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# WAITING FOR CAPITAL:
# DYNAMIC INTERMEDIATION IN ILLIQUID MARKETS

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**ABSTRACT**

We consider a firm with infrequent access to capital markets, continuous access to financing by a risk-averse intermediary, and a cost of holding cash. The intermediary absorbs a fraction of cash-flow risk that decreases with the firm's liquidity reserves and acquires a stake in the firm under distress. Implementing the optimal contract suggests an overlapping pecking order. The firm simultaneously finances shortfalls with cash reserves and a credit line and sells equity to the intermediary when it runs out of cash. The model helps explain empirical facts and trends in financial intermediation, such as the rise of private equity.

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The assumption of frictionless financial markets, while plausible over the long run for most firms, is often at odds with the reality facing many firms. Raising capital at short notice is usually costly, constrained, or both. Indeed, specialized intermediaries often provide short-notice financing, while raising financing in broader capital markets typically takes time. Consequently, we observe that contracts with specialized intermediaries often result in financing arrangements that are more complex than common equity or long term debt, such as credit lines with performance-sensitive interest rates, flexible equity stakes, convertible debt, and so forth.

We construct a tractable continuous-time model to capture the salient features of the disparity between short-notice financing and broader financial markets. A firm owned by risk-neutral shareholders produces risky cash flows and faces financial constraints. As in Hugonnier, Malamud, and Morellec (2015), the firm can issue equity to new risk-neutral and competitive outside investors only at random times and must thus "wait for capital." This assumption reflects capital supply uncertainty or proxies for frictions that cause a delay between the firm's need for financing and its access to broader markets. In the interim, the firm finances operating losses with either internal cash reserves or intermediary funds. However, both liquidity facilities are costly. Cash held in the firm earns a return below the risk-free rate due to an internal carry cost of cash, while the intermediary requires compensation for bearing risk. We then derive the optimal contract between the firm's shareholders and the intermediary that maximizes the firm's equity value.

We first show that we can summarize the state of the firm with a single variable that we term excess liquidity. This variable equals the firm's cash holdings less the intermediary's stake in the firm measured in dollars. In optimum, the firm's equity value solves an ordinary differential equation (ODE) over excess liquidity with two free boundaries. At the upper boundary, the firm pays a dividend to shareholders. At the lower boundary, the firm either liquidates or the intermediary rescues the firm by injecting capital in exchange for a stake in the firm. The intermediary exits its position and cashes out this stake by selling it to risk-neutral outside investors once the firm has access to capital markets and can issue new equity. As such, the intermediary's willingness to rescue the firm increases with prospective financial market access, facilitating exit, but decreases with cash flow risk, making it costly for the intermediary to hold a stake in the firm.

Negative cash flow shocks can induce financial distress because the firm has infrequent access to capital markets and thus faces financial constraints. Therefore, the firm's shareholders are effectively risk-averse, the more so as the firm depletes its liquidity reserves and nears distress. At the same time, the intermediary demands compensation for bearing cash flow risk. As such, the shareholders and the intermediary face a state-dependent risk-sharing problem. The optimal financing agreement thus calls for the intermediary to bear an increasing fraction of cash flow risk as the firm's excess liquidity decreases.

As long as the firm has positive cash holdings, it uses cash and intermediary financing to absorb cash flow shocks, effectively saving and borrowing simultaneously. For example, the firm may cover a $1 operating loss by withdrawing $.35 from its internal cash and raising $.65 in financing from the intermediary. Symmetrically, if the firm makes a $1 operating profit, it may deposit $.35 in its cash account and repay $.65 to the intermediary. The fraction of risk covered by cash reserves and the intermediary depends on the firm's excess liquidity as follows. If the firm runs out of cash, it finances itself by selling an ownership stake to the intermediary, which the firm buys back when it realizes positive cash flows or can raise outside financing upon market access. This arrangement effectively shifts payouts to the intermediary from states in which the firm is financially constrained, i.e., out of cash, to states where the firm is unconstrained, and the intermediary can exit its position. Intuitively, when the firm undergoes financial distress, the intermediary injects capital in exchange for a stake in the firm and keeps the firm alive until the resolution of financial distress when it can sell its stake.

The intermediary ownership dynamics resemble that of a private equity (PE) investor who acquires distressed firms and holds them primarily to realize capital gains upon exit. Thus, we can interpret the intermediary's stake as PE investment in the firm. If the firm we consider is public, this type of financing resembles private investment in public equity (PIPE) of distressed firms. Our model, therefore, suggests that through their provision of financing in distress, PE investors help resolve distress and more efficiently reduce the risk of liquidation of their portfolio firms, in line with the findings in Bernstein, Lerner, and Mezzanotti (2019) and Hotchkiss, Smith, and Strömberg (2021). Notably, the firm's access to PE-like financing is endogenous. It depends on the intermediary's willingness to acquire a stake in the firm, which reflects the resale option value of this stake and hence the firm's prospective financial market access.

Our analysis has implications for when circumstances intermediary financing and PE financing, in particular, are most likely to occur and to generate value. Interestingly, our findings predict that intermediation in the form of PE is most pronounced when the firm's access to financial markets is at intermediate levels. The firm's access to financial markets determines its ability to raise financing from competitive and risk-neutral outside investors. Intuitively, when the firm's market access is good, the intermediary can exit its positions quickly, reducing the stake the intermediary holds on average. On the other hand, when market access is impaired, and a successful exit is difficult, the intermediary is unwilling to acquire a stake in the firm with a low resale option value. As a firm's access to financial markets tends to increase over its life cycle, our model suggests that firms rely on and benefit from PE-like financing the most in their intermediate stages.

We also show that intermediation and PE financing are most pronounced and generate the most value for firms with intermediate levels of cash flow volatility. Not surprisingly, firms with low cash flow volatility have limited financing needs and thus do not rely much on an intermediary. Crucially, firms with very volatile cash flows do not have access to cheap intermediary financing, as the cost of intermediary financing increases with the level of risk. Intuitively, PE firms are reluctant to invest in very risky firms. Therefore, these types of firms accumulate large cash reserves to manage liquidity, reflecting that intermediary financing and cash reserves are substitutes in the firm's liquidity management.

Our model also connects two significant secular trends of the last decades. First, risk-free rates have steadily declined. Second, the PE sector has grown substantially. Our model links these two facts by showing that a decline in interest rates stimulates PE-like intermediation. In particular, we show that as the risk-free rate decreases, the intermediary's willingness to provide financing to the firm in exchange for an equity stake increases because both the intermediary's valuation of the firm's future cash flows and the resale option value of firm ownership increase. The improved access to intermediary financing, in turn, induces firms to reduce precautionary cash holdings.

We show that we can implement the optimal contract can with a credit line and restricted equity that vest upon refinancing, with an overlapping pecking order. When the firm is flush with liquidity, it simultaneously uses internal cash reserves and credit line debt to cover cash flow shortfalls. Moreover, the firm relies increasingly on the credit line as cash reserves dwindle. When the firm exhausts its cash reserves, it either liquidates or finances itself by drawing on the credit line and selling (restricted) equity to the intermediary. Symmetrically, the firm pays back the credit line, buys back restricted equity, and potentially retains earnings following positive cash flows or upon financial market access. Credit line debt effectively leads to debt overhang, undermining shareholders' incentives to raise financing by issuing equity and diluting their ownership because the firm uses newly raised funds to repay the credit line. Thus, the credit line may feature early repayment incentives in the form of a repayment discount to incentivize shareholders to refinance despite debt overhang,

Our model describes different types of intermediaries, including bank-like intermediaries providing credit line financing and PE-like intermediaries providing equity financing under distress. Interestingly, we show that the availability of credit line financing is a prerequisite for the firm to access PE financing. Intuitively, PE investors are only willing to acquire a stake in the firm if the firm also has access to credit line debt (provided by banks). In other words, the availability of debt financing is necessary for and complementary to PE financing, consistent with Ivashina and Kovner (2011). In contrast, increased availability of PE financing crowds out credit line financing. Importantly, our model predicts that declining interest rates and improved access to capital markets cause a shift from bank-like to PE-like intermediation.

Finally, we also consider another model application in which all financing from the intermediary takes the form of credit line debt. Then, the firm finances cash flow shortfalls first with internal cash reserves and unsecured/uncollateralized credit line debt. Once the firm has run out of cash, it can only survive if it has access to secured or collateralized credit line financing, which requires to pledge the firm as collateral. Crucially, the intermediary is only willing to accept the firm as collateral if its resale option value is high; otherwise, the firm is liquidated and defaults. Conditional on survival, the firm effectively undergoes bankruptcy at the lower bound in the state space when it cannot repay its credit line debt without external financing. Then, the intermediary seizes the underlying collateral, takes full ownership of the firm, and keeps the firm alive until it can sell the firm to new equity investors upon market access, in which case bankruptcy is resolved. In this context, our analysis suggests that the firm relies on unsecured debt financing in normal times. It finances with secured debt only under financial distress, in line with the findings in Benmelech, Kumar, and Rajan (2020). Interestingly, financial market access and the decline of interest rates improve the firm's borrowing capacity, reducing the risk of firm liquidation.

Our paper mainly relates to the literature on dynamic corporate liquidity management in continuous time, pioneered by Bolton, Chen, and Wang (2011) and Décamps, Mariotti, Rochet, and Villeneuve (2011). In a unified model of corporate investment, financing, and liquidity management, Bolton et al. (2011) demonstrate how liquidity management and firm financing interacts with a firm's investment decisions. Further contributions in this literature include Gryglewicz (2011), Bolton, Chen, and Wang (2013), Hugonnier et al. (2015); Hugonnier and Morellec (2017), Malamud and Zucchi (2018), and, more recently, Abel and Panageas (2020b,a), Dai, Giroud, Jiang, and Wang (2020), and Bolton, Li, Wang, and Yang (2021). In addition, Bolton, Wang, and Yang (2021) study dynamic liquidity management with short-term debt financing, thereby highlighting the interaction between the endogenous pricing of the debt and the optimal the equity payout and issuance strategies. Our paper differs from these papers, as it adds an intermediary who can provide any type of financing to a dynamic liquidity management model.

Our paper also relates to the corporate finance literature on dynamic contracting without liquidity management. Recent contributions include DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), Green and Taylor (2016), Piskorski and Westerfield (2016), Varas (2018), Marinovic and Varas (2019), Malenko (2019), Gryglewicz, Mayer, and Morellec (2020), Feng and Westerfield (2021), and Feng, Taylor, Westerfield, and Zhang (2021). Similarly, DeMarzo and Urošević (2006), Marinovic and Varas (2021), and Hu and Varas (2021) study the dynamic incentive problems of blockholders and intermediaries. Notably, the intermediary in our model has certain characteristics of private equity (PE) investors. Kaplan and Stromberg (2001) and, more recently, Ewens, Gorbenko, and Korteweg (2021) study venture capital contracts in depth, while Axelson, Strömberg, and Weisbach (2009), Malenko and Malenko (2015), and Gryglewicz and Mayer (2021) focus on leveraged buyouts (LBOs).

In particular, our work relates to the dynamic contracting literature that studies optimal risk-sharing between a principal and an agent under limited commitment, such as Ai and Li (2015), Ai, Kiku, and Li (2019), and Bolton, Wang, and Yang (2019). The closest reference to our paper is Bolton et al. (2019). Our model differs from theirs in that we allow the deep-pocketed but risk-averse intermediary to provide the marginal financing of the firm. Our work is complementary to theirs in that it highlights optimal financing from an under-diversified intermediary in the presence of physical cash constraints and limited commitment. In contrast, their model is driven by the connection between investment, firm scale, and the scale of the manager's outside option in an otherwise complete market world. Rampini and Viswanathan (2010) and Rampini, Sufi, and Viswanathan (2014) provide models in which the optimal allocation of net worth in a complete market is restricted by limited commitment in the form of limited enforcement, which implies a role for net worth in easing financial constraints. In their work, risk management requires net worth that could otherwise be used for productive investment.

Finally, there is an extensive structural and empirical literature on the connection between corporate liquidity management and firm policies. Hennessy and Whited (2005, 2007); Hennessy, Levy, and Whited (2007), Whited and Wu (2006), and Nikolov, Schmid, and Steri (2019) develop structural corporate liquidity management models. Almeida and Campello (2007) and Campello, Graham, and Harvey (2010), amongst many, provide evidence how financial constraints affect corporate investment. In a theoretical setting, Acharya, Almeida, and Campello (2007) show that debt and cash are not perfect substitutes due to different hedging properties and thus may be used simultaneously. They provide empirical evidence supporting this conclusion. Sufi (2009) provides further empirical evidence on the difference between the use of cash and lines of credit for firms. Other related empirical studies on cash holdings and financing choices include Leary and Roberts (2005), Bates, Kahle, and Stulz (2009), Eisfeldt and Muir (2016), and Darmouni and Mota (2020).

# 1 Model Setup

We consider a firm whose assets produce volatile cash flows. The firm is owned by risk-neutral *outside investors*, also referred to as the firm's shareholders. Access to new outside capital is limited, but a risk-averse *intermediary* is available to supply capital continuously at an elevated cost, with the cost scaling with the riskiness of the financing provided. To structure financing from the intermediary, shareholders and the intermediary sign a transfer agreement at time zero. We interpret the role and identity of the intermediary broadly who may represent a group of different intermediaries, including private equity (PE) firms, venture capitalists, banks, or non-bank lenders.

## 1.1 Preferences

Time $t \in [0, \infty)$ is continuous. A common discount rate $r > 0$ applies to all agents, and is equal to the risk-free rate. Outside investors are risk neutral, reflecting their diversification. The intermediary is risk-averse with CARA preferences with risk-aversion of $\rho > 0$, so that the intermediary's instantaneous utility of consumption $c$ is

$$u(c) = -\frac{1}{\rho}\exp(-\rho c). \tag{1}$$

Risk aversion's main role is to generate an elevated cost of funds. It either reflects the notion that the type of intermediaries we consider in this paper typically have undiversified exposure to the firm or that the intermediary is subject to risk-based shadow costs of capital. The risk-aversion coefficient $\rho > 0$ quantifies the cost of financing from the intermediary.

Importantly, the intermediary has limited commitment in that it can always leave the firm, and the payout agreement, if doing so makes it better off. Thus, the intermediary's continuation value in dollars from following the payout agreement must at any time $t \geq 0$ exceed the intermediary's outside option in dollars, which we normalize to zero.[1] Similarly, shareholders also have limited

[1] Ai and Li (2015) and Bolton et al. (2019) present models in which the time-varying outside option plays a key

commitment and a zero outside option (in dollars). Effectively, limited commitment implies that shareholders cannot commit to undertaking actions that lower their continuation value and that their continuation payoff must always exceed their outside option.

The intermediary can maintain savings on its own account, denoted by $S_t$. Savings accrue interest at rate $r$ and are subject to changes induced by transfers to ($dI_t < 0$) and from ($dI_t > 0$) the firm and consumption $c_t$, so

$$dS_t = rS_t dt + dI_t - c_t dt. \tag{2}$$

Endowing the intermediary with the possibility to accumulate savings ensures that it can smooth its consumption beyond liquidation of the firm. Consumption $c_t$ and savings balance $S_t$ can both take positive and negative values. As such, the intermediary has essentially deep pockets, but capital provision by the intermediary is costly because the intermediary is risk-averse.

We normalize the balance of savings at $t = 0^-$ to zero, i.e., $S_{0^-} = 0$, where time $t = 0^-$ denotes the time before the payout agreement is written and any transfers are made. Savings must satisfy the standard transversality condition $\lim_{t\to\infty} \mathbb{E}\left[e^{-rt} S_t\right] = 0$, ruling out Ponzi schemes.

### 1.2 Cash Flows, Cash Holdings, and Transfers

The firm has assets in place that generate cash flows $dX_t$ with constant mean $\mu > 0$ and volatility $\sigma > 0$. That is,

$$dX_t = \mu dt + \sigma dZ_t, \tag{3}$$

where $dZ_t$ is the increment of a standard Brownian Motion. The intermediary and shareholders sign a payout agreement or contract $\mathcal{C}$ at time $t = 0$. This contract, $\mathcal{C} = (Div, I, \Delta M)$, stipulates cumulative dividend payouts $Div_t$ to shareholders, money raised from new shareholders upon financial market access $\Delta M_t$,[2] and cumulative transfers $I_t$ to/from the intermediary.

As in Hugonnier et al. (2015), the firm can only raise external funds from competitive outside investors at Poisson times $d\Pi_t = 1$ that arrive with constant intensity $\pi \geq 0$, with $d\Pi_0 = 1$ to reflect market access at the founding of the firm regardless of the value of $\pi$.[3] This assumption reflects capital supply uncertainty or proxies for frictions that cause a delay between the firm's

role in determining the contract. We highlight a complementary mechanism by assuming a constant outside option.

[2]Here, we are separating payments *to* existing shareholders, $Div_t$, from payments *from* newly arriving outside investors at refinancing, $\Delta M_t$.

[3]Hugonnier et al. (2015) show that it is without loss of generality to assume that newly arriving outside investors are competitive. Any surplus extracted by newly arriving outside investors could be fully captured by appropriately transforming the arrival rate of refinancing opportunities.

need for financing and its access to broader markets.[4] We will refer to such capital market access as refinancing opportunities. We write $dDiv_t \geq 0$ to denote the financing constraint and denote capital infusions by outsiders upon capital market access by $\Delta M_t d\Pi_t \geq 0$. Put differently, the restriction $dDiv_t \geq 0$ means that shareholders have limited liability as in Bolton et al. (2011). The firm's financial constraints, together with the fact that cash flow shocks can be negative, imply that the firm has an incentive to build a buffer stock of cash $M_t$ via retained earnings. Specifically, as the shareholders are unable to inject cash, all cash flow realizations $dX_t$, dividends $dDiv_t \geq 0$, and transfers to/from the intermediary $dI_t$ flow through the firm's internal cash balance $M_t$. We normalize the cash balance at $t = 0^-$ to zero, i.e., $M_{0^-} = 0$. In contrast to the firm's shareholders, the intermediary can inject cash into the firm, $dI_t < 0$, at any time, but this source of financing is costly and possibly limited due to the intermediary's elevated cost of funds induced by the intermediary's risk-aversion. The cash balance held within the firm accrues interest at the rate $r - \lambda$ where $r$ is the common interest rate and $\lambda \in (0, r)$ represents a carrying cost of cash.[5] The dynamics of cash reserves $M_t$ are then given by

$$dM_t = dX_t + (r - \lambda) M_t dt - dDiv_t - dI_t + \Delta M_t d\Pi_t. \tag{4}$$

Cash holdings must remain positive, $M_t \geq 0$ for all $t \geq 0$. This implies that if $M_t$ attains zero, the intermediary must either inject the necessary funds or the firm must liquidate. Liquidation thus occurs at a stopping time $\tau \in [0, \infty]$, and $dDiv_t = dI_t = dX_t = 0$ for all $t > \tau$. For simplicity, we assume that the liquidation value of the firm beyond its current cash holdings $M_\tau$ is zero.[6]

### 1.3 Optimal contracting problem

We call the payout agreement $\mathcal{C}$ incentive compatible (IC) if it respects the intermediary's and the shareholders' limited commitment, and we restrict our attention to IC payout agreements. The firm's founders, i.e., its original shareholders, have full bargaining power and can extract all surplus

[4] One may interpret the time it takes to arrange for financing as caused by (un-modeled) asymmetric information — outside investors take time to verify information. At the same time, the intermediary, being a specialist, circumvents this delay.

[5] The assumption that internal cash reserves earn interest at a rate below the discount rate is standard in the literature (see, e.g., Décamps et al. (2011) and Bolton et al. (2011)) to preclude a degenerate solution in which payouts are indefinitely delayed. Assuming impatient equity investors leads to similar results, but we follow Bolton et al. (2011) and instead assume a carry cost of cash. For simplicity, the intermediary's savings are not subject to the carry cost of cash $\lambda > 0$ we will impose on the firm's cash holdings.

[6] That is, the assets in place stop producing cash flows upon liquidation, while the cash-balance is divided up according to the contract.

from the intermediary when signing the payout agreement with the intermediary at $t = 0$. Let $U_0$ be the intermediary's utility for a given contract $\mathcal{C}$, that is,

$$U_0 = \max_{c_t} \mathbb{E}\left[\int_0^{\infty} e^{-rt} u\left(c_t\right) dt\right] \quad \text{s.t.} \quad (2) \quad \text{and} \quad \lim_{t\to\infty} \mathbb{E}\left[e^{-rt} S_t\right] = 0. \tag{5}$$

Given $\mathcal{C}$, the intermediary chooses consumption $c_t$ to maximize its lifetime utility, with optimal consumption denoted by $c_t^*$. The intermediary's outside option is to stay away from the contract and to consume out of its savings, yielding lifetime utility $\overline{U}$ and the participation constraint $U_0 \geq \overline{U}$. The initial equity value, i.e., the original shareholders' value function $P_{0^-}$ before initial financing at time $t = 0$, is the discounted stream of dividend payouts net the costs of refinancing:

$$P_{0^-} = \max_{\mathcal{C} \text{ is IC}} \mathbb{E}\left[\int_0^{\tau} e^{-rt}(dDiv_t - \Delta M_t d\Pi_t)\right] \quad \text{s.t.} \quad U_0 \geq \overline{U} \quad \text{and} \quad M_{0^-} = 0. \tag{6}$$

Note that upon refinancing, i.e., $d\Pi_t = 1$, the firm raises $\Delta M_t$ dollars from newly arriving competitive outside investors at fair value by issuing $\Delta M_t$ dollars worth of new equity. The firm's existing shareholders, in turn, are diluted as they must give up $\Delta M_t$ dollars worth of equity ownership. At inception at time $t = 0$, the firm's original shareholders are penniless, and the firm's cash holdings are zero, but the firm can raise financing from newly arriving (competitive) outside investors as $d\Pi_0 = 1$. We provide additional discussion of our key modeling assumptions in Section 5.5.

# 2 Model Solution

This section solves the model and derives the optimal payout agreement. First, we derive the state variables. Tractability comes from collapsing the model's state variables to the one-dimensional difference between the firm's cash holdings, and the intermediary's future promised payouts, which we term excess liquidity. Finally, we derive the optimal payout agreement and express all model quantities in terms of the single state variable, excess liquidity.

## 2.1 State Variables

In principle, shareholders' dynamic optimization has three state variables: The firm's cash holdings $M_t$, the intermediary's savings $S_t$, and the intermediary's continuation utility defined as

$$U_t \equiv \mathbb{E}_t\left[\int_t^{\infty} e^{-r(s-t)} u(c_s^*) ds\right]. \tag{7}$$

Further, $U_t$ can be expressed as $W_t$ in certainty equivalent monetary terms, with

$$W_t \equiv \frac{-\ln(-\rho r U_t)}{\rho r}, \tag{8}$$

and we work with $W_t$ instead of $U_t$. Note that $W_t$ is the intermediary's total continuation payoff in monetary terms with law of motion

$$dW_t = \left[\frac{\rho r}{2}(\beta_t \sigma)^2 + \pi\left(\alpha_t - \frac{1-e^{-\rho r \alpha_t}}{\rho r}\right)\right]dt + \beta_t \sigma dZ_t + \alpha_t(d\Pi_t - \pi dt), \tag{9}$$

where $\alpha_t$ and $\sigma\beta_t$ are the loadings of $dW_t$ on the martingales $(d\Pi_t - \pi dt)$ and $dZ_t$ respectively. The first term in the drift of (9) captures the intermediary's required compensation for being exposed to Brownian cash flow risk. The second term captures the intermediary's required compensation for being exposed to shocks $d\Pi_t$. Both terms are unambiguously positive, so that $W_t$ increases in expectation, $\mathbb{E}[dW_t] \geq 0$. We summarize our results so far in the following proposition.

**Proposition 1.** *The intermediary's optimal consumption satisfies $c_t^* = rW_t$. The intermediary's certainty equivalent payoff $W_t$, defined in (8), follows the dynamics (9). $W_t$ is a sub-martingale and increases in expectation, $\mathbb{E}[dW_t] \geq 0$. The intermediary's outside option to stay away from the contract at $t = 0$ and to consume out of its savings yields lifetime utility $\overline{U} = -\frac{1}{\rho r}$. The intermediary's participation constraint $U_0 \geq \overline{U}$ is equivalent to $W_0 \geq 0$.*

Note that the intermediary's certainty equivalent payoff $W_t$ consists of two sources. First, the intermediary has savings $S_t$ it has accumulated up to time $t$. Second, the intermediary expects to receive payouts from the firm after time $t$, which it values at

$$Y_t \equiv W_t - S_t. \tag{10}$$

Combining (2) and (9) and using the optimal consumption choice $c_t = c_t^* = rW_t$, we obtain

$$dY_t = dW_t - dS_t = \left[rY_t + \frac{\rho r}{2}(\beta_t \sigma)^2 - \pi\left(\frac{1-e^{-\rho r \alpha_t}}{\rho r}\right)\right]dt + \beta_t \sigma dZ_t + \alpha_t d\Pi_t - dI_t. \tag{11}$$

We refer to equation (11) as the promise keeping constraint. It means that current transfers $dI_t$ must be accompanied by a commensurate change in future promised transfers $dY_t$ to deliver the promised payoff $Y_t$ to the intermediary. Integrating (11) against time and taking expectations, we obtain

$$Y_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}\left\{dI_s - \left[\frac{\rho r}{2}\left(\beta_s\sigma\right)^2 + \pi\left(\alpha_s - \frac{1-e^{-\rho r\alpha_s}}{\rho r}\right)\right]ds\right\}\right]. \tag{12}$$

In words, $Y_t$ is the intermediary's risk-adjusted present value of future transfers $dI_s$, i.e., the value of its deferred payouts or stake within the firm. Because the intermediary has limited commitment and can always part from the firm, the value of the intermediary's stake $Y_t$ must always be positive, $Y_t \geq 0$; otherwise, the intermediary would be better off leaving the contractual agreement and consuming out of its savings while not receiving any transfers.

