NBER WORKING PAPER SERIES

A GOLDILOCKS THEORY OF FISCAL DEFICITS

Atif R. Mian
Ludwig Straub
Amir Sufi

Working Paper 29707
http://www.nber.org/papers/w29707

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
January 2022

We are grateful to Manuel Amador, Johannes Brumm, Gabriel Chodorow-Reich, Arvind Krishnamurthy, Hanno Lustig, Alberto Martin, Klaus Masuch, Ricardo Reis, Larry Summers, Annette Vissing-Jorgensen, Iván Werning, and seminar participants at ASSA 2022, Berkeley Haas, Chicago Booth, CIREQ conference Montreal, Cornell, Federal Reserve Board, Georgetown, Harvard, NYU Stern, Princeton, Queens University, Stanford GSB, Trinity College Dublin, and UC Irvine. Agustin Barboza, Laurenz DeRosa, Jan Ertl, Pranav Garg, and Keelan Beirne provided excellent research assistance. Straub appreciates support from the Molly and Domenic Ferrante Award. Contact info: Mian: (609) 258 6718, atif@princeton.edu; Straub: (617) 496 9188, ludwigstraub@fas.harvard.edu; Sufi: (773) 702 6148, amir.sufi@chicagobooth.edu

A Goldilocks Theory of Fiscal Deficits
Atif R. Mian, Ludwig Straub, and Amir Sufi
NBER Working Paper No. 29707
January 2022
JEL No. E31,E62,H30,H62,H63

ABSTRACT

This paper proposes a tractable framework to analyze fiscal space and the dynamics of government debt, with a possibly binding zero lower bound (ZLB) constraint. Without the ZLB, a greater primary deficit unambiguously raises debt. However, debt need not explode: When $R < G - \varphi$, where φ is the sensitivity of $R - G$ to debt, a modest permanent increase in the deficit can be sustained forever, a policy we call "free lunch". With the ZLB, the relationship between deficit and debt can become non-monotone. Both high and low deficits can increase debt, as the latter weaken demand and reduce nominal growth at the ZLB. A rise in income inequality expands fiscal space outside the ZLB, but contracts it at the ZLB. Calibrating the model, we find little space for "free lunch" policies for the United States in 2019, but ample space for Japan.

Atif R. Mian
Princeton University
Bendheim Center For Finance
26 Prospect Avenue
Princeton, NJ 08540
and NBER
atif@princeton.edu

Ludwig Straub
Department of Economics
Harvard University
Littauer 211
Cambridge, MA 02139
and NBER
ludwigstraub@fas.harvard.edu

Amir Sufi
University of Chicago
Booth School of Business
5807 South Woodlawn Avenue
Chicago, IL 60637
and NBER
amir.sufi@chicagobooth.edu

1 Introduction

Advanced economies in recent years have been characterized by very low interest rates, also known as "secular stagnation" (Summers 2014). This has led economists and policymakers to challenge the textbook view on the relationship between public debt and (primary) deficits; raising current deficits may no longer have to be offset by lowering future deficits and raising taxes. Instead, when the interest rate R lies below the growth rate G, there may be a "free lunch" (Blanchard 2019), according to which deficits can be increased permanently without causing explosive debt dynamics; and higher debt levels can be sustained without reduced deficits. In other words, when R lies below G, the fiscal cost of increased debt may be zero or even negative.

This paper systematically studies the fiscal cost of borrowing and the joint dynamics of public debt and primary deficits. Our starting point is a tractable model with two main ingredients. First, R can lie below G, and R increases in government debt. We model this simply by assuming government debt provides convenience benefits (e.g., Krishnamurthy and Vissing-Jorgensen 2012, Greenwood, Hanson, and Stein 2015) but also argue that our results likely carry over to other microfoundations. The second ingredient is a zero lower bound (ZLB) constraint on the nominal interest rate R, which allows for the possibility that weak demand reduces output and inflation, and thus also the nominal growth rate G of the economy.

Building on the model, our paper makes four contributions. First, we show that the correct condition for the existence of a free lunch policy is not $R < G$; instead, it is a tighter condition, $R < G - \varphi$, where φ is the sensitivity of $R - G$ to the logarithm of public debt to GDP. As a consequence, even for countries in which $R < G$, borrowing more may not be free. The intuition for why $R < G - \varphi$ is the free lunch condition is the following. Suppose that $R < G$. The government decides to borrow one additional dollar and plans to roll it over forever. This fiscal choice will have two opposing effects on government's budget constraint. On the one hand, the rolling over of additional dollar produces a positive cash flow for the government equal to $G - R$. On the other hand however, the additional borrowed dollar also tightens the budget constraint because of its impact on the interest rate on all infra-marginal outstanding units of debt. This latter effect is precisely captured by φ and combining the two effects gives us $R < G - \varphi$ as the free lunch condition.

Our second contribution is to characterize the dynamics of debt and deficits at or near the ZLB. This is important, as many economies with low R today are close to the ZLB. There, deficits are important instruments to increase aggregate demand (Blanchard and Tashiro 2019, Furman and Summers 2020). We show that this aggregate-demand channel may "invert" the textbook view on deficits and debt at the ZLB: Greater deficits may reduce, rather than increase, debt. This is because greater deficits raise aggregate demand and inflation; higher inflation translates into higher nominal growth rates; this pushes debt down, as it increases the speed at which debt is "inflated away". This indirect effect through the nominal growth rate can be sufficiently strong to overwhelm the direct effect of greater deficits on debt.

The third contribution of our paper is to study the role of inequality and tax progressivity. Inequality matters for government debt since as much as 69% of U.S. government debt held by U.S. households is directly or indirectly held by households in the top 10% of the U.S. wealth distribution (Mian, Straub, and Sufi 2020). To evaluate the role of inequality in our framework, we allow for saver and spender households, as in Campbell and Mankiw [1989], Mankiw [2000], Galí, López-Salido, and Vallés [2007], and Bilbiie [2008]. Using these two types of households, we show that increased inequality, modeled as a greater share of income earned by savers, increases fiscal space and increases the availability of free lunch policies outside the ZLB. We believe that this finding is interesting as it points to a potential conflict between reducing inequality (e.g. via progressive taxation) on the one hand, and funding large deficits on the other. We show that at the ZLB, inequality reduces fiscal space as it reduces aggregate demand and nominal growth rates.

The fourth contribution of our paper is a calibration of our model to the economies of the United States and Japan in December 2019, before the disruptions caused by Covid-19. Our calibration suggests that the United States was just inside the free lunch region at the time, and could sustain a maximum permanent primary deficit of just over 2% of GDP at a stable debt-to-GDP ratio of 110%. These are the free lunch limits implied by the condition $R < G - \varphi$. Notably, the traditional condition of $R < G$ is valid until a debt-to-GDP ratio of 220%. However, higher deficits beyond the free lunch limit of 110% have to be paid for either through higher future taxes or reduced spending. In contrast to the United States, our calibration for Japan as of December 2019 shows ample room for free lunch policies. In fact, we find that Japan is in the "inverted" regime, in which an increase in deficits would reduce debt levels, precisely due to the effect on aggregate demand and inflation.

We make our four contribution by analyzing our model in an intuitive *deficit-debt diagram*, in which a locus characterizes the feasible set of steady state combinations of the primary deficit (or surplus) and debt. The deficit-debt locus is hump-shaped: deficits are zero both for zero debt and when debt is sufficiently large that $R = G$. In between, deficits are positive, consistent with the idea that $R < G$ allows an economy to permanently run positive deficits. The locus characterizes where a free lunch policy is available, namely exactly on the left branch of the locus, to the left of its peak. It also explains how, at the ZLB, the inverted relationship between deficits and debt levels occurs because of a "backward-bending" shape of the deficit-debt locus. While we focus on our tractable model for the most part in our paper, we plot the deficit-debt diagram also for a number of alternative models, to illustrate that our results are likely to generalize.

We provide several extensions to our basic framework that expand the scope of our results. First among these is the introduction of aggregate risk into our model, building on the framework of Mehrotra and Sergeyev [2020]. We prove that, even with aggregate risk, our free lunch condition remains informative. Specifically, when $R < G - \varphi$ holds on average, the probability of a free lunch succeeding can be chosen to be arbitrarily close to 1 by suitable choice of the size of the free lunch policy. On the contrary, when $R > G - \varphi$ on average, a free lunch never succeeds. We confirm that this result numerically also holds for the Blanchard [2019] model.

Our second extension adds capital to the production function and thus allows government debt to crowd out capital, as explored by Blanchard and Weil [2001]. Interestingly, we show that greater crowding out of capital increases fiscal space and makes a free lunch policy more likely to exist as it reduces the sensitivity of the interest rate to the level of government debt.

In our third and final extension, we study the role of the maturity structure of debt. We find that issuing long-term debt generally reduces fiscal space at small debt levels, but increases it for higher debt levels. This suggests that some forms of Quantitative Easing (QE) may have the side effect of constraining fiscal space with rising debt levels among advanced economies.

Related Literature. This study is part of a growing body of theoretical research that has emerged around two important facts on government debt. The first fact is that the nominal interest rate on government debt is lower than the nominal growth rate on average, $R < G$, going back to at least Feldstein [1976].[1] The second, and more recent fact, is that the demand curve for government debt slopes down empirically, that is, the interest rate on government debt rises in the volume of government debt (Engen and Hubbard 2004, Laubach 2009, Krishnamurthy and Vissing-Jorgensen 2012, Greenwood et al. 2015, Presbitero and Wiriadinata 2020). This fact is typically either attributed to effects of government debt on the marginal product of capital (due to crowding out of capital), to

[1] See Bohn [1991], Ball, Elmendorf, and Mankiw [1998], Blanchard [2019], Mehrotra and Sergeyev [2020] for more recent papers documenting the historical patterns of R vs. G.

effects of debt on risk premia, or to effects of debt on its "convenience benefits", capturing regulatory requirements, liquidity premia, and safety premia.

The literature has explored several ways to explain one or both of these facts. Bohn [1995] and Barro [2020] suggest that $R < G$ can naturally occur in complete markets economies with aggregate risk. Due to Ricardian equivalence (Barro 1974), the model suggests that government debt neither affects R, nor can the government run a permanent deficit in each state of the world. According to Jiang, Lustig, Van Nieuwerburgh, and Xiaolan [2019], this approach cannot explain the valuation of U.S. government debt.

The perhaps largest literature on $R < G$ is based on OLG models, going back to Samuelson [1958] and Diamond [1965]. One branch of this literature studies when $R < G$ is a sign of dynamic inefficiency (Abel, Mankiw, Summers, and Zeckhauser 1989, Blanchard and Weil 2001, Ball and Mankiw 2021); another branch evaluates the welfare implications of increased debt levels (Ball et al. 1998, Blanchard 2019, Brumm, Feng, Kotlikoff, and Kubler 2021a,b); the most relevant branch of the OLG literature for us is the one concerned with the possibility of "free lunch" policies (Blanchard and Weil 2001, Blanchard 2019). These papers show that, when $R < G$, a debt-rollover policy is more likely to succeed when the economy is inefficient and production is linear in capital, but no general condition is developed.[2] Our paper develops a precise condition for a free lunch to exist in a deterministic model, $R < G - \varphi$, which is significantly stricter than $R < G$. We show that the condition still has bite with aggregate risk, and even holds in the Blanchard [2019] model itself. Interestingly, in recent work, Aguiar, Amador, and Arellano [2021] find that a similar condition is indicative of the possibility of robust welfare improvements.

The above facts have also been approached using liquidity premia. Woodford [1990] illustrates how liquidity demand by producers or consumers can lead to $R < G$. Angeletos, Collard, and Dellas [2020] microfound a convenience yield function based on liquidity needs to revisit the optimality of the Barro [1979] tax smoothing results.[3] Bayer, Born, and Luetticke [2021] estimate the response of the liquidity premium to fiscal policy shocks empirically and model it with an estimated two-asset HANK model. Domeij and Ellingsen [2018] obtain $R < G$ in a Bewley-Aiyagari model. The closest paper to ours among this class of models is Reis [2021]. The paper microfounds liquidity and safety premia of government debt and shows that a "bubble premium" emerges on public debt, which can be used to

[2]There is also a long literature on the private production of assets when the return on non-government assets is also below the growth rate, see, e.g., Tirole [1985], Kocherlakota [2009], Farhi and Tirole [2012], Hirano and Yanagawa [2016], and Martin and Ventura [2018].

