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ABSTRACT

We develop a dynamic contracting theory of asset- and cash flow-based financing that demonstrates how firm, intermediary, and capital market characteristics jointly shape firms' financing constraints. A firm with imperfect access to equity financing covers financing needs through costly sources: an intermediary and retained cash. The firm's financing capacity is endogenously determined by either the liquidation value of assets (asset-based) or the intermediary's going-concern valuation of the firm's cash flows (cash flow-based). The optimal contract is implemented with defaultable debt — specifically unsecured credit lines and senior-secured debt — and features risk-sharing via bankruptcy. When the firm does well, it repays its debt in full. When it does poorly, distress resolution mirrors U.S. bankruptcy procedures (Chapter 7 and 11). Secured and unsecured debt are complements because risk-sharing via unsecured debt increases secured debt capacity. Debt and equity are dynamic complements because future access to equity financing increases current debt capacity.

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Financing constraints lead firms to manage liquidity risks with internal sources, such as retained earnings, and external sources, such as credit lines. At the same time, the total amount of external liquidity available to the firm depends on financing constraints. Specifically, it depends on the valuation of the firm as a collateral asset from the perspective of a financier who provides that external liquidity. If the financier values the firm's cash flows more than the liquidation value of the firm's assets, then it will extend liquidity based on this going-concern value of those cash flows, generating a distinction between asset-based and cash flow-based financing (Lian and Ma, 2021). Consequently, the key determinant of financing constraints becomes the financier's valuation of the firm in distress, which depends on the firm's future financing arrangements and constraints.

This paper sheds light on the joint determination of financing constraints, the optimal management of liquidity risks, and distress resolution by presenting a theory of corporate finance based on dynamic risk sharing. Our main contribution is to show how defaultable debt and bankruptcy procedures emerge as optimal liquidity risk management tools that facilitate cash flow-based financing within a complete contracts framework. Specifically, our model endogenously generates distress resolution for optimal risk sharing that resembles bankruptcy procedures such as *Chapter 7* and *Chapter 11*, linking the dynamics of bankruptcy to asset-versus cash flow-based financing. Our theory also highlights how debts of different seniority as well as debt and equity can act as complements in a firm's capital structure.

The optimal contract features state-contingent risk sharing between the firm's hand-to-mouth, risk-neutral owners and a deep-pocketed intermediary with limited risk-bearing capacity who provides liquidity. We show that a capital structure consisting of equity, senior-secured short-term debt (or credit line), and an unsecured credit line implements the optimal contract. The firm's secured debt capacity coincides with the intermediary's valuation of the firm as a collateral asset, which is either asset- or cash flow-based. Sharing liquidity risks by using an unsecured credit line and implementing optimal distress resolution in the future increases the firm's collateral value in the present, making secured and unsecured debt complements. This can raise secured debt capacity above the liquidation value of assets and thereby facilitate cash flow-based financing. Allowing default on the unsecured credit line implements optimal risk sharing and distress resolution without necessarily liquidating the firm. The method of resolving distress, whether through liquidation or not, is dynamically linked to whether the firm's secured debt capacity is asset-based or cash flow-based. Anticipating that the firm is not liquidated, the intermediary is willing to lend against cash flows, thereby facilitating distress resolution without liquidation.

In more detail, we model a firm that produces risky cash flows and has imperfect access to capital markets. The firm's current owners cannot inject cash into the firm but, as in

Hugonnier, Malamud, and Morellec (2015), the firm can raise external financing at intermittent random times from new competitive, risk-neutral investors who subsequently become hand-to-mouth. Such intermittent access to broader and competitive capital markets captures capital supply risk. An intermediary stands ready to provide liquidity at any time. Thus, absent market access, the firm finances cash flow shortfalls, i.e., manages liquidity risks, with internal cash reserves and/or funds provided by the intermediary. However, both liquidity facilities are costly. First, cash held in the firm earns a return below the risk-free rate. Second, the intermediary requires compensation for exposure to risk which reflects its own financial or regulatory constraints, limiting its risk-bearing capacity.

We derive the optimal contract under commitment between the firm's owners and the intermediary that maximizes firm value. In the optimal contract, the firm's owners act as shareholders and the firm raises new equity upon capital market access. The intermediary provides liquidity between equity financing rounds. As we show, its claim resembles defaultable debt, so the intermediary may represent a bank or a non-bank lender.

We can summarize the firm's state with a single variable that we term net liquidity. This variable equals the firm's cash holdings less the risk-adjusted present value of future transfers to and from the intermediary. The shareholders' value function solves an ordinary differential equation with two free boundaries. The firm pays dividends to its shareholders when net liquidity achieves the upper boundary. Because the firm has infrequent access to equity financing, negative cash flow shocks can induce financial distress, making the firm's shareholders effectively risk-averse. As such, it becomes optimal to share liquidity risks, in the form of cash flow and capital supply risks, with the intermediary. Specifically, the intermediary bears a fraction of cash flow risk that increases as net liquidity decreases. The intermediary also provides liquidity to the firm in exchange for a payout in the next equity financing round. However, because the firm's current shareholders cannot inject funds into the firm, i.e., they have limited liability, the maximum payout that can be promised to the intermediary equals the total resale value of the firm during an equity financing round.

If the firm runs out of cash, the firm continues operations by relying on financing from the intermediary against promised future payments. However, the firm has a limited financing capacity vis-a-vis the intermediary in that the value of promised repayments to the intermediary cannot exceed some endogenous upper limit, which equals the intermediary's valuation of the firm. Intuitively, due to shareholders' limited liability, repayment promises must be be backed by the firm as a collateral asset and so cannot exceed the intermediary's valuation of it, establishing a lower limit on net liquidity.

When the intermediary values the firm above its liquidation value, financing capacity is cash flow-based, i.e., determined by the intermediary's endogenous going-concern valuation of the firm's cash flows. Otherwise, it is asset-based, i.e., determined by the exogenous liquidation value of the firm's assets. Importantly, the intermediary's going-concern valuation of the firm's cash flows consists of two components. The first is the exogenous risk-adjusted value of the cash flows it will receive prior to selling the firm to new investors upon capital market access. The second component is the expected discounted proceeds from selling the firm to new investors at the fair market value, which endogenously depends on future liquidity risk management and thus on the contract with the intermediary. This second effect means that more efficient risk sharing and, specifically, distress resolution, both in the present and the future, increases cash-flow based financing capacity.

Once the firm exhausts its financing capacity, i.e., when net liquidity approaches its lower boundary, the intermediary effectively takes full ownership of the firm while existing shareholders are wiped out. If financing capacity is cash flow-based and net liquidity is at the lower boundary, the intermediary bears all liquidity risk and allows the firm to continue operations until it can sell the firm to new investors. In contrast, when financing capacity is asset-based, the liquidation value of the firm exceeds the value that the intermediary would obtain by allowing the firm to continue. In this case, the intermediary liquidates the firm once its promised repayments reach the liquidation value of assets. Thus, the nature of financing capacity shapes the resolution of financial distress and vice versa. If the intermediary anticipates that the firm will optimally resolve distress without liquidation in the future, it is willing to provide cash flow-based financing today, which in turn facilitates distress resolution without liquidation.

To implement this contract via standard securities, we first show that the sum of past transfers to and from the intermediary since last market access, compounded at an endogenous rate, is a sufficient statistic for the contract's state. We interpret this sum as the balance of a credit line because it accumulates past transfers, i.e., drawdowns and repayments, and accrued interest, with its balance due upon market access.

Current shareholders raise equity to replenish the cash balance of the firm upon market access, creating value in the process. They also a make payment to the intermediary that resets the credit line balance to zero. If the balance of the credit line is small, then the value of the firm to the new equity investors is high and the firm raises enough cash to achieve the target cash balance, pay off the credit line in full, and allow existing shareholders to maintain a stake in the firm. However, if the balance of the credit line is large, there is insufficient value to achieve all three objectives. The contract then forces the current shareholder to raise the most cash they can by selling their entire stake in the firm. In order to create maximal value, the proceeds are first used to achieve the target cash balance. The remaining proceeds are allocated to the intermediary, and even though they are less than the credit line balance, the balance is reset to zero. Effectively, raising new equity when the firm is in financial distress wipes out existing shareholders and leads to partial default on the credit line.

This repayment schedule is consistent with an absolute priority rule in which the intermediary must be repaid in full before the current shareholders and thus leads to the interpretation of the intermediary's claim as debt. As current shareholders are worse off by raising new equity when the outstanding debt is large, the implementation features a debt overhang problem. To overcome this problem, we allocate control rights in distress states to the intermediary that allow it to force a sale of the firm in case of market access.

The optimal contract involves two key elements, financing against promised repayments and risk sharing. To better delineate these functions, we split the intermediary's claim into senior-secured and unsecured debt. Senior-secured debt or, in short, secured debt, is always repaid in full; it can be implemented as short-maturity term debt or a credit line. The firm's capacity for this form of debt coincides with the intermediary's valuation of the firm as a collateral asset. In contrast, unsecured debt features default in distress states and takes the form of a credit line. The unsecured credit line is used as a liquidity risk management tool over the entire state space. It facilitates risk-sharing with the intermediary because it features default in some states. The secured debt is used to finance cash flow shortfalls once the firm has run out of cash. It does not implement any risk-sharing but merely provides financing against promised future repayments because it earns the market interest rate and is always fully repaid.

The optimal contract resolves financial distress in a manner resembling US bankruptcy procedures. A debt covenant allocates control rights during distress to the creditors, i.e., the intermediary. Without this covenant, the current shareholders would not agree to raise more equity due to debt overhang. When the firm raises new equity outside of distress, it repays all creditors and existing shareholders are better off. Otherwise, the threat of bankruptcy enforces repayment of debt as long as the proceeds from raising equity are sufficient. If it is not possible to fully repay debt, creditors force Chapter 11 bankruptcy while the firm continues operations. Bankruptcy resolution follows one of three cases. First, upon finding new equity investors, the firm repays secured debt, partially defaults on unsecured debt, and wipes out existing equity claims. Second, while in Chapter 11, it may exhaust its financing capacity after a series of negative cash flows. The firm then converts to Chapter 7 and liquidates assets, repays secured debt, and wipes out unsecured debt and existing equity claims. Third, it may emerge from bankruptcy after a series of positive cash flows.

Our model highlights a novel mechanism that links asset- and cash flow-based financing to the mode of bankruptcy. Liquidation and conversion to Chapter 7 bankruptcy might occur under asset-based financing. In contrast, under cash flow-based financing, the firm never exhausts its financing capacity and eventually recovers from Chapter 11 bankruptcy by either locating new investors or earning its way out. Because creditors anticipate this scenario, they are willing to provide secured debt exceeding the liquidation value of assets, which as previously stated facilitates distress resolution without liquidation. Crucially, bankruptcy involves allocating risks to shareholders and unsecured creditors, which insures secured debt and allows access to it in distress.

Finally, our theory highlights how different financing instruments interact in the capital structure. First, unsecured and secured debt are complements. The unsecured credit line allows the firm to share risk with creditors, which can raise the collateral value of the firm. Specifically, by absorbing risk and staving off liquidation in distress, unsecured debt can ensure the repayment of secured debt even when its balance exceeds the liquidation value of assets. This relation allows cash-flow based financing. Second, debt and equity are dynamic complements. Improved access to equity financing increases the likelihood that debt is repaid and raises secured debt capacity, again facilitating cash flow-based financing. Third, our implementation features an overlapping pecking order in which unsecured debt is used in all states, while senior-secured debt is used only when out of cash. When the firm holds cash, it finances cash flow shortfalls with cash reserves and the unsecured credit line. When out of cash, it finances such shocks with both the unsecured credit line and secured debt.

Related Literature. Our paper relates to the literature on corporate liquidity and risk management, as exemplified by Bolton, Chen, and Wang (2011) and Décamps, Mariotti, Rochet, and Villeneuve (2011). Bolton et al. (2011) demonstrate how liquidity management and firm financing interacts with a firm's investment decisions. Bolton, Wang, and Yang (2021) study optimal debt issuance and investment when equity financing is costly. Further contributions in this literature include Gryglewicz (2011); Bolton, Chen, and Wang (2013); Décamps, Gryglewicz, Morellec, and Villeneuve (2016); Hugonnier and Morellec (2017); Malamud and Zucchi (2018), and, more recently, Abel and Panageas (2022); Dai, Giroud, Jiang, and Wang (2020); Bolton, Li, Wang, and Yang (2021). While this literature typically assumes exogenous security design for liquidity management, we characterize optimal liquidity risk management through an optimal long-term contract with a financier. We then implement this optimal contract via standard securities, such as credit lines, and distress resolution procedures, such as the US Bankruptcy Code, rationalizing their use to optimally manage liquidity in practice. Moreover, by endogenizing liquidity management contracts, we uncover a joint determination of liquidity management and financial constraints.

Our work also relates to the dynamic contracting literature that studies optimal risk-sharing between a principal and an agent under limited commitment, such as Ai and Li (2015), Ai, Kiku, and Li (2019), and Bolton, Wang, and Yang (2019). Our model differs in that it highlights optimal financing from an intermediary in the presence of cash constraints and limited liability, resulting in defaultable debt as the optimal contract. Rampini and Viswanathan (2010); Rampini, Sufi, and Viswanathan (2014) provide models in which limited enforcement constrains financing and creates a role for collateral. Rampini and Viswanathan (2020) applies their framework to distinguish between secured and unsecured debt. In these models, the optimal contract takes the form of state-contingent debt, which involves distress resolution without speaking to bankruptcy, liquidation, or changes in ownership. Abel (2018) develops a dynamic trade-off theory in which a (cash flow-based) borrowing constraint prevents shareholders from defaulting immediately. In contrast, in our model the borrowing constraint is optimally chosen to be either cash flow- or asset-based.

Our paper also relates to the extensive literature on corporate bankruptcy and its resolution.[1] Related to our work, Von Thadden, Berglöf, and Roland (2010) derive optimal bankruptcy rules in light of incompatible promises to different creditors in an incomplete contracting framework. Antill and Grenadier (2019) present a model of bankruptcy choice between Chapter 7 and Chapter 11 that is based on dynamic bargaining. We contribute by showing how distress resolution akin to bankruptcy procedures arises as a feature within a complete long-term contract that facilitates optimal risk-sharing.

Finally, our paper also relates to the dynamic contracting literature that studies corporate settings with moral hazard but without liquidity management. Bolton and Scharfstein (1990), and more recently, DeMarzo and Sannikov (2006); Biais, Mariotti, Plantin, and Rochet (2007); DeMarzo and Fishman (2007); Sannikov (2008); DeMarzo, Fishman, He, and Wang (2012); Malenko (2019) show how financing constraints can arise endogenously as the result of a agency conflicts. Using techniques from this literature, we solve for the optimal contract between the firm's shareholders and the intermediary in the absence of agency conflicts but when the firm faces liquidity constraints and therefore must both manage its liquidity and design the contract, inducing endogenous financing capacity.

1 Model Setup

Time $t \geq 0$ is continuous and infinite. We consider a firm whose assets produce cash flows X_t with stationary increments

$$dX_t = \mu dt + \sigma dZ_t, \tag{1}$$

where dZ_t is the increment of a standard Brownian Motion. The firm is owned by risk-neutral *investors* who are hand-to-mouth, i.e., they consume all payouts and do not have

[1] White (2017) reviews this literature which typically takes debt as given or assumes incomplete contracts to derive debt.

any resources to inject into the firm. Access to external financing from newly arriving risk-neutral investors occurs only infrequently. An *intermediary* (distinct from the investors) is available to continuously provide (bridge) financing at a premium. Both the investors and the intermediary discount the future at the risk-free rate of $r > 0$. The intermediary and investors sign a long-term contract $\mathcal{C}$ at time $t = 0$. This contract, $\mathcal{C} = (Div, I, \Delta M)$, stipulates cumulative payouts Div_t to investors, money raised from new investors upon access to external financing ΔM_t, and cumulative transfers I_t to and from the intermediary. Cash flows dX_t are publicly observable, verifiable, and contractible. As will become clear, within the optimal contract, the investors act as the firm's shareholders, who receive dividend payouts $dDiv_t$. In anticipation of this result, we refer to the investors as the *shareholders* and to the value of their stake as the firm's equity value. Further, external financing takes the form of equity injections, which we often refer to as refinancing. However, we emphasize that we do not impose ex-ante restrictions on the form of the contract.

As in Hugonnier et al. (2015), the firm can only access financing from competitive and risk-neutral investors at Poisson times with constant intensity $\pi \geq 0$. Here, $d\Pi_t = 1$ indicates access to external financing or market access at time t. We assume $d\Pi_0 = 1$ so that the initial shareholders (owners) can raise outside financing at inception by selling part of the firm. Once outside investors provide financing, they become hand-to-mouth and thus indistinguishable from the previous shareholders. This assumption reflects capital supply risk or proxies for frictions that cause a delay between the firm's need for financing and its access to broader markets.[2] The key implication of the outside investors becoming hand-to-mouth after providing capital is that dividend payouts to shareholders must be non-negative, i.e., $dDiv_t \geq 0$ at all times $t \geq 0$, including at refinancing times with $d\Pi_t = 1$.

The firm's financial constraints and possibility of negative cash flow shocks imply that the firm has an incentive to accumulate cash M_t via retained earnings. The cash balance held within the firm accrues interest at the rate $r - \lambda$ where r is the common interest rate and $\lambda \in (0, r)$ represents a carry cost of cash.[3] The dynamics of cash reserves M_t are

$$dM_t = dX_t + (r - \lambda) M_t dt - dDiv_t - dI_t + \Delta M_t d\Pi_t. \tag{2}$$

Absent access to equity financing, all cash flow realizations dX_t, payouts to shareholders $dDiv_t \geq 0$, and transfers to/from the intermediary $dI_t \gtrless 0$ flow through the cash balance M_t. Unlike shareholders, the intermediary can provide financing to the firm at any time, so

[2] One may interpret the time it takes to arrange for financing as proxying for asymmetric information — outside investors take time to verify information. The intermediary as a specialist does not face such a delay.

[3] This assumption is standard (see, e.g., Décamps et al. 2011 and Bolton et al. 2011) and prevents the firm from saving itself out of the constraint. Assuming impatient investors leads to similar results.

dI_t can be negative. However, this source of financing is costly, as we formalize below.

The cash balance of the firm at $t = 0^-$, i.e., before the contract is signed, is $M_{0^-} = 0$, and it must remain non-negative throughout, $M_t \geq 0$ for all $t \geq 0$. This constraint implies that if M_t reaches zero, the intermediary must either inject the necessary funds or the firm must liquidate or, equivalently, conduct a fire-sale of assets. Liquidation occurs at a stopping time $\tau \in [0, \infty]$, and $dDiv_t = dI_t = dX_t = 0$ for all $t > \tau$. At liquidation time τ, the firm realizes cash flows equal to the assets' liquidation value, i.e., $dX_\tau = L \in [0, \mu/r)$, so liquidation is costly compared to the first-best. Thus the total liquidation value of the firm is $L + M_\tau$.

1.1 Optimal Contracting Problem

We stipulate that, given a contract $\mathcal{C}$, the intermediary's certainty equivalent continuation value is

$$Y_t = \mathbb{E}_t \left[\int_t^\tau e^{-r(s-t)} \left(dI_s - k_s ds \right) \right]. \tag{3}$$

We refer to Y_t as the intermediary's *promised payments*, or as the intermediary's stake, because it represents a portion of firm value promised to the intermediary. In (3), k_t is the intermediary's endogenous cost of providing financing to the firm. We will discuss k_t and specify its functional form once we characterize the dynamics of Y_t. The intermediary has an outside option normalized to zero. It can always part from the contract and receive its outside option whenever it is privately optimal to do so and is thus subject to limited commitment, i.e., $Y_t \geq 0$ at any time $t \geq 0$.[4]

We denote the firm's equity value, i.e., the shareholders' value function, by P_t. Upon access to new equity financing the firm raises ΔM_t dollars from competitive risk-neutral investors at fair value by issuing ΔM_t dollars worth of new equity. Refinancing changes total equity value from $P_{s^-} \equiv \lim_{s \uparrow t} P_s$ pre-refinancing to $P_t \equiv \lim_{s \downarrow t} P_s$ post-refinancing, while existing shareholders' are diluted and their post-refinancing payoff is $P_t - \Delta M_t$. The shareholders' inability to inject cash into the firm and financial frictions together imply the joint constraints

$$dDiv_t \geq 0 \quad \text{and} \quad \Delta M_t \leq P_t, \tag{4}$$

as current shareholders cannot issue equity worth more than the total value of the firm post-refinancing P_t. We refer to (4) as the shareholders' *limited liability* constraint. If the firm issues the maximal amount $\Delta M_t = P_t$, existing shareholders are completely diluted.

At time t, given a contract $\mathcal{C}$, the equity value of the current shareholders is the expected

[4] This constraint prevents the firm from raising a large amount of cash and saving it with the intermediary, avoiding the internal cost of cash. Qualitative results are unchanged if we assume $Y \geq \underline{Y}$ for $\underline{Y} < 0$.

discounted stream of future dividends less the costs of refinancing via dilution,

$$P_t = \mathbb{E}_t \left[\int_t^\tau e^{-r(s-t)} (dDiv_s - \Delta M_s d\Pi_s) \right]. \tag{5}$$

From (4) it immediately follows that $P_t \geq 0$. For all $t \geq 0$, the optimal contract maximizes

$$P_{0^-} = \max_{\mathcal{C}} \mathbb{E} \left[\int_{0^-}^\tau e^{-rt} (dDiv_t - \Delta M_t d\Pi_t) \right] \quad \text{s.t. limited liability (4), } Y_t, M_t \geq 0, \tag{6}$$

where the intermediary's stake Y_t is given by (3) with initial value $Y_{0^-} = 0$, the cash balance M_t follows (2) with initial balance $M_{0^-} = 0$, and the equity value P_t is given by (5). Table A.1 provides an overview of variables used in the model while Table 1 does so for parameters.

2 Model Solution

We now solve the model and derive the optimal contract. We gain tractability by showing that the difference between the firm's cash holdings and the intermediary's future promised payments is a sufficient statistic for the state of the firm.

2.1 Intermediary Valuation

The following lemma gives the dynamics of total compensation to the intermediary, $Y_t + I_t$.

Lemma 1. *The intermediary's continuation payoff evolves according to*

$$dY_t + dI_t = (rY_t + k_t)\, dt + \beta_t \sigma dZ_t + \alpha_t (d\Pi_t - \pi dt), \tag{7}$$

where β_t captures the intermediary's exposure to Brownian cash flow shocks dZ_t, while α_t captures the intermediary's exposure to the (compensated) market access process $(d\Pi_t - \pi dt)$.

See Appendix A.2 for a proof. We refer to equation (7) as the *promise keeping* constraint. It pins down the sum of current transfers dI_t and changes in promised payments dY_t, but not the split between the two. Our assumptions on liquidation together with (7) require that at the time of liquidation τ, the intermediary receives a lumpy payout of $dI_\tau = Y_{\tau^-}$.

Intermediary cost of financing. Next, we specify k_t as a function of α_t and β_t only:

$$k_t = k(\alpha_t, \beta_t) = \underbrace{\sigma^2 \cdot \left(\frac{\rho r \beta_t^2}{2} \right)}_{\text{Risk-premium } dZ_t \text{ exposure}} + \underbrace{\pi \cdot \left(\alpha_t - \frac{1 - e^{-\rho r \alpha_t}}{\rho r} \right)}_{\text{Risk-premium } (d\Pi_t - \pi dt) \text{ exposure}}. \tag{8}$$

We microfound this specific form of k_t in Appendix B by assuming the intermediary has CARA preferences with risk-aversion ρ and can privately saving/borrow at rate r. We interpret $1/\rho$ as the intermediary's limited risk-bearing capacity due to regulatory or capital constraints. Further, we assume this risk-bearing capacity is constant for two reasons: First, the intermediary, which may represent a group of bank or non-bank lenders investing in many firms, is large relative to the firm. Thus, while the intermediary requires some compensation for bearing firm risk, its risk-bearing capacity is not significantly affected by the performance of one individual firm. Second, this assumption lends tractability to our model; otherwise, one would have to track the additional state variables that drive the intermediary's risk-bearing capacity such as its net worth. [5]

Autarky value. Consider the *autarky* value of the firm to the intermediary if it does not have access to outside financing and continues operations. The firm does not retain cash due to the firm's carry-cost-of-cash, and the intermediary fully absorbs all shocks for an autarky value of

$$Y^A \equiv \frac{\mu}{r} - \frac{\rho}{2}\sigma^2, \tag{9}$$

which is the first-best value, μ/r, less a risk-adjustment term.[6] If $Y^A < L$, rather than operating the firm in autarky for a value Y^A, the intermediary instead immediately liquidates it for a value of L. The value Y^A will play a pivotal role in interpreting our later results.

2.2 Equity Valuation and the HJB Equation

In principle, the dynamic optimization of the shareholders' value function depends on two state variables: the intermediary's promised payments Y_t, and the firm's cash holdings M_t. We heuristically show how to reduce the problem to a single state variable called *net liquidity*

$$C_t \equiv M_t - Y_t. \tag{10}$$

Observe that for any one-time transfer ΔI from the firm to the intermediary

$$C_t = M_t - Y_t = (M_t - \Delta I) - (Y_t - \Delta I) \tag{11}$$

by (7). Thus, C_t is invariant to such transfers so we can freely adjust Y_t freely without affecting net liquidity, which in turn implies that Y_t is a control variable and C_t is the state variable. The dynamics of C are characterized in Appendix A.3 by combining (2) and (7).

[5] We could assume that equity investors are risk-averse, in that they apply a stochastic discount factor. As long as it is not optimal to sell the entire firm to the intermediary, the model's dynamics remain similar.

[6] Setting $dI_s = dX_s$ in (3) and $\beta_s = 1$ and $\alpha_s = 0$ in (8) delivers the result.

In what follows, we omit time subscripts unless necessary.

Limits to Deferred Payments. Even though Y is a control variable, it is constrained: First, the intermediary's limited commitment requires $Y \geq 0$. Second, the definition of net liquidity (10) and the physical constraint on cash $M \geq 0$ together imply that $Y \geq -C$. Combining, we have

$$Y \geq \max\{0, -C\}. \tag{12}$$

Limits to Refinancing. Let C^* be the post-refinancing level of C. When the firm has access to new equity investors, it issues total claims worth ΔM to increase cash holdings from M to M^* and cover payouts to the intermediary dI. By (7), at market access times we have $\alpha = dI + dY = dI + (Y^* - Y)$, i.e., the change in total compensation to the intermediary upon refinancing α is equal to its immediate payouts dI plus any changes in promised payments $Y^* - Y$. At refinancing, the sources and uses of funds accounting identity is given by

$$\underbrace{\Delta M}_{\text{Capital Raised}} = \underbrace{M^* - M}_{\text{Cash Replenishment}} + \underbrace{\alpha + Y - Y^*}_{\text{Payouts to Intermediary}} = C^* - C + \alpha, \tag{13}$$

which implies that optimizing over $(\Delta M, \alpha)$ is equivalent to optimizing over (C^*, α). Combining (13) with the limited liability constraint (4), we derive the following Lemma.

Lemma 2. *Any contract respecting promise keeping* (7) *and limited liability* (4) *must satisfy*

$$\alpha \leq [P(C^*) - C^*] + C. \tag{14}$$

Constraint (14) is a key feature of the model that follows naturally from the assumption that shareholders cannot freely inject capital into the firm. This assumption contrasts with much of the prior literature that is often focused on impediments to the sharing of cash flow-risk, for example due to moral hazard or limited enforcement. In our context, such impediments would manifest as forces that shape β. Constraint (14) instead highlights a restriction on a novel risk-sharing problem: $\alpha > 0$ means that existing equity holders and the intermediary share the risk of delays to market access. Limited liability and promise keeping, however, restrict how much of this risk shareholders can lay off to the intermediary. In our implementation of the optimal contract, this constraint implies a new role for bankruptcy.[7]

HJB Equation. We conjecture and verify that equity value can be expressed as a function of net liquidity only, $P_t = P(C_t)$. To solve the shareholders' dynamic problem (6), we solve for the optimal $P(C)$ for a given level of C. With $P(C)$ in hand, we then determine the

[7]We investigate limits to the shareholders' ability to commit in Section 6.1 by using the generalized constraint $\alpha \leq [P(C^*) - C^*] - [(1-\nu)P(C) - C]$, $\nu \in [0,1]$, where ν measures the strength of commitment.

payoff to the original shareholders (owners) by determining the initial choice of C_0. We conjecture that the firm optimally makes dividend payouts to shareholders at an endogenous upper boundary $C = \overline{C}$, and that it either liquidates or receives sufficient financing to stay alive at some endogenous lower boundary $\underline{C}$. Thus, in the continuation region $C \in \left(\underline{C}, \overline{C}\right)$, given the dynamics of C derived from combining (2) and (7), we have the HJB

$$r \cdot P(C) = \max_{\beta, Y} \left\{ P'(C) \left[\mu + (r - \lambda) C - \lambda Y - \sigma^2 \frac{\rho r}{2} \beta_t^2 \right] + P''(C) \frac{\sigma^2}{2} (1 - \beta)^2 \right\}$$
$$+ \pi \cdot \max_{C^*, \alpha} \left\{ P'(C) \left(\frac{1 - e^{-\rho r \alpha}}{\rho r} \right) + \left[(P(C^*) - C^*) - (P(C) - C) - \alpha \right] \right\}. \quad (15)$$

subject to (12) and (14). The term in square brackets on the second line is the change in existing shareholders value upon refinancing, $P(C^*) - P(C) - \Delta M$, after plugging in (13). Note that the right-hand-side only depends on the state variable C and controls (α, β, Y, C^*) with constraints that only depend on C and the controls themselves. As the boundary conditions below also only depend on the value of C at the boundaries, we can express equity value as a function of C alone, $P_t = P(C_t)$.