Finally, let us introduce two benchmark valuations of the firm that will be useful later. First, consider the firm's value in a frictionless market in which the firm has continual access to outside markets, i.e., $\pi \to \infty$. Then, the firm need not hold any cash and covers all cash flow shortfalls by raising new equity. The value of the firm then is simply its net present value (NPV)

$$NPV \equiv \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}dX_s\right] = \frac{\mu}{r}. \tag{13}$$

Next, consider the value of the firm if the intermediary fully owned it and had no access to outside financing, a situation we term *autarky*. In this case, the intermediary's saving account absorbs all shocks, and cash within the firm is again zero.[7] Setting $dI_s = dX_s$, $\beta_s = 1$, and $\alpha_s = 0$ in equation (12) we calculate the value of the firm to the intermediary as

$$Y^A \equiv \frac{\mu}{r} - \frac{\rho}{2}\sigma^2. \tag{14}$$

We refer to $Y^A$ as the autarky value, and if $Y^A < 0$, the intermediary would be unwilling to operate the firm on its own. As we will see, when $\pi < \infty$, the optimal payout agreement features risk-sharing between the risk-averse but deep-pocketed intermediary and risk-neutral but financially constrained shareholders, which improves upon the autarky solution. The total net value of the firm, which is the sum of the shareholders' value function $P_t$ and the intermediary's stake $Y_t$ minus the current cash-holdings $M_t$, accordingly satisfies $Y^A \leq P_t + Y_t - M_t < NPV$.

## 2.2 Dynamic Program and HJB Equation

We now derive an expression for the equity value, i.e., shareholders' value function, that depends on the three endogenous state variables of the problem: the intermediary's certainty equivalent

[7] As cash within the firm earn less interest than the intermediary's savings, the firm optimally holds zero cash.

payoff $W_t$, the intermediary's savings $S_t$, and the firm's cash holdings $M_t$. We first show how to reduce the problem to a single state variable so that the equity value $P_t$ can be characterized by an ordinary differential equation (ODE).

**Reduction of the state space.** First, note that since the intermediary has CARA preferences and is not financially constrained, the level of its total payoff $W_t$ and its private savings $S_t$ do not separately enter the optimal contracting problem. Instead, the intermediary's stake $Y_t = W_t - S_t$ is a sufficient state variable for the contracting problem.

Next, let us define the variable $C_t$, which we term *excess liquidity*, by

$$C_t \equiv M_t - Y_t. \tag{15}$$

According to (4) and (11), excess liquidity $C$ has the following law of motion:

$$\begin{aligned} dC_t = dM_t - dY_t = &\left[\mu + (r-\lambda)\, C_t - \lambda Y_t - \frac{\rho r}{2}\,(\beta_t \sigma)^2 + \pi \left(\frac{1-e^{-\rho r \alpha_t}}{\rho r}\right)\right] dt \\ &+ \sigma\,(1-\beta_t)\, dZ_t + (C_t^* - C_t)\, d\Pi_t - dDiv_t, \end{aligned} \tag{16}$$

where we define the post-refinancing level of excess liquidity as $C_t^* \equiv \Delta M_t + C_t - \alpha_t$.

Under the promise-keeping constraint (11), a transfer of cash $dI_t$ between the balance $M_t$ of the firm and the intermediary must be accompanied by a commensurate increase or decrease of the intermediary's stake $Y_t$. Thus, $C_t = (M_t - dI_t) - (Y_t - dI_t) = M_t - Y_t$ is invariant to transfers $dI_t$. Moreover, such transfers do not affect the intermediary's total payoff $W_t$ because the promise keeping constraint (11) implies that current payouts, $dI_t = \Delta > 0$, lead to a commensurate decrease in deferred payouts, $dY_t = -\Delta < 0$ and $dW_t$ does not depend on $dI_t$. Since $dI_t$ can be positive or negative, we can use transfers to freely adjust $Y_t$ without affecting excess liquidity $C_t$ or the intermediary's total payoff $W_t$. Thus, $Y_t$ is a choice variable, and the only remaining state variable is $C_t$.

Although $Y_t$ is a choice variable in the dynamic optimization, it is subject to constraints. First, the intermediary's limited commitment requires $Y_t \geq 0$. Next, the definition of excess liquidity $C_t = M_t - Y_t$ and the physical constraint on the firm's cash balance requires $M_t \geq 0$ together imply that $Y_t \geq -C_t$. Thus,

$$Y_t \geq \max\{0, -C_t\}. \tag{17}$$

Note that excess liquidity $C_t$ can become negative if the intermediary's stake $Y_t$ is larger than the firm's cash balance $M_t$. In what follows, we conjecture that equity value $P_t$ can be expressed as a function of excess liquidity only, in that $P_t = P(C_t)$. We verify this conjecture towards the end of this section by deriving the HJB equation that only involves the state variable $C$, control variables, and exogenous model constants. We omit time subscripts unless necessary.

**Shareholders' limited commitment.** Next, we discuss possible restrictions imposed by the limited commitment of shareholders. Consider the change in value to the existing shareholders in case of refinancing, which moves the firm from some level of excess liquidity $C$ before refinancing to the post-refinancing level $C^*$ with a prescribed increase of the intermediary's payoff by $\alpha$. The change in value to the existing shareholders from refinancing is

$$P\left(C^*\right) - \Delta M - P\left(C\right) = P\left(C^*\right) - \left(C^* - C + \alpha\right) - P\left(C\right). \tag{18}$$

Because the loading of $dY + dI$ on $d\Pi$ equals $\alpha$ by (11), the firm needs to raise $\Delta M = C^* - C + \alpha$ dollars to transition from $C$ to $C^*$ in case of refinancing, increasing $Y + I$ by $\alpha$.

The limited commitment of shareholders requires (18) to be non-negative so that shareholders are not made worse off by refinancing; otherwise, shareholders would be better off passing up the refinancing opportunity. That is, shareholders cannot commit to refinancing policies that make them worse off after refinancing. Formally, $\alpha$ is subject to the constraint

$$\alpha \leq P\left(C^*\right) - \left(C^* - C\right) - P\left(C\right). \tag{19}$$

As dividend payouts to shareholders $dDiv_t$ also must be non-negative, limited commitment also implies $P\left(C\right) \geq 0$; that is, $P(C) < 0$ would violate limited commitment. Let $\mathcal{S}\left(C^*, C\right)$ denote the set of all admissible choices for $\alpha$.

**The HJB Equation.** To solve the shareholders' dynamic problem (6), we first maximize the equity value $P\left(C\right)$ for a given level of excess liquidity $C$ and then determine the optimal level of initial excess liquidity $C_0$. We now derive the HJB equation that characterizes equity value. We conjecture that the firm pays out dividends at an upper boundary $C = \overline{C}$, which cause $C$ to reflect at $\overline{C}$, and that it either liquidates or receives sufficient financing to stay alive at some (endogenous) lower boundary $\underline{C}$. For $C \in \left(\underline{C}, \overline{C}\right)$ where $dDiv = 0$, shareholders' value function solves the HJB equation

$$rP(C) = \max_{\beta, Y \geq \max\{0, -C\}} \left\{ P'(C) \left[ \mu + (r - \lambda) C - \lambda Y - \frac{\rho r}{2} (\beta \sigma)^2 \right] + P''(C) \frac{\sigma^2}{2} (1 - \beta)^2 \right\}$$
$$+ \pi \max_{C^*, \alpha \in \mathcal{S}(C^*, C)} \left\{ P'(C) \left( \frac{1 - e^{-\rho r \alpha}}{\rho r} \right) + [P(C^*) - P(C) - (C^* - C + \alpha)] \right\}. \quad (20)$$

Note that the right-hand-side of (20) only depends on the state variable $C$, control variables $(\alpha, \beta, Y, C^*)$ with constraints that depend on $C$, and exogenous constants, confirming the conjecture that we can express equity value as a function of $C$ only. Thus, $P_t = P(C_t)$ in optimum and excess liquidity $C$ is the only payoff-relevant state variable.

Like in related papers (see, e.g., Bolton et al. (2011)), the payout boundary satisfies the standard smooth pasting and super contact conditions:

$$P'(\overline{C}) = 1 \quad \text{and} \quad P''(\overline{C}) = 0. \quad (21)$$

As we show later in Proposition 3, for $C < \overline{C}$, the marginal value of liquidity exceeds one, i.e., $P'(C) > 1$, and the value function is strictly concave, i.e., $P''(C) < 0$. The concavity of equity value reflects that the firm's shareholders are effectively risk-averse since negative cash flow shocks can trigger financial distress. We summarize our findings so far in the following Proposition.

**Proposition 2.** *Excess liquidity $C_t = M_t - Y_t$ evolves according to (16). The firm's equity value can be expressed as function of $C$ only, in that $P_t = P(C_t)$. The function $P(C)$ solves the HJB equation (20) on the endogenous state space $(\underline{C}, \overline{C})$. Optimal dividend payouts take the form $dDiv_t = \max\{C_t - \overline{C}, 0\}$.*

## 2.3 Optimal Control Variables

We now solve the optimization in the HJB equation (20) to obtain a state-dependent characterization of the four control variables: (i) the intermediary's stake $Y$, (ii) the refinancing target $C^*$, (iii) the intermediary's payouts upon refinancing $\alpha$, and (iv) the risk-sharing intensity $\beta$.

First, consider the optimal choice of the intermediary's stake $Y$. As $P'(C) > 0$ and $\lambda > 0$, the optimal contract picks the lowest $Y$ possible subject to constraint (17), so that

$$Y(C) = \max\{-C, 0\} \quad \text{and} \quad M(C) = \max\{C, 0\}. \quad (22)$$

As holding cash is costly, given excess liquidity $C$, it is optimal to minimize cash holdings $M = C+Y$ and therefore minimize $Y$ subject to the intermediary's limited commitment, $Y \geq 0$, and the cash constraint $M \geq 0$. If $C > 0$, the firm holds cash $M = C > 0$, and the intermediary's stake $Y$ is zero. If $C < 0$, the firm holds no cash, but the intermediary's stake $Y = -C$ is positive.

Second, consider the refinancing target $C^*$. The first order condition (FOC) with respect to to the refinancing target $C^*$ yields

$$P'\left(C^*\left(C\right)\right) = 1, \tag{23}$$

so that refinancing occurs up until a point at which the internal value of cash is equalized with the value of paying it out. Note that $P'(\overline{C}) = 1$ at the dividend payout boundary $\overline{C}$ and $P'(C) > 1$ for $C < \overline{C}$, so that $C^* = \overline{C}$.[8]

Third, consider the intermediary's payouts upon refinancing. The optimal choice of $\alpha = \alpha(C)$ depends on whether the shareholders' limited commitment constraint (19) binds or not. Let $\alpha_U(C)$ be the optimal policy when the constraint is not binding, in which case $\alpha$ solves the first-order condition $\frac{\partial P(C)}{\partial \alpha} = 0$, and let $\alpha_{LC}(C)$ be the optimal policy when the constraint is binding. Then, using $C^* = \overline{C}$ from above, we have

$$\alpha_U\left(C\right) \equiv \frac{\ln P'\left(C\right)}{\rho r} \quad \text{and} \quad \alpha_{LC}\left(C\right) \equiv P\left(\overline{C}\right) - \overline{C} - \left[P\left(C\right) - C\right]. \tag{24}$$

Combining, we have

$$\alpha(C) = \min\left\{\alpha_U(C), \alpha_{LC}(C)\right\} \tag{25}$$

and $\alpha(C)$ decreases in $C$ due to concavity of the value function.[9]

Fourth, consider the optimal risk-sharing intensity $\beta$. The first order condition with respect to instantaneous risk-sharing $\beta$ yields

$$\beta\left(C\right) = \frac{P''\left(C\right)}{P''\left(C\right) - \rho r P'\left(C\right)} \in \left[0, 1\right]. \tag{26}$$

Setting $\beta > 0$ transfers risk to the intermediary, reducing the volatility of excess liquidity. In addition, it reduces the drift of excess liquidity because of the risk compensation the firm must pay to the intermediary.

[8] Any $C^* > \overline{C}$ also fulfills the first order condition but leads to an immediate payout $C^* - \overline{C} > 0$ – essentially the firm would raise cash just to immediately pay it back to its shareholders. Setting $C^* = \overline{C}$ minimizes these round-trip transactions.

[9] By concavity of $P(C)$, $\alpha'_U(C) = \frac{P''(C)}{\rho r P'(C)} < 0$ and $\alpha'_{LC}(C) = 1 - P'(C) < 0$ for $C < \overline{C}$.

Finally, we can insert $C^* = \overline{C}$, $P(\overline{C}) - \overline{C} = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C}$, and optimal $\beta(C)$ from (26) into the HJB equation (20) and obtain (after simplifications):

$$rP(C) = P'(C)\left[\mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] \\ + \pi\left[P\left(\overline{C}\right) - P(C) - \left(\overline{C} - C + \alpha(C)\right)\right], \tag{27}$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function, i.e., it is 1 if $\{\cdot\}$ is true and 0 otherwise.

## 2.4 Lower boundary in the state space $\underline{C}$

To begin with, note that $C = M - Y \geq -Y$ and $Y \leq \frac{\mu}{r}$ where $\frac{\mu}{r}$ is the NPV of the firm (the first best value). Thus, $C \geq -\frac{\mu}{r}$, so that excess liquidity is bounded from below. Therefore, there must exist an endogenous lower boundary $\underline{C}$ such that $C_t \geq \underline{C}$ at all times $t$.

Next, let us determine the lower boundary $\underline{C}$. The key question is whether the firm always liquidates when cash runs out at $C = M = 0$ or not. This question is equivalent to asking if the intermediary optimally provides financing at the lower boundary $\underline{C}$ to prevent firm liquidation or not. For $\underline{C}$ to be a lower bound for $C$, it must be that either (i) the firm liquidates at $\underline{C}$, in which case we denote the lower boundary by $\underline{C}^L$, or that (ii) $\underline{C}$ is either a reflection, inaccessible or an absorbing state (absent refinancing), in which case we denote the lower boundary by $\underline{C}^S$.[10]

Let us consider both cases in turn. First, let us consider that there is liquidation once $C$ reaches $\underline{C}$, i.e., case (i). At $C = \underline{C}$, the total liquidation value is the firm's cash $M(C)$ which is split by intermediary and shareholders, so that $P(C) + Y(C) = M(C)$.[11] Since $P(C), Y(C) \geq 0$ due to limited commitment, $M(C) = 0$ implies $P(C) = Y(C) = 0$. Because optimally $Y(C) = \max\{0, -C\}$ and $M(C) = \max\{C, 0\}$, if liquidation occurs, it must occur on the interval $\left[0, \overline{C}\right]$ as otherwise $Y(C) > 0 = M(C) = P(C)$ at the time of liquidation, a violation of promise keeping. Next, liquidation at any $C > 0$ with dividend $C$ to the shareholders is sub-optimal, because $P(C) > C$ for any $C > 0$ due to $P'(C) > 1$ for $C < \overline{C}$. Thus, conditional on liquidating, it is optimal to liquidate at the lowest value $C$ not violating promise-keeping, leading to the liquidation boundary $\underline{C}^L = 0$. Further, provided liquidation is optimal, it occurs the first time $\tau = \inf\{t \geq 0 : C_t = 0\}$ excess liquidity $C$ or, equivalently, cash holdings $M(C)$ reach zero.

[10]The law of motion (16) reveals that there is no impulse control possible to reflect $C$ upward. Our approach to determine the lower boundary in the survival scenario is similar to the one in Bolton et al. (2019).

[11]Recall that we assume that the firm's liquidation beyond its cash holdings is zero.

Next, assume there is no liquidation at $C = \underline{C}$, i.e., case (ii). Then, for $\underline{C}$ to be a lower bound of $C$, something we term survival, it must be that the volatility of excess liquidity $\sigma_C(C) = \sigma(1-\beta(C))$ vanishes, which requires $\beta(C) = 1$), while its drift, denoted $\mu_C(C)$, and the shareholders' value function $P(C)$ both stay non-negative as $C$ approaches $\underline{C}$. The intuition is that at $C = \underline{C}$, the intermediary keeps the firm alive by providing continuous financing, absorbing all cash flow shocks via $\beta(\underline{C}) = 1$. However, it is optimal to delay setting $\beta = 1$ due to the intermediary's risk-aversion, which requires the greatest possible risk compensation to the intermediary, as long as possible. Therefore, the lower boundary $\underline{C}^S$ is determined as the lowest level $C$ at which $\mu_C(C) \geq 0$, $P(C) \geq 0$, and $\beta(C) = 1$ simultaneously hold, so that $\mu_C(\underline{C}^S) = P(\underline{C}^S) = 0$ while $\beta(\underline{C}^S) = 1$. We indeed show that $\beta(C) < 1$ for $C > \underline{C}^S$, and that $\beta(\underline{C}^S) = 1$ is consistent with the optimization in (20) and the expression (26).[12]

Using the law of motion (16) and the HJB equation (20), one can solve $\mu_C(\underline{C}^S) = P(\underline{C}^S) = 0$ and $\beta(\underline{C}^S) = 1$:

$$\underline{C}^S = \frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r} - Y^A, \tag{28}$$

where $w(\cdot)$ is the principal branch of the product logarithm or Lambert's w-function. Additional details on the derivation of (28) can be found in the proof of Proposition 3 in Appendix C.3.2.

We now have to determine the optimal policy between survival and liquidation. First, consider $\underline{C}^S < 0$. Suppose the firm instead liquidates the first time $C$ falls to zero. This is sub-optimal as liquidation at $C = 0$ implies $P(0) = 0$ and $Y(0) = 0$, whereas keeping the firm alive yields the same $Y(0) = 0$ but a higher $P(0) > 0$. Thus, if $\underline{C}^S < 0$, the firm never liquidates. Second, consider $\underline{C}^S > 0$ so that $M(\underline{C}^S) > 0$. Shareholders' value at the boundary is $P(\underline{C}^S) = 0$ while the intermediary's stake is $Y(\underline{C}^S) = 0$. However, liquidating the firm and paying out $M(\underline{C}^S) = \underline{C}^S > 0$ dollars as dividends yield a higher payoff than survival. Thus, when $\underline{C}^S > 0$, the firm optimally liquidates and, as we have shown, optimal liquidation occurs at $\underline{C} = \underline{C}^L = 0$.

Taken together, the lower boundary and the associated value of equity are given by

$$\underline{C} = \min\left\{\underline{C}^S, \underline{C}^L\right\} = \min\left\{\underline{C}^S, 0\right\} \quad \text{with} \quad P\left(\underline{C}\right) = 0. \tag{29}$$

[12] The lower boundary $\underline{C}^S$ and $\beta(\underline{C}^S) = 1$ follows from the requirement that $C$ must be bounded from below under incentive compatible contracts and survival. In principle, it is possible to have $\beta(C') = 1$ for $C' > \underline{C}^S$ in which case $\mu_C(C') > 0 = \sigma_C(C')$ and $C'$ is the *effective* lower boundary of the state space (i.e., $C_t \geq C'$ for all $t$). We do not preclude this case, but Lemma 2 in Appendix C.3 shows that $\beta(C) < 1$ for $C > \underline{C}^S$, so that $\underline{C}^S$ can be viewed as the "optimal lower boundary."

How can $P(\underline{C}) = 0$ hold for a firm that never liquidates and that only pays positive dividends in the future as limited liability holds? Upon refinancing at $C = \underline{C}$, existing shareholders are completely diluted, in that shareholders' limited commitment constraint (19) binds.

### 2.5 The Optimal Contract

We now characterize the optimal financing arrangement by summarizing our previous results.

**Proposition 3.** *The optimal contract is described by the optimal policies (22), $C^* = \overline{C}$, (25), and (26). The shareholders' value function $P(C)$ solves the HJB equation (20) with boundary conditions*

$$P'(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0, \tag{30}$$

*where $\underline{C}$ is given by (29). Dividend payouts cause $C$ to reflect at the payout boundary $\overline{C} > 0$. Under the optimal controls, the HJB equation (20) simplifies to (27). The value function is concave, i.e., $P''(C) < 0$ for $C < \overline{C}$. If $\underline{C} = 0$, the firm liquidates once $C$ reaches the lower boundary $\underline{C}$ and $\beta(C) < 1$ for all $C > \underline{C}$. If $\underline{C} < 0$, the firm never liquidates and $\beta(\underline{C}) = 1$ while $\beta(C) < 1$ for $C > \underline{C}$. Regardless of the value of $\underline{C}$, $\alpha(C)$ decreases in $C$.*

*Finally, optimal transfers to the intermediary are given by*

$$dI = \mu_I dt + \sigma_I dZ + \alpha_I d\Pi = \begin{cases} \left[\frac{\rho r}{2}(\beta\sigma)^2 - \pi\left(\frac{1-e^{-\rho r\alpha}}{\rho r}\right)\right] dt + \beta\sigma dZ + \alpha d\Pi & \text{for} \quad C > 0 \\ \mu dt + \sigma dZ + (\alpha + Y)\, d\Pi & \text{for} \quad C < 0. \end{cases} \tag{31}$$

*Equation (31) implicitly pins down $\mu_I = \mu_I(C)$, $\sigma_I = \sigma_I(C)$, and $\alpha_I = \alpha_I(C)$.*

Note that by the concavity of $P(C)$ and $P'(C) = 1$ for $C \geq \overline{C}$, the firm's initial level of excess liquidity $C_0$ chosen by its founders coincides with the dividend payout boundary $\overline{C}$ as

$$\overline{C} = \arg\max_{C_0} P(C_0) - C_0. \tag{32}$$

Thus, the founders' initial value in (6) is $P_{0^-} = P_0 - \Delta M_0 = P(\overline{C}) - \overline{C}$ as the firm raises $\Delta M_0 = \overline{C}$ from newly arriving investors who pay fair value and the intermediary merely breaks even with $Y_0 = Y(\overline{C}) = 0$. Importantly, the payout threshold $\overline{C}$ is positive.[13]

[13] To see this, note that we can evaluate the ODE (20) or (27) at the payout boundary $\overline{C}$ to obtain $P(\overline{C}) - \overline{C} = \frac{\mu}{r} - \frac{\lambda}{r}\left(\overline{C}\mathbb{1}_{\{\overline{C}\geq 0\}}\right)$. This payoff must be strictly lower than the NPV of the firm, $\frac{\mu}{r}$, which implies $\overline{C} > 0$.

| Parameter | Value | Interpretation |
|---|---|---|
| $r$ | 0.06 | Common discount & interest rate |
| $\lambda$ | 0.01 | Internal carry cost of cash |
| $\mu$ | 0.18 | Drift of cash flow process |
| $\sigma$ | 1.5 | Volatility of cash flow process |
| $\rho$ | 6 | Coefficient of absolute risk aversion |
| $\pi$ | 0.5 | Arrival rate of refinancing opportunities |
| $Y^A = \frac{\mu}{r} - \rho\frac{\sigma^2}{2}$ | $-3.75$ | Autarky value of firm to intermediary |

Table 1: Baseline Parameter Values for all Figures.

We present numerical examples based on the parameters given in Table 1. Following Bolton et al. (2011), we take the interest rate $r = 0.06$, the cash flow rate $\mu = 0.18$, and the carry cost of cash $\lambda = 0.1$. Similar to He (2011), we take the CARA risk-aversion coefficient $\rho = 6$. We set $\sigma = 1.5$ to ensure that $Y^A = \frac{\mu}{r} - \rho\frac{\sigma^2}{2} < 0$ and absent refinancing opportunities, i.e., $\pi = 0$, the firm is liquidated at $C = 0$.[14] This assumption allows us to separately identify the effects of capital market access on firm liquidation or survival. We consider both $\pi = 0$ and $\pi = 0.5$ in which case the expected time until next market access is $1/\pi = 2$ years.[15]

The value function $P(C)$ for $\pi = 0$ (left panel) and $\pi = 0.5$ (right panel) are depicted as the solid black lines in the top row of Figure 1, the payout boundary $\overline{C}$ as the vertical red lines, and lower boundary $\underline{C}$ as the vertical blue lines. As stated in Proposition 3, the value function is increasing and concave in either scenario. Figure 1 reveals that the risk-sharing intensity $\beta(C)$ decreases with $C$. Also, when $C < 0$, the risk-sharing intensity $\beta(C)$ differs from the scaled volatility of the intermediary's transfers, as $\sigma^I(C) = \sigma$ for $C < 0$. In the left panel for $\pi = 0$, there are no refinancing opportunities, so $\underline{C} = 0$ as $Y^A < 0$, and the firm liquidates once it runs out of cash. In the right panel, $\pi = 0.5$ and $\underline{C} < 0$, so the firm never liquidates as the intermediary is willing to provide financing against a stake in the firm $Y$ once cash has run out.

The left panel of Figure 2 displays the firm's cash holdings $M(C)$ and the intermediary's stake $Y(C)$ in the scenario with refinancing opportunities ($\pi = 0.5$). The right panel of Figure 2 displays

[14] Note that our model primarily describes young or private firms with limited access to capital markets which tend to have higher cash volatility than more mature firms. This also motivates our choice of a relatively high value for $\sigma$.

[15] Notably, Hugonnier et al. (2015) show that the effective arrival rate of refinancing opportunities is the product of the physical arrival rate and shareholders' (incumbents) bargaining power in Nash bargaining with newly arriving outside investors. Our parameter choice $\pi = 0.5$, therefore, follows Hugonnier et al. (2015) who assume (in Table 1) a physical arrival rate of refinancing opportunities of 2 and incumbent shareholders' bargaining power of 0.25 so that the effective arrival rate of refinancing opportunities is $0.25 \cdot 2 = 0.5$.