[3]See also Canzoneri, Cumby, and Diba [2016], Bhandari, Evans, Golosov, and Sargent [2017], Azzimonti and Yared [2019].

sustain permanent primary deficits. Different from the analysis in Reis [2021], we focus on the dynamics of debt and deficits, both with and without the ZLB, and show when a free lunch exists.[4]

Mehrotra and Sergeyev [2020] share with our paper the assumption of a convenience utility function $v(b)$ over government debt, which they employ in a model with aggregate risk and a specific deficit rule that yields a particularly tractable law of motion of debt-to-GDP. They use it to show that slower trend growth and higher output risk can increase debt-to-GDP. By comparison, our paper focuses on free lunch policies and the possibility of a binding ZLB. Kocherlakota [2021] microfounds a linear utility from holding bonds with a small probability disaster shock and shows how this allows the government to improve welfare by increasing debt. Michau [2020] presents a model with net wealth in the utility and a ZLB constraint and uses it to study fiscal policy plans that lead the economy away from the ZLB. Guerrieri, Lorenzoni, and Rognlie [2021] argue that a potentially binding ZLB constraint in the future can be a rationale for preserving fiscal space.

Our model is based on the assumption that monetary policy is active in stabilizing inflation and economic activity whenever it is not constraint by the ZLB. A recent branch of the literature explores deviations from this assumption.[5] Brunnermeier, Merkel, and Sannikov [2020a,b] derive a Laffer curve for the rate of inflation in a model with liquidity needs among producers. Sims [2019] argues that fiscal policy should, in general, use this "inflation tax" to generate seignorage-like revenue and reduce distortionary taxes (different from Chari and Kehoe 1999). The deficit-debt schedule that we derive, and on which our phase diagram is based, may seem similar to the inflation Laffer curve, but is quite distinct, as we explain in Section 3.4.

This study is also closely related to the burgeoning literature on the sources and implications of safe asset demand (e.g., Caballero, Farhi, and Gourinchas 2008, Caballero and Farhi 2018b, and Farhi and Maggiori 2018). In their model of the international monetary system, Farhi and Maggiori [2018] explore an equilibrium in which there is large demand for debt issued by a hegemon government. When this is met by too much issuance, default risk emerges. When there is too little issuance, the ZLB may bind. This pattern resembles our deficit-debt diagram with a potentially binding ZLB, albeit it emerges in our case as the steady state locus of a dynamic model, rather than as a one-shot choice of the government as in Farhi and Maggiori [2018].

[4] In Appendix C, we show that the economy in Reis [2021] can also be represented in a deficit-debt diagram.

[5] See also the recent work by Bassetto and Cui [2018] and Bianchi and Melosi [2019]. See Bassetto and Sargent [2020] for an excellent survey.

Finally, the notion of a free lunch formalizes an intuition that is often associated with "Modern Monetary Theory" (MMT). However, unlike common renditions of MMT (see Bisin 2020 for a critical review), our model spells out the exact conditions under which a free lunch policy works or does not work. In line with intuition by Lerner [1943], we find that a free lunch policy always exists if an economy faces a persistent demand shortage at the ZLB.

2 Model

We begin with a stylized model that we extend in later sections. The model runs in continuous time and is deterministic.[6] It consists of a government, a household side with savers and spenders, and a monetary authority. The government issues government debt, spends, and raises lump-sum taxes. Spenders and savers consume and savers draw convenience benefits from holding government debt. The monetary authority targets inflation.

Throughout, we denote by R_t the net nominal interest rate on government debt and by $G_t \equiv \gamma + \pi_t$ the net nominal growth rate, which is equal to real trend growth γ plus inflation π_t. $G^* \equiv \gamma + \pi^*$ corresponds to nominal trend growth, when inflation is at its target π^*.

To save on notation, we will conduct our analysis entirely in the context of a model that is de-trended with the nominal growth rate. Potential output y^* in the de-trended model is constant and we normalize it to one, $y^* \equiv 1$. Any quantities, such as the level of government debt b_t are to be understood as government debt relative to potential GDP. Moreover, we refer to $R_t - G_t$ as the "de-trended rate of return" on government debt, as it is the return R_t net of the re-investment that is necessary to keep a constant ratio of government debt to potential GDP. We abstract from capital in our baseline model, but discuss it at length in Section 7.2.

Households. The economy is populated by a unit mass of savers and a unit mass of spenders, as in Campbell and Mankiw [1989] and Mankiw [2000]. Savers choose paths of consumption c_t and government debt holdings b_t in order to maximize

$$\max_{\{c_t, b_t\}} \int_0^\infty e^{-\rho t} \left\{ \log c_t + v\left(b_t\right) \right\} dt \tag{1}$$

[6] We separately study aggregate risk in Section 6.

subject to the consolidated budget constraint

$$c_t + \dot{b}_t \leq (R_t - G_t)\, b_t + (1-\mu)\, w_t n_t - \tau_t. \tag{2}$$

The objective (1) involves flow utility from consumption $\log c_t$ and a utility $v(b_t)$ from holding government debt (relative to potential GDP). The latter captures safety and liquidity benefits that have been used extensively and are well documented in the literature (e.g. Sidrauski 1967, Krishnamurthy and Vissing-Jorgensen 2012). In line with this literature, we assume that the utility over government debt is twice differentiable, increasing and concave, $v' \geq 0, v'' \leq 0$.[7] Flow utility is discounted using a discount rate ρ.

Each saver has a labor endowment of $1-\mu$, where $\mu \in [0,1)$ captures the income share of spenders. Savers sell a fraction $n_t \leq 1$ of their endowments at real wage w_t each instant. n_t can lie strictly below 1 if there is rationing (see below). Savers pay lump-sum taxes τ_t.

Spenders are hand-to-mouth.[8] Each spender has a labor endowment of μ, and also sells a fraction n_t of it at real wage w_t. Spenders pay lump-sum taxes $\tilde{\tau}_t$. Thus, their consumption is equal to

$$\tilde{c}_t = \mu n_t w_t - \tilde{\tau}_t. \tag{3}$$

Representative firm. We assume that labor is used by a representative firm with linear production technology $y_t = n_t$. The firm sets flexible prices, pinning down the real wage to 1 at all times, $w_t = 1$. In contrast, we assume that nominal wages are downwardly rigid. Similar to Schmitt-Grohé and Uribe [2016], the path of nominal wages W_t satisfies

$$\frac{\dot{W}_t}{W_t} \geq \pi^* - \kappa(1 - n_t). \tag{4}$$

This implies that, whenever labor demand is falling short of the labor endowments, wage inflation will fall short of π^*. The lower labor demand is, the lower wage inflation will be, just like in a standard Phillips curve. $\kappa \geq 0$ parameterizes the slope of the Phillips curve. Price inflation π_t in our de-trended model is equal to wage inflation and therefore determined by (4). Observe that potential output, with $n_t = 1$, is indeed equal to one, $y^* = 1$. The term $1 - n_t$ in (4) is therefore simply equal to the output gap, $(y^* - y_t)/y^*$.

[7] We also assume that the range of v' is given by $[0,\infty)$ or $(0,\infty)$, that $v'' < 0$ whenever $v' > 0$, and that v' is weakly convex.

[8] One can easily microfound this behavior by assuming that spenders do not enjoy any convenience benefits from holding government bonds and are unable to borrow.

Government. The government sets fiscal and monetary policy. Fiscal policy consists of paths $\{x, b_t, \tau_t, \tilde{\tau}_t\}$ of government spending x, government debt b_t and taxes $\tau_t, \tilde{\tau}_t$, subject to the flow budget constraint

$$x + (R_t - G_t)\, b_t \leq \dot{b}_t + \tau_t + \tilde{\tau}_t \tag{5}$$

The primary deficit is given by

$$z_t \equiv x - \tau_t - \tilde{\tau}_t \tag{6}$$

We assume taxes adjust to ensure that z_t follows a given fiscal rule $z_t = \mathcal{Z}(b_t)$. Our baseline assumption is that taxes on spenders are zero $\tilde{\tau} = 0$ and taxes on savers τ_t adjust. We consider the case where $\tilde{\tau} \neq 0$ in Section 3.5. Typically, $\mathcal{Z}(b)$ is downward-sloping in debt b, corresponding to a lower deficit or greater surplus with a higher debt level.

Government debt b_t is short-term and real in our baseline model. We study long-term debt in Section 7.1. Government spending $x \geq 0$ is assumed to be constant for now. Our analysis below is similar to one in which government spending is allowed to vary while taxes are kept fixed.

Monetary policy is "dominant" in our model, that is, it successfully implements the natural allocation whenever feasible. In particular, we denote by $\{R_t^*\}$ the path of the nominal natural interest rate, which would materialize in the absence of nominal rigidities in our model, assuming inflation is constant at its target π^*. We assume that the actual nominal interest rate then follows

$$R_t = \max\{0, R_t^*\}. \tag{7}$$

In particular, whenever the natural interest rate is positive, R_t tracks the natural interest rate R_t^*, the economy is at potential, $y_t = n_t = 1$, and inflation is at its target $\pi_t = \pi^*$.[9] When the natural rate is negative, however, R_t is constrained to be equal to zero by the ZLB. In that case, we will find that the economy falls below potential, $y_t = n_t < 1$. Labor endowments are rationed, equally across the two types of agents.[10]

Equilibrium. We define equilibrium in our model as follows.

Definition 1. Given an initial level of debt b_0 and a fiscal rule $\mathcal{Z}(\cdot)$, a (competitive) equilib-

[9] As is well understood, the inflation target can be implemented by an active Taylor rule $R_t = R_t^* + \phi \pi_t$ with $\phi > 1$. We discuss equilibrium uniqueness in Section 4.

[10] This is similar to the rationing equilibria in Barro and Grossman [1971], Malinvaud [1977], and Benassy [1986].

rium consists of a tuple $\{c_t, \tilde{c}_t, y_t, n_t, b_t, R_t, G_t, \pi_t, \tau_t, \tilde{\tau}_t, z_t, w_t\}$, such that: (a) $\{c_t, b_t\}$ maximizes savers' objective (1) subject to (2), and $\tilde{c}_t$ satisfies (3); (b) the deficit $\{z_t\}$ follows the fiscal rule $\mathcal{Z}$ and taxes are in line with (6); (c) debt evolves in line with the flow budget constraint (5) and remains bounded; (d) monetary policy sets the nominal rate R_t in line with the rule (7); (e) inflation π_t is determined by the Phillips curve (4); (f) output y_t is given by $y_t = n_t$ and the real wage is $w_t = 1$; (g) the goods market clears $c_t + \tilde{c}_t + x = y_t$. A *steady state equilibrium* is an equilibrium in which all quantities, real prices, and inflation are constant.

Features of government debt in the model. There are two ways to interpret the convenience utility $v(b)$, either as coming from the asset supply or the asset demand side.

$v(b)$ as coming from asset supply. According to this view, government debt offers asset-specific benefits, due to liquidity, safety, regulatory requirements, or international institutional demands. These benefits, or some subset of them, are often grouped together as "convenience benefits", and collectively explain why certain government bonds may have a particularly low yield relative to seemingly similar other assets (Krishnamurthy and Vissing-Jorgensen 2012, Caballero, Farhi, and Gourinchas 2017, Jiang, Lustig, Van Nieuwerburgh, and Xiaolan 2020, Koijen and Yogo 2020, Mota 2020). In Appendix D we offer a simple microfoundation for our convenience utility $v(b)$ based on a low-probability disaster shock (as in Barro 2020), after which a government may default on its debt.

$v(b)$ as coming from asset demand. According to this view, savers require higher yields in order to hold greater amounts of government debt. This can be microfounded with life-cycle (as in Diamond 1965, Blanchard 2019) or precautionary saving motives (as in Aiyagari and McGrattan 1998). We demonstrate in Section 5.5 and in Appendix C that results are likely similar when these microfoundations of the asset demand view are used instead of our $v(b)$ utility.

3 Fiscal space without the ZLB

In this section, we focus on the case without a ZLB constraint, so that $R_t = R_t^*$ in all periods, effectively implementing the flexible price allocation. We study the role of the ZLB in Section 4. We begin our analysis by characterizing steady state equilibria.

