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Barney Hartman-Glaser
Simon Mayer
Konstantin Milbradt

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ABSTRACT

We consider a firm with infrequent access to capital markets, continuous access to costly intermediary financing, and a cost of holding cash. In the optimal renegotiation-proof contract, the intermediary absorbs a share of cash-flow risk. This share increases as firm depletes its internal cash, suggesting an overlapping pecking order. If the firm runs out of cash, it liquidates if the cash-flow is too risky, otherwise the intermediary extends financing that resembles either private equity or collateralized debt. The model helps explain trends in financial intermediation, such as the rise of private equity and the use of collateralized debt.

Barney Hartman-Glaser
University of California at Los Angeles
110 Westwood Plaza
Suite C421
Los Angeles, CA 90095
barney.hartman-glaser@anderson.ucla.edu

Simon Mayer
1140 S Wabash Avenue, Apt 1315
Chicago, IL 60605
United States
simon.mayer@chicagobooth.edu

Konstantin Milbradt
Kellogg School of Management
Northwestern University
2001 Sheridan Rd #401
Evanston, IL 60208
and NBER
milbradt@northwestern.edu

The assumption of frictionless financial markets, while plausible over the long run for most firms, is often at odds with the reality facing many firms. Raising capital at short notice is usually costly, constrained, or both. Indeed, specialized intermediaries often provide short-notice financing, while it typically takes for firms to access financing from broader capital markets. Consequently, we observe that contracts with specialized intermediaries often result in financing arrangements that are more complex than common equity or long term debt, such as credit lines with performance-sensitive interest rates, flexible equity stakes, convertible or collateralized debt.

We construct a tractable continuous-time model to capture the salient features of the disparity between short-notice financing and broader financial markets. A firm owned by risk-neutral investors produces risky cash flows and faces financial constraints. As in Hugonnier, Malamud, and Morellec (2015), the firm can raise financing from new risk-neutral and competitive outside investors only at random times and must thus "wait for capital." This assumption reflects capital supply uncertainty or proxies for frictions that cause a delay between the firm's need for financing and its access to broader markets. In the interim, the firm finances operating losses with either internal cash reserves or funds provided by an intermediary. However, both liquidity facilities are costly. Cash held in the firm earns a return below the risk-free rate due to an internal carry cost of cash, while the intermediary requires compensation that increases in the cash-flow risk it bears by providing financing. We derive the optimal contract between the firm's investors and the intermediary that maximizes the value of the firm, subject to two-sided limited commitment.

We show that we can summarize the state of the firm with a single variable that we term excess liquidity. This variable equals the firm's cash holdings less the intermediary's stake in the firm measured in dollars. Under the optimal contract, the investors' value function solves an ordinary differential equation (ODE) over excess liquidity with two free boundaries. The firm makes payouts to its investors only when excess liquidity attains the upper boundary. These payouts thus resemble dividends on common shares and, as such, we can interpret the investors as common shareholders. When the firm runs out of cash, which coincides with excess liquidity hitting zero, it either liquidates or the intermediary rescues the firm by injecting capital in exchange for a stake, allowing excess liquidity to turn negative. Once the firm gains access to capital markets, it raises new capital and pays out the intermediary's stake, replenishing excess liquidity in the process. The intermediary thus provides intertemporal intermediation between current and future investors, that is, it acquires a stake in the firm with the intention of selling this stake on to future investors. The outside investor's limited commitment constraint leads to renegotiation proofness of the optimal contract. Consequently, even though future investors are by definition not party to the current contract, they agree to the same terms when investing in the firm.

The intermediary's stake in the firm consists of deferred payouts, which only feature in the optimal contract if the firm runs out of cash and the intermediary is willing to provide financing. This stake is tightly linked to (partial) ownership of the firm as follows. The maximum stake the intermediary is willing to provide is simply its value from sole-ownership of the firm which is equal to the expected value of receiving all the cash-flows of the firm (net of costs) until market access, at which point the intermediary can sell the firm's to outside investors for market value. Thus the maximum stake the intermediary is willing to provide increases with the frequency of financial market access and decreases with cash flow risk. Moreover, the intermediary is only willing to accumulate a stake in the firm if this maximum stake is positive. In other words, the intermediary only acquires a stake if it is willing to fully own the firm for the time it takes to find new outside investors. The emergence of such *deferred payout financing* to hedge cash-flow shocks is the novel part of the optimal contract that sets our results apart from more traditional models with exogenous hedging costs that do not involve a second type of agent, the intermediary. We offer two interpretations of this type of financing. First, the stake could represent private equity (PE) investment that the intermediary holds in order to realize a capital gain upon exit (market access). Second, the stake could represent a collateralized debt claim in which the intermediary receives ownership of the firm's assets if the firm exhausts its supply of internal cash.

Negative cash flow shocks can induce financial distress because the firm has infrequent access to capital markets. Consequently, the firm's investors are effectively risk averse with respect to cash-flow shocks, the more so as the firm depletes its liquidity reserves and nears distress. At the same time, the intermediary demands compensation for providing financing to cover cash flow shortfalls. As such, the investors and the intermediary face a state-dependent risk-sharing problem. The optimal financing agreement calls for the intermediary to bear an increasing fraction of cash flow risk as the firm's excess liquidity decreases. Further, the prospect of market access allows the firm to shift payouts to the intermediary from states in which the firm is financially constrained to states where the firm is unconstrained. The optimal financing agreement in regard to market access resembles a short-maturity insurance contract that delivers a premium flow to the firm in exchange for a lumpy payout in a specific state, in this case, the state in which the firm has access to new outside investors. Any financing from the intermediary that does not result in a stake can thus be seen as *committed short-term financing*.

The fraction of risk covered by cash reserves and the intermediary depends on the firm's excess liquidity as follows. As long as the firm has positive cash holdings, it uses a combination of cash and intermediary financing to absorb cash flow shocks, effectively saving and borrowing simultaneously, a configuration we call an *overlapping pecking order*. For example, the firm may cover a \$1 operating loss by withdrawing \$.35 from its internal cash and raising \$.65 in financing from the intermediary. Symmetrically, if the firm were to make a \$1 operating profit, it deposits \$.35 in its cash account and repays \$.65 to the intermediary. The share of cash-flow shocks allocated to the intermediary decreases at the excess liquidity of the firm increases. If the firm runs out of cash, it either liquidates or, if possible, it finances itself by gradually granting the intermediary a stake in the firm in exchange for financing, which the firm buys back when it realizes positive cash flows or can raise outside financing upon market access.

Our analysis has implications for when intermediary financing, and deferred-payout financing in particular, is most likely to occur and to generate value. Interestingly, our findings predict that intermediation in the form of deferred-payout financing is most pronounced when the firm's access to financial markets is at intermediate levels. The firm's access to financial markets determines its ability to raise financing from competitive and risk-neutral outside investors. Intuitively, when the firm's market access is good, the intermediary can exit its positions quickly, reducing the stake the intermediary holds on average. On the other hand, when market access is impaired, and a successful exit is difficult, the intermediary is unwilling to acquire a stake in the firm given the low resale option value. As a firm's access to financial markets tends to increase over its life cycle, our model suggests that firms rely on and benefit from deferred-payout financing the most in their intermediate stages. Interestingly, we show that the availability of committed short-term financing is a prerequisite for the firm to access deferred payout financing.

Next, we show that intermediation and deferred-payout financing are most pronounced and generate the most value for firms with intermediate levels of cash flow volatility. On one hand, firms with low cash flow volatility have limited financing needs and thus do not rely much on an intermediary. On the other hand, firms with very volatile cash flows do not have access to cheap intermediary financing, as the cost of intermediary financing increases with the level of risk. Intuitively, intermediaries are reluctant to invest firms with very high risk (relative to returns). Therefore, these types of firms accumulate large cash reserves to manage liquidity, reflecting that intermediary financing and cash reserves are substitutes in the firm's liquidity management.

Our model offers a connection between two significant secular trends of the last decades. First, risk-free rates have steadily declined. Second, deferred-payout financing in the form of PE and/or collateralized debt has grown substantially. Our model links these two facts by showing that a decline in interest rates stimulates PE-like intermediation. In particular, we show that as the risk-free rate decreases, the intermediary's willingness to provide financing to the firm in exchange for an equity stake increases because both the intermediary's valuation of the firm's future cash flows and the resale option value of firm ownership increase. The improved access to intermediary financing, in turn, induces firms to reduce precautionary cash holdings.

One interpretation of the stake is as a Private Equity (PE) investment in the firm. The derived intermediary ownership dynamics resemble that of a PE investor who acquires distressed firms and holds them primarily to realize capital gains upon exit. If the firm we consider is public, this type of financing resembles private investment in public equity (PIPE) of distressed firms. Our model, therefore, suggests that through their provision of financing in distress, PE investors help resolve distress and more efficiently reduce the risk of liquidation of their portfolio firms, in line with the findings in Bernstein, Lerner, and Mezzanotti (2019) and Hotchkiss, Smith, and Strömberg (2021). Notably, the firm's access to PE-like financing is endogenous, as it depends on the intermediary's willingness to acquire a stake in the firm. Further, the availability of debt financing is necessary for and complementary to PE financing, consistent with Ivashina and Kovner (2011). In contrast, increased availability of PE financing crowds out debt like financing that in the PE implenetation takes the form of a credit line.

The optimal contract can be implemented with a credit line and restricted equity that vest upon refinancing, with an overlapping pecking order. We feel this is a natural implementation given the ownership stake that the intermediary may acquire. When the firm is flush with liquidity, it simultaneously uses internal cash reserves and credit line debt to cover cash flow shortfalls. Moreover, the firm relies increasingly on the credit line as cash reserves dwindle. When the firm exhausts its cash reserves, it either liquidates, effectively defaulting on the credit line, or finances itself by drawing on the credit line and selling (restricted) equity to the intermediary. Symmetrically, the firm pays back the credit line, buys back restricted equity, and potentially retains earnings following positive cash flows or upon financial market access.

An alternative interpretation of the stake is as secured / collateralized debt, so that all financing from the intermediary takes the form of debt. The firm finances cash flow shortfalls first with internal cash reserves and unsecured/uncollateralized credit line debt. Once the firm has run out of cash, it can only survive if it has access to secured or collateralized credit line financing, which requires to pledge the firm as collateral. Crucially, the intermediary is only willing to accept the firm as collateral if its resale option value plus cash-flows are high enough; otherwise, the firm is liquidated and defaults. Conditional on survival, the firm effectively undergoes bankruptcy at the lower bound in the state space when it cannot repay its credit line debt without external financing: The intermediary seizes the underlying collateral, takes full ownership of the firm and its cash-flows, and keeps the firm alive until it can sell the firm to new equity investors upon market access, in which case bankruptcy is resolved. In this context, our analysis suggests that the firm relies on unsecured debt financing in normal times, while it finances with secured debt only under financial distress, in line with the findings in Benmelech, Kumar, and Rajan (2020). Interestingly, financial market access and the decline of interest rates improve the firm's borrowing capacity, reducing the risk of firm liquidation.

Finally, to capture that some types of intermediaries (e.g., banks or PE investors) often actively seek to improve the performance of the firms they invest in (e.g., via monitoring), we extend our model to allow the intermediary to exert costly effort which increases the firm's cash flow drift. The intermediary's incentives to exert such effort increase with its exposure to the firm's cash flows, which decreases with the firm's liquidity position and increases with the intermediary's equity stake in the firm. As such, the intermediary exerts high effort when the firm undergoes financial distress, thereby stabilizing firm performance. Moreover, we highlight how the intermediary's incentives to improve firm performance crucially depend on its exit opportunities. Interestingly, overly good exit opportunities undermine the intermediary's incentives to exert high effort to improve cash flows, and the intermediary's effort may increase when exit opportunities worsen (e.g., due to a financial crisis).

Our paper mainly relates to the literature on dynamic corporate liquidity management in continuous time, pioneered by Bolton, Chen, and Wang (2011) and Décamps, Mariotti, Rochet, and Villeneuve (2011). In a unified model of corporate investment, financing, and liquidity management, Bolton et al. (2011) demonstrate how liquidity management and firm financing interacts with a firm's investment decisions. Further contributions in this literature include Gryglewicz (2011), Bolton, Chen, and Wang (2013), Hugonnier et al. (2015); Hugonnier and Morellec (2017), Malamud and Zucchi (2018), and, more recently, Abel and Panageas (2020b,a), Dai, Giroud, Jiang, and Wang (2020), and Bolton, Li, Wang, and Yang (2021). In addition, Bolton, Wang, and Yang (2021) study dynamic liquidity management with short-term debt financing, thereby highlighting the interaction between the endogenous pricing of the debt and the optimal the equity payout and issuance strategies. Our paper differs from these papers, as it adds an intermediary who can provide any type of financing to a dynamic liquidity management model. Intermediary financing arranged through an optimal contract (i) features state-contingent risk-sharing between the firm and the intermediary, and (ii) allows the firm to finance cash flow shortfalls against future promised payments to the intermediary (e.g., by granting the intermediary an equity-like stake in the firm) within limited commitment constraints. While dynamic risk-sharing would also arise in a liquidity management model with access to short-term hedging contracts as say an extension of Bolton et al. (2011) would deliver, result (ii) requires the introduction of a second agent, the intermediary, to sign a long-term contract with. In other words, the novel contribution of our model is that it combines dynamic liquidity management with optimal long-term contracting. As such, our model delivers novel results on the firm's reliance on intermediary equity financing and, more broadly, on endogenous PE-like intermediation.

Our paper also relates to the corporate finance literature on dynamic contracting without liquidity management. Recent contributions include DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), Green and Taylor (2016), Piskorski and Westerfield (2016), Varas (2018), Marinovic and Varas (2019), Malenko (2019), Gryglewicz and Hartman-Glaser (2020), Gryglewicz, Mayer, and Morellec (2020), Feng and Westerfield (2021), and Feng, Taylor, Westerfield, and Zhang (2021). Our paper also relates to works on financial intermediaries and their incentives, such as the seminal work of Holmstrom and Tirole (1997). In contemporaneous work, Gryglewicz and Mayer (2021) study a dynamic contracting model with intermediation, and show that intermediary incentives are highest following poor performance. Different from classical (static) theories of intermediation (e.g., (Holmstrom and Tirole, 1997)) or from the work of Gryglewicz and Mayer (2021), the intermediary in our paper engages in dynamic, intertemporal intermediation between present and future investors, which yields novel results on the effects of intermediary exit opportunities (financial market liquidity) as well as their relation to intermediary incentives. The intermediary in our model also has certain characteristics of private equity (PE) investors; recent theories of private equity include Axelson, Strömberg, and Weisbach (2009), Malenko and Malenko (2015) or Ewens, Gorbenko, and Korteweg (2021).

In particular, our work relates to the dynamic contracting literature that studies optimal risk-sharing between a principal and an agent under limited commitment, such as Ai and Li (2015), Ai, Kiku, and Li (2019), and Bolton, Wang, and Yang (2019). The closest reference to our paper is Bolton et al. (2019). Our model differs from theirs in that we allow a deep-pocketed but costly intermediary to provide the marginal financing of the firm. Our work is complementary to theirs in that it highlights optimal financing from a costly intermediary in the presence of physical cash constraints and limited commitment. In contrast, their model is driven by the connection between investment, firm scale, and the scale of the manager's outside option. Rampini and Viswanathan (2010) and Rampini, Sufi, and Viswanathan (2014) provide models in which financing is restricted by limited commitment in the form of limited enforcement, which implies a role for net worth in easing financial constraints. In their work, risk management requires net worth that could otherwise be used for productive investment.

Finally, there is an extensive structural and empirical literature on the connection between corporate liquidity management and firm policies. Hennessy and Whited (2005, 2007); Hennessy, Levy, and Whited (2007), Whited and Wu (2006), and Nikolov, Schmid, and Steri (2019) develop structural corporate liquidity management models. Almeida and Campello (2007) and Campello, Graham, and Harvey (2010), amongst many, provide evidence how financial constraints affect corporate investment. In a theoretical setting, Acharya, Almeida, and Campello (2007) show that debt and cash are not perfect substitutes due to different hedging properties and thus may be used simultaneously. They provide empirical evidence supporting this conclusion. Sufi (2009) provides further empirical evidence on the difference between the use of cash and lines of credit for firms. Other related empirical studies on cash holdings and financing choices include Leary and Roberts (2005), Bates, Kahle, and Stulz (2009), Eisfeldt and Muir (2016), and Darmouni and Mota (2020).

1 Model Setup

We consider a firm whose assets produce volatile cash flows. The firm is owned by *outside investors*, also referred to as shareholders in anticipation of our results. Access to new outside capital is limited, but an *intermediary* is available to supply capital continuously at a cost. Both the firm's investors and the intermediary discount the future at the risk free rate $r > 0$. To structure financing from the intermediary, investors and the intermediary sign a transfer agreement at time zero. We interpret the role and identity of the intermediary broadly who may in reduced form represent a group of different intermediaries, including banks, or non-bank lenders, private equity (PE) firms, and venture capitalists. We call this agent an intermediary, because as will become apparent below, in some states, the intermediary accumulates a stake in the firm in exchange for financing with the intention to sell this state to later investors and thus provides intertemporal intermediation.

The firm has assets in place that generate cash flows dX_t with constant mean $\mu > 0$ and volatility $\sigma > 0$. That is,

$$dX_t = \mu dt + \sigma dZ_t, \tag{1}$$

where dZ_t is the increment of a standard Brownian Motion. The intermediary and investors sign a payout agreement or contract $\mathcal{C}$ at time $t = 0$. This contract, $\mathcal{C} = (Div, I, \Delta M)$, stipulates cumulative payouts Div_t to investors and we will refer to such as *dividends*, money raised from new investors upon financial market access ΔM_t,[1] and cumulative transfers I_t to/from the intermediary.

As in Hugonnier et al. (2015), the firm can only raise external funds from competitive outside investors at Poisson times $d\Pi_t = 1$ that arrive with constant intensity $\pi \geq 0$, with $d\Pi_0 = 1$ to reflect market access at the founding of the firm.[2] This assumption reflects capital supply uncertainty or proxies for frictions that cause a delay between the firm's need for financing and its access to broader markets.[3] We will refer to such capital market access as refinancing opportunities. We write the financing constraint as $dDiv_t \geq 0$. Capital infusions by outsiders upon capital market access are thus written as $\Delta M_t d\Pi_t \geq 0$. Put differently, the constraint $dDiv_t \geq 0$ means that investors are protected by limited liability as in Bolton et al. (2011). The firm's financial constraints together with the fact that cash flow shocks can be negative imply that the firm has an incentive to build a buffer stock of cash M_t via retained earnings.

Specifically, as the investors are unable to inject cash between refinancing times, all cash flow realizations dX_t, dividends $dDiv_t \geq 0$, and transfers to/from the intermediary dI_t flow through the firm's internal cash balance M_t. We normalize the cash balance at $t = 0^-$ (i.e., before the contract is signed) to zero, i.e., $M_{0^-} = 0$. In contrast to the firm's investors, the intermediary can inject cash into the firm at any time, that is, dI_t can be negative, but this source of financing is costly. In particular, we assume that the intermediary faces a flow cost for providing financing, $k_t \geq 0$. We interpret k_t as a proxy for deeper frictions the intermediary may face, such as regulatory requirements or limited capital of its own. We will specify k_t in more detail below.

The cash balance held within the firm accrues interest at the rate $r - \lambda$ where r is the common

[1] Here, in anticipation of the key financing friction, we are separating payments *to* existing shareholders, Div_t, from payments *from* newly arriving outside investors at refinancing, ΔM_t.

[2] As is well known in the search literature, for example Duffie, Garlenau, and Pedersen (2005), it is without loss of generality to assume that newly arriving outside investors are competitive. Any surplus extracted by newly arriving outside investors could be fully captured by appropriately transforming the arrival rate of refinancing opportunities. We discuss alternatives to this setup in Section 5.3.

[3] One may interpret the time it takes to arrange for financing as caused by (un-modelled) asymmetric information — outside investors take time to verify information, while the intermediary, being a specialist, circumvents this delay.

interest rate and $\lambda \in (0, r)$ represents a carry cost of cash.[4] The dynamics of cash reserves M_t are then given by

$$dM_t = dX_t + (r - \lambda) M_t dt - dDiv_t - dI_t + \Delta M_t d\Pi_t. \tag{2}$$

Next, we impose a "physical" constraint on cash, in that cash holdings must remain non-negative, $M_t \geq 0$ for all $t \geq 0$. This implies that if M_t attains zero, the intermediary must either inject the necessary funds or the firm must liquidate. Liquidation thus occurs at a stopping time $\tau \in [0, \infty]$, and $dDiv_t = dI_t = dX_t = 0$ for all $t > \tau$. For simplicity, we assume that the liquidation value of the firm beyond its current cash holdings M_τ is zero. In other words, liquidation is costly.

1.1 Optimal contracting problem

We call the payout agreement $\mathcal{C}$ incentive compatible (IC) if it respects the intermediary's as well as the shareholders' limited commitment, and we restrict our attention to IC payout agreements. The firm's founders, i.e., its original investors, have full bargaining power, and can extract all surplus from the intermediary upon signing the payout agreement at $t = 0$. Let Y_0 be the intermediary's expected payoff for a given contract $\mathcal{C}$, that is,

$$Y_0 = \mathbb{E}\left[\int_0^\infty e^{-rt}\left(dI_t - k_t dt\right)\right], \tag{3}$$

where k_t is the intermediary's cost of providing financing to the firm.

The original investors' value function P_{0^-} before initial financing at time $t = 0$, is the discounted stream of dividend payouts net the costs of refinancing:

$$P_{0^-} = \max_{\mathcal{C} \text{ is IC}} \mathbb{E}\left[\int_0^\tau e^{-rt}(dDiv_t - \Delta M_t d\Pi_t)\right] \quad \text{s.t.} \quad Y_0 \geq 0 \quad \text{and} \quad M_{0^-} = 0. \tag{4}$$

Note that upon refinancing, i.e., $d\Pi_t = 1$, the firm raises ΔM_t dollars from newly arriving competitive outside investors at fair value by issuing ΔM_t dollars worth of new equity. The firm's existing investors are diluted as they must give up ΔM_t dollars worth of equity ownership. At inception at time $t = 0$, the firm's original investors are penniless and the firm's cash holdings are zero, but the firm can raise financing from newly arriving (competitive) outside investors as $d\Pi_0 = 1$. Importantly, new investors that enter at future refinancing times are not party to the current contract.

[4] The assumption that internal cash reserves earn interest at a rate below the discount rate is standard in the literature (see, e.g., Décamps et al. 2011 and Bolton et al. 2011) to preclude a degenerate solution in which the firm saves itself out of the constraint and payouts are indefinitely delayed. Assuming impatient investors leads to similar results.

However, limited commitment on part of the outside investors results in renegotiation proofness, which in turn implies that any new investors optimally agree to the same terms when investing, as we discuss in more detail in Section 2.5.

We provide additional discussion of, and alternatives to, our key assumptions in Section 5.3.

2 Model Solution

In this section, we solve the model and derive the optimal payout agreement. We gain tractability by showing that a sufficient state-variable of the model is the difference between the firm's cash holdings and the intermediary's future promised payouts, which we term excess liquidity.

2.1 State Variables

In principle, shareholders' dynamic optimization has two state variables: The firm's cash holdings M_t, and the intermediary's continuation utility at time t defined as

$$Y_t = \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} \left(dI_s - k_s ds \right) \right]. \tag{5}$$

We also refer to Y_t as the intermediary's *stake* because it represents the portion of enterprise value promised to the intermediary. Because the intermediary has limited commitment and can always part from the firm, the value of the intermediary's stake must always be positive, $Y_t \geq 0$; otherwise, the intermediary would be better off leaving the contractual agreement.

Next, we derive the dynamics of the intermediary's stake Y_t.

Lemma 1. *The intermediary's continuation payoff evolves according to*

$$dY_t = (rY_t + k_t)\, dt + \beta_t \sigma dZ_t + \alpha_t (d\Pi_t - \pi dt) - dI_t, \tag{6}$$

where β_t captures the intermediary's exposure to Brownian cash flow shocks dZ_t and α_t captures the intermediary's exposure to the jump shocks $d\Pi_t$.

We refer to equation (6) as the promise keeping constraint. It means that current transfers dI_t must be accompanied by a commensurate change in future promised transfers dY_t to deliver the promised payoff Y_t to the intermediary.

In what follows, we stipulate that the cost of intermediary financing increases with its sensitivity $\beta_t\sigma$ to cash flow shocks dZ_t. Specifically, we pick the following quadratic cost specification:

$$k_t = \frac{\rho r}{2}(\beta_t\sigma)^2, \tag{7}$$

for a constant $\rho > 0$. The cost function in (7) implies that requiring on-demand financing, i.e., $\beta_t > 0$, from the intermediary is costly. Moreover, these costs increase in the interest rate r reflecting forgone opportunities of the intermediary.[5] Intuitively, to be able to provide such on-demand financing in response to negative cash flow shocks, the intermediary must maintain liquid reserves which it otherwise could invest in (relatively) illiquid yet profitable investments whose returns scale with r. For simplicity, the cost of intermediary financing in (7) does not depend on uncertainty other than cash flow risks, such as the random refinancing events $d\Pi_t$.[6]

Finally, we introduce two benchmark valuations of the firm. First, consider the value of the firm in a frictionless market in which the firm has continual access to outside markets, i.e., $\pi \to \infty$. Then, the firm need not hold any cash and can cover all cash flow shortfalls by raising new financing. The value of the firm then is simply its first-best net present value (NPV)

$$NPV \equiv \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}dX_s\right] = \frac{\mu}{r}. \tag{8}$$

Next, consider the value of the firm if the intermediary solely owned it and had no access to outside financing, a situation we term *autarky*. In this case, the intermediary would fully absorbs all shocks, and cash within the firm is optimally zero due to the firm's carry-cost-of-cash. Setting $dI_s = dX_s$, $\beta_s = 1$, and $\alpha_s = 0$ in equation (5), we calculate the autarky value of the firm to the intermediary as

$$Y^A \equiv \frac{\mu}{r} - \frac{\rho}{2}\sigma^2. \tag{9}$$

If $Y^A < 0$, the intermediary is unwilling to operate the firm without the prospect of outside financing. As we will see, when $\pi < \infty$, the optimal payout agreement features risk-sharing between the costly but liquid intermediary and the financially constrained investors, which improves upon

[5] Internet Appendix G micro-founds the cost of intermediary financing by assuming that the intermediary is risk-averse with CARA preferences, while we also allow the shareholders to exhibit some risk-aversion. The intermediary's cost of providing financing in response to cash flow shocks dX (that is, the cost of setting $\beta_t > 0$) then becomes k_t from (7), i.e., $\frac{\rho r}{2}(\beta_t\sigma)^2$.

[6] The micro-foundation of intermediary financing cost from Internet Appendix G with CARA preferences also implies some cost w.r.t. the stipulation of α_t. Our results are robust to the specific form of k_t and remain qualitatively unchanged with such fully micro-founded costs of intermediary financing.

the autarky solution. The total net value of the firm is the sum of the shareholders' value function P_t and the intermediary's stake Y_t minus the current cash-holdings M_t, and satisfies

$$Y^A \leq P_t + Y_t - M_t < NPV. \tag{10}$$

2.2 Dynamic Program and HJB Equation

We now derive an expression for the equity value, i.e., investors' value function, that depends on the two endogenous state variables of the problem: the intermediary's continuation payoff Y_t, and the firm's cash holdings M_t. We show how to reduce the problem to a single state variable so that the equity value P_t can be characterized by an ordinary differential equation (ODE).

Reduction of the state space. We define the variable C_t, which we term *excess liquidity*, by

$$C_t \equiv M_t - Y_t. \tag{11}$$

According to (2) and (6), excess liquidity C has the following law of motion:

$$\begin{aligned} dC_t = dM_t - dY_t = \left[\mu + (r-\lambda) C_t - \lambda Y_t - \frac{\rho r}{2} (\beta_t \sigma)^2 + \pi \alpha_t\right] dt \\ + \sigma (1-\beta_t) dZ_t + (C_t^* - C_t) d\Pi_t - dDiv_t, \end{aligned} \tag{12}$$

where we defined the post-refinancing level of excess liquidity as $C_t^* \equiv \Delta M_t + C_t - \alpha_t$.