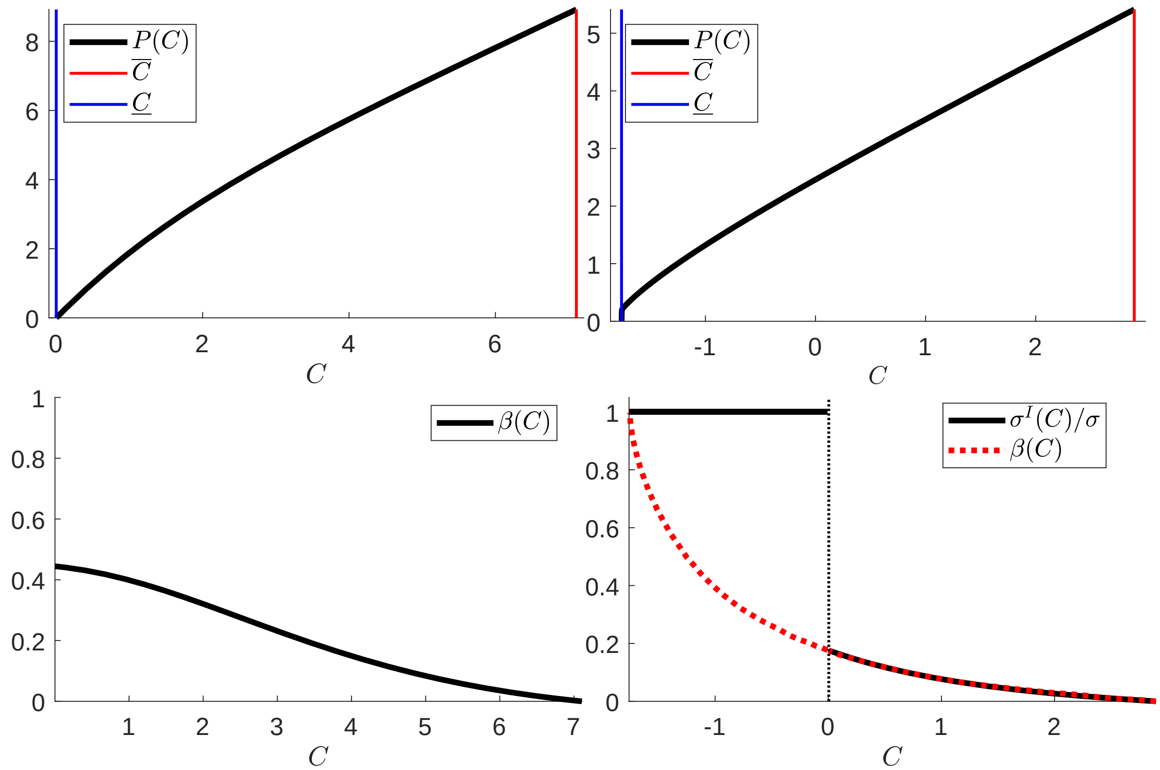


Figure 1: **Value function and risk-sharing:** This figure plots the value function $P(C)$ (upper panels) and risk-sharing $\beta(C)$ (lower panels) against excess liquidity for $\pi = 0$ (left panels) and $\pi = 0.5$ (right panels). The parameters are such that $Y^A < 0$, and consequently the firm is liquidated at $C = 0$ for $\pi = 0$ (left panels) but not for $\pi = 0.5 > 0$ (right panels).

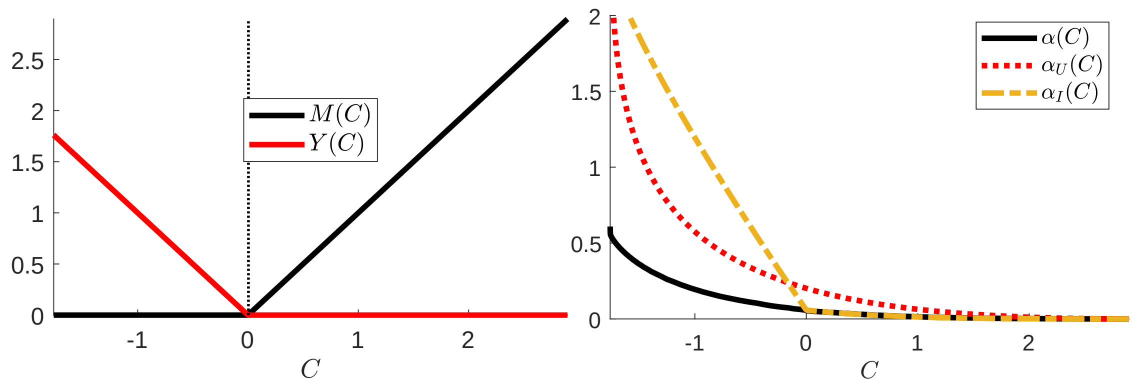


Figure 2: **Optimal Cash, Promised Value, and Payment upon Refinancing**: Numerical illustration of the optimal cash holdings $M(C) = \max\{C, 0\}$ and intermediary stake $Y(C) = \max\{-C, 0\}$ as a function of the state variable (left panel); Numerical illustration of optimal $\alpha$ (solid black), the unconstrained $\alpha_U(C)$ (dotted red), and the total actual payout upon refinancing $\alpha_I(C) = \alpha(C) + Y(C)$ (dashed yellow) (right panel). This figure uses the baseline parameters and $\pi = 0.5$.

optimal $\alpha(C)$ as well as $\alpha_U(C) = \frac{\ln P'(C)}{\rho r}$, the optimal choice of $\alpha$ absent shareholders' limited commitment, and $\alpha_I(C) = \alpha(C) + Y(C)$ as in (31). As can be seen, $\alpha(C) = \alpha_{LC}(C) < \alpha_U(C)$, which means that shareholders' limited commitment constrains the choice of $\alpha$, that is, (19) binds.

# 3 Analysis and Discussion

## 3.1 Contract dynamics and PE financing

In this section, we discuss the dynamics of the contract with the intermediary and establish a connection to private equity (PE) financing. It is important to recall that the intermediary in our model can represent a group of different intermediaries, including PE firms, banks, and non-bank lenders.[16] In other words, the different services the intermediary provides to the firm that are characterized by the optimal choice of $\alpha, \beta$, and $Y$ could be provided by different types of intermediaries in practice.

We start by discussing the dynamics of $\beta(C)$. Recall that $\beta(C)$ captures the firm's reliance on intermediary financing to cover cash flow shocks. Setting $\beta > 0$ transfers cash flow risk to the intermediary, which reduces the volatility of excess liquidity $C$ and is beneficial because the value function is concave. However, risk-sharing with the risk-averse intermediary is costly in that it decreases the drift of $C$ due to the required risk compensation to the intermediary. The optimal choice of $\beta$ in (26) trades off the decrease in volatility versus the decrease in the drift of excess liquidity.

The bottom row of Figure 1 depicts the instantaneous financing policies $\beta(C)$ for $\pi = 0$ (left panel, solid black line) and $\pi = 0.5$ (right panel, dashed red line). As the negative cash flow shocks reduce excess liquidity, the intermediary's risk exposure $\beta(C)$ increases, that is, $\beta(C)$ decreases with $C$. Thus, when $C$ is low, the contract stipulates high risk-sharing $\beta$ between the firm and the intermediary, reducing the volatility of excess liquidity $C$. In other words, when $C$ is low, and $\beta(C)$ is high, the firm covers cash shortfalls to a large extent with intermediary funds and to a lesser extent by drawing on its internal cash reserves. On the other hand, when $C$ is high and $\beta(C)$ is low, the firm mostly relies on its internal cash reserves to cover cash flow shocks but less on intermediary financing. Overall, the intermediary's risk exposure and the firm's reliance on intermediary financing decrease with the level of excess liquidity.

A similar mechanism implies that the intermediary's payouts upon market access, $\alpha(C)$ and

[16] See Jang (2020) for empirical facts on non-bank lenders in PE.

$\alpha_I(C) = \alpha(C) + Y(C)$, also decrease with $C$. Formally, according to (16), the optimal choice of $\alpha$ trades off a decrease in the jump component of liquidity against an increase in the drift of liquidity. Setting $\alpha > 0$ essentially transforms flow payouts to the intermediary today into a (promised) lumpy payout upon refinancing, which is particularly beneficial when the firm is constrained, and its liquidity reserves are low so that $\alpha(C)$ decreases with $C$. However, such a delay of payouts is costly due to the intermediary's risk-aversion and the random nature of the refinancing opportunities. Intuitively, we can understand $\alpha(C)$ as a way the firm finances its contract with the intermediary, specifically the risk-sharing $\beta(C)$ the intermediary provides, by delaying payment until financial market access.

The optimal contract minimizes the deferral of payouts to the intermediary. It picks the lowest possible intermediary stake $Y$ as shown in (22), and as depicted in the left panel of Figure 2. For $C > 0$, the firm's cash holdings are positive, i.e., $M(C) > 0$. An increase in $Y$ via $dI < 0$ implies an equivalent increase in $M$ and therefore leaves $C$ unchanged. But, due to the carry costs of cash, holding cash inside the firm is inefficient. Thus, it is optimal to have the minimum amount of cash consistent with $C > 0$, which yields $Y(C) = 0$. Likewise, because the firm minimizes the carrying cost of cash, $C < 0$ implies that the firm holds no cash, $M(C) = 0$, while the intermediary takes a stake in the firm $Y(C) = -C > 0$. The intuition is that when $C < 0$, the firm has run out of cash, and the intermediary takes a stake in the firm to resolve financial distress.

Next, we argue that for $C < 0$, we can interpret $Y$ as an ownership stake in the firm. Consider the contract dynamics at the optimal lower boundary of excess liquidity $\underline{C} < 0$. At this boundary, the intermediary must absorb all the cash flow risk and $\beta(\underline{C}) = 1$; otherwise, a cash flow shock could push $C$ below $\underline{C}$. Notice that at the lower boundary $\mu_C(\underline{C}) = 0$, so that the boundary $\underline{C}$ is absorbing absent the arrival of a refinancing opportunity. Further, at $C = \underline{C}$, (31) shows that the firm pays all cash flows to the intermediary until a refinancing opportunity. At this point, new risk-neutral outside investors buy out the intermediary and receive $\alpha(C) + Y(C)$ dollars. In other words, the intermediary owns the entire firm at $C = \underline{C}$ and derives value from both the cash flows and the possible exit at which it sells the firm to financial markets.

These dynamics resemble the strategy of PE investors that acquire distressed firms and hold these firms primarily to realize capital gains upon exit. In particular, when the firm undergoes financial distress, the intermediary provides capital in exchange for a stake in the firm to keep the firm alive to resell the firm after the resolution of distress, which is consistent with PE investors' role in firms' financial distress (Hotchkiss et al., 2021). Thus, the intermediary's stake $Y$ can be interpreted as PE investment or as the level of PE-like intermediation. If the firm we consider is public, we can interpret this type of financing as private investment in public equity (PIPE) of distressed firms. Importantly, the firm's access to PE-like financing endogenously depends on the intermediary's willingness to acquire a stake in the firm under distress, which in turn depends on the resale option value of such a stake. If refinancing opportunities are sufficiently common, then the intermediary is willing to assume full ownership of the firm at $C = \underline{C} < 0$, even if the autarky value $Y^A$ is negative precisely because the intermediary can realize a capital gain at exit through the resale option. For $\underline{C} < C < 0$, the dynamics are similar except that the intermediary only holds a partial stake in the firm, which increases following negative cash flow shocks but shrinks following positive cash flow shocks as the firm buys back part of the intermediary's stake.

With the interpretation of the intermediary's stake $Y$ as a PE investment, one can also calculate the fraction of PE ownership of the firm $\frac{Y(C)}{Y(C)+P(C)}$ where $Y(C) + P(C)$ is total firm value. Consistent with the previous arguments, PE investment as a fraction of total decreases with $C$ since $Y(C)$ decreases and $P(C)$ increases with $C$, is positive for $C < 0$, zero for $C \geq 0$, and approaches one as $C$ approaches $\underline{C}$. Given our model of the firm, we think of PE investment mostly as buyouts or growth capital rather than early-stage venture capital financing.[17]

Importantly, the firm faces liquidation risk if and only if $\underline{C} = 0$, that is, if and only if the intermediary is unwilling to take a stake in the firm and $Y = 0$ at all times. In contrast, when $\underline{C} < 0$ and the intermediary takes a stake $Y > 0$ in the firm following negative shocks, the firm is never liquidated, as it has access to sufficient intermediary financing to survive distress. That is, the model implies that PE investors acquire a stake in distressed firms, and so help resolve distress and more efficiently reduce the risk of liquidation of their portfolio firms, in line with the empirical findings in Bernstein et al. (2019), Gompers, Kaplan, and Mukharlyamov (2020), and Hotchkiss et al. (2021) on PE investors' role in the resolution of distress.[18] We emphasize, however, that in our model, the intermediary mitigates a firm's financial distress only through the provision of financing, but not through other channels, such as operational and governance engineering (Kaplan and Stromberg (2009); Bernstein and Sheen (2016)). Our analysis also suggests that PE investors' willingness to invest and resolve the financial distress of portfolio firms crucially depends on capital

[17](Leveraged) buyouts focus on more mature companies where the primary source of risk is the level of cash flows. In contrast, venture capital funds focus on younger firms that might not produce cash flows and in which the risk is primarily about failing or achieving a breakthrough. There is no failure or breakthrough in our model, but cash flows are risky.

[18]Bernstein et al. (2019) and Hotchkiss et al. (2021) present evidence that PE investors inject capital in exchange for ownership stakes, when portfolio firms undergo financial distress.

market access facilitating exit.

Given that access to intermediary financing, specifically PE-like financing, may be sufficient to cover cash flow shortfalls and prevent firm liquidation, why does the firm not reduce cash holdings to zero in all states and simply rely on intermediary financing? Even when $\underline{C} < 0$ and the intermediary is willing to provide sufficient financing, the firm holds cash reserves. The reason is that intermediary financing is costly because the intermediary is risk-averse while the firm's shareholders are risk-neutral. As such, the firm holds cash reserves to reduce its reliance on costly intermediary financing. In other words, cash reserves allow the firm to absorb cash flow risk after good performance and thus engage in risk-sharing with the intermediary.

For the remainder of the paper, we will refer to the intermediary's stake $Y$ as "PE financing," even though it may represent a stake in the firm of other types of intermediaries and not necessarily just PE firms.

## 3.2 Financial Market Access and Liquidity

As we have seen in the previous section, the ability to access capital markets and exit its position in the firm crucially affects the intermediary's incentives to provide financing to the firm in exchange for a stake within the firm. We now consider the impact of enhanced financial market access, as captured by an increase in $\pi$ or a decrease in $1/\pi$. Notice that more mature, larger, or public firms tend to have better market access than early-stage, small, or private firms and may be characterized by a larger value of $\pi$. In this analogy, financial market access $\pi$ tends to increase over the firm's life cycle. Alternatively, one can view changes in financial market access as proxying for financial market development, for example, across different countries, or as a consequence of aggregate financial market conditions (i.e., low or high $\pi$ corresponds to low or high public market liquidity).

The left panel of Figure 3 shows the upper (solid black) and lower boundaries (solid red) as a function of $1/\pi$. Observe that both $\underline{C}$ and $\overline{C}$ increase with $1/\pi$, and $\underline{C} = -\max\{Y^A, 0\}$ for $\pi = 0$. Crucially, as $\underline{C} = 0$, which implies $Y = 0$, for larger values of $1/\pi$, the intermediary is only willing to acquire a stake $Y > 0$ in the firm if financial market access and thus exit opportunities are favorable, that is, for low values of $1/\pi$ for which $\overline{C} < 0$. As $\pi$ increases, the resale option value to holding a stake in the firm increases. As a result, the intermediary's willingness to provide financing in exchange for a stake in the firm also increases. At the same time, $\overline{C}$ decreases with $\pi$. The reason is that better market access, i.e., higher $\pi$, allows the firm to rely more on intermediary financing and to raise new capital more frequently. As a result, the firm faces less severe financing

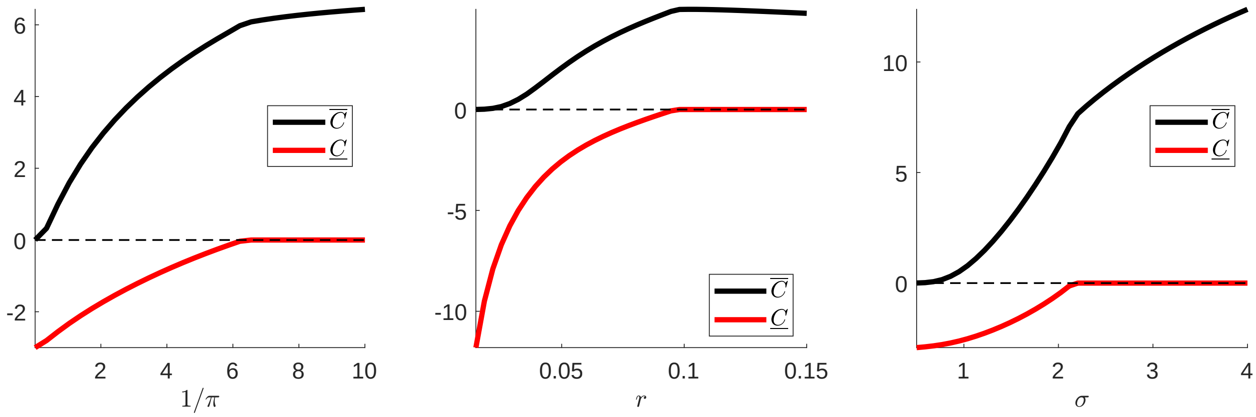


Figure 3: **Boundaries:** Comparative statics of the boundaries $\underline{C}, \overline{C}$ with respect to to the expected time until refinancing $1/\pi$ (left panel), the interest rate $r$ (middle panel), and cash flow volatility (right panel). The parameters follow 1.

frictions, reducing precautionary cash holdings, and, as a result, the payout boundary $\overline{C}$.

As the expected time until refinancing $1/\pi$ decreases, our model implies that — all else equal — we should see a larger proportion of firms with $\underline{C} < 0$ that have access to PE-like financing. When $\underline{C} < 0$, the firm never fails, and the stationary distribution of states $C$ exists. We can use this stationary distribution to calculate the average levels of model quantities in question, for example, to obtain the model-implied level of intermediary, specifically PE, activity. Two distinct measures come to mind: First, we can measure the outstanding *stock* of PE investments $Y_t$, which essentially proxies for the total size and funding of the PE sector. Second, we can measure the deal *flow* of PE investments $\pi \cdot Y_t$, which captures the amount of intermediation done by PE firms at each point in time. These measures are possibly distinct: For example, large PE positions that are not exited frequently would imply a large stock but small flow measurement.

The left panel of Figure 4 shows that both the average PE stake $avg(Y_t)$ as well as the average exit deal flow $\pi \cdot avg(Y_t)$ are hump-shaped in $1/\pi$. Thus, the largest amount of PE-like intermediary activity, either measured in terms of outstanding dollar ownership in the firm or exit deal flow, occurs at intermediate liquidity levels $1/\pi$. We note that the exit deal flow peaks at a lower $1/\pi$ than the average outstanding ownership stake. The intuition behind these findings is as follows. On the one hand, when $1/\pi$ is large and market liquidity is low, the intermediary cannot exit its position quickly and is therefore unwilling to take a large stake $Y$ in the firm in the first place. On the other hand, when $1/\pi$ is small and market liquidity is high, the intermediary would be willing to take a large stake in the firm but exits any position quickly so that $Y$ on average remains low.

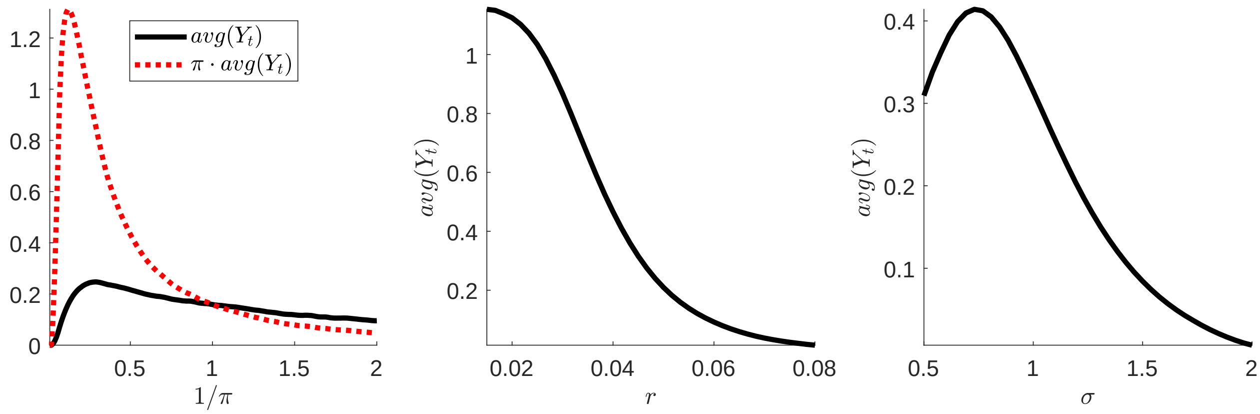


Figure 4: **PE activity in steady state:** This figure plots the average intermediary stake $avg(Y_t)$ against $1/\pi$ (left panel), $r$ (middle panel), and $\sigma$ (right panel). The left panel also plots the average deal flow $\pi \cdot avg(Y_t)$ that is omitted in the other two panels as $\pi$ is constant. The parameters follow 1.

Further, as the average $Y$ decreases faster than $\pi$ increases, the average exit deal flows $\pi \cdot avg(Y_t)$ also declines at some level of $\pi$.

Note that the comparative statics of $avg(Y_t)$ with respect to $1/\pi$ also have implications for financing and PE investment during a firm's life cycle. In its early stages, a firm often faces difficulties accessing capital markets to raise equity, i.e., large $1/\pi$, consequently leading to low average levels of PE investment in the firm. As capital market access improves and $1/\pi$ shrinks over the firm's life cycle, PE investment rises too, and the firm relies more on PE financing. Finally, when the firm has gained sufficient market access, and $1/\pi$ is low, its reliance on PE financing is low again. Taking stock, our model implies that firms' reliance on PE financing is highest in their intermediate stages.

Finally, we investigate the value created by intermediation and the value specifically created by PE, where we interpret the intermediary's stake $Y$ as PE investment. We do this by pitting our full model against two benchmarks: The no-intermediation benchmark, in which $\alpha = \beta = 0$, and the no-PE benchmark, in which $Y = 0$. Importantly, the no-intermediation benchmark $\alpha = \beta = 0$ also precludes PE financing, so $Y = 0$, $C = M$, and the firm is liquidated once it runs out of cash, i.e., $\underline{C} = 0$.[19] In the no-PE benchmark, the firm still relies on intermediary financing through the choice of $\alpha$ and $\beta$. Still, the intermediary does not take a stake in the firm so that $Y = 0$, leading to the lower boundary in the state space $\underline{C} = 0$. If $\mu_C(0) < 0$, the firm is liquidated at

[19] Note that when $\beta = 0$, it is not possible to prevent liquidation at $C = \underline{C}$, in that survival requires $\beta(\underline{C}) = 1$. Thus, the firm must be liquidated at $C = 0$. In fact, the no-intermediation benchmark becomes the solution to the baseline model in the limit case $\rho \to \infty$.

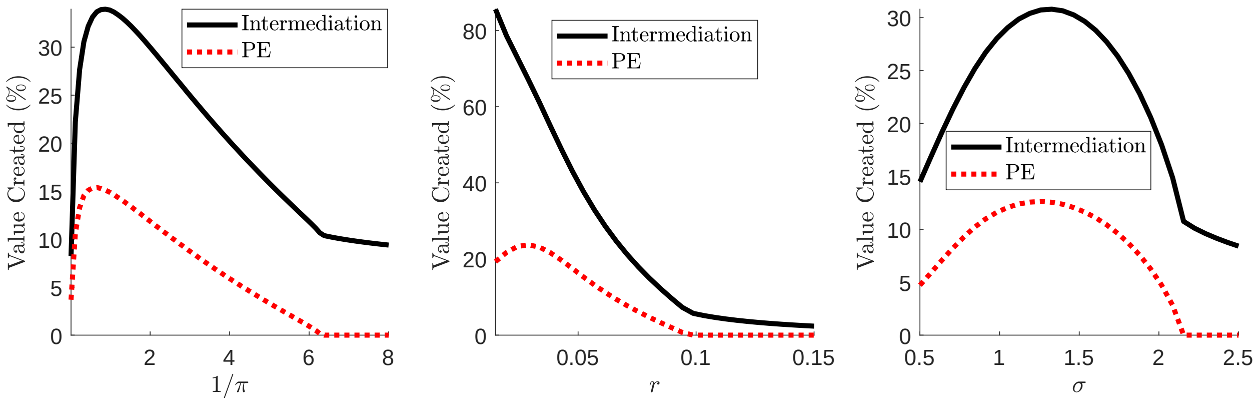


Figure 5: **The value of intermediation and PE**. This figure depicts the proportional value generated by intermediation relative to the no-intermediation case $\left(\frac{\text{Value}}{\text{Value}_{\alpha=\beta=0}} - 1\right)$ (solid black line) and by PE financing $\left(\frac{\text{Value}}{\text{Value}_{Y=0}} - 1\right)$ (dotted red line) for different values of $1/\pi$ (left panel), $r$ (middle panel), and $\sigma$ (right panel). The parameters follow 1.

$C = 0$. Otherwise, if $\mu_C(0) \geq 0$, the firm is never liquidated and survives as $C$ approaches zero, in that $\lim_{C\to 0} \beta(C) = 1$. It turns out that survival prevails in the no-PE benchmark if and only if it prevails in the baseline too; the conditions for survival in the no-PE and baseline solution are therefore characterized in Proposition 3.[20] Conditional on survival, the solution to the no-PE benchmark is characterized by the HJB equation (20) subject to smooth pasting and super contact conditions at the payout boundary as well as the boundary condition $\lim_{C\downarrow 0} P''(C) = -\infty$ which ensures — by means of (26) — $\lim_{C\downarrow 0} \beta(C) = 1$ and survival.