Payout boundary. The payout boundary satisfies smooth pasting and super contact conditions,

$$P'(\overline{C}) = 1 \quad \text{and} \quad P''(\overline{C}) = 0. \quad (16)$$

For now, we assume that a well-behaved, non-negative, and twice continuously differentiable solution to (15) exists on the endogenous state space $(\underline{C}, \overline{C})$ subject to (12), (14), and (16) for any given $\underline{C}$. In Appendix A.12, we establish existence and uniqueness of such a solution. Proposition 1 summarizes our findings, formally proven in Appendix A.4.

Proposition 1. *Equity value under the optimal contract can be expressed as function of C only, $P_t = P(C_t)$, and solves the HJB (15) on the endogenous state space $(\underline{C}, \overline{C})$ subject to (12), (14), and (16). Equity value is strictly concave on $(\underline{C}, \overline{C})$, so that $P''(C) < 0$ and $P'(C) > 1$ for $C < \overline{C}$. Optimal dividend payouts $dDiv$ cause C to reflect at $\overline{C}$ and are zero in the interior of the state space. The payout boundary is strictly positive, $\overline{C} > 0$.*

The concavity of equity value implies that shareholders are effectively risk-averse, since liquidation is inefficient, access to external equity financing limited, and intermediary funds costly. To withstand cash flow shocks absent access to equity financing, the firm accumulates cash and delays payouts until C reaches the payout boundary.

2.3 Optimal Controls & Lower Boundary

Optimal Controls. First, the FOC of the HJB with respect to C^* yields $P'(C^*(C)) = 1$ and without loss of generality[8]

$$C^* = C^*(C) = \overline{C}. \tag{17}$$

The original shareholders' payoff $P(C_0) - C_0$ is maximized via initial equity issuance at $C_0 = \overline{C}$.

Second, due to the carry cost of cash λ, the contract picks the lowest Y possible subject to (12):

$$Y(C) = \max\{-C, 0\} \quad \text{and} \quad M(C) = \max\{C, 0\}. \tag{18}$$

Given the strictly positive payout boundary $\overline{C} > 0$, we can now write the transfer process as

$$dI = \mu_I(C)dt + \sigma_I(C)dZ + \alpha_I(C)d\Pi \quad \text{with} \quad \alpha_I(C) \equiv \alpha(C) + Y(C). \tag{19}$$

Rewriting (14), we see that the payout to the intermediary upon market access is bounded by

$$\underbrace{\alpha(C) + Y(C)}_{\text{Payouts to Intermediary}} \leq \overbrace{\underbrace{P(\overline{C}) - \overline{C}}_{\text{Post-refi Net Value}} + \underbrace{M(C)}_{\text{Pre-refi Cash}}}^{\text{Total Resale Value}}. \tag{20}$$

Third, setting $\beta > 0$ transfers cash flow risk to the intermediary, reducing the volatility of C at the cost of a flow risk-premium of $\sigma^2 \frac{\rho r}{2} \beta^2$ in (7). The FOC with respect to β yields

$$\beta(C) = \frac{P''(C)}{P''(C) - \rho r P'(C)} \in [0, 1]. \tag{21}$$

Fourth, setting $\alpha > 0$ shifts payouts to the intermediary from financially constrained states to financially unconstrained states of the firm, at the cost of a flow risk-premium of $\pi \cdot \left(\alpha - \frac{1-e^{-\rho r \alpha}}{\rho r}\right)$ in (7). The FOC with respect to α when (14) is not binding yields

$$\alpha = \alpha_U(C) \equiv \frac{\ln P'(C)}{\rho r}. \tag{22}$$

Given that (14) may be binding, the optimal $\alpha = \alpha(C)$ is characterized by

$$\alpha(C) = \min\left\{\alpha_U(C), P(\overline{C}) - \overline{C} + C\right\} \geq 0.\text{[9]} \tag{23}$$

[8] Any $C^* > \overline{C}$ also fulfills the FOC but leads to an immediate dividend of $C^* - \overline{C} > 0$ upon refinancing.

[9] As we show in Lemma 3 below, the lower boundary satisfies $\underline{C} \geq -[P(\overline{C}) - \overline{C}]$, which implies $\alpha(C) \geq 0$.

Optimal Lower Boundary. We now argue there exist an endogenous lower boundary $\underline{C}$ such that $C_t \geq \underline{C}$ at all times t. For $\underline{C}$ to be a lower bound for C, it must be that either (i) the firm liquidates at $\underline{C}$, or that (ii) $\underline{C}$ is either inaccessible or absorbing (absent refinancing), in which case we denote the lower boundary by C^S and the firm is not liquidated at $\underline{C} = C^S$.

For (i), given that liquidation is inefficient, the firm delays liquidation as long as possible to the lowest boundary consistent with promise keeping $\underline{C} = -L$. The key to (ii) is that for $\underline{C}$ to be a lower bound of C in the absence of liquidation, something we term continuation, it must not be crossed. This requires that as net liquidity C approaches $\underline{C}$ its volatility must vanish, requiring $\lim_{C \downarrow \underline{C}} \beta(C) = 1$, while its drift and the shareholders' value function $P(C)$ both must stay non-negative. Intuitively, at $\underline{C}$ the intermediary keeps the firm alive by providing continuous financing and absorbing all cash flow risks. Further, it is optimal to delay setting $\beta = 1$ as long as possible due to the intermediary's cost of bearing risk, so that C^S is the lowest of all potential continuation boundaries, pinned down by $P(C^S) = 0$, which requires the drift to vanish.

The following Lemma states the lower boundary in closed form as a function of $\overline{C}$:

Lemma 3. *The lower boundary and the associated value of equity are given by*

$$\underline{C} = \min\left\{C^S, -L\right\} \quad \textit{with} \quad P\left(\underline{C}\right) = 0. \tag{24}$$

For $w(\cdot)$ representing the Lambert-W function, C^S is given by

$$C^S = \frac{1}{\rho r}\left[w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}\right] - Y^A \in \left[-\frac{\mu}{r}, -Y^A\right]. \tag{25}$$

The proof is in Appendix A.5. At the lower boundary $\underline{C}$, the intermediary owns the entire firm and values it at $Y(\underline{C}) = -\underline{C}$. To gain intuition about C^S, consider an approximation to (25) by ignoring the required risk premium for α in (8), i.e., $k(\alpha, \beta) \approx \sigma^2 \frac{\rho r}{2} \beta^2$. Then

$$Y(C^S) = -C^S \approx \frac{r}{r+\pi}\underbrace{Y^A}_{\text{Autarky Value}} + \frac{\pi}{r+\pi}\underbrace{\left[P(\overline{C}) - \overline{C}\right]}_{\text{Resale Value}}. \tag{26}$$

The intermediary's valuation of the entire firm $Y(C^S)$ at the continuation boundary is approximately the weighted average of its (exogenous) autarky valuation Y^A and the (endogenous) gain from selling the firm to outside investors at the fair price $P(\overline{C}) - \overline{C}$ upon the next equity market access as cash-holdings are zero, i.e., $M(C^S) = 0$.

The following Lemma proved in Appendix A.7 will be useful in our implementation.

Lemma 4. *When $\underline{C} = C^S$, the lower boundary is inaccessible.*

Lastly, the following Lemma proved in Appendix A.8 shows that the limited liability constraint (14) always binds on the lower end of $\left[\underline{C}, \overline{C}\right]$ when $\underline{C} = C^S$, in which case the firm is never liquidated and the boundary is inaccessible. In this case, there exists a region in which the proceeds from raising equity are insufficient to implement the desired capital supply risk sharing $\alpha_U(C)$.

Lemma 5. *When $\underline{C} = C^S$, constraint (14) always binds in a neighborhood of $\underline{C}$.*

2.4 Optimal Contract

We now characterize the optimal financing arrangement by summarizing our previous results.

Proposition 2. *Under the optimal contract, $P(\underline{C}) = 0$ where $\underline{C}$ is given by (24). Optimal controls (C^*, Y, β, α) are characterized by (17)-(18) and (21)-(23). If $\underline{C} = -L$, the firm liquidates once C reaches the lower boundary $\underline{C}$ and $\beta(C) < 1$ for all $C \geq \underline{C}$. If $\underline{C} = C^S$, the firm never liquidates and $\beta\left(\underline{C}\right) = 1$ while $\beta(C) < 1$ for all $C > \underline{C}$. The firm's initial liquidity choice coincides with the payout boundary, $C_0 = \overline{C}$.*

The proof is given in Appendix A.9. Section 3 shows a natural implementation of the optimal contract via defaultable debt subject to common bankruptcy procedures. Section 4 presents numerical evaluations of the optimal contract with accompanying figures.

Asset- and Cash Flow-Based Financing Capacity. In the optimal contract, the intermediary provides financing against promised payments, as long as the value of these promises Y does not exceed the firm's *financing capacity* defined as

$$\overline{Y} \equiv Y(\underline{C}) = -\underline{C} = \max\{L, -C^S\}. \tag{27}$$

When $\underline{C} = -L$, financing capacity is *asset-based* and is determined by the liquidation value assets. When $C < 0$, the firm taps into intermediary financing against future promised payments $Y(C) > 0$ and pledges the firm's assets to the intermediary so as maintain promise keeping. Upon liquidation at $\underline{C} = -L$, the intermediary seizes the entire liquidation value to receive its promised payoff $Y(\underline{C}) = L$.

When $\underline{C} = C^S$, financing capacity is *cash flow-based*, and the intermediary provides sufficient financing to prevent liquidation, for which it is compensated via future promised payments. The intermediary effectively obtains a stake in the firm which backs future promised payments. Since the contract cannot allocate a stake in the firm greater than full ownership, the intermediary's valuation of the firm, including the value of future refinancing opportunities, constrains the amount of financing that the intermediary provides against promised payments. Intuitively, for the firm to be able to obtain financing against future cash flows, the firm must not be liquidated beforehand so that these cash flows indeed realize.

A key insight from the model is that a firm's financing capacity $\underline{C}$ depends on intermediary and market characteristics in addition to firm characteristics.

Corollary 1. *When ρ is sufficiently large or $Y^A < L$ and π is sufficiently small, then $\underline{C} = -L$, while for ρ sufficient small or π sufficiently large, $\underline{C} = C^S$.*

First, consider $\rho \to \infty$, so the intermediary has zero risk-bearing capacity, leading to $\beta(C) = \alpha(C) = 0$ and $\underline{C} = -L$ by (21), (23), and (24). The intermediary is still willing to provide financing against promised payments. Thus, although the intermediary covers all cash flow shocks when $C < 0$, it cannot bear cash-flow risk. By (7), $dI + dY = rY dt$, so future promised payments grow at the risk-free interest rate r and are fully backed by the assets' liquidation value L. In this case, financing capacity is asset-based.

Second, consider $\pi = 0$, so there is no further equity market access. From (25), we have $C^S = -Y^A$, and the intermediary's going concern value of the firm is exogenously pinned down by the intermediary's risk-bearing capacity $1/\rho$ and the parameters of the cash-flow process as derived in (9). In other words, at $C = \underline{C}$, the intermediary holds the entire firm. However, since it can never resell the firm, the shape of the future contract does not affect its current going-concern valuation for the firm. Thus, the shape of the future contract only matters when $\pi > 0$ and operates through future resale value, which in turn endogenously affects the intermediary's current going-concern valuation, in that $-C^S > Y^A$. Further, when $Y^A < L < -C^S$, financing capacity is cash-flow based, but absent equity market access it would be asset-based.

2.5 Discussion of the Mechanism (1)

The key force that drives this model's optimal contract and the firm's endogenous financing capacity, i.e., the maximal promised payments Y, is the interplay of the risk-sharing of cash-flow risk β with the novel risk-sharing of capital supply risk α. To illustrate this mechanism, let us examine the optimal contract when we incrementally add the ability of the firm and the intermediary to share cash flow risk and capital supply risk.[10]

First, suppose the firm and the intermediary are unable to share any risks, $\alpha = \beta = 0$. The firm can still use transfers and promise payments so that $Y \geq 0$. However, the contract cannot expose the intermediary to any risk, so just as in the case of $\rho \to \infty$ we discussed above, $\alpha = \beta = 0$ implies $\underline{C} = -L$, and there is no scope for cash flow-based financing. The

[10] We separately discuss the mechanism in relation to the implementation in Section 3.2.

model then reduces to a simple cash-holding problem as in Hugonnier et al. (2015) with the addition of risk-free borrowing up to the firm's exogenous financing capacity of L.

Next, suppose the firm and the intermediary can share capital supply risk, but not cash-flow risk, $\beta = 0$. The firm now must liquidate at $\underline{C}_{\beta=0} = -L$ just as if the firm cannot share any risk with the intermediary, albeit for a more subtle reason. When $\beta = 0$, the intermediary cannot own the firm and all promise payments to the intermediary must be satisfied by transfers dI. Moreover, the contract cannot influence the volatility of C_t, so that if $C_t < L$, there are paths for the firm that lead to promised payouts Y that would exceed the risk-neutral present value of the firm μ/r, a clear violation of promise keeping.

Finally, suppose the firm and the intermediary can share cash flow risk but not capital supply risk, $\alpha = 0$. This effectively implies that the intermediary cannot benefit from the gains of equity issuance.[11] As a consequence, the intermediary's valuation of the firm does not reflect the gains from reselling the firm to new equity investors. That is, the lower boundary associated with continuation is consequently equal to the intermediary's autarky value $C^S_{\alpha=0} = -Y^A$. In other words, the firm's financing capacity does not depend on the intensity of capital market access. While there is scope for cash flow-based financing, the determination of the financing constraint is essentially exogenous.

3 Implementing the Optimal Contract

We implement the optimal contract with a debt claim held by the intermediary. Up to this point, the state variable of the contract is the net liquidity $C = M - Y$ of the firm, which contains the forward looking promise Y to the intermediary. While this is a natural variable for shareholders to consider when deciding on payout and financing policies, standard securities typically do not explicitly specify payouts contingent on future payout promises. We show that we can implement the optimal contract based on the past transfer process dI_t.

3.1 Credit Line, Control Rights, and Bankruptcy

Financial Overhang. Refinancing from C to $\overline{C}$ changes total equity value by an amount $P(\overline{C}) - P(C) > 0$, but by (13) and (14) changes existing shareholders' value by an amount

$$J(C) \equiv P\left(\overline{C}\right) - \Delta M - P\left(C\right) = \left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P\left(C\right) - C\right] - \alpha \geq -P(C). \tag{28}$$

[11]Note $\alpha = 0$ implies by (19) that even though Y is fully paid out upon market access due to promise keeping, no other payments to the intermediary occur.

Figure 1 depicts the optimal $J(C)$ by a solid red line and the negative of the shareholder's value function $P(C)$ by a dotted black line. In particular, for $J(C) < 0$ equity holders lose value through dilution when raising new equity. Intuitively, the firm faces a *financial overhang* problem when it raises new equity in such states: raising equity financing is only possible if shareholders agree to take a loss from dilution. In practice, it may prove difficult for shareholders to directly commit to take such a loss. Thus, in the absence of direct commitment, the contract must allocate control rights to the intermediary for all states C with $J(C) < 0$ to implement refinancing in those states.

Accounting for Past Transfers. For any time $t > 0$, define the last refinancing time $\tau^{\Pi}(t) \equiv \sup\{s \leq t : d\Pi_s = 1\}$. Let T_t be the cumulative net transfers received from the intermediary since the last refinancing time, i.e., over $s \in (\tau^{\Pi}(t), t)$, with each transfer dI_s compounding at some endogenous rate. We now characterize T_t as a function of C_t:

Proposition 3. *For dI defined by the optimal (Y, α, β) given in (18), (21), and (23), let*

$$T_t := -\int_{\tau^{\Pi}(t)}^{t} e^{\int_s^t \hat{r}_u du} dI_s. \tag{29}$$

If $T_t = T(C)$ is Markov, then the pair $(T(C), \hat{r}(C))$ is unique, $T'(C) < 0$,

$$T(C) = \alpha_U(C) + Y(C) \quad \textit{and} \quad \hat{r}(C)T(C) = \frac{\lambda}{\rho r}\mathbb{1}_{\{C \geq 0\}} + rY(C). \tag{30}$$

The proof is given in Appendix A.10. With Proposition 3 in hand, we can change the state variable from forward looking C_t to backward looking $T_t = T(C_t)$.

Defaultable debt as the optimal contract. The proof of Proposition 3 establishes that *any* Markovian contract with respect to C featuring optimal cash flow risk-sharing β and promises Y can be summarized by a unique $T(C)$, and that this quantity is linked to the unconstrained optimal choice of α, $\alpha_U(C)$. Note from (19) that the choice of α determines the payout in the event of refinancing. By (20), the optimal repayment to the intermediary upon refinancing equals $\min\{T(C), P(\overline{C}) - \overline{C} + M(C)\}$, i.e., the minimum of $T(C)$ and the total resale value of the firm to new equity investors upon capital market access. As such, the repayment resembles that of defaultable debt with balance (face-value) $T(C)$, and we will often refer to the intermediary as the *creditors*. Lemma 5 shows that (14) binds in some neighbourhood of $\underline{C}$ under cash flow-based financing. In this case, some repayment shortfall occurs in optimum as a risk-sharing tool. Further, this interpretation identifies financial overhang as a form of debt overhang. Consequently, the optimal contract does not fully avoid debt overhang, necessitating the allocation of control rights to the creditors.

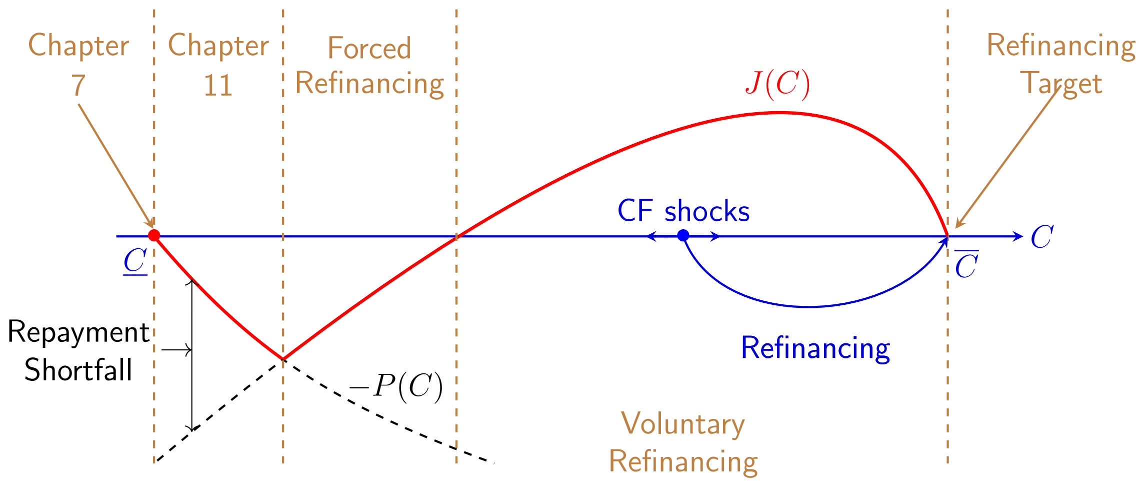

Figure 1: **Bankruptcy and Control Rights.** Schematic graph of the different regions of the value of refinancing to existing shareholders under optimal (Y, β, α).

Bankruptcy and control rights. We achieve this control right allocation via bankruptcy or threat thereof from debt covenant violations. Specifically, we need to structure the covenant so that it is violated whenever $J(C) < 0$. It can be a balance sheet covenant, e.g., a maximum debt-to-asset ratio, or a financial covenant, e.g., an earnings-based covenant which is violated after a sufficient string of negative cash flow realizations. Figure 1 illustrates.

For high C such that $J(C) > 0$ we are in the Voluntary Refinancing region. Shareholders voluntarily refinance because they gain from doing so, and creditors are repaid in full.

For low C such that $J(C) < 0$, three cases arise. First, in the Forced Refinancing region, $J(C) \in (-P(C), 0)$ so that the covenant is violated and existing shareholders face the threat of bankruptcy. However, in these states refinancing proceeds are sufficient to repay the creditors in full while not fully wiping out all equity. Thus, rather than facing bankruptcy and losing all continuation value, shareholders voluntarily pay off the creditors upon finding new equity financing, in the process preserving some value for themselves.

In contrast, in the Chapter 11 region constraint (14) binds and $J(C) = -P(C)$. Recall that this implies the refinancing proceeds are insufficient to fully pay back the creditors, so shareholders have no incentive to refinance on their own. Consequently the creditors enforce the covenant and force the firm into Chapter 11 bankruptcy. If the opportunity to raise new equity arrives in this region, the existing shareholders are fully wiped out.

Finally, in the Chapter 7 region the firm runs out of cash as the credit line is exhausted, and consequently the firm is liquidated. As Chapter 7 can only be reached from Chapter 11, this implies that the bankruptcy is converted from Chapter 11 to Chapter 7. Lemma 4 established that Chapter 7 can only be reached under asset-based financing, i.e., $\underline{C} = -L$.

Credit line repayment and bankruptcy resolution. Given the preceding discussion, there are four cases to credit line repayment. First, credit lines are gradually repaid after positive cash flow realizations. Second, when the firm's liquidity reserves are sufficiently high, i.e., in the Voluntary Refinancing and Forced Refinancing regions in Figure 1, credit lines are repaid in full. Third, when the firm is in distress in the Chapter 11 region, i.e., for low C such that (14) is binding, its creditors force it to enter Chapter 11 bankruptcy and the firm continues operations. If the firm finds new equity investors, the creditors take a partial write-down while the existing equity claims are wiped out. The firm then emerges from bankruptcy under the new ownership, completing the reorganization. The firm may also emerge from bankruptcy following a string of positive cash flow realizations without new equity infusions. Fourth, while in bankruptcy, the firm may hit its financing capacity at Chapter 7 in Figure 1, i.e., for $C = \underline{C}$, in which case it is liquidated, akin to converting Chapter 11 to Chapter 7. In all cases, repayments respect the *absolute priority rule* (APR).[12]

Modes of Financing and Bankruptcy Codes. As Lemma 4 established, Chapter 7 is never reached in the cash-flow based financing case, i.e., $\underline{C} = C^S$. Thus it follows that cash flow-based financing is associated with only Chapter 11 reorganization, while asset-based financing, i.e., $\underline{C} = -L$, is associated with both Chapter 11 reorganization and Chapter 7 liquidation. Different from bargaining-based models of bankruptcy, such as Antill and Grenadier (2019), the current shareholders have full commitment power vis-a-vis the intermediary. Thus, different bankruptcy resolutions are not a consequence of bargaining, but endogenously arise from the shareholders' commitment to the optimal contract.[13] Lastly, bankruptcy and its resolution are ex-ante efficient.

Instantaneous controls and long-term financing. One key observation is that even though (α, β, Y) are instantaneous controls in (15), the optimal contract — via Proposition 3 — allows a long-term financing interpretation. For example, α_t can be seen as instantaneous risk-sharing at time t of the realization of $d\Pi_t$ that trades off receiving a sure cash inflow of $\pi \left(\frac{1-e^{-\rho r \alpha_t}}{\rho r} \right) dt$ (essentially receiving cash in financially constrained states) against a lumpy cash outflow of $\alpha_t d\Pi_t$ that occurs with probability πdt (essentially delivering cash in financially unconstrained states). Importantly, α being a choice variable it has no persistence per se, so the current choice of α need not reflect the path of α. For arbitrary $\beta(C)$, a Markovian $T(C)$ can still be derived, but it has no apparent connection to $\alpha(C)$. However, by the continuity of the optimization in the HJB and the underlying state variable C, (30) implies

[12]The reason is that the states in which there is partial default on the credit line are exactly the states in which (14) is binding, implying existing equity holders are wiped out while the firm survives.

[13]We investigate possible limits to this ability to commit, in the form of violations of APR, in Section 6.1.

that the sequence of optimal (Y_s, α_s, β_s)'s along any path looks like long-term financing with repayment $\alpha_t + Y_t$ linked to the total money drawn down T_t.

3.2 Discussion of Mechanism (2)

The implementation highlights a key role played by risk-sharing over supply uncertainty. As discussed in Section 2.5, it is the interplay of risk-sharing of both cash-flow risk β and capital supply risk α that makes the choice between asset- and cash flow-based financing endogenous. The implementation delivers the next part of our contribution: dynamic risk-sharing with an intermediary optimally takes the form of a debt contract that features common bankruptcy procedures for risk-sharing. Thus, it demonstrates that debt default in bankruptcy plays a distinct role from liquidation; default in bankruptcy implements optimal risk sharing, while liquidation eliminates future cash-flow risk.

To show why specifically risk-sharing over capital supply risk is needed for this result, let us again consider $\alpha = 0$. By (28), we have $J_{\alpha=0}(C) > 0$ everywhere and therefore do not require state-dependent control rights. Thus, while existing shareholders gain upon refinancing, creditors always take a loss equal to $\alpha_U(C)$ regardless of the level of C. Recall from Section 2.5 that $\alpha = 0$ implied that the risk-free debt capacity was pinned down by the intermediary's autarky value of the firm Y^A regardless of π. In this case, the firm could still issue risky debt, but doing so does not improve risk-free debt capacity. Imposing $\alpha = 0$ makes debt equally risky across the range of C while keeping risk-free debt capacity at Y^A. The optimal $\alpha(C)$ makes debt *maximally* risky when the firm is in distress (low C states) and safe otherwise (high C states), and thus expands risk-free debt capacity beyond Y^A. It also implies that APR must hold. Thus, α and the restriction (14) are the key elements that link risk-free debt capacity to bankruptcy and its resolution.

How is the optimal bankruptcy process different from the firm transferring full ownership to the intermediary the first time (14) binds, i.e., when the firm enters the Chapter 11 region in Figure 1? While in this region, the cash-flows to the intermediary are the same in either scenario, as the intermediary completely absorbs all cash flows and upon refinancing the intermediary has full ownership of the firm.[14] Thus, the difference has to arise from cash-flow outside the Chapter 11 region. Under the optimal contract, shareholders retain a stake in the firm even in the Chapter 11 region. This stake is valuable because the firm may emerge from bankruptcy through a sequence of positive cash flows. Moreover, this value allows the the intermediary and the existing equity holders to optimally share cash flow risk even though the equity holders do not receive any cash flows unless the firm earns itself out

[14]This is implied for the optimal contract by the fact that existing shareholders are completely wiped out.

of bankrupcty. Thus, an immediate ownership transfer is sub-optimal and the bankruptcy process optimally delays the ownership transfer to the intermediary for as long as possible.

3.3 Senior-secured and Unsecured Debt

In the following, we present an implementation that splits the balance $T(C)$ in (30) into two separate debt securities, senior-secured debt and an unsecured credit line. This split is such that each of the securities implements specific functions of the optimal contract.

Senior-secured debt can take the form of short-maturity term debt or a credit line. It implements financing against promises, as captured by $Y(C)$; the balance of senior-secured debt coincides with the intermediary's promised payments and is always repaid in full. The unsecured credit line has balance $\alpha_U(C) = T(C) - Y(C)$. It implements contracted risk-sharing $\alpha(C)$ and $\beta(C)$ and thus is risky.

Corollary 2. *The optimal contract can be implemented via two securities that respect APR: (1) Senior-secured debt with balance $Y(C)$, and (2) an unsecured, risky credit line with balance $\alpha_U(C)$ for $C \in (\underline{C}, \overline{C}]$. At the dividend payout boundary, these balances are $Y(\overline{C}) = \alpha_U(\overline{C}) = 0$. For $C \in (\underline{C}, \overline{C})$, the following holds:*

1. *Outstanding senior-secured debt $Y(C)$ grows with interest at rate r and rises and falls with transfers $dI^Y(C)$. Upon refinancing or liquidation, secured debt is repaid in full:*

$$dY(C) = rY(C)dt - dI^Y(C) \tag{31}$$
$$dI^Y(C) = \left\{ \left[\mu - \sigma^2 \frac{\rho r}{2} \beta(C)^2 + \pi \frac{1 - e^{-\rho r \alpha(C)}}{\rho r} \right] dt + \sigma(1 - \beta(C))dZ + Y(C)d\Pi \right\} \mathbb{1}_{\{C \leq 0\}}.$$

2. *The balance of the unsecured credit line $\alpha_U(C)$ increases with a maintenance fee $\frac{\lambda}{\rho r}\mathbb{1}_{\{C \geq 0\}}$, and rises and falls with transfers $dI^{\alpha_U}(C)$:*

$$d\alpha_U(C) = \frac{\lambda}{\rho r}\mathbb{1}_{\{C \geq 0\}} dt - dI^{\alpha_U}(C) - [\alpha_U(C) - \alpha(C)]\, d\Pi \tag{32}$$
$$dI^{\alpha_U}(C) = \left[\sigma^2 \frac{\rho r}{2} \beta(C)^2 - \pi \frac{1 - e^{-\rho r \alpha(C)}}{\rho r} \right] dt + \sigma\beta(C)dZ + \alpha(C)d\Pi.$$

Upon refinancing, if (14) is binding, the firm defaults on $[\alpha_U(C) - \alpha(C)] > 0$ of the unsecured credit line, otherwise the unsecured credit line is repaid in full. Upon liquidation, the unsecured credit line is fully defaulted on.

Creditors have control rights over the firm for all C such that $J(C) \leq 0$ via debt covenants.

The proof is given in Appendix A.11.