Under the promise-keeping constraint (6), a transfer of cash dI_t between the balance M_t of the firm and the intermediary must be accompanied by a commensurate increase or decrease in the intermediary's stake Y_t. Thus, $C_t = (M_t - dI_t) - (Y_t - dI_t) = M_t - Y_t$ is invariant to transfers dI_t. Since dI_t can be positive or negative, we can use transfers to freely adjust Y_t without affecting excess liquidity C_t. Thus, Y_t is a choice variable subject to constraints. First, the intermediary's limited commitment requires $Y_t \geq 0$. Next, the definition of excess liquidity $C_t = M_t - Y_t$ and the physical constraint on cash requires $M_t \geq 0$ together imply that $Y_t \geq -C_t$. Thus,

$$Y_t \geq \max\{0, -C_t\}. \tag{13}$$

Note that excess liquidity C_t can become negative if the intermediary's stake Y_t is larger than the firm's cash balance M_t. In what follows, we conjecture and verify that equity value P_t can be expressed as a function of excess liquidity only, $P_t = P(C_t)$, and consequently that the total net value of the firm also reduces to a function of excess liquidity only, $P_t + Y_t - M_t = P(C_t) - C_t$.

Shareholders' limited commitment. Next, we discuss possible restrictions imposed by the limited commitment of investors. Consider the change in value to the existing investors in case of refinancing, which moves the firm from some level of excess liquidity C before refinancing to the post-refinancing level C^* with a prescribed increase of the intermediary's payoff by α. The change in value to the existing shareholders from refinancing is

$$P\left(C^*\right) - \Delta M - P\left(C\right) = P\left(C^*\right) - \left(C^* - C + \alpha\right) - P\left(C\right). \tag{14}$$

Because the loading of $dY + dI$ on $d\Pi$ equals α by (6), the firm needs to raise $\Delta M = C^* - C + \alpha$ dollars to transition from C to C^* in case of refinancing, increasing $Y + I$ by α.

The limited commitment of investors requires (14) to be non-negative so that current investors are not made worse off by refinancing; otherwise, investors would be better off passing up the refinancing opportunity. That is, investors cannot commit to refinancing policies that make them worse off after refinancing. Formally, α is subject to the constraint

$$\alpha \leq \left[P\left(C^*\right) - C^*\right] - \left[P(C) - C\right]. \tag{15}$$

As dividend payouts to investors $dDiv_t$ also must be non-negative, limited commitment therefore implies $P\left(C\right) \geq 0$. Let $\mathcal{S}\left(C^*, C\right)$ denote the set of all admissible choices for α. Crucially, this limited commitment makes the contract renegotiation proof as discussed in more detail in Section 2.5, and thus implies that the absence of future investors during the initial contracting at $t = 0$ is not an issue: Once any future investors invest, they agree to the same contract.

The HJB Equation. To solve the investors' dynamic problem (4), we first maximize the investors' value $P\left(C\right)$ for a given level of excess liquidity C and then determine the optimal level of initial excess liquidity C_0. We now derive the HJB equation that characterizes investors' value. We conjecture that the firm optimally makes payouts to investors at an endogenous upper boundary $C = \overline{C}$, and that it either liquidates or receives sufficient financing to stay alive at some endogenous lower boundary $\underline{C}$. For $C \in \left(\underline{C}, \overline{C}\right)$ by conjecture $dDiv = 0$, and the investors' value function solves

$$rP(C) = \max_{\beta, Y \geq \max\{0, -C\}} \left\{ P'(C) \left[\mu + (r - \lambda) C - \lambda Y - \frac{\rho r}{2} (\beta \sigma)^2 \right] + P''(C) \frac{\sigma^2}{2} (1 - \beta)^2 \right\}$$
$$+ \pi \max_{C^*, \alpha \in \mathcal{S}(C^*, C)} \left\{ P'(C) \alpha + \left[P(C^*) - P(C) - (C^* - C + \alpha) \right] \right\}. \quad (16)$$

Note that the right-hand-side of (16) only depends on the state variable C, control variables (α, β, Y, C^*) with constraints that depend on C and exogenous constants, confirming the conjecture that we can express equity value as a function of C alone. Thus, $P_t = P(C_t)$.

The payout boundary satisfies the standard smooth pasting and super contact conditions

$$P'(\overline{C}) = 1 \quad \text{and} \quad P''(\overline{C}) = 0. \quad (17)$$

As we show later in Proposition 2, for $C < \overline{C}$, the marginal value of liquidity exceeds one, i.e., $P'(C) > 1$, and the value function is strictly concave, i.e., $P''(C) < 0$. The concavity of equity value reflects that the firm's investors are effectively risk-averse since negative cash flow shocks can trigger financial distress. We summarize our findings so far in the following Proposition.

Proposition 1. *Excess liquidity $C_t = M_t - Y_t$ evolves according to (12). The firm's equity value can be expressed as function of C only, in that $P_t = P(C_t)$. The function $P(C)$ solves the HJB equation (16) on the endogenous state space $(\underline{C}, \overline{C})$. Optimal dividend payouts take the form $dDiv_t = \max\left\{C_t - \overline{C}, 0\right\}$.*

2.3 Optimal Control Variables

We now solve the optimization in the HJB equation (16) to obtain a state-dependent characterization of the four control variables: (i) the intermediary's stake Y, (ii) the refinancing target C^*, (iii) the intermediary's payouts upon refinancing α, and (iv) the risk-sharing intensity β.

First, consider the optimal choice of the intermediary's stake Y. Since $P'(C) > 0$ and $\lambda > 0$, the optimal contract picks the lowest Y possible subject to constraint (13), so that

$$Y(C) = \max\{-C, 0\} \quad \text{and} \quad M(C) = \max\{C, 0\}. \quad (18)$$

As holding cash is costly, given excess liquidity C, it is optimal to minimize cash holdings $M = C + Y$ and therefore minimize Y subject to the intermediary's limited commitment, $Y \geq 0$, and the cash constraint $M \geq 0$. If $C > 0$, the firm holds cash $M = C > 0$, and the intermediary's stake Y is zero. If $C < 0$, the firm holds no cash, but the intermediary's stake $Y = -C$ is positive.

Second, the first order condition (FOC) with respect to the refinancing target C^* yields

$$P'\left(C^*\left(C\right)\right) = 1, \tag{19}$$

so that refinancing occurs up until a point at which the internal value of cash is equalized with the value of paying it out. Recall that $P'(\overline{C}) = 1$ at the dividend payout boundary $\overline{C}$ and $P'(C) > 1$ for $C < \overline{C}$, so that without loss of generality $C^* = \overline{C}$.[7]

Third, consider the intermediary's payouts upon refinancing. Taking the derivative with respect to α in (16), it follows that

$$\frac{\partial P(C)}{\partial \alpha} = \pi\left[P'(C) - 1\right] \geq 0, \tag{20}$$

because $P'(C) \geq 1$ with the inequality being strict for $C < \overline{C}$. The intuition is that raising α is beneficial as it defers the intermediary's payouts from states in which the firm is financially constrained to states in which the firm is flush with cash since it has access to cheap outside financing. Consequently, the optimal choice of α is constrained by the investors' limited commitment constraint (15), so that

$$\alpha(C) = \left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P\left(C\right) - C\right]. \tag{21}$$

Fourth, the first order condition with respect to instantaneous risk-sharing β yields

$$\beta\left(C\right) = \frac{P''\left(C\right)}{P''\left(C\right) - \rho r P'\left(C\right)} \in \left[0, 1\right]. \tag{22}$$

Setting $\beta > 0$ transfers risk to the intermediary, reducing the volatility of excess liquidity. In addition, it reduces the drift of excess liquidity because of the risk compensation the firm must pay to the intermediary.

Finally, we insert $P\left(\overline{C}\right) - \overline{C} = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C}$ and the optimal policies into the HJB equation (16) to get

$$rP\left(C\right) = P'\left(C\right)\left[\mu + \left(r - \lambda \mathbb{1}_{\{C \geq 0\}}\right)C - \frac{\rho r}{2}\beta\left(C\right)\sigma^2 + \pi\alpha(C)\right], \tag{23}$$

where $\mathbb{1}_{\{.\}}$ denotes the indicator function.

[7] Any $C^* > \overline{C}$ also fulfills the FOC but leads to an immediate payout $C^* - \overline{C} > 0$ – essentially the firm would raise cash just to immediately pay it back to its shareholders. Setting $C^* = \overline{C}$ minimizes these round-trip transactions.

2.4 The Lower Boundary for Excess Liquidity

Let us determine the lower boundary $\underline{C}$. The key question is whether the firm always liquidates when cash runs out at $C = M = 0$ or not. This question is equivalent to asking if the intermediary optimally provides financing at the lower boundary $\underline{C}$ to prevent firm liquidation or not. For $\underline{C}$ to be a lower bound for C, it must be that either (i) the firm liquidates at $\underline{C}$, in which case we denote the lower boundary by $\underline{C}^L$, or that (ii) $\underline{C}$ is either a reflection, inaccessible or an absorbing state (absent refinancing), in which case we denote the lower boundary by $\underline{C}^S$.[8][9]

First, let us consider that there is liquidation once C reaches $\underline{C}$, i.e., case (i). At $C = \underline{C}$, the total liquidation value is the firm's cash $M(C)$ which is split by intermediary and shareholders, so that $P(C) + Y(C) = M(C)$.[10] Since $P(C), Y(C) \geq 0$ due to limited commitment, $M(C) = 0$ implies $P(C) = Y(C) = 0$. Because optimally $Y(C) = \max\{0, -C\}$ and $M(C) = \max\{C, 0\}$, if liquidation occurs, it must occur on the interval $\left[0, \overline{C}\right]$ as otherwise $Y(C) > 0 = M(C) = P(C)$ at the time of liquidation, a violation of promise keeping. Next, liquidation at any $C > 0$ with dividend C to the shareholders is sub-optimal, because $P(C) > C$ for any $C > 0$ due to $P'(C) > 1$ for $C < \overline{C}$. Therefore, conditional on liquidating, it is optimal to liquidate at the lowest value C not violating promise-keeping, and thus $\underline{C}^L = 0$.

Next, assume there is no liquidation at $C = \underline{C}$, i.e., case (ii). Then, for $\underline{C}$ to be a lower bound of C, something we term "survival", it must be that the volatility of excess liquidity $\sigma_C(C) = \sigma(1 - \beta(C))$ vanishes as C approaches $\underline{C}$, which requires $\beta(C) = 1$, while its drift $\mu_C(C)$ and the shareholders' value function $P(C)$ both must stay non-negative. The intuition is that at $C = \underline{C}$, the intermediary keeps the firm alive by providing continuous financing, absorbing all cash flow risk via $\beta(\underline{C}) = 1$. However, it is optimal to delay setting $\beta = 1$ due to the intermediary's cost of bearing risk as long as possible. Therefore, the lower boundary $\underline{C}^S$ is determined as the lowest level C at which $\mu_C(C) \geq 0$, $P(C) \geq 0$, and $\beta(C) = 1$ simultaneously hold.[11]

Using the law of motion (12) and the HJB equation (16), imposing $\mu_C(\underline{C}^S) = P(\underline{C}^S) = 0$ and

[8] The law of motion (12) reveals that there is no impulse control possible to reflect C upward. Our approach to determine the lower boundary in the survival scenario is similar to the one in Bolton et al. (2019).

[9] Note that $C = M - Y \geq -Y$ and $Y \leq \frac{\mu}{r}$ where $\frac{\mu}{r}$ is the NPV of the firm (the first best value). Thus, $C \geq -\frac{\mu}{r}$, so that excess liquidity is bounded from below. Therefore, there must exist an endogenous lower boundary $\underline{C}$ such that $C_t \geq \underline{C}$ at all times t.

[10] Recall that we assume that the firm's liquidation beyond its cash holdings is zero.

[11] The lower boundary $\underline{C}^S$ and $\beta(\underline{C}^S) = 1$ follows from the requirement that C must be bounded from below under incentive compatible contracts and survival. In principle, it is possible to have $\beta(C') = 1$ for $C' > \underline{C}^S$ in which case $\mu_C(C') > 0 = \sigma_C(C')$ and C' is the *effective* lower boundary of the state space (i.e., $C_t \geq C'$ for all t). We do not preclude this case, but Lemma 3 in Appendix D.3.1 shows that $\beta(C) < 1$ for $C > \underline{C}^S$, so that $\underline{C}^S$ can be viewed as the "optimal lower boundary."

$\beta(\underline{C}^S) = 1$, we have

$$\underline{C}^S = -\left\{ \frac{r}{r+\pi} Y^A + \frac{\pi}{r+\pi} \left[P\left(\overline{C}\right) - \overline{C} \right] \right\}. \tag{24}$$

Because the intermediary's stake in the firm is $Y(C) = -\min\{C, 0\}$, the value $-\underline{C}^S$ is the maximum value – under survival – of the intermediary's stake in the firm, i.e., the intermediary's valuation for owning the entire firm, including refinancing opportunities: it is the frequency-weighted average of the intermediary's autarky value Y^A and the exit value, i.e., its net value after refinancing $\left[P\left(\overline{C}\right) - \overline{C}\right]$.

Lastly, as we prove in the Appendix C, the optimal lower boundary is the minimum of the survival and and liquidation boundary:

Lemma 2. *The lower boundary and the associated value of equity are given by*

$$\underline{C} = \min\left\{\underline{C}^S, \underline{C}^L\right\} = \min\left\{\underline{C}^S, 0\right\} \quad \textit{with} \quad P\left(\underline{C}\right) = 0. \tag{25}$$

where $\underline{C}^S$ *is given in* (24). *When* $\underline{C} = \underline{C}^S$, *the firm always survives, whereas when* $\underline{C} = \underline{C}^L = 0$, *the firm defaults the first time* C_t *attains* 0, *i.e.,* $\tau = \inf\{t \geq 0 : C_t = 0\}$.

How can $P\left(\underline{C}\right) = 0$ hold for a firm that never liquidates and that only pays positive dividends in the future as limited liability holds? Upon refinancing at $C = \underline{C}$, existing shareholders are completely diluted, in that shareholders' limited commitment constraint (15) binds.

2.5 Renegotiation Proofness

Importantly, the optimal contract is renegotiation proof. Given any future renegotiation time $\hat{\tau}$ with some $C_{\hat{\tau}}$, the same contract is optimal going forward.[12] The result follows from three observations. First, the firm cannot raise external cash outside a refinancing opportunity, and thus a renegotiation can only decrease C: Either any reshuffling between M and Y leads to the same C, or the firm is giving out free promises Y, lowering C which costs $P'(C) > 1$ for only a return of 1 to the intermediary. Further, the contract above already picked the optimal Y to minimize the cost of holding cash, i.e., (18). Thus, no reshuffling can deliver Y in a cheaper way. Second, consider renegotiation *during* a refinancing opportunity. With external financing available, we should renegotiate as long as surplus can be increased, i.e., $\left(P\left(C^*\right) - C^*\right)' > 0$. But as (19) implies, we are already optimally refinancing to the payout boundary, which maximizes surplus.

[12] We follow Strulovici (2020) in defining renegotiation proofness in the presence of an physical state, here C.

Third, current shareholders have no incentive to renege on the contract upon refinancing due to the contract already accounting for their limited commitment via (15).

An important implication of renegotiation proofness is that today's contract with the current intermediary is not affected by a possible switch to and renegotiation with a new intermediary (given appropriate side-payments of value Y between the intermediaries) at a future time, or by the same token the arrival of new future shareholders not currently party to the contract.

2.6 The Optimal Contract

We now characterize the optimal financing arrangement by summarizing our previous results.

Proposition 2. *The optimal contract is described by the optimal policies* (18), $C^* = \overline{C}$, (21), *and* (22). *The shareholders' value function* $P(C)$ *solves the HJB equation* (16) *with boundary conditions*

$$P'(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0, \tag{26}$$

where $\underline{C}$ *is given by* (25). *Dividend payouts cause* C *to reflect at the payout boundary* $\overline{C} > 0$. *Under the optimal controls, the HJB equation* (16) *simplifies to* (23). *The value function is concave, i.e.,* $P''(C) < 0$ *for* $C < \overline{C}$. *If* $\underline{C} = 0$, *the firm liquidates once* C *reaches the lower boundary* $\underline{C}$ *and* $\beta(C) < 1$ *for all* $C > \underline{C}$. *If* $\underline{C} < 0$, *the firm never liquidates and* $\beta\left(\underline{C}\right) = 1$ *while* $\beta(C) < 1$ *for* $C > \underline{C}$. *Regardless of the value of* $\underline{C}$, $\alpha(C)$ *and* $\beta(C)$ *decrease in* C.

Finally, optimal transfers to the intermediary are given by

$$dI = \mu_I dt + \sigma_I dZ + \alpha_I d\Pi = \begin{cases} \left[\frac{\rho r}{2}\left(\beta\sigma\right)^2 - \pi\alpha\right] dt + \beta\sigma dZ + \alpha d\Pi & \text{for} \quad C > 0 \\ \mu dt + \sigma dZ + (\alpha + Y)\, d\Pi & \text{for} \quad C < 0. \end{cases} \tag{27}$$

Equation (27) *implicitly pins down* $\mu_I = \mu_I(C)$, $\sigma_I = \sigma_I(C)$, *and* $\alpha_I = \alpha_I(C)$.

Note that by the concavity of $P\left(C\right)$ and $P'\left(C\right) = 1$ for $C \geq \overline{C}$, the firm's initial level of excess liquidity C_0 chosen by its founders (original investors) coincides with the dividend payout boundary $\overline{C}$ as

$$\overline{C} = \arg\max_{C_0} P\left(C_0\right) - C_0. \tag{28}$$

Thus, the founders' initial value in (4) is $P_{0-} = P_0 - \Delta M_0 = P\left(\overline{C}\right) - \overline{C}$ as the firm raises $\Delta M_0 = \overline{C}$ from newly arriving investors who pay fair value and the intermediary merely breaks even with $Y_0 = Y(\overline{C}) = 0$. Importantly, the payout threshold $\overline{C}$ is positive.[13]

Parameter	Value	Interpretation
r	0.06	Common discount & interest rate
λ	0.01	Internal carry cost of cash
μ	0.18	Drift of cash flow process
σ	1.5	Volatility of cash flow process
ρ	6	Cost intermediary risk-bearing
π	0.5	Arrival rate of refinancing opportunities
$Y^A = \frac{\mu}{r} - \rho\frac{\sigma^2}{2}$	-3.75	Autarky value of firm to intermediary

Table 1: Baseline Parameter Values for all Figures.

We present numerical examples based on the parameters given in Table 1. We follow Bolton et al. (2011) in setting r, μ, λ. We take the cost function coefficient $\rho = 6$ and set $\sigma = 1.5$ to ensure that $Y^A < 0$ and absent refinancing opportunities, i.e., $\pi = 0$, the firm is liquidated at $C = 0$.[14] We compare a completely illiquid market $\pi = 0$ and to a mildly liquid one with $\pi = 0.5$ so that the expected time until next market access is $1/\pi = 2$ years.[15] The model's qualitative outcomes are robust to the choice of these parameters.

The value function $P(C)$ for $\pi = 0$ (left panel) and $\pi = 0.5$ (right panel) are depicted as the solid black lines in the top row of Figure 1, the payout boundary $\overline{C}$ as the vertical red lines, and lower boundary $\underline{C}$ as the vertical blue lines. The bottom row of Figure 1 shows how the risk-sharing intensity $\beta(C)$ decreases with C. In the left panels for $\pi = 0$, there are no refinancing opportunities, so $\underline{C} = 0$ as $Y^A < 0$, and the firm liquidates once it runs out of cash, with $\beta(\underline{C}) < 1$. In the right panels, $\pi = 0.5$ and $\underline{C} < 0$, so the firm never liquidates as the intermediary is willing to provide financing against a stake in the firm Y once cash has run out. Further, when $C < 0$, the risk-sharing intensity $\beta(C)$ differs from the scaled volatility of the intermediary's transfers, as $\sigma^I(C) = \sigma$ for $C < 0$, and only coincides again at $C = \underline{C}$ when $\beta(\underline{C}) = 1$.

[13] To see this, note that we can evaluate the ODE (16) or (23) at the payout boundary $\overline{C}$ to obtain $P(\overline{C}) - \overline{C} = \frac{\mu}{r} - \frac{\lambda}{r}\left(\overline{C}\mathbb{1}_{\{\overline{C}\geq 0\}}\right)$. This payoff must be strictly lower than the NPV of the firm, $\frac{\mu}{r}$, which implies $\overline{C} > 0$.

[14] Note that our model primarily describes young or private firms with limited access to capital markets which tend to have higher cash volatility than more mature firms. This also motivates the relatively high value of σ.

[15] Our parameter choice $\pi = 0.5$ follows Hugonnier et al. (2015) who assume (in their Table 1) a physical arrival rate of refinancing opportunities of 2 and incumbent shareholders' bargaining power of 0.25 so that the effective arrival rate of refinancing opportunities is $0.25 \cdot 2 = 0.5$.

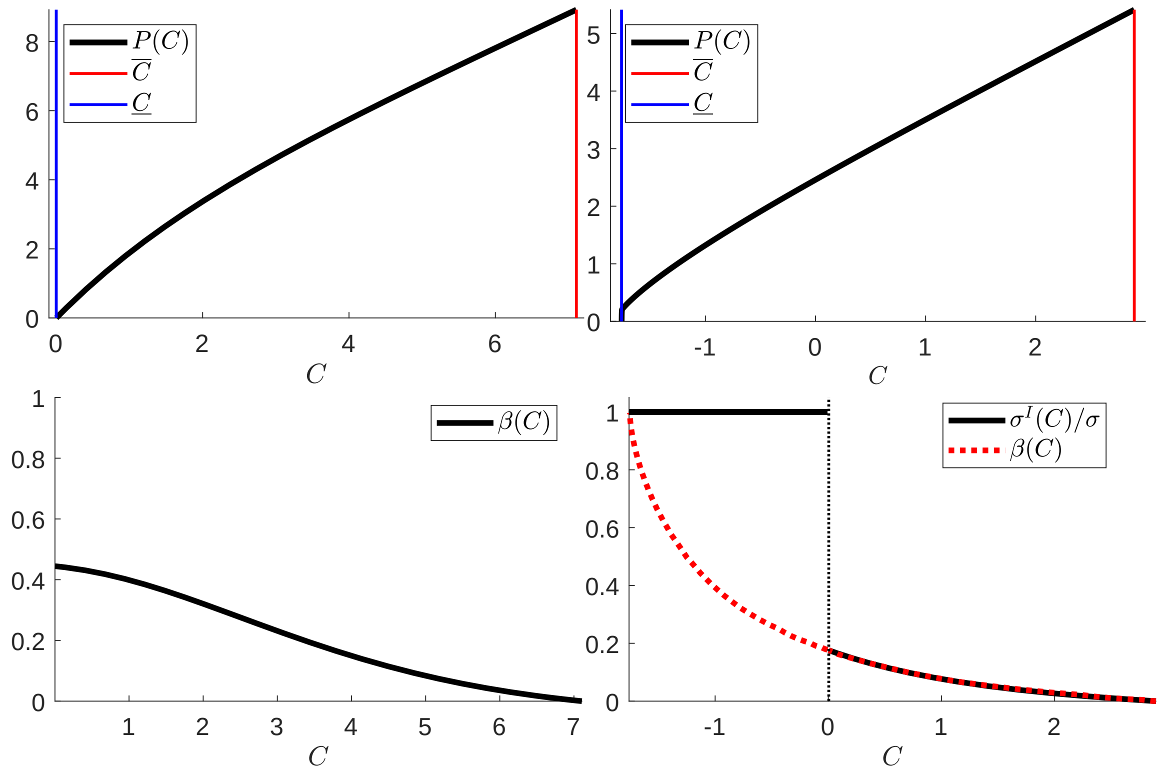

Figure 1: **Value function and risk-sharing:** This figure plots the value function $P(C)$ (upper panels) and risk-sharing $\beta(C)$ (lower panels) against excess liquidity for $\pi = 0$ (left panels) and $\pi = 0.5$ (right panels). The parameters are such that $Y^A < 0$, and consequently the firm is liquidated at $C = 0$ for $\pi = 0$ (left panels) but not for $\pi = 0.5 > 0$ (right panels).

3 Analysis and Discussion

3.1 Contract dynamics

In this section, we discuss the dynamics of the contract with the intermediary. It is important to recall that the intermediary in our model can represent a group of different intermediaries, including banks, non-bank lenders, and PE firms.[16] In other words, the different services the intermediary provides to the firm that are characterized by the optimal choice of α, β, and Y could be provided by different types of intermediaries in practice.

We start by discussing the dynamics of $\beta(C)$. Recall that $\beta(C)$ captures the firm's reliance on intermediary financing to cover cash flow shocks. Setting $\beta > 0$ transfers risk to the intermediary, which reduces the volatility of excess liquidity C and is beneficial because the value function is concave. However, such risk-sharing is costly in that it decreases the drift of C due to the required

[16] See Jang (2020) for empirical facts on non-bank lenders in PE.

risk compensation to the intermediary, as by (7) its cost k_t scale with volatility. The optimal choice of β in (22) trades off the decrease in volatility versus the decrease in the drift of excess liquidity.

The bottom row of Figure 1 depicts the instantaneous financing policies $\beta(C)$ for $\pi = 0$ (left panel, solid black line) and $\pi = 0.5$ (right panel, dashed red line). As the negative cash flow shocks reduce excess liquidity, the intermediary's risk exposure $\beta(C)$ increases, that is, $\beta(C)$ decreases with C. Thus, when C is low, the contract stipulates high risk-sharing β between the firm and the intermediary, reducing the volatility of excess liquidity C. In other words, when C is low but positive, and $\beta(C)$ is high, the firm covers cash shortfalls to a large extent with intermediary funds and to a lesser extent by drawing on its internal cash reserves. When C is low and negative, the firm covers all its cash-flow risk with intermediary funds, but the intermediary continuation value, i.e., its stake in the firm, does not move one-for-one with the cash provided because $\beta(C) < 1$. On the other hand, when C is high and $\beta(C)$ is low, the firm mostly relies on its internal cash reserves to cover cash flow shocks but less on intermediary financing. Overall, the intermediary's risk exposure and the firm's reliance on intermediary financing decrease with excess liquidity.

A similar mechanism implies that the intermediary's payouts upon market access, $\alpha_I(C) = \alpha(C) + Y(C)$, also decrease with C. As we can see from (12), α trades off the jump component of liquidity versus the drift of liquidity. Setting $\alpha > 0$ essentially transforms flow payouts to the intermediary today into a (promised) lumpy payout upon refinancing. As the HJB equation shows delaying payouts via α is unambiguously valuable to the investors because of the absence of costs on α in (7), but is constrained by their limited commitment. Intuitively, we can understand $\alpha(C)$ as a way the firm partially finances its contract with the intermediary by delaying payment until financial market access.

Finally, the optimal contract minimizes the deferral of payouts to the intermediary, as it picks the lowest possible intermediary stake Y as shown in (18). For $C > 0$, due to the carry costs of cash, it is optimal to have the minimum amount of cash consistent with C, which yields $Y(C) = 0$. Thus, the firm's cash holdings are $M(C) = C > 0$. Likewise, because the firm minimizes the carry cost of cash, $C < 0$ implies that the firm holds no cash, $M(C) = 0$, while the intermediary takes a stake in the firm $Y(C) = -C > 0$.

3.2 Interpreting Y

The emergence of a stake $Y > 0$ is the novel part of the optimal contract that sets our results apart from more traditional setups with exogenous hedging costs that do not involve a second agent, e.g., Bolton et al. (2011). Recall that a stake $Y > 0$ only arises when $\underline{C} < 0$. Strictly speaking, $Y > 0$ indicates deferred payouts, and we can refer to financing in return for Y as *deferred payout financing*. We show how the stake Y can be contractually delivered to the intermediary in terms of financial securities in detail in Section 4. There, we propose two implementations, one based on private equity (PE), the other on collateralized debt.

In contrast to $Y > 0$, note that $\alpha > 0$ is not a deferred payout. Rather, it is a fairly-priced instantaneous contract that delivers receipt of a dt flow against a state-dependent lumpy payment based on $d\Pi_t$. As such, it is similar to say a short-maturity insurance contract that requires a premium flow and promises a lumpy payout in case of $d\Pi = 1$. In other words, as α is not a state-variable, it has no persistence, and thus setting a high α for an instant does not restrict the choice of α going forward except through its impact on C. A similar rational holds for β, except it is not fairly-priced due to the cost k_t of the intermediary. Thus, we will refer to financing not involving Y as *committed short-term financing*.

