_a, \lambda^*)$ where $(\lambda_a, \lambda^*) \in \Phi$.

Let τ_a and τ_b be firm a's and b's stochastic entry time associated with the strategy pair $(\lambda_a, \lambda_b) = (\lambda_a, \lambda^*)$, where $\lambda_a \neq \lambda^*$. Let $\tau := \min\{\tau_a, \tau_b\}$.

For both Subcase A_1 and Subcase A_2, $V_*(x)$ is twice continuously differentiable except at finite points and is globally continuously differentiable (see Lemma 3.) Applying Itô's Lemma to $e^{-rs}V_*(X_s)$ for $s \in [t, \tau]$ and taking expectations at time t, we obtain the following expression for $V_*(x)$:

$$V_*(x) = \mathbb{E}_t^x[e^{-r(\tau - t)}V_*(X_\tau)] - \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right]. \tag{C.9}$$

Recall that $V_*(x)$ satisfies the variational inequality (34), we have $V_*(x) \geq L(x) - K_L$, $\forall x > 0$.

[41] According to Lemma 3, the variational-inequality problem (34)-(36) admits a unique solution, $V_*(x)$, given by (41)-(42) for Subcase A_1 and (C.3)-(C.6) for Subcase A_2, respectively.

Substituting it into the right side of (C.9), we obtain the following inequality:

$$V_*(x) \geq \mathbb{E}_t^x[e^{-r(\tau-t)}(L(X_\tau) - K_L)] - \mathbb{E}_t^x[\int_t^\tau e^{-r(s-t)} \mathcal{A}V_*(X_s)ds]. \tag{C.10}$$

Note that

$$\begin{aligned} J_a(x;\lambda_a,\lambda^*) &= \mathbb{E}_t^x\Big[e^{-r(\tau-t)}\left[\mathbf{1}_{\tau_a<\tau_b}(L(X_\tau)-K_L)+\mathbf{1}_{\tau_a>\tau_b}F(X_\tau)\right]\Big] \\ &= \mathbb{E}_t^x[e^{-r(\tau-t)}(L(X_\tau)-K_L)] - \mathbb{E}_t^x\Big[\mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau-t)}(L(X_\tau)-K_L-F(X_\tau))\Big], \end{aligned} \tag{C.11}$$

where the second equality follows from the property: $\mathbf{1}_{\tau_a=\tau_b} = 0$ almost surely. Using (C.11) and (C.10), we obtain

$$J_a(x;\lambda_a,\lambda^*) \leq V_*(x) + \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds - \mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau-t)}(L(X_\tau)-K_L-F(X_\tau))\right]. \tag{C.12}$$

We can simplify the first term on the right side of (C.12) as follows:

$$\begin{aligned} \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right] =& \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}f(X_s)ds\right] \\ =& \mathbb{E}_t^x\left[\int_t^\infty \int_t^{\tau_a\wedge z} e^{-r(s-t)}f(X_s)\lambda^*(X_z)e^{-\int_t^z \lambda^*(X_u)du}dsdz\right] \\ =& \mathbb{E}_t^x\left[\int_t^{\tau_a}\int_s^\infty \lambda^*(X_z)e^{-\int_t^z\lambda^*(X_u)du}dze^{-r(s-t)}f(X_s)ds\right] \\ =& \mathbb{E}_t^x\left[\int_t^{\tau_a} e^{-\int_t^s(r+\lambda^*(X_u))du}f(X_s)ds\right] \\ =& \mathbb{E}_t^x\left[\int_t^{\tau_a} e^{-\int_t^s(r+\lambda^*(X_u))du}\lambda^*(X_s)[L(X_s)-K_L-F(X_s)]ds\right] \\ =& \mathbb{E}_t^x\left[\mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau_b-t)}[L(X_{\tau_b})-K_L-F(X_{\tau_b})]\right] \end{aligned} \tag{C.13}$$

using (C.7), Tonelli's Theorem (to interchange the integration order in the third equality as $f(x) \leq 0$ and $\lambda^*(x) \geq 0$ for any $x > 0$), integration by parts, and (C.8). Combining (C.12) and (C.13) yields $J_a(x;\lambda_a,\lambda^*) \leq V_*(x)$.