Senior-Secured Debt. The purpose of the senior-secured debt is to implement financing against future promised payment, as characterized by $Y(C)$. For brevity, in the following we refer to senior-secured debt simply as secured debt. This debt can take the form of short-maturity term debt, which matures instantaneously as in Abel (2018), or a credit line. Crucially, outstanding secured debt $Y(C)$ is always repaid in full. In our setting with constant firm fundamentals and continuous, stationary cash flows, secured debt does not provide any risk-sharing services. As such, it has an interest rate equal to the risk-free rate r. The firm relies on secured debt only when out of cash, i.e., it is a financing instrument of last resort. In other words, cash and secured debt are substitutes in our setting, in that they are not used simultaneously.

The firm's secured debt capacity, i.e., the maximum amount of secured debt available, equals $\overline{Y} = -\underline{C}$ and thus coincides with the firm's financing capacity in (27). Therefore, secured debt capacity is either asset-based, i.e., determined by the liquidation of the firm's assets in that $\overline{Y} = L$, or cash flow-based, i.e., determined the firm's going concern value in that $\overline{Y} = -C^S$. Since the secured debt capacity matches the intermediary's valuation of the firm, it is natural to see senior debt as backed by the firm as collateral. The key function of the collateralized nature of secured debt is to implement the appropriate seniority structure.

Unsecured Credit Line. The firm's unsecured, risky credit line implements risk-sharing, that is, α and β. Repayments and drawdowns on the unsecured credit line in response to cash flow shocks are proportional to $\beta(C)$: Upon a negative cash flow shock of \$1, the firm draws down the credit line by \$$\beta(C)$. Similarly, as can be seen from (32), the speed with which the intermediary draws on the credit line, that is, $-dI^{\alpha_U}(C)$, increases with $\alpha(C)$, while the balance of the credit line is paid back by the amount $\alpha(C)$ (including possible default) upon refinancing $d\Pi = 1$. Thus, the higher $\alpha(C)$, the larger the transfer from the intermediary to the firm. Again, from the discussion following Corollary 1, as the risk-bearing capacity of the intermediary vanishes, i.e., $\rho \to \infty$, there is no more risk-sharing and risky lending disappears, i.e., $\lim_{\rho\to\infty} \alpha_U(C) = 0$.

Risk-sharing through the unsecured credit line occurs through lowered repayments in distress situations. First, there is default on the unsecured credit line when the firm raises new equity financing in distress when $\alpha(C) < \alpha_U(C)$, corresponding to bankruptcy. Second, the credit line features a preferential interest rate and a state-dependent maintenance fee, which is waived for $C < 0$. In our setting, due to our focus on a Markovian credit line implementation, this preferential interest rate equals zero, as it generates the optimal state-dependent repayments.

The interest rate of zero also reflects the lack of other frictions. For example, in Section 6.1, we show that in the absence of full commitment, the unsecured credit line carries an interest rate that is related to the expected loss from default.

The complementarity between secured and unsecured debt. Secured and unsecured debt in our implementation exhibit a form of complementarity, in that usage of one debt instrument facilitates usage of the other one. First, the use of the unsecured credit line as a liquidity risk management tool enhances risk-sharing and therefore increases the collateral value of the firm, thereby expanding the firm's secured debt capacity and potentially making it cash flow-based. Second, the unsecured credit line, which implements risk-sharing α and β, is necessary to prevent liquidation at $\underline{C}$ via $\beta(\underline{C}) = 1$ and thus for the secured credit line to have capacity beyond L, i.e., for financing capacity to be cash flow-based. Intuitively, unsecured debt allows the firm to offload risk to prevent liquidation, ensuring repayment of secured debt when it exceeds the liquidation value of assets. In distress, the unsecured credit line must take a loss and buffer negative cash flow shocks to facilitate the continuation of the firm, i.e., distress resolution without liquidation akin to Chapter 11 reorganization.

The complementarity between debt and equity. As stated in Corollary 1, higher π leads to can lead to a larger secured debt capacity by allowing the firm to raise new equity capital more frequently. An immediate consequence is that the better access to equity financing, the more likely the firm's financing capacity is cash flow-based rather than asset-based. On one hand, when the firm raises new equity financing, it repays the intermediary; that is, equity financing substitutes for debt financing. On the other hand, the prospect of future access to equity financing improves secured debt capacity by ensuring that secured debt is repaid. Thus, our results imply that secured debt and equity financing are static substitutes but dynamic complements.

An overlapping pecking-order. When the firm has cash reserves, it finances negative cash flow shocks with internal cash and unsecured debt. When the firm has run out of cash, it finances such shocks with unsecured and secured debt. That is, the firm uses unsecured debt financing in all states, while it resorts to secured debt financing only when out of cash. This is consistent with the practice of debtor-in-possession financing where secured loans allow the firm to continue operations in Chapter 11 bankruptcy.[15]

One might expect the opposite, that is, the firm first uses its available collateral to pledge for secured debt financing and, once collateral is exhausted, the firm raises unsecured debt financing. This intuition does not apply in our context because secured and unsecured credit

[15] Because secured debt is always repaid in full, the relative security of secured debt claims does not matter. Thus, one could without loss of generality label part of the outstanding secured debt as super senior secured.

Parameter	Value/Equation	Interpretation
r	0.06	Common discount & interest rate
λ	0.01	Internal carry cost of cash
μ	0.18	Drift of cash flow process
σ	1	Volatility of cash flow process
$1/\rho$	0.167	Intermediary's risk-bearing capacity
π	0.5	Arrival rate of equity financing opportunities
L	1	Liquidation value

Table 1: **Definitions and Baseline Parameter Values for all Figures.** The implied first-best value of the firm is $\frac{\mu}{r} = 3$ and the autarky value to intermediary is $Y^A = \frac{\mu}{r} - \frac{\sigma^2}{2} = 0$.

line debt serve different functions. Unsecured debt is used to share liquidity risk with the intermediary. Such risk sharing is valuable, even when the firm has cash. Put differently, cash or secured debt and the unsecured credit line are complementary liquidity management tools. Secured debt is a substitute to cash and only used when the firm has none.

4 Numerical Analysis of the Contract

To illustrate the model, we present numerical examples based on the parameters given in Table 1. We follow Bolton et al. (2011) in setting r, μ, and λ. We set the liquidation value to $L = 1$, which is about 33.3% of the firm's first-best value $\mu/r = 3$ in line with the liquidation values of non-financial firms reported in Kermani and Ma (2023). We take the intermediary's risk bearing capacity to be $1/\rho = 0.167$, similar to He (2011), and normalize $\sigma = 1$, which also normalizes $Y^A = 0$. In the absence of refinancing opportunities, that is, $\pi = 0$, the firm is liquidated at $C = -L$. In the baseline, we pick $\pi = 0.5$ (that is, expected time until the next market access is $1/\pi = 2$ years) leading to $\underline{C} < -L$ and the firm does not liquidate; we also provide analysis for $\pi = 0.01$ in which case the firm is liquidated at $C = -L$.[16] The model's qualitative outcomes are robust to the choice of these parameters.

4.1 Risk-sharing Dynamics

Figure 2 illustrates the contract dynamics by plotting $\alpha(C), \alpha_U(C)$, and $\beta(C)$ against C in the state space $(\underline{C}, \overline{C})$, both for $\pi = 0.5$ (see Panels A and C) and $\pi = 0.01$ (see Panels B and D). The boundaries $\underline{C}$ and $\overline{C}$ are indicated as a vertical blue and red line respectively. In

[16] Our choice $\pi = 0.5$ follows Hugonnier et al. (2015) who assume an arrival rate of market access of 2 and incumbent shareholders' bargaining power of 0.25, resulting in an effective arrival rate of $0.25 \times 2 = 0.5$.

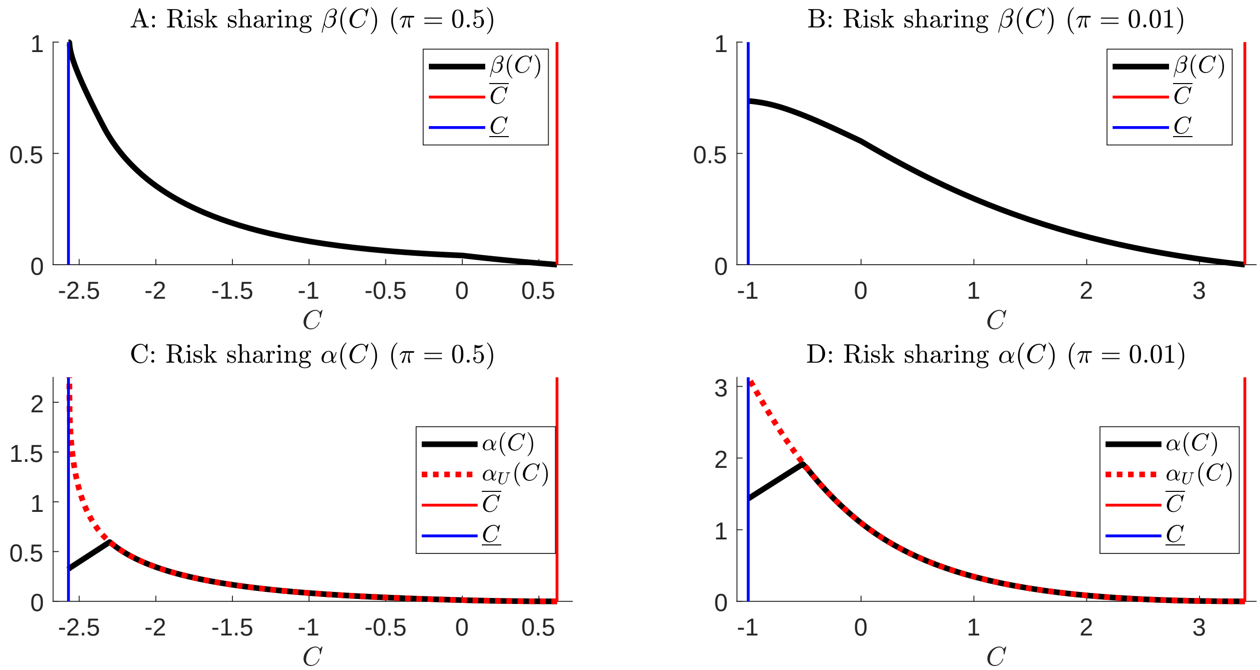

Figure 2: **Contract Dynamics.** This figure plots $\beta(C)$ and $\alpha(C)$ against C both for $\pi = 0.5$ (Panels A and C) and $\pi = 0.01$ (Panels B and D). The parameters follow Table 1.

Panels A and C, $\pi = 0.5$ and $C^S < -L$, so the firm never liquidates and $\lim_{C \to \underline{C}} \beta(C) = 1$. In Panels B and D, for $\pi = 0.01$, the firm liquidates at $-L$, with $\beta(\underline{C}) < 1$.

Panels A and B of Figure 2 show that the intensity of risk-sharing $\beta(C)$ decreases with C or, alternatively, increases with how financially constrained the firm is. Due to financial constraints, shareholders are effectively risk-averse, i.e., $P''(C) < 0$. It is thus optimal to share risk with the intermediary through $\beta(C) > 0$, which is costly due to the intermediary's limited risk-bearing capacity. Because the effective risk aversion of shareholders decreases in C, while the intermediary's risk-bearing capacity $1/\rho$ is constant, $\beta(C)$ decreases with C.

Panels C and D of Figure 2 show that, provided (14) does not bind, $\alpha(C) = \alpha_U(C)$ smoothly decreases with C. Recall that α can be understood as a costly financing or risk-sharing instrument. Setting $\alpha(C) > 0$ essentially transforms flow payouts to the intermediary today, i.e., from states in which the firm is constrained, into a promised lumpy payout upon refinancing in the future, i.e., a state in which the firm is financially unconstrained. This transfer of promised payments to the future is relatively more beneficial when net liquidity is lower and the firm is more constrained, but exposing the intermediary to the market access shock is costly. The optimal choice of α trades off relaxation of financial constraints now versus larger payments to the intermediary in the future, so unconstrained $\alpha_U(C)$ decreases with C. However, limited liability limits what amount $\alpha(C)$ can be promised, as shown in the constraint (14). It is binding when C is low and close to $\underline{C}$. In this case, $\alpha(C)$ mechanically

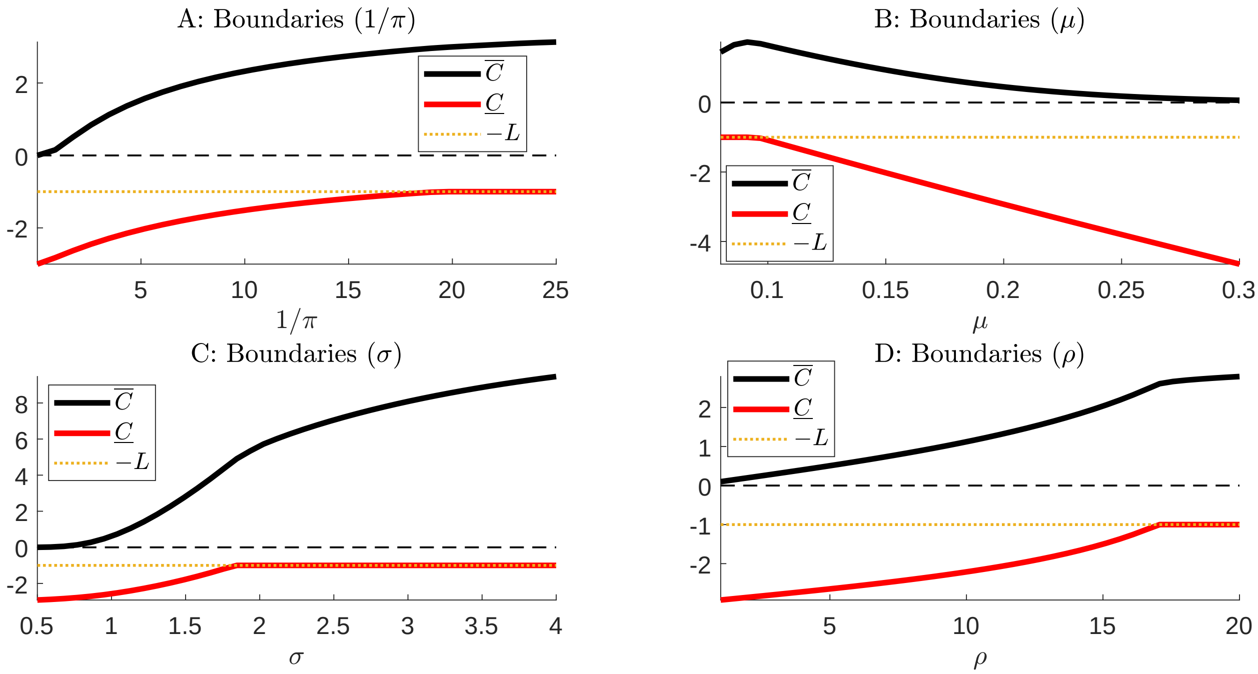

Figure 3: **Boundaries:** Comparative statics of the boundaries $\underline{C}, \overline{C}$ with respect to to the expected time until refinancing $1/\pi$ (top left panel), profitability μ (top right panel), cash flow volatility σ (bottom left panel), intermediary risk aversion ρ (bottom right panel). The parameters follow Table 1.

increases with C, as additional excess liquidity C relaxes the constraint (14).

4.2 Determinants of Financing Capacity and Utilization

We examine the determinants of the firm's financing and, specifically, when and whether it is cash flow-based or asset-based. Recall that in light of our implementation from Section 3, the firm's financing capacity also represents the firm's secured debt capacity.

Figure 3 plots $\underline{C}$ and $\overline{C}$ against the expected time to refinancing $1/\pi$ (Panel A), cash flow drift μ (Panel B), cash flow volatility σ (Panel C), and intermediary CARA coefficient ρ (Panel D). Figure 3 shows that the firm's financing capacity $\overline{Y} = -\underline{C}$ increases with access to equity financing π, i.e., $\underline{C}$ increases with $1/\pi$, and the firm's cash flow rate μ, while it decreases with cash flow volatility σ and intermediary CARA coefficient ρ (recall that $1/\rho$ is the intermediary's risk-bearing capacity). Thus, financing capacity is cash flow-based for low values of $1/\pi$, σ, ρ or when μ is high. Otherwise financing capacity is asset-based. Intuitively, intermediaries are more willing to provide financing against cash flow for firms with high profitability μ and less volatile cash flows σ. At the same time, low values of μ

and large values of σ are associated with larger target cash holdings $\overline{C}$.[17]

Related to Corollary 1, Panel D of Figure 3 shows that financing capacity decreases with ρ and is cash flow-based only for low values of ρ, while target cash holdings $\overline{C}$ increase with ρ. That is, the model predicts that intermediaries with higher risk-bearing capacity are more willing to provide cash flow-based financing, specifically, secured cash flow-based debt.[18]

As discussed in Section 3.3, Panel A shows that lower $1/\pi$ increases financing capacity and reduces the reliance on costly precautionary cash holdings as captured by $\overline{C}$ by allowing the firm to raise new equity capital more frequently. As shown in the Panel, the better the access to equity financing, the more likely the firm's financing capacity is cash flow-based rather than asset-based, highlighting the dynamic complementarity between debt and equity.

Finally, Figure 4 illustrates how the firm's average secured debt balance $avg(Y_t)$, the average utilization of secured debt capacity $avg(Y_t/\overline{Y})$, and the average leverage ratio defined as $avg(Leverage_t) = avg\left(\frac{Y_t+\alpha_{U,t}}{P_t+Y_t+\alpha_{U,t}}\right)$[19] change with $1/\pi$ (left column), ρ (middle column), and σ (right column). Observe that these quantities are hump-shaped in $1/\pi$ and σ, in that firms rely the most on (secured) debt when access to equity financing and cash flow volatility take intermediate values.

The intuition for the hump-shaped patterns in Panels A and C is that firms obtain relatively little secured debt and debt in general when (i) their secured debt capacity is high but they do not need it due to low cash flow risk or frequent access to equity financing or (ii) when their secured debt capacity is low. The intuition for this pattern carrying through to secured debt utilization in Panels D and F is more subtle. Firms with low cash flow volatility or easy access to equity financing have high secured debt capacity, but do not need it, explaining the left side of the graphs. In contrast, firms with high cash flow volatility or little access to equity financing try to avoid using secured debt altogether by accumulating greater cash balances before paying dividends, i.e., self-insuring via cash. Finally, Panels G and I show that the hump-shape in secured-debt usage and utilization extends to overall leverage as well, with similar intuition as above.

Panels B, E, and H show that the firm's reliance on debt financing increases when the intermediary exhibits a higher risk-bearing capacity. The intuition is that when ρ is low, the cost of risk-sharing is low, which enhances access to debt financing in general and also

[17]More volatile cash flows raise the need for precautionary cash holdings (Décamps et al., 2011). $\overline{C}$ is non-monotone in μ because an increase in μ raises profitability and thus makes liquidation more costly while reducing the need for precautionary cash. The liquidation effect dominates for low μ because financing capacity is asset-based. Otherwise the precautionary saving effect dominates.

[18]Different types of intermediaries may exhibit different risk-bearing capacities. Because banks face regulatory capital constraints, they likely have lower risk-bearing capacity. Non-bank lenders face less regulatory constraints than banks but still might be capital-constrained, so they likely have higher risk-bearing capacity.

[19]Note that equity is measured at its market value P_t, while debt is measured at its book value $Y_t + \alpha_{U,t}$.

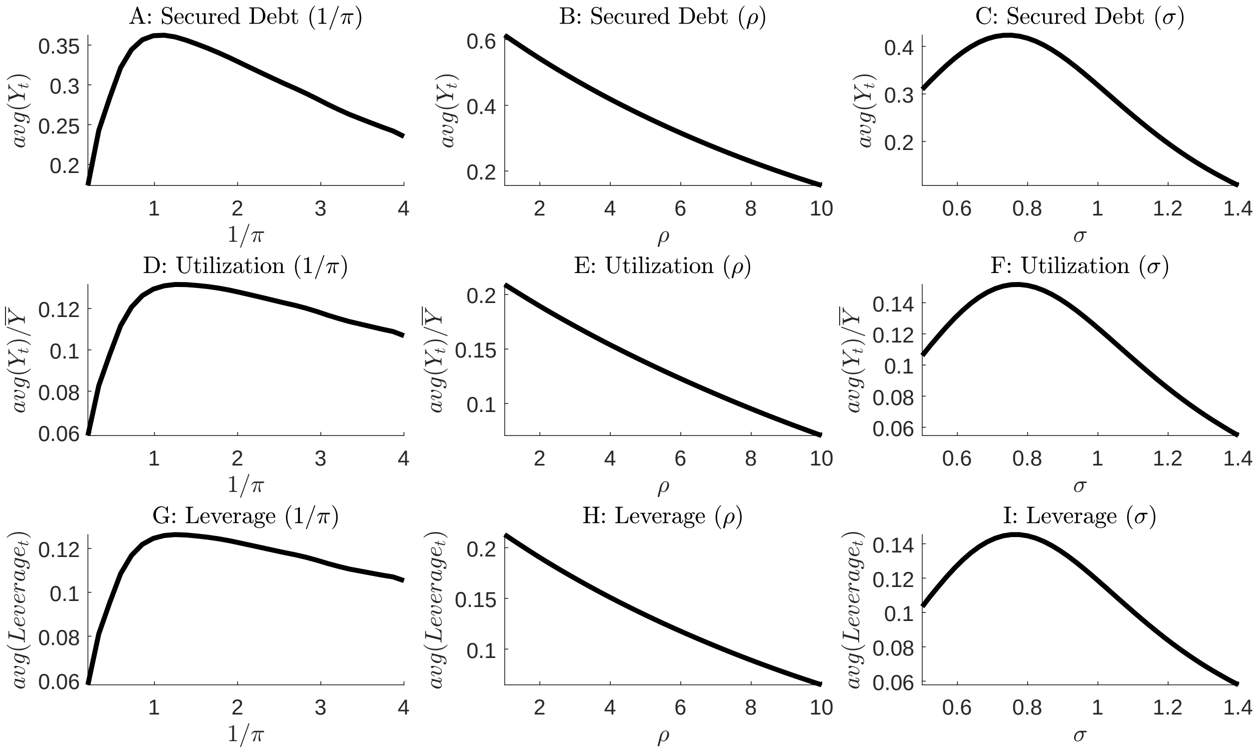

Figure 4: **Debt Financing.** This figure plots the average secured debt $avg(Y_t)$ (top row), the average utilization of secured debt capacity $avg(Y_t)/\overline{Y}$ (middle row), and the average leverage $avg(Leverage_t) = avg\left(\frac{Y_t+\alpha_{U,t}}{P_t+Y_t+\alpha_{U,t}}\right)$ (bottom row) against $1/\pi$ (left column), ρ (middle column), and σ (right column). The parameters follow Table 1.

increases its usage. This effect implies that firms use more secured debt and, specifically, cash flow-based secured debt, when creditors exhibit a higher risk-bearing capacity.

5 Empirical Implications

In this section, we give an overview of the key empirical implications of our theory. We also explain how these implications are novel and unique to our setting, which combines optimal contracting and dynamic liquidity management.

Before describing these implications in detail, let us briefly discuss the distinction between debt capacity in the model and the empirical measurement for the basis of debt. As discussed in Lian and Ma (2021), secured debt can be either asset-based or cash flow-based. Asset-based debt is secured by specific assets, while cash flow-based debt is unsecured or secured by a blanket lien on the firm. For asset-based debt, the final recourse of the creditor is to sell the asset at scrap value, which in our model is the case for $\overline{Y} = L$. For cash flow-based debt, the final recourse of the creditor is to take ownership and sell the firm to new investors, occurring in our model if and only if $\overline{Y} > L$. In particular, when $Y \in (L, \overline{Y})$, part of the firm's secured debt is necessarily cash flow-based, i.e., backed by the firm's going-concern value. Thus, a cash flow-based secured debt capacity $\overline{Y}$ implies the use of cash flow-based secured debt in some states. These observations link the distinction between cash flow-based and asset-based secured debt capacity in our setting to practice.

Financing Instruments and Secured vs. Unsecured Debt. In our implementation, the firm uses multiple securities to manage liquidity at the same time, leading to an overlapping pecking order. This strategy facilitates state-dependent risk sharing. Outside of distress, the firm uses cash, i.e. retained earnings, and unsecured debt simultaneously. This pattern is supported by the rich interaction between leverage and cash management policies documented in DeAngelo, Gonçalves, and Stulz (2022) who show that (net of cash) leverage increases as cash balances decrease. The firm uses unsecured debt in all states to share liquidity risk with creditors, consistent with the practice of using credit lines as liquidity management tools (see, e.g., Acharya, Almeida, Ippolito, and Perez 2014; Brown, Gustafson, and Ivanov 2021). The firm also relies on secured debt, but primarily in distress situations, consistent with Rauh and Sufi (2010); Benmelech, Kumar, and Rajan (2024, 2022).

Our model demonstrates that debt (in general) is not negative cash. When the firm runs out of cash, it needs to issue the most "negative cash"-like security it can. Secured debt fills this role because, in our context, it is the least risky among all debt securities. Therefore, unlike unsecured debt and cash, secured debt and cash act as substitutes. Our model further predicts that the use of unsecured debt allows the firm to issue more secured debt, making unsecured debt and secured debt complements. The mechanism is that unsecured debt absorbs cash flow risk and therefore raises the collateral value of cash flows, thereby allowing secured debt to be cash flow based.

Our predictions about different financing instruments arise from combining dynamic liquidity management with optimal contracting. As a result, unsecured credit lines are optimal for risk-sharing, while secured debt acts as a cash substitute. Endogenous security design is crucial for understanding the use of these instruments. Unlike other models that treat cash as negative debt (see, e.g., Bolton et al. 2011, 2021), we show that unsecured debt complements both cash and secured debt, which are substitutes. This explains why firms use different financing tools simultaneously to manage liquidity and optimize risk-sharing.

Debt and Equity as Dynamic Complements. In our model, improved access to equity financing, such as that enjoyed by public firms, increases secured debt capacity and access to cash flow-based debt. This prediction is supported by the finding of Lian and Ma (2021) that public firms use more cash flow-based debt than private firms.[20] Similarly, private firms backed by Private Equity (PE) funds have better access to equity financing, and thus our model predicts that they can obtain higher leverage compared to otherwise similar firms. This explanation for increased leverage of PE backed firms differs from ones focused on tax benefits or leveraging investments. Further, our analysis suggests that PE-backed firms use more cash flow-based debt than non PE-backed firms, and that PE sponsorship relaxes financing constraints with lenders (Ivashina and Kovner, 2011; Demiroglu and James, 2010).

These empirical predictions rest upon a key insight from our model: Creditors are willing to offer cash flow-based debt because they anticipate that the firm will survive long enough to locate new equity investors. At the same time, the firm is only able to survive long enough because it can obtain cash flow-based debt. Improved access to equity financing strengthens this feedback effect, thereby raising secured debt capacity and facilitating access to cash flow-based debt. This mechanism requires a fully dynamic model in which the firms future prospects, i.e., continuation or liquidation, are endogenous to the contract.

Bankruptcy. Our model offers a novel perspective on bankruptcy by showing how it emerges as a tool for optimal risk sharing. Rather than an immediate change in ownership or liquidation, bankruptcy is a *process* that dynamically resolves either through continuation and reorganization (Chapter 11) or liquidation (Chapter 7). In particular, this process involves allocating all risk to shareholders through dilution and to unsecured creditors through debt default. This approach protects secured creditors and ultimately facilitates debtor-in-possession financing, i.e., the use of senior-secured debt in Chapter 11 bankruptcy, allowing the firm to continue operations during bankruptcy.

In line with distress resolution in our model, Ivashina, Iverson, and Smith (2016) document that bankruptcy commonly involves the write-down of subordinate debt and equity claims. Further, our model predicts that distress resolution akin to Chapter 11 bankruptcy and debtor-in-possession financing become more prevalent when the firm has better access to equity financing. For private firms, the anticipation of equity injections from PE sponsors raises the creditors' willingness to provide secured debt, specifically, cash flow-based secured debt, which allows the firm to continue operations when in distress and in bankruptcy. This effect implies that PE-backed firms are less likely to be liquidated in distress, as documented in Hotchkiss, Smith, and Strömberg (2021).

By jointly determining the financing constraint and distress resolution, our model highlights an endogenous link between modes of financing and bankruptcy. When creditors anticipate that distress is resolved without liquidation, they are willing to lend against cash

[20] Such cash flow-based debt is often secured by a blanket lien, i.e., a security interest in the firm's going concern value minus assets separately pledged in other secured debt contracts.