Next, we argue that $Y > 0$ is tightly related to a notion of ownership of the firm. Suppose the intermediary is the sole owner of the assets. The intermediary then realizes expected cash flows net of costs $\left(\mu - \rho r \frac{\sigma^2}{2}\right)$ per unit of time until it is able to enter public markets. This occurs with intensity π, at which point the intermediary sells the firm for $\left[P(\overline{C}) - \overline{C}\right]$ to new outside investors. With τ_r equal to the next refinancing time, the expected value of the firm to the intermediary is

$$\overline{Y} \equiv \mathbb{E}_0 \left[\int_0^{\tau_r} e^{-rt} \left(\mu - \rho r \frac{\sigma^2}{2}\right) dt + e^{-r\tau_r} \left[P(\overline{C}) - \overline{C}\right]\right] = \frac{r}{r+\pi} Y^A + \frac{\pi}{r+\pi} \left[P\left(\overline{C}\right) - \overline{C}\right] \tag{29}$$

which by (24) equals $-\underline{C}^S$. One can thus interpret the stake $Y \in \left(0, \overline{Y}\right)$ as partial ownership of the firm: Consider the fraction of intermediary ownership of the firm be $Y(C)/[Y(C) + P(C)]$ where $Y(C) + P(C)$ is total firm value (including cash). Consistent with the previous arguments, intermediary ownership as a fraction of total decreases with C, approaches one as C approaches $\underline{C} < 0$, and is zero for $C \geq 0$. In other words, for $C > 0$ the intermediary simply provides short-term costly hedging services, and the presence of the intermediary as an agent is irrelevant. However, the presence of the intermediary as an agent is a key for $C < 0$, states that could not be reached in an exogenous hedging costs model without the intermediary.

Importantly, when $\underline{C}^S > 0$ so that $\underline{C} = 0$ the intermediary is unwilling to take any stake in the firm. This implies that the autarky value of the firm is too negative and dominates the positive exit value. Why don't the investors simply promise a sufficient amount to the intermediary to make it willing to temporarily hold the firm? They do not do so because they face a limited commitment constraint as follows. Recall that the payment to the intermediary is given by $\alpha_I = \alpha + Y$. Without their limited commitment constraint, the investors could always commit to sufficiently large payments α to the intermediary in case of refinancing, making the intermediary willing to extend financing in all states of the world, staving off costly liquidation. However, the investors' ability to promise α is restricted by (15) and is binding in equilibrium. Plugging this binding constraint into $\alpha_I(C)$, we see that the payment conditional on refinancing is bounded by

$$\alpha_I(C) = \left[P(\overline{C}) - \overline{C}\right] - P(C) + C\mathbb{1}_{\{C>0\}} \leq \left[P(\overline{C}) - \overline{C}\right] = \alpha_I(\underline{C}) \tag{30}$$

In words, the payment to the intermediary upon refinancing is bounded by the net value of the firm post refinancing. If this frequency-weighted net value $\pi/(r+\pi) \cdot \left[P(\overline{C}) - \overline{C}\right]$ is insufficient to compensate for the negative flow value of the firm to the intermediary, $r/(r+\pi) \cdot Y^A$, the intermediary is unwilling to extend financing against any stake, and $\underline{C}^S > 0$. Consequently $Y = 0$, and the firm has to inefficiently liquidate at $C = 0$ due to a lack of funds.

Given that access to intermediary financing may be sufficient to cover cash flow shortfalls and prevent firm liquidation, why does the firm not reduce cash holdings to zero in all states and simply rely on intermediary financing? Even when $\underline{C} < 0$ and the intermediary is willing to provide sufficient financing, the firm accumulates cash reserves until C reaches $\overline{C} > 0$. The reason is that intermediary financing is costly, and the firm holds cash reserves to reduce its reliance on such costly financing. In other words, cash reserves allow the firm to absorb cash flow risk after good performance and thus engage in risk-sharing with the intermediary.

3.3 The Effects of Financial Market Access

As we have seen in the previous section, the ability to access capital markets and exit its position in the firm crucially affects the intermediary's willingness to provide financing to the firm in exchange for a stake. We now consider the impact of enhanced financial market access, as captured by an increase in π or a decrease in $1/\pi$, with the latter corresponding to the expected time until refinancing. Notice that more mature, larger, or public firms tend to have better market access than early-stage, small, or private firms and may be characterized by a larger value of π. In this analogy, financial market access π tends to increase over the firm's life cycle. Alternatively, one can view changes in financial market access as proxying for broader financial market development, for

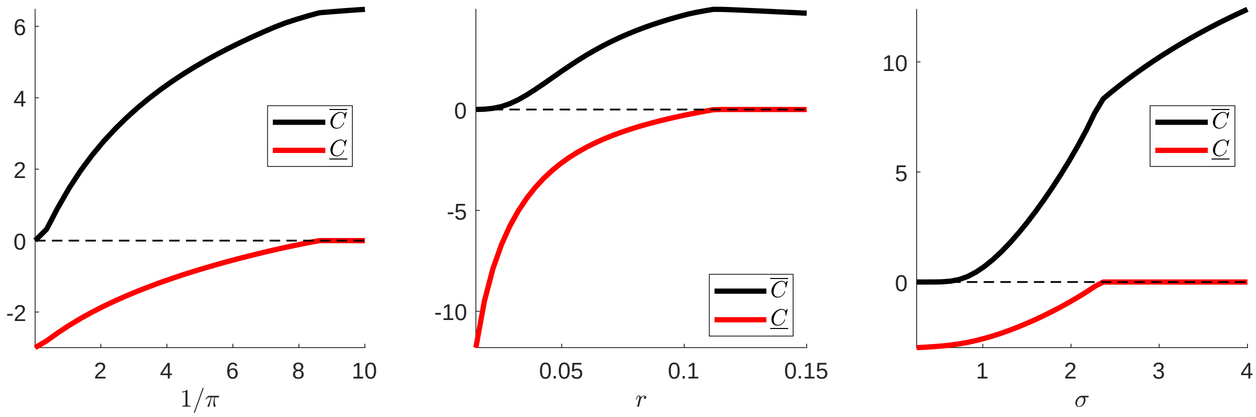

Figure 2: **Boundaries:** Comparative statics of the boundaries $\underline{C}, \overline{C}$ with respect to to the expected time until refinancing $1/\pi$ (left panel), the interest rate r (middle panel), and cash flow volatility (right panel). The parameters follow Table 1.

example, across different countries, or as a consequence of aggregate financial market conditions, i.e., low (high) π corresponds to low (high) public market liquidity.

The left panel of Figure 2 shows the upper (solid black) and lower boundaries (solid red) as a function of $1/\pi$. Observe that both $\underline{C}$ and $\overline{C}$ increase with $1/\pi$, and by our baseline parameter specification, $\underline{C} = -\max\left\{Y^A, 0\right\} = 0$ for $\pi = 0$, i.e., liquidiation prevails in completely illiquid markets. The intermediary is only willing to acquire a stake $Y > 0$ in the firm if financial market access and thus exit opportunities are sufficiently favorable, that is, for sufficiently low values of $1/\pi$ for which $\overline{C} < 0$. As π increases, the resale option value to holding a stake in the firm in (24) increases, both in value terms conditional on being able to exercise the option, i.e., $\left[P(\overline{C}) - \overline{C}\right]$ increases, as well as the frequency with which the option can be exercised, i.e., $\pi/\left(r+\pi\right)$ increases. As a result, the intermediary's willingness to provide financing in exchange for a stake in the firm also increases. Thus, higher π allows the firm to raise new capital more frequently but also to rely more on intermediary financing, possibly including deferred payout financing, in between refinancing episodes. Consequently, the firm faces less severe financing frictions, reducing precautionary cash holdings and lowering the payout boundary $\overline{C}$.

As the expected time until refinancing $1/\pi$ decreases, our model implies that — all else equal — we should see a larger proportion of firms with $\underline{C} < 0$ that are using deferred payment financing. When $\underline{C} < 0$, the firm never fails, and a stationary distribution over C exists. We can use this stationary distribution to calculate the average levels of model quantities in question. Let us measure intermediary activity, specifically deferred payout financing, in the model. Two distinct

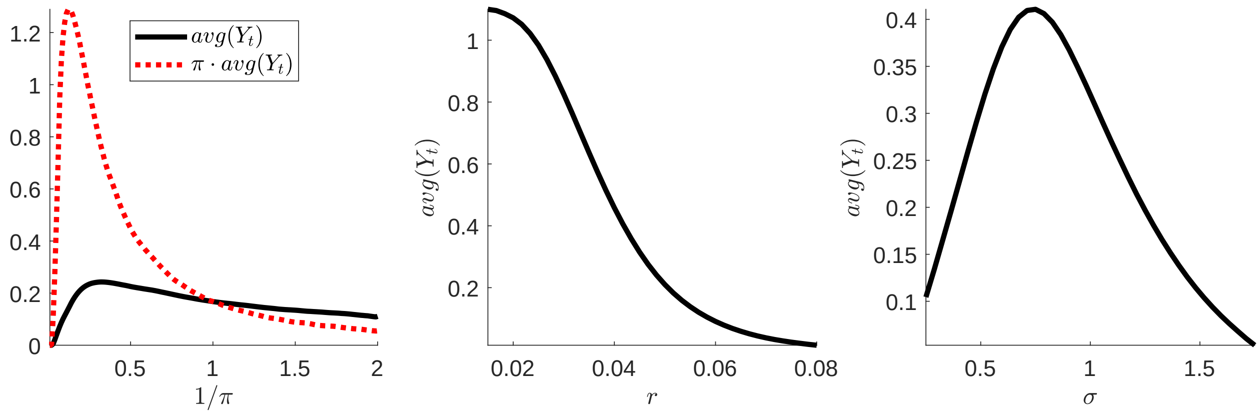

Figure 3: **Deferred payout financing in steady state:** This figure plots the average intermediary stake $avg(Y_t)$ against $1/\pi$ (left panel), r (middle panel), and σ (right panel). The left panel also plots the turnover $\pi \cdot avg(Y_t)$ that is omitted in the other two panels as π is constant. The parameters follow Table 1.

measures come to mind: First, we can measure the outstanding *stock* of intermediary stakes Y_t, which essentially proxies for the total size of the deferred payment agreements. Second, we can measure the *turnover/flow* of (value-weighted) intermediary stakes $\pi \cdot Y_t$, which captures the amount of intermediation done via deferred payments at each point in time. These measures are possibly distinct: For example, large stakes that are not exited frequently would imply a large stock but small flow measurement.

The left panel of Figure 3 shows that both the average stake $avg(Y_t)$ as well as the average turnover $\pi \cdot avg(Y_t)$ are hump-shaped in $1/\pi$. Thus, the largest amount of deferred-payout financing, either measured in terms of outstanding dollar stake in the firm or value-weighted turnover, occurs at intermediate liquidity levels $1/\pi$. We note that the value-weighted turnover peaks at a lower $1/\pi$ than the average outstanding intermediary stake. The intuition behind these findings is as follows. On the one hand, when $1/\pi$ is large and market liquidity is low, the intermediary cannot exit its position quickly and is therefore unwilling to take a large stake Y in the firm in the first place. On the other hand, when $1/\pi$ is small and market liquidity is high, the intermediary would be willing to take a large stake in the firm but exits any position quickly so that Y on average remains low. Further, as the average Y decreases faster than π increases, the value-weighted turnover $\pi \cdot avg(Y_t)$ also declines at some level of π.

Note that the comparative statics of $avg(Y_t)$ with respect to $1/\pi$ also have implications for financing, and specifically for deferred payout agreements, during a firm's life cycle. In its early stages, a firm often faces difficulties accessing capital markets to raise financing, i.e., large $1/\pi$.

Because the willingness of the intermediary to agree to a deferred payout financing depends on successful exit, this also leads to low average levels of such in the firm. As capital market access improves and $1/\pi$ shrinks over the firm's life cycle, the firm relies more on deferred payout financing. Finally, when the firm has gained sufficient market access, and $1/\pi$ is low, its reliance on deferred payout financing is low again. Taking stock, our model implies that firms' reliance on deferred payouts financing is highest in their intermediate stages.

Finally, we investigate the value created by intermediation and the value specifically created by the deferred payout financing, i.e., the intermediary providing financing in return for a stake Y. We do this by pitting our full model against two benchmarks: The no-intermediation benchmark, in which $\alpha = \beta = 0$, and the no-deferred-payout benchmark. Importantly, the no-intermediation benchmark $\alpha = \beta = 0$ also precludes any deferred payout financing, so $Y = 0$, $C = M$, and the firm is liquidated once it runs out of cash, i.e., $\underline{C} = 0$.[17] In the no-deferred-payout benchmark, the firm still relies on intermediary financing through the choice of α and β. However, the intermediary does not take a stake in the firm so that $Y = 0$, leading to the lower boundary in the state space $\underline{C} = 0$. If $\mu_C(0) < 0$, the firm is liquidated at $C = 0$. Otherwise, if $\mu_C(0) \geq 0$, the firm is never liquidated and survives as C approaches zero, in that $\lim_{C \to 0} \beta(C) = 1$. It turns out that survival prevails in the no-deferred-payout benchmark if and only if it prevails in the baseline too; the conditions for survival in the no-deferred-payout and baseline solution are therefore characterized in Proposition 2.[18] Conditional on survival, the solution to the no-deferred-payout benchmark is characterized by the HJB equation (16) subject to smooth pasting and super contact conditions at the payout boundary as well as the boundary condition $\lim_{C \downarrow 0} P''(C) = -\infty$ which ensures — by means of (22) — $\lim_{C \downarrow 0} \beta(C) = 1$ and survival.

The value creation of deferred payout financing over and above the no-deferred-payouts case with $Y = 0$ thus does not come from a reduction of the firm's liquidation risk, as the set of firms that do not face liquidation is the same. Rather, the additional value of deferred payout financing comes from improved risk-sharing. Consider the $Y = 0$ solution with a lower boundary $\underline{C}_{Y=0} = 0$, some positive payout boundary $\overline{C} > 0$, and firm survival, i.e., $\underline{C}^S < 0$. Now consider decreasing $\underline{C}$ to slightly below zero. The extra slack $[\underline{C}, 0]$ optimally decreases the sensitivity β

[17] Note that when $\beta = 0$, it is not possible to prevent liquidation at $C = \underline{C}$, in that survival requires $\beta(\underline{C}) = 1$. The no-intermediation benchmark becomes the solution to the baseline model in the limit case $\rho \to \infty$.

[18] Heuristically, suppose that the firm survives in the baseline so that $\underline{C} < 0$ (unless in the knife-edge case $\underline{C} = 0$). This implies $\mu_C(\underline{C}) = 0$, $\beta(\underline{C}) = 1$, and $P(\underline{C}) = 0$ by means of Proposition 2. But then, it is possible to set $\beta(0) = 1$ and $\mu_C(0) \geq 0$ to achieve $P(0) > 0$, which is higher than the value attained from liquidation at $C = 0$. Then, the firm prefers survival at $C = 0$ over liquidation in the no-deferred-payout benchmark.

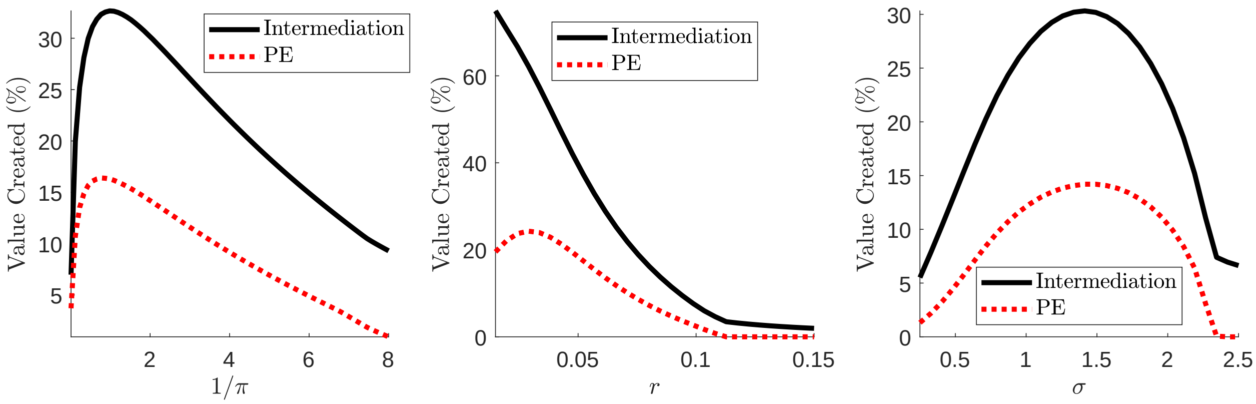

Figure 4: **The value of intermediation and deferred payouts**. This figure depicts the proportional value generated by intermediation relative to the no-intermediation case $\left(\frac{\text{Value}}{\text{Value}_{\alpha=\beta=0}} - 1\right)$ (solid black line) and by deferred-payout financing $\left(\frac{\text{Value}}{\text{Value}_{Y=0}} - 1\right)$ (dotted red line) for different values of $1/\pi$ (left panel), r (middle panel), and σ (right panel). The parameters follow Table 1.

of the intermediary's continuation utility to cash flow shocks in a neighborhood of $C = 0$, i.e., $\beta(0) < 1 = \beta_{Y=0}(0)$. Consequently, the firm has to pay less risk compensation to the intermediary. A similar argument holds for $[0, \overline{C}]$. Thus, deferred payout financing relegates high levels of risk-sharing, i.e., high levels of β, to less likely states, i.e., low and negative values of C. This shift in risk allocation creates value as transferring risk to the intermediary is expensive.

Let us measure the relative value creation at the inception of the firm. Recall that firm value at inception is given by Value $= \left[P\left(\overline{C}\right) - \overline{C}\right]$. Figure 4 depicts relative value creation for changes in $1/\pi$ (left panel), r (middle panel), and σ (right panel). The solid black lines depict $\left(\frac{\text{Value}}{\text{Value}_{\alpha=\beta=0}} - 1\right)$ and the dotted red lines depict $\left(\frac{\text{Value}}{\text{Value}_{Y=0}} - 1\right)$ in percentage terms. Focusing on the comparative statics with respect to financial market access $1/\pi$, i.e., the left panel of Figure 4, we find that intermediation in general and deferred payout financing in particular generate the most value for intermediate levels of $1/\pi$. Firms with sufficient market access, i.e., low $1/\pi$, do not rely much on intermediary financing, as they can easily raise capital from outside investors. Conversely, firms with low market access, i.e., high $1/\pi$, cannot benefit from any deferred payout financing, as poor market access undermines the intermediary's exit opportunities and thus the intermediary's willingness to take a stake in the firm. If $1/\pi$ decreases over a firm's life cycle, the firms benefiting the most from deferred payout financing are in the intermediate stages of their life cycle.

3.4 The Effects of the Risk Free Rate

As the previous section has shown, financial market development has ambiguous effects on intermediation. We next want to consider the impact of the interest rate environment on intermediation, again with an eye towards deferred payout financing.

Over the last decades, an important secular trend has been the continuous decline in the risk-free rate. We now assess how a decline in the interest rate r changes the nature of intermediation in our model. A priori, a change in r has two effects. First, a decrease in r increases the NPV (8) of the project, thus making all projects more attractive for a given μ. Second, a change in the interest rate changes the required compensation of risk-exposure to the intermediary by definition of k_t in (7). Intuitively, a low interest rate r implies a low cost of capital for the intermediary, as the return on outside opportunities for the intermediary decrease. Consequently, all else equal, risk-sharing with the intermediary becomes less costly when r declines.

The middle panel in Figure 2 shows that both the payout boundary $\overline{C}$ and the lower boundary $\underline{C}$ increase with the risk-free rate r as long as $\underline{C} < 0$. The payout boundary $\overline{C}$ starts decreasing in r once $\underline{C} = 0$. Importantly, the difference $\overline{C} - \underline{C}$ monotonically decreases in r throughout. When interest rates are high, $\underline{C} = 0$, and our model implies there is no deferred payout financing. When interest rates are sufficiently low, the lower boundary $\underline{C} < 0$, implying that deferred payout financing, i.e., $Y > 0$, can occur. Intuitively, the intermediary is willing to take a larger stake in the firm in a low interest environment since low interest rates increase both the intermediary's valuation and the value of the resale option. As r declines, the firm's access to deferred payout financing improves, allowing the firm to reduce precautionary cash holdings. Therefore, $\overline{C}$ increases with r. However, for high enough levels of r, further increases in r lead to a decrease of $\overline{C}$ as the cost of risk-sharing effect dominates.

The middle panel in Figure 3 depicts the average intermediary stake $avg(Y_t)$, which is proportional to the turnover $\pi \cdot avg(Y_t)$ as π remains constant, for different levels of the interest rate r and for firms that do not face the risk of liquidation. Note that the existence of a stationary distribution of C requires this restriction. A decline in interest rate r increases the average intermediary stake and the average value-weighted turnover, spurring deferred payout financing.

Finally, let us investigate the relative value creation at the firm's inception. The middle panel of Figure 4 depicts the relative value creation with respect to the risk free rate r. The black line implies that the value of intermediation increases in r. This effect is intuitive: as r declines, the project's

NPV and the value loss due to liquidation increase, thus increasing the value of intermediation. However, the red line implies that the value of deferred payout financing is non-monotone in r. As r shrinks, the required compensation for risk-exposure to the intermediary $\frac{\rho r}{2}(\beta\sigma)^2$ becomes cheaper. Consequently, a sufficiently low interest rate limits the value of allowing C to become negative and thereby, all else equal, lowering β on $[0, \overline{C}]$.

3.5 The Effects of Cash Flow Volatility

We now investigate the effect of cash flow volatility σ on the model. As the right panel of Figure 2 shows, the boundaries $\underline{C}$ and $\overline{C}$ are both increasing in σ, with $\overline{C}$ strictly so. When cash flow volatility is sufficiently high, $\underline{C} = 0$, the firm has no access to deferred payout financing and faces the risk of liquidation. The intuition is that sufficiently risky firms do not have access to intermediary financing that prevents liquidation. Indeed, it is inefficient for the intermediary to acquire a stake in such firms due to the compensation it demands to bear risk.

Next, the right panel of Figure 3 indicates that the deferred payout activity as measured by $avg(Y_t)$, or equivalently $\pi \cdot avg(Y_t)$ as π remains constant, is hump-shaped in cash flow volatility σ. The intuition for why it decreases in σ for high levels of σ is as above, that is, the intermediary is unwilling to have a stake in sufficiently risky firms. The intuition for why it increases in σ for low levels of σ is more nuanced. In the limit, as $\sigma \to 0$, the firm does not require any intermediary capital, nor any cash-buffers, to stave off liquidation, as the cash flows are risk-free and positive. Thus, even though $\underline{C} < 0$, the need to dip into deferred payout financing vanishes, thus $Y = 0$. As σ increases, the need to buffer shocks increases, increasing the value of both intermediation and cash holdings. For low σ, this effect dominates the increase in the cost of intermediation induced by σ. Therefore, even though $\underline{C}$ increases with σ, shrinking the maximum intermediary stake, the average use of such stakes increases due to the increase in shocks.

Finally, the right panel of Figure 4 shows that both the value of intermediation and deferred payout financing is hump-shaped in cash flow volatility. The intuition why firms with intermediate levels of cash flow volatility benefit the most from intermediary and deferred payout financing is as follows. First, the cost of intermediary financing increases with cash flow risk as in (7). Thus, firms with very volatile cash flow only have access to very expensive intermediary financing and therefore manage liquidity mostly by accumulating cash reserves. Second, firms with low cash flow volatility have access to relatively inexpensive intermediary financing but face limited financing needs and, therefore, mechanically do not have to rely on intermediary financing. Thus, deferred payout financing is most likely to occur in firms with an intermediate level of cash flow volatility. In contrast, the model predicts low levels of deferred payout financing in firms with low or high cash flow volatility.

4 Implementation — An Overlapping Pecking Order

We now provide two ways to interpret the stake Y, with details of the full implementation provided in subsequent sections. Regardless of the interpretation, an *overlapping pecking order* of financing arises: For $C > 0$, the firm uses internal cash and committed short-term financing, in that $\alpha > 0$ and $\beta \in (0, 1)$ but $Y = 0$. Thus, no stake or deferred payouts are accumulated. The firm's cash holdings absorb a fraction $(1 - \beta(C))$ and committed short-term financing absorbs a fraction $\beta(C)$ of cash flow risk. For $\underline{C} < C < 0$ (if feasible), the firm has depleted its cash reserves, and thus relies on committed short-term financing and deferred payouts. Even though $\sigma_I(C) = \sigma$, we have $\sigma_C(C) = \sigma(1 - \beta(C))$, so that short-term financing is absorbing a fraction $\beta(C)$ while deferred payouts absorb a fraction $(1 - \beta(C))$ of the cash-flow risk.

There are two notable differences to the traditional pecking order theory of Jensen and Meckling (1976). First, the pecking order is characterized in terms of the firm's liquidity reserves, not the firm's life cycle. Second, the pecking order suggests an intensity of use of different financing modes and not strict dominance of one mode over another. Specifically, the firm always uses two financing modes simultaneously.

A Private Equity (PE) interpretation. First, we can interpret Y as resembling a private equity stake. PE investors often acquire distressed firms and hold these firms primarily to realize capital gains upon exit, which here would correspond to situations in which $Y^A < 0$ but $\underline{C} < 0$. In particular, when the firm undergoes financial distress, the intermediary provides capital in exchange for a stake in the firm to profit of a future refinancing opportunity, which is consistent with PE investors' role in firms' financial distress (Hotchkiss et al., 2021). If the firm we consider is public, we can interpret this type of financing as private investment in public equity (PIPE) of distressed firms. Importantly, the firm's access to PE-like financing endogenously depends on the intermediary's willingness to acquire a stake in the firm under distress, which in turn depends on the resale option value of such a stake. If refinancing opportunities are sufficiently common, then the intermediary is willing to assume full ownership of the firm at $C = \underline{C} < 0$ even if the autarky value Y^A is negative, precisely because the intermediary can realize a sufficiently positive capital gain at exit through the resale option. For $\underline{C} < C < 0$, the dynamics are similar except that the intermediary only holds a partial stake in the firm, which increases following negative cash flow shocks but shrinks following positive cash flow shocks as the firm buys back part of the intermediary's stake. Given our model of the firm, we think of PE investment mostly as buyouts or growth capital rather than early-stage venture capital financing.[19] Details of the full implementation for the private equity interpretation are given in Section 4.1.

A collateralized debt interpretation. Second, we can interpret Y as reflecting collateralized debt. We interpret any financing for $C < 0$ to be fully collateralized by the firm's assets. Thus, if the firm declares bankruptcy, the intermediary, as a collateralized creditor, takes possession of the firm, wiping out the equity holders, but not liquidating the assets. The intermediary then realizes expected cash flows net of costs $\left(\mu - \rho r \frac{\sigma^2}{2}\right)$ per unit of time until it is able to emerge from bankruptcy. Emergence from bankruptcy here is the refinancing event and thus occurs with intensity π, at which point the intermediary sells the firm for $\left[P(\overline{C}) - \overline{C}\right]$ to new outside investors. The current investors never voluntarily enter bankruptcy before $\underline{C}$ because at any point prior they still have positive value of keeping the firm alive, i.e., $P(C) > 0$ as long as $C > \underline{C}$, and thus pay down the intermediary's claim upon refinancing. Thus, in this interpretation, the firm enters bankruptcy only when $C = \underline{C} < 0$. Details of the full implementation for the collateralized debt interpretation are given in Section 4.2.

4.1 A PE implementation

We introduce an implementation of the optimal contract with common financial instruments, here a credit line provided by the intermediary, cash holdings by the firm, common equity held by shareholders, and restricted equity held by the intermediary. This specific implementation is "natural" in our minds due to previous arguments establishing Y as an equity-like stake of the intermediary. Interestingly, this implementation suggests an overlapping pecking order that depends on the level of the firm's liquidity. When $C > 0$ and the firm holds cash, it uses both the credit line and its cash reserves to cover cash flow shortfalls. Symmetrically, following positive cash flow realizations, the firm retains earnings to grow its cash reserves and pays back the credit line. Importantly, $\beta(C)$ quantifies the extent of credit line usage, in that high value of $\beta(C)$ indicates that the firm covers

[19] (Leveraged) buyouts focus on more mature companies where the primary source of risk is the level of cash flows. In contrast, venture capital funds focus on younger firms that might not produce cash flows and in which the risk is primarily about failing or achieving a breakthrough. There is no failure or breakthrough in our model, but cash flows are risky.

negative cash flow shocks to a large extent by drawing on the credit line. Note that credit line usage $\beta(C)$ decreases with liquidity C, as shown in Figure 1.