3.1 Steady state equilibria

Our model admits a set of steady state equilibria, indexed by the level of steady state debt $b \geq 0$. For each b, one can find a primary deficit z such that $\dot{b} = 0$ and the economy remains steady at that level of debt b.

The interest rate is equal to the natural rate, $R_t = R_t^*$, output and employment are at potential, $y_t = n_t = 1$, inflation is at its target, $\pi_t = \pi^*$, and the nominal growth rate is equal to nominal trend growth, $G_t = G^*$.

To see how the natural rate is determined, consider the savers' Euler equation

$$\frac{\dot{c}_t}{c_t} = R_t^* - G^* - \rho + v'(b_t)c_t \tag{8}$$

Here, $v'(b_t)$ enters as it is the marginal convenience utility from saving one more unit in government bonds. It enters with the opposite sign as the discount rate ρ and therefore effectively makes the household more patient when saving in government bonds.

In a steady state, savers' consumption is constant and equal to $1 - x - \mu$ by goods market clearing, where x is government spending and μ consumption of spenders. This lets us solve (8) for the natural interest rate,

$$R^*(b) = \rho + G^* - \underbrace{v'(b) \cdot (1 - x - \mu)}_{\text{convenience yield}}. \tag{9}$$

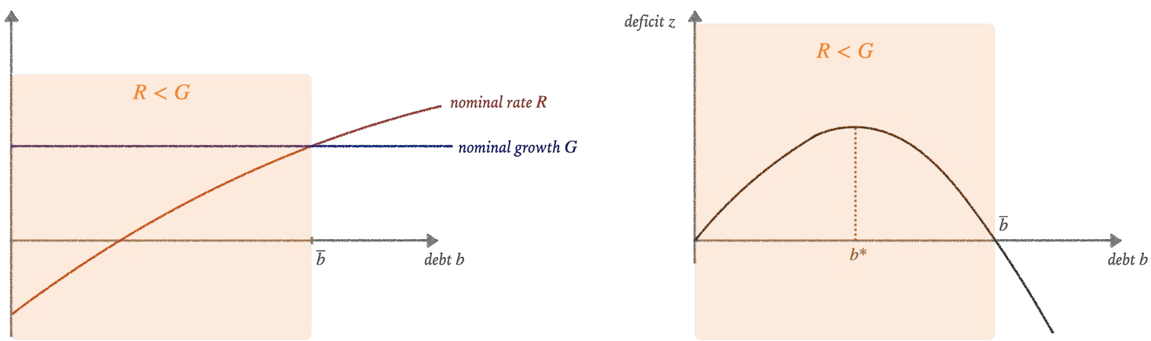
This expression for the natural interest rate on government debt is intuitive. The natural rate is equal to $\rho + G^*$, which would be the steady state return on any non-convenience-bearing assets, minus the steady state convenience yield $v'(b) \cdot (1 - x - \mu)$. The expression already suggests how R^* moves with debt. As v is a concave utility function, R^* weakly increases in government debt b.

3.2 Steady state deficits and deficit-debt diagram

It is useful to represent $R^*(b)$ and G^* in a diagram, Figure 1(a). $R^*(b)$ is increasing in debt levels as higher debt levels reduce the convenience yield. The threshold for $R = G$ is determined by $v'(\overline{b})(1 - x - \mu) = \rho$. It is positive, $\overline{b} > 0$, if $v'(0)\,(1 - x - \mu) > \rho$. One noteworthy implication of the equation for the upper bound is that the size of $\overline{b}$ can be large, and is in no meaningful way constrained by existing household or private wealth of agents. In fact, with $\rho \to 0$, $\overline{b}$ would diverge to infinity, allowing the government to run

Figure 1: Interest rate, growth rate, and deficits. Case without ZLB.

(a) Interest rate and growth rate **(b)** Deficit debt diagram



permanent deficits even for very large debt levels. In this limit, private wealth relative to potential GDP could become unboundedly large.

For any given level of debt b we can then compute the primary deficit that keeps debt constant at b by setting $\dot{b} = 0$ in (5),

$$z(b) = (G^* - R^*(b))\, b. \tag{10}$$

We plot $z(b)$ in Figure 1(b). We refer to this diagram as the *deficit-debt diagram* and we will use it extensively in this paper. Each point (b, z) on the locus shown corresponds to a steady state equilibrium with constant debt level b and constant primary deficit z. The locus is naturally hump-shaped. If $R(0)$ is finite, the steady state primary deficit is zero when debt is zero, as well as when $R = G$, $z(0) = z(\bar{b}) = 0$. Between 0 and $\bar{b}$, the primary deficit is positive.

The deficit-debt diagram is an intuitive description of an economy's "fiscal space": for any given initial level of debt, it exactly shows what the primary deficit needs to be in order for debt to remain put.

To characterize the shape further, we define the semi-elasticity of the convenience yield as follows,

$$\varphi(b) \equiv -(1 - x - \mu)\frac{\partial v'(b)}{\partial \log b} = -(1 - x - \mu)v''(b)b.$$

φ is effectively the inverse (semi-)elasticity of savers' demand for government debt. With φ at hand, we then have the following result.

Proposition 1. *If $z(b)$ has a local maximum at some $b^* \in (0, \bar{b})$, we have that*

$$R^*(b^*) = G^* - \varphi(b^*). \tag{11}$$

If, in addition, $\varphi(b)$ is weakly increasing in b, then b^* is the unique local (and global) maximum, with primary deficit $z^* \equiv \varphi(b^*)b^*$.

Proof. From (10) we see that $z'(b) = G^* - R^*(b) - R^{*\prime}(b)b$. Substituting (9) into $z'(b)$ yields that $z'(b^*) = 0$ holds precisely when (11). If $\varphi(b) = R^{*\prime}(b)b$ is weakly increasing, $z'(b)$ is strictly decreasing and hence b^* is the unique global maximum. □

The proposition characterizes the maximum of the primary deficit schedule $z(b)$ as being at a point at which $R^* = G^* - \varphi$, with a maximum primary deficit of $z^* = \varphi b^*$. This emphasizes that $\varphi(b)$ is a crucial determinant of the shape of the deficit-debt diagram. The fact that the maximum is in the interior of $[0, \bar{b}]$ echoes a similar finding in Bassetto and Sargent [2020] in an OLG setting.

3.3 The free lunch condition $R < G - \varphi$

One idea that has garnered considerable attention in the literature surrounding $R < G$ (see, e.g., Blanchard 2019) is that the condition seemingly allows economies to run larger deficits temporarily, and then simply "grow out" of the resulting increased debt levels without a need to raise taxes. We refer to this idea as the "free lunch" property of higher deficits. A stronger version of the "free lunch" idea is that *permanent* increases in deficits do not require tax increases going forward, even if they lead to permanently greater (non-explosive) debt levels.

Both versions of the free lunch idea can easily be derived from the government budget constraint (5), under the assumption of a constant interest rate R and a constant growth rate $G > R$. Then,

$$\dot{b}_t = -(G - R)\, b_t + z \tag{12}$$

describes a stable differential equation for debt b. This implies that temporary increases in deficits of arbitrary magnitude, leading to greater debt levels, can always be grown out of over time. Also, a permanent increase in deficits by some Δz simply raises steady state debt levels by $\Delta z/(G - R)$, with no need for a reduction in deficits, i.e. an increase in taxes, at any point. Both versions of the free lunch property are satisfied with exogenous R and G in (12). This is clearly a stylized example but it captures one, if not *the* most, important reason why fiscal policy in a world with $R < G$ is thought to be so different from fiscal policy with $R > G$.

We next investigate the extent to which the free lunch idea holds true in our model. What distinguishes our model from the stylized analysis in (12), is that R is endogenous to the debt level. To understand the dynamics of b_t, it is crucial to incorporate this endogeneity.[11] To do so, we first describe the behavior of the debt level for a general exogenous path z_t of primary deficits. Then, we feed in the specific paths for deficits that correspond to the two versions of the free lunch property.

Even along transitions, savers' consumption remains constant at $1 - x - \mu$. Thus, the natural rate $R^*(b_t)$ is still given by (9),

$$R^*(b_t) = \rho + G^* - v'(b_t) \cdot (1 - x - \mu).$$

Therefore, the dynamics of the debt level simply follow

$$\dot{b}_t = -\left(G^* - R^*(b_t)\right) b_t + z_t \tag{13}$$

for an exogenous path of deficits z_t.[12] Notably, the dynamics of debt are perfectly backward looking, despite households being forward looking with rational expectations. This stems from the fact that consumption is constant even along transitions due to the goods market clearing condition, pinning down the natural interest rate in each instant.

Representing transitions in the deficit-debt diagram. A useful diagram to study the effects of temporary or permanent changes in deficits is the deficit-debt diagram. In Figure 2 we indicate with arrows the direction the economy travels in when deficits are moved above or below the steady state locus.

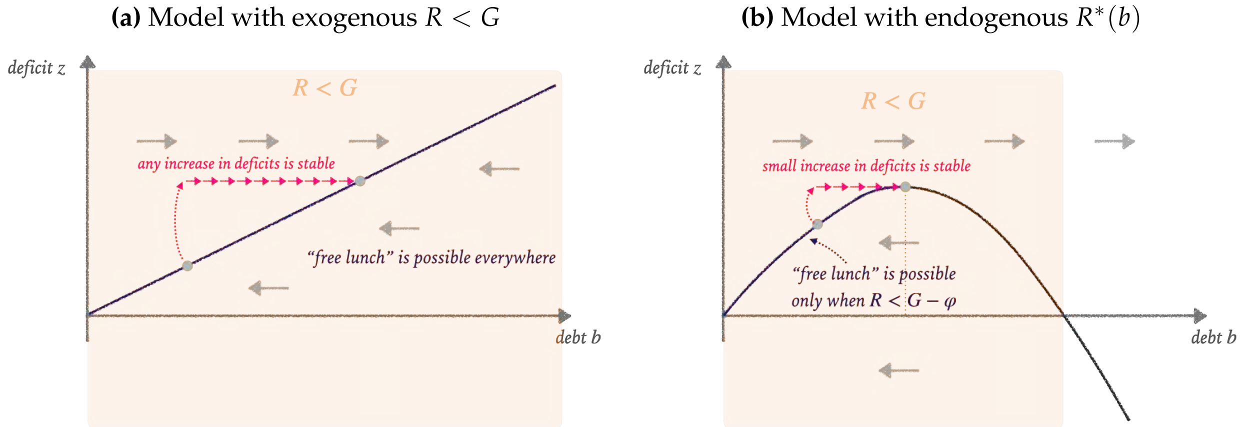
As the figure shows, when deficits are raised above the steady state locus, debt grows, until either the steady state locus is hit, or until, at some point in the future, the deficit is reduced again down to the steady state locus. When deficits are reduced below the steady state locus, debt falls over time. Mathematically, this behavior follows immediately from (13).

Figure 2(a) plots the evolution of debt in a model with exogenous R and G. As one can see, in this case, any increase in deficits is stable. A free lunch policy is always available. This would be the outcome of a model with a linear convenience utility $v(b)$, and is analyzed in Kocherlakota [2021], who microfounds the utility over bonds by allowing for a small probability disaster state with high and constant marginal utility of wealth. This is also why the debt rollover experiments in Blanchard and Weil [2001] and Blanchard [2019] are stable with linear technology.

[11] See also the response of Miller and Sargent [1984] to Darby [1984] for a similar conceptual point.

[12] If deficits followed a fiscal rule $z_t = \mathcal{Z}(b_t)$ instead, one would simply have to replace z_t in (13).

Figure 2: Transitions when changing the deficit



The free lunch region in our model. By contrast, Figure 2(b) allows us to see the region of the state space in which the government can obtain a "free lunch" in our model, with endogenous R. Indeed, any steady state on the increasing part to the left of the peak at b^* allows for some form of a free lunch. For example, starting at any of these steady states, a permanent increase in the deficit to any value below or equal to z^* can be sustained indefinitely. If the deficit increase is temporary, it can exceed z^*, as long as it is reduced back to z^* or below in time. We show an example transition along these lines in Figure 2(b).