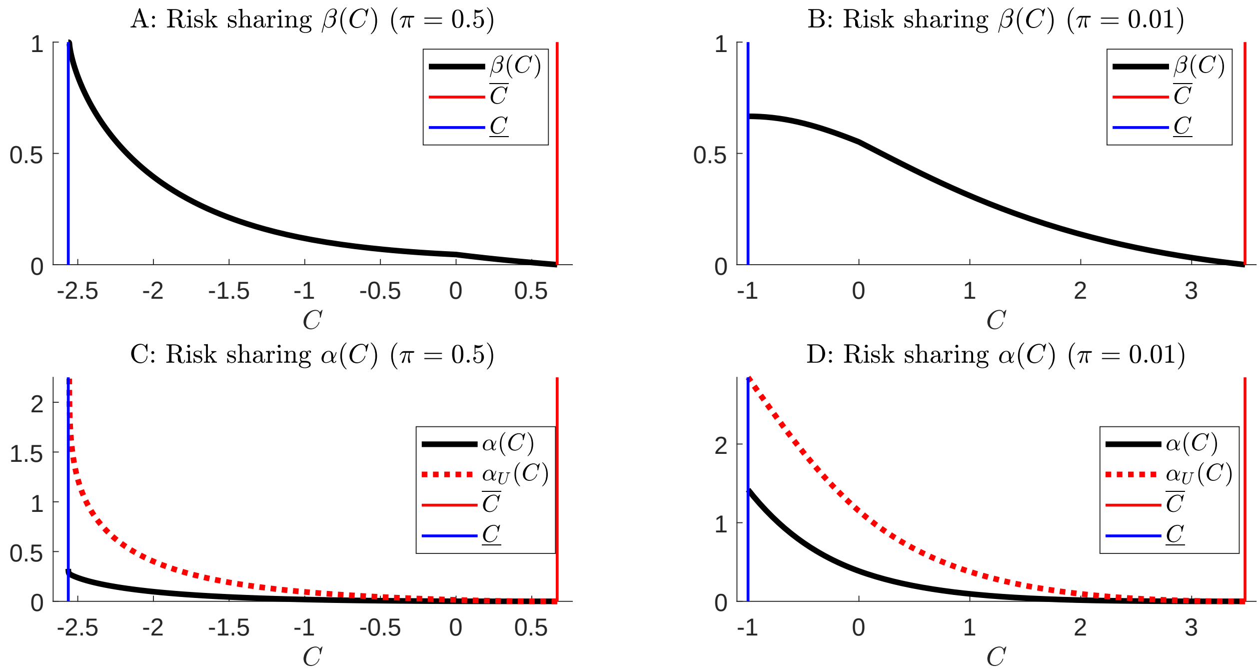

Figure 5: **Contract Dynamics under limited creditor protection.** This figure plots $\beta(C)$ and $\alpha(C)$ against C both for $\pi = 0.5$ (Panels A and C) and $\pi = 0$ (Panels B and D). The parameters follow Table 1 and $\nu = 0$ in refinancing constraint (33).

flows, enhancing the firm's secured debt capacity. This allows the firm to obtain secured debt to continue operations in bankruptcy, which ultimately prevents liquidation. Thus, cash flow-based financing is tied to the distress resolution via Chapter 11 reorganization, while asset-based financing is linked to distress resolution via both Chapter 11 reorganization and Chapter 7 liquidation. This rationalizes the empirical findings in Lian and Ma (2021) that cash flow–based lending is more prevalent in countries with Chapter 11–type corporate bankruptcy systems.

6 Extensions

6.1 Refinancing and Weak Creditor Rights

Recall that constraint (14) implies that the shareholders' continuation payoff is always nonnegative by limiting the amount α that can be promised to the intermediary upon refinancing. However, there are states C in which existing shareholders are so heavily diluted upon refinancing that they are worse off than "just before" refinancing, i.e., $J(C) < 0$.

For $\nu \in [0,1]$, consider the following generalized constraint on α that nests our base case:

$$\alpha(C) \leq [P(\overline{C}) - \overline{C}] - [(1-\nu)P(C) - C] \quad \Longleftrightarrow \quad J(C) \geq -\nu P(C). \tag{33}$$

Here, $\nu = 1$ yields our baseline constraint (14) implying full commitment, while $\nu = 0$ yields a tighter constraint we term monotonicity, as refinancing cannot make current shareholders worse off via dilution. Thus, ν describes shareholders' strength of commitment to refinance. The solution of this model variant follows the baseline solution, except that constraint (14) is replaced by constraint (33). The boundary conditions remain the same, specifically the expressions for the lower bound (24), as due to $P(\underline{C}) = 0$, we have $J(\underline{C}) = 0$ regardless of ν. The following proposition shows that weaker commitment, i.e., smaller ν, increases the payout boundary and lowers cash flow-based financing capacity.

Proposition 4. *The payout boundary $\overline{C}$ and lower boundary under continuation C^S both decrease in ν.*

The proof is given in Appendix A.13. Notably, the solution, implementation, and the key results for $\nu < 1$ remain qualitatively similar to the baseline. Figure 5 illustrates the dynamics of the optimal contract under $\nu = 0$ using the same parameters as in Figure 2. The outcomes in Figure 5 are broadly similar to Figure 2, with the differences being that the constraint (33) now binds for all C and that $\underline{C}$ and $\overline{C}$ are larger than in the baseline case.

Our implementation from Proposition 3 and Corollary 2 via secured and unsecured debt applies after a change to the interest rate on $d\alpha_U(C)$ and the bankruptcy rules for $\nu < 1$.[21] The key difference to our baseline implementation is that for $\nu < 1$, existing shareholders are not fully wiped out when creditors take a loss. In other words, APR may fail in bankruptcy, and we thus interpret ν as a proxy for creditor protection. In sum, weak creditor protection — which limits the extent of dilution of equity in distress resolution — limits the availability and feasibility of cash flow-based secured debt.

Weaker creditor rights, i.e., lower ν, lead to a decrease in financing capacity, a shift from cash flow-based financing toward asset-based financing, and a shift from Chapter 11 bankruptcy with reorganization toward Chapter 7 bankruptcy and liquidation for distress resolution. Consistent with these predictions, Antill (2022) documents that weakened creditor protection is associated with inefficient liquidation in bankruptcy.

[21] By (A.46), the interest rate on the unsecured credit line changes as $\alpha_C(C)$ is redefined by (33), so that

$$d\alpha_U(C) = \left[\frac{\lambda}{\rho r}\mathbb{1}_{\{C\geq 0\}} + \pi\,(1-\nu)\left(\frac{e^{\rho r[\alpha_U(C)-\alpha(C)]}-1}{\rho r}\right)\right] dt - dI^{\alpha_U}(C) - [\alpha_U(C) - \alpha(C)]\, d\Pi. \tag{34}$$

Note that the term involving ν, i.e., the second term in $[\cdot]$, is always positive. The unsecured interest rate is positive to compensate the unsecured creditors for the higher expected losses $\alpha_U(C) - \alpha(C)$ from bankruptcy.

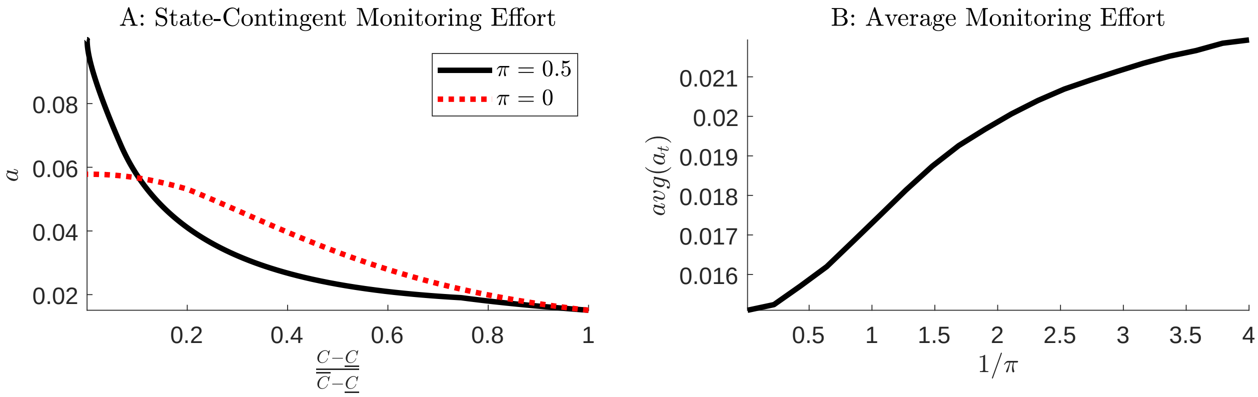

Figure 6: **Intermediary Incentives**. The left panel of this figure plots the intermediary's effort $a = \beta/\kappa$ both for $\pi = 0.5$ (solid black line) and for $\pi = 0$ (dotted red line) in the transformed state space against $(C - \underline{C})/(\overline{C} - \underline{C})$. The right panel plots average effort $avg(a_t)$ against $1/\pi$. The parameters follow Table 1, and we use $\kappa = 10$.

6.2 Active Intermediaries and Monitoring

Financial intermediaries, such as banks or direct lenders, often monitor borrowers to contain credit risk and improve operational performance, or actively engage in financial distress resolution. We extend our model to account for such intermediary actions by allowing the intermediary to affect cash flows via effort. The intermediary's incentives to exert effort increase with the its exposure to cash flow shocks ("skin-in-the-game") β_t.

As described in Appendix C.1, we now assume that cash flows evolve according to

$$dX_t = (\mu + a_t)dt + \sigma dZ_t, \tag{35}$$

where $a_t \geq 0$ is the intermediary's non-contractible and privately observable effort which entails a cost $\frac{\kappa a_t^2}{2}dt$ for a constant $\kappa > 0$.[22] Effort boosts the firm's cash flow drift, which could capture the intermediary's active role in firm operations or its role in disciplining management through monitoring. The intermediary's incentives to exert effort are determined by the incentive condition $a_t = \frac{\beta_t}{\kappa}$, and thus increase with the intermediary's exposure to cash flow shocks β_t. That is, there is a moral hazard with regard to intermediary's monitoring effort.

Panel A of Figure 6 illustrates the dynamics of effort a in two scenarios, (i) $\pi = 0.5$ in which case $\underline{C} = C^S$ and the firm is never liquidated and (ii) $\pi = 0$ in which case the firm is liquidated once $C = -L$.[23] Intermediary effort increases upon negative cash flow shocks and is highest when C is low. The intermediary exerts particularly high effort when $\beta(C) \approx 1$,

[22] Note that an infinite cost of effort, i.e., $\kappa \to \infty$, implies $a_t = 0$, thus giving our baseline case.

[23] To facilitate comparison across scenarios, we plot intermediary effort a against the adjusted liquidity position $(C - \underline{C})/(\overline{C} - \underline{C}) \in [0, 1]$ both for $\pi = 0.5$ (solid black line) and for $\pi = 0$ (dotted red line).

which occurs at $C = \underline{C}$ in the case of cash flow-based financing. Thus, cash flow-based financing is associated with more intense monitoring under financial distress.

Panel B of Figure 6 plots the intermediary's average effort against $1/\pi$, the expected time to refinancing. Better access to equity markets reduces the intermediary's incentives to monitor, so that $avg(a_t)$ increases with $1/\pi$. That is, that lenders monitor less when borrowers have better access to equity financing. On the contrary, when liquidity dries up, e.g., in a financial crisis, intermediaries providing debt financing exert more effort to improve firm operations or, similarly, engage more in monitoring.

Finally, cash flow-based financing capacity generally decreases with κ and so increases with the intermediary's monitoring ability. Intuitively, when the intermediary can add value to the firm through monitoring, it has a higher valuation for the firm and so is more willing to provide cash flow-based financing.

7 Conclusion

We develop a dynamic contracting theory of asset- and cash flow-based financing that demonstrates how firm, intermediary, and capital market characteristics jointly shape firms' financing constraints. A firm with imperfect access to equity financing covers financing needs through costly sources: an intermediary and retained cash. The firm's financing capacity is endogenously determined by either the liquidation value of assets (asset-based) or the intermediary's going-concern valuation of the firm's cash flows (cash flow-based). The optimal contract is implemented with defaultable debt — specifically an unsecured credit line and senior-secured debt — and features optimal risk-sharing via bankruptcy. When the firm does well, it repays its debt in full. When it does poorly, distress resolution mirrors U.S. bankruptcy procedures (Chapter 7 vs 11). Secured and unsecured debt are complements because risk-sharing via unsecured debt increases secured debt capacity. Debt and equity are dynamic complements because access to equity financing increases debt capacity.

By providing a micro-foundation of financing constraints through the lens of dynamic contracting theory, our paper can provide guidance on reduced-form financing constraints, for instance, in dynamic macroeconomic models. While most papers in the macroeconomic literature focus on collateral, i.e., asset-based, constraints as key financing constraint (Kiyotaki and Moore, 1997; Bernanke, Gertler, and Gilchrist, 1999; Dávila and Korinek, 2018), Greenwald (2019), Drechsel and Kim (2022), and Drechsel (2023) introduce cash flow-based financing constraints in general equilibrium models. Our findings provide guidance how these constraints are related to capital market and intermediary characteristics.

References

Abel, A. B. (2018). Optimal debt and profitability in the trade-off theory. *The Journal of Finance 73*(1), 95–143.

Abel, A. B. and S. Panageas (2022). Precautionary saving in a financially-constrained firm. *forthcoming, Review of Financial Studies*.

Acharya, V., H. Almeida, F. Ippolito, and A. Perez (2014). Credit lines as monitored liquidity insurance: Theory and evidence. *Journal of financial economics 112*(3), 287–319.

Ai, H., D. Kiku, and R. Li (2019). A quantitative model of dynamic moral hazard. *Available at SSRN 2801116*.

Ai, H. and R. Li (2015). Investment and ceo compensation under limited commitment. *Journal of Financial Economics 116*(3), 452–472.

Antill, S. (2022). Do the right firms survive bankruptcy? *Journal of Financial Economics 144*(2), 523–546.

Antill, S. and S. R. Grenadier (2019). Optimal capital structure and bankruptcy choice: Dynamic bargaining versus liquidation. *Journal of Financial Economics 133*(1), 198–224.

Benmelech, E., N. Kumar, and R. Rajan (2022). The secured credit premium and the issuance of secured debt. *Journal of Financial Economics 146*(1), 143–171.

Benmelech, E., N. Kumar, and R. Rajan (2024). The decline of secured debt. *The Journal of Finance 79*(1), 35–93.

Bernanke, B. S., M. Gertler, and S. Gilchrist (1999). The financial accelerator in a quantitative business cycle framework. *Handbook of macroeconomics 1*, 1341–1393.

Biais, B., T. Mariotti, G. Plantin, and J.-C. Rochet (2007). Dynamic security design: Convergence to continuous time and asset pricing implications. *Review of Economic Studies 74*(2), 345–390.

Bolton, P., H. Chen, and N. Wang (2011). A unified theory of tobin's q, corporate investment, financing, and risk management. *Journal of Finance 66*(5), 1545–1578.

Bolton, P., H. Chen, and N. Wang (2013). Market timing, investment, and risk management. *Journal of Financial Economics 109*(1), 40–62.

Bolton, P., Y. Li, N. Wang, and J. Yang (2021). Dynamic banking and the value of deposits. Technical report, National Bureau of Economic Research.

Bolton, P. and D. S. Scharfstein (1990). A theory of predation based on agency problems in financial contracting. *The American economic review*, 93–106.

Bolton, P., N. Wang, and J. Yang (2019). Optimal contracting, corporate finance, and valuation with inalienable human capital. *Journal of Finance 74*(3), 1363–1429.

Bolton, P., N. Wang, and J. Yang (2021). Leverage dynamics under costly equity issuance. *Working Paper, Columbia University*.

Brown, J. R., M. T. Gustafson, and I. T. Ivanov (2021). Weathering cash flow shocks. *The Journal of Finance 76*(4), 1731–1772.

Brunnermeier, M. K. and Y. Sannikov (2014). A macroeconomic model with a financial sector. *American Economic Review 104*(2), 379–421.

Dai, M., X. Giroud, W. Jiang, and N. Wang (2020). A q theory of internal capital markets. Technical report, National Bureau of Economic Research.

Dávila, E. and A. Korinek (2018). Pecuniary externalities in economies with financial frictions. *The Review of Economic Studies 85*(1), 352–395.

DeAngelo, H., A. S. Gonçalves, and R. M. Stulz (2022). Leverage and cash dynamics. *Review of Finance 26*(5), 1101–1144.

Décamps, J.-P., S. Gryglewicz, E. Morellec, and S. Villeneuve (2016). Corporate policies with permanent and transitory shocks. *The Review of Financial Studies*, hhw078.

Décamps, J.-P., T. Mariotti, J.-C. Rochet, and S. Villeneuve (2011). Free cash flow, issuance costs, and stock prices. *Journal of Finance 66*(5), 1501–1544.

DeMarzo, P. M. and M. J. Fishman (2007). Optimal long-term financial contracting. *Review of Financial Studies 20*(6), 2079–2128.

DeMarzo, P. M., M. J. Fishman, Z. He, and N. Wang (2012). Dynamic agency and the q theory of investment. *Journal of Finance 67*(6), 2295–2340.

DeMarzo, P. M. and Y. Sannikov (2006). Optimal security design and dynamic capital structure in a continuous-time agency model. *Journal of Finance 61*(6), 2681–2724.

Demiroglu, C. and C. M. James (2010). The role of private equity group reputation in lbo financing. *Journal of Financial Economics 96*(2), 306–330.

Drechsel, T. (2023). Earnings-based borrowing constraints and macroeconomic fluctuations. *American Economic Journal: Macroeconomics 15*(2), 1–34.

Drechsel, T. and S. Kim (2022). Macroprudential policy with earnings-based borrowing constraints. *Working Paper*.

Greenwald, D. (2019). Firm debt covenants and the macroeconomy: The interest coverage channel. *Working Paper*.

Gryglewicz, S. (2011). A theory of corporate financial decisions with liquidity and solvency concerns. *Journal of Financial Economics 99*(2), 365–384.

He, Z. (2011). A model of dynamic compensation and capital structure. *Journal of Financial Economics 100*(2), 351–366.

Hotchkiss, E. S., D. C. Smith, and P. Strömberg (2021). Private equity and the resolution of financial distress. *The Review of Corporate Finance Studies 10*(4), 694–747.

Hugonnier, J., S. Malamud, and E. Morellec (2015). Capital supply uncertainty, cash holdings, and investment. *Review of Financial Studies 28*(2), 391–445.

Hugonnier, J. and E. Morellec (2017). Bank capital, liquid reserves, and insolvency risk. *Journal of Financial Economics 125*(2), 266–285.

Ivashina, V., B. Iverson, and D. C. Smith (2016). The ownership and trading of debt claims in chapter 11 restructurings. *Journal of Financial Economics 119*(2), 316–335.

Ivashina, V. and A. Kovner (2011). The private equity advantage: Leveraged buyout firms and relationship banking. *The Review of Financial Studies 24*(7), 2462–2498.

Kermani, A. and Y. Ma (2023). Asset specificity of nonfinancial firms. *The Quarterly Journal of Economics 138*(1), 205–264.

Kiyotaki, N. and J. Moore (1997). Credit cycles. *Journal of political economy 105*(2), 211–248.

Lian, C. and Y. Ma (2021). Anatomy of corporate borrowing constraints. *The Quarterly Journal of Economics 136*(1), 229–291.

Malamud, S. and F. Zucchi (2018). Liquidity, innovation, and endogenous growth. *Journal of Financial Economics*.

Malenko, A. (2019). Optimal dynamic capital budgeting. *The Review of Economic Studies 86*(4), 1747–1778.

Rampini, A. A., A. Sufi, and S. Viswanathan (2014). Dynamic risk management. *Journal of Financial Economics 111*(2), 271–296.

Rampini, A. A. and S. Viswanathan (2010). Collateral, risk management, and the distribution of debt capacity. *The Journal of Finance 65*(6), 2293–2322.

Rampini, A. A. and S. Viswanathan (2020). Collateral and secured debt. *Unpublished working paper, Duke University*.

Rauh, J. D. and A. Sufi (2010). Capital structure and debt structure. *The Review of Financial Studies 23*(12), 4242–4280.

Sannikov, Y. (2008). A continuous-time version of the principal-agent problem. *Review of Economic Studies 75*(3), 957–984.

Von Thadden, E.-L., E. Berglöf, and G. Roland (2010). The design of corporate debt structure and bankruptcy. *Review of Financial Studies 23*(7), 2648–2679.

White, M. J. (2017). Economics of bankruptcy. *Oxford Handbook of Law and Economics II 447.*

Online Appendix

A Proofs & Omitted Derivations

A.1 Summary of Variables used in Model and Notation

Variable	Equation(s)	Interpretation
M	(2) & (18)	Cash holdings of firm
I	(19)	Intermediary transfer process
Y	(3) & (18)	Promised continuation value to intermediary (coincides with senior-secured debt balance)
$I+Y$	(7)	Total compensation process defining α, β
$k(\alpha, \beta)$	(8)	Required flow risk-premium by intermediary
Y^A	(9)	Autarky value of firm to intermediary
C	(10) & (A.3)	Excess liquidity $M - Y$
ΔM	(13)	Equity issuance proceeds (aka money raised upon refinancing)
$P(C)$	(5) & (15)	Total equity value (including cash)
$\alpha(C)$	(23)	Optimal capital-supply risk
$\alpha_U(C)$	(22)	Optimal capital-supply risk absent LL constraint (14) (coincides with unsecured credit-line balance)
$\beta(C)$	(21)	Optimal cash-flow risk
$\overline{C}$	(16)	Upper boundary (aka dividend payout boundary) (coincides with refinancing target C^*)
$\underline{C}$	(24)	Lower boundary
C^S	(25)	Lower boundary under continuation
$\overline{Y}$	(27)	Maximum promised continuation value to intermediary
$J(C)$	(28)	Value change of existing shareholders upon refinancing
$T(C)$	(30)	Total credit-line balance $Y(C) + \alpha_U(C)$

Table A.1: **Variable definitions.** See Table 1 for parameter definitions.

A.2 Proof of Lemma 1 (Dynamics of Y)

Recall from (3) that $Y_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}\,(dI_s - k_s ds)\right]$, and define

$$A_t = \mathbb{E}_t\left[\int_0^\infty e^{-rs}(dI_s - k_s ds)\right] = \int_0^t e^{-rs}(dI_s - k_s ds) + e^{-rt}Y_t. \tag{A.1}$$

By construction, $A = \{A_t\}$ is a martingale. By the martingale representation theorem, there exist stochastic processes $\alpha = \{\alpha_t\}$ and $\beta = \{\beta_t\}$ such that

$$e^{rt} dA_t = \beta_t \big(dX_t - \mu dt\big) + \alpha_t (d\Pi_t - \pi dt), \tag{A.2}$$

where $dZ_t = \frac{dX_t - \mu dt}{\sigma}$ is the increment of a standard Brownian Motion and $(d\Pi_t - \pi dt)$ is the increment of a compensated Poisson process (a martingale). We differentiate (A.1) with respect to time t to obtain an expression for dA_t, then plug this expression into (A.2) and solve (A.2) to get $dY_t = (rY_t + k_t)dt + \beta_t \sigma dZ_t + \alpha_t (d\Pi_t - \pi dt)$.

A.3 Dynamics of C

Combining (2) and (7), excess liquidity C has the following law of motion:

$$\begin{aligned} dC_t = dM_t - dY_t = \mu_{C,t} dt + \sigma_{C,t} dZ_t + (C_t^* - C_{t^-})\, d\Pi_t - dDiv_t, \\ \mu_{C,t} = \mu + (r - \lambda)\, C_t - \lambda Y_t - \frac{\rho r}{2} (\beta_t \sigma)^2 + \pi \left(\frac{1 - e^{-\rho r \alpha_t}}{\rho r} \right), \\ \sigma_{C,t} = \sigma (1 - \beta_t), \end{aligned} \tag{A.3}$$

where C_t^* is the post-refinancing level of C from raising ΔM given in (13).

A.4 Proof of Proposition 1

We prove Proposition 1 in several parts. First, we provide formal arguments for the reduction in state space and show that equity value can be expressed as function of C_t only and solves the HJB equation (15). Second we prove the concavity of the value function, assuming a well-behaved solution exists in the state space. Third, we provide the formal verification argument that under the optimal contract, the value function indeed solves (15). We impose the regularity condition that sensitivities are bounded, i.e., $|\alpha_t|, |\beta_t| \leq M$ for arbitrarily large $0 < M < \infty$ (see, e.g., Sannikov (2008)). This assumption is needed in the formal verification proof, but we pick M sufficiently large so that this constraint never binds in optimum. We can therefore ignore it in the follow-up analysis.

Further, throughout the Appendix, we define for convenience

$$k_\Pi(\alpha_t) = \alpha_t - \frac{1 - e^{-\rho r \alpha_t}}{\rho r} \tag{A.4}$$

$$k_Z(\beta_t) = \frac{\rho r}{2} (\beta_t)^2. \tag{A.5}$$

This notation allows us to abbreviate certain expressions.

For convenience and to limit the number of distinct cases to deal with, we already conjecture that the lower boundary of the endogenous state space $(\underline{C}, \overline{C})$ satisfies $\underline{C} \geq -[P(\overline{C}) - \overline{C}]$. This conjecture will be verified in the proof of Lemma 3 in Appendix A.5. Further, we impose that dividend payouts must satisfy $dDiv_t \leq C_t - \underline{C}$. In Appendix A.6, we show that $dDiv_t > C_t - \underline{C}$ would lead to a violation of promise-keeping, i.e., an inconsistency with (7), giving rise to the constraint $dDiv_t \leq C_t - \underline{C}$ that applies under a full-commitment contract $\mathcal{C}$. It turns out that this constraint never binds.

Notably, all arguments in this proof are carried out under the assumption that a well-behaved, non-negative, and twice continuously differentiable solution $P(C)$ to (15) exists on the endogenous state space $(\underline{C}, \overline{C})$ (subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$). We formally establish existence of such a solution in Appendix A.12.

To proceed, we rewrite the law of motion of dC_t from (A.3) as $dC_t = \mu_C(C_t)dt + \sigma_C(C_t)dZ_t + (C_t^* - C_t)\, d\Pi_t - dDiv_t$, where denote the drift and volatility of dC_t by

$$\mu_C(C_t) = \left[\mu + (r-\lambda)\, C_t - \lambda Y_t - \frac{\rho r}{2}\,(\beta_t \sigma)^2 + \pi \left(\frac{1 - e^{-\rho r \alpha_t}}{\rho r}\right)\right];\ \sigma_C(C_t) = \sigma(1-\beta_t). \quad \text{(A.6)}$$

As we verify, α_t and β_t will be functions of C_t in optimum, i.e., we can write $\beta_t = \beta(C_t)$ and $\alpha_t = \alpha(C_t)$.

A.4.1 Part I

The endogeneous state space is two dimensional, with two state variables (M, Y). The state space is contained in $\{(M, Y) \in \mathbb{R}^2 : M, Y \geq 0\}$, due to $Y \geq 0$ (intermediary limited commitment) and $M \geq 0$ (non-negativity constraint on cash). Take $C := M - Y$ and rotate the state space by considering (C, Y) rather than (M, Y). To respect $M \geq 0$, the state space must be a subset of $\{(C, Y) \in \mathbb{R}^2 : Y \geq -C\}$. Given $\mathcal{C}$, the time-t equity value is

$$P_t = P(C_t, Y_t) = \mathbb{E}_t \left[\int_t^{\tau} e^{-r(s-t)}(dDiv_s - \Delta M_s d\Pi_s)\Big| C_t = C, Y_t = Y\right].$$

By the dynamic programming principle, $P_t = P(C, Y)$ solves the Hamilton-Jacobi-Bellman (HJB) equation, i.e., the partial differential equation (PDE)

$$rP\,(C, Y))\, dt = \max_{dI, \Delta M, dDiv \geq 0} dDiv + \mathbb{E}[dP(C, Y) - \Delta M d\Pi]. \quad \text{(A.7)}$$

In the following, we prove that $P(C, Y)$ only depends on C, in that $\frac{\partial P(C,Y)}{\partial Y} = \frac{\partial^2 P(C,Y)}{\partial C \partial Y} = \frac{\partial^2 P(C,Y)}{\partial C^2} = 0$, and we can write with a slight abuse of notation $P(C, Y) = P(C)$, and denote $P'(C) = P_C(C, Y)$ as well as $P''(C) = P_{CC}(C, Y)$.

First, fix a state (C, Y) with $M = C + Y \geq 0$ as well as $Y \geq 0$. By (A.30) and (A.3), payouts to the intermediary dI do not change the level of C, but change the level of Y by amount $-dI$. It is always possible to stipulate (negative) payouts $dI = -\varepsilon < 0$ for $\varepsilon > 0$ to the intermediary, which by (A.30) moves the intermediary's continuation payoff from Y to $Y + \varepsilon$ and moves the firm's cash balance from $M = C + Y$ to $M + \varepsilon = C + Y + \varepsilon > 0$. In state (C, Y), payouts to the intermediary $dI = -\varepsilon < 0$ are possible but not necessarily optimal, so that $P(C, Y) \geq P(C, Y + \varepsilon)$. Likewise, in state $(C, Y + \varepsilon)$, positive payouts to the intermediary $dI = \varepsilon > 0$, which move intermediary's continuation payoff from $Y + \varepsilon$ to Y, are possible but not necessarily optimal, so $P(C, Y + \varepsilon) \geq P(C, Y - \varepsilon + \varepsilon) = P(C, Y)$. As a result, in the entire state space, we obtain $P(C, Y + \varepsilon) = P(C, Y)$ for $\varepsilon > 0$. Thus, above relationship implies $P_Y(C, Y) = 0$ and $P_{CY}(C, Y) = P_{YY}(C, Y) = 0$, whenever the respective derivatives exist. Thus, $P(C,Y)$ is constant in the Y-dimension, i.e., does not depend on the state of Y. Thus, we write from now on with slight abuse of notation $P(C,Y) = P(C)$.

Second, using $P_Y(C,Y) = P_{CY}(C,Y) = P_{YY}(C,Y) = 0$ and Ito's Lemma, we can expand the right-hand-side of (A.7) to obtain:

$$rP(C,Y)dt = \max_{dI,\Delta M,dDiv\geq 0}\Big\{dDiv + P'(C)\big[\mu_C(C)dt - dDiv\big] + \frac{P''(C)\sigma_C(C)^2 dt}{2} + \pi[P(C^*) - \Delta M]dt\Big\}, \tag{A.8}$$

where the post-refinancing level C^* satisfies $C^* = \Delta M + C - \alpha$. As in related papers (e.g., Bolton et al. (2011)), dividend payouts are optimal only if $P'(C) \leq 1$, occur at a payout boundary $\overline{C}$, and follow a barrier strategy, that is, they cause C to reflect at $\overline{C}$. As such, we have $P(C) = P(\overline{C}) + C - \overline{C}$ and $P'(C) = 1$ for $C > \overline{C}$. The location of the payout boundary is determined by smooth pasting and super contact conditions, that is, $P'(\overline{C}) = 1$ and $P''(\overline{C}) = 0$. We verify the optimality of this dividend payout strategy in the verification argument in Part III. Due to the (downward) reflection of C at $\overline{C}$, the (endogenous) state space can be written as an interval $(\underline{C}, \overline{C})$, with the boundaries to be characterized later on.