The value of PE over and above the no-PE case with $Y = 0$ thus does not come from a reduction of the firm's liquidation risk, as the set of firms that do not face liquidation is the same. Rather, the value of PE must come from improved risk-sharing. Consider the $Y = 0$ solution with a lower boundary $\underline{C}_{Y=0} = 0$, some positive payout boundary $\overline{C} > 0$, and firm survival, i.e., $\underline{C}^S < 0$. Now consider decreasing $\underline{C}$ to slightly below zero. The extra slack $[\underline{C}, 0]$ optimally decreases the sensitivity $\beta$ of the intermediary's continuation utility to cash flow shocks in a neighborhood of $C = 0$, i.e., $\beta(0) < 1 = \beta_{Y=0}(0)$. Consequently, the firm has to pay less risk compensation to the intermediary. A similar argument holds for $[0, \overline{C}]$. Thus, PE financing relegates high levels of risk-sharing, i.e., high levels of $\beta$, to less likely states, i.e., low and negative values of $C$. This shift in risk allocation creates value as exposing the intermediary to risk is expensive.

[20] Heuristically, suppose that the firm survives in the baseline so that $\underline{C} < 0$ (unless in the knife-edge case $\underline{C} = 0$). This implies $\mu_C(\underline{C}) = 0$, $\beta(\underline{C}) = 1$, and $P(\underline{C}) = 0$ by means of Proposition 3. But then, it is possible to set $\beta(0) = 1$ and $\mu_C(0) \geq 0$ to achieve $P(0) > 0$, which is higher than the value attained from liquidation at $C = 0$. Then, the firm prefers survival at $C = 0$ over liquidation in the no-PE benchmark.

Let us investigate the relative value creation at the inception of the firm. Recall that firm value at inception is given by Value $= P(\overline{C}) - \overline{C}$. Figure 5 depicts relative value creation for changes in $1/\pi$ (left panel), $r$ (middle panel), and $\sigma$ (right panel). The solid black lines depict $\left(\frac{\text{Value}}{\text{Value}_{\alpha=\beta=0}} - 1\right)$ and the dotted red lines depict $\left(\frac{\text{Value}}{\text{Value}_{Y=0}} - 1\right)$ in percentage terms.[21] Focusing on the comparative statics with respect to financial market access $\pi$, i.e., the left panel of Figure 5, we find that intermediation in general and PE-like intermediation, in particular, generate the most value for intermediate levels of $1/\pi$ or $\pi$. Firms with sufficiently good market access, i.e., high $\pi$, do not rely much on intermediary financing, as they can easily raise capital from outside investors. Conversely, firms with low market access $\pi$ cannot benefit from any PE financing, as poor market access undermines the intermediary's exit opportunities and thus the intermediary's willingness to take a stake in the firm. If we think of $\pi$ changing over the firm's life cycle, the firms benefiting the most from PE financing are in the intermediate stages of their life cycle.

### 3.3 Intermediary Activity and the Secular Decline in Interest Rates

As the previous section has shown, financial market development has ambiguous effects on intermediation. We next want to consider the impact of the interest rate environment on intermediation, again with an eye towards PE investment.

Over the last decades, an important secular trend has been the continuous decline in the risk-free rate. We now assess how a decline in interest rate $r$ changes the nature of intermediation in our model. A priori, a change in $r$ has two effects. First, a decrease in $r$ increases the NPV (13) of the project, thus making all projects more attractive. Second, a change in the interest rate changes the required compensation of risk-exposure to the intermediary given the term $\frac{\rho r}{2}(\beta\sigma)^2$ in the drift of (16). Intuitively, a low interest rate $r$ implies a low cost of capital for the intermediary, which allows the intermediary to absorb cash flow shocks by drawing on its savings account and so reduces the intermediary's required compensation. Consequently, all else equal, risk-sharing with the intermediary becomes less costly when $r$ is lower.

The middle panel in Figure 3 shows that both the payout boundary $\overline{C}$ and the lower boundary $\underline{C}$ increase with the risk-free rate $r$ as long as $\underline{C} < 0$. The payout boundary $\overline{C}$ starts decreasing in $r$ once $\underline{C} = 0$. Importantly, the difference $\overline{C} - \underline{C}$ monotonically decreases in $r$ throughout. When interest rates are high, $\underline{C} = 0$, and our model implies there is no PE investment. When the interest

[21] Figure F.2 provides complete graphs w.r.t. $\pi$ for initial cash-endowments, firm values, and relative value creation, i.e., $\overline{C}$ (left panel), $P(\overline{C}) - \overline{C}$ (middle panel), as well as replicating the left panel of Figure 5 (right panel).

rate $r$ is sufficiently low, the lower boundary $\underline{C} < 0$, implying that PE investment, i.e., $Y > 0$, can occur. Intuitively, the intermediary is willing to take a larger stake in the firm in a low interest environment since low interest rates increase both the intermediary's valuation and the value of the resale option. As $r$ declines, the firm's access to PE financing improves, allowing the firm to reduce precautionary cash holdings. Therefore, $\overline{C}$ increases with $r$. However, for high enough levels of $r$, further increases in $r$ lead to a decrease of $\overline{C}$ as the cost of risk-sharing effect dominates.

The middle panel in Figure 4 depicts the average intermediary stake $avg(Y_t)$, which is proportional to the deal flow $\pi \cdot avg(Y_t)$ as $\pi$ remains constant, for different levels of the interest rate $r$ and for firms that do not face the risk of liquidation. Note that the existence of a stationary distribution of $C$ requires this restriction. A decline in interest rate $r$ increases the average intermediary stake and the average exit deal flow, spurring PE-like intermediation.

Next, let us investigate the relative value creation at the firm's inception. The middle panel of Figure 5 depicts the relative value creation with respect to the interest rate $r$.[22] The black line implies that the value of intermediation increases in $r$. This effect is intuitive: as $r$ declines, the project's NPV and the value loss due to liquidation increase, thus increasing the value of intermediation. However, the red line implies that the value of PE is non-monotone in $r$. As $r$ shrinks, the required compensation for risk-exposure to the intermediary $\frac{\rho r}{2}(\beta\sigma)^2$ becomes cheaper. Consequently, a sufficiently low interest rate limits the value of allowing $C$ to become negative and thereby, all else equal, lowering $\beta$ on $[0, \overline{C}]$.

### 3.4 The Effects of Cash Flow Volatility

Lastly, let us investigate the effect of cash flow volatility $\sigma$ on the model. As the right panel of Figure 3 shows, the boundaries $\underline{C}$ and $\overline{C}$ are both increasing in $\sigma$, with $\overline{C}$ strictly so. When cash flow volatility is sufficiently high, $\underline{C} = 0$, the firm has no access to PE-like financing and faces the risk of liquidation. The intuition is that sufficiently risky firms do not have access to intermediary financing that prevents liquidation. Indeed, it is inefficient for the intermediary to acquire a stake in such firms due to the compensation it demands to bear risk.

Next, the right panel of Figure 4 indicates that the intermediation activity as measured by $avg(Y_t)$, or equivalently $\pi \cdot avg(Y_t)$ as $\pi$ is constant, is hump-shaped in cash flow volatility $\sigma$. The intuition for why it decreases in $\sigma$ for high levels of $\sigma$ is as above, that is, the intermediary is

[22] Figure F.3 provides complete graphs w.r.t. $r$ for initial cash-endowments, firm values, and relative value creation, i.e., $\overline{C}$ (left panel), $P(\overline{C}) - \overline{C}$ (middle panel), as well as replicating the middle panel of Figure 5 (right panel).

unwilling to take ownership of sufficiently risky firms. The intuition for why it increases in $\sigma$ for low levels of $\sigma$ is more nuanced. In the limit, as $\sigma \to 0$, the firm does not require any intermediary capital, nor any cash-buffers, to stave off liquidation, as the cash flows are risk-free. Thus, even though $\underline{C} < 0$, the need to dip into an intermediary equity stake vanishes, thus $Y = 0$. As $\sigma$ increases, the need to buffer shocks increases, increasing the value of both intermediation and cash holdings. For low $\sigma$, this effect dominates the increase in the cost of intermediation induced by $\sigma$. Therefore, even though $\underline{C}$ increases with $\sigma$, shrinking the maximum intermediary equity stake, the average use of such stakes increases due to the increase in shocks.

Finally, the right panel of Figure 5 shows that both the value of intermediation and PE is hump-shaped in cash flow volatility.[23] The intuition why firms with intermediate levels of cash flow volatility benefit the most from intermediary and PE financing is as follows. First, note that because the intermediary is risk-averse, the cost of intermediary financing increases with cash flow risk. Thus, firms with very volatile cash flow only have access to very expensive intermediary financing and therefore manage liquidity mostly by accumulating cash reserves. Second, firms with low cash flow volatility have access to relatively inexpensive intermediary financing but face limited financing needs and, therefore, mechanically do not have to rely on intermediary financing. Thus, the above arguments suggest that PE financing is most likely to occur in firms with an intermediate level of cash flow volatility. In particular, the model predicts low levels of PE investment in firms with low or high cash flow volatility.

## 4 Implementation — An Overlapping Pecking Order

We introduce an implementation of the optimal contract with common financial instruments, here a credit line provided by the intermediary, cash holdings by the firm, common equity held by shareholders, and restricted equity held by the intermediary. Interestingly, our implementation below suggests an overlapping pecking order that depends on the level of the firm's liquidity. When $C > 0$ and the firm holds cash, it uses both the credit line and its cash reserves to cover cash flow shortfalls. Symmetrically, following positive cash flow realizations, the firm retains earnings to grow its cash reserves and pays back the credit line. Importantly, $\beta(C)$ quantifies the extent of credit line usage, in that high value of $\beta(C)$ indicates that the firm covers negative cash flow shocks to a large extent by drawing on the credit line. Note that credit line usage $\beta(C)$ decreases with

[23] Figure F.4 provides complete graphs w.r.t. $\sigma$ for initial cash-endowments, firm values, and relative value creation, i.e., $\overline{C}$ (left panel), $P(\overline{C}) - \overline{C}$ (middle panel), as well as replicating the right panel of Figure 5 (right panel).

liquidity $C$, as shown in Figure 1.

Next, the firm does not use restricted equity until all cash reserves have been exhausted at $M = C = 0$ in that $Y(C) = 0$ for $C \geq 0$. When $C < 0$ and the firm's cash reserves are exhausted, the firm finances cash flow shortfalls by drawing on the credit line and selling restricted equity to the intermediary at the same time. The dollar value of restricted equity held by the intermediary is $Y(C)$ and increases following negative cash flow shocks, i.e., $Y(C)$ decreases with $C$. The firm uses positive cash flow realizations to repay the credit line and repurchase restricted equity. Upon market access, the firm raises financing by issuing common equity to new outside investors to replenish cash reserves, repurchase restricted equity from the intermediary, and pay back the credit line.

Thus, an overlapping pecking order of financial instruments arises. When the firm is flush with liquidity, the firm's internal cash reserves are used with the highest intensity to cover negative cash flow shocks. As liquidity reserves dwindle, the firm relies increasingly on credit line financing, and, as a last resort, the firm finances by selling equity to intermediaries. There are two notable differences to the traditional pecking order theory of Jensen and Meckling (1976). First, the pecking order is characterized in terms of the firm's liquidity reserves, not the firm's life cycle. Second, the pecking order suggests an intensity of use of different instruments and not strict dominance of one instrument over another. Specifically, the firm always simultaneously uses two instruments. That is, it always uses the credit line with either internal cash reserves or restricted equity to manage its liquidity.

To further illustrate how a credit line is a natural implementation of the optimal contract, we derive the credit line balance, $T(C)$. This balance records cumulative undiscounted transfers to and from the intermediary in response to Brownian cash flow shocks $dZ$ since the last time $C$ has reached $\underline{C}$ or $\overline{C}$. Thus, the volatility of $T(C)$, denoted $\sigma_T(C)$, must match $-\sigma_I(C)$, the negative of the volatility of transfers from the firm to the intermediary. In other words, total contributions increase one-for-one with transfers from the intermediary caused by cash flow shocks $dZ$. Using $\sigma_I(C) = \sigma\beta(C) + \sigma(1-\beta(C))\mathbb{1}_{\{C<0\}}$ and $\sigma_T(C) = T'(C)\sigma(1-\beta(C))$, we have

$$\sigma_T(C) = -\sigma_I(C) \iff T'(C) = -\frac{\beta(C)}{1-\beta(C)} - \mathbb{1}_{\{C<0\}}. \tag{33}$$

Plugging in $\beta(C)$ from (26), integrating, and imposing $T\left(\overline{C}\right) = 0$, we solve[24]

$$T(C) = \frac{\ln P'(C)}{\rho r} + Y(C) = \alpha_U(C) + Y(C). \tag{34}$$

As we have argued before, $Y(C)$ can be viewed as the intermediary's (equity) stake in the firm. We now argue that $\alpha_U(C)$ can be understood as a credit line balance. First, note that when $\underline{C} = 0$, $Y = 0$, so there is never an equity stake. Consider now a situation in which the limited commitment constraint (19) is not binding, i.e., $\alpha(C) = \alpha_U(C)$. Then, at refinancing, the firm *exactly* pays back the cumulative undiscounted transfers, $\alpha_I(C) = T(C)$, which looks exactly like repaying debt of face-value $D(C) = \alpha_U(C)$. We thus interpret $D(C)$ as a credit line balance.

A few observations are in order. First, as $D'(C) = -\frac{\beta(C)}{1-\beta(C)} < 0$, the firm partially covers negative cash flow shocks by drawing on its credit line, while it pays back some of the credit line after positive cash flows. Moreover, $\beta(C)$ captures the intensity of credit line usage, which decreases in the level of liquidity $C$. Second, upon financial market access, the firm raises financing to replenish its cash reserves, pay back the credit line, and repurchase the intermediary's stake in the firm, in that $D(\overline{C}) = Y(\overline{C}) = 0$ after refinancing. Third, the credit line may stipulate "early repayment incentives." Upon refinancing, the firm pays the intermediary $Y(C)$ dollars to buy back the intermediary's stake at fair value and $\alpha(C)$ dollars to repay the credit line $D(C)$ while existing shareholders are diluted. When $D(C)$ is large, the gains from refinancing mostly accrue to the intermediary and, in particular, to the credit line owners (i.e., creditors), effectively leading to "debt overhang." To incentivize shareholders to seek refinancing despite this debt overhang, the firm may pay back the credit line at a discount when $\alpha(C) < D(C) = \alpha_U(C)$, that is, when shareholders' limited commitment constraint (19) binds. Crucially, the interest rate and fees compensate the intermediary for potential default or repayment at a discount.

We complete the characterization of this implementation in Appendix D, where we discuss in detail how to determine the fees and interest of the credit line and how to implement the intermediary's stake as (restricted) equity.

## 4.1 Alternative Implementation

In this section, we consider an alternative implementation in which *all* financing from the intermediary is arranged as a credit line. That is, we interpret $T(C) = \alpha_U(C) + Y(C)$ from (33) as

[24]In more detail, $\frac{1}{\rho r}\frac{P''(C)}{P'(C)} = -\frac{\beta(C)}{1-\beta(C)} = \frac{d}{dC}\frac{\ln P'(C)}{\rho r}$.

the firm's credit line balance. When the firm holds cash $C = M(C) > 0$, it covers negative cash flow shocks by drawing on its cash reserves and the credit line. The credit line is not secured or collateralized, and the intermediary holds no stake in the firm, that is, $Y(C) = 0$. When the firm's cash reserves are exhausted, the firm defaults on its credit line if $\underline{C} = 0$ in which case the firm has no access to secured/collateralized credit line debt. Otherwise, if $\underline{C} < 0$, the intermediary provides further credit line financing but requires collateral. This collateral takes the form of a stake in the firm, $Y(C) > 0$. The intermediary is only willing to accept the firm as collateral if the cash-flows are sufficiently valuable or the firm's resale option value is high, in which case $\underline{C} < 0$. Once the credit line reaches its endogenous limit at $C = \underline{C}$, the entire firm serves as the underlying collateral.

The firm repays the credit following positive cash flow shocks or access to financial markets to raise equity. The situation $C = \underline{C}$ is akin to bankruptcy. With no external financing, the firm cannot recover from financial distress and repay the credit line. When $\underline{C} = 0$, the firm defaults and must liquidate. When $\underline{C} < 0$, the intermediary seizes the collateral, assumes full ownership of the firm, and keeps the firm alive until bankruptcy resolution. Bankruptcy is resolved upon market access $d\Pi_t = 1$ when the intermediary sells the firm to newly arriving outside investors. As before, the credit line stipulates "early repayment incentives" or debt forgiveness to incentivize shareholders to seek outside equity financing and to dilute their stake in light of a debt overhang problem.

In this interpretation, the intermediary resembles a specialized lender or bank, and its risk-aversion may reflect regulatory requirements. This implementation suggests an overlapping pecking order again, albeit a slightly different one: first, the firm finances cash flow shortfalls with internal cash reserves and unsecured/uncollateralized credit line debt. Only under financial distress has to rely on secured/collateralized credit line debt. The result that the firm relies on secured debt only under distress but on unsecured debt in normal times is consistent with the findings in Benmelech et al. (2020). Also in line with our model's predictions, Rauh and Sufi (2010) find that debt financing of high-credit-quality firms, which may correspond in our model to the ones with high liquidity reserves, predominantly takes the form of unsecured debt. In contrast, debt financing of firms with lower credit quality (i.e., with low liquidity reserves) involves some secured debt.

Interestingly, a firm's access to equity financing $\pi$ increases the firm's debt capacity, i.e., decreases $\underline{C}$, as it improves the resale option value of the underlying collateral (see Figure 3). Likewise, a decrease in the interest rate $r$ increases collateral value decreases $\underline{C}$, and so increases the credit line capacity, reducing the likelihood of default. Finally, the use of secured debt, as captured by

$avg(Y_t)$, is most pronounced when market access $\pi$ is at intermediate levels, a firm's cash flow volatility is intermediate, and interest rates are low (see Figure 4).

# 5 Discussion and Further Results

## 5.1 Renegotiation Proofness

Importantly, all contracts we derive above are renegotiation proof. Given any future renegotiation time $\hat{\tau}$ with some $C_{\hat{\tau}}$, the same contract is optimal going forward.[25] The result follows from two observations. First, the firm cannot raise external cash outside a refinancing opportunity, and thus a renegotiation can only decrease $C$: Either any reshuffling between $M$ and $Y$ leads to the same $C$, or the firm is giving out free promises $Y$, lowering $C$ which costs $P'(C) > 1$ for only a return of 1 to the intermediary. Further, the contract above already picked the optimal $Y$ to minimize the cost of holding cash, i.e., (22). Thus, no reshuffling of the delivery of $Y$ would deliver such continuation value in a cheaper way, i.e., $\alpha, \beta$ are already optimal. Second, consider renegotiation *during* a refinancing opportunity. With external cash available, we should renegotiate as long as surplus can be increased, i.e., $(P(C^*) + Y(C^*) - M(C^*))' = (P(C^*) - C^*)' > 0$. But as (23) implies, we are optimally refinancing to the payout boundary, which maximizes surplus. An important implication of renegotiation proofness is that today's contract with the current intermediary is not affected by a possible switch to and renegotiation with a new intermediary (given appropriate side-payments of value $Y$ between the intermediaries) at a future time.

## 5.2 Different Types of Intermediation and Secular Trends

As stated initially, we can view the intermediary in our model as representing a group of different types of intermediaries. The abstract intermediary provides different services to the firm characterized by the control variables. The implementation has also shown that the optimal financing arrangement involves credit line financing and equity-like financing provided by the intermediary. The extent of equity-like financing from the intermediary is quantified by $Y$, which we can as PE investment. Credit line financing, in turn, is traditionally provided by banks.[26] Note that $\beta$ captures the firm's reliance on credit line financing where high $\beta$ means that the firm covers a large part of negative cash flow shocks by drawing on the credit line.

[25] We follow Strulovici (2020) in defining renegotiation proofness in the presence of an exogenous state, here $C$.

[26] Admittedly, a recent trend is that PE-backed firms borrow from non-bank lenders too (Jang (2020)).

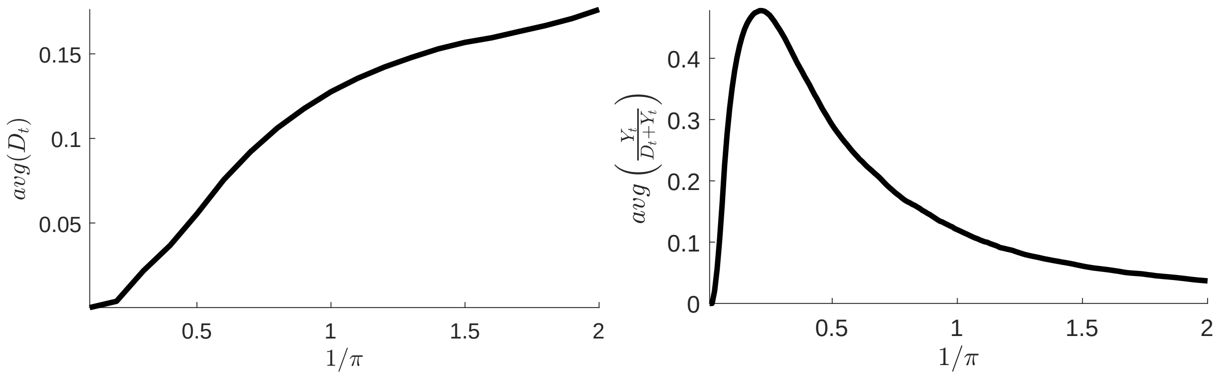


Figure 6: **Intermediation in steady state:** This figure plot the average credit line balance $avg(D_t)$ (left panel) and $avg\left(\frac{Y_t}{D_t+Y_t}\right)$ (right panel) against $1/\pi$. The parameters follow 1.

That is, we can split intermediation in our model into two parts: PE-like intermediation and bank-like intermediation (i.e., credit line financing). One key prediction of the paper is that the firm first relies on credit line financing and internal reserves to cover cash shortfalls. Only after the firm's liquidity reserves are sufficiently low does the firm seeks financing provided by PE investors. In addition, our model implies that the availability of credit line financing is necessary for PE financing to arise. In other words, PE investors are only willing to take a stake $Y > 0$, e.g., in the form of restricted equity, if the firm also has sufficient access to credit line financing. Formally, this result reflects that $Y > 0$ can only arise if $\underline{C} < 0$, in which case $\beta(\underline{C}) = 1$ and the firm never liquidates. If, by contrast, we impose $\beta = 0$ in the optimization (20) to capture that the firm has no access to credit line financing, the firm liquidates the first time its cash holdings reach zero, which occurs in the state $C = 0$. That is, the absence of credit line financing $\beta = 0$ would imply $C \geq 0$ and therefore $Y = 0$, thereby precluding PE investment $Y > 0$. Interpreted more broadly, the rise of PE investments also relies on the presence of other intermediaries providing credit line financing, in that credit line financing and PE investments are complementary.

In the following, we examine how changes in interest rate $r$ or capital market access $\pi$ shape the nature of financial intermediation. The prevalence of PE-like and bank-like intermediation are captured in our model by the average levels of $Y_t$ and $D_t$ (credit line balance): $avg(Y_t)$ was depicted in Figure 4. Figure 6 depicts $avg(D_t)$ and $avg\left(\frac{Y_t}{D_t+Y_t}\right)$ for different levels of $1/\pi$ (upper two panels) and different levels of $r$ (lower two panels). We interpret $avg\left(\frac{Y_t}{D_t+Y_t}\right)$ as the fraction of intermediation coming from PE-like intermediaries as opposed to traditional bank-like intermediaries.

First, consider the impact of financial market access, as indexed by $\pi$. The left panel of Figure 4

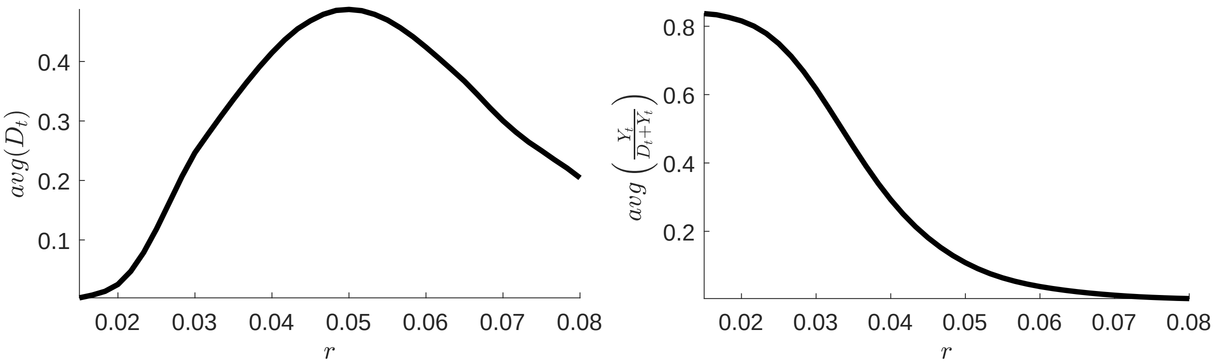


Figure 7: **Intermediation in steady state:** This figure plots the average credit line balance $avg(D_t)$ (left panel) and $avg\left(\frac{Y_t}{D_t+Y_t}\right)$ (right panel) against the interest rate $r$. The parameters follow 1.

showed the average equity stake is hump-shaped with respect to $1/\pi$. Next, the left panel Figure 6 shows that the average credit line balance increases with $1/\pi$. That is, better access to financial markets or improved financial market development imply that the firm relies less on credit line financing and, interpreted broadly, less on financing provided by banks. In contrast, the right panel of Figure 6 shows the fraction of intermediation coming from PE-like intermediaries is hump-shaped with $1/\pi$. In other words, the model implies that improvements in financial market access at first suggest a shift from traditional bank financing to PE financing. In contrast, once market access is "good" enough, a shift back to bank financing, although on a vanishing level of intermediation as $avg(D_t + Y_t)$ converges to zero.