Next, the firm does not use restricted equity until all cash reserves have been exhausted at $M = C = 0$ in that $Y(C) = 0$ for $C \geq 0$. When $C < 0$ and the firm's cash reserves are exhausted, the firm finances cash flow shortfalls by drawing on the credit line and selling restricted equity to the intermediary at the same time. The dollar value of restricted equity held by the intermediary is $Y(C)$ and increases following negative cash flow shocks, i.e., $Y(C)$ decreases with C. The firm uses positive cash flow realizations to repay the credit line and repurchase restricted equity. Upon market access, the firm raises financing by issuing common equity to new outside investors to replenish cash reserves, repurchase restricted equity from the intermediary at market value, and pay back the credit line.

Thus, an overlapping pecking order of financial instruments arises. When the firm is flush with liquidity, the firm's internal cash reserves are used with the highest intensity to cover negative cash flow shocks. As liquidity reserves dwindle, the firm relies increasingly on credit line financing, and, as a last resort, the firm finances by selling equity to intermediaries. That is, it always uses the credit line with either internal cash reserves or restricted equity to manage its liquidity.

Importantly, the firm faces liquidation risk if and only if $\underline{C} = 0$, which corresponds to the intermediary being unwilling to fully absorb all cash-flow risks of the firm, and consequently $Y = 0$ at all times. In contrast, when $\underline{C} < 0$, the firm is never liquidated, as it always has access to sufficient intermediary financing. That is, the model implies that by acquiring a stake in distressed firms, the intermediary helps resolve distress and more efficiently reduce the risk of liquidation of their portfolio firms, in line with the empirical findings in Bernstein et al. (2019), Gompers, Kaplan, and Mukharlyamov (2020), and Hotchkiss et al. (2021) on PE investors' role in the resolution of distress.[20] Our analysis also suggests that PE investors' willingness to invest and resolve the financial distress of portfolio firms crucially depends on capital market access facilitating exit.

To further illustrate how a credit line is a natural implementation of the optimal contract, we introduce the balance $T(C)$ which records *cumulative undiscounted transfers* to and from the intermediary in response to Brownian cash flow shocks dZ since the last time C reached $\underline{C}$ or $\overline{C}$. Thus, the volatility of $T(C)$, denoted $\sigma_T(C)$, must match $-\sigma_I(C)$, the negative of the volatility of transfers from the firm to the intermediary. In other words, total contributions increase one-

[20] Bernstein et al. (2019) and Hotchkiss et al. (2021) present evidence that PE investors inject capital in exchange for ownership stakes, when portfolio firms undergo financial distress.

for-one with transfers from the intermediary caused by cash flow shocks dZ. Using $\sigma_I(C) = \sigma\beta(C) + \sigma(1-\beta(C))\mathbb{1}_{\{C<0\}}$ and $\sigma_T(C) = T'(C)\sigma(1-\beta(C))$, we have

$$\sigma_T(C) = -\sigma_I(C) \iff T'(C) = -\frac{\beta(C)}{1-\beta(C)} - \mathbb{1}_{\{C<0\}}. \tag{31}$$

Noting that (22) implies $\frac{1}{\rho r}\frac{P''(C)}{P'(C)} = -\frac{\beta(C)}{1-\beta(C)}$, integrating, and imposing $T(\overline{C}) = 0$, we solve

$$T(C) = \frac{\ln P'(C)}{\rho r} + Y(C). \tag{32}$$

$Y(C)$ is the intermediary's "equity" stake in the firm. We interpret the remainder $D(C) \equiv \frac{\ln P'(C)}{\rho r}$ as a credit line face-value or balance, in that it records all payments to/from the intermediary in response to cash flow shocks dZ that are not compensated by granting the intermediary an (equity) stake in the firm.

A few observations are in order. First, as $D'(C) < 0$, the firm partially covers negative cash flow shocks by drawing on its credit line, while it pays back some of the credit line after positive cash flows. Moreover, $\beta(C)$ captures the intensity of credit line usage, which decreases in the level of liquidity C. Second, upon financial market access, the firm raises financing to replenish its cash reserves, pays back the credit line, and repurchase the intermediary's stake in the firm, in that $D(\overline{C}) = Y(\overline{C}) = 0$ after refinancing.

We complete the characterization of this implementation in Internet Appendix F, where we discuss in detail how to determine the fees and interest of the credit line and how to implement the intermediary's stake as (restricted) equity.

4.2 A Collateralized Debt Implementation

In this section, we consider an alternative implementation in which *all* financing from the intermediary is arranged as a credit line. That is, we interpret $T(C) = \alpha_U(C) + Y(C)$ from (31) as the firm's credit line balance. When the firm holds cash $C = M(C) > 0$, it covers negative cash flow shocks by drawing on its cash reserves and the credit line. The credit line is not secured or collateralized, and the intermediary holds no stake in the firm, that is, $Y(C) = 0$. When the firm's cash reserves are exhausted, the firm defaults on its credit line if $\underline{C} = 0$ if the firm has no access to secured/collateralized credit line debt. Otherwise, if $\underline{C} < 0$, the intermediary provides further credit line financing but requires collateral. This collateral takes the form of a stake in the firm,

$Y(C) > 0$. The intermediary is only willing to accept the firm as collateral if the cash-flows are sufficiently valuable or the firm's resale option value is high, in which case $\underline{C} < 0$.

The firm repays the credit following positive cash flow shocks or access to financial markets to raise equity. The situation $C = \underline{C}$ is akin to bankruptcy. With no external financing, the firm cannot recover from financial distress and repay the credit line. When $\underline{C} = 0$, the firm defaults and must liquidate. When $\underline{C} < 0$, the intermediary seizes the collateral, assumes full ownership of the firm, and keeps the firm alive until bankruptcy resolution, deriving a value $\overline{Y}$ given in (29). Bankruptcy is resolved upon market access $d\Pi_t = 1$ when the intermediary sells the firm to newly arriving outside investors. As before, the credit line stipulates "early repayment incentives" or debt forgiveness prior to bankruptcy to incentivize shareholders to seek outside equity financing and to dilute their stake in light of a debt overhang problem.

In this interpretation, the intermediary resembles a specialized lender or bank. This implementation suggests an overlapping pecking order again, albeit a slightly different one: first, the firm finances cash flow shortfalls with internal cash reserves and unsecured/uncollateralized credit line debt. Only under financial distress does it rely on secured/collateralized credit line debt. The result that the firm relies on secured debt only under distress but on unsecured debt in normal times is consistent with the findings in Benmelech et al. (2020). Also in line with our model's predictions, Rauh and Sufi (2010) find that debt financing of high-credit-quality firms, which may correspond in our model to the ones with high liquidity reserves, predominantly takes the form of unsecured debt. In contrast, debt financing of firms with lower credit quality (i.e., with low liquidity reserves) involves some secured debt.

Under this implementation, a firm's access to equity financing π increases the firm's debt capacity, i.e., decreases $\underline{C}$, as it improves the resale option value of the underlying collateral (see Figure 2). Likewise, a decrease in the interest rate r increases collateral value, decreases $\underline{C}$, and thus increases the credit line capacity, reducing the likelihood of default and improving risk-sharing. Finally, the use of secured debt, as captured by $avg(Y_t)$, is most pronounced when market access π is at intermediate levels, a firm's cash flow volatility is intermediate, and interest rates are low (see Figure 3).

5 Discussion and Extensions

In this section, we discuss our assumptions and extend our model to incorporate additional functions of the intermediary.

5.1 Intermediary Effort

In practice, some types of financial intermediaries or financiers such as PE funds actively seek to improve firm performance and cash flows, for instance, through their monitoring, through directly affecting firm operations, or restructuring a firm under distress. In this section, we extend our model to account for such intermediary actions by allowing the intermediary to affect cash flows dX_t via its effort a_t. In particular, we stipulate that cash flows evolve according to

$$dX_t = (\mu + a_t)dt + \sigma dZ_t, \tag{33}$$

where $a_t \geq 0$ is the intermediary's non-contractible and privately observable effort which comes against quadratic flow cost $\frac{\kappa a_t^2}{2}$ for a constant $\kappa > 0$. Notice that $\kappa \to \infty$ implies $a_t = 0$ and thus yields our baseline setting. The intermediary's effort boosts the firm's cash flows, which could capture the intermediary's active role in firm operations (or restructuring) or its role in disciplining management through monitoring or designing managerial contract terms. Importantly, the intermediary's incentives to exert effort are determined by the incentive condition (the first order condition w.r.t. a_t)

$$a_t = \frac{\beta_t}{\kappa}, \tag{34}$$

and thus increase with the intermediary's exposure to cash flow shocks. The remainder of the solution is similar to the baseline and is discussed in greater detail in Appendix E. Since $\beta(C) < 1$, we have $a(C) < 1/\kappa$, and, as $\beta(C)$ decreases with C (see Figure 5 below), the intermediary's incentives and effort decrease with the firm's liquidity position C and so are highest under financial distress (when C is low). Loosely speaking, when the firm undergoes distress, the intermediary possesses high incentives to exert effort to actively improve firm, to monitor firm management, or to restructure the firm.

Notably, the intermediary exerts particularly high effort when $\beta(C)$ is close to one, which occurs in equilibrium if the intermediary takes a stake in the firm $Y > 0$ and $\underline{C} < 0$. In a similar vein, since the intermediary's effort incentives are highest when Y is largest, the model also predicts a

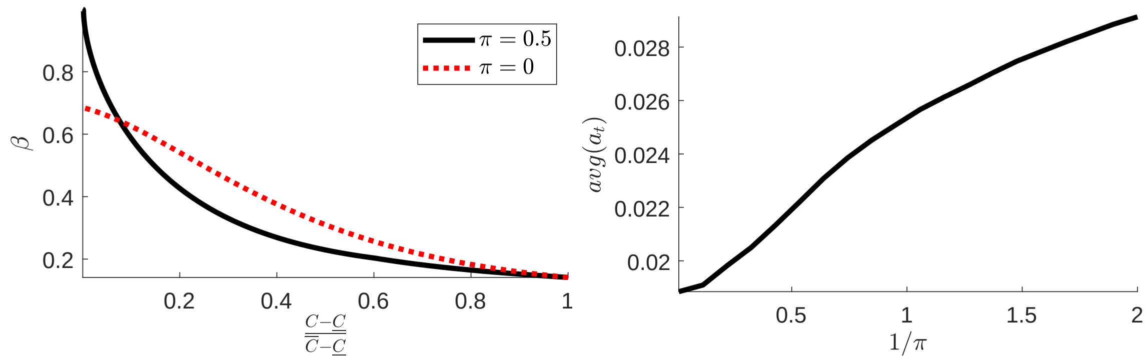

Figure 5: **Intermediary Incentives**. The left panel of this figure plots the intermediary's effort incentives $\beta = \kappa a$ both for $\pi = 0.5$ (solid black line) and for $\pi = 0$ (dotted red line) in the state space; to ensure comparability across different parameterizations, we normalize the state space. The right panel plots average effort $avg(a_t)$ against $1/\pi$ under parameters that ensure $\underline{C} < 0$ (so that a stationary distribution exists). The parameters follow Table 1, and we use $\kappa = 7.5$.

positive relationship between intermediary ownership and intermediary effort a_t. Next, recall that when exit opportunities are sufficiently good, i.e., π is sufficiently large, the intermediary is willing to take a stake in the firm, in that $\underline{C} < 0$ and $Y(C) > 0$ for $C \in (\underline{C}, 0)$, and therefore exerts high effort. As such, sufficiently good exit opportunities are necessary for an intermediary stake to arise and thus for high intermediary effort, whereas bleak exit opportunities, i.e., low π, discourage intermediary ownership and limit intermediary effort.

To further illustrate the effects of exit opportunities on intermediary ownership and effort, the left panel of Figure 5 plots the intermediary's effort incentives $\beta = a\kappa$ against the firm's relative liquidity position $(C - \underline{C})/(\overline{C} - \underline{C}) \in [0, 1]$ both for $\pi = 0.5$ (solid black line) and for $\pi = 0$ (dotted red line). Note that intermediary ownership (i.e., $\underline{C} < 0$) only arises for $\pi = 0.5$ but not for $\pi = 0$. While β decreases with C in both scenarios, we note that an increase in π (from 0 to 0.5) increases intermediary effort under distress for low C but reduces it when the firm is flush with liquidity. When π is large, the intermediary takes a stake under distress, prevents liquidation and exerts high effort in bad times, which reduces the need to stipulate high β and effort in good times. In contrast, when π is low, the firm faces the risk of liquidation and therefore transfers relatively more risk to the intermediary in good times, thereby endowing the intermediary with larger effort incentives in good times.

Next, we examine whether, conditional on $\underline{C} < 0$, exit opportunities improve or undermine the intermediary's incentives to exert effort. To do so, the right panel of Figure 5 plots average effort $avg(a_t)$ against $1/\pi$. Interestingly, $avg(a_t)$ increases with $1/\pi$ and thus decreases with financial market liquidity π and with better exit opportunities. The intuition is that a decrease in $1/\pi$ allows the intermediary to exit its position in the firm and curbs the intermediary's incentives to improve firm operations and cash flows. Loosely speaking, high liquidity and the possibility to exit the firm quickly causes some form of short-termism, in a sense that they limit the intermediary's investment horizon in the firm and reduce its incentives to exert effort improving firm performance. We emphasize that this finding is conditional on the intermediary being willing to take a stake in the firm in the first place. Overly bleak exit opportunities (i.e., low π) undermine the intermediary's incentives to take a stake in the firm, which also curbs effort incentives especially under distress.

Focusing on the set of firms with access to intermediary equity financing (i.e., $\underline{C} < 0$), the model implies that in a financial crisis when liquidity dries up and π is low, intermediaries seek more intensely to improve firm operations or, similarly, engage more in monitoring management. Related, intermediary incentives also tend to be highest when the firm's liquid reserves C are low and the firm undergoes financial distress.

5.2 Empirical Implications

Besides the direct implications of the implementation previously discussed, the model delivers several empirical predictions and implications. We frame these empirical predictions by assuming the the stake Y represents PE financing, however similar predictions apply if Y represents collateralized debt.

Prediction. *PE-supported firms are less likely to be liquidated.*

This prediction follows from the result that the firm does not face liquidation risk when $\underline{C} < 0$. Further, when $\underline{C} < 0$, the intermediary infuses capital in exchange for a stake $Y > 0$ in the firm following negative cash flow realizations and under financial distress. Consistent with our theoretical findings, Bernstein et al. (2019) and Hotchkiss et al. (2021) present evidence that PE investors inject capital in exchange for ownership stakes when portfolio firms undergo financial distress. Through the provision of financing under distress, more efficiently reduce the risk of liquidation and the exposure to negative shocks (Bernstein, Lerner, Sorensen, and Strömberg (2017)).

The implementation of the optimal contract suggests an overlapping pecking order in which credit line financing substitutes financing via internal cash reserves and financing via equity. However, the model also predicts that the availability of credit line financing is necessary for PE investments to arise. This insight leads to the following prediction.

Prediction. *Credit line financing and PE investments are complementary.*

Formally, the possibility of $\beta > 0$ and, in particular, $\beta = 1$ is necessary for $Y > 0$ to occur in equilibrium. If there is no credit line financing, i.e., $\beta = 0$, the firm would be liquidated once it runs out of cash, and the intermediary would never be willing to take a stake $Y > 0$ in the firm. Conversely, PE investment, i.e., $Y > 0$ and $\underline{C} < 0$, imply $\lim_{C \to \underline{C}} \beta(C) = 1$ and $\sup\{\beta(C) : C \in (\underline{C}, \overline{C})\} = 1$, while it generally holds that $\lim_{C \to \underline{C}} \beta(C) < 1$ and $\sup\{\beta(C) : C \in (\underline{C}, \overline{C})\} < 1$ when $\underline{C} = 0$. That is, PE investment is associated with intense reliance on credit line financing as captured by $\sup\{\beta(C) : C \in (\underline{C}, \overline{C})\}$; interpreted broadly, PE investment raises a firm's debt capacity. These findings suggest a complementarity between PE financing and credit line financing, and are consistent with the fact that PE investors typically engage in financial engineering (Kaplan and Stromberg (2009)), which includes making portfolio firms borrow from banks to increase their leverage. In related work, Ivashina and Kovner (2011) document the importance of private equity firms' bank relationships, suggesting complementarities between bank-like intermediation (responsible for credit line financing) and PE-like intermediation.

The next prediction concerns the equilibrium levels of intermediation and PE-like financing.

Prediction. *PE investments are low when either capital markets are very liquid or very illiquid.*

Recall that PE investment, as captured by Y, only occur when the intermediary is willing to keep the firm alive in financial distress. This willingness increases with financial market access π, which suggests that periods with booming financial markets (characterized by a large value of π) also experienced greater private equity fundraising as documented in Axelson, Jenkinson, Strömberg, and Weisbach (2013). However, the average PE investment Y shrinks once π increases sufficiently, as the positions are exited quickly. The reason is that the intermediary's increased exit implies that positions never grow very large, even though intermediaries would be willing to build up large positions Y. In any case, our model highlights that exit prospects are one key determinant for PE investors' willingness to take ownership in distressed firms.

The model also predicts how the level of PE investment depends on the firm's cash flow volatility.

Prediction. *PE investments are highest in firms with intermediate levels of cash flow volatility.*

This prediction stems from the result that the average intermediary stake, $avg(Y_t)$, is hump-shaped in cash flow volatility. In particular, the model suggests that PE firms tend not to invest in the riskiest firms but only in those that are sufficiently, but not too, risky. These firms also benefit the most from the availability of PE financing.

Prediction. *PE investors engage more actively in portfolio firms (via operational or governance engineering) when the firm undergoes financial distress or when financial market liquidity is low (e.g., due to a financial crisis.*

This prediction is in line with the empirical results in Bernstein et al. (2019) or Gompers et al. (2020). Related, Cornelli, Kominek, and Ljungqvist (2013) find that boards of PE-backed firms monitor firm managers to discipline them after poor performance, which suggests that PE investors tend to improve portfolio firm performance after poor performance through monitoring. The finding that PE investors seek to actively improve firm performance under financial distress or crisis times that PE ownership stabilizes firm cash flows and performance, consistent with the findings of Bernstein et al. (2017).

5.3 Discussion of Key Assumptions

Let us discuss two key assumptions that deviate from the literature and contrast them with alternative choices.

Infrequent capital market access. As opposed to Décamps et al. (2011) and Bolton et al. (2011), who assume fixed and variable cost of equity issuance, we follow Hugonnier et al. (2015) in assuming that the firm is unable to raise capital outside random, i.e., Poisson, times, capturing capital supply uncertainty or delays in raising financing. For instance, this assumption may reflect (un-modeled) adverse selection, i.e., arranging financing takes time as dispersed outside investors need to verify information. Indeed, many empirical studies document that firms often face uncertainty regarding their future access to capital markets.[21] Further, one can interpret our assumption as a Markov chain variant in which the costs are infinite except for very short periods at which the costs vanish (Bolton et al., 2013).

Second, we can relatively easily incorporate fixed costs ϕ of equity issuance in addition to infrequent capital market access. Then, the firm refinances in state C upon capital market access if and only if the total gains from refinancing $\left[P(\overline{C}) - P(C) - (\overline{C} - C) - \phi\right]$ are positive, and $\alpha(C)$ is set such that the firm's shareholders realize zero gains from refinancing.[22] This results in no refinancing on some $\left[\tilde{C}, \overline{C}\right]$ for some $\tilde{C} \in \left[\underline{C}, \overline{C}\right]$, even when market access is available. The lower

[21]See, e.g., Campello et al. (2010), Duchin, Ozbas, and Sensoy (2010), or Lemmon and Roberts (2010) and other relevant empirical references cited in Hugonnier et al. (2015).

[22]If $\left[P(\overline{C}) - P(C) - (\overline{C} - C) - \phi\right] < 0$, the firm does not refinance upon capital market access, so the choice of $\alpha(C)$ is immaterial.

boundary now becomes[23]

$$\underline{C} = \min\left\{0, \underline{C}^S\right\} \mathbb{1}_{\{[P(\overline{C})-\overline{C}]>\phi\}} \quad \text{where} \quad \underline{C}^S = -\left\{\frac{r}{r+\pi}Y^A + \frac{\pi}{r+\pi}\left(\left[P\left(\overline{C}\right) - \overline{C}\right] - \phi\right)\right\} \tag{35}$$

and we see that the boundary conditional on survival is still a weighted average of autarky value and net firm value after costs. Fixed costs further constrain α, and thus lead to even stronger debt overhang issues, as the value gain from refinancing now has to cover ϕ as well as α. Even when $\pi \to \infty$, $\overline{C} > 0$ as both intermediary financing and refinancing in capital markets are costly. Provided ϕ is sufficiently low, the firm refinances at a lower boundary $\underline{C} < 0$. Thus, intermediary exit now occurs "deterministically" at the lower bound $\underline{C}$. In other words, when $\pi \to \infty$ and refinancing is costly ($\phi > 0$) but only mildly so, then $\underline{C} < 0$ and $Y > 0$ may arise in which case the firm finances cash flow shortfalls against future promises (for $C < 0$) and the intermediary takes a stake in the firm and exits upon refinancing. Therefore, we expect our qualitative results to go through as long as refinancing from outside investors is costly (e.g., $\phi > 0$) or not frequently available ($\pi < \infty$) or both.[24]

In sum, we feel that our assumptions are more relevant for the purposes of this paper and, in particular, for describing young and private firms that have limited market access.[25]

Costly intermediary financing. We want to capture the reality that specialized intermediary financing is costly. Moreover, the riskier the intermediary's payouts, the more expensive this financing will be. To capture this mechanism, we could have introduced an intermediary that is (i) cash-constrained or (ii) has a regulatory cost-of-capital linked to the riskiness of its financial agreements. Note that modeling the cost of intermediary financing via (i) would result in a higher cost of funds for riskier financing arrangements, as higher risk makes the intermediary more likely to hit its own funding constraint. But this would introduce an additional state variable, the cash holdings or net worth of the intermediary. Modeling it via (ii) would require us to take a stand

[23] For a derivation, notice that at the lower boundary $\underline{C} < 0$ (conditional on survival), we have $\alpha(\underline{C}) = P(\overline{C}) - \overline{C} + \underline{C} - \phi$, as $P(\underline{C}) = 0$. Inserting this expression for α as well as $\beta(\underline{C}) = 0$ into (12) and setting the drift of dC at $C = \underline{C}$ to zero, we can solve for $\underline{C}$.

[24] For completeness, one could also add — next to the fixed cost of refinancing ϕ — a variable "flotation" cost of refinancing $\hat{\phi}$, as in Bolton et al. (2011). Under these circumstances, the firm would choose a refinancing target $C^* < \overline{C}$, but the remainder of the findings would likely remain similar.

[25] Berger and Udell (1995) show that the length of a banking relationship plays a key role in the level of credit lines provided to a small firm, reflecting information asymmetries. Thus, the speed at which a firm can raise outside funding and the ability of the intermediary to provide immediate financing may reflect both the speed with which the outside investors can overcome such information asymmetries and the privileged position of the intermediary as an informed relationship lender.

on how the riskiness of financing translates into the regulatory cost of funds. We view our cost assumption as reflecting such costs while maintaining tractability, i.e., not introducing additional state-variables into the problem. Moreover, we emphasize that our results also go through, if both the intermediary and the shareholders are risk-averse. To this end, Internet Appendix G presents a model variant in which both the shareholders and the intermediary are risk-averse (in particular, the intermediary has CARA preferences); we show that our results remain qualitatively unchanged compared with the baseline.

6 Conclusion

We consider a firm with infrequent access to capital markets but continuous access to costly intermediary financing. As intermediary financing is costly, the firm holds internal cash reserves to manage its liquidity. Under the optimal financing agreement, the intermediary absorbs a fraction of cash flow risk that decreases in the firm's excess liquidity position. Importantly, the optimal contract is renegotiation proof, implying that future shareholders that are not party to the current contract will select the same contract upon investing. Once the firm depletes its cash reserves, the firm either liquidates or the intermediary rescues the firm by absorbing all cash-flow risk in exchange for a stake in the firm. Crucially, the intermediary's willingness to rescue the firm increases with its prospective financial market access. Indeed, this access allows the intermediary to exit its position and resell the stake to risk-neutral and competitive investors. Further, such a strategy to take a stake in the firm under financial distress and sell off this stake after the resolution of distress resembles the strategy of PE investors. Importantly, such PE-like intermediation helps resolve distress and reduce the risk of liquidation more efficiently. Our model implies that the extent of such PE-like intermediation crucially depends on financial market access, facilitating the exit of the intermediary's position in the firm. We find that PE-like intermediation is most pronounced when the firm's market access and cash flow volatility lie at intermediate levels. Interestingly, our findings suggest that a decline in interest rate spurs PE financing. Finally, we show how to implement the optimal contract with a credit line provided by the intermediary and restricted equity held by the intermediary. Our implementation suggests an overlapping pecking order of financial instruments. The firm simultaneously finances cash flow shortfalls with cash reserves and a credit line and only sells equity to the intermediary in distress. We also discuss an alternative implementation of the optimal contract, which calls for the firm to finance cash flow shortfalls first with an unsecured credit line and cash reserves, while it relies on a secured credit line only under distress.

References

Abel, A. B. and S. Panageas (2020a). An analytic framework for interpreting investment regressions in the presence of financial constraints. Technical report, National Bureau of Economic Research.

Abel, A. B. and S. Panageas (2020b). Precautionary saving in a financially-constrained firm. *Working Paper, University of Pennsylvania*.

Acharya, V. V., H. Almeida, and M. Campello (2007). Is cash negative debt? a hedging perspective on corporate financial policies. *Journal of financial intermediation 16*(4), 515–554.

Ai, H., D. Kiku, and R. Li (2019). A quantitative model of dynamic moral hazard. *Available at SSRN 2801116*.

Ai, H. and R. Li (2015). Investment and ceo compensation under limited commitment. *Journal of Financial Economics 116*(3), 452–472.

Almeida, H. and M. Campello (2007). Financial constraints, asset tangibility, and corporate investment. *Review of Financial Studies 20*(5), 1429–1460.

Axelson, U., T. Jenkinson, P. Strömberg, and M. S. Weisbach (2013). Borrow cheap, buy high? the determinants of leverage and pricing in buyouts. *The Journal of Finance 68*(6), 2223–2267.

Axelson, U., P. Strömberg, and M. S. Weisbach (2009). Why are buyouts levered? the financial structure of private equity funds. *The Journal of Finance 64*(4), 1549–1582.

Bates, T. W., K. M. Kahle, and R. M. Stulz (2009). Why do us firms hold so much more cash than they used to? *Journal of Finance 64*(5), 1985–2021.

Benmelech, E., N. Kumar, and R. Rajan (2020). The decline of secured debt. Technical report, National Bureau of Economic Research.

Berger, A. N. and G. F. Udell (1995). Relationship lending and lines of credit in small firm finance. *Journal of business*, 351–381.

Bernstein, S., J. Lerner, and F. Mezzanotti (2019). Private equity and financial fragility during the crisis. *The Review of Financial Studies 32*(4), 1309–1373.

Bernstein, S., J. Lerner, M. Sorensen, and P. Strömberg (2017). Private equity and industry performance. *Management Science 63*(4), 1198–1213.

Biais, B., T. Mariotti, G. Plantin, and J.-C. Rochet (2007). Dynamic security design: Convergence to continuous time and asset pricing implications. *Review of Economic Studies 74*(2), 345–390.

Bolton, P., H. Chen, and N. Wang (2011). A unified theory of tobin's q, corporate investment, financing, and risk management. *Journal of Finance 66*(5), 1545–1578.

Bolton, P., H. Chen, and N. Wang (2013). Market timing, investment, and risk management. *Journal of Financial Economics 109*(1), 40–62.

Bolton, P., Y. Li, N. Wang, and J. Yang (2021). Dynamic banking and the value of deposits. Technical report, National Bureau of Economic Research.

Bolton, P., N. Wang, and J. Yang (2019). Optimal contracting, corporate finance, and valuation with inalienable human capital. *Journal of Finance 74*(3), 1363–1429.

Bolton, P., N. Wang, and J. Yang (2021). Leverage dynamics under costly equity issuance. *Working Paper, Columbia University*.

Campello, M., J. R. Graham, and C. R. Harvey (2010). The real effects of financial constraints: Evidence from a financial crisis. *Journal of financial Economics 97*(3), 470–487.

Cornelli, F., Z. Kominek, and A. Ljungqvist (2013). Monitoring managers: Does it matter? *The Journal of Finance 68*(2), 431–481.

Dai, M., X. Giroud, W. Jiang, and N. Wang (2020). A q theory of internal capital markets. Technical report, National Bureau of Economic Research.

Darmouni, O. and L. Mota (2020). The financial assets of non-financial firms. *Available at SSRN 3543802*.