Third, because payouts to the intermediary dI are always possible (when $M > 0$), do not change the level of C, but change the level of Y by amount $-dI$, controlling payouts dI implies controlling the level of Y. Therefore, Y becomes a control variable in the dynamic optimization, as its level can be freely adjusted via dI. More generally, the payout process dI is fully characterized by Y, α, and β, due to (7). Thus, instead of working with dI as control variable, we work with α, β, and Y. Once we have solved for the optimal contract and the optimal level of Y, we can back out the payout process dI (see (A.31)). Likewise, due to $C^* + \alpha - C = \Delta M$, controlling ΔM is equivalent to controlling C^*; in what follows, we work with C^* rather than ΔM as control variable.

For $C \in (\underline{C}, \overline{C})$ where $dDiv = 0$, the HJB equation (A.8) reduces to the ODE:

$$rP(C) = \max_{\alpha,\beta,Y,C^*}\Big\{P'(C)\left[\mu + (r-\lambda)C - \lambda Y - \sigma^2 \cdot k_Z(\beta) + \pi\left(\alpha - k_\Pi(\alpha)\right)\right] + P''(C)\frac{\sigma^2}{2}(1-\beta)^2 + \pi\left[P(C^*) - P(C) - (C^* - C + \alpha)\right]\Big\},$$

subject to $Y \geq \max\{-C, 0\}$ and $\alpha \in \mathcal{S}(C^*, C)$. The above ODE is equivalent to (15) after rewriting and spelling out the constraints on α and Y into the max operator.

We henceforth assume that a twice continuously differentiable and non-negative solution $P(C)$ to (15) exists on the endogenous state space $(\underline{C}, \overline{C})$ (subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$). We formally establish existence of such a solution in Appendix A.12. Next, we prove $\overline{C} > 0$. To see this, note that we can evaluate the ODE (15) at the payout boundary $\overline{C}$ to obtain $P\left(\overline{C}\right) - \overline{C} = \frac{\mu}{r} - \frac{\lambda}{r}\left(\overline{C}\mathbb{1}_{\{\overline{C}\geq 0\}}\right)$. This payoff must be strictly lower than the NPV of the firm, $\frac{\mu}{r}$, which implies $\overline{C} > 0$. Section 2.3 in the main text goes through the maximization of the HJB equation (15), and derives the optimal control variables as functions of excess liquidity C, that is, $Y = Y(C)$, $M = M(C)$, $\alpha = \alpha(C)$, $\beta = \beta(C)$, and $C^* = C^*(C)$.

A.4.2 Part II — Concavity of Value Function

For convenience, we already conjecture a lower boundary $\underline{C}$ satisfying $\underline{C} \geq -(P(\overline{C}) - \overline{C})$. Further, we can conjecture and verify that $P'(C) \geq 1$ on the state space.

We can solve the optimization in the HJB equation (15) (see also Section 2.3 in the main text). We have $\beta(C) = \frac{P''(C)}{P''(C) - \rho r P'(C)}$ if $P''(C) < 0$ and $\beta(C)$ if $P''(C) \geq 0$. As, by assumption, $P(C)$ is twice continuously differentiable, we have that $P''(C) > -\infty$ for any $C \in (\underline{C}, \overline{C})$, so that $\beta(C) \in [0, 1)$.

Under the optimal choice of the refinancing target $C^* = \overline{C}$ which is independent of C, we define

$$J(C) \equiv P\left(\overline{C}\right) - \Delta M - P\left(C\right) = \left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P\left(C\right) - C\right] - \alpha. \tag{A.9}$$

The jump in the value function upon refinancing $J(C)$ is also defined in (28). The constraint (14) can be rewritten as $J(C) \geq P(C)$. When $J(C)$ is differentiable (which is the case when $\alpha(C)$ is differentiable), then $J'(C) = 1 - P'(C) - \alpha'(C)$. We now rewrite the HJB equation (15) as

$$rP\left(C\right) = \max_{\beta \in [0,1]} \left\{ P'(C)\mu_C(C) + \frac{P''(C)}{2}\sigma^2(1 - \beta(C))^2 + \pi \cdot J(C) \right\}, \tag{A.10}$$

under the optimal choice of α in (23), β, $Y = \max\{-C, 0\}$ and $C^* = \overline{C}$, and with $\mu_C(C)$ from (A.6).

When $P''(C)$ and $\alpha(C)$ are differentiable, we can use the envelope theorem and differentiate the HJB equation (A.10) under the optimal $\beta = \beta(C) \in [0, 1)$, satisfying $(1 - \beta(C))\sigma^2 > 0$, with respect to C and rearrange to obtain

$$P'''(C) = \frac{2}{(1 - \beta(C))^2\sigma^2} \left(P'(C)\lambda \mathbb{1}_{\{C \geq 0\}} - P''(C)\mu_C(C) - \pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right) \right), \tag{A.11}$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function which is equal to one if $\{\cdot\}$ is true and is equal to zero otherwise. The set of points at which either $P''(C)$ or $\alpha(C)$ is not differentiable is countable; therefore, for any C, the limits $\lim_{x \uparrow C} P'''(C), \lim_{x \downarrow C} P'''(C)$ and $\lim_{x \uparrow C} \alpha'(C), \lim_{x \downarrow C} \alpha'(C)$ exist and are well-defined.

Suppose that $\alpha(C)$ is differentiable, and recall $\alpha(C) = \min\{\alpha_C(C), \alpha_U(C)\}$. If $\alpha(C) = \alpha_C(C) = P(\overline{C}) - \overline{C} + C$, then $\alpha'(C) = 1$ and $J'(C) = -P'(C) < 0$. As such,

$$\pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right) = \pi P'(C) \left(e^{-\rho r \alpha(C)} - 1 \right) \leq 0. \tag{A.12}$$

When $\alpha(C) = \alpha_U(C) = \frac{\ln P'(C)}{\rho r}$, then

$$\begin{aligned} \pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right) &= \pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + 1 - P'(C) - \alpha'(C) \right) \\ &= \pi \left(\alpha'(C) + 1 - P'(C) - \alpha'(C) \right) = \pi(1 - P'(C)) \leq 0. \end{aligned} \tag{A.13}$$

where it was used that $J'(C) = 1 - P'(C) - \alpha'(C)$ and $e^{-\rho r \alpha(C)} = 1/P'(C)$ as well as

$P'(C) \geq 1$. Thus, altogether,

$$\pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right) \leq 0, \tag{A.14}$$

provided $\alpha(C)$ is differentiable.

At the payout boundary, it therefore holds that $P'(\overline{C}) = 1$, and $P''(\overline{C}) = 0$ and

$$\lim_{C \uparrow \overline{C}} \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right) = \lim_{C \uparrow \overline{C}} (1 - P'(C)) = 0. \tag{A.15}$$

As $\overline{C} > 0$, $P''(\overline{C}) = 0$, $P'(\overline{C}) = 1$, (A.11) and (A.15) imply $\lim_{C \uparrow \overline{C}} P'''(C) > 0$. Because $\lim_{C \uparrow \overline{C}} P'''(C) > 0$ for $\overline{C} > 0$, we obtain $P'''(C) > 0$, and therefore $P''(C) < 0$ and $P'(C) > 1$ and in a left-neighbourhood of $\overline{C}$. Suppose to the contrary that there exists $C' \in (\underline{C}, \overline{C})$, with $P''(C') \geq 0$. Define $\hat{C} = \sup\{C \in (\underline{C}, \overline{C}) : P''(C) \geq 0\}$; note that $\hat{C} < \overline{C}$ and $P''(C) < 0$ for $C \in (\hat{C}, \overline{C})$. By continuity of $P''(C)$, $P''(\hat{C}) = 0$, and $P'(\hat{C}) > 1$.

By (A.14), $\lim_{C \downarrow \hat{C}} \pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right)$ is weakly negative. Hence, (A.11) implies $\lim_{C \downarrow \hat{C}} P'''(C) \geq 0$, where the inequality is strict if $\hat{C} \geq 0$. Consider $\hat{C} \geq 0$. Due to $\lim_{C \downarrow \hat{C}} P'''(C) > 0$, there exists $C' > \hat{C}$ with $P''(C') > 0$, contradiction the definition of $\hat{C}$.

Next, take $\hat{C} < 0 < \overline{C}$. Distinguish two cases. First, when $\pi J(\hat{C}) > 0 \geq -\pi P(\hat{C})$, then (14) does not bind in a neighbourhood of $\hat{C}$ and $\alpha(C) = \alpha_U(C)$. Thus, by (A.13):

$$\lim_{C \downarrow \hat{C}} \pi \left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) \right) = \pi(1 - P'(\hat{C})) < 0.$$

Then, (A.11) implies $\lim_{C \downarrow \hat{C}} P'''(C) > 0$. Thus, there exists $C' > \hat{C}$ so that $P''(C') > 0$, which contradicts the definition of $\hat{C}$.

Second, suppose $\pi J(\hat{C}) \leq 0$. Because $\hat{C} \in (\underline{C}, \overline{C})$, $P'(\hat{C}) > 1$, and $P(C) \geq 0$ for all $C \in (\underline{C}, \overline{C})$, it follows that $P(\hat{C}) > 0$ which — by (A.10) — implies that $\mu_C(\hat{C}) > 0$. By definition of $\hat{C}$ and because $\hat{C} < \overline{C}$, there must exist $\epsilon > 0$ such that $P''(C) < 0$, $\mu_C(C) > 0$, and $P'''(C)$ exists with $P'''(C) < 0$ for $C \in (\hat{C}, \hat{C} + \epsilon)$. But, using $P''(C) < 0$ and $\mu_C(C) > 0$ as well as (A.14), we obtain from (A.11) that $P'''(C) > 0$ for $C \in (\hat{C}, \hat{C} + \epsilon)$, a contradiction.

Either way, it follows that $P''(C) < 0$ for all $C \in (\underline{C}, \overline{C})$, which concludes the proof.

A.4.3 Part III — Verification Argument

Let $\mathcal{C}$ be the contract which implements the controls according to the optimization in the HJB equation (15) and under which equity value $P_t = P(C_t)$ solves (15) on $(\underline{C}, \overline{C})$ with dividend payout boundary $\overline{C}$ subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$. And, consider any other contract $\hat{\mathcal{C}}$ that respects the intermediary's and the investors' limited commitment and $dDiv_t \leq C_t - \underline{C}$. We show that contract $\mathcal{C}$ yields higher payoff at $t = 0$ than any other admissible contract $\hat{\mathcal{C}}$.

For $t \leq \tau$ (possibly $\tau = \infty$), define $\mathcal{L}P(C_t) = P'(C_t)\mu_C(C_t) + \frac{\sigma_C(C_t)^2 P''(C_t)}{2}$, with $\mu_C(C_t)$ and $\sigma_C(C_t)$ from (A.6), and

$$G_t^P = \int_0^t e^{-rs}(dDiv_s - \Delta M_s d\Pi_s) + e^{-rt} P(C_t) \mathbb{I}_{\{t<\tau\}},$$

where $\Delta M_t = C_t^* - C_t + \alpha_t$. Note that G_t^P is the equity value when the contract $\hat{\mathcal{C}}$ is followed up to time t and after time t the contract $\mathcal{C}$ is followed. By Itô's Lemma for $t < \tau$:

$$\begin{aligned} e^{rt} dG_t^P &= \{-rP(C_t) + \mathcal{L}P(C_t)\}\, dt + (P(C_t^*) - P(C_t))d\Pi_t \\ &\quad - \Delta M_t d\Pi_t + \big(1 - P'(C_t)\big) d\hat{D}iv_t + \sigma_{Ct} P'(C_t) dZ_t \\ &\equiv \mu_t^G dt + \big(1 - P'(C_t)\big) dDiv_t + \sigma_{Ct} P'(C_t) dZ_t + (P(C_t^*) - P(C_t) - \Delta M_t)(d\Pi_t - \pi dt). \end{aligned}$$

We define μ_t^G as

$$\mu_t^G = -rP(C_t) + \mathcal{L}P(C_t) + \pi(P(C_t^*) - P(C_t) - \Delta M_t).$$

Next, note that we can rewrite the HJB equation (15) as

$$rP(C_t) = \max_{\alpha_t, \beta_t, Y_t, C^*} \{\mathcal{L}P(C_t) + \pi\,[P(C_t^*) - P(C_t) - \Delta M_t]\}\,, \tag{A.16}$$

subject to all relevant constraints, where we use time subscripts in this part of the proof.

As a result, the HJB equation (A.16) implies that the drift term of $e^{rt} dG_t^P$ — that is, $\mu_t^G = -rP(C_t) + \mathcal{L}P(C_t) + \pi(P(C_t^*) - P(C_t) - \Delta M_t)$ — is zero under the controls obtained via the optimization in the HJB equation (15). Moreover, any other strategy $\hat{\mathcal{C}}$ and choice of $(\alpha_t, \beta_t, Y_t, C_t^*)$ makes this term (weakly) negative, so that $\mu_t^G \leq 0$ for $t < \tau$. Because of $dDiv_t \geq 0$, $P'(C_t) > 1$ for $C_t < \overline{C}$, and $P'(C_t) = 1$ for $C_t \geq \overline{C}$, the term $\big(1 - P'(C_t)\big)$ is (weakly) negative under any dividend payout policy $dDiv_t$ and zero under the dividend payout policy $dDiv_t$ that causes C_t to reflect at $\overline{C}$.

Next, our regularity conditions ensure that α_t, β_t are bounded. Thus, $\sigma_{Ct} = \sigma(1 - \beta_t)$ is bounded too. In addition, $P'(C)$ and $P(C)$ are bounded over $(\underline{C}, \overline{C})$, as $P(C)$ is twice continuously differentiable on the same interval. Thus, $\mathbb{E}\Big[\int_0^t e^{-rs} \sigma_{Cs} P'(C_s) dZ_s\Big] = 0$. for all $t \leq \tau$. As α_t and $P(C_t)$ are bounded, we obtain likewise $\mathbb{E}\Big[\int_0^t e^{-rs}(P(C_s^*) - P(C_s) - \Delta M_s)(d\Pi_s - \pi ds)\Big] = 0$. Therefore, the process $\{G_t^P\}$ follows a supermartingale (i.e., decreases in expectation) up to τ. By the optional stopping theorem, the process $\{G_{t\wedge\tau}^P\}$ follows a supermartingale too, so $\mathbb{E}[G_{t\wedge\tau}^P] \leq G_0^P$. Taking the limit $t \to \infty$ yields

$$P(C_0) - C_0 = G_0^P \geq \lim_{t\to\infty} \mathbb{E}[G_{t\wedge\tau}^P] = \mathbb{E}[G_\tau^P] = \mathbb{E}\left[\int_0^\tau e^{-rs}\Big(dDiv_s - \Delta M_s d\Pi_s\Big)\right].$$

At inception, the firm is penniless and has access to equity financing, i.e., $d\Pi_0 = 1$ and $\Delta M_0 = C_0$, so ex-ante payoff under $\mathcal{C}$ is $P(C_0) - C_0$. Because contract $\mathcal{C}$ yields equity value $G_0^P = P(C_0) - C_0$, it maximizes the equity value over all admissible contracts that respect the intermediary's and the investors' limited commitment.

A.5 Proof of Lemma 3

The proof is split in three parts. Part I presents auxiliary results and conditions that the lower boundary satisfies in the survival scenario. Part II derives (25). Part III demonstrates that the firm optimally never liquidates and $\underline{C} = C^S$ if and only if $C^S \leq -L$; otherwise, liquidation occurs in finite time and $\underline{C} = -L$. In either scenario, $P(\underline{C}) = 0$.

A.5.1 Part I — Auxiliary Results

Lemma 6. *Suppose the firm never liquidates (i.e., $\tau = \infty$) and $P(C)$ solves the HJB equation* (15)*. Define $C^S = \inf\{C \geq -\frac{\mu}{r} : \mu_C(C) \geq 0, P(C) \geq 0, \sigma_C(C) = 0\}$. Then, $\mu_C(C^S) = \sigma_C(C^S) = 0$, $\beta(C^S) = 1$, $P(C^S) = 0$, and $\alpha(C^S) = P(\overline{C}) - \overline{C} + C^S$.*

Proof of Lemma 6. Rewrite the HJB equation (15) under the optimal controls $\alpha(C)$ (see (23)), $\beta(C)$ (see (21)), $Y(C) = \max\{-C, 0\}$, $J(C)$ (see (A.9)), and $C^* = \overline{C}$ as

$$rP(C) = P'(C)\mu_C(C) + \frac{P''(C)(\sigma_C(C))^2}{2} + \pi J(C). \tag{A.17}$$

Holding $\alpha = \alpha(C)$ fixed, $J(C)$ decreases with C, as $P'(C) \geq 1$.

Note that $\sigma_C(C) = \sigma(1 - \beta(C)) = 0$ is equivalent to $\beta(C) = 1$. As $\mu_C(C)$ increases with C, decreases with $\beta(C)$, and increases with $\alpha(C)$, as $J(C)$ decreases with $\alpha(C)$, as the right hand side of (14) (with $C^* = \overline{C}$) increases with C, and as equity value $P(C)$ is characterized by (A.17), it follows that $\sigma_C(C^S) = 0 \iff \beta(C^S) = 1$, $J(C^S) = 0$, $\mu_C(C^S) = 0$, and $P(C^S) = 0$ (so $J(C^S) = -P(C^S)$ and (14) is tight at $C = C^S$). In more detail, if it were $\mu_C(C^S) > 0$, there would exist $C' < C^S$ such that the contract could implement $\sigma_C(C') = 0$ and $\mu_C(C') \geq 0$ with the same choice of β (i.e., $\beta(C') = \beta(C^S) = 1$) and α such that $J(C') \geq 0 \geq -P(C')$ holds and $P(C') \geq 0$ due to (A.17), contradicting the definition of C^S.

Likewise, if it were $J(C^S) > -P(C^S)$, then there would exist $C' < C^S$, such that the contract can stipulate $J(C') \geq -P(C')$, $\alpha(C') \geq \alpha(C^S) \geq 0$, $\beta(C') = 1$, and $\mu_C(C') \geq 0$ as $\mu_C(C)$ increases with α which leads to $P(C') \geq 0$, contradicting the definition of C^S. Finally, $P(C^S) > 0$ while $\beta(C^S) = 1$ would imply $\mu_C(C^S) > 0$ or $J(C^S) > 0$, again leading to a contradiction. Thus, $\sigma_C(C^S) = 0$, $J(C^S) = 0$, $\mu_C(C^S) = 0$, and $P(C^S) = 0$.

Next, we show that $P(C^S) = 0$ implies $\beta(C^S) = 1$, $J(C^S) = 0$, and $\mu_C(C^S) = 0$ under the optimal controls from (15). According to the optimization in (15), the optimal choice of $\alpha(C^S)$ and $\beta(C^S)$ induces $P(C^S) = 0$. Setting $\beta(C^S) = 1$ and $\alpha(C^S)$ such that $J(C^S) = 0$ implies, by definition of C^S, $\mu_C(C^S) = \sigma_C(C^S) = J(C^S) = 0$ and therefore $rP(C^S) = P'(C^S)\mu_C(C^S) + \frac{P''(C^S)(\sigma_C(C^S))^2}{2} + \pi J(C^S) = 0$. Thus, setting $\beta(C^S) = 1$ and $\alpha(C^S)$ such that $J(C^S) = 0$ is optimal and consistent with the optimization in (15). □

A.5.2 Part II — Derivation of (25)

As shown in Lemma 6, we have $\mu_C(\underline{C}) = P(\underline{C}) = 0$ and $\beta(\underline{C}) = 1$ for $\underline{C} = C^S$. To derive an expression for C^S, one first uses (A.3) to calculate the drift of excess liquidity under the optimal choice of Y derived in the previous section (that is, $Y(C) = \max\{-C, 0\}$):

$$\mu_C(C) = \mu + \left(r - \lambda \mathbb{1}_{\{C \geq 0\}}\right) C - \frac{\rho r}{2}\sigma^2 \beta(C)^2 + \pi \left(\frac{1 - e^{-\rho r \alpha(C)}}{\rho r}\right), \tag{A.18}$$

where $\mathbb{1}_{\{.\}}$ denotes the indicator function, i.e., it is 1 if $\{\cdot\}$ is true and 0 otherwise. The HJB equation (15) evaluated under the optimal controls $\alpha(C)$ and $\beta(C)$ as well as $C^* = \overline{C}$ can be rewritten as in (A.17). Due to $\mu_C(\underline{C}) = P(\underline{C}) = \sigma_C(\underline{C}) = 0$, we have by means of (A.17) that $J(\underline{C}) = 0$ and therefore $\alpha(\underline{C}) = P(\overline{C}) - [\overline{C} - \underline{C}] = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C} + \underline{C}$. The last equality uses that at the payout boundary $\overline{C} > 0$, the HJB equation (15) implies $P(\overline{C}) = \frac{\mu}{r} + \overline{C} - \frac{\lambda \overline{C}}{r}$, due to $\beta(\overline{C}) = \alpha(\overline{C}) = P'(\overline{C}) - 1 = P''(\overline{C}) = 0$.

Substituting in for the optimal policies, and using $\alpha(\underline{C})$ from above in $\mu_C(\underline{C}) = 0$ while using that $\sigma_C(\underline{C}) = 0 \iff \beta(\underline{C}) = 1$, we have

$$0 = \mu_C(\underline{C}) = \mu + r\underline{C} - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\left(1 - e^{-\rho r\left[\frac{\mu}{r} - \frac{\lambda}{r}\overline{C} + \underline{C}\right]}\right). \tag{A.19}$$

We use the following Lemma to solve for $\underline{C}$:

Lemma 7. *The solution to*

$$0 = a + x + e^{(b + c \cdot x)} \tag{A.20}$$

is given by

$$x = -\frac{w(c \cdot \exp\{b - a \cdot c\}) + a \cdot c}{c}, \tag{A.21}$$

where $w(\cdot)$ is the primary branch of the Lambert-w function.

Proof of Lemma 7. Define $z \equiv c \cdot \exp\{b - ac\}$. Plugging in proposed solution (A.21) into (A.20), we have

$$\begin{aligned}
0 &= a + \left(-\frac{w(c \cdot \exp\{b - ac\})}{c} - a\right) + \exp\{b - w(c \cdot \exp\{b - ac\}) - ac\} \\
&= -\frac{w(c \cdot \exp\{b - ac\})}{c} + \exp\{b - ac\}\exp\{-w(c \cdot \exp\{b - ac\})\} \\
&= -w(c \cdot \exp\{b - ac\}) + c \cdot \exp\{b - ac\}\exp\{-w(c \cdot \exp\{b - ac\})\} \\
&= -w(z) + z\exp(-w(z))
\end{aligned}$$

where we multiplied through by $c \neq 0$ in the second-to-last line. The last line equals zero by definition of the Lambert-w function $w(z)e^{w(z)} = z \iff w(z) = z \cdot e^{-w(z)}$. □

Next, we rewrite (A.19) as

$$0 = \frac{-\rho r}{\pi}\left(\mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\right) - \frac{\rho r^2}{\pi}\underline{C} + e^{-\rho(\mu - \lambda\overline{C}) - \frac{\pi}{r}\frac{\rho r^2}{\pi}\underline{C}}. \tag{A.22}$$

Define $a \equiv \frac{-\rho r}{\pi}\left(\mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\right)$, $b \equiv -\rho(\mu - \lambda\overline{C})$, $c \equiv \frac{\pi}{r}$, and $x \equiv -\frac{\rho r^2}{\pi}\underline{C}$. We now apply the above lemma to solve (A.22) for x and to thus obtain (25), that is,

$$\underline{C} = C^S = \frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r} - Y^A, \tag{A.23}$$

where $w(\cdot)$ is the Lambert function (i.e., $w(z)$ is the principal-branch solution to $we^w = z$). Note that when $\pi = 0$, then $C^S = -Y^A$, where Y^A is the autarky value defined in (9).

Finally, we show that $-\underline{C} \leq P(\overline{C}) - \overline{C}$ and as such $\alpha(\underline{C}) \geq 0$ as well as $\alpha(C) = \min\{\alpha_C(C), \alpha_U(C)\} \geq 0$ for all C. For this sake, note that $P(\overline{C}) - \overline{C}$ must weakly exceed Y^A. When $\pi = 0$, then $-\underline{C} = Y^A$ and the claim follows. Next, consider $\pi > 0$. Suppose to the contrary $-\underline{C} > P(\overline{C}) - \overline{C}$, i.e., $\alpha(\underline{C}) < 0$. Then, $\pi\left(\frac{1-e^{-\rho r\alpha(\underline{C})}}{\rho r}\right) < 0$ and, by (A.19), we obtain $-\underline{C} < Y^A \leq P(\overline{C}) - \overline{C}$, a contradiction.

A.5.3 Part III

We determine when liquidation or survival (i.e., $\tau = \infty$) scenario applies. We distinguish between i) $C^S < -L$ and ii) $C^S > -L$. The knife-edge case $C^S = -L$ follows analogously.

Suppose that $C^S < -L \leq 0$. We conjecture and verify that the survival scenario prevails (so that $\tau = \infty$). Suppose that $P(C)$ solves (15) subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = P(C^S) = 0$. Previous results imply that $P'(C) > 1$ on $(C^S, \overline{C})$. Note that $Y(C) = \max\{0, -C\}$ implies $Y(C) \leq -C^S$. Due to $P'(C) > 1$ for $C < \overline{C}$ and $P(C^S) = 0$, it follows that $P(C) > C - C^S$. If in state $C \geq C^S$ the firm is liquidated and all cash holdings $M(C)$ are paid out (to shareholders and intermediary), total firm value "just before" liquidation is the sum of cash balance $M(C)$ and liquidation value L, which is split between intermediary and shareholders. Thus, shareholders would obtain upon liquidation in state $C > C^S$, $M(C) + L - Y(C)$, while the intermediary receives $Y(C)$ (dollars). Note that

$$M(C) + L = C + Y(C) + L \leq C - C^S + Y(C) < P(C) + Y(C),$$

where the first inequality used that $L \leq -C^S$, and the second that $P(C) > C - C^S$. As a result, $P(C) > M(C) + L - Y(C)$, and liquidation in state $C > C^S$ is not optimal for shareholders. As, in addition, liquidation in state $C = C^S$ would violate $Y_\tau = -C^S \leq L$, the survival scenario prevails in optimum, i.e., $\tau = \infty$.

Suppose that $C^S > -L$. It follows that $Y(C) = \max\{0, -C\} = 0$ for $C \geq C^S$, and $M(C) = \max\{C, 0\}$. Conditional on survival, i.e., no liquidation at the lower boundary $\underline{C} = C^S$ and $\tau = \infty$, the boundary condition $P(C^S) = 0$ applies for the hypotehtical value function. However, survival cannot be optimal for shareholders. Liquidating the firm at $C = C^S$ and paying out $M(C^S) = \max\{0, C^S\} \geq 0$ dollars as dividends yields value $L + \max\{0, C^S\} > 0$ for shareholders. As such, the liquidation scenario prevails.

It remains to show that liquidation occurs the first time C falls to $-L$ so that $\underline{C} = -L \leq 0$. To start with, note that liquidation at $C < -L \leq 0$ with $M(C) = 0$ is not possible because at the time of liquidation, $Y(C) \leq L$ must hold to ensure promise-keeping, and $C < -L$ would imply $Y(C) = -C > L$. Thus, liquidation can only occur in states $C \geq -L$. Next, suppose the firm is liquidated at $C = -L$, so the intermediary receives a payout of

$Y(C) = -C = L$ and $P(-L) = 0$ at liquidation. With $P(C)$ the value function solving (15) subject to $P(-L) = P'(\overline{C}) - 1 = P''(\overline{C}) = 0$, we have that $P'(C) > 1$. This implies for $C < \overline{C}$ that $P(C) > C + L$ for $C > -L$. Consider $C > -L \iff -C < L$. If the firm were liquidated in state $C > -L$, then shareholders and intermediary would jointly receive

$$M(C) + L = C + Y(C) + L < P(C) + Y(C),$$

where the inequality uses $P(C) > C + L$. Thus, liquidation at $C > -L$ is not optimal. As liquidation must occur for $C \geq -L$, it follows that optimal liquidation occurs at $C = \underline{C} = -L$, i.e., $\tau = \inf\{t \geq 0 : C_t = -L\}$. Taken together, $\underline{C} = \min\left\{C^S, -L\right\}$ and $P\left(\underline{C}\right) = 0$.

A.6 Dividend Payouts and Promise-Keeping

Lemma 8. *In a full-commitment contract $\mathcal{C}$, dividends must satisfy $dDiv_t \leq C_t - \underline{C}$.*

Proof. Due to $P(C) \geq 0$, $Y(C) \leq M(C) + \frac{\mu}{r}$ and $C \geq -\frac{\mu}{r}$. Suppose to the contrary that at time t, the firm pays $dDiv_t > C_t - \underline{C}$ and so causes C_t to drop to value $C' < \underline{C} = \min\{-L, C^S\} \leq 0$. After the dividend payout, we have $Y = Y' > L$. By definition of C^S and Lemma 6, we have for any $C < C^S$ that $\mu_C(C) < 0$ or $\sigma_C(C) \neq 0$ in case the firm does not liquidate. Liquidation in any state $C < -L$ violates promise-keeping (as we have $Y > L$). Suppose the firm liquidates in state $C'' < \underline{C}$. Then, with strictly positive probability, there exists time $T > t$ such that C_T reaches C'', leading to liquidation and a violation of promise-keeping. Next, consider that the firm does not liquidate in any state $C < -L$. As a result, with strictly positive probability, there exists time $T > t$ such that $C_T < -\frac{\mu}{r}$, a contradiction. Thus, Y_t must exhibit a discrete downward jump upon the payout $dDiv_t = C_t - \underline{C}$ ensuring $C' \geq \underline{C}$, which is inconsistent with the dynamics (7) and violates promise-keeping. Thus, dividend payouts satisfy $dDiv_t \leq C_t - \underline{C}$. □

A.7 Proof of Lemma 4

Part I derives a sufficient condition for the stationary distribution to exist. In particular, under this condition, the lower boundary $\underline{C} = C^S$, while conditionally absorbing, is not attainable. Parts II and III show that this condition is satisfied.