However, while the diagram in Figure 2(b) illustrates how a "free lunch" policy is indeed possible, it also makes the limits of such a policy very clear. For example, if deficits are increased by too much and / or for too long, a free lunch cannot be obtained.

More fundamentally, a free lunch policy cannot work if the initial debt level already exceeds b^*, that is, the initial steady state lies on the downward-sloping branch of the deficit-debt locus in Figure 2(b).[13] In this case, any deficit increases, however temporary, must ultimately be met by reduced deficits (or surpluses). In other words, taxes must rise. Crucially, this logic applies despite the fact that the economy displays $R < G$ throughout.

How is this possible? The aspect of our theory that is responsible for this result is the endogeneity of interest rates $R^*(b)$ to the debt level. As the debt level increases, the convenience yield of government debt falls, raising the interest cost on all (infra-marginal) outstanding debt positions. This can undo the positive effect of a greater debt position on the government budget constraint when $R < G$ that we highlighted at the beginning of

[13]Strictly speaking, there could be multiple local maxima of $z(b)$ in our model. The condition for the absence of a free lunch policy is that there can be no steady state with a greater debt level and a greater or equal deficit z.

this section. In fact, as Figure 2(a) illustrates, this precisely happens for debt levels greater than b^*. If the level of debt is larger than b^*, the economics behind the financing of fiscal deficits are entirely conventional: greater debt must be repaid by raising taxes. Whether $R < G$ or $R > G$ is totally irrelevant for the question whether debt is above or below b^*. As the following corollary shows, based on our results in Proposition 1, the correct threshold is not G, but $G - \varphi$.

Corollary 1. *Assume the deficit-debt diagram $z(b)$ is single-peaked. Then, there is a free-lunch policy available at a steady state with debt level $b_0 > 0$ if and only if $R^*(b_0) < G^* - \varphi(b_0)$.*

Proof. If $z(b)$ is single peaked, the unique global maximum b^* is also the unique local maximum. All points $b_0 < b^*$ are then necessarily characterized by $z'(b_0) > 0$, or equivalently, $R^*(b_0) < G^* - \varphi(b_0)$. For any such point, a permanent deficit increase by $\Delta z \equiv z(b_1) - z(b_0) > 0$, for some $b_1 \in (b_0, b^*)$ is a free lunch policy. Any point $b_0 \geq b^*$ does not allow for a free lunch as $z'(b_0) \leq 0$ there. □

Is a free lunch policy always Pareto-improving? We largely refrain from making welfare statements in this paper, partly because different microfoundations for the convenience utility $v(b)$ exist, and they carry different welfare implications.

If the model in Section 2 is taken literally, then a free lunch policy always constitutes a Pareto improvement. It is easy to see why: consumption of both agents remains unchanged in all periods, while debt increases. Since debt enters the utility of savers, welfare increases. In fact, for a similar logic, increases in the debt level even beyond the upper bound b^* of the free lunch region can be welfare improving.

This becomes a bit more nuanced if one assumes that both agents are paying taxes, e.g. for simplicity $\tau_t = \tilde{\tau}_t$ with both taxes adjusting in response to the policy. Now, a free lunch policy is still a Pareto-improvement since it is associated with tax reductions for both agents, and higher interest rates for savers. However, raising debt beyond b^* generally is no longer Pareto improving, echoing results in Aguiar et al. [2021].

3.4 Discussion

Transversality condition. The transversality condition of the saver associated with utility maximization problem (1) is given by $e^{-\rho t} c_t^{-1} b_t \to 0$. This is clearly satisfied in the equilibria described above, as $c_t = 1 - x$ and b_t always converges to a finite value. The transversality condition rules out paths along which debt levels explode.

Present value vs. flow budget constraint. Our analysis illustrates the usefulness of working with the government's flow budget constraint. We have found the present value budget constraint of the government to be somewhat less practical. To see why, let us discount the flow budget constraint (5) at some arbitrary rate θ_t. We obtain

$$\int_0^T e^{-\int_0^t \theta_u du} z_t dt + b_0 \leq e^{-\int_0^T \theta_u du} b_T + \int_0^T e^{-\int_0^t \theta_u du} \left(R(b_t) - G^* - \theta_t \right) b_t dt \tag{14}$$

(14) is equivalent to the flow budget constraint (5). However, (14) is less useful than typical present value budget constraints. This is because in (5), the interest rate $R(b_t)$ is a function of the stock of debt b_t. Irrespective of how θ_t is chosen, this means that the path of debt b_t cannot be eliminated from (14), which defeats one of the main purposes of writing a present value constraint. If θ_t is chosen to be entirely unrelated to $R(b_t) - G^*$, e.g. equal to the household discount rate ρ, the dependence on b_t enters in the final term in (14); if, instead, θ_t is chosen to be equal to $R(b_t) - G^*$, the final term in (14) disappears but the dependence on b_t enters in (14) through θ_t. Moreover, if $\theta_t < 0$, one cannot take the limit $T \to \infty$ in (14). This is why we prefer to work with the flow budget constraint (5) instead.

If one were to work with (14), one natural choice for θ_t is the marginal cost of borrowing, $\theta_t = R(b_t) - G^* + \varphi(b_t)$, which includes $\varphi(b_t)$. Locally around a steady state with debt b_{ss}, interest rate $R_{ss} = R(b_{ss})$ and $\varphi = \varphi(b_{ss})$, we then find a present value constraint

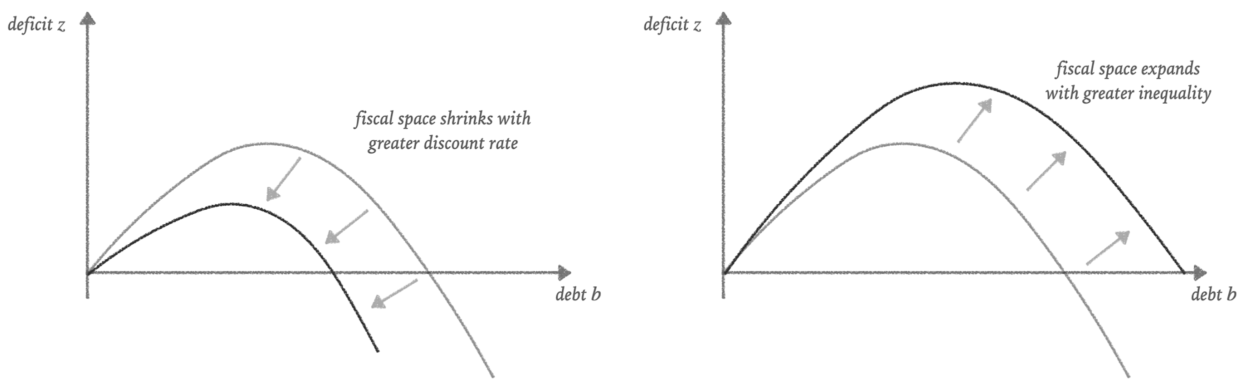
$$\int_0^\infty e^{-(R_{ss} - G + \varphi)t} z_t dt + b_{ss} \leq \frac{\varphi}{R_{ss} - G + \varphi} b_{ss} \tag{15}$$

(15) is well-defined whenever there is no free lunch, that is, $R_{ss} > G - \varphi$. Relative to a standard present value condition, it includes an extra term (one may call this a "debt revenue term" as in Reis 2021) on the right hand side of (15), capturing additional fiscal space afforded by the convenience utility. The fact that discounting includes φ means that this condition is well defined even if $R_{ss} < G_{ss}$, so long as there is no free lunch, $R_{ss} > G_{ss} - \varphi$. If there is a free lunch, (15) is not well-defined as, locally, there are no constraints on deficits z_t around a free lunch steady state.

Comparison with a Sidrauski [1967] money-in-the-utility model. Our model is similar to money-in-the-utility models in that a real asset enters the utility function directly. As we show in Appendix C.2, a straightforward money-in-the-utility version of our model would give a steady state first order condition $\rho + G^* = v'(M/P) \cdot (1 - x - \mu)$ where M is money supply and P is the price level. While this looks almost like the first order

Figure 3: What determines fiscal space?

(a) Discount rate ρ **(b)** Income inequality $1 - \mu$



condition (9) in our model, there are crucial differences. The main difference is that, in the flexible-price money-in-the-utility model, the supply of money determines the price level P. In Appendix C.2, we show that this allows the government to move M/P up or down instantaneously, wherever it starts from, and thus realize a free lunch from any initial value of M/P. This starkly differs from the flexible-price version of our model, where the price level grows at exogenous rate π^* and the supply of bonds instead influences the interest rate R_t. A free lunch in our model is only available when $R < G - \varphi$.

3.5 What determines fiscal space?

What does the size and shape of the deficit-debt locus depend on? This section investigates the role of four factors: discount rates, trend growth, income inequality, and tax policy.

Discount rates. A greater discount rate means savers are more impatient and would like to spend more and save less, which raises R^*. Vice versa, a smaller discount rate means savers are more patient and would like to spend less and save more, which lowers R^*. A lower discount rate can capture a reduction in aggregate demand.

Figure 3(a) sketches the deficit-debt diagram, for two values of ρ. For higher ρ, we see that fiscal space shrinks, as R^* is increased and $G^* - R^*$ falls. We confirm this in the following result.

Corollary 2. *An increase in the discount rate ρ strictly reduces fiscal space.*

Proof. See (16) below. □

The result follows directly by substituting out the interest rate (9) in the equation for the primary deficit (10),

$$z(b) = \left(v'(b)\,(1 - x - \mu) - \rho\right) b \tag{16}$$

The locus $z(b)$ shifts down with higher ρ. In fact, if ρ rises above $v'(0)\,(1 - x - \mu)$, the government has to run a primary surplus at any positive level of debt.

Trend growth. A reduction in nominal trend growth G^*—whether caused by a productivity growth slowdown, falling inflation expectations, or declining population growth—seems like it may tighten fiscal space by moving G^* closer to R. But this is not obvious as slower growth rates lead to a greater desire for saving by households, pushing R^* down alongside G^*. With log preferences over consumption as in (1), R^* falls one for one with G^*, as in (9), leaving $G^* - R^*$ unchanged. This is why, without a ZLB in our model, growth does not affect steady state deficits. We revisit this comparative static below in the model with a potentially binding ZLB.

Income inequality. Inequality is relevant for fiscal sustainability, as it is mainly richer households that, directly or indirectly, own government debt. The top 10% of the wealth distribution in the United States hold 69% of the government debt outstanding held by the U.S. household sector. The bottom 50% of the wealth distribution hold almost no government debt at all (Mian et al. 2020). The willingness or ability of richer households to save may thus be a primary factor in the determination of interest rates on government debt.

Our model naturally speaks to these issues. To see how, notice that greater inequality, in the form of a reduced income share of spenders $\mu \downarrow$, increases the locus $z(b)$ in (16).

Corollary 3. *Absent a ZLB constraint, greater inequality, $\mu \downarrow$, expands fiscal space.*

Proof. See (16) above. □

Thus, greater income inequality unambiguously expands fiscal space without ZLB. This is intuitive. Savers in our model have a greater propensity to save out of an increase in permanent income compared to spenders, so that any increase in inequality reduces R^* and thus increases fiscal space as $z = (G^* - R^*)b$. Figure 3(b) illustrates these findings.

The model provides intuition behind the observation that rising income inequality has been accompanied by rising fiscal deficits and government debt levels in many advanced economies. Rising income inequality allows governments to borrow more cheaply from savers.

Tax policy. Similar to changes in the income distribution, tax policy also affects fiscal space. To see how, allow for nonzero taxes (or transfers) on spenders, $\tilde{\tau} \neq 0$, as well as consumption taxes τ^c paid by both types of agents and capital income taxes τ^b. The budget constraint of savers is then given by

$$(1+\tau^c)\, c_t + \dot{b}_t \leq \left(\left(1-\tau_t^b\right) R_t^{pre} - G_t\right) b_t + (1-\mu)\, w_t n_t - \tau_t$$

where we use R_t^{pre} as the pre-tax interest rate. We use R_t and R_t^* to denote post-tax interest rates. This changes the Euler equation of savers, leading to an updated equation for the (post-tax) natural interest rate,

$$R^*(b_t) = \rho + G^* - v'(b_t)\left((1+\tau^c)(1-x) - \mu + \tilde{\tau}\right). \tag{17}$$

The relationship between R^* and $\tau^c, \tilde{\tau}$ then gives us the following result.