Décamps, J.-P., T. Mariotti, J.-C. Rochet, and S. Villeneuve (2011). Free cash flow, issuance costs, and stock prices. *Journal of Finance 66*(5), 1501–1544.

DeMarzo, P. M. and Y. Sannikov (2006). Optimal security design and dynamic capital structure in a continuous-time agency model. *Journal of Finance 61*(6), 2681–2724.

Duchin, R., O. Ozbas, and B. A. Sensoy (2010). Costly external finance, corporate investment, and the subprime mortgage credit crisis. *Journal of financial economics 97*(3), 418–435.

Duffie, D., N. Garlenau, and L. H. Pedersen (2005, November). Over-the-counter markets. *Econometrica 73*(6), 1815–1847.

Eisfeldt, A. L. and T. Muir (2016). Aggregate external financing and savings waves. *Journal of Monetary Economics 84*, 116–133.

Ewens, M., A. Gorbenko, and A. Korteweg (2021). Venture capital contracts. *Journal of Financial Economics forthcoming*.

Feng, F. Z., C. R. Taylor, M. M. Westerfield, and F. Zhang (2021). Setbacks, shutdowns, and overruns. *Available at SSRN 3775340*.

Feng, F. Z. and M. M. Westerfield (2021). Dynamic resource allocation with hidden volatility. *Journal of Financial Economics 140*(2), 560–581.

Gompers, P. A., S. N. Kaplan, and V. Mukharlyamov (2020). Private equity and covid-19. Technical report, National Bureau of Economic Research.

Green, B. and C. R. Taylor (2016). Breakthroughs, deadlines, and self-reported progress: Contracting for multistage projects. *American Economic Review 106*(12), 3660–99.

Gryglewicz, S. (2011). A theory of corporate financial decisions with liquidity and solvency concerns. *Journal of Financial Economics 99*(2), 365–384.

Gryglewicz, S. and B. Hartman-Glaser (2020). Investment timing and incentive costs. *The Review of Financial Studies 33*(1), 309–357.

Gryglewicz, S. and S. Mayer (2021). Dynamic contracting with intermediation: Operational, governance, and financial engineering. *Available at SSRN 3175528*.

Gryglewicz, S., S. Mayer, and E. Morellec (2020). Agency conflicts and short-versus long-termism in corporate policies. *Journal of Financial Economics 136*(3), 718–742.

Hennessy, C. A., A. Levy, and T. M. Whited (2007). Testing q theory with financing frictions. *Journal of Financial Economics 83*(3), 691–717.

Hennessy, C. A. and T. M. Whited (2005). Debt dynamics. *The journal of finance 60*(3), 1129–1165.

Hennessy, C. A. and T. M. Whited (2007). How costly is external financing? evidence from a structural estimation. *The Journal of Finance 62*(4), 1705–1745.

Holmstrom, B. and J. Tirole (1997). Financial intermediation, loanable funds, and the real sector. *the Quarterly Journal of economics 112*(3), 663–691.

Hotchkiss, E. S., D. C. Smith, and P. Strömberg (2021). Private equity and the resolution of financial distress. *The Review of Corporate Finance Studies 10*(4), 694–747.

Hugonnier, J., S. Malamud, and E. Morellec (2015). Capital supply uncertainty, cash holdings, and investment. *Review of Financial Studies 28*(2), 391–445.

Hugonnier, J. and E. Morellec (2017). Bank capital, liquid reserves, and insolvency risk. *Journal of Financial Economics 125*(2), 266–285.

Ivashina, V. and A. Kovner (2011). The private equity advantage: Leveraged buyout firms and relationship banking. *The Review of Financial Studies 24*(7), 2462–2498.

Jang, Y. S. (2020). Five facts about direct lending to middle-market buyouts. *Available at SSRN 3741678*.

Jensen, M. C. and W. H. Meckling (1976). Theory of the firm: Managerial behavior, agency costs and ownership structure. *Journal of Financial Economics 3*(4), 305–360.

Kaplan, S. N. and P. Stromberg (2009). Leveraged buyouts and private equity. *Journal of economic perspectives 23*(1), 121–46.

Leary, M. T. and M. R. Roberts (2005). Do firms rebalance their capital structures? *Journal of Finance 60*(6), 2575–2619.

Lemmon, M. and M. R. Roberts (2010). The response of corporate financing and investment to changes in the supply of credit. *Journal of Financial and quantitative analysis 45*(3), 555–587.

Malamud, S. and F. Zucchi (2018). Liquidity, innovation, and endogenous growth. *Journal of Financial Economics*.

Malenko, A. (2019). Optimal dynamic capital budgeting. *The Review of Economic Studies 86*(4), 1747–1778.

Malenko, A. and N. Malenko (2015). A theory of lbo activity based on repeated debt-equity conflicts. *Journal of Financial Economics 117*(3), 607–627.

Marinovic, I. and F. Varas (2019). Ceo horizon, optimal pay duration, and the escalation of short-termism. *Journal of Finance 74*(4), 2011–2053.

Nikolov, B., L. Schmid, and R. Steri (2019). Dynamic corporate liquidity. *Journal of Financial Economics 132*(1), 76–102.

Piskorski, T. and M. M. Westerfield (2016). Optimal dynamic contracts with moral hazard and costly monitoring. *Journal of Economic Theory 166*, 242–281.

Rampini, A. A., A. Sufi, and S. Viswanathan (2014). Dynamic risk management. *Journal of Financial Economics 111*(2), 271–296.

Rampini, A. A. and S. Viswanathan (2010). Collateral, risk management, and the distribution of debt capacity. *The Journal of Finance 65*(6), 2293–2322.

Rauh, J. D. and A. Sufi (2010). Capital structure and debt structure. *The Review of Financial Studies 23*(12), 4242–4280.

Strulovici, B. (2020). Renegotiation-proof contracts with persistent private information. *Working Paper, Northwestern University*.

Sufi, A. (2009). Bank lines of credit in corporate finance: An empirical analysis. *The Review of Financial Studies 22*(3), 1057–1088.

Varas, F. (2018). Managerial short-termism, turnover policy, and the dynamics of incentives. *The Review of Financial Studies 31*(9), 3409–3451.

Whited, T. M. and G. Wu (2006). Financial constraints risk. *Review of Financial Studies 19*(2), 531–559.

Appendix

A Proof of Lemma 1

Take the intermediary's continuation value from (5), i.e.,

$$Y_t = \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} \left(dI_s - k_s ds \right) \right].$$

Define

$$A_t = \mathbb{E}_t \left[\int_0^\infty e^{-rs} (dI_s - k_s ds) \right] = \int_0^t e^{-rs} (dI_s - k_s ds) + e^{-rt} Y_t. \tag{A.1}$$

By construction, $A = \{A_t\}$ is a martingale. By the martingale representation theorem, there exist stochastic processes $\alpha = \{\alpha_t\}$ and $\beta = \{\beta_t\}$ such that

$$e^{rt} dA_t = \beta_t \big(dX_t - \mu dt \big) + \alpha_t (d\Pi_t - \pi dt), \tag{A.2}$$

where $dZ_t = \frac{dX_t - \mu dt}{\sigma}$ is the increment of a standard Brownian Motion under the probability measure and $(d\Pi_t - \pi dt)$ is the increment of a compensated Poisson process which is a martingale.

We differentiate (A.1) with respect to time t to obtain an expression for dA_t, then plug this expression into (A.2) and solve (A.2) to get

$$dY_t = (rY_t + k_t) dt + \beta_t \sigma dZ_t + \alpha_t (d\Pi_t - \pi dt),$$

which is (6) as desired.

B Proof of Proposition 1

Recall (2), that is,

$$dM_t = \mu dt + \sigma dZ_t + (r - \lambda) M_t dt - dDiv_t - dI_t + \Delta M_t d\Pi_t, \tag{B.1}$$

and (6), that is,

$$dY_t = [rY_t + k_t] dt + \beta_t \sigma dZ_t + \alpha_t (d\Pi_t - \pi dt).$$

Next, we combine (2) and (6) to calculate for $C_t = M_t - Y_t$:

$$\begin{aligned} dC_t = dM_t - dY_t = \mu dt + r(M_t - Y_t) dt + \lambda (Y_t - M_t) dt - \lambda Y_t dt - k_t dt + \pi \alpha_t dt \\ + \sigma (1 - \beta_t) dZ_t - dDiv_t + (\Delta M_t - \alpha_t) d\Pi_t. \end{aligned} \tag{B.2}$$

We define the "post-refinancing" level of excess liquidity

$$C_t^* = C_t + \Delta M_t - \alpha_t, \tag{B.3}$$

so that

$$\Delta M_t = C_t^* - C_t + \alpha_t \quad \text{and} \quad \Delta M_t - \alpha_t = C_t^* - C_t. \tag{B.4}$$

Using these relations, we obtain

$$\begin{aligned} dC_t = & [\mu + (r-\lambda) C_t - \lambda Y_t - k_t + \pi \alpha_t] \, dt \\ & + \sigma (1-\beta_t) \, dZ_t + (C_t^* - C_t) \, d\Pi_t - dDiv_t, \end{aligned} \tag{B.5}$$

which is (12) as desired after inserting (7), that is, $k_t = \frac{\rho r}{2}(\beta_t \sigma)^2$.

We denote the drift of dC_t by

$$\mu_C(C_t) = \Big[\mu + (r-\lambda) C_t - \lambda Y_t - \frac{\rho r}{2} (\beta_t \sigma)^2 + \pi \alpha_t \Big] \tag{B.6}$$

and the volatility of dC_t by

$$\sigma_C(C_t) = \sigma(1-\beta_t). \tag{B.7}$$

Next, we conjecture and verify that the equity value can be expressed as function of C_t only (i.e., $P_t = P(C_t)$), while $Y_t = Y$ is a control variable. Indeed, according to (6) and (12), it is always possible to increase or decrease Y by picking $dI < 0$ or $dI > 0$, whilst leaving the value of excess liquidity C unchanged. Given the Markovian representation, we omit time subscripts unless necessary.

Given $\mathcal{C}$, the equity value at time t (i.e., shareholders' value function) reads

$$P_t = P(C_t) = \mathbb{E}_t \left[\int_t^\tau e^{-r(s-t)} (dDiv_s - \Delta M_s d\Pi_s) \Big| C_t = C \right].$$

By the dynamic programming principle, the equity value $P(C)$ must then solve the HJB equation

$$rP(C)dt = \max_{\beta, Y, \alpha, C^*, dDiv \geq 0} \Big\{ dDiv + \mathbb{E}[dP(C) - \Delta M d\Pi] \Big\}, \tag{B.8}$$

subject to shareholders' and intermediary's limited commitment constraints (15), $Y \geq 0$, the constraint that cash holdings remain positive, $M \geq 0$, and the limited commitment constraint (15). Invoking Ito's Lemma, we can calculate

$$dP(C) = P'(C)\mu_C(C)dt + \frac{P''(C)\sigma_C(C)^2}{2} dt + P'(C)\sigma_C(C)dZ + (P(C^*) - P(C))d\Pi_t - P'(C)dDiv. \tag{B.9}$$

Thus,

$$\mathbb{E}[dP(C)] = P'(C)\mu_C(C)dt + \frac{P''(C)\sigma_C(C)^2}{2} dt + (P(C^*) - P(C))\pi dt - P'(C)dDiv. \tag{B.10}$$

Using this relation and $\Delta M_t = C_t^* - C_t + \alpha_t$, we can write the HJB equation (G.40) as

$$\begin{aligned} rP(C)dt = \max_{\beta, Y, \alpha, C^*, dDiv \geq 0} \Big\{ & \big[1 - P'(C)\big] \, dDiv + P'(C)\mu_C(C)dt + \frac{P''(C)\sigma_C(C)^2}{2} dt \\ & + \pi \big(P(C^*) - P(C) - (C^* - C) - \alpha \big) dt \Big\}, \end{aligned} \tag{B.11}$$

which is solved subject to shareholders' and intermediary's limited commitment constraints (15) and $Y \geq 0$ as well as the constraint that cash holdings remain positive, $M \geq 0$.

As $dDiv \geq 0$, it is optimal to stipulate dividend payouts if and only if $P'(C) \geq 1$. As in related papers (e.g., Bolton et al. 2011), dividend payouts occur at a payout boundary $\overline{C}$ and follow a barrier strategy, that is, $dDiv = \max\{C - \overline{C}, 0\}$, and dividend payouts cause C to reflect at $\overline{C}$. The location of the payout boundary is determined by smooth pasting and super contact conditions, that is,

$$P'(\overline{C}) = 1 \quad \text{and} \quad P''(\overline{C}) = 0. \tag{B.12}$$

For $C < \overline{C}$, there are no dividend payouts and the HJB equation (G.44) simplifies to

$$rP(C) = \max_{\beta, Y, \alpha, C^*} \left\{ P'(C)\mu_C(C) + \frac{P''(C)\sigma_C(C)^2}{2} + \pi\big[P(C^*) - P(C) - (C^* - C) - \alpha\big] \right\}, \tag{B.13}$$

subject to shareholders' limited commitment constraint (15) , the intermediary's limited commitment $Y \geq 0$, and the physical cash constraint $M \geq 0$. The above HJB equation becomes (16) after inserting above expressions for $\mu_C(C)$ and $\sigma_C(C)$ respectively (see (B.6) and (B.7)). The right-hand side of the HJB equation only depends on C, the control variables, and exogenous model constants, and so does the left-hand side. In addition, because $P(C)$ is a function of C, the derivatives $P'(C)$ and $P''(C)$ are functions of C too, and so are the control variables. As a result, we have verified that equity value can be expressed as a function of C only so that C is the only payoff-relevant state variable.

C Proof of Lemma 2

First, consider $\underline{C}^S < 0$. Suppose the firm instead liquidates the first time C falls to zero. This is sub-optimal as liquidation at $C = 0$ implies $P(0) = 0$ and $Y(0) = 0$, whereas keeping the firm alive yields the same $Y(0) = 0$ but a higher $P(0) > 0$. Thus, if $\underline{C}^S < 0$, the firm never liquidates. Second, consider $\underline{C}^S > 0$ so that $M(\underline{C}^S) > 0$. Shareholders' value at the boundary is $P(\underline{C}^S) = 0$ while the intermediary's stake is $Y(\underline{C}^S) = 0$. However, liquidating the firm and paying out $M(\underline{C}^S) = \underline{C}^S > 0$ dollars as dividends yield a higher payoff than survival. Thus, when $\underline{C}^S > 0$, the firm optimally liquidates and, as we have shown, optimal liquidation occurs at $\underline{C} = \underline{C}^L = 0$.

D Proof of Proposition 2

In what follows, we make the following technical regularity assumption.

Assumption 1. *Fix $\underline{C}$, and take a constant $K \geq 0$. The HJB equation (16) with the boundary conditions $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$ and $P(\underline{C}) = K$ admits a unique solution $P(C)$ on the endogenous state space $(\underline{C}, \overline{C})$, with payout boundary $\overline{C}$. The solution $P(C)$ is twice continuously differentiable on $(\underline{C}, \overline{C})$, which implies that $P'(C)$ and $P''(C)$ exist and are continuous on the interval $(\underline{C}, \overline{C})$. The set of points $C \in (\underline{C}, \overline{C})$ at which either $P''(C)$ or $\alpha(C)$ is not differentiable is countable.*

To start with, note that the optimal control variables are derived in the main text in Section 2.3 by going through the otpimization in the HJB equation (16). That is, Section 2.3 in the main text derives the optimal control variables as functions of excess liquidity C, that is, $Y = Y(C)$, $M = M(C)$, $\alpha = \alpha(C)$, $\beta = \beta(C)$, and $C^* = C^*(C)$.

Here, we prove the remaining claims of Proposition 2. We split the proof of Proposition 2 into several parts. Part I establishes the concavity of the equity value (i.e., $P''(C) \leq 0$) and shows that the payout boundary is strictly positive. Part II proves that the HJB equation (16) simplifies to (23). Part III characterizes the lower boundary in the state space $\underline{C}$ and, in particular, establishes (24) and (25). Part IV characterizes the optimal transfer process dI. Importantly, we prove Proposition 2 under the technical Assumption 1.

D.1 Part I — Concavity of Value Function and $\overline{C} > 0$

Define the jump in the value function upon refinancing as

$$J(C) \equiv P\left(\overline{C}\right) - P\left(C\right) - \left(\overline{C} - C + \alpha(C)\right), \tag{D.1}$$

under the optimal choice of the refinancing target $C^* = \overline{C}$, and note that under the optimal choice of $\alpha(C)$ from (21), we have $J(C) = 0$ as well as $\alpha'(C) \leq 0$.

We now rewrite the HJB equation (16) as

$$rP\left(C\right) = \max_{\beta \in [0,1]} \left\{ P'(C)\mu_C(C) + \frac{P''(C)}{2}\sigma^2(1-\beta(C))^2 \right\}, \tag{D.2}$$

under the optimal choice of α in (21), $Y = \max\{-C, 0\}$, $C^* = \overline{C}$, and with $\mu_C(C)$ from (B.6).

When $P''(C)$ is differentiable, we can use the envelope theorem and differentiate the HJB equation (D.2) under the optimal β with respect to C and rearrange to obtain

$$P'''(C) = \frac{2}{(1-\beta(C))^2\sigma^2}\left(P'(C)\lambda \mathbb{1}_{\{C \geq 0\}} - P''(C)\mu_C(C) - \pi P'(C)\alpha'(C)\right), \tag{D.3}$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function which is equal to one if $\{\cdot\}$ is true and is equal to zero otherwise. The set of points at which $P''(C)$ is not differentiable is countable; therefore, for any C, the limits $\lim_{x \uparrow C} P'''(C), \lim_{x \downarrow C} P'''(C)$ exist and are well-defined.

We now show that $\overline{C} > 0$. Note that we can evaluate the ODE (16) or (23) at the payout boundary $\overline{C}$ to obtain $P(\overline{C}) = \frac{\mu}{r} + \overline{C} - \frac{\overline{C}\lambda \mathbb{1}_{\{C \geq 0\}}}{r}$, with $\alpha(\overline{C}) = \beta(\overline{C}) = 0$. Thus, the initial payoff of outside investors equals $P(\overline{C}) - \overline{C} = \frac{\mu}{r} - \frac{\overline{C}\lambda \mathbb{1}_{\{C \geq 0\}}}{r}$. This payoff must be strictly lower than the NPV of the firm, $\frac{\mu}{r}$, which implies $\overline{C} > 0$.

As $\overline{C} > 0$, $P''(\overline{C}) = 0$, $P'(\overline{C}) = 1$, (D.3) and $\alpha'(C) \leq 0$ imply $\lim_{C \uparrow \overline{C}} P'''(C) > 0$. Because $\lim_{C \uparrow \overline{C}} P'''(C) > 0$ for $\overline{C} > 0$, continuity implies $P'''(C) > 0$ in a left-neighbourhood of $\overline{C}$. As a result, there exists $\varepsilon > 0$ such that $P''(C) < 0$ and $P'(C) > 1$ on the interval $(\overline{C} - \varepsilon, \overline{C})$. Define $\hat{C} = \sup\{C \geq 0 : P''(C) \geq 0\}$ and suppose to the contrary that $\hat{C} < \overline{C}$. As $P''(C) < 0$ in a neighbourhood of $\overline{C}$, it follows by continuity that $P''(\hat{C}) = 0$. Since $P''(C) < 0$ for $C \in (\hat{C}, \overline{C})$, it follows that $P'(\hat{C}) > 1$.

As $P'(\hat{C}) > 1$ and $\alpha'(C) \leq 0$, (D.3) implies $\lim_{C \downarrow \hat{C}} P'''(C) > 0$. Due to $\lim_{C \downarrow \hat{C}} P'''(C) > 0$, there exists $C' > \hat{C}$ so that $P''(C') > 0$, which contradicts the definition of $\hat{C}$. Therefore, $\hat{C} = \overline{C}$ and $P''(C) < 0$ for all $C < \overline{C}$, which was to show.

D.2 Part II — Simplified HJB Equation (23)

Note that

$$1 - \beta(C) = 1 - \frac{P''(C)}{P''(C) - \rho r P'(C)} = \frac{-\rho r P'(C)}{P''(C) - \rho r P'(C)}. \tag{D.4}$$

Thus, $\frac{P''(C)}{-\rho r P'(C)} = \frac{\beta(C)}{1-\beta(C)}$ and $1-\beta(C) = \frac{-\rho r P'(C)}{P''(C)}\beta(C)$ as well as $\beta(C) = \frac{P''(C)}{-\rho r P'(C)}(1-\beta(C))$. As a result, we can calculate

$$-\frac{\rho r}{2}\beta(C)^2 P'(C) + \frac{P''(C)}{2}((1-\beta(C))^2 = -\frac{\rho r}{2}\beta(C)^2 P'(C) + \frac{(\rho r P'(C))^2}{2P''(C)}(\beta(C))^2 \tag{D.5}$$
$$= -\frac{\rho r}{2}\beta(C)^2 P'(C)\left(1 - \frac{\rho r P'(C)}{P''(C)}\right) = -\frac{\rho r}{2}\beta(C)P'(C),$$

where the last equality uses that

$$\frac{1}{\beta(C)} = \frac{P''(C) - \rho r P'(C)}{P''(C)} = 1 - \frac{\rho r P'(C)}{P''(C)}. \tag{D.6}$$

We can insert relation (D.5) as well as $C^* = \overline{C}$ into (16) to obtain

$$rP(C) = P'(C)\left[\mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\alpha(C)\right]$$
$$+ \pi\left[P\left(\overline{C}\right) - P(C) - \left(\overline{C} - C + \alpha(C)\right)\right], \tag{D.7}$$

which is (23). Here $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function which is equal to one if $\{\cdot\}$ is true and is equal to zero otherwise. Using (21) so that the term $\pi\left[P\left(\overline{C}\right) - P(C) - \left(\overline{C} - C + \alpha(C)\right)\right]$ in (D.7) is identically zero, we obtain

$$rP(C) = P'(C)\left[\mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\alpha(C)\right],$$

which is (23).

D.3 Lower Boundary

D.3.1 Part III.A — Auxiliary Results

Lemma 3. *Conditional on survival, the lower boundary is* $\underline{C} = \underline{C}^S$ *and satisfies* $\mu_C(\underline{C}^S) = 0$, $\beta(\underline{C}^S) = 1$, $P(\underline{C}^S) = 0$, *and* $\alpha(\underline{C}^S) = \left[P(\overline{C}) - \overline{C}\right] - \left[P(\underline{C}^S) - \underline{C}^S\right]$. *It holds that* $\beta(C) < 1$ *for* $C > \underline{C}^S$ *with* $\lim_{C\downarrow \underline{C}^S}\beta(C) = 1$.

Proof of Lemma 3. We consider that the firm is never liquidated (i.e., survival). Note that $M \geq 0$, $C = M - Y \geq -Y$ and $Y \leq \frac{\mu}{r}$ where $\frac{\mu}{r}$ is the first best value. As such, $C \geq -\frac{\mu}{r}$, so that excess liquidity is bounded from below. Therefore, there must exist a lower boundary $\underline{C}$ such that $C_t \geq \underline{C}$ at all times t. Under survival (i.e., the firm is never liquidated), C follows (12) at all times t and thus has drift $\mu_C(C)$ (see (B.6)) and volatility $\sigma_C(C)$ (see (B.7)). Under survival, the lower boundary $\underline{C}$ of the state space must therefore satisfy $\mu_C(\underline{C}) \geq 0$ and $\sigma_C(\underline{C}) = 0$ so that C does not drop below $\underline{C}$.[26]

The HJB equation (16) evaluated under the optimal controls $\alpha(C)$ (see (21)), $\beta(C)$ (see (22)), $Y(C) = \max\{-C, 0\}$ as well as $C^* = \overline{C}$ can be rewritten as

$$rP(C) = P'(C)\mu_C(C) + \frac{P''(C)(\sigma_C(C))^2}{2}, \tag{D.8}$$

[26]Clearly, if C is bounded from below by $\underline{C}$, then it bounded from below by $\underline{C} - \varepsilon$ too. Unless otherwise mentioned, we consider the tightest lower bound.

where we used (15) binding, and where $\mu_C(C)$ and $\sigma_C(C)$ are drift and volatility of dC in (12) defined in (B.6) and (B.7).

We define $\underline{C}^S$ as the lowest value of C such that the payout agreement can implement $\mu_C(C) \geq 0$, $P(C) \geq 0$, and $\sigma_C(C) = 0$, and the value function solves (D.8). Note that $\sigma_C(C) = \sigma(1-\beta(C)) = 0$ is equivalent to $\beta(C) = 1$. Because $\beta(\underline{C}^S) = 1 \iff \sigma_C(\underline{C}^S) = 0$ and $\mu(\underline{C}^S) \geq 0$, the payout agreement implements $C_t \geq \underline{C}^S$, thereby satisfying that C is bounded from below under survival. Because $\mu_C(C)$ increases with C, decreases with $\beta(C)$, and increases with $\alpha(C)$, because $\alpha(C)$ decreases with C, and because equity value $P(C)$ is characterized by (D.8), it follows that $\sigma_C(\underline{C}^S) = 0 \iff \beta(\underline{C}^S) = 1$, $\mu_C(\underline{C}^S) = 0$, and $P(\underline{C}^S) = 0$.

In more detail, if it were $\mu_C(\underline{C}^S) > 0$, there would exist $C' < \underline{C}^S$ such that the payout agreement could implement $\sigma_C(C') = 0$ and $\mu_C(C') \geq 0$ with the same choice of α and β (i.e., $\alpha(C') = \alpha(\underline{C}^S)$ and $\beta(C') = \beta(\underline{C}^S) = 1$) and $P(C') \geq 0$ due to (D.8), contradicting the definition of $\underline{C}^S$. Likewise, if it were $P(\underline{C}^S) > 0$ while $\beta(\underline{C}^S) = 1$, then (D.8) would imply $\mu_C(\underline{C}^S) > 0$, leading to a contradiction. Thus, $\sigma_C(\underline{C}^S) = 0 \iff \beta(\underline{C}^S) = 1$, and $\mu_C(\underline{C}^S) = 0$, which — by (D.8) — readily implies $P(\underline{C}^S) = 0$.

Next, we show that $P(\underline{C}^S) = 0$ implies $\beta(\underline{C}^S) = 1$ and $\mu_C(\underline{C}^S) = 0$. According to the dynamic programming principle and the optimization in the HJB equation (16), the optimal choice of $\alpha(\underline{C}^S)$ and $\beta(\underline{C}^S)$ induces $P(\underline{C}^S) = 0$. As shown above, under the optimal controls this HJB equation simplifies to (D.8). Setting $\beta(\underline{C}^S) = 1$ and $\alpha(\underline{C}^S)$ according to (21) implies, by definition of $\underline{C}^S$, $\mu_C(\underline{C}^S) = \sigma_C(\underline{C}^S) = 0$ and therefore $rP(\underline{C}^S) = P'(\underline{C}^S)\mu_C(\underline{C}^S) + \frac{P''(\underline{C}^S)(\sigma_C(\underline{C}^S))^2}{2} = 0$. As a result, setting $\beta(\underline{C}^S) = 1$ and $\alpha(\underline{C}^S)$ according to (21) is optimal and consistent with the optimization in the HJB equation (16). As the optimization with respect to α and β in the HJB equation (16) yields the unique solutions (21) and (22), it follows that setting $\beta(\underline{C}^S) = 1$ is strictly optimal. Taken together, we have established the equivalence $P(\underline{C}^S) = 0 \iff \beta(\underline{C}^S) = 1 \;\wedge\; \mu(\underline{C}^S) = 0$.

Importantly, under survival, the payout agreement must satisfy $\beta(\underline{C}^S) = 1$, $\mu_C(\underline{C}^S) = 0$, and $P(\underline{C}^S) = 0$ to ensure that C is bounded from below. That is, if C reaches $\underline{C}^S$, it must be that $\beta(\underline{C}^S) = 1$ to ensure that C is bounded from below with probability one. Crucially, the stipulation of the boundary condition(s) $P(\underline{C}^S) = 0$ and $\beta(\underline{C}^S) = 1$ to solve the HJB equation (16) is not an optimality result but a consequence of the requirement C must be bounded from below under incentive compatible contracts and survival; the stipulation of the boundary condition $P(\underline{C}^S) = 0$ does not per-se preclude $\beta(C') = 1$ for $C' > \underline{C}^S$. In particular, it is always possible to set $\beta(C') = 1$ for $C' > \underline{C}^S$ in which case $\mu_C(C') > 0$ and $\sigma_C(C') = 1$ and $C_t \geq C'$ at all times t (with certainty). In other words, it is always possible to implement a different effective lower bound $C' > \underline{C}^S$ on C through the choice of the control variable β.