A.7.1 Part I

In the interior of the state space for $C \in (\underline{C}, \overline{C})$ when $\sigma_C(C)$ is twice differentiable, the stationary density $g(C)$ — provided its existence — satisfies the Kolmogorov forward (Fokker Planck) equation: $\pi \cdot g(C) = -\frac{\partial}{\partial C}\left[\mu_C(C)g(C)\right] + \frac{1}{2}\frac{\partial^2}{\partial C^2}\left[\sigma_C(C)^2 g(C)\right]$. We can integrate over C to obtain

$$\pi G(C) = G(\underline{C}) - \mu_C(C)g(C) + \frac{1}{2}\frac{\partial}{\partial C}\left[\sigma_C(C)^2 g(C)\right]. \tag{A.24}$$

with stationary distribution function $G(C) = \int_{\underline{C}}^{x} g(x)dx$. When $\underline{C}$ is not accessible, (A.24) is solved subject to $G(\overline{C}) = 1$ and $G(\underline{C}) = 0$. Define the scaled stationary density $\hat{g}(C) =$

$\sigma_C(C)^2 g(C)$, so that $\hat{g}'(C) = 2\pi G(C) + \mu_C(C)g(C) = 2\pi G(C) + 2\hat{g}(C)\left(\frac{\mu_C(C)}{\sigma_C(C)^2}\right)$. Equivalently, the log scaled density has derivative $\frac{d\ln\hat{g}(C)}{dC} = \frac{\hat{g}'(C)}{\hat{g}(C)} = \frac{2\pi G(C)}{\hat{g}(C)} + 2\left(\frac{\mu_C(C)}{\sigma_C(C)^2}\right)$. The boundary $\underline{C}$ is absorbing conditionally on no jumps. A non-degenerate stationary density, with the absorbing boundary at $\underline{C}$, exists if the boundary condition $\hat{g}(\underline{C}) = 0$ can be satisfied together with $\hat{g}(\hat{C}) > 0$ for $\hat{C} > \underline{C}$; in this case, the boundary $\underline{C}$ is never reached or inaccessible. For this to happen, we need that

$$\ln\hat{g}(C) = \ln\hat{g}(\hat{C}) - 2\int_C^{\hat{C}} \frac{2\pi G(c)}{\hat{g}(c)}dc - 2\int_C^{\hat{C}} \frac{\mu_C(c)}{\sigma_C(c)^2}dc$$

tends to $-\infty$, as $C \to \underline{C}$; see Brunnermeier and Sannikov (2014) for an analogous argument in a similar context. A sufficient condition is Feller's test for explosions, i.e.,

$$\lim_{C\to\underline{C}} \int_C^{\hat{C}} \frac{\mu_C(c)}{\sigma_C(c)^2}dc = +\infty. \tag{A.25}$$

In the following two parts, we show that (A.25) is met, which then implies that $\underline{C}$ is never reached and a stationary distribution of states exists.

A.7.2 Part II

Define $\Gamma(C) = -P''(C)/P'(C)$ and rewrite

$$\beta(C) = 1 - \frac{\rho r}{\Gamma(C) + \rho r} \iff \sigma_C(C) = \sigma(1 - \beta(C)) = \frac{\sigma\rho r}{\Gamma(C) + \rho r}. \tag{A.26}$$

Next, we focus on states $C < 0$ and in which $P''(C)$ is differentiable. For $C < 0$, the drift of C becomes

$$\begin{aligned} \mu_C(C) &= \mu + rC - \frac{\sigma^2\rho r}{2}\left[1 - \frac{2\sigma_C(C)}{\sigma} + \left(\frac{\sigma_C(C)}{\sigma}\right)^2\right] + \pi\left[\alpha(C) - k_\Pi(\alpha(C))\right] \qquad \text{(A.27)} \\ &= r(C - \underline{C}) + \pi\big[\alpha(C) - k_\Pi(\alpha(C)) - \alpha(\underline{C}) + k_\Pi(\alpha(\underline{C}))\big] + \frac{\rho r\sigma_C(C)}{2}\left(\frac{2}{\sigma} - \sigma_C(C)\right), \end{aligned}$$

where in the second line we subtracted $\mu_C(\underline{C}) = \mu + r\underline{C} - \frac{\sigma^2\rho r}{2} + \pi\left[\alpha(\underline{C}) - k_\Pi(\alpha(\underline{C}))\right] = 0$. Recall $k_\Pi(\alpha)$ from (A.4). For C sufficiently close to $\underline{C}$ all of these three terms are positive, since $\sigma_C(C)$ tends to zero as C approaches $\underline{C}$. Therefore, a sufficient condition for (A.25) is

$$\lim_{C\to\underline{C}} \int_C^{\hat{C}} \frac{(c - \underline{C})}{\sigma_C(c)^2}dc = +\infty. \tag{A.28}$$

Using (A.26), a sufficient condition for (A.28), and thus for (A.25), is

$$\lim_{C\to\underline{C}} \int_C^{\hat{C}} (c - \underline{C})\Gamma(c)^2 dc = \lim_{C\to\underline{C}} \int_C^{\hat{C}} \frac{(c - \underline{C})}{1/(\Gamma(c))^2}dc = +\infty.$$

In Part III, we will show that there exists constant $\mathcal{K} > 0$ such that $1/\Gamma(C) < \mathcal{K}(C - \underline{C})$ in a neighbourhood $(\underline{C}, \underline{C} + \varepsilon)$ with $\varepsilon > 0$ of $\underline{C}$. Then, we can pick $\hat{C} \in (\underline{C}, \underline{C} + \varepsilon)$ and calculate

$$\begin{aligned}\lim_{C \to \underline{C}} \int_C^{\hat{C}} \frac{(c - \underline{C})}{1/(\Gamma(c))^2} dc \geq \lim_{C \to \underline{C}} \int_C^{\hat{C}} \frac{(c - \underline{C})}{\mathcal{K}^2 (c - \underline{C})^2} dc = \frac{1}{\mathcal{K}^2} \lim_{C \to \underline{C}} \int_C^{\hat{C}} \frac{1}{c - \underline{C}} dc \\ = \frac{1}{\mathcal{K}^2} \lim_{C \to \underline{C}} \left[\ln(\hat{C} - \underline{C}) - \ln(C - \underline{C}) \right] = +\infty.\end{aligned}$$

A.7.3 Part III

Since $\underline{C} = \underline{C}^S$, we have $\lim_{C \to \underline{C}} \beta(C) = 0$, $\lim_{C \to \underline{C}} \sigma_C(C) = 0$, as well as $\lim_{C \to \underline{C}} P''(C) = -\infty$ and $\lim_{C \to \underline{C}} \Gamma(C) = +\infty$. To show $1/\Gamma(C) < \mathcal{K}(C - \underline{C})$ in a neighbourhood of $\underline{C}$, we assume sufficient differentiability and conduct a Taylor expansion around $\underline{C}$ (noting that $1/\Gamma(\underline{C}) = 0$ with a slight abuse of notation):

$$\frac{1}{\Gamma(C)} = -\frac{\Gamma'(C)}{\Gamma(C)^2}(C - \underline{C}) + o((C - \underline{C})^2). \tag{A.29}$$

Calculate $\Gamma'(C) = \frac{-P'''(C)P'(C) + P''(C)^2}{P'(C)^2} = \frac{-P'''(C)}{P'(C)} + \Gamma(C)^2$. The proof is complete if we are able to show that

$$\frac{\Gamma'(C)}{\Gamma(C)^2} = 1 - \frac{P'''(C)}{P'(C)\Gamma(C)^2}$$

is bounded in a neighbourhood of $\underline{C}$; this term is bounded if and only if $\frac{P'''(C)}{P'(C)\Gamma(C)^2}$ is bounded.

Next, recall (A.11), which implies for $C < 0$:

$$\begin{aligned}P'''(C) &= \frac{2}{\sigma_C(C)^2} \left(-P''(C)\mu_C(C) - \pi \left[e^{-\rho r \alpha(C)} P'(C) \alpha'(C) + J'(C) \right] \right) \\ &= \frac{2 \left[\Gamma(C) + \rho r \right]^2}{(\sigma \rho r)^2} \left(-P''(C)\mu_C(C) - \pi \left[e^{-\rho r \alpha(C)} P'(C) \alpha'(C) + J'(C) \right] \right),\end{aligned}$$

where $J(C)$ is from (A.9).

If the constraint (14) binds, we have $\pi \left[e^{-\rho r \alpha(C)} P'(C) \alpha'(C) + J'(C) \right] = \pi P'(C) \left(e^{-\rho r \alpha(C)} - 1 \right)$; see (A.12) for a derivation. If it does not bind, we have $\pi \left[e^{-\rho r \alpha(C)} P'(C) \alpha'(C) + J'(C) \right] = \pi(1 - P'(C))$; see (A.13) for a derivation. Either way, it follows that the expression $\chi(C) := \frac{\pi \left[e^{-\rho r \alpha(C)} P'(C) \alpha'(C) + J'(C) \right]}{P'(C)}$ is bounded, specifically in a neighbourhood of $\underline{C}$.

Thus,

$$\frac{P'''(C)}{P'(C)} = \frac{2 \left[\Gamma(C) + \rho r \right]^2}{(\sigma \rho r)^2} \left[\Gamma(C) \mu_C(C) - \chi(C) \right]$$

and

$$\frac{P'''(C)}{P'(C)\Gamma(C)^2} = \frac{2}{(\sigma \rho r)^2} \left[\Gamma(C) \mu_C(C) - \chi(C) \right] + o\left(\frac{1}{\Gamma(C)} \right)$$

As discussed above, the term $\chi(C)$ is bounded, and so are the remainder terms of order

$1/\Gamma(C)$, since $\lim_{C\to\underline{C}}\Gamma(C)=+\infty$.

Suppose now that $\Gamma(C)\mu_C(C)$ is bounded in a neighbourhood of $\underline{C}$. Then, $\frac{P'''(C)}{P'(C)\Gamma(C)^2}$ and therefore also $-\frac{\Gamma'(C)}{\Gamma(C)^2}=\frac{P'''(C)P'(C)}{P''(C)^2}-1$ are bounded in a neighbourhood of $\underline{C}$. Thus, there exists $\mathcal{K}>0$ such that $1/\Gamma(C)<\mathcal{K}(C-\underline{C})$ in a neighbourhood of $\underline{C}$, as desired.

Next, consider that $\Gamma(C)\mu_C(C)$ is *not* bounded in a neighbourhood of $\underline{C}$. In particular, $\lim_{C\to\underline{C}}\Gamma(C)\mu_C(C)=+\infty$. Using (A.27), we obtain for $C<0$ close to $\underline{C}$:

$$\begin{aligned}\Gamma(C)\mu_C(C)&=r\Gamma(C)(C-\underline{C})+\pi\Gamma(C)\big[\alpha(C)-k_\Pi(\alpha(C))-\alpha(\underline{C})+k_\Pi(\alpha(\underline{C}))\big]\\&\quad+\frac{(\rho r)^2\sigma\Gamma(C)}{2(\Gamma(C)+\rho r)}\left(\frac{2}{\sigma}-\sigma_C(C)\right)\\&\le r\Gamma(C)(C-\underline{C})+\pi\Gamma(C)\big[\alpha(C)-k_\Pi(\alpha(C))-\alpha(\underline{C})+k_\Pi(\alpha(\underline{C}))\big]+\rho^2r^2.\end{aligned}$$

Next, note that by (A.4), we have $\alpha(C)-k_\Pi(\alpha)=\frac{1-e^{-\rho r\alpha(C)}}{\rho r}$, which is clearly bounded between 0 and $1/(\rho r)$; the derivative of this term with respect to C is bounded too. Specifically, we can find a constant $\mathcal{K}_1\in(0,\infty)$ such that $\pi\big[\alpha(C)-k_\Pi(\alpha(C))-\alpha(\underline{C})+k_\Pi(\alpha(\underline{C}))\big]<\mathcal{K}_1(C-\underline{C})$ in a neighbourhood of $\underline{C}$. Accordingly, in a neighbourhood of $\underline{C}$, we have $\Gamma(C)\mu_C(C)\le\mathcal{K}_2\Gamma(C)(C-\underline{C})$ for an appropriate constant $\mathcal{K}_2\in(0,\infty)$. Due to $\lim_{C\to\underline{C}}\Gamma(C)\mu_C(C)=+\infty$, we have $\lim_{C\to\underline{C}}\Gamma(C)(C-\underline{C})=+\infty$. In particular, there exists constant $\mathcal{K}>0$ such that $\Gamma(C)(C-\underline{C})>\frac{1}{\mathcal{K}}$, i.e., $\frac{1}{\Gamma(C)}<\mathcal{K}(C-\underline{C})$, in a neighbourhood of $\underline{C}$, concluding the proof.

A.8 Proof of Lemma 5

We show that for $\lim_{C\to\underline{C}}\alpha_U(C)=\infty$, when $\underline{C}=C^S$. This proves the claim by continuity, since $\lim_{C\to\underline{C}}\alpha_C(C)<+\infty$, implying that the constraints (14) binds in a right-neighbourhood of $\underline{C}$. In the proof, we use results from the proof of the preceding Lemma A.7, since some arguments are identical.

Define $\Gamma(C)=-P''(C)/P'(C)$. Note that

$$\alpha_U'(C)=\frac{P''(C)}{\rho rP'(C)}=\frac{-\Gamma(C)}{\rho r}.$$

Thus, for $\hat{C}>\underline{C}$ and $C\in(\underline{C},\hat{C})$, we have

$$\mathcal{D}(C):=\alpha_U(C)-\alpha(\hat{C})=\int_{\hat{C}}^{C}\frac{-\Gamma(c)}{\rho r}dc=\int_{C}^{\hat{C}}\frac{\Gamma(c)}{\rho r}dc.$$

Note that $\lim_{C\to\underline{C}}\alpha_U(C)=+\infty$ is equivalent to $\lim_{C\to\underline{C}}\mathcal{D}(C)=+\infty$. Thus, it suffices to show that $\lim_{C\to\underline{C}}\int_C^{\hat{C}}\Gamma(c)dc=+\infty$ for some $\hat{C}>\underline{C}$.

By the Proof of Lemma 4 in Appendix A.7 (see Part III in particular), we can pick $\hat{C}$ sufficiently close to $\underline{C}$, so that there exists constant $\mathcal{K}>0$ with $1/\Gamma(C)<\mathcal{K}(C-\underline{C})$ for all $C\in(\underline{C},\hat{C})$. That is, when $\hat{C}$ is sufficiently close to $\underline{C}$, then $\Gamma(C)>\frac{1}{\mathcal{K}(C-\underline{C})}$ for $C\in(\underline{C},\hat{C})$.

Hence, for $\hat{C}$ sufficiently close to $\underline{C}$ and $C \in (\underline{C}, \hat{C})$, we calculate

$$\int_C^{\hat{C}} \Gamma(c)dc > \frac{1}{\mathcal{K}} \int_C^{\hat{C}} \frac{1}{c - \underline{C}} dc = \frac{1}{\mathcal{K}} \big[\ln(\hat{C} - \underline{C}) - \ln(C - \underline{C}) \big].$$

As a result,

$$\lim_{C \to \underline{C}} \int_C^{\hat{C}} \Gamma(c)dc \geq \frac{1}{\mathcal{K}} \lim_{C \to \underline{C}} \big[\ln(\hat{C} - \underline{C}) - \ln(C - \underline{C}) \big] = +\infty.$$

Thus, $\lim_{C \to \underline{C}} \alpha_U(C) = +\infty$ and therefore $\alpha_U(C) > \alpha_C(C)$, in that the constraint (14) binds, in a right-neighbourhood of $\underline{C}$.

A.9 Proof of Proposition 2

The claims of Proposition 2 follow from the previous results presented in the main text and proven in the Appendix. The optimal control variables — that is, $Y = Y(C)$, $M = M(C)$, $\alpha = \alpha(C)$, $\beta = \beta(C)$, and $C^* = C^*(C)$ — are derived in the main text in Section 2.3 by going through the optimization in the HJB equation (15).

Appendix A.12 establishes existence of a (twice differentiable) non-negative solution $P(C)$ to the system (15) subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ with $\underline{C}$ from (24). Given this existence result, it follows $\beta(C) < 1$ for $C \in (\underline{C}, \overline{C})$. To see this, recall that the optimal choice of $\beta(C)$ is determined according to the optimization in (15) and therefore satisfies (21), with $\beta(C) \in [0, 1]$ due to concavity on $(\underline{C}, \overline{C})$. It follows that $\beta(C) \to 1$ only if $P''(C) \to -\infty$, as $P'(C) \geq 1$. However, because the value function $P(C)$ is twice continuously differentiable on $(\underline{C}, \overline{C})$, there cannot exist $C' \in (\underline{C}, \overline{C})$ such that $\lim_{C \to C'} P''(C) = -\infty$. As such, there cannot exist $C' \in (\underline{C}, \overline{C})$ such that $\lim_{C \to C'} \beta(C) = 1$. Thus, $\beta(C) < 1$ for $C > \underline{C}$.

That is, while it is always possible to set $\beta(C') = 1$ for $C' > C^S$ to ensure $\mu_C(C') > 0$ and $\sigma_C(C') = 1$ and $C_t \geq C'$ at all times t (with certainty), this is not optimal.[24] So, all states C within $(\underline{C}, \overline{C})$ are reached with positive probability.

By (A.3) and (3), changes in promised payments to the intermediary follow

$$dY = \begin{cases} 0 & \text{for} \quad C > 0 \\ [rY - \mu + \sigma^2 k_Z(\beta) - \pi (\alpha - k_\Pi(\alpha))] \, dt - (1 - \beta)\sigma dZ - Y d\Pi & \text{for} \quad C < 0. \end{cases} \tag{A.30}$$

Recall that $k_\Pi(\alpha)$ and $k_Z(\beta)$ are defined in (A.4). Combining this with the promise keeping constraint (7), we have that for any α, β the transfers to/from the intermediary before liquidation are given by

$$dI = \begin{cases} [\sigma^2 k_Z(\beta) - \pi (\alpha - k_\Pi(\alpha))] \, dt + \beta \sigma dZ + \alpha d\Pi & \text{for} \quad C > 0 \\ \mu dt + \sigma dZ + (\alpha + Y) \, d\Pi & \text{for} \quad C < 0. \end{cases} \tag{A.31}$$

[24] Note that the stipulation of the boundary condition(s) $P(C^S) = 0$ and $\beta(C^S) = 1$ to solve the HJB equation (15) is not an optimality result but a consequence of the requirement C must be bounded from below under incentive compatible contracts and survival; the stipulation of the boundary condition $P(C^S) = 0$ does not per-se preclude $\beta(C') = 1$ for $C' > C^S$.

Alternatively, we write (A.31) as $dI = \mu_I dt + \sigma_I dZ + \alpha_I d\Pi$ with drift $\mu_I = \mu_I(C)$, volatility $\sigma_I = \sigma_I(C)$, and $\alpha_I = \alpha_I(C) = \alpha(C) + Y(C)$. We note that upon liquidation at time τ (with $\tau = \infty$ possible), promise-keeping (7) requires that the intermediary receives a lumpy payout dI_τ equal to the value of promised payments Y_τ at or "just before" liquidation.

A.10 Proof of Proposition 3

First, the full statement of the proposition is as follows:

Proposition 3 (Full statement) For any dI defined by some Markovian (α, β) and optimal Y given in (18), let

$$T_t := -\int_{\tau^\Pi(t)}^{t} e^{\int_s^t \hat{r}_u du} dI_s. \tag{A.32}$$

If $T_t = T(C_t)$ Markov, then the pair $(T(C), \hat{r}(C))$ is unique, $\hat{r}(C)T(C) = \mu_T(C) + \mu_I(C)$,

$$T(C) = \int_C^{\overline{C}} \frac{\beta(x)}{1-\beta(x)} dx + Y(C) \tag{A.33}$$

and $T'(C) < 0$. Further, under the optimal $\beta(C)$ given in (21), $T(C)$ simplifies to

$$T(C) = \alpha_U(C) + Y(C). \tag{A.34}$$

Finally, under both the optimal $\beta(C)$ and $\alpha(C)$ given in (21) and (23), respectively, $\hat{r}(C)$ simplifies to

$$\hat{r}(C)T(C) = \frac{\lambda}{\rho r} \mathbb{1}_{\{C \geq 0\}} + rY(C). \tag{A.35}$$

A.10.1 Part I

Let $\tau^\Pi(t)$ be the most recent refinancing time, and define T_t as

$$T_t := -\int_{\tau^\Pi(t)}^{t} e^{\int_s^t \hat{r}(C_u) du} dI_s. \tag{A.36}$$

We are looking for functions $\hat{r}(C)$ such that T_t can be represented as a Markovian function $T(C_t)$. Doing so, we show that there exists (unique) $\hat{r}(C)$ such that T_t is Markovian and depends on C_t only, in that we can write $T_t = T(C_t)$. We also solve for $T(C_t)$, thereby showing its uniqueness. First, we differentiate T_t for $t > \tau^\Pi(t)$ in (A.36) w.r.t. t to get

$$dT_t = \hat{r}(C_t)T_t dt - dI_t - T_{t-} d\Pi_t = \hat{r}(C_t)T_t dt - \mu_I(C_t)dt + \sigma_I(C_t)dZ_t + \alpha_I(C_t)d\Pi_t - T_{t-}d\Pi_t, \tag{A.37}$$

where dI_t follows (A.31) and $\mu_I(C_t)$, $\sigma_I(C_t)$, $\alpha_I(C_t)$ are implicitly defined in (A.31). When $d\Pi_t = 1$, then T_t is reset to zero, where $T_{t-} := \lim_{s \uparrow t} T_s$ denotes the left-limit of T_t.

Applying Ito's Lemma to $T(C_t)$ for $C_t \in (\underline{C}, \overline{C})$ we get:

$$dT(C_t) = \left(T'(C_t)\mu_C(C_t) + \frac{T''(C_t)\sigma_C(C_t)^2}{2}\right) dt + T'(C_t)\big[\sigma_C(C_t)dZ_t - dDiv_t\big] + \big[T(\overline{C}) - T(C_t)\big] d\Pi_t. \tag{A.38}$$

By conjecture $T_t = T(C_t)$, so $dT_t = dT(C_t)$ for $C_t \in (\underline{C}, \overline{C})$. We now match the terms in (A.37) and (A.38). First, matching the exposure to Poisson shocks $d\Pi_t$ in (A.37) and (A.38), we obtain $T(\overline{C}) = 0$. Second, matching the exposure to Brownian shocks, we obtain $T'(C)\sigma_C(C) = -\sigma_I(C)$. Plugging in $\sigma_I(C)$ from (A.31) — that is, $\sigma_I(C) = \sigma$ for $C < 0$ and $\sigma_I(C) = \beta(C)\sigma$ for $C \geq 0$ — and $\sigma_C(C) = \sigma(1 - \beta(C))$, we have the first-order ODE for $C \in (\underline{C}, \overline{C})$

$$T'(C) = -\left(\frac{\big[\mathbb{1}_{\{C<0\}} + \mathbb{1}_{\{C\geq 0\}}\beta(C)\big]}{[1 - \beta(C)]}\right) \tag{A.39}$$

with boundary condition $T(\overline{C}) = 0$. As $\beta(C) < 1$ for $C \in (\underline{C}, \overline{C})$, the right-hand-side of (A.39) is well-defined on $C \in (\underline{C}, \overline{C})$, with potentially degenerate boundary behavior as $C \to \underline{C}$. The solution of this first-order ODE with boundary condition $T(\overline{C}) = 0$ is unique for integrable $\beta(C)$, and is given by

$$T(C) = \int_C^{\overline{C}} \frac{\mathbb{1}_{\{C<0\}} + \mathbb{1}_{\{C>0\}}\beta(x)}{1 - \beta(x)} dx = \int_C^{\overline{C}} \frac{\beta(x)}{1 - \beta(x)} dx + Y(C) \tag{A.40}$$

for some unique $\hat{r}(C)$ that is a function of both $\alpha(C)$ and $\beta(C)$. In other words, there is a unique pair $(T(C), \hat{r}(C))$ that yields the Markovian representation, with the key being that $T(C)$ is pinned down by $\beta(C)$ alone by the primacy of the dZ shocks, while $\hat{r}(C)$ is pinned down by both α and β. Under the optimal $\beta(C)$, we can solve (A.40) for $T(C) = \frac{\ln P'(C)}{\rho r} + Y(C)$, i.e., (30).

Finally, we denote the drift term of $dT(C_t)$ by $\mu_T(C_t) = T'(C_t)\mu_C(C_t) + \frac{T''(C_t)\sigma_C(C_t)^2}{2}$. By matching drift terms, we obtain $\hat{r}(C)T(C_t) = \mu_T(C) + \mu_I(C)$.

A.10.2 Part II

The next part of the proof shows that for $C \in (\underline{C}, \overline{C})$ and the optimal α, β:

$$\mu_T(C) + \mu_I(C) = \frac{\lambda}{\rho r}\mathbb{1}_{\{C\geq 0\}} + rY(C). \tag{A.41}$$

We assume that that $\alpha(C)$ and $\beta(C)$ are differentiable in state C. We establish (A.41) for points C at which $\alpha(C)$ and $\beta(C)$ are differentiable. Because the set of points at which $\alpha(C)$ or $\beta(C)$ are not differentiable is countable, (A.41) then holds for all C in $(\underline{C}, \overline{C})$

Part II.A — Auxiliary Result: Simplified HJB under optimal $\beta(C)$. Assume that $\alpha(C)$ and $\beta(C)$ are differentiable. Note that $1 - \beta(C) = 1 - \frac{P''(C)}{P''(C) - \rho r P'(C)} = \frac{-\rho r P'(C)}{P''(C) - \rho r P'(C)}$.