Corollary 4. *Absent a ZLB constraint, increased regressive income taxes $\tilde{\tau}$ and consumption taxes τ^c expand fiscal space. Increased capital income taxes τ^b leave fiscal space unchanged.*

Proof. The post-tax interest rate $R^*(b)$ in (17) is the correct one to use in the government budget constraint. From (17), we see that for any b, $R^*(b)$ falls in $\tau^c, \tilde{\tau}$, and is independent of τ^b. Substituting (17) into (10) then proves the statements. □

Corollary 4 studies the effects of raising taxes on the deficit-debt schedule. To interpret the results, we bear in mind that increased taxes $\tilde{\tau}, \tau^c, \tau^b$, are by construction met by a reduction in lump-sum taxes on savers τ.

Raising taxes $\tilde{\tau}$ on spenders is regressive, reducing demand and thus natural interest rates R^* in (17). This acts like an increase in income inequality, and increases fiscal space absent a binding ZLB constraint. It may be surprising that raising consumption taxes τ^c is similarly regressive in our model, even though both agents pay them. This is because savers trade off consuming and saving in their Euler equation, and their saving is not just driven by future consumption, but also by convenience benefits from holding bonds. Greater consumption taxes τ^c tilt savers towards saving, expanding fiscal space. Increased capital income taxes τ^b are irrelevant for fiscal policy in our model, as the before-tax return on government debt immediately adjusts upwards to keep the after-tax return $R^*(b)$ constant.[14]

[14] Two caveats are in place here. First, with longer-duration debt or large surprise taxes at date $t=0$, some initial expropriation occurs, which can be used for a one-time reduction in government debt. Second, the only source of capital income in our model is interest income from government debt. If other types of capital

These results have three important implications. First, they suggest a potential dilemma. Large redistributive programs may reduce fiscal space, potentially limiting the extent to which such programs can be deficit-financed. Second, regressive taxation is able to finance a greater level of government debt than progressive taxation, holding fixed the overall tax burden. Governments with sufficiently large debt levels and interest rates R near or above G may thus be forced to resort to such regressive taxation.[15]

The third implication concerns financial repression. Financial repression can be thought of as the government imposing a lower bound on the required bond holdings of the saver, $b_t \geq \underline{b}$, thereby allowing it to reduce the interest rate it pays on government debt, from the market rate R_t down to some $R_t - \xi_t$, where ξ_t measures the shadow value of marginally relaxing the constraint. Modeled this way, financial repression corresponds to nothing other than a tax on bondholders. When large debt positions are financed this way, a significant amount of repression is necessary. Since it acts like a tax on savers, it reduces the resources of savers, reduces their demand for bonds and thus requires even more stringent financial repression.

4 Fiscal space near the ZLB

We are now ready to re-introduce the ZLB constraint. As evidenced by the extended period of time many advanced economies have spent at the ZLB (or a similar effective lower bound) over the past decade or more, this is a real binding constraint that needs to be analyzed jointly with fiscal policy.

4.1 Deficit debt diagram with ZLB

To derive the deficit debt diagram, the slope of the Phillips curve κ turns out to be a crucial parameter. The larger it is, the weaker is nominal growth in the liquidity trap at the ZLB, which changes the dynamics and the shape of the deficit debt diagram. Below, we derive the relevant threshold for κ to be equal to $\hat{\kappa} \equiv \frac{1-\mu}{1-\mu-x}(\rho + G^*)$.

Case (A): $\kappa < \hat{\kappa}$. We begin with the case where $\kappa < \hat{\kappa}$. In this case, whether the ZLB is binding can be read off from the formula for R^*, (9). R^* is negative precisely when

income, such as dividends, were present, the capital income tax would adopt some of the properties of a tax on savers' income.

[15]This argument can, in principle, be taken even further. Any policy instrument that discourages demand reduces natural interest rates R^* and thus has beneficial effects on the government's interest expenses.

$b < b^{ZLB}$, where b^{ZLB} is implicitly defined by $v'(b^{ZLB}) = \frac{\rho+G^*}{1-\mu-x}$. At the ZLB, output is demand determined, via the goods market clearing condition,

$$c_t + \tilde{c}_t + x = y_t. \tag{18}$$

Here, consumption of spenders is simply $\tilde{c}_t = \mu y_t$ (assuming $\tilde{\tau} = 0$ as before), while consumption of savers still follows an Euler equation much like (8), only that now, the interest rate is zero, $R_t = 0$, and the growth rate is endogenous, due to the endogeneity of inflation, $G_t = G^* - \kappa(1 - y_t)$. Substituting this into the Euler equation, we find

$$\frac{\dot{c}_t}{c_t} = -G^* + \kappa(1 - y_t) - \rho + v'(b_t)c_t. \tag{19}$$

Jointly solving (18) and (19), we find an expression for the nominal growth rate of the economy at the ZLB,

$$G(b) = G^* - \frac{\kappa}{v'(b)\,(1-\mu) - \kappa}(-R^*(b)). \tag{20}$$

This emphasizes a key theme that distinguishes fiscal space considerations at the ZLB vs those outside the ZLB: Rather than R being endogenous to the level of debt, it is now the growth rate G. We highlight this in Figure 4(a).

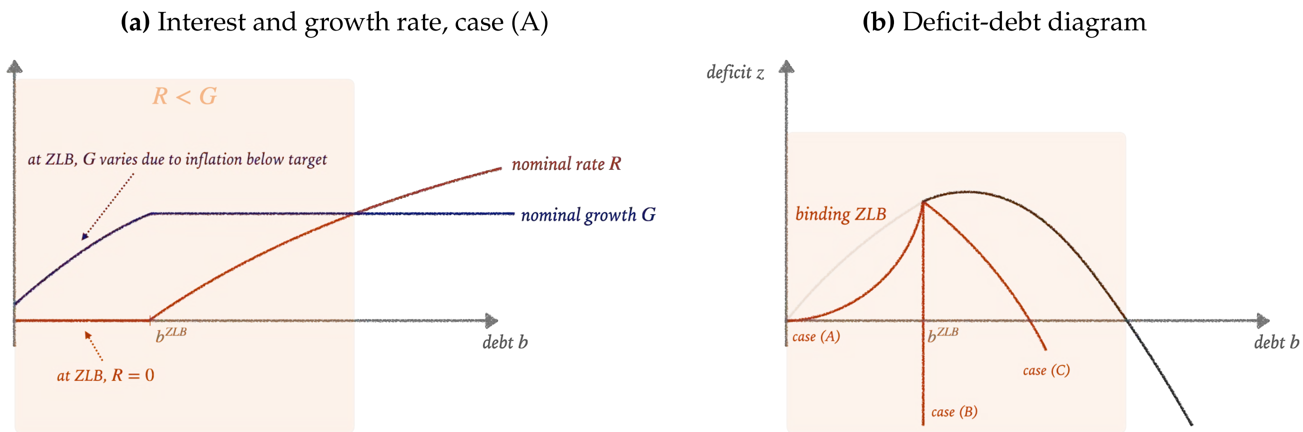
This has important implications for the deficit-debt locus $z(b)$. Without the ZLB, low natural rates R^* unambiguously increase fiscal space. Now, however, as soon as R^* falls below zero, the nominal rate R stops decreasing, and it is instead the nominal growth rate G that declines. This has the opposite effect on fiscal space: the permanent deficit $z(b) = (G(b) - R(b))b$ that the economy can run is reduced by the binding ZLB relative to the case without ZLB. This can be seen in Figure 4(b).[16]

Case (B): $\kappa = \hat{\kappa}$. In this case, the reduction in nominal growth at the ZLB is sufficiently large that no steady state left of b^{ZLB} exists. Instead, the Euler equation (19) can be simplified to

$$\frac{\dot{c}_t}{c_t} = c_t\left(v'(b_t) - v'(b^{ZLB})\right).$$

[16]This result also holds with $\kappa = 0$, where both R and G are constant at the ZLB. Also, the fact that our steady state analysis assumes a permanently binding ZLB should be viewed as an abstraction. Similar results hold when it is simply the frequency of ZLB incidence over the business cycle that increases as an economy has a lower average natural interest rate.

Figure 4: Fiscal space with ZLB constraint



When $\kappa = \hat{\kappa}$, all ZLB steady states lie on a vertical line with debt level $b = b^{ZLB}$, as shown in Figure 4(b).

Case (C): $\kappa > \hat{\kappa}$. In this case, nominal growth at the ZLB is sufficiently weak that even for debt levels $b = b^{ZLB}$, debt still trends to the right. The steady state locus is now "bending backwards", as shown in Figure 4(b). There is no steady state for $b < b^{ZLB}$, and for $b > b^{ZLB}$, growth rates are given by

$$G(b) = G^* - \frac{\kappa}{\kappa - v'(b)(1-\mu)} R^*(b) < G^*$$

which lie below G^* since $R^*(b)$ is positive in this range and $\kappa > v'(b)(1-\mu)$.

How is it possible that the deficit-debt locus bends backwards when $\kappa > \hat{\kappa}$? For an intuition, imagine the economy starts on the top black branch of the locus and the government reduces its primary deficit. This has two effects on the dynamics of debt:

$$\dot{b} = \underbrace{z}_{\text{direct effect}} \underbrace{-(G-R)\,b}_{\text{indirect effect}}. \tag{21}$$

Reducing the primary deficit z at a given level of debt b has a negative direct effect on $\dot{b}$. It also indirectly reduces the natural interest rate R^*. In the absence of a ZLB constraint, the indirect effect unambiguously raises the gap between G and R, implying that both effects contribute to reducing debt. This is why, without the ZLB, even large deficit reductions will always lead to faster deleveraging.

This is different in the presence of the ZLB. Now, current (and future) reductions in R^*

do not materialize as lower R, but instead weaker nominal growth G. This pushes against the direct effect of lower z in (21). If $\kappa > \hat{\kappa}$, the indirect effect dominates for sufficiently small deficits, and further deficit reductions no longer lead to any deleveraging.

4.2 Free lunch at the ZLB

We next revisit the question of when free lunch policies exist. To do so, we study transitional dynamics. Different from the analysis in Section 3, consumption is now no longer constant along the transition. Then, the economy is governed by a system of two differential equations, the Euler equation

$$\frac{\dot{c}_t}{c_t} = \kappa \left(1 - \frac{x + c_t}{1 - \mu}\right) - G^* - \rho + v'(b_t)c_t \tag{22}$$

in addition to the government budget constraint

$$\dot{b}_t = \left(\kappa \left(1 - \frac{x + c_t}{1 - \mu}\right) - G^*\right) b_t + z_t. \tag{23}$$

While this system is harder to represent in the deficit-debt diagram, it is still straightforward to determine the availability of a free lunch. In fact, a free lunch always exists for any ZLB steady state.[17]

To see this for case (A), notice that the ZLB region is always on the left branch of the hump-shaped deficit-debt locus. An increase in the primary deficit to, say, the maximum level z^* ensures both a free lunch and an exit out of the liquidity trap. For cases (B) and (C), it is even simpler. A second non-ZLB steady state exists, with the same debt level but a greater deficit.

One issue to point out here is that in the economy with a potentially binding ZLB and with $\kappa > 0$, multiple equilibria (with the same initial debt level) can emerge. This is well known from standard models with active interest rate rules and a ZLB constraint (Benhabib, Schmitt-Grohé, and Uribe 2001). Michaillat and Saez [2019] show that a convenience utility function can resolve multiplicity when there bonds are in zero net supply and $\kappa < v'(0)$.

Our model allows bonds b to be a state variable, different from zero, which can introduce multiplicity even if $\kappa < v'(0)$. For example, starting at one of the (non-ZLB) steady states on the right branch of the deficit-debt locus, there may be a second, non-steady-state

[17] If an economy is not literally in a steady state with a binding ZLB constraint, this result is to be understood as: The closer an economy is to the ZLB, and the more frequently it hits it, the more likely it becomes that a permanent deficit expansion is possible.

equilibrium, along which the economy converges towards one of the ZLB steady states on the left.