Step 2: We prove $V_*(x) = J_a(x;\lambda^*,\lambda^*)$.

Recall that τ_a^* and τ_b^* are firm a's and b's stochastic entry time, respectively, associated with strategy $(\lambda_a(x), \lambda_b(x)) = (\lambda^*(x), \lambda^*(x))$ and $\tau^* := \min\{\tau_a^*, \tau_b^*\}$. Because $\lambda^*(x) = 0$ for any $x \in (0,\infty) \setminus \mathcal{R}^E$, we have $X_{\tau^*} \in \mathcal{R}^E$, which implies $V_*(X_{\tau^*}) = L(X_{\tau^*}) - K_L$. Therefore, we can see that (C.10)-(C.12) hold with equality if λ_a, τ_a, τ_b and τ therein are set to λ^*, τ_a^*, τ_b^* and τ^*, respectively. We have thus shown $V_*(x) = J_a(x;\lambda^*,\lambda^*)$.

In sum, combining our analyses in Steps 1 and 2, we obtain $J_a(x;\lambda^*,\lambda^*) \geq J_a(x;\lambda_a,\lambda^*)$. By symmetry, we also have $J_b(x;\lambda^*,\lambda^*) \geq J_b(x;\lambda^*,\lambda_b)$ for $(\lambda^*,\lambda_b) \in \Phi$. $\square$

Proof of Proposition 2: This is implied by Theorem 1 and part (i) in Lemma 3. $\square$

Proof of Proposition 3: This is implied by Theorem 1 and part (ii) in Lemma 3. $\square$

Proof of Theorem 2:

Let $\tau_a^* = \tau_b^* = \widehat{\tau} := \inf\{s \geq t : X_s \geq \widehat{x}_L\}$. We prove that (τ_a^*, τ_b^*) is the equilibrium strategy pair in three steps.

First, because $L(x)-K_L \geq F(x)$ holds in $x \geq \widehat{x}_L$ region, it is optimal for firms to compete to enter as Leader in this region. Leader is selected randomly via the rent-equalization principle of Grenadier (1996), which implies $V_i(x) = (L(x) - K_L + F(x))/2$.

Second, we analyze the solution in the $x \in (0, \widehat{x}_L)$ region. As both firms wait in the $(0, \widehat{x}_L)$ region and compete to enter only when $\{X_s; s \geq 0\}$ exceeds $\widehat{x}_L$, firm i's value is given by

$$V_i(x) = \mathbb{E}_t^x\left[e^{-r(\widehat{\tau}-t)}\frac{L(X_{\widehat{\tau}}) - K_L + F(X_{\widehat{\tau}})}{2}\right] = \mathbb{E}_t^x\left[e^{-r(\widehat{\tau}-t)}F(X_{\widehat{\tau}})\right] = F(x), \tag{C.14}$$

where the first equality is due to definition (18), the second equality follows from $L(\widehat{x}_L) - K_L = F(\widehat{x}_L)$ (see Proposition 1) and $X_{\widehat{\tau}} = \widehat{x}_L$, and the last equality follows from the property that Follower's present value is a martingale in its pre-entry region.

Third, we show that firms have no incentives to deviate from the strategy pair (τ_a^*, τ_b^*). Suppose firm a chooses its entry time τ_a, deviating from τ_a^*, and firm b chooses $\tau_b = \tau_b^*$. Let $\tau := \min\{\tau_a, \tau_b^*\}$. Using the definition of $J_i(x)$ given in (18), we obtain

$$J_a(x) \leq \mathbb{E}_t^x\left[e^{-r(\tau-t)}F(X_\tau)\right] = F(x) = V_a(x), \tag{C.15}$$

where the inequality in (C.15) follows from 1.) $\tau_b = \widehat{\tau}$ and 2.) the property that $X(s) \leq \widehat{x}_L$ and $L(X_s) - K_L \leq F(X_s)$ hold for any $s \in [t,\tau]$ (see Proposition 1), the first equality follows from the property that $\{F(X_s); s \geq 0\}$ is a martingale in the pre-entry region $X_s \leq x_F$ and the second equality follows from (C.14). Therefore, firm a has no incentives to deviate from τ_a^*. The same analysis holds for firm b. We thus have proven (τ_a^*, τ_b^*) is the equilibrium strategy pair. $\square$