Thus, $\frac{P''(C)}{-\rho r P'(C)} = \frac{\beta(C)}{1-\beta(C)}$ and $1-\beta(C) = \frac{-\rho r P'(C)}{P''(C)}\beta(C)$ and $\beta(C) = \frac{P''(C)}{-\rho r P'(C)}(1-\beta(C))$. As a result, we can calculate

$$-\frac{\rho r}{2}\beta(C)^2 P'(C) + \frac{P''(C)}{2}((1-\beta(C))^2 = -\frac{\rho r}{2}\beta(C)^2 P'(C) + \frac{(\rho r P'(C))^2}{2P''(C)}(\beta(C))^2 \tag{A.42}$$
$$= -\frac{\rho r}{2}\beta(C)^2 P'(C)\left(1 - \frac{\rho r P'(C)}{P''(C)}\right) = -\frac{\rho r}{2}\beta(C)P'(C),$$

where the last equality uses that $\frac{1}{\beta(C)} = \frac{P''(C)-\rho r P'(C)}{P''(C)} = 1 - \frac{\rho r P'(C)}{P''(C)}$. We can insert relation (A.42) as well as $C^* = \overline{C}$ into (15) to obtain

$$rP(C) = P'(C)\left[\mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right]$$
$$+ \pi\left[P\left(\overline{C}\right) - P(C) - \left(\overline{C} - C + \alpha(C)\right)\right]. \tag{A.43}$$

Part II.B — Auxiliary Result: Slope of optimal risk-sharing $\beta'(C)$. Consider $C \neq 0$ and that $\alpha(C)$ and $\beta(C)$ are differentiable. We differentiate both sides of the ODE (A.43) with respect to C:

$$rP'(C) = P''(C)\left[\mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right]$$
$$+ P'(C)\left[\left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right) - \frac{\rho r}{2}\beta'(C)\sigma^2 + \pi\alpha'(C)e^{-\rho r\alpha(C)}\right] + \pi\left[1 - \alpha'(C) - P'(C)\right].$$

Rearranging, we have

$$\left[\pi\left(1 - \alpha'(C)e^{-\rho r\alpha(C)}\right) + \lambda \mathbb{1}_{\{C\geq 0\}} + \frac{\rho r}{2}\beta'(C)\sigma^2\right]P'(C) \tag{A.44}$$
$$= P''(C)\left[\mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] + \pi\left[1 - \alpha'(C)\right].$$

Dividing through by $\rho r P'(C)$ and solving for $\frac{\sigma^2}{2}\beta'(C)$, we have

$$\frac{\sigma^2}{2}\beta'(C) = \frac{P''(C)}{\rho r P'(C)}\left[\mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right]$$
$$+ \frac{\pi}{\rho r}\frac{\left[1-\alpha'(C)\right]}{P'(C)} - \frac{\pi}{\rho r}\left[1 - \alpha'(C)e^{-\rho r\alpha(C)}\right] - \frac{\lambda \mathbb{1}_{\{C\geq 0\}}}{\rho r}$$
$$= -\frac{\beta(C)}{1-\beta(C)}\left[\mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right]$$
$$+ \frac{\pi}{\rho r}\frac{\left[1-\alpha'(C)\right]}{P'(C)} - \frac{\pi}{\rho r}\left[1 - \alpha'(C)e^{-\rho r\alpha(C)}\right] - \frac{\lambda \mathbb{1}_{\{C\geq 0\}}}{\rho r},$$

where the second equality uses $\frac{1}{\rho r}\frac{P''(C)}{P'(C)} = -\frac{\beta(C)}{1-\beta(C)}$. Using the expression for the drift of excess liquidity $\mu_C(C)$ in (A.6), we get

$$\frac{\sigma^2}{2}\beta'(C) = -\frac{\beta(C)}{1-\beta(C)}\mu_C(C) + \frac{\rho r}{2}\beta^2(C)\sigma^2 + \frac{\pi}{\rho r}\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] - \frac{\pi}{\rho r}\left(1 - \frac{1}{P'(C)}\right) - \frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r} \tag{A.45}$$

Part II.C Derivation of (A.41). Consider $C \neq 0$ and that $\alpha(C)$ and $\beta(C)$ are differentiable. We can calculate

$$\begin{aligned}
\mu_T(C) =& T'(C)\mu_C(C) + T''(C)\frac{\sigma_C^2(C)}{2} = T'(C)\mu_C(C) - \frac{\beta'(C)}{[1-\beta(C)]^2}\frac{\sigma^2[1-\beta(C)]^2}{2} \\
=& T'(C)\mu_C(C) - \frac{\sigma^2}{2}\beta'(C) \\
=& -\left[\frac{\rho r}{2}\beta^2(C)\sigma^2 + \frac{\pi}{\rho r}\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] - \frac{\pi}{\rho r}\left(1 - \frac{1}{P'(C)}\right) - \frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r}\right] \\
=& \mu_C\mathbb{1}_{\{C\geq 0\}}(C) - \left[\mu + \left(r - \lambda\mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r}\right] \\
& - \frac{\pi}{\rho r}\left\{\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] - \left(1 - \frac{1}{P'(C)}\right) + \left(1 - e^{-\rho r\alpha(C)}\right)\right\},
\end{aligned}$$

where the second equality uses $T''(C) = -\frac{\beta'(C)}{[1-\beta(C)]^2}$, and the fourth equality uses (A.45) to substitute in for $\frac{\sigma^2}{2}\beta'(C)$. We can simplify to obtain

$$\mu_T(C) = \left[\frac{\lambda}{\rho r} + \mu_C(C)\right]\mathbb{1}_{\{C\geq 0\}} - \left[\mu + \left(r - \lambda\mathbb{1}_{\{C\geq 0\}}\right)C\right] - \frac{\pi}{\rho r}\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right][1 - \alpha'(C)].$$

Utilizing $\mu_I(C) = \mu + [(r-\lambda)C - \mu_C]\mathbb{1}_{\{C\geq 0\}}$, we have

$$\hat{r}(C)T(C) = \mu_T(C) + \mu_I(C) = \frac{\lambda}{\rho r}\mathbb{1}_{\{C\geq 0\}} + rY(C) - \frac{\pi}{\rho r}\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right][1 - \alpha'(C)]. \tag{A.46}$$

When $\alpha(C) = \alpha_U(C) = \frac{\ln P'(C)}{\rho r}$, we have $e^{-\rho r\alpha(C)} = 1/P'(C)$ and $\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right] = 0$. When $\alpha(C) = \alpha_C(C) = P(\overline{C}) - \overline{C} + C$, then $\alpha'(C) = 1$. Either way, under the optimal contract, $\frac{\pi}{\rho r}\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right][1 - \alpha'(C)] = 0$ and (A.46) simplifies to (A.41).

A.11 Proof of Corollary 2

By construction, balances of the credit lines add up to $T(C)$, i.e., $T(C) = D(C) + Y(C)$. The sum of interest payments on the secured credit line, $rY(C)$, plus maintenance $\frac{\lambda}{\rho r}\mathbb{1}_{\{C\geq 0\}}$ equals $\hat{r}(C)T(C)$. We verify $dI^D(C) + dI^Y(C) = dI(C)$. When $C > 0$, then $dI^Y(C)$ and it can be readily seen from (A.31) and the expression in Corollary 2 that $dI^D(C) = dI(C)$, as desired. When $C \le 0$, we can calculate $dI^D(C) + dI^Y(C) = \mu dt + \sigma dZ + (Y(C) + \alpha(C))d\Pi = dI(C)$.

A.12 Existence of Solution

We consider $\varepsilon > 0$ and impose the constraint $\beta_t \in [0, 1-\varepsilon]$ to deal with the problem of vanishing volatility $\sigma_C(C)$. We also impose the constraint $\alpha \ge 0$ which never binds in optimum in the interior of the state space. We establish that for any ε, a solution to the system considered exists. Lastly, we take the limit $\varepsilon \to 0$ to show that there exists a solution $(P(C), \overline{C}, \underline{C})$ to the system (15) with (24) and $P'(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$.

The proof proceeds in five parts. Part I establishes existence and uniqueness to an auxiliary ODE system, with exogenous boundaries $\overline{C}_1 \ge 0$ and boundary $\underline{C}_1$. Part II shows that this solution is concave, and strictly concave for $\overline{C}_1 > 0$. Using these results, Part III and VI show that there exists a solution to the system (15) subject to $P(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ and with (24). Part V concludes, and argues that a unique solution to the dynamic contracting problem (6) exists.

A.12.1 Part I

Recall $k_Z(\beta)$ and $k_\Pi(\alpha)$ from (A.4). For any exogenous $\overline{C}_1 \ge 0 > \underline{C}_1$, consider the auxiliary function $P_{\varepsilon,\overline{C}_1}(C)$ solving the HJB

$$\begin{aligned} &rP_{\varepsilon,\overline{C}_1}(C) = \qquad\qquad (A.47)\\ &\max_{\beta\in[0,1-\varepsilon], Y\ge\max\{0,-C\}} \left\{ P'_{\varepsilon,\overline{C}_1}(C)\left[\mu + (r-\lambda)C - \lambda Y - \sigma^2 k_Z(\beta)\right] + P''_{\varepsilon,\overline{C}_1}(C)\frac{\sigma^2}{2}(1-\beta)^2 \right\} \\ &+ \pi \max_{\alpha} \left\{ P'_{\varepsilon,\overline{C}_1}(C)\left[\alpha - k_\Pi(\alpha)\right] + \left[P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - P_{\varepsilon,\overline{C}_1}(C) - \left(\overline{C}_1 - C + \alpha\right)\right] \right\}, \end{aligned}$$

with α satisfying $0 \le \alpha \le P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1 + C$ and subject to the boundary conditions $P'_{\varepsilon,\overline{C}_1}(\overline{C}_1) - 1 = P''_{\varepsilon,\overline{C}_1}(\overline{C}_1) = 0$.

In other words, we are solving a second-order ODE which is pinned down by two boundary conditions at $\overline{C}_1$. Note that above HJB equation becomes akin to (15) upon setting $\varepsilon = 0$. We can then calculate

$$Y(C) = \max\{-C, 0\},\ \alpha(C) = \min\left\{ \max\left\{ 0, \frac{\ln P'_{\varepsilon,\overline{C}_1}(C)}{\rho r} \right\}, P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1 + C \right\}. \quad (A.48)$$

We now prove uniqueness and existence of a solution to (A.47) adopting the argument in

Sannikov (2008). Inserting optimal $Y = Y(C)$ and $\alpha = \alpha(C)$, we can rewrite (A.47):

$$P''_{\varepsilon,\overline{C}_1}(C) = \min_{\beta\in[0,1-\varepsilon]} \frac{2}{\sigma^2(1-\beta)^2}\Bigg[rP_{\varepsilon,\overline{C}_1}(C) - P'_{\varepsilon,\overline{C}_1}(C)\left[\mu + (r-\lambda)C - \lambda Y(C) - \sigma^2 k_Z(\beta)\right]$$
$$- \pi\left\{P'_{\varepsilon,\overline{C}_1}(C)\left[\alpha(C) - k_\Pi(\alpha(C))\right] + \left[P_{\varepsilon,\overline{C}_1}\left(\overline{C}_1\right) - P_{\varepsilon,\overline{C}_1}(C) - \left(\overline{C}_1 - C + \alpha(C)\right)\right]\right\}\Bigg], \tag{A.49}$$

where due to the constraint $\beta \in [0, 1-\varepsilon]$: $\frac{2}{\sigma^2} \geq \frac{2}{\sigma^2(1-\beta)^2} \geq \frac{2}{\sigma^2\varepsilon^2} > 0$. Next, we can rewrite (A.49) as $P''_{\varepsilon,\overline{C}_1}(C) = \min_{\beta\in[0,1-\varepsilon]} \mathcal{H}_\beta(C, P_{\varepsilon,\overline{C}_1}(C), P'_{\varepsilon,\overline{C}_1}(C))$, where $\mathcal{H}_\beta(C, P_{\varepsilon,\overline{C}_1}(C), P'_{\varepsilon,\overline{C}_1}(C))$ is implicitly defined via (A.49). Note that for all $\beta \in [0, 1-\varepsilon]$, functions $H_\beta(\cdot)$ are Lipschitz-continuous in the arguments. It follows the solution to (A.49) with $P'_{\varepsilon,\overline{C}_1}(\overline{C}_1) - 1 = P''_{\varepsilon,\overline{C}_1}(\overline{C}) = 0$ exists on $(\underline{C}_1, \overline{C}_1)$ and is unique and continuous in boundary conditions and $\varepsilon, \overline{C}_1$.

A.12.2 Part II — Concavity of $P_{\varepsilon,\overline{C}_1}(C)$

We restrict now attention to $\overline{C}_1 > 0$ and $\underline{C}_1 \geq -\frac{\mu}{r} + \frac{\lambda\overline{C}_1}{r}$. We show that, in this case, $P_{\varepsilon,\overline{C}_1}(C)$ is strictly concave on $(\underline{C}_1, \overline{C}_1)$, satisfying $P'_{\varepsilon,\overline{C}_1}(C) > 1$. We already conjecture (and then verify) that $P'_{\varepsilon,\overline{C}_1}(C) > 1$. By continuity, it then follows that the solution is at least weakly concave for any $\overline{C}_1 \geq 0$. To begin with note that at $P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1 = \frac{\mu}{r} + \frac{\lambda\overline{C}_1}{r}$. Thus, for $C > \underline{C}_1$, we have $\alpha(C) > 0$ by (A.48).

Define the jump in the value function upon refinancing as

$$J(C) \equiv P_{\varepsilon,\overline{C}_1}\left(\overline{C}_1\right) - P_{\varepsilon,\overline{C}_1}(C) - \left(\overline{C}_1 - C + \alpha(C)\right). \tag{A.50}$$

When $P''_{\varepsilon,\overline{C}_1}(C)$ and $\alpha(C)$ are differentiable, we can use the envelope theorem and differentiate the HJB equation (A.10) under the optimal $\beta = \beta(C) \in [0, 1-\varepsilon]$ with respect to C and rearrange to obtain

$$P'''_{\varepsilon,\overline{C}_1}(C) = \frac{2}{(1-\beta(C))^2\sigma^2}\Bigg(P'_{\varepsilon,\overline{C}_1}(C)\lambda\mathbb{1}_{\{C\geq 0\}}$$
$$- P''_{\varepsilon,\overline{C}_1}(C)\mu_C(C) - \pi\left(e^{-\rho r\alpha(C)}P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right)\Bigg), \tag{A.51}$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function which is equal to one if $\{\cdot\}$ is true and is equal to zero otherwise. The set of points at which either $P''_{\varepsilon,\overline{C}_1}(C)$ or $\alpha(C)$ is not differentiable is countable; therefore, for any C, the limits $\lim_{x\uparrow C} P'''_{\varepsilon,\overline{C}_1}(C)$, $\lim_{x\downarrow C} P'''_{\varepsilon,\overline{C}_1}(C)$, and $\lim_{x\uparrow C}\alpha'(C)$, $\lim_{x\downarrow C}\alpha'(C)$ exist and are well-defined.

Suppose that $\alpha(C)$ is differentiable, and recall (A.48). If $\alpha(C) = P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1 + C > 0$, then $\alpha'(C) = 1$ and $J'(C) = -P'_{\varepsilon,\overline{C}_1}(C) < 0$. As such,

$$\pi\left(e^{-\rho r\alpha(C)}P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right) = \pi P'_{\varepsilon,\overline{C}_1}(C)\left(e^{-\rho r\alpha(C)} - 1\right) \leq 0.$$

When $\alpha(C) = \frac{\ln P'_{\varepsilon,\overline{C}_1}(C)}{\rho r}$, then

$$\begin{aligned}\pi\left(e^{-\rho r\alpha(C)}P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right) &= \pi\left(e^{-\rho r\alpha(C)}P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + 1 - P'_{\varepsilon,\overline{C}_1}(C) - \alpha'(C)\right) \\ &= \pi\left(\alpha'(C) + 1 - P'_{\varepsilon,\overline{C}_1}(C) - \alpha'(C)\right) = 1 - P'_{\varepsilon,\overline{C}_1}(C) \leq 0.\end{aligned}$$

where it was used that $J'(C) = 1 - P'_{\varepsilon,\overline{C}_1}(C) - \alpha'(C)$ and $e^{-\rho r\alpha(C)} = 1/P'_{\varepsilon,\overline{C}_1}(C)$ and $P'_{\varepsilon,\overline{C}_1}(C) \geq 1$. Thus, $\pi\left(e^{-\rho r\alpha(C)}P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right) \leq 0$, if $\alpha(C)$ is differentiable.

At the payout boundary, it therefore holds that $P'_{\varepsilon,\overline{C}_1}(\overline{C}_1) = 1$, and $P''_{\varepsilon,\overline{C}_1}(\overline{C}_1) = 0$ and

$$\lim_{C\uparrow\overline{C}_1}\left(e^{-\rho r\alpha(C)}P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right) = \lim_{C\uparrow\overline{C}}(1 - P'_{\varepsilon,\overline{C}_1}(C)) = 0. \tag{A.52}$$

Recall $\overline{C}_1 > 0$. Then, $P''_{\varepsilon,\overline{C}_1}(\overline{C}_1) = 0$, $P'_{\varepsilon,\overline{C}_1}(\overline{C}_1) = 1$, we get $\lim_{C\uparrow\overline{C}_1} P'''_{\varepsilon,\overline{C}_1}(C) > 0$. Because $\lim_{C\uparrow\overline{C}_1} P'''_{\varepsilon,\overline{C}_1}(C) > 0$ for $\overline{C}_1 > 0$, continuity implies $P'''(C) > 0$, $P'(C) > 1$, and $P''(C) < 0$ in a left-neighbourhood of $\overline{C}_1$.

Suppose to the contrary that there exists $C' \in (\underline{C}_1, \overline{C}_1)$ with $P''_{\varepsilon,\overline{C}_1}(C) \geq 0$. Define $\hat{C} = \sup\{C \in (\underline{C}_1, \overline{C}_1) : P''_{\varepsilon,\overline{C}_1}(C) \geq 0\}$ and suppose to the contrary that $\hat{C} < \overline{C}$. As $P''_{\varepsilon,\overline{C}_1}(C) < 0$ in a neighbourhood of $\overline{C}_1$, it follows that $\hat{C} < \overline{C}_1$ and, by continuity of $P''_{\varepsilon,\overline{C}_1}(C)$, that $P''_{\varepsilon,\overline{C}_1}(\hat{C}) = 0$. Since $P''_{\varepsilon,\overline{C}_1}(C) < 0$ for $C \in (\hat{C}, \overline{C}_1)$, it follows that $P'_{\varepsilon,\overline{C}_1}(\hat{C}) > 1$. Note that the term $\lim_{C\downarrow\hat{C}} \pi\left(e^{-\rho r\alpha(C)}P'_{\varepsilon,\overline{C}_1}(C)\alpha'(C) + J'(C)\right)$ is weakly negative. As such, $\lim_{C\downarrow\hat{C}} P'''_{\varepsilon,\overline{C}_1}(C) \geq 0$, where the inequality is strict if $\hat{C} > 0$.

Consider now $\hat{C} \geq 0$. Due to $\lim_{C\downarrow\hat{C}} P'''_{\varepsilon,\overline{C}_1}(C) > 0$, there exists $C' > \hat{C}$ so that $P''_{\varepsilon,\overline{C}_1}(C') > 0$, which contradicts the definition of $\hat{C}$. Next, suppose that $\hat{C} < 0$. If $\lim_{C\downarrow\hat{C}} P'''_{\varepsilon,\overline{C}_1}(C) > 0$, we achieve a contradiction similar to above. Consider $\lim_{C\downarrow\hat{C}} P'''_{\varepsilon,\overline{C}_1}(C) = 0$. Due to $P''_{\varepsilon,\overline{C}_1}(\hat{C}) = 0$, there exists then $\delta > 0$ such that the (unique) solution to (A.49) on $(\hat{C}, \hat{C}+\delta)$ satisfies $P'''_{\varepsilon,\overline{C}_1}(C) \geq 0$. Thus, there exists $C' > \hat{C}$ with $P''_{\varepsilon,\overline{C}_1}(C') = 0$, a contradiction.

Either way, it follows that $P''_{\varepsilon,\overline{C}_1}(C) < 0$ for all $C \in (\underline{C}_1, \overline{C}_1)$, which concludes the proof.

A.12.3 Part III — $P_{\varepsilon,\overline{C}_1}(C)$ has a Root

It follows that $P_{\varepsilon,\overline{C}_1}(C)$ is strictly concave, so that $P'_{\varepsilon,\overline{C}_1}(C) \geq 1$. Take $\overline{C}_1 \geq 0$ and $\underline{C}_1 = -\frac{\mu}{r} + \lambda\overline{C}_1$. Then, $P_{\varepsilon,\overline{C}_1}(C)$ is strictly concave on $(\underline{C}_1, \overline{C}_1)$ with $P'_{\varepsilon,\overline{C}_1}(C) \geq 1$.

Using the boundary conditions at $\overline{C}_1$, i.e., $P'_{\varepsilon,\overline{C}_1}(C) - 1 = P''_{\varepsilon,\overline{C}_1}(C) = 0$, and strict concavity on $(\underline{C}_1, \overline{C}_1)$, we bound the function $P_{\varepsilon,\overline{C}_1}(C)$ via a linear function $\overline{P}(C; \overline{C}_1)$:

$$P_{\varepsilon,\overline{C}_1}(C) \leq \overline{P}(C; \overline{C}_1) \equiv \frac{\mu + (r-\lambda)\overline{C}_1}{r} + C - \overline{C}_1 = \frac{\mu - \lambda \overline{C}_1}{r} + C.$$

By construction, $P(\overline{C}_1; \overline{C}_1) = P_{\varepsilon,\overline{C}_1}\left(\overline{C}_1\right)$ and $\overline{P}(C; \overline{C}_1) = 0$ for $C = \hat{C}(\overline{C}_1) = -\frac{\mu}{r} + \frac{\lambda \overline{C}_1}{r}$. Thus, there exists a unique $\hat{C}_\varepsilon(\overline{C}_1) \geq \hat{C}(\overline{C}_1) = -\frac{\mu}{r} + \frac{\lambda \overline{C}_1}{r}$ at which $P_{\varepsilon,\overline{C}_1}(\hat{C}_\varepsilon(\overline{C}_1)) = 0$. The function $\hat{C}_\varepsilon(\overline{C}_1)$ is continuous in $\overline{C}_1$ and ε. For $\overline{C}_1$ sufficiently large, $\hat{C}_\varepsilon(\overline{C}_1) \geq \hat{C}(\overline{C}_1) \geq 0$.

Next, consider $\overline{C}_1 = 0$. Thus, by (A.51), we obtain after using (A.52) and $P''_{\varepsilon,\overline{C}_1}(0) = 0$ that $\lim_{C\uparrow 0} P'''_{\varepsilon,\overline{C}_1}(C) = 0$. Therefore, the solution to (A.47) satisfies $P_{\varepsilon,\overline{C}_1} = \frac{\mu}{r} + C$ on $(-\frac{\mu}{r}, 0)$, so that $\hat{C}_\varepsilon(0) = \frac{-\mu}{r}$. Take

$$\underline{C}_\varepsilon(\overline{C}_1) = \min\left\{ \frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C}_1 + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r} - Y^A, -L \right\}.$$

Observe that by Lemma 3, we have $\underline{C}_\varepsilon(\overline{C}_1) \in [-[P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1], -L]$, with $P_{\varepsilon,\overline{C}_1}(\overline{C}_1) - \overline{C}_1 = \frac{\mu}{r} - \frac{\lambda \overline{C}_1 \mathbb{I}_{\{\overline{C}_1 > 0\}}}{r}$. Further, $\underline{C}_\varepsilon(\overline{C}_1)$ increases with $\overline{C}_1$ and is continuous in $\overline{C}_1 \geq 0$.

The image of the continuous function $\hat{C}_\varepsilon : [0, \infty) \to \mathbb{R}$ contains $[-\frac{\mu}{r}, 0]$, i.e., $[-\frac{\mu}{r}, 0] \subseteq \hat{C}_\varepsilon([0, \infty])$. Since, in addition, for $\overline{C}_1$ sufficiently large, $\hat{C}(\overline{C}_1) \geq 0$, it must be that there exists $\overline{C}_\varepsilon \geq 0$ such that $\underline{C}_\varepsilon = \underline{C}_\varepsilon(\overline{C}_\varepsilon) = \hat{C}(\overline{C}_\varepsilon)$. If there exist multiple values C' with $\underline{C}_\varepsilon(C') = \hat{C}(C')$, we define $\overline{C}_\varepsilon$ as the lowest intersection point.

Thus, the function $P_{\varepsilon,\overline{C}_\epsilon}(C)$ solves (A.47) in $(\underline{C}_\varepsilon, \overline{C}_\varepsilon)$ subject to $P_{\varepsilon,\overline{C}_\varepsilon}(\underline{C}_\varepsilon) = P'_{\varepsilon,\overline{C}_\varepsilon}\left(\overline{C}_\varepsilon\right) - 1 = P''_{\varepsilon,\overline{C}_\varepsilon}\left(\overline{C}_\varepsilon\right) = 0$. As $\underline{C}_\varepsilon \geq \frac{\mu}{r} - \frac{\lambda \overline{C}_\varepsilon}{r}$, it follows from Part II that $P_{\varepsilon,\overline{C}_\varepsilon}$ is concave on $(\underline{C}_\varepsilon, \overline{C}_\varepsilon)$, and strictly so if $\overline{C}_\varepsilon > 0$.

A.12.4 Part IV

Take $\overline{C}_\varepsilon$ and $\underline{C}_\varepsilon = \underline{C}(\overline{C}_\varepsilon)$ with $\underline{C}(\cdot)$ from (24) as well as the function $P_{\varepsilon,\overline{C}_\varepsilon}(C)$ solving (A.47) on $(\underline{C}_\varepsilon, \overline{C}_\varepsilon)$ subject to $P_{\varepsilon,\overline{C}_\varepsilon}(\underline{C}_\varepsilon) = P'_{\varepsilon,\overline{C}_\varepsilon}\left(\overline{C}_\varepsilon\right) - 1 = P''_{\varepsilon,\overline{C}_\varepsilon}\left(\overline{C}_\varepsilon\right) = 0$. The functions $P_{\varepsilon,\overline{C}_\varepsilon}$ are strictly concave and increasing on $(\underline{C}_\varepsilon, \overline{C}_\varepsilon)$.

We can take the limit $\varepsilon \to 0$ to obtain $\overline{C} := \lim_{\varepsilon\to 0} \overline{C}_\varepsilon$ and $\underline{C} := \lim_{\varepsilon\to 0} \underline{C}_\varepsilon$. As $\underline{C}_\varepsilon \in [-\frac{\mu}{r}, -L]$, $\underline{C}$ is well-defined with $\underline{C} \in [-\frac{\mu}{r}, -L]$. Define $P(C) := \lim_{\varepsilon\to 0} P_{\varepsilon,\overline{C}_\varepsilon}(C)$ on $(\underline{C}, \overline{C})$ as well as $P'(C) := \lim_{\varepsilon\to 0} P'_{\varepsilon,\overline{C}_\varepsilon}(C)$ and $P''(C) := \lim_{\varepsilon\to 0} P''_{\varepsilon,\overline{C}_\varepsilon}(C)$. We show now that for all $C \in (\underline{C}, \overline{C})$, these limits are finite (and exist).

For a given value $C \in (\underline{C}, \overline{C})$, there exist $\delta > 0$ and $\bar{\varepsilon} > 0$ such that for all $0 < \varepsilon < \bar{\varepsilon}$, we have $C > \underline{C}_\varepsilon + \delta$ and, due to $P'_{\varepsilon,\overline{C}_\varepsilon}(\cdot) \geq 1$, consequently $P_{\varepsilon,\overline{C}_\varepsilon}(C) > \delta$. Therefore $P(C) = \lim_{\varepsilon\to 0} P_{\varepsilon,\overline{C}_\varepsilon}(C) > 0$. As $P_{\varepsilon,\overline{C}_\varepsilon}(C)$ is concave and increasing, it follows that, if $P(C)$ is not finite, then $P(C) = -\infty$. However, due to $P(C) > 0$, $P(C)$ must be finite.

If $P'(C)$ is not finite, then $P'(C) = +\infty$. Suppose $P'(C) = +\infty$. Pick $C' \in (\underline{C}, C)$. Then, there is ε_1 such that for all $\varepsilon < \varepsilon_1$ we have $C', C \in (\underline{C}_\varepsilon, \overline{C}_\varepsilon)$ and $P'_{\varepsilon,\overline{C}_\varepsilon}(C'') \geq P'_{\varepsilon,\overline{C}_\varepsilon}(C)$ for $C'' \in [C', C]$, due to concavity. As a consequence, $P_{\varepsilon,\overline{C}_\varepsilon}(C') \leq P_{\varepsilon,\overline{C}_\varepsilon}(C) - (C - C')P'_{\varepsilon,\overline{C}_\varepsilon}(C)$. Owing to $P'(C) = +\infty$, for any $R > 0$, there exists $0 < \bar{\varepsilon} \leq \varepsilon_1$ such that $P'_{\varepsilon,\overline{C}_\varepsilon}(C) > R$ for all $\varepsilon < \bar{\varepsilon}$. That is, $P_{\varepsilon,\overline{C}_\varepsilon}(C') < P_{\varepsilon,\overline{C}_\varepsilon}(C) - (C - C')R$ for all $\varepsilon < \bar{\varepsilon}$. Pick $R > P(C)/(C - C')$ and let $\varepsilon \to 0$ to obtain $P(C') < 0$, a contradiction. Thus, $P'(C)$ must be finite.

If $P''(C)$ is not finite, then $P''(C) = -\infty$. Suppose to the contrary $P''(C) = -\infty$. We pick C as the largest value on $(\underline{C}, \overline{C})$ with $P''(C) = -\infty$; thus, $P''(c) > -\infty$ for $c \in (C, \overline{C})$. Then, it follows that $P'''(C) = +\infty$. Pick $C' \in (\underline{C}, C)$ such that $P'''(\cdot) > 0$ on (C', C). Note that, as we have shown, $P'(C') \geq 1$ is finite. Then, there is ε_1 such that for all $\varepsilon < \varepsilon_1$ we have $C', C \in (\underline{C}_\varepsilon, \overline{C}_\varepsilon)$, and $P'''_{\varepsilon,\overline{C}_\varepsilon}(\cdot) \geq 0$ on (C', C). As such, we have $P''_{\varepsilon,\overline{C}_\varepsilon}(C'') \leq P''_{\varepsilon,\overline{C}_\varepsilon}(C)$ for $C'' \in [C', C]$. Because $P''(C) = -\infty$, for any $R > 0$ there exists $0 < \bar{\varepsilon} < \varepsilon_1$ such that $P''_{\varepsilon,\overline{C}_\varepsilon}(C) < -R$ for $\varepsilon < \bar{\varepsilon}$. Thus, $P'_{\varepsilon,\overline{C}_\varepsilon}(C) \leq P'_{\varepsilon,\overline{C}_\varepsilon}(C') + (C - C')P''_{\varepsilon,\overline{C}_\varepsilon}(C)$. Take $R > P'(C')/(C - \underline{C})$, and take the limit $\varepsilon \to 0$ to obtain $P'(C) = \lim_{\varepsilon\to 0} P'_{\varepsilon,\overline{C}_\varepsilon}(C) \leq 0$, a contradiction. Thus, $P''(C)$ is finite.

Having established that the limit expressions $P(C), P'(C), P''(C)$ are finite for any $C \in (\underline{C}, \overline{C})$, we take the limit ε in (A.47) on $(\underline{C}, \overline{C}) = \lim_{\varepsilon\to 0}(\underline{C}_\varepsilon, \overline{C}_\varepsilon)$.We obtain that $P(C) = \lim_{\varepsilon\to 0} P_{\varepsilon,\overline{C}_\varepsilon}(C)$ solves on $(\underline{C}, \overline{C})$ the HJB equation:

$$
\begin{aligned}
&rP(C) = \lim_{\varepsilon\to 0} rP_{\varepsilon,\overline{C}_\varepsilon}(C) = \\
&\lim_{\varepsilon\to 0} \max_{\beta\in[0,1-\varepsilon], Y\geq \max\{0,-C\}} \left\{ P'_{\varepsilon,\overline{C}_\varepsilon}(C)\left[\mu + (r-\lambda)C - \lambda Y - \sigma^2 k_Z(\beta)\right] + P''_{\varepsilon,\overline{C}_\varepsilon}(C)\frac{\sigma^2}{2}(1-\beta)^2 \right\} \\
&+ \lim_{\varepsilon\to 0} \pi \max_{\alpha\in\mathcal{S}(C,\overline{C})} \left\{ P'_{\varepsilon,\overline{C}_\varepsilon}(C)\left[\alpha - k_\Pi(\alpha)\right] + \left[P_{\varepsilon,\overline{C}_\varepsilon}\left(\overline{C}_\varepsilon\right) - P_{\varepsilon,\overline{C}_\varepsilon}(C) - \left(\overline{C}_\varepsilon - C + \alpha\right)\right] \right\} \\
&= \max_{\beta, Y\geq \max\{0,-C\}} \left\{ P'(C)\left[\mu + (r-\lambda)C - \lambda Y - \sigma^2 k_Z(\beta)\right] + P''(C)\frac{\sigma^2}{2}(1-\beta)^2 \right\} \\
&+ \pi \max_\alpha \left\{ P'(C)\left[\alpha - k_\Pi(\alpha)\right] + \left[P\left(\overline{C}\right) - P(C) - \left(\overline{C} - C + \alpha\right)\right] \right\}.
\end{aligned}
$$

Thus, $(P(C), \overline{C}, \underline{C})$ is a solution to the system (15) subject to $P(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ and with (24). Given $\overline{C}$, there exists — by construction — no other function that solves (15) subject to $P(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ with $\underline{C} = \underline{C}(\overline{C})$ from (24).