As we have argued, we can interpret improvements in financial market access as the result of the firm's life cycle. In that case, these findings imply that early on in a firm's life cycle, i.e., for low $\pi$, the firm mostly relies on credit line or bank financing. As market access improves over the firm's life cycle, the firm's usage of credit line financing declines and its usage on PE financing increases, i.e., $avg(D_t)$ increases with $1/\pi$ and $avg(Y_t)$ decreases in $1/\pi$ for large values of $1/\pi$. Mature firms with sufficient access to capital markets rely mostly on equity financing provided by outside investors and less on a credit line or PE financing as $avg(D_t)$ and $avg(Y_t)$ are low for low values of $1/\pi$.

Next, let us consider the impact of the interest rate environment on financing arrangements. As the middle panel of Figure 4 illustrates, a decline in interest rate $r$ stimulates PE investments and intermediation, in that the average PE investment $avg(Y_t)$ decreases with $r$. The left panel of Figure 7 shows that the average credit line balance $avg(D_t)$ — a proxy for intermediation activity by banks — is hump-shaped in $r$. Intuitively, high and low interest rates curb intermediation activity by banks. When the interest rate $r$ is high, the intermediary's cost of capital is high. It is costly for the intermediary to cover negative shocks by drawing on its savings account, which boosts the intermediary's required risk compensation $\frac{\rho r}{2}(\beta\sigma)^2$ and so the cost of credit line financing. When $r$ is low, holding cash is costly, and firms heavily rely on intermediary financing. But as low interest rates favor PE financing, the firm mostly relies on financing provided by PE intermediaries and only to a lesser extent on credit line financing. That is, the decline in interest rates favors PE intermediaries but crowds out bank-like intermediaries. The right panel of Figure 7 supports this intuition by plotting $avg\left(\frac{Y_t}{D_t+Y_t}\right)$ against $r$. We see that the fraction of PE intermediation monotonically decreases with the interest $r$ so that for low interest rates, PE investors play a majority role in providing financing to firms.

To summarize, both the secular decline in interest rates and improvements in financial market access can explain the recent rise of intermediation activity conducted by PE firms instead of banks.

### 5.3 Other Results

Appendix F presents comparative statics that are omitted from the main text. For instance, we find that average cash holdings increase with $1/\pi$, with the interest rate $r$, and with cash flow volatility $\sigma$ (see Figure F.1). Figure F.2, Figure F.3, and Figure F.4, display the payout boundary and initial value under the baseline, the no-intermediation benchmark (i.e., $\alpha = \beta = 0$), and the no-PE benchmark (i.e., $Y = 0$) for different values of $1/\pi$, $r$, and $\sigma$ respectively.

### 5.4 Empirical Implications for PE

Besides the direct implications of the implementation previously discussed, the model delivers several empirical predictions and implications.

**Prediction.** *PE-supported firms are less likely to be liquidated.*

This prediction follows from the result that the firm does not face liquidation risk when $\underline{C} < 0$. Further, when $\underline{C} < 0$, the intermediary infuses capital in exchange for a stake $Y > 0$ in the firm following negative cash flow realizations and under financial distress. Consistent with our theoretical findings, Bernstein et al. (2019) and Hotchkiss et al. (2021) present evidence that PE investors inject capital in exchange for ownership stakes when portfolio firms undergo financial distress. Through the provision of financing under distress, more efficiently reduce the risk of liquidation and the exposure to negative shocks (Bernstein, Lerner, Sorensen, and Strömberg (2017)).

The implementation of the optimal contract suggests an overlapping pecking order in which credit line financing substitutes financing via internal cash reserves and financing via equity. However, the model also predicts that the availability of credit line financing is necessary for PE investments to arise. This insight leads to the following prediction.

**Prediction.** *Credit line financing and PE investments are complementary.*

Formally, the possibility of $\beta > 0$ and, in particular, $\beta = 1$ is necessary for $Y > 0$ to occur in equilibrium. If there is no credit line financing, i.e., $\beta = 0$, the firm would be liquidated once it runs out of cash, and the intermediary would never be willing to take a stake $Y > 0$ in the firm. This finding is consistent with the fact that PE investors typically engage in financial engineering (Kaplan and Stromberg (2009)), which includes making portfolio firms borrow from banks to increase their leverage. In related work, Ivashina and Kovner (2011) document the importance of private equity firms' bank relationships, suggesting complementarities between bank-like intermediation (responsible for credit line financing) and PE-like intermediation.

The next prediction concerns the equilibrium levels of intermediation activity and PE-like financing.

**Prediction.** *PE placements are less likely when either capital markets are very liquid or very illiquid.*

Recall that PE placements, as captured by $Y$, only occur when the intermediary is willing to keep the firm alive in financial distress. This willingness increases with financial market access $\pi$, which suggests that periods with booming financial markets (characterized by a large value of $\pi$) also experienced greater private equity fundraising as documented in Axelson, Jenkinson, Strömberg, and Weisbach (2013). However, the average outstanding placement shrinks once $\pi$ increases sufficiently, as the positions are exited quickly. The reason is that the intermediary's increased exit implies that positions never grow very large, even though intermediaries would be willing to build up large positions $Y$. In any case, our model highlights that exit prospects are one key determinant for PE investors' willingness to take ownership in distressed firms.

The model also predicts how the level of PE investment depends on the firm's cash flow volatility.

**Prediction.** *PE investments are highest in firms with intermediate levels of cash flow volatility.*

This prediction stems from the result that the average intermediary stake, $avg(Y_t)$, is hump-shaped in cash flow volatility. In particular, the model suggests that PE firms tend not to invest in the riskiest firms but only in those that are sufficiently, but not too, risky. These firms also benefit the most from the availability of PE financing.

The last prediction concerns equilibrium levels of PE intermediation in response to the secular decline of interest rates.

**Prediction.** *A decline in interest rate spurs PE investments.*

As we have shown in Section 3.3, the secular decline in interest rate can be one reason driving the simultaneous expansion of the size of the PE sector.

### 5.5 Discussion of Key Assumptions

Let us discuss two key assumptions that deviate from the literature and contrast them with alternative choices.

**Infrequent capital market access.** As opposed to Décamps et al. (2011) and Bolton et al. (2011), who model fixed and variable cost of equity issuance, we follow Hugonnier et al. (2015) in assuming that the firm is unable to raise capital outside random, i.e., Poisson, times, capturing capital supply uncertainty or delays in raising financing. For instance, this assumption may reflect (un-modeled) adverse selection, i.e., arranging financing takes time as dispersed outside investors need to verify information. Indeed, many empirical studies document that firms often face uncertainty regarding their future access to capital markets.[27] First, note that one can interpret our assumption as a Markov chain variant in which the costs are infinite except for very short periods at which the costs vanish (Bolton et al. (2013)). Second, and more importantly, the key difference to an issuance cost setup lies with how each of these assumptions link to possible firm liquidation: Under a costly equity issuance assumption, the fixed and variable costs parameters determine the firm's survival. Under our assumptions, the firm's survival rests on the willingness of the intermediary to take a stake in the firm and bridge the time until market access. Thus, our assumptions are more relevant for the purposes of this paper and, in particular, for describing young and private firms that have limited market access.[28]

[27] See, e.g., Campello et al. (2010), Duchin, Ozbas, and Sensoy (2010), or Lemmon and Roberts (2010) and other relevant empirical references cited in Hugonnier et al. (2015).

[28] Berger and Udell (1995) show that the length of a banking relationship plays a key role in the level of credit lines provided to a small firm, reflecting information asymmetries. Thus, the speed at which a firm can raise outside

**Costly intermediary financing via CARA.** We want to capture the reality that specialized intermediary financing is costly. Moreover, the riskier the intermediary's payouts, the more expensive this financing will be. To capture this mechanism, we could have introduced an intermediary that is (i) cash-constrained or (ii) has a regulatory cost-of-capital linked to the riskiness of its financial agreements. Note that modeling the cost of intermediary financing via (i) would result in a higher cost of funds for riskier financing arrangements, as higher risk makes the intermediary more likely to hit its own funding constraint. But this would introduce an additional state variable, the cash holdings of the intermediary. Modeling it via (ii) would require us to take a stand on how the riskiness of financing translates into the regulatory cost of funds. We view CARA utility as a parsimonious way of introducing a cost of capital that varies with its riskiness while not requiring an additional state variable. In other words, we use CARA utility as a proxy for (i) and (ii) without the technical difficulties that either of these choices entails.

## 6 Conclusion

We consider a firm with infrequent access to capital markets but continuous access to financing by a risk-averse intermediary. As intermediary financing is costly, the firm holds internal cash reserves to manage its liquidity. Under the optimal financing agreement, the intermediary absorbs a fraction of cash flow risk that decreases in the firm's excess liquidity position. Once the firm depletes its cash reserves, the firm either liquidates or the intermediary rescues the firm by injecting capital in exchange for a stake in the firm. Crucially, the intermediary's willingness to rescue the firm increases with its prospective financial market access. Indeed, this access allows the intermediary to exit its position and resell the stake to risk-neutral and competitive investors. The intermediary's strategy to take a stake in the firm under financial distress and sell off this stake after the resolution of distress resembles the strategy of PE investors. Importantly, such PE-like intermediation helps resolve distress and reduce the risk of liquidation more efficiently. Our model implies that the extent of such PE-like intermediation crucially depends on financial market access, facilitating the exit of the intermediary's position in the firm. We find that PE-like intermediation is most pronounced when the firm's market access and cash flow volatility lie at intermediate levels. Interestingly, our findings suggest that a decline in interest rate spurs PE financing. Finally, we show how to

funding and the ability of the intermediary to provide immediate financing may reflect both the speed with which the outside investors can overcome such information asymmetries and the privileged position of the intermediary as an informed relationship lender.

implement the optimal contract with a credit line provided by the intermediary and restricted equity held by the intermediary. Our implementation suggests an overlapping pecking order of financial instruments. The firm simultaneously finances cash flow shortfalls with cash reserves and a credit line and only sells equity to the intermediary in distress. We also discuss an alternative implementation of the optimal contract, which calls for the firm to finance cash flow shortfalls first with an unsecured credit line and cash reserves, while it relies on a secured credit line only under distress.

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# Appendix

## A Proof of Proposition 1

We split the proof into two parts. The first part characterizes the intermediary's consumption smoothing and optimal consumption. The second part derives the dynamics of $W_t$.

### A.1 Part I — Optimal Consumption

We first state an auxiliary Lemma:

**Lemma 1.** *Take a process $\hat{I}$ and $s_1, s_2 \in \mathbb{R}$. Consider the problem*

$$U_t := U_t(c) = \max_{\{c_s\}_{s\geq t}} \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} u(c_s) ds \right] \tag{A.1}$$
$$\text{subject to } dS_s(c) = rS_s(c)ds + d\hat{I}_s - c_s ds, \; S_t(c) = s_1, \text{ and } \lim_{s\to\infty} \mathbb{E}e^{-r(s-t)} S_s(c) = 0,$$

*where we explicitly denote the dependence of savings $S$ on the consumption path $c$. Next, consider the problem*

$$\tilde{U}_t := \tilde{U}_t(\tilde{c}) = \max_{\{\tilde{c}_s\}_{s\geq t}} \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} u(\tilde{c}_s) ds \right] \tag{A.2}$$
$$\text{subject to } d\tilde{S}_s(\tilde{c}) = r\tilde{S}_s(\tilde{c})ds + d\hat{I}_s - \tilde{c}_s ds, \; \tilde{S}_t(\tilde{c}) = s_2, \text{ and } \lim_{s\to\infty} \mathbb{E}e^{-r(s-t)} \tilde{S}_s(\tilde{c}) = 0.$$

*Then, for $\Delta^S := s_2 - s_1$, the optimal consumption processes $c$ and $\tilde{c}$, solving (A.1) and (A.2) respectively, satisfy $\tilde{c}_t = c_t + r\Delta^S$ so that $\tilde{U}_t = e^{-\rho r \Delta^S} U_t$.*

*Proof.* To start with, note that with $\tilde{c}_s = c_s + r\Delta^S$,

$$\tilde{U}_t(\tilde{c}) = \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} u(c_s + r\Delta^S) ds \right] = e^{-\rho r \Delta^S} \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} u(c_s) ds \right] = e^{-\rho r \Delta^S} U_t(c), \tag{A.3}$$

where the first equality uses $\tilde{c}_s = c_s + r\Delta^S$ and the second equality uses

$$u(c_s + r\Delta^S) = -\frac{e^{-\rho(c_s + r\Delta^S)}}{\rho} = e^{-\rho r \Delta^S} \left( -\frac{e^{-\rho c_s}}{\rho} \right) = e^{-\rho r \Delta^S} u(c_s). \tag{A.4}$$

Next, suppose to the contrary that there exists a different consumption process $c' \neq \tilde{c}$, solving problem (A.2), with

$$\tilde{U}_t(c') > \tilde{U}_t(\tilde{c}) = e^{-\rho r \Delta^S} U_t(c), \tag{A.5}$$

and the transversality condition $\lim_{s\to\infty} \mathbb{E}e^{-r(s-t)} \tilde{S}_s(c') = 0$ holds under the consumption process $c'$. Define the consumption process $c''$ via $c''_t = c'_t - r\Delta^S$. As $c'$ is different from $\tilde{c}$, it follows that $c''$ is different from $c$. As under the consumption path $c'$ the transversality condition $\lim_{s\to\infty} \mathbb{E}e^{-r(s-t)} \tilde{S}_s(c') = 0$ holds, it follows that under the consumption path $c''$ the transversality condition $\lim_{s\to\infty} \mathbb{E}e^{-r(s-t)} S_s(c'') = 0$ holds too. In addition, note that the payoff under the consumption path $c''$ equals

$$U_t(c'') := \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} u(c''_s) ds \right] = e^{\rho r \Delta^S} \tilde{U}_t(c') > e^{\rho r \Delta^S} e^{-\rho r \Delta^S} U_t(c) = U_t(c), \tag{A.6}$$

where the second equality applies (A.4), which yields $u(c'_s) = e^{-\rho r \Delta^S} u(c''_s)$ and $u(c''_s) = e^{\rho r \Delta^S} u(c'_s)$, and the inequality uses (A.5). However, $U_t(c'') > U_t(c)$ contradicts the fact that $c$ solves problem (A.1). The assertion follows. □

Using Lemma 1, we can now complete the argument by showing that optimal consumption satisfies $u(c_t) = rU_t$ and $c_t = rW_t$. According to Lemma 1, the marginal value of an additional unit of savings $S_t$ at time $t$ for the intermediaryis given by

$$\left[\frac{\partial}{\partial \Delta^S} e^{-\rho r \Delta^S} U_t\right] |_{\Delta^S=0} = -\rho r U_t. \tag{A.7}$$

The intermediary's optimal consumption smoothing implies that along the optimal path the first order condition

$$u'(c_t) = \left[\frac{\partial}{\partial \Delta^S} e^{-\rho r \Delta^S} U_t\right] |_{\Delta^S=0} \tag{A.8}$$

has to hold at all times $t \geq 0$. That is, in optimum, the intermediary's marginal utility $u'(c_t)$ has to be equal to the marginal value of an additional unit of savings, $\left[\frac{\partial}{\partial \Delta^S} e^{-\rho r \Delta^S} U_t\right] |_{\Delta^S=0}$.

Next, observe that $u'(c_t) = -\rho u(c_t)$ and use (A.7), so that (A.8) becomes $u(c_t) = rU_t$. Inverting the relation $u(c_t) = rU_t$ and solving for $c_t$ yields $c_t = rW_t$, with

$$W_t := \frac{-\ln(-\rho r U_t)}{\rho r}, \tag{A.9}$$

which is (8).

Finally, we examine the intermediary's outside option $\bar{U}$ from staying away from the contract and perpetually consuming out of its savings which we normalize at inception to zero, that is, $S_{0^-} = 0$. When the intermediary stays away from the contract, its consumption is constant over time because the intermediary's discount rate is equal to the savings/borrowings rate $r$. As the intermediary's initial savings are normalized to zero, the intermediary then perpetually consumes zero to satisfy the transversality constraint, $\lim_{t\to\infty} \mathbb{E} e^{-rt} S_t = 0$. As such, when the intermediary stays away from the contract, then $c_t = rW_t = 0$ by our previous arguments, thus $W_t = W_0 = 0$. Moreover, using $u(0) = -1/\rho$ and evaluating the integral expression (5), we can solve $\bar{U} = \frac{-1}{\rho r}$. The participation constraint $U_0 \geq \bar{U}$ is therefore equivalent to $W_0 \geq 0$.

### A.1.1 Martingale Representation and Law of Motion (9)

Take the intermediary's continuation value

$$U_t = \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(c_s) ds\right], \tag{A.10}$$

under any consumption process $c_t$ (possibly, $c_t = c_t^*$). Define

$$A_t = \mathbb{E}_t \left[\int_0^\infty e^{-rs} u(c_s) ds\right] = \int_0^t e^{-rs} u(c_s) ds + e^{-rt} U_t. \tag{A.11}$$

By construction, $A = \{A_t\}$ is a martingale. By the martingale representation theorem, there exist stochastic processes $\hat{\alpha} = \{\hat{\alpha}_t\}$ and $\beta = \{\beta_t\}$ such that

$$e^{rt} dA_t = (-\rho r U_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho r U_t)\hat{\alpha}_t(d\Pi_t - \pi dt), \tag{A.12}$$

where $dZ_t = \frac{dX_t - \mu dt}{\sigma}$ is the increment of a standard Brownian Motion under the probability measure and $(d\Pi_t - \pi dt)$ is the increment of a compensated Poisson process which is a martingale.

We differentiate (A.11) with respect to time $t$ to obtain an expression for $dA_t$, then plug this expression into (A.12) and solve (A.12) to get

$$dU_t = rU_t dt - u(c_t)dt + (-\rho r U_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho r U_t)\hat{\alpha}_t(d\Pi_t - \pi dt). \tag{A.13}$$

With the optimal consumption policy $c_t = c_t^*$, satisfying $u(c_t) = rU_t$, equation (A.13) simplifies to

$$dU_t = (-\rho r U_t)\beta_t\big(dX_t - \mu dt\big) + (-\rho r U_t)\hat{\alpha}_t(d\Pi_t - \pi dt), \tag{A.14}$$

which is a martingale in that $\mathbb{E}[dU_t] = 0$.

Next, we derive the law of motion of

$$W_t = W(U_t) := \frac{-\ln(-\rho r U_t)}{\rho r}. \tag{A.15}$$

To do so, note that

$$W'(U) = \frac{1}{-\rho r U} \quad \text{and} \quad W''(U) = \frac{1}{\rho r U^2} \tag{A.16}$$

and

$$W(U - \rho r U \hat{\alpha}) - W(U) = W(U(1 - \rho r \hat{\alpha})) - W(U) = -\frac{\ln(1 - \rho r \hat{\alpha})}{\rho r}. \tag{A.17}$$

Next, we use Itô's Lemma in its version for jump processes and calculate via (A.14)

$$\begin{aligned} dW_t = dW(U_t) &= W'(U_t)\rho r U_t \pi \hat{\alpha}_t dt + W'(U_t)(-\rho r U_t)\beta_t \sigma dZ_t \\ &\quad + W''(U_t)\left(\frac{(\rho r U_t)^2(\beta_t\sigma)^2}{2}\right)dt + \big[W(U_t - \rho r U_t \hat{\alpha}_t) - W(U_t)\big]d\Pi_t \\ &= -\pi\hat{\alpha}_t dt + \beta_t\sigma dZ_t + \frac{\rho r}{2}(\beta_t\sigma)^2 dt - \frac{\ln(1 - \rho r \hat{\alpha}_t)}{\rho r} d\Pi_t \end{aligned} \tag{A.18}$$

Next, we set

$$\alpha_t := -\frac{\ln(1 - \rho r \hat{\alpha}_t)}{\rho r} \quad \Longleftrightarrow \quad \hat{\alpha}_t = \frac{1 - e^{-\rho r \alpha_t}}{\rho r}. \tag{A.19}$$

Thus,

$$dW_t = \frac{\rho r}{2}(\beta_t\sigma)^2 dt - \pi\left(\frac{1 - e^{-\rho r \alpha_t}}{\rho r}\right)dt + \beta_t \sigma dZ_t + \alpha_t d\Pi_t, \tag{A.20}$$

which we can rewrite as

$$dW_t = \left[\frac{\rho r}{2}(\beta_t\sigma)^2 + \pi\left(\alpha_t - \frac{1 - e^{-\rho r \alpha_t}}{\rho r}\right)\right]dt + \beta_t\sigma dZ_t + \alpha_t(d\Pi_t - \pi dt). \tag{A.21}$$

Above expression for $dW_t$ is (9), as desired.

Next, we study the drift of $dW_t$ in (9). Clearly, the first term, $\frac{\rho r}{2}(\beta_t\sigma)^2$, is positive and increases with $\beta_t$. For the second term, we calculate the derivatives $\frac{\partial}{\partial \alpha_t}\left(\alpha_t - \frac{1-e^{-\rho r \alpha_t}}{\rho r}\right) = 1 - e^{-\rho r \alpha_t}$ and $\frac{\partial^2}{\partial \alpha_t^2}\left(\alpha_t - \frac{1-e^{-\rho r \alpha_t}}{\rho r}\right) = \rho r e^{-\rho r \alpha_t} > 0$, so that $\left(\alpha_t - \frac{1-e^{-\rho r \alpha_t}}{\rho r}\right)$ is strictly convex in $\alpha_t$. Moreover, note that $\left(\alpha_t - \frac{1-e^{-\rho r \alpha_t}}{\rho r}\right)|_{\alpha_t=0} = 0$ and $\frac{\partial}{\partial \alpha_t}\left(\alpha_t - \frac{1-e^{-\rho r \alpha_t}}{\rho r}\right)|_{\alpha_t=0} = 0$, so that $\left(\alpha_t - \frac{1-e^{-\rho r \alpha_t}}{\rho r}\right)$ has a unique minimum at $\alpha_t = 0$ and is zero at this point. Thus, $\left(\alpha_t - \frac{1-e^{-\rho r \alpha_t}}{\rho r}\right) \geq 0$ for all $\alpha_t$ and, therefore, $\mathbb{E}[dW_t] \geq 0$, i.e., $W_t$ is a sub-martingale and increases in expectation.

# B Proof of Proposition 2

Recall (4), that is,

$$dM_t = \mu dt + \sigma dZ_t + (r - \lambda) M_t dt - dDiv_t - dI_t + \Delta M_t d\Pi_t, \tag{B.1}$$

Using (2) and (9) as well as $c_t = rW_t$ under the optimal consumption, we can calculate for $Y_t = W_t - S_t$:

$$dY_t = dW_t - dS_t = \left[rY_t + \frac{\rho r}{2}(\beta_t \sigma)^2 - \pi\left(\frac{1-e^{-\rho r \alpha_t}}{\rho r}\right)\right] dt + \beta_t \sigma dZ_t + \alpha_t d\Pi_t - dI_t, \tag{B.2}$$

which is (11). Next, we combine (4) and (11) to calculate for $C_t = M_t - Y_t$:

$$\begin{aligned} dC_t = dM_t - dY_t = \mu dt + r(M_t - Y_t)dt + \lambda(Y_t - M_t)dt - \lambda Y_t dt - \frac{\rho r}{2}(\beta_t \sigma)^2 dt + \pi\left(\frac{1-e^{-\rho r \alpha_t}}{\rho r}\right) dt \\ + \sigma(1-\beta_t)dZ_t - dDiv_t + (\Delta M_t - \alpha_t)d\Pi_t. \end{aligned} \tag{B.3}$$

We define the "post-refinancing" level of excess liquidity

$$C_t^* = C_t + \Delta M_t - \alpha_t, \tag{B.4}$$

so that

$$\Delta M_t = C_t^* - C_t + \alpha_t \quad \text{and} \quad \Delta M_t - \alpha_t = C_t^* - C_t. \tag{B.5}$$

Using these relations, we obtain

$$\begin{aligned} dC_t = \left[\mu + (r-\lambda) C_t - \lambda Y_t - \frac{\rho r}{2}(\beta_t \sigma)^2 + \pi\left(\frac{1-e^{-\rho r \alpha_t}}{\rho r}\right)\right] dt \\ + \sigma(1-\beta_t) dZ_t + (C_t^* - C_t) d\Pi_t - dDiv_t, \end{aligned} \tag{B.6}$$

which is (16) as desired.

We denote the drift of $dC_t$ by

$$\mu_C(C_t) = \left[\mu + (r-\lambda) C_t - \lambda Y_t - \frac{\rho r}{2}(\beta_t \sigma)^2 + \pi\left(\frac{1-e^{-\rho r \alpha_t}}{\rho r}\right)\right] \tag{B.7}$$

and the volatility of $dC_t$ by

$$\sigma_C(C_t) = \sigma(1-\beta_t). \tag{B.8}$$

Next, we conjecture and verify that the equity value can be expressed as function of $C_t$ only (i.e., $P_t = P(C_t)$), while $Y_t = Y$ is a control variable. Indeed, according to (11) and (16), it is always possible to increase or decrease $Y$ by picking $dI < 0$ or $dI > 0$, whilst leaving the value of excess liquidity $C$ unchanged. Given the Markovian representation, we omit time subscripts unless necessary.