Before proving Theorem 3, we introduce the following lemma for the variational-inequality equation (34) for Case B.

Lemma 4 *Let $V_*(x)$ be the unique solution to the variational-inequality problem (34) in the $x > \widehat{x}_F$ region subject to the boundary conditions (36) and (54). Let $\Theta(x; a, b)$ for any $x \in [a, b]$ be given by (49) and let $\mathcal{R}^E$ denote the probabilistic entry domain defined in (55).*

(i) For **Subcase B$_1$** *where $R_{B_1B_2} < R \le R_{AB}$, we have*

$$V_*(x) = L(x) - K_L, \quad x \in [\widehat{x}_F, \underline{x}], \tag{C.16}$$

$$V_*(x) = \Theta(x; \underline{x}, \overline{x}), \quad x \in (\underline{x}, \overline{x}), \tag{C.17}$$

$$V_*(x) = \frac{\Pi(x)}{2} - K_L, \quad x \ge \overline{x}, \tag{C.18}$$

where the cutoff, $\widehat{x}_F$, is given in Lemma 1 and the cutoffs, $\underline{x}$ and $\overline{x}$, are given in Lemma 2. The $\mathcal{R}^E$ domain is the union of two disconnected regions: $\mathcal{R}^E = (\widehat{x}_F, \underline{x}] \cup [\overline{x}, \infty)$.

(ii) For **Subcase B$_2$** *where $1 < R \le R_{B_1B_2}$, we have*

$$V_*(x) = \Theta(x; \widehat{x}_F, \overline{x}), \quad x \in [\widehat{x}_F, \overline{x}), \tag{C.19}$$

$$V_*(x) = \frac{\Pi(x)}{2} - K_L, \quad x \ge \overline{x}, \tag{C.20}$$

where $\widehat{x}_F$ is given in Lemma 1 and $\overline{x}$ is the unique solution of the following equation:

$$\Gamma(\widehat{x}_F, y) = F(\widehat{x}_F) \tag{C.21}$$

in the $y > 2x_M$ region, where $\Gamma(x, y)$ for $x > 0$ and $y > 0$ is defined as follows:

$$\Gamma(x, y) = \frac{\frac{1}{2}(1-\gamma)\Pi(y) + \gamma K_L}{\beta - \gamma}\left(\frac{x}{y}\right)^{\beta} + \frac{\frac{1}{2}(\beta-1)\Pi(y) - \beta K_L}{\beta - \gamma}\left(\frac{x}{y}\right)^{\gamma} \quad . \tag{C.22}$$

Finally, the probabilistic entry region is given by $\mathcal{R}^E = [\overline{x}, \infty)$.

The proof of Lemma 4 is available upon request.

Proof of Theorem 3: Using Lemma 4 and similar arguments as in Theorems 1 and 2, we can prove Theorem 3. $\square$

Proof of Proposition 4: This is implied by Theorem 3 and parts (i)-(ii) of Lemma 4. $\square$

Proof of Theorem 4: We prove that $\mathcal{E}_a^* = [x_L, \infty)$ and $\mathcal{E}_b^* = \emptyset$ form an asymmetric pure-strategy entry equilibrium in two steps.

Step 1: We show that Leader has no incentives to deviate its strategy from $\mathcal{E}_a^* = [x_L, \infty)$ to another strategy $\mathcal{E}_a$.