We conjecture and verify that conditional on survival, $\beta(C) < 1$ for $C > \underline{C}^S$ while we have $\lim_{C \to \underline{C}^S} \beta(C) = 1$ in that $\underline{C} = \underline{C}^S$ is the (tightest) lower bound in the state space. To do so, take as lower bound $\underline{C} = \underline{C}^S$ and impose $P(\underline{C}) = 0$ to solve the HJB equation (16) on $(\underline{C}, \overline{C})$ subject to $P(\underline{C}) = P'(\overline{C}) - 1 = P''(\overline{C}) = 0$. By Assumption 1, a unique solution $P(C)$ to (16) exists, and is twice continuously differentiable on $(\underline{C}, \overline{C})$, with endogenous payout boundary $\overline{C} > \underline{C}$. Recall that the optimal choice of $\beta(C)$ is determined according to the optimization in (16) and therefore satisfies (22). It follows that $\beta(C) \to 1$ only if $P''(C) \to -\infty$, as $P'(C) \geq 1$. However, because — by Assumption 1 — the value function $P(C)$ is twice continuously differentiable on $(\underline{C}, \overline{C})$, there cannot exist $C' \in (\underline{C}, \overline{C})$ such that $\lim_{C \to C'} P''(C) = -\infty$. As such, there cannot exist $C' \in (\underline{C}, \overline{C})$ such that $\lim_{C \to C'} \beta(C) = 1$. Thus, $\beta(C) < 1$ for $C > \underline{C}^S = \underline{C}$ with $\lim_{C \to \underline{C}^S} \beta(C) = 1$. Thus, indeed $\underline{C} = \underline{C}^S$ is the tightest lower boundary in the state space (conditional on survival) in that C there exists no value $C' > \underline{C}^S$ such that $C_t \geq C'$ with certainty. □

D.3.2 Part III.B — Solving for (24)

Next, we derive an expression for $\underline{C}^S$. As shown in Lemma 3, we have $\mu_C(\underline{C}) = P(\underline{C}) = 0$ and $\beta(\underline{C}) = 1$ for $\underline{C} = \underline{C}^S$. To derive an expression for $\underline{C}^S$, one first uses (12) to calculate the drift of excess liquidity under the optimal choice of Y derived in the previous section (that is, $Y(C) = \max\{-C, 0\}$):

$$\mu_C(C) = \mu + \left(r - \lambda \mathbb{1}_{\{C \geq 0\}}\right) C - \frac{\rho r}{2} \sigma^2 \beta(C)^2 + \pi \alpha(C), \tag{D.9}$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function, i.e., it is 1 if $\{\cdot\}$ is true and 0 otherwise. The HJB equation (16) evaluated under the optimal controls $\alpha(C)$ and $\beta(C)$ as well as $C^* = \overline{C}$ can be rewritten as

$$rP(C) = P'(C)\mu_C(C) + \frac{P''(C)(\sigma_C(C))^2}{2},$$

which is (D.8). Due to $P(\underline{C}) = 0$, we have therefore

$$\alpha(\underline{C}) = P(\overline{C}) - [\overline{C} - \underline{C}] = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C} + \underline{C}. \tag{D.10}$$

The last equality uses that at the payout boundary $\overline{C} > 0$, the HJB equation (16) implies

$$P(\overline{C}) = \frac{\mu}{r} + \overline{C} - \frac{\lambda \overline{C}}{r},$$

due to $\beta(\overline{C}) = \alpha(\overline{C}) = P'(\overline{C}) - 1 = P''(\overline{C}) = 0$.

Substituting in for the optimal policies, and using $\alpha(\underline{C})$ from above in $\mu_C(\underline{C}) = 0$ while using that $\sigma_C(\underline{C}) = 0 \iff \beta(\underline{C}) = 1$, we have

$$0 = \mu_C(\underline{C}) = \mu + r\underline{C} - \frac{\rho r}{2}\sigma^2 + \pi\left(\frac{\mu}{r} - \frac{\lambda}{r}\overline{C} + \underline{C}\right). \tag{D.11}$$

Solving this linear equation for $\underline{C}$ we have (24) as desired.

D.3.3 Part III.C — Final Arguments and Proof of (25)

We now determine under what circumstances the liquidation or survival (i.e., the firm never liquidates) scenario applies. We distinguish two different cases, i) $\underline{C}^S \leq 0$ and ii) $\underline{C}^S > 0$.

Suppose that $\underline{C}^S \leq 0$. We conjecture and verify that the survival scenario prevails (so that $\tau = \infty$). Then, Lemma 3 states that conditional on survival, $\underline{C} = \underline{C}^S$ is the lower boundary in the state space and $\beta(C) < 1$ for $C > \underline{C}^S$. Note that $Y(C) = \max\{0, -C\}$ implies $Y \leq -\underline{C}^S$. Due to $P'(C) > 1$ for $C < \overline{C}$ and $P(\underline{C}^S) = 0$, it follows that $P(C) > C - \underline{C}^S$. If in state $C > \underline{C}^S$ the firm is liquidated and all cash holdings $M(C)$ are paid out (to shareholders or intermediary), total firm value "just before" liquidation is the cash balance $M(C)$. Note that $M(C) = C + Y(C) \leq C - \underline{C}^S < P(C) + Y(C)$, where the first inequality used that $Y \leq -\underline{C}^S$ and the second that $P(C) > C - \underline{C}^S$. As a result, liquidation is not optimal and therefore the survival scenario prevails in optimum. Thus, the lower boundary is $\underline{C} = \underline{C}^S$, the boundary condition is $P(\underline{C}) = 0$, and it holds that $\beta(\underline{C}) = 1 > \beta(C)$ for $C > \underline{C}$, leading to $\mu_C(\underline{C}) = \sigma_C(\underline{C}) = 0$.

Suppose that $\underline{C}^S > 0$. It follows that $Y(C) = \max\{0, -C\} = 0$ for $C \geq \underline{C}^S$, and $M(C) = C$. Conditional on survival, the boundary condition $P(\underline{C}^S) = 0$ applies. However, survival cannot be optimal for shareholders. Liquidating the firm at $C = \underline{C}^S$ and paying out $M(\underline{C}^S) = \underline{C}^S > 0$ dollars as dividends yield value $\underline{C}^S > 0$ for shareholders. As such, the liquidation scenario then prevails. It remains to show that liquidation occurs the first time C falls to zero so that $\underline{C} = 0$. To start with, note that liquidation at $C < 0$ is not possible because at the time of liquidation, $Y(C) = 0$ must hold, and $C < 0$ would imply $Y(C) > 0$. More in detail, suppose to the contrary liquidation occurs at some value $C < 0$, so $M(C) = 0$ and $Y = Y(C) = -C > 0$. The fact that the firm holds no cash upon liquidation also precludes any positive transfers to the intermediary upon liquidation. Now, $Y(C) > 0$ implies that upon liquidation in state $C = 0$, promise-keeping (i.e., the requirement that $Y = 0$ at the time of liquidation) is violated, a contradiction. Thus, liquidation can only occur in states $C \geq 0$.

Next, suppose that the firm is liquidated at $C = 0$, so that $P(0) = 0$. Note that $P'(C) > 1$ for $C < \overline{C}$ implies $P(C) > C = M(C)$ for $C > 0$ (clearly, the payout boundary is positive). If the firm is liquidated in state $C > 0$ and all cash is paid out as dividends, then shareholders receive $M(C) = C < P(C)$, so that liquidation at $C > 0$ is not optimal. As liquidation must occur for $C \geq 0$, it follows that optimal liquidation occurs at $C = \underline{C} = 0$, i.e., $\tau = \inf\{t \geq 0 : C_t = 0\}$.

At liquidation at $C = 0$, $P(0) = 0$. As $P(C)$ is by Assumption 1 twice continuously differentiable on $(\underline{C}, \overline{C})$, there cannot exist $C' > 0$ such that $\lim_{C \to C'} P''(C) = -\infty$ and $\lim_{C \to C'} \beta(C) = 1$. That is, $\beta(C) < 1$ for all $C > 0$.

Taken together, the lower boundary and the associated value of equity are given by

$$\underline{C} = \min\left\{\underline{C}^S, 0\right\} \quad \text{with} \quad P\left(\underline{C}\right) = 0, \tag{D.12}$$

which is (25) as desired.

D.3.4 Part IV — Proof that $\beta(C)$ decreases with C

Finally, we use (23) to prove that $\beta(C)$ decreases with C on $(\underline{C}, \overline{C})$. Suppose that $P''(C)$ is differentiable; then, $\beta(C)$ is also differentiable with respect to C. Differentiating both sides of (23) with respect to C and rearranging, we obtain

$$\begin{aligned} &P'(C)\left(\lambda \mathbb{1}_{\{C \geq 0\}} - \pi\alpha'(C) + \frac{\rho r \sigma^2 P'(C)}{2}\beta'(C)\right) \\ &= P''(C)\left[\mu + \left(r - \lambda \mathbb{1}_{\{C \geq 0\}}\right) C - \frac{\rho r}{2}\beta\left(C\right)\sigma^2 + \pi\alpha(C)\right]. \end{aligned} \tag{D.13}$$

As $P(C) > 0$ and $P'(C) \geq 1$ for $C > \underline{C}$, (23) implies

$$\left[\mu + \left(r - \lambda \mathbb{1}_{\{C \geq 0\}}\right) C - \frac{\rho r}{2}\beta\left(C\right)\sigma^2 + \pi\alpha(C)\right] > 0.$$

As such, the second line of (D.13) is negative, and so is the first line. Since $\alpha'(C) < 0$, it follows that $\beta'(C) < 0$ on $(\underline{C}, \overline{C})$.

Since the points C at which $P''(C)$ is not differentiable is countable and not dense in the state space, it follows, that $\beta(C)$ decreases with C for all $C \in (\underline{C}, \overline{C})$.

D.3.5 Part V — Optimal transfer process

We postulate that the dynamics of the intermediary's cumulative transfers I are

$$dI_t = \mu_I(C_t)dt + \sigma_I(C_t)dZ_t + \alpha_I(C_t)d\Pi_t + \xi_I dDiv, \tag{D.14}$$

with endogenous drift $\mu_I = \mu_I(C)$, volatility $\sigma_I = \sigma_I(C)$, and sensitivity to refinancing and dividend payouts $\alpha_I = \alpha_I(C)$ and ξ_I, all as functions of the state variable C. We omit time subscripts henceforth.

Having characterized the optimal control variables (Y, C^*, α, β) for each state C, we can now characterize the transfer process dI_t in (27). For $C > 0$, we optimally have $Y = 0$ which implies $dY = 0$. Thus, by the expression for dY in (6), we can solve for

$$dI = \left[\frac{\rho r}{2}(\beta\sigma)^2 - \pi\alpha\right]dt + \beta\sigma dZ + \alpha d\Pi \quad \text{for} \quad C > 0. \tag{D.15}$$

Note that $\sigma_I = \sigma\beta$, exactly canceling the volatilityh in dY. This is intuitive, as there is no change in deferred payouts $Y = 0$.

For $C < 0$, we optimally have zero cash, i.e., $M = C + Y = 0$. Absent refinancing, $dM = dC + dY = 0$ with $dM = 0$, while upon a refinancing opportunity $C^*(C) + Y^*(C) = \overline{C} > 0$, which implies

$$dI = \mu dt + \sigma dZ + (\alpha + Y)\, d\Pi \ \text{ for } \ C < 0. \tag{D.16}$$

In words, on $C < 0$ the intermediary completely absorbs any cash flow shocks $dX_t = \mu dt + \sigma dZ_t$, while gaining $\alpha_t + Y_t$ upon refinancing. Note that on $C \in (\underline{C}, 0)$, we have $\beta(C) < 1$ and therefore $\beta\sigma < \sigma = \sigma_I$, i.e., the volatility of dI does not completely eliminate the volatility in dY as part of the continuation value is delivered via deferred payouts, i.e., changes in Y. Combining these results yields (27), which concludes the argument.

E Derivations and Solution Steps for Section 5.1

We now solve the model variant with endogenous intermediary effort a_t. For simplicity, we do not distinguish between actual effort levels and effort levels anticipated by outside investors, and simply write a_t for the optimal effort.

E.1 State Variables

In this model variant, the intermediary's continuation value reads

$$Y_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}\left(dI_s - k_s ds - \frac{\kappa a_s^2}{2} ds\right)\right]. \tag{E.1}$$

By the martingale representation theorem, there exists endogenous processes α and β such that

$$dY_t = \left[rY_t + k_t - \frac{\kappa a_t^2}{2}\right]dt + \beta_t(dX_t - \mu - a_t)dt + \alpha_t(d\Pi_t - \pi dt). \tag{E.2}$$

Standard arguments imply that the intermediary chooses at each time t its effort $a_t \geq 0$ to solve

$$\max_{a_t}\left(\beta_t a_t - \frac{\kappa a_t^2}{2}\right),$$

leading to $\beta_t = a_t\kappa$ and the incentive condition (34). As in the baseline, we set k_t according to (7).

Next, the firm's cash balance evolves according to (2), i.e.,

$$dM_t = dX_t + (r - \lambda) M_t dt - dDiv_t - dI_t + \Delta M_t d\Pi_t.$$

Excess liquidity has then the law of motion

$$dC_t = dM_t - dY_t = \left[\mu + a_t + (r - \lambda) C_t - \lambda Y_t - \frac{\rho r}{2} (\beta_t \sigma)^2 - \frac{\kappa a_t^2}{2} + \pi \alpha_t\right] dt$$
$$+ \sigma (1 - \beta_t) dZ_t + (C_t^* - C_t) d\Pi_t - dDiv_t, \quad \text{(E.3)}$$

where we define the post-refinancing level of excess liquidity as $C_t^* \equiv \Delta M_t + C_t - \alpha_t$.

Finally, we determine the intermediary's autarky value, Y^A. In autarky $\beta_t = 1$ so that $a_t = 1/\kappa$. As such,

$$Y^A = \frac{\mu}{r} - \frac{\rho \sigma^2}{2} + \frac{1}{2\kappa r}. \quad \text{(E.4)}$$

E.2 HJB Equation and Optimization

As in the baseline, C_t is the only state variable, and dividend payouts $dDiv_t$ cause C_t to reflect at the endogenous upper boundary $\overline{C}$ satisfying smooth pasting and super contact conditions (Bolton et al., 2011):

$$P'(\overline{C}) - 1 = P''(\overline{C}) = 0.$$

The state space is bounded from below by $\underline{C}$ which we classify later in more detail.

In the interior of the state space (i.e., for $C \in (\underline{C}, \overline{C})$ the value function $P(C)$ solves the following HJB equation:

$$rP(C) = \max_{\beta, Y} \left\{ P'(C) \left[\mu + \frac{2\beta - \beta^2}{2\kappa} + (r - \lambda) C - \lambda Y - \frac{\rho r}{2} (\beta \sigma)^2\right] + P''(C) \frac{\sigma^2}{2} (1 - \beta)^2 \right\}$$
$$+ \pi \max_{C^*, \alpha \in \mathcal{S}(C^*, C)} \left\{ P'(C) \alpha + [P(C^*) - P(C) - (C^* - C + \alpha)] \right\}, \quad \text{(E.5)}$$

where we already inserted the incentive condition (34), that is, $a_t = \beta_t / \kappa$. We now solve the optimization in (E.5) for the optimal controls.

First, we determine optimal Y which must exceed $\max\{0, -C\}$. As in the baseline, it follows that so that

$$Y(C) = \max\{-C, 0\} \ \text{ and } \ M(C) = \max\{C, 0\},$$

which is (18).

Second, consider the refinancing target C^*. he first order condition (FOC) with respect to to the refinancing target C^* yields

$$P'(C^*(C)) = 1,$$

so that refinancing occurs up until a point at which the internal value of cash is equalized with the value of paying it out. Note that $P'(\overline{C}) = 1$ at the dividend payout boundary $\overline{C}$ and $P'(C) > 1$ for $C < \overline{C}$, so that $C^* = \overline{C}$.

Third, due to $P'(C) \geq 0$, it follows from (E.5) that the optimal choice of $\alpha = \alpha(C)$ is constrained by the shareholders' limited commitment constraint (15), so that

$$\alpha(C) = P(\overline{C}) - \overline{C} - [P(C) - C],$$

which is (21). Due to concavity of the value function, $\alpha(C)$ decreases with C.

Fourth, the first order condition in (E.5) with respect to β reads

$$\frac{1-\beta}{\kappa} - \rho r \sigma^2 P'(C)\beta - P''(C)\sigma^2(1-\beta) = 0,$$

which can be solved for

$$\beta = \beta(C) = \frac{1-\kappa\sigma^2 P''(C)}{1+\rho r\kappa\sigma^2 P'(C) - \kappa\sigma^2 P''(C)}. \tag{E.6}$$

Notice that the expression for $\beta(C)$ in (E.6) becomes (22) in the limit $\kappa \to \infty$.

Finally, notice that at the payout boundary $\overline{C}$, we have $P''(\overline{C})$ and $P'(\overline{C}) = 1$ so that

$$\beta(\overline{C}) = \frac{1}{1+\rho r\kappa\sigma^2} \quad \text{and} \quad a(\overline{C}) = \frac{1}{\kappa + \rho r\kappa^2\sigma^2},$$

so that $a(\overline{C})$ is smaller than the autarky effort $1/\kappa$.

E.3 Lower Boundary $\underline{C}$

We determine the lower boundary in the state space $\underline{C}$ using arguments analogous to the ones from the main text. To begin with, suppose that $\underline{C} < 0$ in which case the firm is never liquidated. As such, the following conditions are satisfied.

First, it must be that $\beta(\underline{C}) = 1$ so that $a(\underline{C}) = \frac{1}{\kappa}$. Second, the drift of C in (E.3), denoted $\mu_C(C)$, must be zero, i.e., $\mu_C(\underline{C}) = 0$. Third, $P(\underline{C}) = 0$.

As a result, using $P(\underline{C}) = 0$ and (21), we obtain

$$\alpha(\underline{C}) = P\left(\overline{C}\right) - \overline{C} + \underline{C}.$$

Inserting this expression for $\alpha(\underline{C})$, $\beta(\underline{C}) = 1$, $Y(\underline{C}) = -\underline{C}$ as well as $a(\underline{C}) = 1/\kappa$ into (E.3), we obtain

$$\mu_C(\underline{C}) = \mu + \frac{1}{2\kappa} - \frac{\rho r\sigma^2}{2} + r\underline{C} + \pi\big(P(\overline{C}) - \overline{C} + \underline{C}\big).$$

As such, we can readily solve $\mu_C(\underline{C}) = 0$ for

$$\underline{C}^S = -\left(\frac{r}{r+\pi}Y^A + \frac{\pi}{r+\pi}[P(\overline{C}) - \overline{C}]\right),$$

which is (24) form the baseline, but of course with a different Y^A.

As $\underline{C} \leq 0$, it follows that

$$\underline{C} = \min\left\{\left(\frac{r}{r+\pi}Y^A + \frac{\pi}{r+\pi}[P(\overline{C}) - \overline{C}]\right), 0\right\}.$$

When $\underline{C} = 0$, the firm is liquidated at $C = \underline{C}$; when $\underline{C} < 0$, the firm is never liquidated.

Taking stock, to solve for the optimal contract, one needs to solve the HJB equation (E.5) subject to the following boundary conditions:

$$P''(\overline{C}) = P'(\overline{C}) - 1 = P(\underline{C}) = 0.$$

Internet Appendix

F Details of the PE Implementation

This Section provides a detailed derivation of the implementation in Section 4.

F.1 Auxiliary function: Cumulative transfers since refinancing

To begin, we recall (27) characterizes that optimal transfers to the intermediary, and we introduce the auxiliary function $T(C)$ that records cumulative transfers from the intermediary to the firm in response to Brownian cash flow shocks dZ since the last C has either reached $\underline{C}$ or $\overline{C}$ (which is reached, for instance, upon refinancing). That is, $T(C)$ is defined on the interval $(\underline{C}, \overline{C})$. Unless otherwise mentioned, we consider below that $C \in (\underline{C}, \overline{C}$; we discuss the details prevailing for $C = \underline{C}$ at the end.

By Ito's Lemma, calculate

$$\begin{aligned} dT(C) =& \mu_T(C)\,dt + \sigma_T(C)\,dZ - \alpha_T(C)\,d\Pi \\ =& \left[T'(C)\,\mu_C(C) + \frac{1}{2}T''(C)\,\sigma_C^2(C)\right]dt + T'(C)\,\sigma_C(C)\,dZ + \left[T\left(\overline{C}\right) - T(C)\right]d\Pi, \end{aligned} \quad \text{(F.1)}$$

and so obtain drift $\mu_T(C)$, volatility $\sigma_T(C)$, and loading on $d\Pi$, $\alpha_T(C)$, whereby $\mu_C(C)$ is the drift and $\sigma_C(C) = (1-\beta(C))\sigma$ is the volatility of excess liquidity in (12).[27] As the process has to reset upon refinancing due to its Markovian nature, we impose $T\left(\overline{C}\right) = 0$. Note $T(C_t)$ is subtly different from $-I_t$, as I_t turns out to be non-Markovian.[28]

For $T(C)$ to record cumulative contributions of the intermediary in response to Brownian cash flow shocks dZ, its volatility $\sigma_T(C)$ loading must match $-\sigma_I(C)$, the negative of the volatility of transfers from the firm to the intermediary. In other words, total contributions increase (decrease) one-for-one with transfers from (to) the intermediary caused by cash flow shocks dZ. Matching volatilities, and using (27) — that is, $\sigma_I(C) = \sigma\beta(C) + \sigma(1-\beta(C))\mathbb{1}_{\{C<0\}}$ — and $\sigma_T(C) = T'(C)\sigma(1-\beta(C))$ (by Itô's Lemma), we have for $C \in (\underline{C}, \overline{C})$

$$\sigma_T(C) + \sigma_I(C) = 0 \iff T'(C) = -\frac{\beta(C)}{1-\beta(C)} - \mathbb{1}_{\{C<0\}}, \quad \text{(F.3)}$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function which equals one if $\{\cdot\}$ is true and zero otherwise.

Plugging in $\beta(C)$ from (22), integrating, and imposing $T\left(\overline{C}\right) = 0$, we have for $C \in (\underline{C}, \overline{C})$[29]

$$T(C) = \frac{\ln P'(C)}{\rho r} + Y(C) = \alpha_U(C) + Y(C). \quad \text{(F.4)}$$

where we defined $\alpha_U(C) \equiv \frac{\ln P'(C)}{\rho r}$ as the balance of the credit line. Due to the non-Markovian

[27] An expression for the drift of excess liquidity under the optimal controls is

$$\mu_C(C) = \mu + \left(r - \lambda\mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\sigma^2\beta(C)^2 + \pi\alpha \quad \text{(F.2)}$$

where it is imposed that $Y(C) = \max\{-C, 0\}$.

[28] For example, past refinancing events are recorded in I_t, but not in $T(C_t)$.

[29] In more detail, $\frac{1}{\rho r}\frac{P''(C)}{P'(C)} = -\frac{\beta(C)}{1-\beta(C)} = \frac{d}{dC}\frac{\ln P'(C)}{\rho r}$.

nature of I, there is no guarantee that imposing $\sigma_T(C) + \sigma_I(C) = 0$ implies matching drifts, i.e., $\mu_T(C) + \mu_I(C) = 0$. Noting that $T''(C) = -\frac{\beta'(C)}{[1-\beta(C)]^2}$, after deriving $\beta'(C)$ and some algebra, we show in Internet Appendix F.2 below:

$$\mu_T(C) + \mu_I(C) = \frac{\lambda}{\rho r}\mathbb{1}_{\{C>0\}} + rY + \pi\left\{\left[\frac{e^{\rho r \alpha_U(C)} - 1}{\rho r} - \alpha_U(C)\right] + [\alpha_U(C) - \alpha(C)]\right\}. \tag{F.5}$$

The right-hand-side of (F.5) is made up of three terms: The first term reflects a constant drift part linked to the carry-cost-of-cash, λ. The second term reflects the cost of delaying the payout of the equity stake, which is linked to the discount rate of the intermediary. The third term reflects the additional compensation required to the intermediary from refinancing is related to (i) the level of the balance $\alpha_U(C)$, $\left[\frac{e^{\rho r\alpha_U(C)}-1}{\rho r} - \alpha_U(C)\right] \geq 0$, which features cost of volatility as the level correlates with volatility transfers, and (ii) the possible restriction on α by shareholders' limited commitment constraint (15) if $[\alpha_U(C) - \alpha(C)] \geq 0$. Interest rates and other fees have to absorb this difference in any Markov implementation.

As the firm always refinances to the payout boundary $\overline{C}$, and $T(\overline{C}) = 0$ by definition, the loading on $d\Pi$ is simply given by

$$\alpha_T(C) = T(C). \tag{F.6}$$

F.2 Derivation of (F.5)

We start with an auxiliary Lemma deriving the slope of $\beta(C)$.

Lemma 4. *Suppose that $\alpha(C)$ and $\beta(C)$ are differentiable in state C. Then:*

$$-\frac{\sigma^2}{2}\beta'(C) = \frac{\beta(C)}{1-\beta(C)}\mu_C(C) + \frac{\lambda\mathbb{1}_{\{C\geq 0\}} - \pi\cdot\alpha'(C)}{\rho r} - \beta(C)^2\frac{\sigma^2}{2}\rho r \tag{F.7}$$

Proof of Lemma 4. To start with, note that from (22) we have $\frac{1}{\rho r}\frac{P''(C)}{P'(C)} = -\frac{\beta(C)}{1-\beta(C)}$.

To derive the postulated expression for $\beta'(C)$, we first differentiate both sides of the ODE (23) with respect to C:

$$\begin{aligned} rP'(C) = & P''(C)\left[\mu + \left(r - \lambda\mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\sigma^2}{2}\rho r\beta(C) + \pi\cdot\alpha(C)\right] \\ & + P'(C)\left[\left(r - \lambda\mathbb{1}_{\{C\geq 0\}}\right) - \frac{\sigma^2}{2}\rho r\beta'(C) + \pi\cdot\alpha'(C)\right] \end{aligned}$$

Rearranging, we have

$$P'(C)\left[\lambda\mathbb{1}_{\{C\geq 0\}} + \frac{\sigma^2}{2}\rho r\beta'(C) - \pi\cdot\alpha'(C)\right] = P''(C)\left[\mu + \left(r - \lambda\mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\sigma^2}{2}\rho r\beta(C) + \pi\cdot\alpha(C)\right]$$

Dividing through by $\rho r P'(C)$ and solving for $\frac{\sigma^2}{2}\beta'(C)$, we have

$$
\begin{aligned}
-\frac{\sigma^2}{2}\beta'(C) =& \frac{\beta(C)}{1-\beta(C)}\left[\mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\sigma^2}{2}\rho r \beta(C) + \pi\cdot\alpha(C)\right] + \frac{\lambda \mathbb{1}_{\{C\geq 0\}} - \pi\cdot\alpha'(C)}{\rho r} \\
=& \frac{\beta(C)}{1-\beta(C)}\left[\mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\sigma^2}{2}\rho r \beta^2(C) + \pi\cdot\alpha(C)\right] + \frac{\lambda \mathbb{1}_{\{C\geq 0\}} - \pi\cdot\alpha'(C)}{\rho r} \\
&+ \frac{\beta(C)}{1-\beta(C)}\left[\frac{\sigma^2}{2}\rho r\beta^2(C) - \frac{\sigma^2}{2}\rho r \beta(C)\right] \qquad (F.8)\\
=& \frac{\beta(C)}{1-\beta(C)}\mu_C(C) + \frac{\lambda \mathbb{1}_{\{C\geq 0\}} - \pi\cdot\alpha'(C)}{\rho r} - \beta(C)^2\frac{\sigma^2}{2}\rho r,
\end{aligned}
$$

where the first equality uses $\frac{1}{\rho r}\frac{P''(C)}{P'(C)} = -\frac{\beta(C)}{1-\beta(C)}$ and the third equality uses the expression for the drift of excess liquidity $\mu_C(C)$ in (B.6). □

Next, recall (F.4), that is,

$$
T(C) = \frac{\ln P'(C)}{\rho r} + Y(C) = \alpha_U(C) + Y(C),
$$

and, from (27) and (B.6),

$$
\begin{aligned}
\mu_I = \mu_I(C) &= \mu + [(r-\lambda)C - \mu_C]\,\mathbb{1}_{\{C\geq 0\}} \\
\mu_C = \mu_C(C) &= \mu + \left(r - \lambda\mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta^2(C)\sigma^2 + \pi\alpha.
\end{aligned}
$$

Thus, we can calculate

$$
\begin{aligned}
\mu_T(C) &= T'(C)\mu_C + T''(C)\frac{\sigma_C^2(C)}{2} \\
&= T'(C)\mu_C - \frac{\beta'(C)}{[1-\beta(C)]^2}\frac{\sigma^2}{2}[1-\beta(C)]^2 \\
&= T'(C)\mu_C - \beta'(C)\frac{\sigma^2}{2},
\end{aligned}
$$

where the second equality uses $T''(C) = -\frac{\beta'(C)}{[1-\beta(C)]^2}$. Differentiating the HJB w.r.t. C, and solving for $\beta'(C)$, from (F.7) we have

$$
-\frac{\sigma^2}{2}\beta'(C) = \frac{\beta(C)}{1-\beta(C)}\mu_C(C) + \frac{\lambda\mathbb{1}_{\{C\geq 0\}} - \pi\cdot\alpha'(C)}{\rho r} - \frac{\rho r}{2}\beta(C)^2\sigma^2
$$

Plugging in for $\mu_I(C) = \mu + [(r-\lambda)C - \mu_C]\,\mathbb{1}_{\{C\geq 0\}}$, we have

$$
\mu_T(C) + \mu_I(C) = rY - \frac{\lambda}{\rho r}\mathbb{1}_{\{C<0\}} + \frac{\lambda}{\rho r} + \pi\left[-\frac{\alpha'(C)}{\rho r} - \alpha(C)\right],
$$

Finally, plugging in for $\alpha'(C) = -[P'(C) - 1]$ and using $P'(C) = e^{\rho r \alpha_U(C)}$, we have the result, that is, (F.2).