Given $\mathcal{C}$, the equity value at time $t$ (i.e., shareholders' value function) reads

$$P_t = P(C_t) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} (dDiv_s - \Delta M_s d\Pi_s) \Big| C_t = C \right].$$

By the dynamic programming principle, the equity value $P(C)$ must then solve the HJB equation

$$rP(C)dt = \max_{\beta, Y, \alpha, C^*, dDiv \geq 0} \Big\{ dDiv + \mathbb{E}[dP(C) - \Delta M d\Pi] \Big\}, \tag{B.9}$$

subject to shareholders' and intermediary's limited commitment constraints (19), $Y \geq 0$, the constraint that cash holdings remain positive, $M \geq 0$, and the limited commitment constraint (19). Invoking Ito's Lemma, we can calculate

$$dP(C) = P'(C)\mu_C(C)dt + \frac{P''(C)\sigma_C(C)^2}{2}dt + P'(C)\sigma_C(C)dZ + (P(C^*) - P(C))d\Pi_t - P'(C)dDiv. \tag{B.10}$$

Thus,

$$\mathbb{E}[dP(C)] = P'(C)\mu_C(C)dt + \frac{P''(C)\sigma_C(C)^2}{2}dt + (P(C^*) - P(C))\pi dt - P'(C)dDiv. \tag{B.11}$$

Using this relation and $\Delta M_t = C_t^* - C_t + \alpha_t$, we can write the HJB equation (B.9) as

$$\begin{aligned} rP(C)dt = \max_{\beta, Y, \alpha, C^*, dDiv \geq 0} \Big\{ \left[1 - P'(C)\right] dDiv + P'(C)\mu_C(C)dt + \frac{P''(C)\sigma_C(C)^2}{2}dt \\ + \pi\big(P(C^*) - P(C) - (C^* - C) - \alpha\big)dt \Big\}, \end{aligned} \tag{B.12}$$

which is solved subject to shareholders' and intermediary's limited commitment constraints (19) and $Y \geq 0$ as well as the constraint that cash holdings remain positive, $M \geq 0$.

As $dDiv \geq 0$, it is optimal to stipulate dividend payouts if and only if $P'(C) \geq 1$. As in related papers (e.g., Bolton et al. (2011)), dividend payouts occur at a payout boundary $\overline{C}$ and follow a barrier strategy, that is, $dDiv = \max\{C - \overline{C}, 0\}$, and dividend payouts cause $C$ to reflect at $\overline{C}$. The location of the payout boundary is determined by smooth pasting and super contact conditions, that is,

$$P'(\overline{C}) = 1 \quad \text{and} \quad P''(\overline{C}) = 0, \tag{B.13}$$

again, as in Bolton et al. (2011).

For $C < \overline{C}$, there are no dividend payouts and the HJB equation (B.12) simplifies to

$$rP(C) = \max_{\beta, Y, \alpha, C^*} \Big\{ P'(C)\mu_C(C) + \frac{P''(C)\sigma_C(C)^2}{2} + \pi\big(P(C^*) - P(C) - (C^* - C) - \alpha\big) \Big\}, \tag{B.14}$$

subject to shareholders' and intermediary's limited commitment, $Y \geq 0$, $M \geq 0$, and the limited commitment constraint (19). The above HJB equation becomes (20) after inserting above expressions for $\mu_C(C)$ and $\sigma_C(C)$ respectively (see (B.7) and (B.8)). The right-hand side of the HJB equation only depends on $C$, the control variables, and exogenous model constants, and so does the left-hand side. In addition, because $P(C)$ is a function of $C$, the derivatives $P'(C)$ and $P''(C)$

are functions of $C$ too, and so are the control variables. As a result, we have verified that equity value can be expressed as a function of $C$ only so that $C$ is the only payoff-relevant state variable.

Section 2.3 in the main text goes through the maximization of the HJB equation (20), and derives the optimal control variables as functions of excess liquidity $C$, that is, $Y = Y(C)$, $M = M(C)$, $\alpha = \alpha(C)$, $\beta = \beta(C)$, and $C^* = C^*(C)$.

In what follows, we make the following technical regularity assumption.

**Assumption 1.** *Fix $\underline{C}$, and take a constant $K \geq 0$. The HJB equation (20) with the boundary conditions $P'(\overline{C})-1 = P''(\overline{C}) = 0$ and $P(\underline{C}) = K$ admits a unique solution $P(C)$ on the endogenous state space $(\underline{C}, \overline{C})$, with payout boundary $\overline{C}$. The solution $P(C)$ is twice continuously differentiable on $(\underline{C}, \overline{C})$, which implies that $P'(C)$ and $P''(C)$ exist and are continuous on the interval $(\underline{C}, \overline{C})$. The set of points $C \in (\underline{C}, \overline{C})$ at which either $P''(C)$ or $\alpha(C)$ is not differentiable is countable.*

# C Proof of Proposition 3

To start with, note that the optimal control variables are derived in the main text in Section 2.3 by going through the otpimization in the HJB equation (20). That is, Section 2.3 in the main text derives the optimal control variables as functions of excess liquidity $C$, that is, $Y = Y(C)$, $M = M(C)$, $\alpha = \alpha(C)$, $\beta = \beta(C)$, and $C^* = C^*(C)$.

Here, we prove the remaining claims of Proposition 3. We split the proof of Proposition 3 into several parts. Part I establishes the concavity of the equity value (i.e., $P''(C) \leq 0$) and shows that the payout boundary is strictly positive. Part II proves that the HJB equation (20) simplifies to (27). Part III characterizes the lower boundary in the state space $\underline{C}$ and, in particular, establishes (28) and (29). Part IV characterizes the optimal transfer process $dI$. Importantly, we prove Proposition 3 under the technical Assumption 1.

## C.1 Part I — Concavity of Value Function and $\overline{C} > 0$

Define the jump in the value function upon refinancing as

$$J(C) \equiv P\left(\overline{C}\right) - P\left(C\right) - \left(\overline{C} - C + \alpha(C)\right), \tag{C.1}$$

under the optimal choice of the refinancing target $C^* = \overline{C}$. When $J(C)$ is differentiable (which is the case when $\alpha(C)$ is differentiable), then $J'(C) = 1 - P'(C) - \alpha'(C)$. We now rewrite the HJB equation (20) as

$$rP\left(C\right) = \max_{\beta \in [0,1]} \left\{ P'(C)\mu_C(C) + \frac{P''(C)}{2}\sigma^2(1-\beta(C))^2 + \pi J(C) \right\}, \tag{C.2}$$

under the optimal choice of $\alpha$ in (25), $Y = \max\{-C, 0\}$ and $C^* = \overline{C}$, and with $\mu_C(C)$ from (B.7).

When $P''(C)$ and $\alpha(C)$ are differentiable, we can use the envelope theorem and differentiate the HJB equation (C.2) under the optimal $\beta$ with respect to $C$ and rearrange to obtain

$$P'''(C) = \frac{2}{(1-\beta)^2\sigma^2}\left(P'(C)\lambda \mathbb{1}_{\{C\geq 0\}} - P''(C)\mu_C(C) - \pi\left(e^{-\rho r\alpha(C)}P'(C)\alpha'(C) + J'(C)\right)\right), \tag{C.3}$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function which is equal to one if $\{\cdot\}$ is true and is equal to zero otherwise. The set of points at which either $P''(C)$ or $\alpha(C)$ is not differentiable is countable;

therefore, for any $C$, the limits $\lim_{x\uparrow C} P'''(C), \lim_{x\downarrow C} P'''(C)$ and $\lim_{x\uparrow C} \alpha'(C), \lim_{x\downarrow C} \alpha'(C)$ exist and are well-defined.

Suppose that $\alpha(C)$ is differentiable. If $\alpha(C) = \alpha_{LC}(C) = P(\overline{C}) - \overline{C} - (P(C) - C)$, then $\alpha'(C) = 1 - P'(C) \leq 0$ and $J'(C) = 0$. When $\alpha(C) = \alpha_U(C) = \frac{\ln P'(C)}{\rho r}$, then $\pi\left(e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C)\right) = 1 - P'(C) \leq 0$. Thus, altogether,

$$\pi\left(e^{-\rho r\alpha(C)} P'(C)\alpha'(C) + J'(C)\right) \leq 0, \tag{C.4}$$

provided $\alpha(C)$ is differentiable.

At the payout boundary, it therefore holds that $P'(\overline{C}) = 1$, and $P''(\overline{C}) = 0$ and

$$\lim_{C\uparrow\overline{C}} \left(e^{-\rho r\alpha(C)} P'(C)\alpha'(C) + J'(C)\right) = \lim_{C\uparrow\overline{C}} (1 - P'(C)) = 0. \tag{C.5}$$

We now show that $\overline{C} > 0$. Suppose to the contrary that $\overline{C} \leq 0$, and that $\underline{C} < \overline{C} \leq 0$. We argue that the firm is never liquidated in the scenario $\overline{C} \leq 0$. Observe that at the time of liquidation, $Y = 0$ (i.e., $Y_\tau = 0$). Suppose to the contrary that liquidation occurs at some value $C < 0$, so $M(C) = 0$ and $Y = Y(C) = -C > 0$. At liquidation, the total liquidation value is the firm's cash $M(C)$ which is split by intermediary and shareholders, so that $P(C) + Y(C) = M(C)$. Since $P(C), Y(C) \geq 0$ due to limited commitment, $M(C) = 0$ readily implies $P(C) = Y(C) = 0$, a contradiction. Thus, liquidation can only occur in states $C \geq 0$, and cannot occur when $\underline{C} < \overline{C} \leq 0$.

Next, using (C.5) and (C.3), we obtain $\lim_{C\uparrow\overline{C}} P'''(C) = 0$. It follows that $P''(C) = P'''(C) = 0$ and $P'(C) = 1$ for $C < \overline{C}$. As a result, (26) implies $\beta(C) = 0$ for $C \leq \overline{C}$. Moreover, $Y(C) = -C$. Clearly, total firm value $Y(C) + P(C)$ is bounded from above by the firm's net present value $\frac{\mu}{r}$. Because $P(C) \geq 0$ (shareholders' limited commitment), it follows that $Y(C)$ is bounded from above by $\frac{\mu}{r}$ (i.e., $Y(C) \leq \frac{\mu}{r}$ with certainty). However, due to $\beta(C) = 0$ for $C \leq \overline{C}$ and the fact that the firm is not liquidated due to $\overline{C} \leq 0$, the law of motion (16) implies that $C$ is not bounded from below and thus that $Y(C) = -C$ is not bounded from above, a contradiction. Thus, $\overline{C} > 0$.[29]

As $\overline{C} > 0$, $P''(\overline{C}) = 0$, $P'(\overline{C}) = 1$, (C.3) and (C.5) imply $\lim_{C\uparrow\overline{C}} P'''(C) > 0$. Because $\lim_{C\uparrow\overline{C}} P'''(C) > 0$ for $\overline{C} > 0$, continuity implies $P'''(C) > 0$ in a left-neighbourhood of $\overline{C}$. As a result, there exists $\varepsilon > 0$ such that $P''(C) < 0$ and $P'(C) > 1$ on the interval $(\overline{C} - \varepsilon, \overline{C})$. Define $\hat{C} = \sup\{C \geq 0 : P''(C) \geq 0\}$ and suppose to the contrary that $\hat{C} < \overline{C}$. As $P''(C) < 0$ in a neighbourhood of $\overline{C}$, it follows by continuity that $P''(\hat{C}) = 0$. Since $P''(C) < 0$ for $C \in (\hat{C}, \overline{C})$, it follows that $P'(\hat{C}) > 1$.

Note that by (C.4), the term $\lim_{C\downarrow\hat{C}} \pi\left(e^{-\rho r\alpha(C)} P'(C)\alpha'(C) + J'(C)\right)$ is weakly negative. As such, (C.3) implies $\lim_{C\downarrow\hat{C}} P'''(C) > 0$. Due to $\lim_{C\downarrow\hat{C}} P'''(C) > 0$, there exists $C' > \hat{C}$ so that $P''(C') > 0$, which contradicts the definition of $\hat{C}$. Therefore, $\hat{C} = \overline{C}$ and $P''(C) < 0$ for all $C < \overline{C}$, which was to show.

[29] An alternative argument for $\overline{C} > 0$ runs as follows. Note that we can evaluate the ODE (20) or (27) at the payout boundary $\overline{C}$ to obtain $P(\overline{C}) = \frac{\mu}{r} + \overline{C} - \frac{\overline{C}\lambda\mathbb{1}_{\{C\geq 0\}}}{r}$, with $\alpha(\overline{C}) = \beta(\overline{C}) = 0$. Thus, the initial payoff of outside investors equals $P(\overline{C}) - \overline{C} = \frac{\mu}{r} - \frac{\overline{C}\lambda\mathbb{1}_{\{C\geq 0\}}}{r}$. This payoff must be strictly lower than the NPV of the firm, $\frac{\mu}{r}$, which implies $\overline{C} > 0$.

### C.2 Part II — Simplified HJB Equation (27)

Note that

$$1-\beta(C)=1-\frac{P''(C)}{P''(C)-\rho r P'(C)}=\frac{-\rho r P'(C)}{P''(C)-\rho r P'(C)}. \tag{C.6}$$

Thus, $\frac{P''(C)}{-\rho r P'(C)}=\frac{\beta(C)}{1-\beta(C)}$ and $1-\beta(C)=\frac{-\rho r P'(C)}{P''(C)}\beta(C)$ as well as $\beta(C)=\frac{P''(C)}{-\rho r P'(C)}(1-\beta(C))$. As a result, we can calculate

$$\begin{aligned}-\frac{\rho r}{2}\beta(C)^2P'(C)+\frac{P''(C)}{2}((1-\beta(C))^2&=-\frac{\rho r}{2}\beta(C)^2P'(C)+\frac{(\rho r P'(C))^2}{2P''(C)}(\beta(C))^2 \\ &=-\frac{\rho r}{2}\beta(C)^2P'(C)\left(1-\frac{\rho r P'(C)}{P''(C)}\right)=-\frac{\rho r}{2}\beta(C)P'(C),\end{aligned} \tag{C.7}$$

where the last equality uses that

$$\frac{1}{\beta(C)}=\frac{P''(C)-\rho r P'(C)}{P''(C)}=1-\frac{\rho r P'(C)}{P''(C)}. \tag{C.8}$$

We can insert relation (C.7) as well as $C^*=\overline{C}$ into (20) to obtain

$$\begin{aligned}rP(C)=P'(C)&\left[\mu+\left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)C-\frac{\rho r}{2}\beta(C)\sigma^2+\pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] \\ &+\pi\left[P\left(\overline{C}\right)-P(C)-\left(\overline{C}-C+\alpha(C)\right)\right],\end{aligned} \tag{C.9}$$

which is (27). Here $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function which is equal to one if $\{\cdot\}$ is true and is equal to zero otherwise. When the shareholders' limited commitment constraint (19) binds, the second line of (C.9), i.e., the term $\pi\left[P\left(\overline{C}\right)-P(C)-\left(\overline{C}-C+\alpha(C)\right)\right]$, is identically zero, and

$$rP(C)=P'(C)\left[\mu+\left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)C-\frac{\rho r}{2}\beta(C)\sigma^2+\pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right]. \tag{C.10}$$

### C.3 Part III — Lower Boundary (29)

#### C.3.1 Part III.A — Auxiliary Results

**Lemma 2.** *Conditional on survival, the lower boundary is $\underline{C}=\underline{C}^S$ and satisfies $\mu_C(\underline{C}^S)=0$, $\beta(\underline{C}^S)=1$, $P(\underline{C}^S)=0$, and $\alpha(\underline{C}^S)=P(\overline{C})-\overline{C}-(P(\underline{C}^S)-\underline{C}^S)$. It holds that $\beta(C)<1$ for $C>\underline{C}^S$ with $\lim_{C\downarrow\underline{C}^S}\beta(C)=1$.*

*Proof of Lemma 2.* We consider that the firm is never liquidated (i.e., survival). Note that due to $M\geq 0$, $C=M-Y\geq -Y$ and $Y\leq\frac{\mu}{r}$ where $\frac{\mu}{r}$ is the NPV of the firm (the first best value). As such, $C\geq-\frac{\mu}{r}$, so that excess liquidity is bounded from below. Therefore, there must exist a lower boundary $\underline{C}$ such that $C_t\geq\underline{C}$ at all times $t$. Under survival (i.e., the firm is never liquidated), $C$ follows (16) at all times $t$ and thus has drift $\mu_C(C)$ (see (B.7)) and volatility $\sigma_C(C)$ (see (B.8)). Under survival, the lower boundary $\underline{C}$ of the state space must satisfy $\mu_C(\underline{C})\geq 0$ and $\sigma_C(\underline{C})=0$ so that $C$ does not drop below $\underline{C}$.[30]

[30]Clearly, if $C$ is bounded from below by $\underline{C}$, then it bounded from below by $\underline{C}-\varepsilon$ too. Unless otherwise mentioned, we consider the tightest lower bound.

The HJB equation (20) evaluated under the optimal controls $\alpha(C)$ (see (25)), $\beta(C)$ (see (26)), $Y(C) = \max\{-C, 0\}$ as well as $C^* = \overline{C}$ can be rewritten as

$$rP(C) = P'(C)\mu_C(C) + \frac{P''(C)(\sigma_C(C))^2}{2} + \pi J(C), \tag{C.11}$$

with jump in the value function upon refinancing

$$J(C) \equiv P(\overline{C}) - \overline{C} - P(C) + C - \alpha(C). \tag{C.12}$$

Note that $J(C) = 0$ is equivalent to a binding constraint (19) in which case $\alpha(C) = P(\overline{C}) - \overline{C} - (P(C) - C)$. And, holding $\alpha = \alpha(C)$ fixed, $J(C)$ decreases with $C$, as $P'(C) \geq 1$.

We define $\underline{C}^S$ as the lowest value of $C$ such that the payout agreement can implement $\mu_C(C) \geq 0$, $P(C) \geq 0$, and $\sigma_C(C) = 0$, where $\mu_C(C)$ and $\sigma_C(C)$ are drift and volatility of $dC$ in (16) defined in (B.7) and (B.8) and the value function solves (C.11). Note that $\sigma_C(C) = \sigma(1 - \beta(C)) = 0$ is equivalent to $\beta(C) = 1$. Because $\mu_C(C)$ increases with $C$, decreases with $\beta(C)$, and increases with $\alpha(C)$, because $J(C)$ decreases with $\alpha(C)$, because the right hand side of (19) (with $C^* = \overline{C}$) decreases with $C$, and because equity value $P(C)$ is characterized by (C.11), it follows that $\sigma_C(\underline{C}^S) = 0 \iff \beta(\underline{C}^S) = 1$, $J(\underline{C}^S) = 0$, $\mu_C(\underline{C}^S) = 0$, and $P(\underline{C}^S) = 0$.

In more detail, if it were $\mu_C(\underline{C}^S) > 0$, there would exist $C' < \underline{C}^S$ such that the payout agreement could implement $\sigma_C(C') = 0$ and $\mu_C(C') \geq 0$ with the same choice of $\alpha$ and $\beta$ (i.e., $\alpha(C') = \alpha(\underline{C}^S)$ and $\beta(C') = \beta(\underline{C}^S) = 1$), whilst $J(C') \geq 0$ (as $J(C)$ decreases with $C$ given fixed $\alpha(C)$) and $P(C') \geq 0$ due to (C.11), contradicting the definition of $\underline{C}^S$. Likewise, if it were $J(\underline{C}^S) > 0$, then there would exist $C' < \underline{C}^S$, such that the contract can stipulate $J(C') \geq 0$, $\alpha(C') \geq \alpha(\underline{C}^S) \geq 0$, $\beta(C') = 1$, and $\mu_C(C') \geq 0$ as $\mu_C(C)$ increases with $\alpha$ which leads to $P(C') \geq 0$, contradicting the definition of $\underline{C}^S$. Finally, $P(\underline{C}^S) > 0$ while $\beta(\underline{C}^S) = 1$ would imply $\mu_C(\underline{C}^S) > 0$ or $J(\underline{C}^S) > 0$, again leading to a contradiction. Thus, $\sigma_C(\underline{C}^S) = 0 \iff \beta(\underline{C}^S) = 1$, $J(\underline{C}^S) = 0$, $\mu_C(\underline{C}^S) = 0$, and $P(\underline{C}^S) = 0$. Because $\beta(\underline{C}^S) = 1$ and $\mu(\underline{C}^S) \geq 0$, the payout agreement implements $C_t \geq \underline{C}^S$, thereby satisfying that $C$ is bounded from below under survival.

Next, we show that $P(\underline{C}^S) = 0$ implies $\beta(\underline{C}^S) = 1$, $J(\underline{C}^S) = 0$, and $\mu_C(\underline{C}^S) = 0$. According to the dynamic programming principle and the optimization in the HJB equation (20), the optimal choice of $\alpha(\underline{C}^S)$ and $\beta(\underline{C}^S)$ induces $P(\underline{C}^S) = 0$. As shown above, under the optimal controls this HJB equation simplifies to (C.11). Setting $\beta(\underline{C}^S) = 1$ and $\alpha(\underline{C}^S)$ such that $J(\underline{C}^S) = 0$ implies, by definition of $\underline{C}^S$, $\mu_C(\underline{C}^S) = \sigma_C(\underline{C}^S) = J(\underline{C}^S) = 0$ and therefore $rP(\underline{C}^S) = P'(\underline{C}^S)\mu_C(\underline{C}^S) + \frac{P''(\underline{C}^S)(\sigma_C(\underline{C}^S))^2}{2} + \pi J(\underline{C}^S) = 0$. As a result, setting $\beta(\underline{C}^S) = 1$ and $\alpha(\underline{C}^S)$ such that $J(\underline{C}^S) = 0$ is optimal and consistent with the optimization in the HJB equation (20). As the optimization with respect to $\alpha$ and $\beta$ in the HJB equation (20) yield the unique solutions (25) and (26), it follows that setting $\beta(\underline{C}^S) = 1$ (and $J(\underline{C}^S) = 0$) is strictly optimal. Taken together, we have established the equivalence $P(\underline{C}^S) = 0 \iff \beta(\underline{C}^S) = 1 \wedge J(\underline{C}^S) = 0$.

Importantly, under survival, the payout agreement must satisfy $\beta(\underline{C}^S) = 1$, $\mu_C(\underline{C}^S) = 0$, and $P(\underline{C}^S) = 0$ to ensure that $C$ is bounded from below. That is, if $C$ reaches $\underline{C}^S$, it must be that $\beta(\underline{C}^S) = 1$ to ensure that $C$ is bounded from below with probability one. Crucially, the stipulation of the boundary condition(s) $P(\underline{C}^S) = 0$ and $\beta(\underline{C}^S) = 1$ to solve the HJB equation (20) is not an optimality result but a consequence of the requirement $C$ must be bounded from below under incentive compatible contracts and survival; the stipulation of the boundary condition $P(\underline{C}^S) = 0$ does not per-se preclude $\beta(C') = 1$ for $C' > \underline{C}^S$. In particular, it is always possible to set $\beta(C') = 1$ for $C' > \underline{C}^S$ in which case $\mu_C(C') > 0$ and $\sigma_C(C') = 1$ and $C_t \geq C'$ at all times $t$ (with certainty). In other words, it is always possible to implement a different effective lower bound $C' > \underline{C}^S$ on $C$

through the choice of the control variable $\beta$.

We conjecture and verify that conditional on survival, $\beta(C) < 1$ for $C > \underline{C}^S$ while we have $\lim_{C \to \underline{C}^S} \beta(C) = 1$ in that $\underline{C} = \underline{C}^S$ is the (tightest) lower bound in the state space. To do so, take as lower bound $\underline{C} = \underline{C}^S$ and impose $P(\underline{C}) = 0$ to solve the HJB equation (20) on $(\underline{C}, \overline{C})$ subject to $P(\underline{C}) = P'(\overline{C}) - 1 = P''(\overline{C}) = 0$. By Assumption 1, a unique solution $P(C)$ to (20) exists, and is twice continuously differentiable on $(\underline{C}, \overline{C})$, with endogenous payout boundary $\overline{C} > \underline{C}$. Recall that the optimal choice of $\beta(C)$ is determined according to the optimization in (20) and therefore satisfies (26). It follows that $\beta(C) \to 1$ only if $P''(C) \to -\infty$, as $P'(C) \geq 1$. However, because — by Assumption 1 — the value function $P(C)$ is twice continuously differentiable on $(\underline{C}, \overline{C})$, there cannot exist $C' \in (\underline{C}, \overline{C})$ such that $\lim_{C \to C'} P''(C) = -\infty$. As such, there cannot exist $C' \in (\underline{C}, \overline{C})$ such that $\lim_{C \to C'} \beta(C) = 1$. Thus, $\beta(C) < 1$ for $C > \underline{C}^S = \underline{C}$ with $\lim_{C \to \underline{C}^S} \beta(C) = 1$. Thus, indeed $\underline{C} = \underline{C}^S$ is the tightest lower boundary in the state space (conditional on survival) in that $C$ there exists no value $C' > \underline{C}^S$ such that $C_t \geq C'$ with certainty. □

#### C.3.2 Part III.B — Derivation of (28)

As shown in Lemma 2, we have $\mu_C(\underline{C}) = P(\underline{C}) = 0$ and $\beta(\underline{C}) = 1$ for $\underline{C} = \underline{C}^S$. To derive an expression for $\underline{C}^S$, one first uses (16) to calculate the drift of excess liquidity under the optimal choice of $Y$ derived in the previous section (that is, $Y(C) = \max\{-C, 0\}$):

$$\mu_C(C) = \mu + \left(r - \lambda \mathbb{1}_{\{C \geq 0\}}\right) C - \frac{\rho r}{2} \sigma^2 \beta(C)^2 + \pi \left( \frac{1 - e^{-\rho r \alpha(C)}}{\rho r} \right), \tag{C.13}$$

where $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function, i.e., it is 1 if $\{\cdot\}$ is true and 0 otherwise. The HJB equation (20) evaluated under the optimal controls $\alpha(C)$ and $\beta(C)$ as well as $C^* = \overline{C}$ can be rewritten as

$$rP(C) = P'(C)\mu_C(C) + \frac{P''(C)(\sigma_C(C))^2}{2} + \pi J(C),$$

with jump in the value function upon refinancing

$$J(C) \equiv P(\overline{C}) - \overline{C} - P(C) + C - \alpha(C), \tag{C.14}$$

which is (C.11). Due to $\mu_C(\underline{C}) = P(\underline{C}) = \sigma_C(\underline{C}) = 0$, we have by means of (C.11) that $J(\underline{C}) = 0$ and therefore

$$\alpha(\underline{C}) = P(\overline{C}) - \left[\overline{C} - \underline{C}\right] = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C} + \underline{C}. \tag{C.15}$$

The last equality uses that at the payout boundary $\overline{C} > 0$, the HJB equation (20) implies

$$P(\overline{C}) = \frac{\mu}{r} + \overline{C} - \frac{\lambda \overline{C}}{r},$$

due to $\beta(\overline{C}) = \alpha(\overline{C}) = P'(\overline{C}) - 1 = P''(\overline{C}) = 0$.