Let $\tau_a = \inf\{s \geq t : X_s \in \mathcal{E}_a\}$. As $\mathcal{E}_b^* = \emptyset$, to prove Leader has no incentive to deviate, it is sufficient to prove

$$\mathbb{E}_t^x \left[e^{-r(\tau_L^* - t)}(L(X_{\tau_L^*}) - K_L)\right] \geq \mathbb{E}_t^x \left[e^{-r(\tau_a - t)}(L(X_{\tau_a}) - K_L)\right]. \tag{C.23}$$

Recall that $V_*(x) \in \mathcal{C}^2(\mathbb{R}_+ \setminus \{\overline{x}\}) \cap \mathcal{C}^1(\mathbb{R}_+)$. Applying Itô's Lemma to $e^{-rs}V^*(X_s)$ for $s \in [t, \tau_a]$, we obtain

$$\begin{aligned} V_*(x) =& \mathbb{E}_t^x[e^{-r(\tau_a - t)}V_*(X_{\tau_a})] - \mathbb{E}_t^x \left[\int_t^{\tau_a} e^{-r(s-t)}\mathcal{A}V_*(X_s)ds\right] \\ \geq& \mathbb{E}_t^x \left[e^{-r(\tau_a - t)}(L(X_{\tau_a}) - K_L)\right], \end{aligned} \tag{C.24}$$

where the inequality follows from $\mathcal{A}V_*(x) \leq 0$ and $V_*(x) \geq L(x) - K_L$ (see Lemma 3-(i)). Also note that when $\tau_a = \tau_L^*$, (C.24) holds with equality. This is because $x_L = \overline{x}$, $\mathcal{A}V_*(x) = 0$ for $x < \overline{x}$, and $V_*(x) = L(x) - K_L$ for $x \geq \overline{x}$. Therefore, the inequality given in (C.23) holds.

Step 2: We show that Follower has no incentives to deviate its strategy from $\mathcal{E}_b^* = \emptyset$ to another strategy $\mathcal{E}_b$.

Using the definition of $P_F(x)$ given in (57), we obtain $J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = P_F(x)$. Let $\tau_a^* := \inf\{s \geq t : X_s \geq \overline{x}\}$, $\tau_b := \inf\{s \geq t : X_s \in \mathcal{E}_b\}$, and $\tau := \min\{\tau_a^*, \tau_b\}$. For any $x \geq \overline{x}$, we conclude from the properties that $F(x) \geq L(x) - K_L$ and $F(X_s)$ is a supermartingale that $J_b(x; \mathcal{E}_a^*, \mathcal{E}_b) \leq \mathbb{E}_t^x[e^{-r(\tau - t)}F(X_\tau)] \leq F(x) = P_F(x)$,

For any $x \in (0, \overline{x})$, applying Itô's Lemma to $e^{-rs}P_F(X_s)$ where $s \in [t, \tau]$, we obtain

$$\begin{aligned} P_F(x) =& \mathbb{E}_t^x[e^{-r(\tau - t)}P_F(X_\tau)] = \mathbb{E}_t^x \left[e^{-r(\tau - t)}\Big(F(X_{\tau_a^*})\mathbf{1}_{\tau_a^* \leq \tau_b} + P_F(X_\tau)\mathbf{1}_{\tau_a^* > \tau_b}\Big)\right] \\ \geq& \mathbb{E}_t^x \left[e^{-r(\tau - t)}\Big(F(X_{\tau_a^*})\mathbf{1}_{\tau_a^* \leq \tau_b} + (L(X_\tau) - K_L)\mathbf{1}_{\tau_a^* > \tau_b}\Big)\right] \geq J_b(x; \mathcal{E}_a, \mathcal{E}_b^*), \end{aligned}$$

where the second equality uses $X_{\tau_a^*} \geq \overline{x}$ and $P_F(x) = F(x)$ for all $x \geq \overline{x}$, the first inequality follows from $X_\tau \leq \overline{x}$ and $P_F(x) \geq L(x) - K_L$ for any $x \leq \overline{x}$, and the last inequality uses the result: $F(x) \geq L(x) - K_L$ for any $x > 0$. □