Numerically, it turns out that such multiplicity is not easy to get. We have verified for our two calibrated economies in Section 5 that no equilibrium multiplicity exists with a deficit policy $\mathcal{Z}(b)$ whose slope is not too steep, such as the constant deficit policies we have discussed before. More generally, we show in Appendix A.1 that by suitable choice of $\mathcal{Z}(b)$, any steady state can be established as a unique equilibrium.

4.3 What determines fiscal space near the ZLB?

Next, we revisit the role of some of the drivers of fiscal space, only now allowing for a potentially binding ZLB constraint. For simplicity, we present illustrations only for the non-backward-bending case (A), that is, $\kappa < \hat{\kappa}$. The other cases behave similarly.

Growth slowdown. A slowdown in trend growth G^* did not have an effect fiscal space outside the ZLB since the natural rate R^* shifts down one for one with G^*, so the gap $G^* - R^*$ is unchanged. Yet, at the ZLB, a reduction in the natural rate R^* does not translate into a reduction in policy rate R; instead the nominal growth rate G falls. We summarize this formally below.

Corollary 5. *With a binding ZLB constraint, a reduction in trend growth G^* reduces fiscal space: $z(b) = (G(b) - R(b))b$ falls with lower G^*.*

Proof. This follows from the fact that $z(b) = G(b)b$ at the ZLB. Differentiating (20) we find $\partial G(b)/\partial G^* = v'(b)(1-\mu)/(v'(b)(1-\mu)-\kappa) > 0$. □

Income inequality. Rising income inequality (falling μ) unambiguously increases fiscal space without the ZLB (Section 3.5). With a potentially binding ZLB constraint, we have the following result.

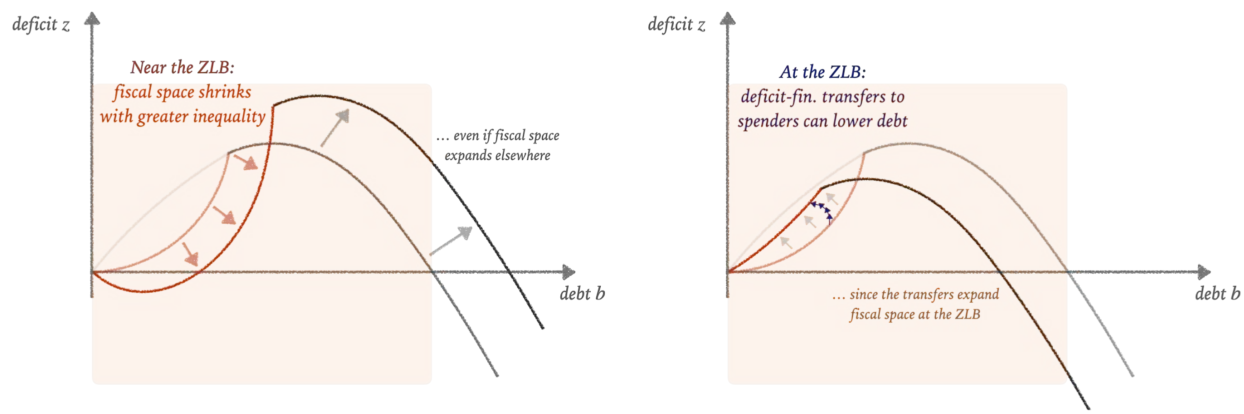
Proposition 2. *When the economy is at the ZLB with debt level b, increased inequality locally reduces fiscal space $z(b)$ if $\kappa > 0$.*

Proof. Differentiating (20) with respect to μ, we see that

$$\frac{\partial G}{\partial \mu} = -\frac{\kappa v'(b)}{(v'(b)(1-\mu)-\kappa)^2}(-R^*(b)) + \frac{\kappa}{v'(b)(1-\mu)-\kappa}v'(b)$$

Figure 5: Drivers of fiscal space at the ZLB

(a) Increasing inequality can reduce fiscal space **(b)** Stimulus checks can reduce debt / GDP



After some algebra, and using $\kappa > 0$, $\partial G/\partial \mu > 0$ is equivalent to $v'(b)x > \kappa - \rho - G^*$. For any $b < b^{ZLB}$, this is implied if $v'(b^{ZLB})x > \kappa - \rho - G^*$. Using $v'(b^{ZLB}) = \frac{\rho+G^*}{1-\mu-x}$, this is equivalent to $\kappa < \hat{\kappa}$, which we assume here. Thus, $\partial G/\partial \mu > 0$. From $z(b) = G(b)b$ we see that inequality reduces fiscal space. □

Proposition 2 shows that income inequality can reduce fiscal space, that is, the permanent primary deficit $z(b)$, at the ZLB. This is because income inequality reduces R^*, which at the ZLB, reduces the nominal growth rate G, rather than the nominal interest rate R. This reduces fiscal space.

Tax policy. An immediate implication of this result is that, at the ZLB, more progressive taxation (which here is identical to redistribution) increases fiscal space.

Corollary 6. *At the ZLB, redistribution raises fiscal space $z(b)$ if $\kappa > 0$.*

In particular, this result implies that greater redistribution at the ZLB reduces the debt level. It turns out that even deficit-financed stimulus checks may ultimately reduce the debt, by way of increased nominal growth. We spell this out in the next result.

Proposition 3. *Starting from a ZLB steady state, a permanent increase in transfers to spenders, $\tilde{\tau} < 0$, without change in taxes on savers τ, reduces the debt level in the long run if $\kappa > (1-\mu)\, v'(b) / \left(v'(b)b + 1\right)$. A necessary and sufficient condition for this to hold for some $b < b^{ZLB}$ is $\kappa > \hat{\kappa} / \left(v'(b^{ZLB}) b^{ZLB} + 1\right)$.*

Proof. A small transfer of $d\tilde{\tau} < 0$ with an associated increase in the primary deficit of $dz = -d\tilde{\tau} > 0$ leads to a reduction in debt if $d\dot{b} = -dz + bdG > 0$. Here, dG is the change in nominal growth, $dG = \kappa v'(b) / (v'(b)(1-\mu) - \kappa) dz$, which follows by substituting $R^*(b_t) = \rho + G^* - v'(b_t)(1 - x - \mu + \tilde{\tau})$ (a special case of (17)) into (20). So, $d\dot{b} > 0$ iff $b \cdot dG/dz > 1$, which is equivalent to $\kappa > \frac{v'(b)}{v'(b)b+1}(1-\mu)$. This condition is loosest among all debt levels with a binding ZLB when $b = b^{ZLB}$, for which case we note that $v'(b^{ZLB})(1-\mu) = \hat{\kappa}$. This proves the results in the proposition. □

This result is a close cousin of case (C) in Figure 4(b). In that figure, the locus bent backwards because deficit-financed transfers to savers at the ZLB can ultimately lower the debt level. We showed that this happens when $\kappa > \hat{\kappa}$.

Proposition 3 highlights that deficit-financed transfers to spenders can also reduce the debt level. In fact, this happens under a looser condition on κ. Figure 5(b) reveals that this is because a transfer to spenders increases fiscal space at the ZLB and thus makes it more likely that the debt level falls, even if the economy is in case (A) where $\kappa < \hat{\kappa}$.

Once again, the intuition is based on the nominal growth rate response: if the inflation response to stimulus (i.e. κ) is sufficiently large, a permanent increase in transfers increases nominal growth sufficiently for debt to remain constant, or even fall. A similar logic applies to temporary stimulus programs.

This intuition works at the ZLB in our model since it is a region in our model in which changes in aggregate demand influence G rather than R. However, it also applies to any situation in which the monetary authority temporarily or permanently keeps the nominal interest rate unchanged while fiscal policy stimulates.

To see whether this effect is realistic and to which economies it applies most likely, consider the following calculation. Imagine a permanent increase in the deficit by Δz raises output at the ZLB by some multiplier $\frac{\Delta y}{\Delta z}$ times Δz.[18] Using (21), $\dot{b}$ falls precisely when

$$\kappa \frac{\Delta y}{\Delta z} b > 1.$$

This "sufficient statistic" condition is more likely to hold when κ, the multiplier $\frac{\Delta y}{\Delta z}$, and the initial debt level are large. For realistic κ's in the range 0.1 to 0.3 (see below), and a debt level as large as Japan's (225%), this can be satisfied for multipliers $\frac{\Delta y}{\Delta z}$ the range $1.5 - 2$.

[18] This is a "permanent stimulus" multiplier which one would expect to be significantly greater than multipliers from transitory stimulus.

5 Quantifying fiscal space in the United States and Japan

In this section we calibrate the model and measure the shape of the deficit-debt locus for the United States and Japan as well as possible. For both countries, we are particularly interested in determining the maximum permanent deficit z^*; the associated debt level b^*, beyond which a free lunch policy ceases to be feasible; the level of debt $\overline{b}$ at which the interest rate R rises above the growth rate G; and, in the case of Japan, the location of ZLB steady states. We focus on the economies at the end of 2019, before the Covid pandemic, to avoid misinterpreting temporary shifts due to Covid as shifts in long-run steady states.

5.1 Functional forms for the convenience yield

The crucial object to calibrate is the shape of the convenience yield $v'(b)(1-\mu-x)$. Our calibration strategy proceeds in two steps. We first assume a parametric family of functional forms for $v'(b)$ and then determine the parameters that match a given steady state with debt b_0 (one of the two economies in 2019) as well as estimates of the local (semi-)elasticity of the convenience yield $\varphi(b_0)$. We henceforth abbreviate $\varphi(b_0)$ by φ. Since matching the elasticity only provides accuracy in a neighborhood of b_0, we provide analyses of robustness with respect to the functional forms below.

We consider two functional forms for $v'(b)$ for which the empirical literature has documented a good empirical fit (e.g., Krishnamurthy and Vissing-Jorgensen 2012, Presbitero and Wiriadinata 2020). The first is a linear specification, which will be our baseline,

$$\text{linear:} \quad v'(b)\,(1-\mu-x) = v'(b_0)\,(1-\mu-x) - \varphi\frac{b-b_0}{b_0}. \tag{24}$$

The second is a log-linear specification,

$$\text{log-linear:} \quad v'(b)\,(1-\mu-x) = v'(b_0)\,(1-\mu-x) - \varphi\log\frac{b}{b_0}. \tag{25}$$

In both cases, we set $v'(b) = 0$ for any b sufficiently large to cause the right hand side to move below zero. The intercept $v'(b_0)\,(1-\mu-x)$ is determined by the initial steady state, for which the Euler equation pins down the convenience yield $v'(b_0)\,(1-\mu-x)$ as

$$v'(b_0)\,(1-\mu-x) = \rho + G^* - R_0. \tag{26}$$

For both functional forms, we can explicitly solve the three main quantities of interest.

Proposition 4. *For the linear specification* (24)*, we have*

$$\frac{\bar{b}}{b_0} = 1 + \frac{1}{\varphi}\left(G^* - R_0\right), \qquad \frac{b^*}{b_0} = \frac{1}{2}\frac{\bar{b}}{b_0}, \qquad z^* = \varphi\frac{\left(b^*\right)^2}{b_0}.$$

For the log-linear specification, we have

$$\log\frac{\bar{b}}{b_0} = \frac{1}{\varphi}\left(G^* - R_0\right), \qquad \log\frac{b^*}{b_0} = \log\frac{\bar{b}}{b_0} - 1, \qquad z^* = \varphi b^*.$$

Proof. We have three equations for the three objects: $v'(\bar{b})\left(1-\mu-x\right) = \rho$, $\rho - v'(b^*)\left(1-\mu-x\right) = -(1-x-\mu)v''(b^*)b^*$, and $z^* = \left(v'(b^*)\left(1-\mu-x\right)-\rho\right)b^*$. Deriving the expressions in the proposition based on (24) and (25) is straightforward algebra. □

These are simple expressions that allow us to translate empirical estimates of φ directly into the three objects of interest. Interestingly, the objects are pinned down by only four statistics: the elasticity φ, the initial debt level b_0, nominal trend growth G^*, and the initial interest rate R_0. The elasticity φ takes a crucial role here, which is why we discuss its measurement next.