A.12.5 Part V — Uniqueness of Solution to (6)

We were able to establish existence of a solution to the ODE system subject to endogenous boundary conditions. Moreover, by construction, the function $P(C)$ that solves (15) subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ is unique, but there could exist multiple boundary values consistent with such a solution; nevertheless, there exists a unique solution to the optimization problem (6). If there exist multiple solutions $\mathcal{S}_i := (P_i(C), \overline{C}_i, \underline{C}_i)$ to the system (15) subject to $P(\overline{C}_i) - 1 = P_i''(\overline{C}_i) = P(\underline{C}_i) = 0$ and with $\underline{C}_i = \underline{C}(\overline{C}_i)$ from (24), we can index these solutions by $i \in \mathcal{I}$ and denote their set by $\Omega = \{(P_i(C), \overline{C}_i, \underline{C}_i) : i \in \mathcal{I}\}$.

Notice that at time $t = 0$, shareholders payoff under solution i is $P(\overline{C}_i) - \overline{C}_i = \frac{\mu}{r} - \frac{\lambda\overline{C}_i}{r}$, because $\overline{C}_i > 0$, so that ex-ante payoff decreases with $\overline{C}_i$. As a result, the optimal contract is obtained by picking the solution with the lowest upper and lower boundary. Formally, take $i^* = \arg\min_{i \in \mathcal{I}} \overline{C}_i$, and set $\overline{C} := \overline{C}_{i^*}$ and $\underline{C} = \underline{C}_{i^*}$ as well as $P(C) = P_{i^*}(C)$. Then, under the optimal contract, shareholders' value function $P(C)$ is the unique solution to (15) subject to $P(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0$ and with (24). Consequently, a unique solution to shareholders' optimization — that is, the optimal contracting problem in (6) — exists.

A.13 Proof of Proposition 4

Proof. As the HJB (15) and boundary conditions remain unchanged besides the new constraint (33), at the payout boundary the HJB implies $P(\overline{C}) - \overline{C} = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C}$. Notice that ν affects the value function only via the constraint (33), so $P(\overline{C}) - \overline{C}$ must increase with ν, as a larger ν relaxes the constraint (33). Thus, $\frac{\partial [P(\overline{C}) - \overline{C}]}{\partial \nu} \geq 0 \iff \frac{\partial \overline{C}}{\partial \nu} \leq 0$, with strict inequality if the constraint is binding on a set with non-zero measure of the state space $(\underline{C}, \overline{C})$. Under survival, $\underline{C} = C^S$, the constraint is binding in a neighbourhood of $\underline{C}$ regardless of ν, by (25), $\frac{\partial C^S}{\partial \nu} < 0$. □

B Micro-foundation of k_t from CARA Preferences

We now present a micro-foundation of the intermediary's payoff in (3) as well as the cost function (8). The intermediary is risk-averse with CARA preferences over consumption c with risk-aversion of $\rho > 0$, $u(c) = -\frac{1}{\rho}\exp(-\rho c)$. The intermediary can maintain savings on its own account, denoted by S_t. Savings accrue interest at rate r and are subject to changes induced by transfers to ($dI_t < 0$) and from ($dI_t > 0$) the firm and consumption c_t, so that

$$dS_t = rS_t dt + dI_t - c_t dt, \tag{B.1}$$

where the payout process dI_t is stipulated by the contract $\mathcal{C}$.[25] As such, the intermediary has essentially deep pockets, but capital provision by the intermediary is costly in a sense that the intermediary is risk-averse. We normalize the balance of savings at $t = 0^-$ to zero, i.e., $S_{0^-} = 0$, where time $t = 0^-$ denotes the time before the contract is written and any transfers are made. Savings must satisfy the standard transversality condition $\lim_{t\to\infty} \mathbb{E}\left[e^{-rt} S_t\right] = 0$, ruling out Ponzi schemes. Let U_0 be the intermediary's utility for a given contract $\mathcal{C}$, that is,

$$U_0 = \max_{c_t} \mathbb{E}\left[\int_0^\infty e^{-rt} u\left(c_t\right) dt\right] \quad \text{s.t.} \quad \text{(B.1)} \quad \text{and} \quad \lim_{t\to\infty} \mathbb{E}\left[e^{-rt} S_t\right] = 0. \tag{B.2}$$

Given $\mathcal{C}$, the intermediary chooses consumption c_t to maximize its lifetime utility, with optimal consumption denoted by c_t^*.

[25] Endowing the intermediary with the possibility to accumulate savings ensures that it can smooth consumption and consume at a rate c_t even if payouts dI_t are not smooth. Consumption c_t and savings balance S_t can both take positive and negative values.

Intermediary Consumption Problem. Given $\mathcal{C}$, the intermediary's continuation utility defined as $U_t \equiv \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}u(c_s^*)ds\right]$. Further, U_t can be expressed as W_t in certainty equivalent monetary terms, with $W_t \equiv \frac{-\ln(-\rho r U_t)}{\rho r}$, and we work with W_t instead of U_t. Note that W_t is the intermediary's total continuation payoff in monetary terms with law of motion derived below in Appendix B.2

$$dW_t = \left[\frac{\rho r}{2}\left(\beta_t\sigma\right)^2 + \pi\left(\alpha_t - \frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)\right]dt + \beta_t\sigma dZ_t + \alpha_t(d\Pi_t - \pi dt), \tag{B.3}$$

where α_t and $\sigma\beta_t$ are the loadings of dW_t on the martingales $(d\Pi_t - \pi dt)$ and dZ_t respectively. The first term in the drift of (B.3) captures the intermediary's required compensation for being exposed to Brownian cash flow risk, while the second term captures the intermediary's required compensation for being exposed to shocks $d\Pi_t$. Both terms are unambiguously positive, so that W_t increases in expectation, $\mathbb{E}[dW_t] \geq 0$. Summarizing, we have:

Proposition 5. *The intermediary's optimal consumption satisfies $c_t^* = rW_t$. The intermediary's certainty equivalent payoff W_t, defined abovem, follows the dynamics* (B.3)*.*

Appendix B.1 and Appendix B.2 provide the proof of Proposition 5 in two parts: Part I analyzes the intermediary's optimal consumption and Part II derives the law of motion (B.3). The intermediary's certainty equivalent payoff W_t consists of two sources. First, the intermediary has savings S_t it has accumulated up to time t. Second, it expects to receive payouts from the firm after t, which it values at $Y_t \equiv W_t - S_t$. If the intermediary leaves the firm at t, dollar continuation payoff is S_t. If it stays with the firm and follows the contract, its expected continuation payoff is $W_t = Y_t + S_t$. The intermediary is better off leaving the firm if and only if $Y_t < 0$, so the optimal contract must respect $Y_t \geq 0$.

Combining (B.1) and (B.3) and using optimal consumption $c_t = c_t^* = rW_t$, we obtain

$$dY_t = dW_t - dS_t = \left[rY_t + \frac{\rho r}{2}\left(\beta_t\sigma\right)^2 - \pi\left(\frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)\right]dt + \beta_t\sigma dZ_t + \alpha_t d\Pi_t - dI_t. \tag{B.4}$$

Integrating (B.4) against time and taking expectations, we obtain

$$Y_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}\left\{dI_s - \left[\frac{\rho r}{2}\left(\beta_s\sigma\right)^2 + \pi\left(\alpha_s - \frac{1-e^{-\rho r\alpha_s}}{\rho r}\right)\right]ds\right\}\right]. \tag{B.5}$$

As desired, the integral representation for Y_t in (B.5) coincides with the integral representation for Y_t in (3) upon choosing k_s according to (8). Likewise, when k_t is determined according to (8), then the law of motion of Y_t in (7) coincides with (B.4).

B.1 Proof of Proposition 5 Part I — Optimal Consumption

We first state an auxiliary Lemma:

Lemma 9. *Take a process $\hat{I}$ and $s_1, s_2 \in \mathbb{R}$. Consider the problem*

$$U_t := U_t(c) = \max_{\{c_s\}_{s\geq t}} \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(c_s) ds \right] \tag{B.6}$$
$$\text{subject to } dS_s(c) = rS_s(c)ds + d\hat{I}_s - c_s ds, \; S_t(c) = s_1, \text{ and } \lim_{s\to\infty} \mathbb{E}e^{-r(s-t)} S_s(c) = 0,$$

where we explicitly denote the dependence of savings S on the consumption path c. Next, consider the problem

$$\tilde{U}_t := \tilde{U}_t(\tilde{c}) = \max_{\{\tilde{c}_s\}_{s\geq t}} \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(\tilde{c}_s) ds \right] \tag{B.7}$$
$$\text{subject to } d\tilde{S}_s(\tilde{c}) = r\tilde{S}_s(\tilde{c})ds + d\hat{I}_s - \tilde{c}_s ds, \; \tilde{S}_t(\tilde{c}) = s_2, \text{ and } \lim_{s\to\infty} \mathbb{E}e^{-r(s-t)} \tilde{S}_s(\tilde{c}) = 0.$$

Then, for $\Delta^S := s_2 - s_1$, the optimal consumption processes c and $\tilde{c}$, solving (B.6) and (B.7) respectively, satisfy $\tilde{c}_t = c_t + r\Delta^S$ so that $\tilde{U}_t = e^{-\rho r \Delta^S} U_t$.

Proof. To start with, note that with $\tilde{c}_s = c_s + r\Delta^S$,

$$\tilde{U}_t(\tilde{c}) = \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(c_s + r\Delta^S) ds \right] = e^{-\rho r \Delta^S} \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(c_s) ds \right] = e^{-\rho r \Delta^S} U_t(c), \tag{B.8}$$

where the first equality uses $\tilde{c}_s = c_s + r\Delta^S$ and the second equality uses

$$u(c_s + r\Delta^S) = -\frac{e^{-\rho(c_s + r\Delta^S)}}{\rho} = e^{-\rho r \Delta^S} \left(-\frac{e^{-\rho c_s}}{\rho} \right) = e^{-\rho r \Delta^S} u(c_s). \tag{B.9}$$

Next, suppose there exists a different consumption process $c' \neq \tilde{c}$, solving (B.7), with

$$\tilde{U}_t(c') > \tilde{U}_t(\tilde{c}) = e^{-\rho r \Delta^S} U_t(c), \tag{B.10}$$

and the transversality condition $\lim_{s\to\infty} \mathbb{E}e^{-r(s-t)} \tilde{S}_s(c') = 0$ holds under the consumption process c'. Define the consumption process c'' via $c''_t = c'_t - r\Delta^S$. As c' is different from $\tilde{c}$, it follows that c'' is different from c. As under the consumption path c' the transversality condition $\lim_{s\to\infty} \mathbb{E}e^{-r(s-t)} \tilde{S}_s(c') = 0$ holds, it follows that under the consumption path c'' the transversality condition $\lim_{s\to\infty} \mathbb{E}e^{-r(s-t)} S_s(c'') = 0$ holds too. In addition, note that the payoff under the consumption path c'' equals

$$U_t(c'') := \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(c''_s) ds \right] = e^{\rho r \Delta^S} \tilde{U}_t(c') > e^{\rho r \Delta^S} e^{-\rho r \Delta^S} U_t(c) = U_t(c), \tag{B.11}$$

where the second equality applies (B.9), which yields $u(c'_s) = e^{-\rho r \Delta^S} u(c''_s)$ and $u(c''_s) = e^{\rho r \Delta^S} u(c'_s)$, and the inequality uses (B.10). However, $U_t(c'') > U_t(c)$ contradicts the fact that c solves problem (B.6). The assertion follows. □

Using Lemma 9, we can now complete the argument by showing that optimal consumption satisfies $u(c_t) = rU_t$ and $c_t = rW_t$. According to Lemma 9, the marginal value of an additional unit of savings S_t at time t for the intermediary is given by $\left[\frac{\partial}{\partial \Delta^S} e^{-\rho r \Delta^S} U_t\right]|_{\Delta^S=0} = -\rho r U_t$. Optimal consumption smoothing implies that along the optimal path the first order $u'(c_t) = \left[\frac{\partial}{\partial \Delta^S} e^{-\rho r \Delta^S} U_t\right]|_{\Delta^S=0}$. has to hold at all times $t \geq 0$. That is, in optimum, the intermediary's marginal utility $u'(c_t)$ has to be equal to the marginal value of an additional unit of savings, $\left[\frac{\partial}{\partial \Delta^S} e^{-\rho r \Delta^S} U_t\right]|_{\Delta^S=0}$. Next, observe that $u'(c_t) = -\rho u(c_t)$ and use the above relations to obtain $u(c_t) = rU_t$. Inverting $u(c_t) = rU_t$ yields $c_t = rW_t$.

B.2 Proof of Proposition 5 Part II — Martingale Representation

Take the intermediary's continuation value $U_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)} u(c_s) ds\right]$ under any consumption process c_t (possibly, $c_t = c_t^*$). Define

$$A_t = \mathbb{E}_t\left[\int_0^\infty e^{-rs} u(c_s) ds\right] = \int_0^t e^{-rs} u(c_s) ds + e^{-rt} U_t. \tag{B.12}$$

By construction, $A = \{A_t\}$ is a martingale. By the martingale representation theorem, there exist stochastic processes $\hat{\alpha} = \{\hat{\alpha}_t\}$ and $\beta = \{\beta_t\}$ such that

$$e^{rt} dA_t = (-\rho r U_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho r U_t)\hat{\alpha}_t(d\Pi_t - \pi dt), \tag{B.13}$$

where $dZ_t = \frac{dX_t - \mu dt}{\sigma}$ is the increment of a standard Brownian Motion and $(d\Pi_t - \pi dt)$ is the increment of a compensated Poisson process. We differentiate (B.12) w.r.t. time t to obtain an expression for dA_t, then plug this expression into (B.13) and solve (B.13) to get $dU_t = rU_t dt - u(c_t)dt + (-\rho r U_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho r U_t)\hat{\alpha}_t(d\Pi_t - \pi dt)$. With the optimal consumption policy $c_t = c_t^*$, satisfying $u(c_t) = rU_t$, this simplifies to

$$dU_t = (-\rho r U_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho r U_t)\hat{\alpha}_t(d\Pi_t - \pi dt), \tag{B.14}$$

which is a martingale in that $\mathbb{E}[dU_t] = 0$. Next, to derive the law of motion of $W_t = W(U_t) := \frac{-\ln(-\rho r U_t)}{\rho r}$, note that $W'(U) = \frac{1}{-\rho r U}$, $W''(U) = \frac{1}{\rho r U^2}$, and $W(U - \rho r U \hat{\alpha}) - W(U) = -\frac{\ln(1-\rho r \hat{\alpha})}{\rho r}$. We now use Itô's Lemma in its version for jump processes and calculate via (B.14)

$$\begin{aligned} dW(U_t) &= W'(U_t)\rho r U_t \pi \hat{\alpha}_t dt + W'(U_t)(-\rho r U_t)\beta_t \sigma dZ_t + W''(U_t)\left(\frac{(\rho r U_t)^2(\beta_t\sigma)^2}{2}\right) dt \\ &+ \big[W(U_t - \rho r U_t \hat{\alpha}_t) - W(U_t)\big] d\Pi_t = -\pi \hat{\alpha}_t dt + \beta_t \sigma dZ_t + \frac{\rho r}{2}(\beta_t \sigma)^2 dt - \frac{\ln(1-\rho r \hat{\alpha}_t)}{\rho r} d\Pi_t. \end{aligned}$$

Setting $\alpha_t := -\frac{\ln(1-\rho r \hat{\alpha}_t)}{\rho r} \iff \hat{\alpha}_t = \frac{1-e^{-\rho r \alpha_t}}{\rho r}$ and rewriting, (B.3) follows.

C Robustness & Extensions

C.1 Active Intermediaries and Monitoring

We now solve the model variant with endogenous intermediary effort a_t. For simplicity, we do not distinguish between actual effort levels and effort levels anticipated by outside investors, and simply write a_t for the optimal effort.

C.1.1 State Variables

In this model variant, the intermediary's continuation value reads

$$Y_t = \mathbb{E}_t \left[\int_t^{\infty} e^{-r(s-t)} \left(dI_s - k_s ds - \frac{\kappa a_s^2}{2} ds \right) \right]. \tag{C.1}$$

By the martingale representation theorem, there exists processes α and β such that

$$dY_t = \left[rY_t + k_t - \frac{\kappa a_t^2}{2} \right] dt + \beta_t (dX_t - \mu - a_t) dt + \alpha_t (d\Pi_t - \pi dt). \tag{C.2}$$

The intermediary chooses at each time t its effort $a_t \geq 0$ to solve $\max_{a_t} \left(\beta_t a_t - \frac{\kappa a_t^2}{2} \right)$, leading to $\beta_t = a_t \kappa$. As in the baseline, we set k_t according to (8). Next, note that the firm's cash balance M_t evolves according to (2) Excess liquidity has then the law of motion $dC_t = dM_t - dY_t$ with

$$\begin{aligned} dC_t = & \left[\mu + a_t + (r - \lambda) C_t - \lambda Y_t - \sigma^2 k_Z(\beta) - \frac{\kappa a_t^2}{2} + \pi \left(\alpha_t - k_\Pi(\alpha_t) \right) \right] dt \\ & + \sigma (1 - \beta_t) dZ_t + (C_t^* - C_t) d\Pi_t - dDiv_t, \end{aligned} \tag{C.3}$$

where we define the post-refinancing level of excess liquidity as $C_t^* \equiv \Delta M_t + C_t - \alpha_t$. Recall $k_Z(\beta)$ and $k_\Pi(\alpha)$ from (A.4).

Finally, note that in autarky $\beta_t = 1$, so $a_t = 1/\kappa$. As such, $Y^A = \frac{\mu}{r} - \frac{\rho\sigma^2}{2} + \frac{1}{2\kappa r}$.

C.1.2 HJB Equation and Optimization

As in the baseline, C_t is the only state variable, and dividend payouts $dDiv_t$ cause C_t to reflect at the endogenous upper boundary $\overline{C}$ satisfying smooth pasting and super contact conditions $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$. In the interior of the state space (i.e., for $C \in (\underline{C}, \overline{C})$ the value function $P(C)$ solves the following HJB equation:

$$\begin{aligned} rP(C) = & \max_{\beta, Y} \left\{ P'(C) \left[\mu + \frac{2\beta - \beta^2}{2\kappa} + (r - \lambda) C - \lambda Y - \frac{\rho 2 \beta^2}{2} \right] + P''(C) \frac{\sigma^2}{2} (1 - \beta)^2 \right\} \\ & + \pi \max_{C^*, \alpha \in \mathcal{S}(C^*, C)} \left\{ P'(C) \left(\frac{1 - e^{-\rho r \alpha}}{\rho r} \right) + \left[P(C^*) - P(C) - (C^* - C + \alpha) \right] \right\}, \end{aligned} \tag{C.4}$$

where we already inserted the incentive condition $a_t = \beta_t/\kappa$. One can show that the value function $P(C)$ is strictly concave on $(\underline{C}, \overline{C})$ (omitted here), with $P'(C) > 1$.

The controls Y, C^*, and α are determined analogously to the baseline. That is, we have $Y(C) = \max\{-C, 0\}$ and $M(C) = \max\{C, 0\}$ which is (18). The firm chooses refinancing target $C^* = \overline{C}$. The optimal choice of $\alpha = \alpha(C)$ is characterized in (23), so $\alpha(C) = \max\{\alpha_U(C), \alpha_C(C)\}$.

Next, the first order condition in (C.4) with respect to β reads $\frac{1-\beta}{\kappa} - \rho r \sigma^2 P'(C)\beta - P''(C)\sigma^2(1-\beta) = 0$, which can be solved for

$$\beta = \beta(C) = \frac{1 - \kappa\sigma^2 P''(C)}{1 + \rho r \kappa \sigma^2 P'(C) - \kappa \sigma^2 P''(C)}. \tag{C.5}$$

Notice that the expression for $\beta(C)$ in (C.5) becomes (21) in the limit $\kappa \to \infty$.

Finally, notice that at the payout boundary $\overline{C}$, we have $P''(\overline{C})$ and $P'(\overline{C}) = 1$ so that $\beta^B := \beta(\overline{C}) = \frac{1}{1+\rho r \kappa \sigma^2}$ and $a^B := a(\overline{C}) = \frac{1}{\kappa + \rho r \kappa^2 \sigma^2}$, so $a(\overline{C})$ is smaller than the autarky effort $1/\kappa$. As a result, the value function at the payout boundary satisfies

$$P(\overline{C}) - \overline{C} = \frac{1}{r}\left(\mu - \lambda\overline{C} + a^B - \frac{\kappa(a^B)^2 - \rho r \sigma^2 (\beta^B)^2}{2}\right). \tag{C.6}$$

C.1.3 Lower Boundary $\underline{C}$

We determine the lower boundary in the state space $\underline{C}$ using arguments analogous to the ones from the main text. To begin with, suppose that $\underline{C} = C^S < -L$ in which case the firm is never liquidated. The following conditions are satisfied. First, it must be that $\beta(\underline{C}) = 1$ so that $a(\underline{C}) = \frac{1}{\kappa}$. Second, the drift of C in (C.3), denoted $\mu_C(C)$, must be zero, i.e., $\mu_C(\underline{C}) = 0$. Third, $P(\underline{C}) = 0$. Furthermore, these requirements can be met jointly only if $\alpha(\underline{C}) = P\left(\overline{C}\right) - \overline{C} - P(\underline{C}) = P\left(\overline{C}\right) - \overline{C}$. Inserting this expression for $\alpha(\underline{C})$, $\beta(\underline{C}) = 1$, $Y(\underline{C}) = -\underline{C}$ as well as $a(\underline{C}) = 1/\kappa$ into (C.3), we obtain

$$\mu_C(\underline{C}) = \mu + \frac{1}{2\kappa} - \frac{\rho r \sigma^2}{2} + r\underline{C} + \pi\left(\frac{1 - e^{-\rho r[P(\overline{C}) - \overline{C}]}}{\rho r}\right).$$

Notice that $P(\overline{C}) - \overline{C}$ is characterized above in (C.6), so that

$$\mu_C(\underline{C}) = \mu + \frac{1}{2\kappa} - \frac{\rho r \sigma^2}{2} + r\underline{C} + \pi\left(\frac{1 - e^{-\rho[\mu - \lambda\overline{C} + a^B - 0.5\kappa(a^B)^2 - 0.5\rho r \sigma^2(\beta^B)^2]}}{\rho r}\right).$$

Using Lemma 7, we can solve $\mu_C(\underline{C}) = 0$ (after some algebra) for

$$C^S = \underline{C} = \frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\chi}{r} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r} - Y^A, \tag{C.7}$$

where $w(\cdot)$ is the primary branch of the Lambert-w function and $\chi = \frac{1}{2\kappa} - \left(a^B - \frac{\kappa (a^B)^2}{2} - \frac{\rho r \sigma^2 (\beta^B)^2}{2} \right)$. Above expression (C.7) simplifies to C^S from the baseline, when $\kappa \to \infty$.

It follows that $\underline{C} = \min \left\{ C^S, -L \right\}$. When $\underline{C} = -L$, the firm is liquidated at $C = \underline{C}$ and $\tau < \infty$ almost surely. When $\underline{C} < -L$, the firm is never liquidated and $\tau = \infty$.

Finally, it is intuitive that cash flow-based financing capacity $-C^S$ decreases with κ, as higher monitoring cost implies that the intermediary can add less value to the firm and reduces the intermediary's valuation of the firm.

C.2 Refinancing Costs

Similar to Décamps et al. (2011) or Bolton et al. (2011), we can incorporate fixed costs $\phi \in (0, \infty)$ of equity issuance in addition to infrequent capital market access. The jump in existing shareholders value upon refinancing is now $J(C) = P(\overline{C}) - P(C) - (\overline{C} - C) - \phi - \alpha(C)$. We require, as in the baseline, $J(C) \geq -P(C)$, due to shareholders' limited liability. With fixed equity financing costs, the firm raises equity in state C upon refinancing if and only if the total gains from refinancing in the HJB equation (15) are positive, i.e.,

$$\max_{C^*, \alpha \in \mathcal{S}(C^*, C)} \left\{ P'(C) \left[\alpha - k_\Pi(\alpha) \right] + \left[P(C^*) - P(C) - (C^* - C + \alpha) \right] \right\} \geq 0 \tag{C.8}$$

It follows that $C^* = \overline{C}$. In this case, $\alpha(C)$ must satisfy $\alpha(C) \leq P(\overline{C}) - (\overline{C} - C) - \phi$. If (C.8) does not hold, the firm does not refinance upon capital market access and the choice of $\alpha(C)$ becomes irrelevant. Overall, we can write $\alpha(C) = \min\{P(\overline{C}) - (\overline{C} - C) - \phi, \alpha_U(C)\}$. This results in no refinancing upon $d\Pi = 1$ on some interval $\left[\tilde{C}, \overline{C} \right]$ for some $\tilde{C} \in \left[\underline{C}, \overline{C} \right]$.

We now derive the lower boundary C^S under survival. Notice that, as in the baseline with $\phi = 0$, we have $P(C^S) = 0$, $\mu_C(C^S) = 0$, and $\beta(C^S) = 1$. In addition, when the firm finds it optimal to raise equity upon capital market access $d\Pi = 1$, i.e., when (C.8) holds at $C = C^S$, then $J(C^S) = 0$ which pins down $\alpha(C^S) = P(\overline{C}) - \overline{C} + C^S$. We can then solve for the lower boundary boundary conditional on survival:[26]

$$C^S = \min \left\{ \frac{w\left(\frac{\pi}{r} \exp\left\{ \rho r \left[\frac{\lambda}{r} \overline{C} + \phi + \frac{\pi}{\rho r^2} - \frac{\rho}{2} \sigma^2 \right] \right\} \right) - \frac{\pi}{r}}{\rho r}, 0 \right\} - Y^A.$$

The lower boundary is then $\underline{C} = \min\{C^S, -L\}$. When $\underline{C} = C^S$, the firm is never liquidated and raises new equity financing in state $\underline{C}$ upon market access. When $\underline{C} = -L$, the firm liquidates once C reaches $-L$. In the limit $\pi \to \infty$ (continuous access to equity financing at fixed cost ϕ), we get $\underline{C} = -\max\{[P(\overline{C}) - \overline{C}] - \phi, L\}$, but $\overline{C} > 0$ as both intermediary financing and refinancing are costly. If ϕ and L are sufficiently small, the firm refinances at a lower boundary $\underline{C} < -L$. If $\underline{C} = C^S$, the firm raises refinances once C reaches $\underline{C}$. The firm finances cash flow shortfalls against future promises Y for $C < 0$. Y may exceed the liquidation value of assets L, leading to cash flow-based financing. Our qualitative results go

[26]When $\{\cdot\} = 0$, then $C^S = -Y^A$ and the firm does not raise equity financing at C^S upon $d\Pi = 1$.

through as long as refinancing is costly (e.g., $\phi > 0$) or not frequently available ($\pi < \infty$).[27]

C.3 Linear Cost k_t and Stochastic Discount Factor

Consider $k_Z(\beta) = \rho\beta$ with $\beta \geq 0$ and $k_\Pi(\alpha) = 0$, so $k_t = \sigma^2 k_Z(\beta_t)$. This can be micro-founded by assuming that the intermediary applies a stochastic discount factor S_t with price of risk $\rho \geq 0$, i.e., $dS_t = S_t(-rdt - \rho\sigma dZ_t)$, so that $Y_t = \mathbb{E}_t\left[\int_t^\tau \frac{S_s}{S_t} dI_s\right]$. We obtain

$$Y_t = \mathbb{E}_t\left[\int_t^\tau \frac{S_s}{S_t} dI_s\right] = \mathbb{E}_t^Q\left[\int_t^\tau e^{-r(s-t)} dI_s\right] = \mathbb{E}_t\left[\int_t^\tau e^{-r(s-t)}\left(dI_s - \sigma^2\rho\beta_s ds\right)\right].$$

The expectation $\mathbb{E}_t^Q$ is taken under the risk-neutral measure with $d\tilde{Z}_t = dZ_t - \rho\sigma dt$ as the increment of a standard Brownian motion; $\mathbb{E}_t$ is taken under the physical measure with dZ_t as increment of a Brownian motion. The remainder of the analysis remains similar as in the baseline. For instance, the HJB equation (15) applies for any $k_Z(\beta)$ and $k_\Pi(\alpha)$, including the specification of this section.

[27] For completeness, one could also add — next to the fixed cost of refinancing ϕ — a variable "flotation" cost of refinancing $\hat{\phi}$, as in Bolton et al. (2011). Under these circumstances, the firm would choose a refinancing target $C^* < \overline{C}$, but the remainder of the findings would likely remain similar.