Substituting in for the optimal policies, and using $\alpha(\underline{C})$ from above in $\mu_C(\underline{C}) = 0$ while using that $\sigma_C(\underline{C}) = 0 \iff \beta(\underline{C}) = 1$, we have

$$0 = \mu_C(\underline{C}) = \mu + r\underline{C} - \frac{\rho r}{2}\sigma^2 + \pi\left(\frac{1 - e^{-\rho r \alpha(\underline{C})}}{\rho r}\right)$$
$$= \mu + r\underline{C} - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\left(1 - e^{-\rho r\left[\frac{\mu}{r} - \frac{\lambda}{r}\overline{C} + \underline{C}\right]}\right). \tag{C.16}$$

We use the following Lemma to solve for $\underline{C}$:

**Lemma 3.** *The solution to*

$$0 = a + x + e^{(b + c\cdot x)} \tag{C.17}$$

*is given by*

$$x = -\frac{w(c\cdot \exp\{b - a\cdot c\}) + a\cdot c}{c}. \tag{C.18}$$

*Proof.* Define

$$z \equiv c\cdot \exp\{b - ac\}$$

Plugging in the proposed solution (C.18) into the equation (C.17), we have

$$0 = a + \left(-\frac{w(c\cdot\exp\{b - ac\})}{c} - a\right) + \exp\{b - w(c\cdot\exp\{b - ac\}) - ac\}$$
$$= -\frac{w(c\cdot\exp\{b - ac\})}{c} + \exp\{b - ac\}\exp\{-w(c\cdot\exp\{b - ac\})\}$$
$$= -w(c\cdot\exp\{b - ac\}) + c\cdot\exp\{b - ac\}\exp\{-w(c\cdot\exp\{b - ac\})\}$$
$$= -w(z) + z\exp(-w(z))$$

where we multiplied through by $c \neq 0$ in the second-to-last line. The last line is identically equal to zero by the definition of the Lambert-w function

$$w(z)\,e^{w(z)} = z \iff w(z) = z\cdot e^{-w(z)}$$

□

Next, we rewrite (C.16) as

$$0 = \underbrace{\frac{-\rho r}{\pi}\left(\mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\right)}_{\equiv a}\underbrace{-\frac{\rho r^2}{\pi}\underline{C}}_{\equiv x} + e^{-\rho(\mu - \lambda\overline{C}) + \frac{\pi}{r}x}, \tag{C.19}$$

where we define $a \equiv \frac{-\rho r}{\pi}\left(\mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\right)$, $b \equiv -\rho(\mu - \lambda\overline{C})$, $c = \frac{\pi}{r}$, and $x \equiv -\frac{\rho r^2}{\pi}\underline{C}$ We now apply the above lemma to solve (C.19) for $x$ and to thus obtain (28), that is,

$$\underline{C} = \underline{C}^S = \frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r} - Y^A, \tag{C.20}$$

where $w(\cdot)$ is the Lambert function (i.e., $w(z)$ is the principal-branch solution to $we^w = z$). Finally, note that when $\pi = 0$, then $\underline{C}^S = -Y^A$, where $Y^A$ is the autarky value defined in (14).

### C.3.3 Part III.C — Final Arguments and Proof of (29)

We now determine under what circumstances the liquidation or survival (i.e., the firm never liquidates) scenario applies. We distinguish two different cases, i) $\underline{C}^S \le 0$ and ii) $\underline{C}^S > 0$.

Suppose that $\underline{C}^S \le 0$. We conjecture and verify that the survival scenario prevails (so that $\tau = \infty$). Then, Lemma 2 states that conditional on survival, $\underline{C} = \underline{C}^S$ is the lower boundary in the state space and $\beta(C) < 1$ for $C > \underline{C}^S$. Note that $Y(C) = \max\{0, -C\}$ implies $Y \le -\underline{C}^S$. Due to $P'(C) > 1$ for $C < \overline{C}$ and $P(\underline{C}^S) = 0$, it follows that $P(C) > C - \underline{C}^S$. If in state $C > \underline{C}^S$ the firm is liquidated and all cash holdings $M(C)$ are paid out (to shareholders or intermediary), total firm value "just before" liquidation is the cash balance $M(C)$. Note that $M(C) = C + Y(C) \le C - \underline{C}^S < P(C) + Y(C)$, where the first inequality used that $Y \le -\underline{C}^S$ and the second that $P(C) > C - \underline{C}^S$. As a result, liquidation is not optimal and therefore the survival scenario prevails in optimum. Thus, the lower boundary is $\underline{C} = \underline{C}^S$, the boundary condition is $P(\underline{C}) = 0$, and it holds that $\beta(\underline{C}) = 1 > \beta(C)$ for $C > \underline{C}$, leading to $\mu_C(\underline{C}) = \sigma_C(\underline{C}) = 0$.

Suppose that $\underline{C}^S > 0$. It follows that $Y(C) = \max\{0, -C\} = 0$ for $C \ge \underline{C}^S$, and $M(C) = C$. Conditional on survival, the boundary condition $P(\underline{C}^S) = 0$ applies. However, survival cannot be optimal for shareholders. Liquidating the firm at $C = \underline{C}^S$ and paying out $M(\underline{C}^S) = \underline{C}^S > 0$ dollars as dividends yield value $\underline{C}^S > 0$ for shareholders. As such, the liquidation scenario prevails. It remains to show that liquidation occurs the first time $C$ falls to zero so that $\underline{C} = 0$. To start with, note that liquidation at $C < 0$ is not possible because at the time of liquidation, $Y(C) = 0$ must hold, and $C < 0$ would imply $Y(C) > 0$. More in detail, suppose to the contrary liquidation occurs at some value $C < 0$, so $M(C) = 0$ and $Y = Y(C) = -C > 0$. The fact that the firm holds no cash upon liquidation also precludes any positive transfers to the intermediary upon liquidation. Now, $Y(C) > 0$ implies that upon liquidation in state $C = 0$, promise-keeping (i.e., the requirement that $Y = 0$ at the time of liquidation) is violated, a contradiction. Thus, liquidation can only occur in states $C \ge 0$.

Next, suppose that the firm is liquidated at $C = 0$, so that $P(0) = 0$. Note that $P'(C) > 1$ for $C < \overline{C}$ implies $P(C) > C = M(C)$ for $C > 0$ (clearly, the payout boundary is positive). If the firm is liquidated in state $C > 0$ and all cash is paid out as dividends, then shareholders receive $M(C) = C < P(C)$, so that liquidation at $C > 0$ is not optimal. As liquidation must occur for $C \ge 0$, it follows that optimal liquidation occurs at $C = \underline{C} = 0$, i.e., $\tau = \inf\{t \ge 0 : C_t = 0\}$.

At liquidation at $C = 0$, $P(0) = 0$. As $P(C)$ is by Assumption 1 twice continuously differentiable on $(\underline{C}, \overline{C})$, there cannot exist $C' > 0$ such that $\lim_{C \to C'} P''(C) = -\infty$ and $\lim_{C \to C'} \beta(C) = 1$. That is, $\beta(C) < 1$ for all $C > 0$.

Taken together, the lower boundary and the associated value of equity are given by

$$\underline{C} = \min\left\{\underline{C}^S, 0\right\} \quad \text{with} \quad P\left(\underline{C}\right) = 0, \tag{C.21}$$

which is (29) as desired.

### C.3.4 Part IV — Optimal transfer process

We postulate that the dynamics of the intermediary's cumulative transfers $I$ are

$$dI_t = \mu_I(C_t)dt + \sigma_I(C_t)dZ_t + \alpha_I(C_t)d\Pi_t + \xi_I dDiv, \tag{C.22}$$

with endogenous drift $\mu_I = \mu_I(C)$, volatility $\sigma_I = \sigma_I(C)$, and sensitivity to refinancing and dividend payouts $\alpha_I = \alpha_I(C)$ and $\xi_I$, all as functions of the state variable $C$. We omit time subscripts henceforth.

Having characterized the optimal control variables $(Y, C^*, \alpha, \beta)$ for each state $C$, we can now characterize the transfer process $dI_t$ in (31). For $C > 0$, we optimally have $Y = 0$ which implies $dY = 0$. Thus, by the expression for $dY$ in (11), we can solve for

$$dI = \left[\frac{\rho r}{2}(\beta\sigma)^2 - \pi\left(\frac{1-e^{-\rho r\alpha}}{\rho r}\right)\right]dt + \beta\sigma dZ + \alpha d\Pi \quad \text{for} \quad C > 0. \tag{C.23}$$

Note that $\sigma_I = \sigma\beta$, and the volatility of $dW$, given by $\sigma\beta$, and $dI$ coincide for $C > 0$. This is intuitive, as there is no change in deferred payouts $Y = 0$, and thus all changes in $W$ lead to equivalent cash flows, i.e., they must coincide with changes in $I$.

For $C < 0$, we optimally have zero cash, i.e., $M = C + Y = 0$. Absent refinancing, $dM = dC + dY = 0$ with $dM = 0$, while upon a refinancing opportunity $C^*(C) + Y^*(C) = \overline{C} > 0$, which implies

$$dI = \mu dt + \sigma dZ + (\alpha + Y)\, d\Pi \text{ for } C < 0. \tag{C.24}$$

In words, on $C < 0$ the intermediary completely absorbs any cash flow shocks $dX_t = \mu dt + \sigma dZ_t$, while gaining $\alpha_t + Y_t$ upon refinancing. Note that on $C \in (\underline{C}, 0)$, we have $\beta(C) < 1$ and therefore $\beta\sigma < \sigma = \sigma_I$, i.e., the volatility of $dW$ and $dI$ diverge as part of the continuation value is delivered via deferred payouts, i.e., changes in $Y$. The refinancing payout $\alpha_I(C) = \alpha(C) + Y(C) \geq \alpha(C)$ can be larger than the change in $W$ because $Y$ is optimally paid out during refinancing. Combining these results yields (31), which concludes the argument.

# D Details on the Implementation

This Section provides a detailed derivation of the implementation in Section 4.

## D.1 Auxiliary function: Cumulative transfers since refinancing

To begin, we recall (31) characterizes that optimal transfers to the intermediary, and we introduce the auxiliary function $T(C)$ that records cumulative transfers from the intermediary to the firm in response to Brownian cash flow shocks $dZ$ since the last $C$ has either reached $\underline{C}$ or $\overline{C}$ (which is reached, for instance, upon refinancing). That is, $T(C)$ is defined on the interval $(\underline{C}, \overline{C})$. Unless otherwise mentioned, we consider below that $C \in (\underline{C}, \overline{C}$; we discuss the details prevailing for $C = \underline{C}$ at the end.

By Ito's Lemma, calculate

$$\begin{aligned} dT(C) =& \mu_T(C)\, dt + \sigma_T(C)\, dZ - \alpha_T(C)\, d\Pi \\ =& \left[T'(C)\,\mu_C(C) + \frac{1}{2}T''(C)\,\sigma_C^2(C)\right]dt + T'(C)\,\sigma_C(C)\, dZ + \left[T\left(\overline{C}\right) - T(C)\right]d\Pi, \end{aligned} \tag{D.1}$$

and so obtain drift $\mu_T(C)$, volatility $\sigma_T(C)$, and loading on $d\Pi$, $\alpha_T(C)$, whereby $\mu_C(C)$ is the drift and $\sigma_C(C) = (1 - \beta(C))\sigma$ is the volatility of excess liquidity in (16).[31] As the process has to reset

[31] An expression for the drift of excess liquidity under the optimal controls is

$$\mu_C(C) = \mu + \left(r - \lambda \mathbb{1}_{\{C \geq 0\}}\right)C - \frac{\rho r}{2}\sigma^2\beta(C)^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right) \tag{D.2}$$

where it is imposed that $Y(C) = \max\{-C, 0\}$.

upon refinancing due to its Markovian nature, we impose $T\left(\overline{C}\right)=0$. Note $T(C_t)$ is subtly different from $-I_t$, as $I_t$ turns out to be non-Markovian.[32]

For $T\left(C\right)$ to record cumulative contributions of the intermediary in response to Brownian cash flow shocks $dZ$, its volatility $\sigma_T(C)$ loading must match $-\sigma_I(C)$, the negative of the volatility of transfers from the firm to the intermediary. In other words, total contributions increase (decrease) one-for-one with transfers from (to) the intermediary caused by cash flow shocks $dZ$. Matching volatilities, and using (31) — that is, $\sigma_I(C)=\sigma\beta(C)+\sigma(1-\beta(C))\mathbb{1}_{\{C<0\}}$ — and $\sigma_T(C)=T'(C)\sigma(1-\beta(C))$ (by Itô's Lemma), we have for $C\in(\underline{C},\overline{C})$

$$\sigma_T\left(C\right)=-\sigma_I(C)\iff T'\left(C\right)=-\frac{\beta\left(C\right)}{1-\beta\left(C\right)}-\mathbb{1}_{\{C<0\}},\tag{D.3}$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function which equals one if $\{\cdot\}$ is true and zero otherwise.

Plugging in $\beta(C)$ from (26), integrating, and imposing $T\left(\overline{C}\right)=0$, we have for $C\in(\underline{C},\overline{C})$[33]

$$T\left(C\right)=\frac{\ln P'\left(C\right)}{\rho r}+Y\left(C\right)=\alpha_U\left(C\right)+Y\left(C\right).\tag{D.4}$$

Due to the non-Markovian nature of $I$, there is no guarantee that imposing $\sigma_T(C)+\sigma_I(C)=0$ implies matching drifts, i.e., $\mu_T\left(C\right)+\mu_I(C)=0$. Noting that $T''\left(C\right)=-\frac{\beta'(C)}{[1-\beta(C)]^2}$, after deriving $\beta'\left(C\right)$ and some algebra, we show in Appendix E.2:

$$\mu_T\left(C\right)+\mu_I(C)=\frac{\lambda}{\rho r}\mathbb{1}_{\{C\geq0\}}+rY\left(C\right)+\pi\left(\frac{e^{\rho r[\alpha_U(C)-\alpha(C)]}-1}{\rho r}\right).\tag{D.5}$$

The right-hand-side of (D.5) is made up of three terms: The first term reflects a constant drift part linked to the carry-cost-of-cash, $\lambda$. The second term reflects the cost of delaying the payout of the equity stake, which is linked to the discount rate. The third term reflects the additional compensation required to the intermediary from the possible restriction on $\alpha$ by shareholders' limited commitment constraint (19), in that $\alpha_U(C)-\alpha(C)\geq 0$. Interest rates and other fees have to absorb this difference in any Markov implementation.

As the firm always refinances to the payout boundary $\overline{C}$, and $T\left(\overline{C}\right)=0$ by definition, the loading on $d\Pi$ is simply given by

$$\alpha_T\left(C\right)=T\left(C\right).\tag{D.6}$$

## D.2 Credit line

Absent shareholders' limited commitment constraint, the optimal contract has $\alpha(C)=\alpha_U(C)$, which implies $\alpha_I(C)=\alpha_U(C)+Y(C)=T(C)$. In other words, $T(C)$ *exactly* equals the actual payment upon refinancing, leading us to a credit line and equity stake interpretation: here, $Y(C)$ is the current equity stake, which leaves $D(C)=\alpha_U(C)$ as the current balance of the credit line.[34]

When the limited commitment constraint (19) is binding so $\alpha\left(C\right)=\alpha_{LC}(C)<\alpha_U\left(C\right)$, the payoff to the intermediary upon refinancing is given by $\alpha_I\left(C\right)=\alpha_{LC}(C)+Y(C)<T(C)$ for

[32] For example, past refinancing events are recorded in $I_t$, but not in $T(C_t)$.

[33] In more detail, $\frac{1}{\rho r}\frac{P''(C)}{P'(C)}=-\frac{\beta(C)}{1-\beta(C)}=\frac{d}{dC}\frac{\ln P'(C)}{\rho r}$.

[34] Importantly, maintaining that the credit line balance records net-transfers coming from credit line usage, $Y(C)$ has to be generated from payments separate from the credit line.

$C < \overline{C}$. In words, the contract specifies that the intermediary optimally demands payment of *less* than its total cumulative transfers since the last refinancing. This feature is due to incentives: demanding the full repayment $T(C) = \alpha_U(C) + Y(C)$ violates limited commitment (19), and thus would be vetoed by the shareholders. As discussed above, we will take the credit line balance to be $D(C) = \alpha_U(C)$, and thus require "early repayment incentives" of amount $\alpha_U(C) - \alpha(C) > 0$, while $Y(C)$ is separately generated via restricted equity.

The following result shows that the limited commitment constraint (19) is always binding if the firm never defaults. Therefore the prospect of any PE investment *always* occurs jointly with early repayment incentives:

**Proposition 4.** *If* $\underline{C} < 0$*, then shareholders' limited commitment constraint* (19) *is always binding, i.e.,* $\alpha(C) = \alpha_{LC}(C) < \alpha_U(C)$ *for* $C \in [\underline{C}, \overline{C})$*.*

Any implementation via a general credit line balance of $D(C)$ together with a portfolio of other *non-interest* bearing instruments — in our interpretation restricted equity $Y(C)$ — by construction must have volatility matching $T(C)$. However, the drifts of $I$ and $T$ do not cancel due to $I$ not being Markovian. To balance the drifts, i.e., match the $dt$ dynamics, we introduce the Markovian interest rate $r_D(C)$ on the balance $D(C)$, as well as a constant maintenance fee $f$ so that

$$\mu_T(C_t) + \mu_I(C_t) = r_D(C_t)\, D(C_t) + f. \tag{D.7}$$

We set $f$ to absorb any constant payouts at $C = \overline{C}$, and let the interest rate $r_D(C)$ capture the remaining variable difference.[35] At the payout boundary we have $\alpha_U(\overline{C}) = \alpha(\overline{C}) = Y(\overline{C}) = 0$, so

$$f = \frac{\lambda}{\rho r} \quad \text{and} \quad r_D(C) = \frac{rY(C) - \frac{\lambda}{\rho r}\mathbb{1}_{\{C<0\}}}{D(C)} + \frac{\pi \frac{e^{\rho r[\alpha_U(C) - \alpha(C)]} - 1}{\rho r}}{D(C)}. \tag{D.8}$$

First, when (19) is not binding, which can only happen when liquidation is possible, i.e., $\underline{C} = \underline{C}^L = 0$, we have $r_D(C) = 0$, and only maintenance fee payments are required. Next, note that when LC is binding, $r_D(C)$ is *state-dependent* and has a jump at $C = 0$ as the equity stake $Y(C)$ enters the picture and the inefficiency of internal cash-holding $\lambda$ disappears.[36] We see that the second term of $r_D(C)$ reflects the possible limited commitment restriction on $\alpha$, as it records the (risk-adjusted) required compensation for the early repayment incentive of $\alpha_U(C) - \alpha(C) \geq 0$ upon refinancing.

## D.3 Restricted equity

Recall that we argued in Section 3.1 that $Y(C)$ can naturally be understood as an equity stake. Let us construct the terms of this stake. For an equity stake to be used, we need $\underline{C} < 0$, which in turn implies (19) is always binding (see Proposition 4). Suppose that the firm allows the intermediary to trade equity internally at the given price schedule $P_I(C)$, but only allows equity shares to trade on the open market at refinancing opportunities. In essence, we will interpret the intermediary's equity shares as restricted equity that vests upon refinancing. The intermediary holds $g_E(C)$ restricted equity shares that if sold at post-issuance share-price $P_E(C)$ achieve the payout of $Y(C)$ upon

[35] Note that $\mu_T(C)$ summarizes all non-interest movements in $T(C)$, which by assumption our portfolio of instruments, here credit line and restricted equity, replicates.

[36] By L'Hopital's rule, the interest rate on the credit line at the payout boundary is positive, and given by $r_D(\overline{C}) = \pi$ as $\alpha''(\overline{C}) = 0$ but $\alpha''_U(\overline{C}) \neq 0$. This is intuitive – the intermediary is approximately risk-neutral towards the very small loss of $\alpha_U(C) - \alpha(C)$, and thus is only compensated for the arrival rate of this loss, $\pi$.

financial market access:

$$Y(C) = g_E(C)\, P_E(C) \iff g_E(C) = \frac{Y(C)}{P(C)}, \tag{D.9}$$

where we used $P_E(C) = P(C)$ due to shareholders' limited commitment.[37] Further, the portfolio of credit line and restricted equity, absent fees and interest, has to match the cumulative transfers, $T(C)$. As $T(C) = \alpha_U(C) + Y(C)$, and the credit line covers $D(C) = \alpha_U(C)$, the value $Y(C)$ has to be generated by the intermediaries' "trading" gains and losses of restricted equity, that is

$$Y(C) = \int_C^0 \left(-g_E'(x)\right) P_I(x)\, dx. \tag{D.10}$$

Setting both expressions for $Y(C)$ equal and differentiating pins down $P_I(C)$:

$$\left(g_E(C)\, P(C)\right)' = -1 = g_E'(C)\, P_I(C) \iff P_I(C) = -\frac{1}{g_E'(C)}. \tag{D.11}$$

Further, writing out $\left(g_E(C)\, P_E(C)\right)'$ and dividing through by $g_E'(C)$, we see that

$$P_I(C) = P(C) - \left(\frac{g_E(C)}{-g_E'(C)}\right) P'(C) < P(C). \tag{D.12}$$

Because $g_E'(C) < 0$ and $P'(C) > 0$, the internal price is always below the external price, resulting in strict incentives for the intermediary to sell its shares on the open market upon refinancing.

## D.4 Refinancing via common equity issuance

Next, we investigate the details of the refinancing. First, we normalize the current number of outstanding shares to unity and let $g$ be the number of new shares issued upon refinancing.[38] Then, the post-issuance equity price is given by $P_E(C) = \frac{P(\overline{C})}{1+g}$. As the proceeds from $g$ must cover the total cash needed to replenish the firm's cash holdings, $\overline{M} - M(C)$, as well as the transfers to the intermediary, $\alpha_I(C) = \alpha(C) + Y(C)$, we have

$$\frac{P\left(\overline{C}\right)}{1+g} g = \overline{M} - M(C) + \alpha(C) + Y(C) \iff g(C) = \frac{\overline{C} - C + \alpha(C)}{P\left(\overline{C}\right) - \left[\overline{C} - C + \alpha(C)\right]}, \tag{D.13}$$

where we used the fact that $\overline{M} = \overline{C}$ and the definition $C = M - Y$. Due to shareholders' limited commitment and its implied constraint on $\alpha$, (19), the denominator is non-negative (strictly so unless $C = \underline{C} < 0$), and we can always implement the optimal allocation via common equity issuance consistent with limited commitment.

The post-issuance price is given by

$$P_E(C) = P\left(\overline{C}\right) - \left[\overline{C} - C + \alpha(C)\right]. \tag{D.14}$$

Note that $P_E'(C) = 1 - \alpha'(C) \geq 0$, with strict inequality for $C < \overline{C}$. Further, when (19) is binding,

[37] See next subsection for details.

[38] Without the normalization, $g$ can be interpreted as the required growth rate in outstanding shares.

the post issuance price corresponds to the pre-issuance price, so that

$$P_E(C) = P(C) \quad \text{and} \quad g(C) = \frac{P\left(\overline{C}\right) - P\left(C\right)}{P\left(C\right)}. \tag{D.15}$$

As $C$ approaches the lower boundary $\underline{C}$, existing outside shareholders are completely diluted upon refinancing, i.e., $\lim_{C \to \underline{C}} g(C) = \infty$, while the cash amount raised stays finite. In our restricted equity implementation, the firm itself only issues $g(C) - g_E(C)$ new shares, while the intermediary sells its $g_E(C)$ vesting shares.

### D.5 Implementation at the lower boundary $\underline{C}$

We now discuss the lower boundary $\underline{C}$. If $\underline{C} = 0$ and the firm is liquidated at $C = \underline{C}$, then $Y(C) = 0$ and the firm defaults on its credit line.

Next, consider survival, that is, $\underline{C} < 0$, and the firm is never liquidated. Note that because $\mu_C(\underline{C}) = \sigma_C(\underline{C}) = 0$ under survival, the state $C = \underline{C}$ is absorbing (if reached) until the next refinancing event in which case $C$ jumps up to $\overline{C}$. Even though we do not provide a formal proof, we expect that $C = \underline{C}$ is an inaccessible state too in that $C_t > \underline{C}$ implies that $C_s > \underline{C}$ with probability one, provided $\underline{C} < 0$. As $\beta(\underline{C}) = 1$ or, more generally, $\lim_{C \to \underline{C}} \beta(C) = 1$, the ODE (D.3), characterizing the process $T(C)$, is not well-defined for $C = \underline{C}$. Loosely speaking, being in state $C = \underline{C}$, the credit line balance is potentially unbounded and no more Markovian.

To deal with the possibility of $C = \underline{C}$, we assume that in the state $C = \underline{C}$, the change of credit line is set to zero and restricted equity pays "dividends" $dX_t = \mu dt + \sigma dZ_t$. That is, the intermediary assumes full ownership of the firm in the state $C = \underline{C}$, covers all cash flow shortfalls by injecting cash, and pays out all positive cash flows as dividends to herself. At the same time, the credit line balance remains constant until the next refinancing event. These assumptions ensure that the credit line balance and the dollar value of restricted equity are indeed Markovian. Again, we do not believe that these assumptions have any major implications.

## E Additional Results

### E.1 Proof of Proposition 4

The proof proceeds by proving two consecutive Lemmatas, Lemma 4 and Lemma 5.

We start with an auxiliary Lemma deriving the slope of $\beta(C)$.