5.2 Measuring the elasticity φ

There are different ways to estimate the elasticity φ that are equivalent within the context of the model. By (26), the convenience yield is nothing other than $\rho + G - R$, so we can write

$$\varphi = -\frac{\partial\left(\rho + G - R\right)}{\partial \log b} = -b_0\frac{\partial\left(\rho + G - R\right)}{\partial b}. \tag{27}$$

Alternative ways to obtain φ are given by

$$\varphi = -\frac{\partial\left(G - R\right)}{\partial \log b} = -b_0\frac{\partial\left(G - R\right)}{\partial b} \tag{28}$$

because, in the model, ρ is independent of b. As both (27) and (28) are valid ways to obtain φ, we will compare estimates across these specifications. Note, however, that specifications estimating (28) are slightly more robust, as they do not hinge on a convenience-yield interpretation of $R^*(b)$ and apply equally well to the alternative models in Appendix C.

The derivatives in equations (27) and (28) have been estimated in the literature, and we summarize these estimates in Table 1.[19] For equation (27), Krishnamurthy and Vissing-

[19] A detailed explanation of the exact specifications used from the existing literature to construct Table 1 is

Jorgensen [2012] focus on estimates of $\frac{\partial(\rho+G-R)}{\partial \log b}$. This derivative measures how the difference between the rate of return on government debt R and the return on other assets $\rho + G$ varies with a change in the log government debt to GDP ratio. Krishnamurthy and Vissing-Jorgensen [2012] use the yield spread difference between corporate bonds rated Baa and 10-year Treasuries as the measure of $\rho + G - R$, and they show a semi-elasticity of -1.3% to -1.7%, depending on the sample. This implies that a 10 percent increase in debt to GDP leads to a 13 to 17 basis point decline in the convenience yield. Alternatively, one can use the Krishnamurthy and Vissing-Jorgensen [2012] estimates to measure $b_0 \frac{\partial(\rho+G-R)}{\partial b}$, which gives estimates between -1.1% and -1.8% when using the average debt to GDP ratio over the relevant sample period for b_0.[20] Finally, Jiang et al. [2020] provide estimates of the effect of government debt to GDP ratios on convenience yields for Eurozone countries from 2002 to 2020. The implied estimate of $b_0 \frac{\partial(\rho+G-R)}{\partial b}$ from their main specification is -0.8%.

There is also a literature estimating the derivative in equation (28), which is $b_0 \frac{\partial(G-R)}{\partial b}$. In particular, the recent study by Presbitero and Wiriadinata [2020] estimate this derivative in a sample of 56 countries from 1950 to 2019. They provide estimates of $\frac{\partial(G-R)}{\partial b}$ for 17 advanced economies and for the full sample. After multiplying these estimates by b_0, which is the average debt to GDP ratio in each of the respective samples, the implied estimates of $b_0 \frac{\partial(G-R)}{\partial b}$ are -1.4%. For this study, we replicated the Presbitero and Wiriadinata [2020] results for the 17 advanced economies and also for the Group of 7 (G7) countries, and the coefficient estimate ranges are also reported in Table 1. The appendix shows the full results from the regressions. The estimates of interest are robust to the inclusion of both time and country fixed effects. Overall, most of the estimates across the different samples and the two different objects fit between -1.0% and -2.5%.

An alternative technique to estimate $\frac{\partial(G-R)}{\partial \log b}$ is an analysis of the 2021 Georgia Senate run-off elections that took place on January 5th in the United States. Ex-ante, there was about an even probability of the two Democrat candidates winning their elections as there was that at least one of the two winning candidates was Republican. In the event of a Democrat win, Democrats would obtain the Senate majority, and would likely pass the \$1.9 trillion deficit-financed stimulus package already proposed by President-elect Biden. This was unlikely to happen otherwise. As shown in Figure 17 in Appendix E, the wins by both

in Appendix E. We thank Sam Hanson, Andrea Presbitero, Quentin Vandeweyer, and Ursula Wiriadinata for helpful discussions.

[20] Two other studies in the literature use short-term T-bills and more high frequency data. Greenwood et al. [2015] find estimates for $b_0 \frac{\partial(\rho+G-R)}{\partial b}$ in this range, around -1.4%, whereas Vandeweyer [2019] finds an estimate of -0.4%. The estimates in these two studies should be regarded as a lower bound as they are based on a local estimate of the demand for T-bills as opposed to demand for all government debt.

Table 1: How does government debt to GDP affect convenience yield and $G - R$?

Study	Countries	Sample	Object	Estimated φ
Convenience yield: $\rho + G - R$				
Krishnamurthy and Vissing-Jorgensen [2012]	USA	1926-2008	$b_0 \frac{\partial(\rho+G-R)}{\partial b}$	-0.011
Krishnamurthy and Vissing-Jorgensen [2012]	USA	1969-2008	$b_0 \frac{\partial(\rho+G-R)}{\partial b}$	-0.018
Krishnamurthy and Vissing-Jorgensen [2012]	USA	1926-2008	$\frac{\partial(\rho+G-R)}{\partial \log b}$	-0.013
Krishnamurthy and Vissing-Jorgensen [2012]	USA	1969-2008	$\frac{\partial(\rho+G-R)}{\partial \log b}$	-0.017
Greenwood et al. [2015]	USA	1983-2007	$b_0 \frac{\partial(\rho+G-R)}{\partial b}$	-0.014
Vandeweyer [2019] (natural experiment)	USA	2014-2016	$\frac{\partial(\rho+G-R)}{\partial \log b}$	-0.009
Jiang et al. [2020]	Eurozone	2002-2020	$b_0 \frac{\partial(\rho+G-R)}{\partial b}$	-0.008
Growth minus Interest Rate: $G - R$				
Presbitero and Wiriadinata [2020]	17 AEs	1950-2019	$b_0 \frac{\partial(G-R)}{\partial b}$	-0.014
Presbitero and Wiriadinata [2020]	31 AEs & 25 EMs	1950-2019	$b_0 \frac{\partial(G-R)}{\partial b}$	-0.013
This paper	17 AEs	1950-2019	$\frac{\partial(G-R)}{\partial \log b}$	-0.015 to -0.031
This paper	G7	1950-2019	$\frac{\partial(G-R)}{\partial \log b}$	-0.020 to -0.028
This paper	USA, Senate	Jan 2021	$\frac{\partial(G-R)}{\partial \log b}$	-0.022
Negative Real Interest Rate: $-R$				
Laubach [2009]	USA	1976-2006	$b_0 \frac{\partial(\pi-R)}{\partial b}$	-0.015 to -0.022*
Engen and Hubbard [2004]	USA	1976-2003	$b_0 \frac{\partial(\pi-R)}{\partial b}$	-0.015*

Notes. This table summarizes estimates from the existing literature of the effect of government debt to GDP ratios on convenience yields (upper panel) and $G - R$ (lower panel). All of the details on the exact specifications used are in the appendix. Further details on the country-year panel regressions done in this study and the evaluation of the Georgia Senate election results of January 2021 are also in the appendix.

* Estimates in Laubach [2009] and Engen and Hubbard [2004] are stated in terms of $\frac{\partial(-R)}{\partial b}$. To obtain $b_0 \frac{\partial(-R)}{\partial b}$, estimates were multiplied by $b_0 = 0.5$, the average level of total federal debt to GDP over the sample period.

Democrats in Georgia led to a significant persistent increase in real 10 year Treasury yields of about 8 basis points. The effect is concentrated right after the election. It is unlikely that the election was associated with a change in long-term growth prospects; as a result, we interpret the evidence as suggesting that the prospect of the $1.9 trillion stimulus package, approximately corresponding to 7.4% of outstanding debt, led to a persistent 8 basis point reduction in $G - R$. As this the Democrat win was anticipated with 50% likelihood, this gives $\frac{\partial(G-R)}{\partial \log b} = -2.2\%$. The natural experiment yields an effect of government debt on $G - R$ that is in the same range as the estimates from the existing literature. Please see the Appendix E for details on this calculation.

Finally, Laubach [2009] and Engen and Hubbard [2004] estimate the effect of government debt to GDP on real interest rates, finding effects in the range 3% to 4.4%. The average level of government debt (total federal debt) to GDP over their sample period was about 50%. Together, this gives an estimate of φ of $b_0 \frac{\partial(G-R)}{\partial b} \approx -1.5\%$ to -2.2% under the assumption that the real growth rate is unaffected by government debt.

Overall, while there is some variation, most of the implied elasticity estimates φ lie in the range 1.1% – 2.5%. We pick the average estimate $\varphi = 1.7\%$ for both countries as our baseline parameter but explore robustness to $\varphi = 1.2\%$ and $\varphi = 2.2\%$ below.

5.3 Calibrating other parameters

We calibrate the remaining model parameters as follows, broadly in line with the U.S. and Japanese economies in December 2019, before the pandemic recession of 2020/21.

For the United States, we set the initial debt level to $b_0 = 100\%$ of GDP, assume government consumption expenditure of $x = 14\%$ (in line with its value 2019Q4), and choose an initial nominal rate of $R_0 = 1.5\%$ (in line with nominal interest rates in December 2019).[21] We set the nominal trend growth rate to $G^* = 3.5\%$ (real growth $\gamma = 1.5\%$), equal to the average peak-to-peak growth rate from 2008 through 2019. In line with $G^* - R_0 = 2\%$, the United States was indeed running a primary deficit of about 2% before the pandemic. We set the discount rate to $\rho = 3.5\%$, in line with about a $\rho + \gamma = 5\%$ real return on business equity during this period (see Mian, Straub, and Sufi 2021 and sources therein). We assume savers are the top 10% of the U.S. income distribution. We set their share of $1 - \mu$ to 50%, in line with evidence from Piketty, Saez, and Zucman [2018].

For Japan, we set initial debt to $b_0 = 238\%$ of GDP (2019 value) and assume government consumption expenditure of $x = 20\%$ (2019 value). The economy is at the ZLB, $R_0 = 0$,

[21] The effective federal funds rate was 1.55%, the 5-year Treasury yield was 1.68%, the 10-year yield just above that. The implied 5-year 5-year forward rate was 2.04%.

Table 2: Baseline calibration

Parameter	Description	Value U.S.	Value Japan
b_0	initial debt to GDP	100%	238%
x	gov. spending to GDP	14%	20%
R_0	initial nominal rate	1.5%	0%
G^*	nominal trend growth	3.5%	2.3%
ρ	discount rate	3.5%	1.4%
μ	income share spenders	50%	55%
κ	slope of Phillips curve	0.10	0.10

and had a peak-to-peak nominal growth rate G_0 of 0.6% from 2008 through 2019. During this time, inflation in Japan was 1.7% below the target of 2%. We therefore set nominal trend growth to $G^* = 2.3\%$ (real growth $\gamma = 0.3\%$ plus inflation at 2%). We set $\rho = 1.4\%$ in line with a dividend yield of the Nikkei 225 stock market index before the pandemic of around 1.4 (5-year average 2015 through 2019). The top 10% income share $1 - \mu$ in Japan is set to 45% based on estimates from the World Inequality Database.

For both countries, we also need to set the slope of the Phillips curve κ. Hazell, Herreno, Nakamura, and Steinsson [2020] estimate two values for the slope κ of their reduced form Phillips curve, 0.1 and 0.3. We choose the lower estimate, $\kappa = 0.1$. This is conservative regarding the "inverted" dynamics at the ZLB, and is also closer to the structural Phillips curve estimates of Hazell et al. [2020] applied to our setting.[22]

We study robustness to different calibrations and different functional forms for the convenience yield in Appendix B (for simplicity only for the case of the United States).

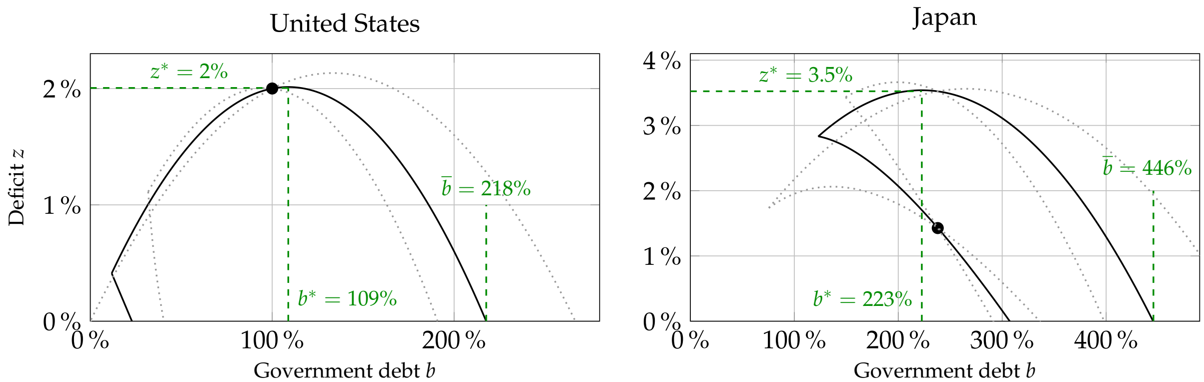
5.4 Calibrated deficit-debt diagrams

Figure 6 shows the calibrated deficit-debt diagrams for the United States and Japan. For both, the figure also plots two gray dotted lines, corresponding to the two alternative values of φ, 1.2% and 2.3%.