F.3 Credit line

As the limited commitment constraint (15) is always binding, the payoff to the intermediary upon refinancing is given by $\alpha_I(C) = \alpha(C) + Y(C)$ which for all of our numerical results is less than $T(C)$ for $C < \overline{C}$. In words, the contract specifies that the intermediary optimally demands payment of *less* than its total cumulative transfers since the last refinancing. This feature is due to incentives: demanding the full repayment $T(C) = \alpha_U(C) + Y(C)$ violates limited commitment (15) when $\alpha(C) < \alpha_U(C)$, and thus would be vetoed by the shareholders. As discussed above, we will take the credit line balance to be $D(C) = \alpha_U(C)$, and thus require "early repayment incentives" of amount $\alpha_U(C) - \alpha(C) > 0$, while $Y(C)$ is separately generated via restricted equity.

Any implementation via a general credit line balance of $D(C)$ together with a portfolio of other *non-interest* bearing instruments — in our interpretation restricted equity $Y(C)$ — by construction must have volatility matching $T(C)$. However, the drifts of I and T do not cancel due to I not being Markovian. To balance the drifts, i.e., match the dt dynamics, we introduce the Markovian interest rate $r_D(C)$ on the balance $D(C)$, as well as a constant maintenance fee f so that

$$\mu_T(C_t) + \mu_I(C_t) = r_D(C_t) D(C_t) + f. \tag{F.9}$$

We set f to absorb any constant payouts at $C = \overline{C}$, and let the interest rate $r_D(C)$ capture the remaining variable difference.[30] At the payout boundary we have $\alpha_U(\overline{C}) = \alpha(\overline{C}) = Y(\overline{C}) = 0$, so

$$f = \frac{\lambda}{\rho r} \quad \text{and} \quad r_D(C) = \frac{rY(C) - \frac{\lambda}{\rho r}\mathbb{1}_{\{C<0\}}}{D(C)} + \pi \frac{\left[\frac{e^{\rho r \alpha_U(C)} - 1}{\rho r} - \alpha_U(C)\right] + [\alpha_U(C) - \alpha(C)]}{D(C)}. \tag{F.10}$$

First, when liquidation is possible, i.e., $\underline{C} = \underline{C}^L = 0$, we have $r_D(C) = 0$, only maintenance fee payments and variable payments based on $\alpha_U(C)$ and $\alpha(C)$ are required. Next, there is jump at $C = 0$ as the equity stake $Y(C)$ enters the picture and the inefficiency of internal cash-holding λ disappears.[31] We see that the last term of $r_D(C)$ reflects the possible limited commitment restriction on α, as it records the required compensation for the early repayment incentive of $\alpha_U(C) - \alpha(C) \geq 0$ upon refinancing.

F.4 Restricted equity

Recall that we argued in Section 3.1 that $Y(C)$ can naturally be understood as an equity stake. Let us construct the terms of this stake. For an equity stake to be used, we need $\underline{C} < 0$. Suppose that the firm allows the intermediary to trade equity internally at the given price schedule $P_I(C)$, but only allows equity shares to trade on the open market at refinancing opportunities. In essence, we will interpret the intermediary's equity shares as restricted equity that vests upon refinancing. The intermediary holds $g_E(C)$ restricted equity shares that if sold at post-issuance share-price $P_E(C)$ achieve the payout of $Y(C)$ upon financial market access:

$$Y(C) = g_E(C) P_E(C) \iff g_E(C) = \frac{Y(C)}{P(C)}, \tag{F.11}$$

[30] Note that $\mu_T(C)$ summarizes all non-interest movements in $T(C)$, which by assumption our portfolio of instruments, here credit line and restricted equity, replicates.

[31] By L'Hopital's rule, the interest rate on the credit line at the payout boundary is positive, and given by $r_D(\overline{C}) = \pi$ as $\alpha''(\overline{C}) = 0$ but $\alpha_U''(\overline{C}) \neq 0$. This is intuitive – the intermediary is only compensated for the arrival rate of the loss $\alpha_U(C) - \alpha(C)$, π.

where we used $P_E(C) = P(C)$ due to shareholders' limited commitment.[32] Further, the flows of the portfolio of credit line and restricted equity, absent fees and interest, has to match the cumulative transfers, $T(C)$. As $T(C) = \alpha_U(C) + Y(C)$, and the credit line covers $D(C) = \alpha_U(C)$, the value $Y(C)$ has to be generated by the intermediaries' "trading" gains and losses of restricted equity, that is

$$Y(C) = \int_C^0 \left(-g_E'(x)\right) P_I(x)\, dx. \tag{F.12}$$

Setting both expressions for $Y(C)$ equal and differentiating pins down $P_I(C)$:

$$\left(g_E(C)\, P(C)\right)' = -1 = g_E'(C)\, P_I(C) \iff P_I(C) = -\frac{1}{g_E'(C)}. \tag{F.13}$$

Further, writing out $\left(g_E(C)\, P_E(C)\right)'$ and dividing through by $g_E'(C)$, we see that

$$P_I(C) = P(C) - \left(\frac{g_E(C)}{-g_E'(C)}\right) P'(C) < P(C). \tag{F.14}$$

Because $g_E'(C) < 0$ and $P'(C) > 0$, the internal price is always below the external price, resulting in strict incentives for the intermediary to sell its shares on the open market upon refinancing.

F.5 Refinancing via common equity issuance

Next, we investigate the details of the refinancing. First, we normalize the current number of outstanding shares to unity and let g be the number of new shares issued upon refinancing.[33] Then, the post-issuance equity price is given by $P_E(C) = \frac{P(\overline{C})}{1+g}$. As the proceeds from g must cover the total cash needed to replenish the firm's cash holdings, $\overline{M} - M(C)$, as well as the transfers to the intermediary, $\alpha_I(C) = \alpha(C) + Y(C)$, we have

$$\frac{P\left(\overline{C}\right)}{1+g} g = \overline{M} - M(C) + \alpha(C) + Y(C) \iff g(C) = \frac{\overline{C} - C + \alpha(C)}{P\left(\overline{C}\right) - \left[\overline{C} - C + \alpha(C)\right]}, \tag{F.15}$$

where we used the fact that $\overline{M} = \overline{C}$ and the definition $C = M - Y$. Due to shareholders' limited commitment and its implied constraint on α, (15), the denominator is non-negative (strictly so unless $C = \underline{C} < 0$), and we can always implement the optimal allocation via common equity issuance consistent with limited commitment.

The post-issuance price is given by

$$P_E(C) = P\left(\overline{C}\right) - \left[\overline{C} - C + \alpha(C)\right]. \tag{F.16}$$

Note that $P_E'(C) = 1 - \alpha'(C) \geq 0$, with strict inequality for $C < \overline{C}$. Further, as (15) is binding, the post issuance price corresponds to the pre-issuance price, so that

$$P_E(C) = P(C) \quad \text{and} \quad g(C) = \frac{P\left(\overline{C}\right) - P(C)}{P(C)}. \tag{F.17}$$

As C approaches the lower boundary $\underline{C}$, existing outside shareholders are completely diluted upon refinancing, i.e., $\lim_{C \to \underline{C}} g(C) = \infty$, while the cash amount raised stays finite. In our restricted

[32] See next subsection for details.

[33] Without the normalization, g can be interpreted as the required growth rate in outstanding shares.

equity implementation, the firm itself only issues $g(C) - g_E(C)$ new shares, while the intermediary sells its $g_E(C)$ vesting shares.

F.6 Implementation at the lower boundary $\underline{C}$

We now discuss the lower boundary $\underline{C}$. If $\underline{C} = 0$ and the firm is liquidated at $C = \underline{C}$, then $Y(C) = 0$ and the firm defaults on its credit line.

Next, consider survival, that is, $\underline{C} < 0$, and the firm is never liquidated. Note that because $\mu_C(\underline{C}) = \sigma_C(\underline{C}) = 0$ under survival, the state $C = \underline{C}$ is absorbing (if reached) until the next refinancing event in which case C jumps up to $\overline{C}$. Even though we do not provide a formal proof, we expect that $C = \underline{C}$ is an inaccessible state too in that $C_t > \underline{C}$ implies that $C_s > \underline{C}$ with probability one, provided $\underline{C} < 0$. As $\beta(\underline{C}) = 1$ or, more generally, $\lim_{C \to \underline{C}} \beta(C) = 1$, the ODE (F.3), characterizing the process $T(C)$, is not well-defined for $C = \underline{C}$. Loosely speaking, being in state $C = \underline{C}$, the credit line balance is potentially unbounded and no longer Markovian.

To deal with the possibility of $C = \underline{C}$, we assume that in the state $C = \underline{C}$, the change of credit line is set to zero and restricted equity pays "dividends" $dX_t = \mu dt + \sigma dZ_t$. That is, the intermediary assumes full ownership of the firm in the state $C = \underline{C}$, covers all cash flow shortfalls by injecting cash, and pays out all positive cash flows as dividends to herself. At the same time, the credit line balance remains constant until the next refinancing event. These assumptions ensure that the credit line balance and the dollar value of restricted equity are indeed Markovian. Again, we do not believe that these assumptions have any major implications.

G Model Variant with Risk Aversion

We now present a model variant with risk-averse intermediary which serves as a micro-foundation for the flow cost of intermediary financing k_t. Notably, we also allow shareholders to be risk-averse in a sense that they apply a stochastic discount factor when evaluating payoffs.

Specifically, the intermediary is risk-averse with CARA preferences with risk-aversion of $\rho > 0$, so that the intermediary's instantaneous utility of consumption c is

$$u(c) = -\frac{1}{\rho} \exp(-\rho c). \tag{G.1}$$

The intermediary can maintain savings on its own account, denoted by S_t. Savings accrue interest at rate r and are subject to changes induced by transfers to ($dI_t < 0$) and from ($dI_t > 0$) the firm and consumption c_t, so

$$dS_t = rS_t dt + dI_t - c_t dt. \tag{G.2}$$

Endowing the intermediary with the possibility to accumulate savings ensures that it can smooth its consumption beyond liquidation of the firm. Consumption c_t and savings balance S_t can both take positive and negative values. As such, the intermediary has essentially deep pockets, but capital provision by the intermediary is costly in a sense that the intermediary is risk-averse.

We normalize the balance of savings at $t = 0^-$ to zero, i.e., $S_{0^-} = 0$, where time $t = 0^-$ denotes the time before the payout agreement is written and any transfers are made. Savings must satisfy the standard transversality condition $\lim_{t \to \infty} \mathbb{E}\left[e^{-rt} S_t\right] = 0$, ruling out Ponzi schemes.

As in the main text, the firm's cash holdings evolve according to

$$dM_t = dX_t + (r - \lambda) M_t dt - dDiv_t - dI_t + \Delta M_t d\Pi_t. \tag{G.3}$$

Here, $dDiv_t \geq 0$ are the payouts to the shareholders, where the constraint $dDiv_t \geq 0$ reflects limited liability of shareholders.

G.1 Optimal Contracting Problem

We call the payout agreement $\mathcal{C}$ incentive compatible (IC) if it respects the intermediary's as well as the shareholders' limited commitment, and we restrict our attention to IC payout agreements. The firm's founders, i.e., its original shareholders, have full bargaining power, and can extract all surplus from the intermediary when signing the payout agreement with the intermediary at $t = 0$. Let U_0 be the intermediary's utility for a given contract $\mathcal{C}$, that is,

$$U_0 = \max_{c_t} \mathbb{E}\left[\int_0^\infty e^{-rt} u\left(c_t\right) dt\right] \quad \text{s.t.} \quad \text{(G.2)} \quad \text{and} \quad \lim_{t\to\infty} \mathbb{E}\left[e^{-rt} S_t\right] = 0. \tag{G.4}$$

Given $\mathcal{C}$, the intermediary chooses consumption c_t to maximize its lifetime utility, with optimal consumption denoted by c_t^*. The intermediary's outside option is to stay away from the contract and to consume out of its savings, yielding lifetime utility $\overline{U}$ and the participation constraint $U_0 \geq \overline{U}$.

The initial equity value, i.e., the original shareholders' value function P_{0^-} before initial financing at time $t = 0$, is the discounted stream of dividend payouts net the costs of refinancing:

$$P_{0^-} = \max_{\mathcal{C} \text{ is IC}} \mathbb{E}\left[\int_0^\tau \frac{\Lambda_s}{\Lambda_t}\big(dDiv_t - \Delta M_t d\Pi_t\big)\right] \quad \text{s.t.} \quad U_0 \geq \overline{U} \quad \text{and} \quad M_{0^-} = 0, \tag{G.5}$$

where Λ_t is shareholders' stochastic discount factor evolving according to

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - \eta dZ_t. \tag{G.6}$$

is shareholders' stochastic discount factor with "risk premium" $\eta \geq 0$. Thus, we account for shareholder risk aversion with respect to cash flow risks by assuming that shareholders evaluate payoffs using a stochastic discount factor. For simplicity, the stochastic discount factor has no loading on $d\Pi_t$, but one could easily introduce that witout altering our key findings. Note that while η is constant over time, shareholders' effective risk aversion induced by financial constraints — just like in the baseline — will be time-varying.

Note that upon refinancing, i.e., $d\Pi_t = 1$, the firm raises ΔM_t dollars from newly arriving competitive outside investors at fair value by issuing ΔM_t dollars worth of new equity. The firm's existing shareholders, in turn, are diluted as they must give up ΔM_t dollars worth of equity ownership. At inception at time $t = 0$, the firm's original shareholders are penniless and the firm's cash holdings are zero, but the firm can raise financing from newly arriving (competitive) outside investors as $d\Pi_0 = 1$.

To solve the model and derive the optimal payout agreement, we first derive the state variables and their dynamics. Tractability comes from collapsing the state-variables of the model to the one-dimensional difference between the the firm's cash holdings and the intermediary's future promised payouts, which we term excess liquidity. Finally, we derive the optimal payout agreement, and express all model quantities in terms of the single state variable, excess liquidity.

G.2 Intermediary Consumption Problem

In principle, shareholders' dynamic optimization has three state variables: The firm's cash holdings M_t, the intermediary's savings S_t, and the intermediary's continuation utility defined as

$$U_t \equiv \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(c_s^*) ds \right]. \tag{G.7}$$

Further, U_t can be expressed as W_t in certainty equivalent monetary terms, with

$$W_t \equiv \frac{-\ln(-\rho r U_t)}{\rho r}, \tag{G.8}$$

and we work with W_t instead of U_t. Note that W_t is the intermediary's total continuation payoff in monetary terms with law of motion

$$dW_t = \left[\frac{\rho r}{2} (\beta_t \sigma)^2 + \pi \left(\alpha_t - \frac{1 - e^{-\rho r \alpha_t}}{\rho r} \right) \right] dt + \beta_t \sigma dZ_t + \alpha_t (d\Pi_t - \pi dt), \tag{G.9}$$

where α_t and $\sigma \beta_t$ are the loadings of dW_t on the martingales $(d\Pi_t - \pi dt)$ and dZ_t respectively. The first term in the drift of (G.9) captures the intermediary's required compensation for being exposed to Brownian cash flow risk, while the second term captures the intermediary's required compensation for being exposed to shocks $d\Pi_t$. Both terms are unambiguously positive, so that W_t increases in expectation, $\mathbb{E}[dW_t] \geq 0$. We summarize our results so far in the following proposition.

Proposition 3. *The intermediary's optimal consumption satisfies $c_t^* = rW_t$. The intermediary's certainty equivalent payoff W_t, defined in (G.8), follows the dynamics (G.9). W_t is a sub-martingale and increases in expectation, $\mathbb{E}[dW_t] \geq 0$. The intermediary's outside option to stay away from the contract at $t = 0$ and to consume out of its savings yields lifetime utility $\overline{U} = -\frac{1}{\rho r}$. The intermediary's participation constraint $U_0 \geq \overline{U}$ is equivalent to $W_0 \geq 0$.*

Section G.2.1 and Section G.2.2 provide the proof of Proposition 3 in two parts (Part I analyzes the intermediary's optimal consumption and Part II derives the law of motion (G.9)). Without loss, the reader can proceed to Section G.3.

G.2.1 Proof of Proposition 3 — Optimal Consumption

We first state an auxiliary Lemma:

Lemma 5. *Take a process $\hat{I}$ and $s_1, s_2 \in \mathbb{R}$. Consider the problem*

$$U_t := U_t(c) = \max_{\{c_s\}_{s \geq t}} \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(c_s) ds \right] \tag{G.10}$$

$$\text{subject to } dS_s(c) = rS_s(c)ds + d\hat{I}_s - c_s ds, \; S_t(c) = s_1, \text{ and } \lim_{s \to \infty} \mathbb{E} e^{-r(s-t)} S_s(c) = 0,$$

where we explicitly denote the dependence of savings S on the consumption path c. Next, consider the problem

$$\tilde{U}_t := \tilde{U}_t(\tilde{c}) = \max_{\{\tilde{c}_s\}_{s \geq t}} \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(\tilde{c}_s) ds \right] \tag{G.11}$$

$$\text{subject to } d\tilde{S}_s(\tilde{c}) = r\tilde{S}_s(\tilde{c})ds + d\hat{I}_s - \tilde{c}_s ds, \; \tilde{S}_t(\tilde{c}) = s_2, \text{ and } \lim_{s \to \infty} \mathbb{E} e^{-r(s-t)} \tilde{S}_s(\tilde{c}) = 0.$$

Then, for $\Delta^S := s_2 - s_1$, the optimal consumption processes c and $\tilde{c}$, solving (G.10) *and* (G.11) *respectively, satisfy $\tilde{c}_t = c_t + r\Delta^S$ so that $\tilde{U}_t = e^{-\rho r \Delta^S} U_t$.*

Proof. To start with, note that with $\tilde{c}_s = c_s + r\Delta^S$,

$$\tilde{U}_t(\tilde{c}) = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)} u(c_s + r\Delta^S) ds\right] = e^{-\rho r \Delta^S}\mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)} u(c_s) ds\right] = e^{-\rho r \Delta^S} U_t(c), \tag{G.12}$$

where the first equality uses $\tilde{c}_s = c_s + r\Delta^S$ and the second equality uses

$$u(c_s + r\Delta^S) = -\frac{e^{-\rho(c_s + r\Delta^S)}}{\rho} = e^{-\rho r \Delta^S}\left(-\frac{e^{-\rho c_s}}{\rho}\right) = e^{-\rho r \Delta^S} u(c_s). \tag{G.13}$$

Next, suppose to the contrary that there exists a different consumption process $c' \neq \tilde{c}$, solving problem (G.11), with

$$\tilde{U}_t(c') > \tilde{U}_t(\tilde{c}) = e^{-\rho r \Delta^S} U_t(c), \tag{G.14}$$

and the transversality condition $\lim_{s\to\infty} \mathbb{E} e^{-r(s-t)}\tilde{S}_s(c') = 0$ holds under the consumption process c'. Define the consumption process c'' via $c''_t = c'_t - r\Delta^S$. As c' is different from $\tilde{c}$, it follows that c'' is different from c. As under the consumption path c' the transversality condition $\lim_{s\to\infty} \mathbb{E} e^{-r(s-t)}\tilde{S}_s(c') = 0$ holds, it follows that under the consumption path c'' the transversality condition $\lim_{s\to\infty} \mathbb{E} e^{-r(s-t)} S_s(c'') = 0$ holds too. In addition, note that the payoff under the consumption path c'' equals

$$U_t(c'') := \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)} u(c''_s) ds\right] = e^{\rho r \Delta^S}\tilde{U}_t(c') > e^{\rho r \Delta^S} e^{-\rho r \Delta^S} U_t(c) = U_t(c), \tag{G.15}$$

where the second equality applies (G.13), which yields $u(c'_s) = e^{-\rho r \Delta^S} u(c''_s)$ and $u(c''_s) = e^{\rho r \Delta^S} u(c'_s)$, and the inequality uses (G.14). However, $U_t(c'') > U_t(c)$ contradicts the fact that c solves problem (G.10). The assertion follows. □

Using Lemma 5, we can now complete the argument by showing that optimal consumption satisfies $u(c_t) = rU_t$ and $c_t = rW_t$. According to Lemma 5, the marginal value of an additional unit of savings S_t at time t for the intermediaryis given by

$$\left[\frac{\partial}{\partial \Delta^S} e^{-\rho r \Delta^S} U_t\right]|_{\Delta^S = 0} = -\rho r U_t. \tag{G.16}$$

The intermediary's optimal consumption smoothing implies that along the optimal path the first order condition

$$u'(c_t) = \left[\frac{\partial}{\partial \Delta^S} e^{-\rho r \Delta^S} U_t\right]|_{\Delta^S = 0} \tag{G.17}$$

has to hold at all times $t \geq 0$. That is, in optimum, the intermediary's marginal utility $u'(c_t)$ has to be equal to the marginal value of an additional unit of savings, $\left[\frac{\partial}{\partial \Delta^S} e^{-\rho r \Delta^S} U_t\right]|_{\Delta^S = 0}$.

Next, observe that $u'(c_t) = -\rho u(c_t)$ and use (G.16), so that (G.17) becomes $u(c_t) = rU_t$. Inverting the relation $u(c_t) = rU_t$ and solving for c_t yields $c_t = rW_t$, with

$$W_t := \frac{-\ln(-\rho r U_t)}{\rho r}, \tag{G.18}$$

which is (G.8).

Finally, we examine the intermediary's outside option $\bar{U}$ from staying away from the contract and perpetually consuming out of its savings which we normalize at inception to zero, that is, $S_{0^-} = 0$. When the intermediary stays away from the contract, its consumption is constant over time because the intermediary's discount rate is equal to the savings/borrowings rate r. As the intermediary's initial savings are normalized to zero, the intermediary then perpetually consumes zero to satisfy the transversality constraint, $\lim_{t\to\infty} \mathbb{E}e^{-rt}S_t = 0$. As such, when the intermediary stays away from the contract, then $c_t = rW_t = 0$ by our previous arguments, thus $W_t = W_0 = 0$. Moreover, using $u(0) = -1/\rho$ and evaluating the integral expression (G.4), we can solve $\bar{U} = \frac{-1}{\rho r}$. The participation constraint $U_0 \geq \bar{U}$ is therefore equivalent to $W_0 \geq 0$.

G.2.2 Proof of Proposition 3 Part II — Martingale Representation and (G.9)

Take the intermediary's continuation value

$$U_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}u(c_s)ds\right], \tag{G.19}$$

under any consumption process c_t (possibly, $c_t = c_t^*$). Define

$$A_t = \mathbb{E}_t\left[\int_0^\infty e^{-rs}u(c_s)ds\right] = \int_0^t e^{-rs}u(c_s)ds + e^{-rt}U_t. \tag{G.20}$$

By construction, $A = \{A_t\}$ is a martingale. By the martingale representation theorem, there exist stochastic processes $\hat{\alpha} = \{\hat{\alpha}_t\}$ and $\beta = \{\beta_t\}$ such that

$$e^{rt}dA_t = (-\rho rU_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho rU_t)\hat{\alpha}_t(d\Pi_t - \pi dt), \tag{G.21}$$

where $dZ_t = \frac{dX_t - \mu dt}{\sigma}$ is the increment of a standard Brownian Motion under the probability measure and $(d\Pi_t - \pi dt)$ is the increment of a compensated Poisson process which is a martingale.

We differentiate (G.20) with respect to time t to obtain an expression for dA_t, then plug this expression into (G.21) and solve (G.21) to get

$$dU_t = rU_t dt - u(c_t)dt + (-\rho rU_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho rU_t)\hat{\alpha}_t(d\Pi_t - \pi dt). \tag{G.22}$$

With the optimal consumption policy $c_t = c_t^*$, satisfying $u(c_t) = rU_t$, equation (G.22) simplifies to

$$dU_t = (-\rho rU_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho rU_t)\hat{\alpha}_t(d\Pi_t - \pi dt), \tag{G.23}$$

which is a martingale in that $\mathbb{E}[dU_t] = 0$.

Next, we derive the law of motion of

$$W_t = W(U_t) := \frac{-\ln(-\rho rU_t)}{\rho r}. \tag{G.24}$$

To do so, note that

$$W'(U) = \frac{1}{-\rho rU} \quad \text{and} \quad W''(U) = \frac{1}{\rho rU^2} \tag{G.25}$$

and

$$W(U-\rho rU\hat{\alpha})-W(U)=W(U(1-\rho r\hat{\alpha}))-W(U)=-\frac{\ln(1-\rho r\hat{\alpha})}{\rho r}. \tag{G.26}$$

Next, we use Itô's Lemma in its version for jump processes and calculate via (G.23)

$$\begin{aligned} dW_t = dW(U_t) &= W'(U_t)\rho rU_t\pi\hat{\alpha}_t dt + W'(U_t)(-\rho rU_t)\beta_t\sigma dZ_t \\ &\quad + W''(U_t)\left(\frac{(\rho rU_t)^2(\beta_t\sigma)^2}{2}\right)dt + \left[W(U_t-\rho rU_t\hat{\alpha}_t)-W(U_t)\right]d\Pi_t \\ &= -\pi\hat{\alpha}_t dt + \beta_t\sigma dZ_t + \frac{\rho r}{2}(\beta_t\sigma)^2 dt - \frac{\ln(1-\rho r\hat{\alpha}_t)}{\rho r}d\Pi_t \end{aligned} \tag{G.27}$$

Next, we set

$$\alpha_t := -\frac{\ln(1-\rho r\hat{\alpha}_t)}{\rho r} \quad\Longleftrightarrow\quad \hat{\alpha}_t = \frac{1-e^{-\rho r\alpha_t}}{\rho r}. \tag{G.28}$$

Thus,

$$dW_t = \frac{\rho r}{2}(\beta_t\sigma)^2 dt - \pi\left(\frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)dt + \beta_t\sigma dZ_t + \alpha_t d\Pi_t, \tag{G.29}$$

which we can rewrite as

$$dW_t = \left[\frac{\rho r}{2}(\beta_t\sigma)^2 + \pi\left(\alpha_t - \frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)\right]dt + \beta_t\sigma dZ_t + \alpha_t(d\Pi_t - \pi dt). \tag{G.30}$$

Above expression for dW_t is (G.9), as desired.

Next, we study the drift of dW_t in (G.9). Clearly, the first term, $\frac{\rho r}{2}(\beta_t\sigma)^2$, is positive and increases with β_t. For the second term, we calculate the derivatives $\frac{\partial}{\partial\alpha_t}\left(\alpha_t - \frac{1-e^{-\rho r\alpha_t}}{\rho r}\right) = 1-e^{-\rho r\alpha_t}$ and $\frac{\partial^2}{\partial\alpha_t^2}\left(\alpha_t - \frac{1-e^{-\rho r\alpha_t}}{\rho r}\right) = \rho r e^{-\rho r\alpha_t} > 0$, so that $\left(\alpha_t - \frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)$ is strictly convex in α_t. Moreover, note that $\left(\alpha_t - \frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)|_{\alpha_t=0} = 0$ and $\frac{\partial}{\partial\alpha_t}\left(\alpha_t - \frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)|_{\alpha_t=0} = 0$, so that $\left(\alpha_t - \frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)$ has a unique minimum at $\alpha_t = 0$ and is zero at this point. Thus, $\left(\alpha_t - \frac{1-e^{-\rho r\alpha_t}}{\rho r}\right) \geq 0$ for all α_t and, therefore, $\mathbb{E}[dW_t] \geq 0$, i.e., W_t is a sub-martingale and increases in expectation.