**Lemma 4.** *Suppose that $\alpha(C)$ and $\beta(C)$ are differentiable in state $C$. Then:*

$$\begin{aligned}\frac{\sigma^2}{2}\beta'\left(C\right) = &-\frac{\beta\left(C\right)}{1-\beta\left(C\right)}\mu_C\left(C\right) + \frac{\rho r}{2}\beta^2\left(C\right)\sigma^2 + \frac{\pi}{\rho r}\alpha'\left(C\right)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'\left(C\right)}\right] \\ &-\frac{\pi}{\rho r}\left(1 - \frac{1}{P'\left(C\right)}\right) - \frac{\lambda \mathbb{1}_{\{C \geq 0\}}}{\rho r}\end{aligned} \tag{E.1}$$

*Proof of Lemma 4.* To start with, note that

$$1 - \beta(C) = 1 - \frac{P''(C)}{P''(C) - \rho r P'(C)} = \frac{-\rho r P'(C)}{P''(C) - \rho r P'(C)}, \tag{E.2}$$

so $\frac{1}{\rho r}\frac{P''(C)}{P'(C)} = -\frac{\beta(C)}{1-\beta(C)}$.

To derive the postulated expression for $\beta'(C)$, we first differentiate both sides of the ODE (27) with respect to $C$:

$$\begin{aligned} rP'(C) =& P''(C)\left[\mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] \\ &+ P'(C)\left[\left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right) - \frac{\rho r}{2}\beta'(C)\sigma^2 + \pi\alpha'(C)e^{-\rho r\alpha(C)}\right] \\ &+ \pi\left[1-\alpha'(C) - P'(C)\right]. \end{aligned} \tag{E.3}$$

Rearranging, we have

$$\begin{aligned} &\left[\pi\left(1-\alpha'(C)e^{-\rho r\alpha(C)}\right) + \lambda\mathbb{1}_{\{C\geq 0\}} + \frac{\rho r}{2}\beta'(C)\sigma^2\right]P'(C) \\ &= P''(C)\left[\mu + \left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] + \pi\left[1-\alpha'(C)\right]. \end{aligned} \tag{E.4}$$

Dividing through by $\rho r P'(C)$ and solving for $\frac{\sigma^2}{2}\beta'(C)$, we have

$$\begin{aligned} \frac{\sigma^2}{2}\beta'(C) =& \frac{P''(C)}{\rho r P'(C)}\left[\mu + \left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] \\ &+ \frac{\pi}{\rho r}\frac{\left[1-\alpha'(C)\right]}{P'(C)} - \frac{\pi}{\rho r}\left[1-\alpha'(C)e^{-\rho r\alpha(C)}\right] - \frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r} \\ =& -\frac{\beta(C)}{1-\beta(C)}\left[\mu + \left(r-\lambda\mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] \\ &+ \frac{\pi}{\rho r}\frac{\left[1-\alpha'(C)\right]}{P'(C)} - \frac{\pi}{\rho r}\left[1-\alpha'(C)e^{-\rho r\alpha(C)}\right] - \frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r} \\ =& -\frac{\beta(C)}{1-\beta(C)}\mu_C(C) + \frac{\rho r}{2}\beta^2(C)\sigma^2 + \frac{\pi}{\rho r}\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] \\ &- \frac{\pi}{\rho r}\left(1-\frac{1}{P'(C)}\right) - \frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r}, \end{aligned} \tag{E.5}$$

where the second equality uses $\frac{1}{\rho r}\frac{P''(C)}{P'(C)} = -\frac{\beta(C)}{1-\beta(C)}$ and the third equality uses

$$-\frac{\beta(C)}{1-\beta(C)}\cdot(-\beta(C)) = \frac{\beta(C)^2}{1-\beta(C)} + \frac{\beta(C)^3}{1-\beta(C)} - \frac{\beta(C)^3}{1-\beta(C)} = \beta(C)^2 - \frac{\beta(C)}{1-\beta(C)}\cdot(-\beta(C)^2) \tag{E.6}$$

as well as the expression for the drift of excess liquidity $\mu_C(C)$ in (B.7). □

Using the auxiliary Lemma 4, we can now establish that equity holders' limited commitment constraint binds under survival.

**Lemma 5.** *Under survival (i.e.,* $\underline{C}^S \leq 0$*), the limited commitment constraint of equity holders is always binding, so that* $\alpha(C) = \alpha_{LC}(C)$.

*Proof of Lemma 5.* Recall the expressions for $\alpha_U(C)$ and $\alpha_{LC}(C)$ in (24). Define the difference function $f(C)$ and calculate its derivatives (assuming they exist):

$$f\left(C\right) \equiv \alpha_U\left(C\right) - \alpha_{LC}\left(C\right) = \frac{\ln P'\left(C\right)}{\rho r} - \left\{\left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P\left(C\right) - C\right]\right\}, \tag{E.7}$$

$$f'\left(C\right) = \frac{P''\left(C\right)}{\rho r P'\left(C\right)} + P'\left(C\right) - 1 = -\frac{\beta\left(C\right)}{1-\beta\left(C\right)} + P'\left(C\right) - 1, \tag{E.8}$$

$$f''\left(C\right) = -\frac{\beta'\left(C\right)}{\left[1-\beta\left(C\right)\right]^2} + P''\left(C\right) = -\frac{1}{1-\beta\left(C\right)}\left[\frac{\beta'\left(C\right)}{1-\beta\left(C\right)} + \rho r P'\left(C\right)\beta\left(C\right)\right], \tag{E.9}$$

where we used $\frac{P''(C)}{\rho r P'(C)} = -\frac{\beta(C)}{1-\beta(C)}$.

Throughout the proof, we follow the convention that whenever $f''(C)$ does not exist, then — with some abuse of notation — $f''(C)$ denotes the left-limit $\lim_{x\uparrow C} f''(x)$. This limit exists as $f''(C)$ exists at all values $C \in (\underline{C}, \overline{C}) - \{0\}$. The set of points at which either $P''(C)$ or $\alpha(C)$ is not differentiable is countable; therefore, for any $C$, the limits $\lim_{x\uparrow C} f''(C), \lim_{x\downarrow C} f''(C)$ and $\lim_{x\uparrow C} \alpha'(C), \lim_{x\downarrow C} \alpha'(C)$ exist and are well-defined.

First, note that $f\left(\overline{C}\right) = f'\left(\overline{C}\right) = 0$. Regardless of whether $\alpha(C) = \alpha_U(C)$ or $\alpha(C) = \alpha_{LC}(C)$, we know that $\alpha'(\overline{C}) = 0$ and $\beta(\overline{C}) = 0$. Then, expression (E.1) in Lemma 4 implies that $\beta'\left(\overline{C}\right) = -\frac{2}{\sigma^2}\frac{\lambda}{\rho r}$, so that $f''\left(\overline{C}\right) = \frac{2}{\sigma^2}\frac{\lambda}{\rho r} > 0$. Thus, using a Taylor expansion around $C = \overline{C}$, we have

$$f\left(\overline{C} - \varepsilon\right) \approx \frac{1}{2} f''\left(\overline{C}\right)\varepsilon^2 > 0. \tag{E.10}$$

Thus, $f(C) > 0$ in a left-neighbourhood of $\overline{C}$. To show that $f\left(C\right) > 0$ for all $C \in \left(\underline{C}, \overline{C}\right])$, we proceed by proof of contradiction.

Suppose to the contrary that there exists a point $C_0 \in \left(\underline{C}, \overline{C}\right)$ at which $f\left(C_0\right) \leq 0$ — if there are several such points, we the pick the one closest to $\overline{C}$ — which satisfies (by continuity) $f(C_0) = 0$ and has to be an up-crossing, in that $f'(C_0) > 0$. This implies that there exists a local maximum point $\hat{C} \in \left(C_0, \overline{C}\right)$ at which $f(\hat{C}) > 0$ so $\alpha(\hat{C}) = \left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P\left(\hat{C}\right) - \hat{C}\right]$ applies, the first order condition

$$f'\left(\hat{C}\right) = 0 \iff P'\left(\hat{C}\right) = \frac{1}{1-\beta\left(\hat{C}\right)}, \tag{E.11}$$

holds and the second order condition $f''\left(\hat{C}\right) < 0$ holds. Note that in case $f'(C)$ is not differentiable at $C = \hat{C}$, then the second order condition becomes $\lim_{x\uparrow \hat{C}} f''(x) < 0$ and $\lim_{x\downarrow \hat{C}} f''(C) < 0$. Recall that in this case, we write with some abuse of notation $f''(\hat{C}) = \lim_{x\uparrow \hat{C}} f''(x) < 0$. Evaluating $f''(C)$ at $C = \hat{C}$ and using $P'(\hat{C}) = 1/(1-\beta(\hat{C})$, we have

$$f''\left(\hat{C}\right) = -\frac{1}{\left[1-\beta\left(\hat{C}\right)\right]^2}\left[\beta'\left(\hat{C}\right) + \rho r \beta\left(\hat{C}\right)\right]. \tag{E.12}$$

To get a contradiction, we need to show $f''\left(\hat{C}\right) > 0 \iff \beta'\left(\hat{C}\right) + \rho r \beta\left(\hat{C}\right) < 0$.

Note that by (E.1) from Lemma 4 with $\alpha'(\hat{C}) = 1 - P'(\hat{C})$:

$$
\begin{aligned}
\frac{\sigma^2}{2}\beta'\left(\hat{C}\right) =& -\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\mu_C\left(C\right) + \frac{\rho r}{2}\beta^2\left(\hat{C}\right)\sigma^2 - \frac{\pi}{\rho r}\left[P'\left(\hat{C}\right)-1\right]e^{-\rho r\alpha\left(\hat{C}\right)} - \frac{\lambda \mathbb{1}_{\{\hat{C}\geq 0\}}}{\rho r} \\
=& -\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda\mathbb{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r}\left[1 - e^{-\rho r\alpha\left(\hat{C}\right)}\right]\right\} \\
&+ \frac{1}{1-\beta\left(\hat{C}\right)}\frac{\rho r}{2}\beta^2\left(\hat{C}\right)\sigma^2 - \frac{\pi}{\rho r}\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}e^{-\rho r\alpha\left(\hat{C}\right)} - \frac{\lambda \mathbb{1}_{\{\hat{C}\geq 0\}}}{\rho r} \\
=& -\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda\mathbb{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r}\right\} + \frac{1}{1-\beta\left(\hat{C}\right)}\frac{\rho r}{2}\beta^2\left(\hat{C}\right)\sigma^2 - \frac{\lambda \mathbb{1}_{\{\hat{C}\geq 0\}}}{\rho r},
\end{aligned}
\tag{E.13}
$$

where the first equality inserts $\alpha'(\hat{C}) = 1 - P'(\hat{C})$ into (E.1) and simplifies, the second line inserts the expression (B.7) for $\mu_C(C)$ and uses the the first order condition (E.11) or equivalently $P'(\hat{C}) - 1 = \frac{\beta(\hat{C})}{1-\beta(\hat{C})}$, and the third equality simplifies and collects terms.

Combining, we have

$$
\begin{aligned}
\beta'\left(\hat{C}\right) + \rho r\beta\left(\hat{C}\right) =& \frac{2}{\sigma^2}\Bigg\{ -\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda\mathbb{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r}\right\} \\
&+ \frac{1}{1-\beta\left(\hat{C}\right)}\frac{\rho r}{2}\beta^2\left(\hat{C}\right)\sigma^2 - \frac{\lambda \mathbb{1}_{\{\hat{C}\geq 0\}}}{\rho r}\Bigg\} + \rho r\beta\left(\hat{C}\right) \\
=& \frac{2}{\sigma^2}\left\{ -\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda\mathbb{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r}\right\} - \frac{\lambda \mathbb{1}_{\{\hat{C}\geq 0\}}}{\rho r}\right\} \\
&+ \left[\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)} + 1\right]\rho r\beta\left(\hat{C}\right) \\
=& \frac{2}{\sigma^2}\left\{ -\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda\mathbb{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r}\right\} - \frac{\lambda \mathbb{1}_{\{\hat{C}\geq 0\}}}{\rho r}\right\} + \frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\rho r \\
=& -\frac{2}{\sigma^2}\left\{ \frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda\mathbb{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r} - \frac{\sigma^2}{2}\rho r\right\} + \frac{\lambda \mathbb{1}_{\{\hat{C}\geq 0\}}}{\rho r}\right\}.
\end{aligned}
\tag{E.14}
$$

The term

$$
\mu + \left(r - \lambda\mathbb{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r} - \frac{\sigma^2}{2}\rho r \tag{E.15}
$$

is increasing in $\hat{C}$. Evaluated at $\hat{C} = \underline{C} = \underline{C}^S < 0$, we have

$$Y^A + \underline{C} + \frac{\pi}{\rho r^2} = \frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right)}{\rho r} > 0. \tag{E.16}$$

Thus, we have $\beta'\left(\hat{C}\right) + \rho r \beta\left(\hat{C}\right) < 0 \iff f''\left(\hat{C}\right) > 0$ under survival, a contradiction. Consequently, we have $f(C) \geq 0$ and therefore $\alpha(C) = \alpha_{LC}(C)$ for $C \in \left[\underline{C}, \overline{C}\right]$ under survival. □

## E.2 Derivation of (D.5)

First, recall (D.4), that is,

$$T(C) = \frac{\ln P'(C)}{\rho r} + Y(C) = \alpha_U(C) + Y(C), \tag{E.17}$$

and, from (31) and (B.7),

$$\mu_I = \mu_I(C) = \mu + [(r-\lambda)C - \mu_C]\, \mathbb{1}_{\{C\geq 0\}} \tag{E.18}$$

$$\mu_C = \mu_C(C) = \mu + \left(r - \lambda \mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta^2(C)\sigma^2 + \frac{\pi}{\rho r}\left[1 - e^{-\rho r \alpha(C)}\right]. \tag{E.19}$$

Thus, we can calculate

$$\begin{aligned}
\mu_T(C) =& T'(C)\mu_C(C) + T''(C)\frac{\sigma_C^2(C)}{2}\\
=& T'(C)\mu_C(C) - \frac{\beta'(C)}{[1-\beta(C)]^2}\frac{\sigma^2[1-\beta(C)]^2}{2}\\
=& T'(C)\mu_C(C) - \frac{\sigma^2}{2}\beta'(C)\\
=& -\left[\frac{\rho r}{2}\beta^2(C)\sigma^2 + \frac{\pi}{\rho r}\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] - \frac{\pi}{\rho r}\left(1 - \frac{1}{P'(C)}\right) - \frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r}\right]\\
=& \mu_C \mathbb{1}_{\{C\geq 0\}}(C) - \left[\mu + \left(r - \lambda\mathbb{1}_{\{C\geq 0\}}\right)C - \frac{\lambda\mathbb{1}_{\{C\geq 0\}}}{\rho r}\right]\\
&- \frac{\pi}{\rho r}\left\{\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] - \left(1 - \frac{1}{P'(C)}\right) + \left(1 - e^{-\rho r\alpha(C)}\right)\right\}\\
=& \left[\frac{\lambda}{\rho r} + \mu_C(C)\right]\mathbb{1}_{\{C\geq 0\}} - \left[\mu + \left(r - \lambda\mathbb{1}_{\{C\geq 0\}}\right)C\right] - \frac{\pi}{\rho r}\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right]\left[1 - \alpha'(C)\right],
\end{aligned} \tag{E.20}$$

where the second equality uses $T''(C) = -\frac{\beta'(C)}{[1-\beta(C)]^2}$, and the fourth equality uses (E.1) to substitute in for $\frac{\sigma^2}{2}\beta'(C)$. Plugging in for $\mu_I(C) = \mu + [(r-\lambda)C - \mu_C]\, \mathbb{1}_{\{C\geq 0\}}$, we have

$$\mu_T(C) + \mu_I(C) = \frac{\lambda}{\rho r}\mathbb{1}_{\{C\geq 0\}} + rY(C) - \frac{\pi}{\rho r}\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right]\left[1 - \alpha'(C)\right], \tag{E.21}$$

which was to show.

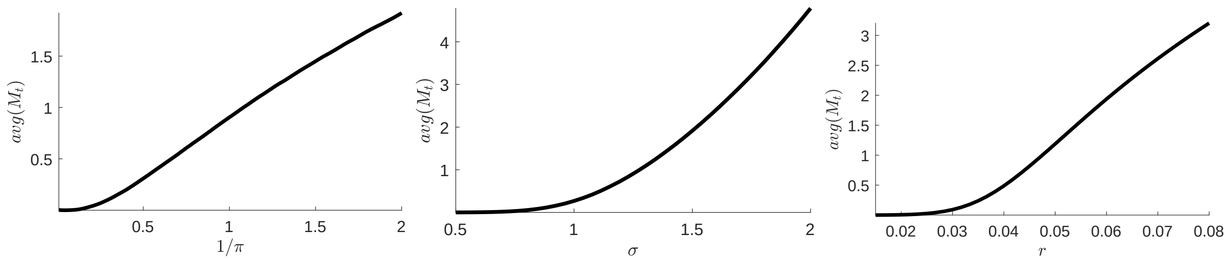


Figure F.1: **Average levels of $M$ in steady state:** This Figure presents average cash holdings, $avg(M_t)$, as a function of $1/\pi$ (left panel), $\sigma$ (middle panel), and $r$ (right panel). The parameters are such that the firm is never liquidated and a stationary distribution exists.

# F Additional Figures

This Section presents additional and omitted numerical results. Whenever we simulate and calculate average quantities, we consider parameter configurations that admit a stationary distribution, in that $\underline{C} < 0$.

Figure F.1 shows that, on average, cash holdings $M_t$ increase with $1/\pi$, $r$, and $\sigma$.

Figure F.2 displays the payout boundary $\overline{C}$ (left panel) and $P(\overline{C}) - \overline{C}$ (middle panel) which is the value extracted by the founders, i.e., the initial shareholders, at time $t = 0$ for different values of $1/\pi$ under the baseline scenario (solid black line), the no-PE benchmark (dotted red line), and the no-intermediation benchmark (dashed yellow line). The right panel depicts the percentage value generated through (i) intermediation relative to the no-intermediation benchmark (solid black line) and (ii) PE financing relative to the no-PE benchmark (dotted red line).

The left panel of Figure F.2 shows that intermediary financing and holding cash are substitutes in the firm's optimal liquidity management. Any restriction in access to intermediary financing implies a higher target cash level $M(\overline{C}) = \overline{C}$, in that the solid black line lies below the dotted red line, which lies below the dashed yellow line. At the same time, a restriction in intermediary financing decreases the total firm value at time $t = 0$ depicted in the middle panel.

Next, the left panel of Figure F.3 displays the boundary $\overline{C}$ in the baseline case (black solid), the no-PE case (red dotted), and the no-intermediation case (yellow dashed). The middle panel displays the initial payoff $P(\overline{C}) - \overline{C}$. The right panel depicts the percentage value generated by intermediation relative to the no-intermediation case (solid black line) and by PE financing (dotted red line). Interestingly, the payout boundary is non-monotonic in $r$ in the baseline and no-PE cases but decreases with $r$ in the no-intermediation case. The value generated by intermediation and PE financing tends to be highest for low interest rates, $r$. Thus, the decline in interest rates boosts the importance of intermediation.

Similarly, Figure F.4 plots the payout boundary (left panel) and total initial payoff (middle panel) against cash flow volatility $\sigma$ under all three scenarios. The right panel depicts the percentage value generated by intermediation (solid black line) and PE-like intermediation (dotted red line). The left panel shows the standard result that the payout boundary increases with cash flow volatility, regardless of the firm's access to intermediary financing. Intuitively, higher cash flow volatility implies the need for higher cash reserves to withstand cash flow shocks. Not surprisingly, the firm's total value at inception decreases with cash flow volatility (middle panel), again regardless of the firm's access to intermediary financing. Surprisingly, note that the value created through intermediation and PE financing in the right panel is hump-shaped in cash flow volatility.

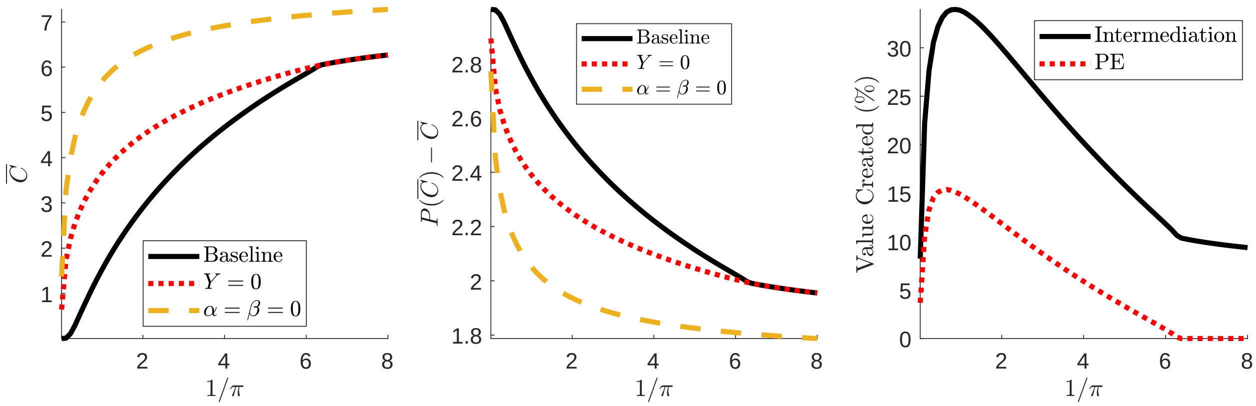


Figure F.2: **The value of intermediation for different values of** $1/\pi$. The left panel displays the boundary $\overline{C}$ in the baseline case (black solid), the no-PE case (red dotted), and the no-intermediation case (yellow dashed). The middle panel displays the initial payoff $P(\overline{C}) - \overline{C}$. The right panel depicts the percentage value generated by intermediation relative to the no-intermediation case (solid black line) and by PE financing (dotted red line).

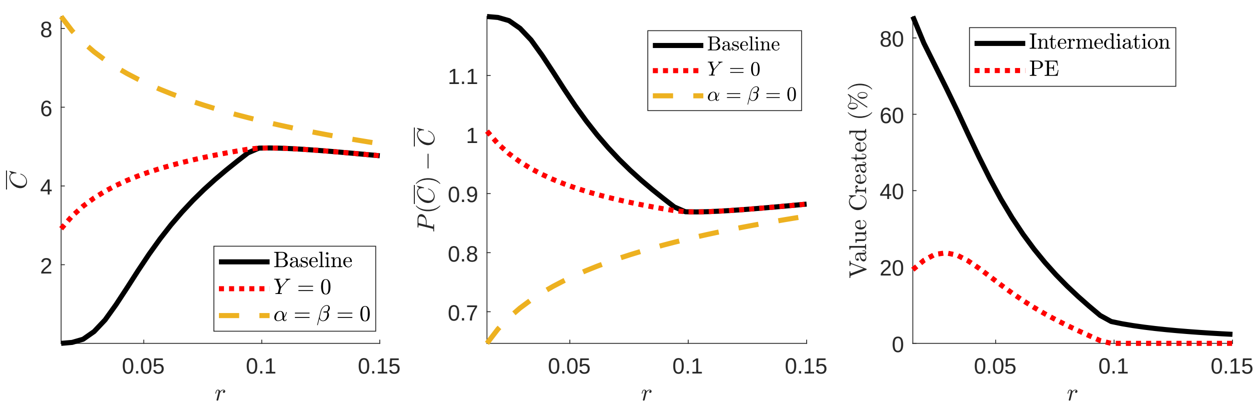


Figure F.3: **The value of intermediation for different values of the interest rate** $r$. The left panel displays the boundary $\overline{C}$ in the baseline case (black solid), the no-PE case (red dotted), and the no-intermediation case (yellow dashed). The middle panel displays the initial payoff $P(\overline{C}) - \overline{C}$. The right panel depicts the percentage value generated by intermediation relative to the no-intermediation case (solid black line) and by PE financing (dotted red line).

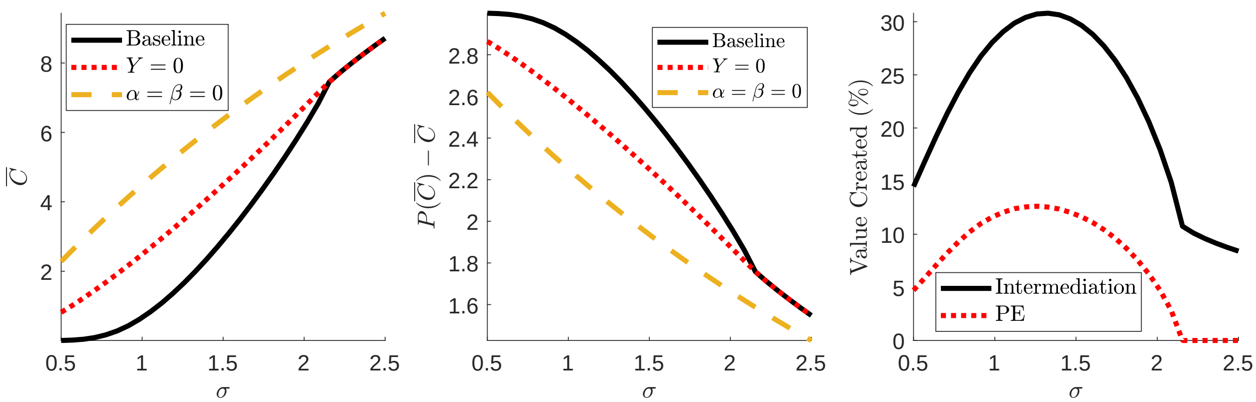


Figure F.4: **The value of intermediation for different values of cash flow volatility,** $\sigma$. The left panel displays the boundary $\overline{C}$ in the baseline case (black solid), the no-PE case (red dotted), and the no-intermediation case (yellow dashed). The middle panel displays the initial payoff $P(\overline{C}) - \overline{C}$. The right panel displays the percentage value generated by intermediation relative to the no-intermediation case (solid black line) and by PE financing (dotted red line).