G.3 State Variables and HJB Equation

In principle, the problem has three state variables: (i) the intermediary continuation value W_t, (ii) the intermediary's savings S_t, and (iii), the firm's cash balance.

Next, note that the intermediary's certainty equivalent payoff W_t consists of two sources. First, the intermediary has savings S_t it has accumulated up to time t. Second, the intermediary expects to receive payouts from the firm after time t, which it values at

$$Y_t \equiv W_t - S_t. \tag{G.31}$$

Because the intermediary has limited commitment and can always part from the firm, the value of the intermediary's stake Y_t must always be positive, $Y_t \geq 0$; otherwise, the intermediary would be better off leaving the contractual agreement and consuming out of its savings whilst not receiving any transfers.

Due to CARA preferences, there are no wealth effects, so the exact values of W_t and S_t do not matter but only their difference Y_t, which leads to a first reduction in dimensionality. Using (G.2) and (G.9) as well as $c_t = rW_t$ under the optimal consumption, we can calculate for $Y_t = W_t - S_t$:

$$dY_t = dW_t - dS_t = \left[rY_t + \frac{\rho r}{2}\left(\beta_t\sigma\right)^2 - \pi\left(\frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)\right]dt + \beta_t\sigma dZ_t + \alpha_t d\Pi_t - dI_t. \tag{G.32}$$

We can integrate this equation over time to obtain

$$Y_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}\left\{dI_s - \left[\frac{\rho r}{2}\left(\beta_s\sigma\right)^2 + \pi\left(\alpha_s - \frac{1-e^{-\rho r\alpha_s}}{\rho r}\right)\right]ds\right\}\right]. \tag{G.33}$$

In words, Y_t is the intermediary's present value of future transfers dI_s, adjusted for the cost (or risk-compensation)

$$k_t = \frac{\rho r}{2}\left(\beta_t\sigma\right)^2 + \pi\left(\alpha_t - \frac{1-e^{-\rho r\alpha_t}}{\rho r}\right).$$

Notice that the term involving β_t is the same as in (7). With CARA preferences, one can therefore micro-found the cost of providing capital β, but it additionally imposes cost in relation to delaying payouts α.

Next, we combine (G.3) and (G.32) to calculate for $C_t = M_t - Y_t$:

$$\begin{aligned} dC_t = {} & \mu dt + r(M_t - Y_t)dt + \lambda(Y_t - M_t)dt - \lambda Y_t dt - \frac{\rho r}{2}\left(\beta_t\sigma\right)^2 dt + \pi\left(\frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)dt \\ & + \sigma(1-\beta_t)dZ_t - dDiv_t + (\Delta M_t - \alpha_t)d\Pi_t. \end{aligned} \tag{G.34}$$

We define the "post-refinancing" level of excess liquidity

$$C_t^* = C_t + \Delta M_t - \alpha_t, \tag{G.35}$$

so that

$$\Delta M_t = C_t^* - C_t + \alpha_t \quad \text{and} \quad \Delta M_t - \alpha_t = C_t^* - C_t. \tag{G.36}$$

Using these relations, we obtain

$$\begin{aligned} dC_t = {} & \left[\mu + (r-\lambda)\,C_t - \lambda Y_t - \frac{\rho r}{2}\left(\beta_t\sigma\right)^2 + \pi\left(\frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)\right]dt \\ & + \sigma\left(1-\beta_t\right)dZ_t + \left(C_t^* - C_t\right)d\Pi_t - dDiv_t. \end{aligned} \tag{G.37}$$

We denote the drift of dC_t by

$$\mu_C(C_t) = \left[\mu + (r-\lambda)\,C_t - \lambda Y_t - \frac{\rho r}{2}\left(\beta_t\sigma\right)^2 + \pi\left(\frac{1-e^{-\rho r\alpha_t}}{\rho r}\right)\right] \tag{G.38}$$

and the volatility of dC_t by

$$\sigma_C(C_t) = \sigma(1-\beta_t). \tag{G.39}$$

Next, we conjecture and verify that the equity value can be expressed as function of C_t only (i.e., $P_t = P(C_t)$), while $Y_t = Y$ is a control variable. Indeed, according to (6) and (12), it is always possible to increase or decrease Y by picking $dI < 0$ or $dI > 0$, whilst leaving the value of excess liquidity C unchanged. Given the Markovian representation, we omit time subscripts unless necessary.

We now derive the shareholders' value function and HJB equation. We conduct the analysis under the risk-neutral measure, with expectation operator $\hat{\mathbb{E}}$. By Girsanov's theorem, $d\hat{Z}_t = dZ_t + \eta dt$ is the increment of a standard Brownian Motion under the risk-neutral measure so that

$$\hat{\mathbb{E}}[dZ_t] = -\eta dt.$$

By the dynamic programming principle, the equity value $P(C)$ must then solve the HJB equation

$$rP(C)dt = \max_{\beta,Y,\alpha,C^*,dDiv\geq 0} \Big\{dDiv + \hat{\mathbb{E}}[dP(C) - \Delta M d\Pi]\Big\}, \tag{G.40}$$

Invoking Ito's Lemma, we can calculate

$$dP(C) = P'(C)\mu_C(C)dt + \frac{P''(C)\sigma_C(C)^2}{2}dt + P'(C)\sigma_C(C)dZ + [P(C^*) - P(C)]d\Pi_t - P'(C)dDiv. \tag{G.41}$$

Thus,

$$\hat{\mathbb{E}}[dP(C)] = P'(C)\mu_C(C)dt - P'(C)[1-\beta(C)]\eta\sigma dt + \frac{P''(C)\sigma_C(C)^2}{2}dt \tag{G.42}$$

$$+ [P(C^*) - P(C)]\pi dt - P'(C)dDiv. \tag{G.43}$$

Using this relation and $\Delta M_t = C_t^* - C_t + \alpha_t$, we can write the HJB equation (G.40) as

$$rP(C)dt = \max_{\beta,Y,\alpha,C^*,dDiv\geq 0} \Big\{ \big[1 - P'(C)\big]\, dDiv + P'(C)\mu_C(C)dt + \frac{P''(C)\sigma_C(C)^2}{2}dt$$

$$- \eta\sigma[1-\beta(C)]P'(C)dt + \pi\big[P(C^*) - P(C) - (C^* - C) - \alpha\big]dt\Big\}. \tag{G.44}$$

As in the baseline, dividend payouts follow a barrier strategy and cause C to reflect at $\overline{C}$ with payout boundary $\overline{C}$ satisfying

$$P'(\overline{C}) - 1 = P''(\overline{C}) = 0.$$

For $C \in (\underline{C}, \overline{C})$, the HJB equation become s

$$rP\left(C\right) = \max_{\beta,Y} \left\{P'\left(C\right)\left[\mu - \eta\sigma(1-\beta) + \left(r-\lambda\right)C - \lambda Y - \frac{\rho r}{2}\left(\beta\sigma\right)^2\right] + \frac{\sigma^2 P''\left(C\right)\left(1-\beta\right)^2}{2}\right\}$$

$$+ \pi \max_{C^*,\alpha}\left\{P'\left(C\right)\left(\frac{1-e^{-\rho r\alpha}}{\rho r}\right) + \left[P\left(C^*\right) - P\left(C\right) - \left(C^* - C + \alpha\right)\right]\right\}, \tag{G.45}$$

which is solved subject to the intermediary's limited commitment, that is, $Y \geq \max\{0, -C\}$ and shareholders' limited commitment constraint (15). We expect the value function to be concave (without formal proof for the sake of brevity).

G.4 Optimal Controls

First, consider the optimal choice of the intermediary's stake Y. As $P'\left(C\right) > 0$ and $\lambda > 0$, the optimal contract picks the lowest Y possible subject to constraint (13), so that

$$Y\left(C\right) = \max\left\{-C, 0\right\} \;\text{ and }\; M\left(C\right) = \max\left\{C, 0\right\}. \tag{G.46}$$

As holding cash is costly, it is optimal, given excess liquidity C, to minimize cash holdings $M = C + Y$ and therefore to minimize Y subject to the intermediary's limited commitment, $Y \geq 0$, and the cash constraint $M \geq 0$. If $C > 0$, the firm holds cash $M = C > 0$ and the intermediary's stake Y is zero. If $C < 0$, the firm holds no cash but the intermediary's stake $Y = -C$ is positive.

Second, consider the refinancing target C^*. The first order condition (FOC) with respect to to the refinancing target C^* yields

$$P'\left(C^*\left(C\right)\right) = 1, \tag{G.47}$$

so that refinancing occurs up until a point at which the internal value of cash is equalized with the value of paying it out. Note that $P'(\overline{C}) = 1$ at the dividend payout boundary $\overline{C}$ and $P'(C) > 1$ for $C < \overline{C}$, so that $C^* = \overline{C}$.[34]

Third, consider the intermediary's payouts upon refinancing. The optimal choice of $\alpha = \alpha(C)$ depends on whether the shareholders' limited commitment constraint (15) binds or not. Let $\alpha_U(C)$ be the optimal policy when the constraint is not binding, in which case α solves the first order condition $\frac{\partial P(C)}{\partial \alpha} = 0$, and let $\alpha_{LC}(C)$ be the optimal policy when the constraint is binding. Then, using $C^* = \overline{C}$ from above, we have

$$\alpha_U\left(C\right) \equiv \frac{\ln P'\left(C\right)}{\rho r} \quad \text{and} \quad \alpha_{LC}\left(C\right) \equiv \left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P\left(C\right) - C\right]. \tag{G.48}$$

Combining, we have

$$\alpha(C) = \min\left\{\alpha_U(C), \alpha_{LC}(C)\right\} \tag{G.49}$$

and $\alpha(C)$ decreases in C due to concavity of the value function.[35]

Fourth, consider the optimal risk-sharing intensity β. The first order condition with respect to instantaneous risk-sharing β yields

$$\beta\left(C\right) = \frac{P''\left(C\right) - \frac{\eta}{\sigma}P'(C)}{P''\left(C\right) - \rho r P'\left(C\right)}. \tag{G.50}$$

Finally, we assume that $\beta(C) \leq 1$ in optimum. That is,

$$\frac{\eta}{\sigma} < \rho r.$$

This assumption essentially states that giving the intermediary full exposure to the firm's cash flows (i.e., $\beta = 1$) is inefficient. If it were $\frac{\eta}{\sigma} \geq \rho r$, the optimal solution would be to sell the firm to the intermediary, who values it at Y^A, and the problem would become trivial.

G.5 Lower Boundary (25)

We characterize the lower boundary in the state space $\underline{C}$.

Suppose that $\underline{C} < 0$ and the firm is not liquidated. Then, we have $\mu_C(\underline{C}) = P(\underline{C}) = 0$ and $\beta(\underline{C}) = 1$ for $\underline{C} = \underline{C}^S$. To derive an expression for $\underline{C}^S$, one first uses (12) to calculate the drift of excess liquidity under the optimal choice of Y derived in the previous section (that is,

[34] Any $C^* > \overline{C}$ also fulfills the FOC but leads to an immediate payout $C^* - \overline{C} > 0$ – essentially the firm would raise cash just to immediately pay it back to its shareholders. Setting $C^* = \overline{C}$ minimizes these round-trip transactions.

[35] By concavity of $P(C)$, $\alpha'_U(C) = \frac{P''(C)}{\rho r P'(C)} < 0$ and $\alpha'_{LC}(C) = 1 - P'(C) < 0$ for $C < \overline{C}$.

$Y(C) = \max\{-C, 0\}$):

$$\mu_C(C) = \mu + \left(r - \lambda \mathbb{1}_{\{C \geq 0\}}\right) C - \frac{\rho r}{2} \sigma^2 \beta(C)^2 + \pi \left(\frac{1 - e^{-\rho r \alpha(C)}}{\rho r} \right), \tag{G.51}$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function, i.e., it is 1 if $\{\cdot\}$ is true and 0 otherwise. The HJB equation (G.45) evaluated under the optimal controls $\alpha(C)$ and $\beta(C)$ as well as $C^* = \overline{C}$ can be rewritten as

$$rP(C) = P'(C)\mu_C(C) - \eta\sigma[1 - \beta(C)]P'(C) + \frac{P''(C)(\sigma_C(C))^2}{2} + \pi J(C), \tag{G.52}$$

with jump in the value function upon refinancing

$$J(C) \equiv P(\overline{C}) - \overline{C} - P(C) + C - \alpha(C). \tag{G.53}$$

Due to $\mu_C(\underline{C}) = P(\underline{C}) = \sigma_C(\underline{C}) = 0$, we have by means of (G.52) that $J(\underline{C}) = 0$ and therefore

$$\alpha(\underline{C}) = P(\overline{C}) - \left[\overline{C} - \underline{C}\right] = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C} - \frac{\eta\sigma[1 - \beta(\overline{C})] + \frac{1}{2}\rho r(\beta(\overline{C}))^2}{r} + \underline{C}. \tag{G.54}$$

The last equality uses that at the payout boundary $\overline{C} > 0$, the HJB equation (G.45) implies

$$P(\overline{C}) = \frac{\mu}{r} + \overline{C} - \frac{\lambda\overline{C}}{r} - \frac{\eta\sigma[1 - \beta(\overline{C})] + \frac{1}{2}\rho r(\beta(\overline{C}))^2}{r},$$

due to $\alpha(\overline{C}) = P'(\overline{C}) - 1 = P''(\overline{C}) = 0$ and with

$$\beta(\overline{C}) = \frac{\frac{\eta}{\sigma}}{\rho r} \in [0, 1).$$

Define

$$\chi := \frac{\eta\sigma[1 - \beta(\overline{C})] + \frac{1}{2}\rho r(\beta(\overline{C}))^2}{r} \quad \text{so that} \quad P(\overline{C}) - \overline{C} = \frac{\mu}{r} + \overline{C} - \frac{\lambda\overline{C}}{r} - \chi.$$

Substituting in for the optimal policies, and using $\alpha(\underline{C})$ from above in $\mu_C(\underline{C}) = 0$ while using that $\sigma_C(\underline{C}) = 0 \iff \beta(\underline{C}) = 1$, we have

$$\begin{aligned} 0 = \mu_C(\underline{C}) &= \mu + r\underline{C} - \frac{\rho r}{2}\sigma^2 + \pi \left(\frac{1 - e^{-\rho r \alpha(\underline{C})}}{\rho r} \right) \\ &= \mu + r\underline{C} - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\left(1 - e^{-\rho r\left[\frac{\mu}{r} - \frac{\lambda}{r}\overline{C} + \underline{C} - \chi\right]}\right). \end{aligned} \tag{G.55}$$

We use the following Lemma to solve for $\underline{C}$:

Lemma 6. *The solution to*

$$0 = a + x + e^{(b + c \cdot x)} \tag{G.56}$$

is given by

$$x = -\frac{w(c \cdot \exp\{b - a \cdot c\}) + a \cdot c}{c}. \tag{G.57}$$

Proof. Define

$$z \equiv c \cdot \exp\{b - ac\}$$

Plugging in the proposed solution (G.57) into the equation (G.56), we have

$$
\begin{aligned}
0 &= a + \left(-\frac{w\left(c\cdot\exp\left\{b-ac\right\}\right)}{c} - a\right) + \exp\left\{b - w\left(c\cdot\exp\left\{b-ac\right\}\right) - ac\right\} \\
&= -\frac{w\left(c\cdot\exp\left\{b-ac\right\}\right)}{c} + \exp\left\{b-ac\right\}\exp\left\{-w\left(c\cdot\exp\left\{b-ac\right\}\right)\right\} \\
&= -w\left(c\cdot\exp\left\{b-ac\right\}\right) + c\cdot\exp\left\{b-ac\right\}\exp\left\{-w\left(c\cdot\exp\left\{b-ac\right\}\right)\right\} \\
&= -w\left(z\right) + z\exp\left(-w\left(z\right)\right)
\end{aligned}
$$

where we multiplied through by $c \neq 0$ in the second-to-last line. The last line is identically equal to zero by the definition of the Lambert-w function

$$
w\left(z\right)e^{w(z)} = z \iff w\left(z\right) = z\cdot e^{-w(z)}
$$

□

Next, we rewrite (G.55) as

$$
0 = \underbrace{\frac{-\rho r}{\pi}\left(\mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\right)}_{\equiv a}\underbrace{-\frac{\rho r^2}{\pi}\underline{C}}_{\equiv x} + e^{-\rho(\mu-\lambda\overline{C})+\frac{\pi}{r}x}, \tag{G.58}
$$

where we define $a \equiv \frac{-\rho r}{\pi}\left(\mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\right)$, $b \equiv -\rho(\mu - \lambda\overline{C} - r\chi)$, $c = \frac{\pi}{r}$, and $x \equiv -\frac{\rho r^2}{\pi}\underline{C}$ We now apply the above lemma to solve (G.58) for x and to thus obtain (24), that is,

$$
\underline{C} = \underline{C}^S = \frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \chi + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r} - Y^A, \tag{G.59}
$$

where $w(\cdot)$ is the Lambert function (i.e., $w(z)$ is the principal-branch solution to $we^w = z$). Finally, note that when $\pi = 0$, then $\underline{C}^S = -Y^A$, where Y^A is the autarky value defined in (9), i.e., $Y^A = \frac{\mu}{r} - \frac{\rho\sigma^2}{2}$.

Taken together, the lower boundary and the associated value of equity are given by

$$
\underline{C} = \min\left\{\underline{C}^S, 0\right\} \quad \text{with} \quad P\left(\underline{C}\right) = 0, \tag{G.60}
$$

where $\underline{C}^S$ is from (G.59).

G.6 Analysis

We now present numerical results under the model variant with CARA preferences, and show that these are qualitatively similar to the ones obtained under the baseline. We restrict attention to the main findings from Section 3. We use the same parameters as in the baseline (see Table 1). Furthermore, we normalize $\eta = 0$. The model's qualitative implications are robust to the choice of these parameters.

Figure G.1 is the analogue to Figure 1 from the main text. As can be seen, the outcomes are qualitatively similar under both specifications.

Next, Figure G.2 presents the comparative statics of the lower boundary $\underline{C}$ (solid red line) and the upper boundary boundary $\overline{C}$ (solid black line) with respect to $1/\pi$ (left panel), r (middle panel),

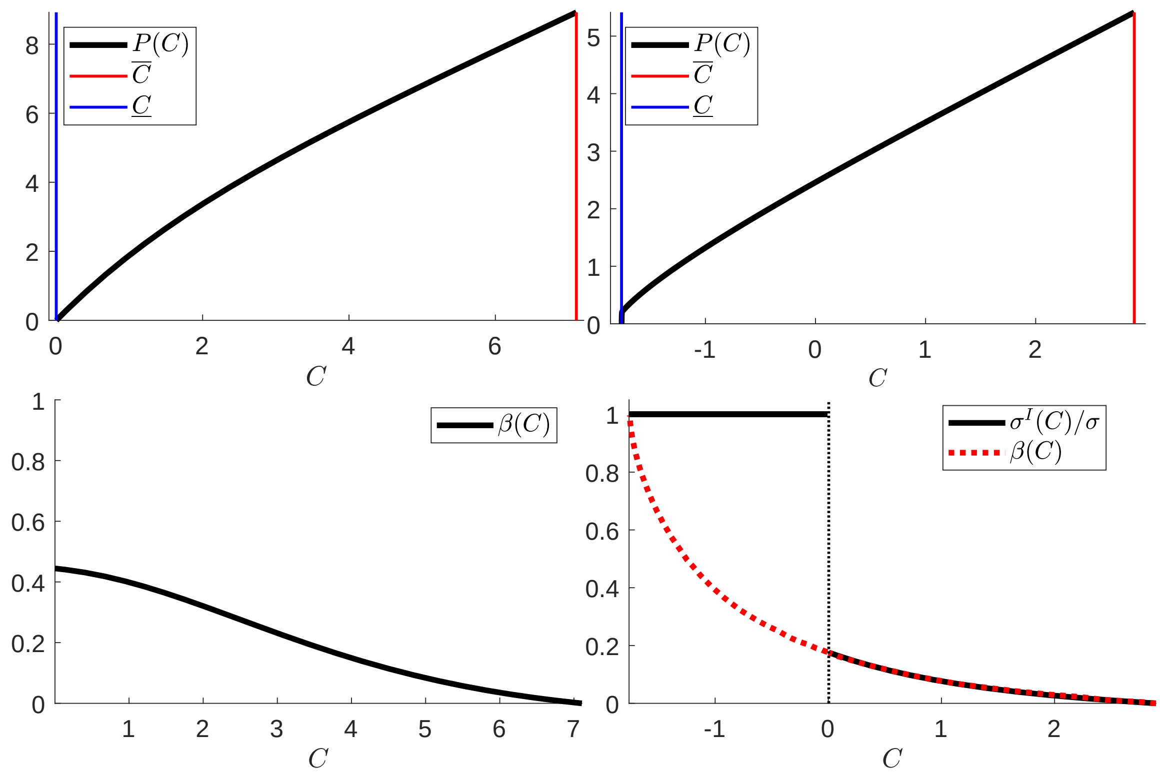

Figure G.1: **Value function and risk-sharing:** This figure plots the value function $P(C)$ (upper panels) and risk-sharing $\beta(C)$ (lower panels) against excess liquidity for $\pi = 0$ (left panels) and $\pi = 0.5$ (right panels). The parameters are such that $Y^A < 0$, and consequently the firm is liquidated at $C = 0$ for $\pi = 0$ (left panels) but not for $\pi = 0.5 > 0$ (right panels).

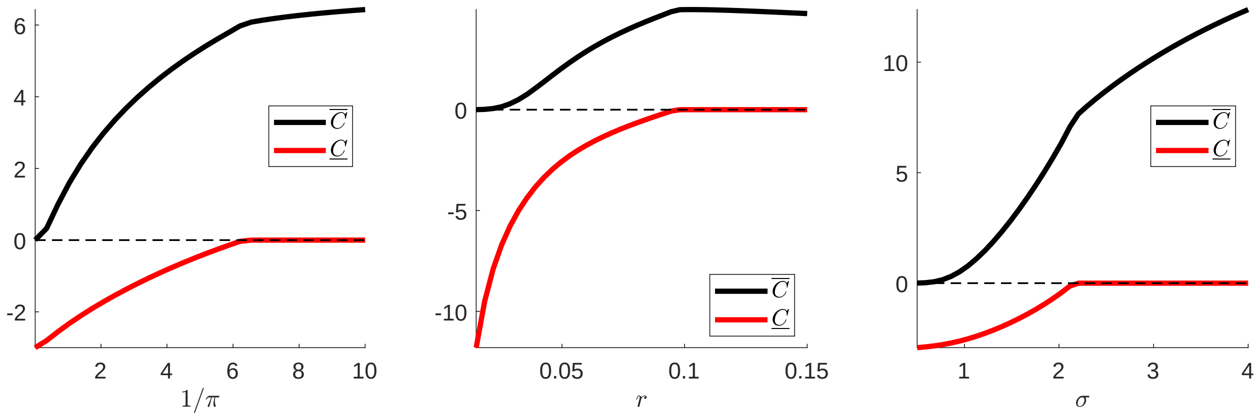

Figure G.2: **Boundaries:** Comparative statics of the boundaries $\underline{C}, \overline{C}$ with respect to to the expected time until refinancing $1/\pi$ (left panel), the interest rate r (middle panel), and cash flow volatility (right panel). The parameters follow 1.

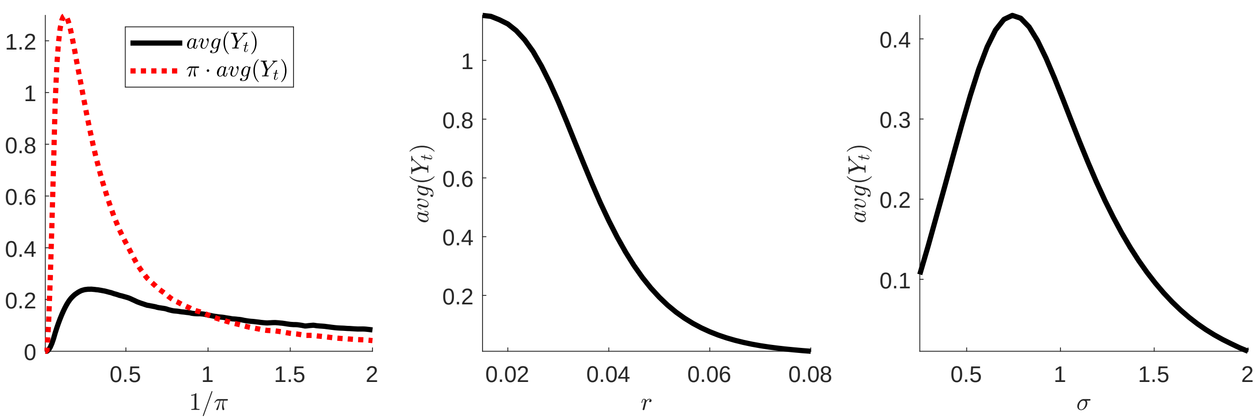

Figure G.3: **PE activity in steady state:** This figure plots the average intermediary stake $avg(Y_t)$ against $1/\pi$ (left panel), r (middle panel), and σ (right panel). The left panel also plots the average deal flow $\pi \cdot avg(Y_t)$ that is omitted in the other two panels as π is constant. The parameters follow 1.

and σ (right panel). The results are qualitatively similar to Figure 2 from the baseline.

Figure G.3 plots the average intermediary stake $avg(Y_t)$ against $1/\pi$ (left panel), r (middle panel), and σ (right panel). The left panel also plots the average deal flow $\pi \cdot avg(Y_t)$ that is omitted in the other two panels as π is constant. As can be see, $avg(Y_t)$ is hump-shaped in $1/\pi$ (left panel) and σ (right panel) and decreases with r. As such, the findings are qualitatively similar to those of Figure 3 from the main text.

Finally, Figure G.4 plots the value of intermediation and PE against $1/\pi$ (left panel), r (middle panel), and σ (right panel). Notably, the value of PE and intermediation is hump-shaped in $1/\pi$ and σ. The results are qualitatively the same as in Figure 4 from the main text. Overall, we have shown that the model results remain unchanged when we micro-found the intermediary cost of financing by assuming CARA preferences.

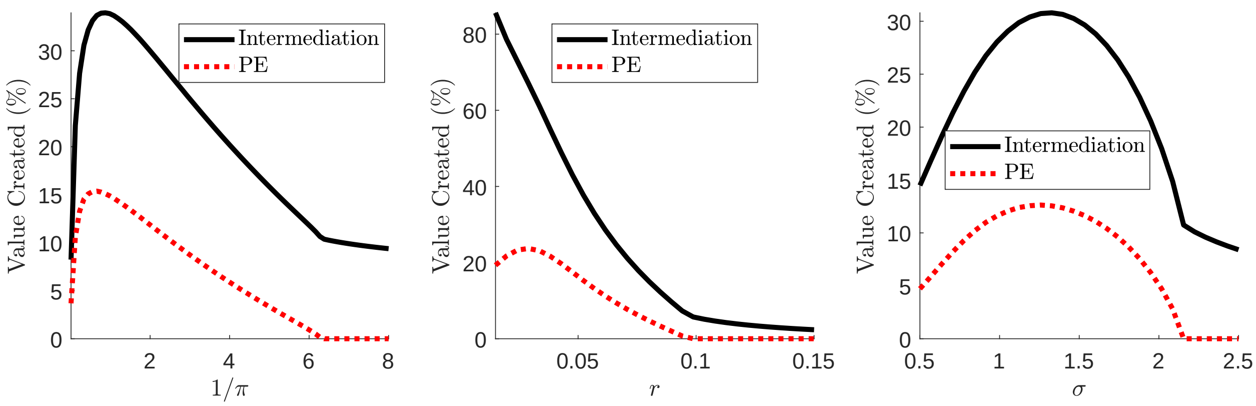

Figure G.4: **The value of intermediation and PE**. This figure depicts the proportional value generated by intermediation relative to the no-intermediation case $\left(\frac{\text{Value}}{\text{Value}_{\alpha=\beta=0}} - 1\right)$ (solid black line) and by PE financing $\left(\frac{\text{Value}}{\text{Value}_{Y=0}} - 1\right)$ (dotted red line) for different values of $1/\pi$ (left panel), r (middle panel), and σ (right panel). The parameters follow 1.