The diagram for the United States suggests that the highest permanent primary deficit

[22] Their estimate of the slope in a New-Keynesian Phillips curve is 0.0063. When shocks are permanent, with a discount rate in the Phillips curve equal to the U.S. real return on equity of 5%, this gives a slope of $0.0063/(1 - 0.05) = 0.126$.

Figure 6: Calibrated deficit-debt diagrams



that the United States can run is $z^* = 2.0\%$ of GDP, at a debt level of $b^* = 109\%$ of GDP. The debt level at which R rises above G is $\overline{b} = 218\%$ of GDP. According to this calibration, the United States was running its fiscal policy within the free-lunch zone as of December, 2019. Not only was the United States running a primary deficit of about 2% of GDP at the time, but CBO was projecting longer run six to ten year out primary deficit to be 2.2% of GDP in August, 2019. The calibration thus suggests that projected U.S. fiscal policy more or less sustainable on the eve of the Covid-19 recession. Of course, the Covid-19 shock increased the debt level by a considerable amount. To the extent that the U.S. economy recovers to the pre-Covid fundamentals, the same diagram in Figure 6 can be used to evaluate fiscal sustainability going forward.

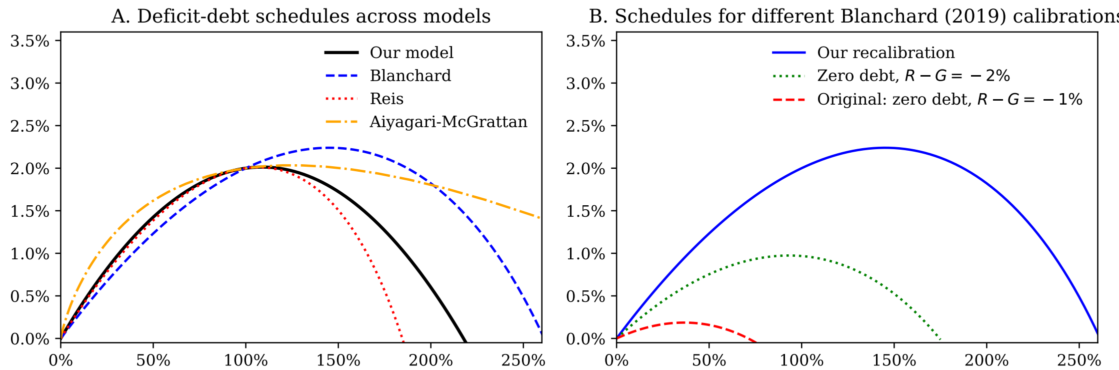
The diagram for Japan is very different. It suggests a markedly higher maximum permanent deficit of $z^* = 3.5\%$, achieved at $b^* = 223\%$. R doesn't rise above G until $\overline{b} = 446\%$ of GDP. The most important difference, however, is the ZLB region. For the United States, there is only a small ZLB region at the far left end of the plot. For Japan, there is a large ZLB region, which is "backward bending" as in case (C) of Figure 4 instead.

This has important implications for fiscal policy in Japan. It suggests that modestly raising the deficit may reduce, rather than increase, the debt level in the long run.[23] Vice versa, regressive policies, such as the increase in the Japanese consumption tax passed in 2012, may increase the debt level. Moreover, redistribution increases fiscal space and allows the government to reduce its debt.

The deficit-debt diagram of Japan in Figure 6 is the namesake of our title: Here, both deficits that are too low and deficits that are too high increase the debt. Intermediate

[23] An interesting extension of our framework would be to include inflation inertia. In that case, increasing the deficit in Japan would first raise the debt to GDP level, as the direct effect in (21) dominates; and then, as inflation picks up and the indirect effect starts dominating the direct effect, the debt to GDP level would fall again.

Figure 7: Deficit-debt schedules across models



deficits keep the debt constant or reduce it.

5.5 Comparison across models

As we mentioned above, we view the convenience utility $v(b)$ as a convenient reduced form description of an increasing relationship between b and R. We next investigate what the deficit diagram would look like in more complex, but less "reduced form" settings.

In Appendix C, we describe how one can compute deficit-debt schedules in three alternative models, based on Aiyagari and McGrattan [1998], Blanchard [2019] and Reis [2021]. For each, we attempt to hit the same calibration targets as those mentioned above.[24] While Aiyagari and McGrattan [1998] and Reis [2021] do not have aggregate risk, Blanchard [2019] does. For Blanchard [2019], therefore, we compute the "risky steady state" (see, e.g. Coeurdacier, Rey, and Winant 2011), that is, the steady state along a path at which the aggregate shock realizes at its mean forever.

Figure 7(A) shows the resulting schedules and compares them to our benchmark schedule. The schedules are similar across models. This suggests that the lessons drawn from our stylized model are robust, and also apply to other models. In particular, while it is harder to prove theoretically, we have numerically verified that a free lunch exists in the (deterministic) Aiyagari and McGrattan [1998] and Reis [2021] models also precisely when $R < G - \varphi$, that is, to the left of the peak of the deficit-debt schedule.

One interesting observation is that the original Blanchard [2019] model assumes no initial debt, and matches $R - G = -1\%$. Figure 7(B) shows the deficit-debt schedule of the original calibration, along with a version that keeps a zero-debt calibration but moves to

[24] We do not calibrate φ in the Blanchard [2019] model as it lacks a free parameter to do so. We also exclude the ZLB for this cross-model comparison.

$R - G = -2\%$ (which we argued above captures better the United States in 2019), with the same risk premium as in Blanchard [2019], and compares them to our preferred calibration of Blanchard [2019]. We see that it is crucial to calibrate that model with a positive level of debt.

6 Fiscal space under aggregate risk

So far, we have analyzed a purely deterministic economy. We now introduce aggregate risk and study its implications for fiscal space and the viability of free lunch policies. To keep things tractable, we omit the ZLB in this section. Our model with aggregate risk builds on the representative-agent model of Mehrotra and Sergeyev [2020].

6.1 Introducing aggregate shocks

Instead of constant (de-trended) potential output, we now assume that (de-trended) potential output y_t is risky and follows a geometric Brownian motion,

$$d \log y_t = \sigma dZ_t$$

where we now allow explicitly for aggregate risk with standard deviation σ. Z_t is a standard Brownian motion. We assume government spending is a fixed share of y_t, xy_t. Furthermore, we allow agents to have a relative risk aversion different from 1, denoted by $\nu > 0$. Savers thus maximize expected lifetime utility,

$$\max_{\{C_t, B_t\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left\{ \frac{C_t^{1-\nu}}{1-\nu} + y_t^{1-\nu} v\left(\frac{B_t}{y_t}\right) \right\} dt \tag{29}$$

where we denote real consumption and debt by C_t and B_t. We continue to use $c_t = C_t / y_t$ and $b_t = B_t / y_t$ for normalized consumption and debt. The utility function in (29) is set up to be scale invariant, as in Mehrotra and Sergeyev [2020]. The budget constraint is

$$C_t + \dot{B}_t \leq (R_t - G^*) B_t + (1 - \mu) w_t n_t - T_t \tag{30}$$

where T_t are unnormalized taxes on savers. We continue to use $\tau_t \equiv T_t / y_t$. Normalized consumption c_t in this economy follows

$$\nu \frac{\dot{c}_t}{c_t} = R_t - G^* - \rho + \frac{1}{2}\nu^2\sigma^2 + c_t^\nu v'(b_t) \tag{31}$$

where, in equilibrium, $c_t = 1 - x - \mu$ as before, which determines the interest rate as

$$R_t = R(b_t) = G^* + \rho - \frac{1}{2}\nu^2\sigma^2 - (1 - x - \mu)^\nu \, v'(b_t). \tag{32}$$

The new term $-\frac{1}{2}\nu^2\sigma^2$ in (32) relative to (9) captures the role of aggregate risk, which all else equal, reduces the interest rate on government debt due to a precautionary motive. Since the new term is constant, $R(b_t)$ has the same functional form as (9).

We denote the primary deficit to GDP ratio by $z_t = x - \tau_t$. The normalized government budget constraint is then given by

$$db_t = z_t dt + \left(R(b_t) - G^* + \frac{\sigma^2}{2} \right) b_t dt - b_t \sigma dZ_t. \tag{33}$$

The transversality condition for savers in this economy is given by $\mathbb{E}_0 \left[e^{-\rho t} C_t^{-\nu} B_t \right] \to 0$. One sufficient condition for it to hold is that there exists an $\epsilon > 0$ such that asymptotically for large t,

$$R(b_t) - G^* + \frac{z_t}{b_t} + \frac{1}{2}\nu^2\sigma^2 < \rho - \epsilon. \tag{34}$$

6.2 Deficit-debt diagram and free lunch

Just like before, we can plot the locus $z(b) \equiv (G^* - R(b))\, b$. Given that $R(b_t)$ has the same functional form, and we calibrated both $R(b_0)$ and $R'(b_0)$, this locus looks identical to those we plotted before. The interpretation is different, however. Before, sitting on the locus $z_t = z(b_t)$ ensured a steady state equilibrium. Here, $z_t = z(b_t)$ only ensures that log government debt remains unchanged *in expectation*, $\mathbb{E}_t [d \log b_t] = 0$. In that sense, the locus here corresponds to a "risky steady state" in the spirit of Coeurdacier et al. [2011]. Just like before, when the economy is above the locus, $z_t > z(b_t)$, log debt rises on average. As in Bohn [1998], a fiscal rule is necessary here to avoid violating the transversality condition (34). As before, we write it as $z_t = \mathcal{Z}(b_t)$.

To study the analogue of a "free lunch" in this economy, we fix a fiscal rule that is consistent with (34) and leads to a well-defined stationary distribution of debt to GDP levels. Fix an initial debt level b_0 and denote the stochastic process of primary deficits implied by the fiscal rule by z_t. We construct the following counterfactual path of government debt b_t^Δ: It starts at in increased initial debt level $b_0^\Delta = b_0 + \Delta$, where $\Delta > 0$, but otherwise follows the exact same deficit path,

$$db_t^\Delta = z_t dt + \left(R(b_t^\Delta) - G^* + \frac{\sigma^2}{2} \right) b_t^\Delta dt - b_t^\Delta \sigma dZ_t.$$

In other words, b_t^Δ is the path of government debt that arises when the government runs a one-time deficit Δ at date 0, but otherwise keeps its deficit unchanged. We refer to the probability that the shifted path b_t^Δ converges back to the original debt level b_t, $P(b_t^\Delta \to b_t)$, as the *success probability of a free lunch policy.* While before, any free lunch had a success probability of 1, this is no longer the case with aggregate risk.

Despite these differences, the following result shows that our condition $R < G - \varphi$ is still relevant with aggregate risk.

Proposition 5. *Denote by $\mathcal{F}(b)$ the cdf of the stationary distribution of debt to GDP in the model with aggregate risk. Assume the convenience yield is of the form (24). Denote by $\overline{R} \equiv \int R(b)\mathcal{F}(db)$ the average interest rate and by $\overline{\varphi}$ the average semi-elasticity $\overline{\varphi} \equiv \int \frac{\partial (R-G)}{\partial \log b} \mathcal{F}(db)$. The success probability of a free lunch policy of size Δ approaches 1 for small Δ if $\overline{R} < G^* - \overline{\varphi}$,*

$$\lim_{\Delta \to 0} P(b_t^\Delta \to b_t) = 1.$$

By contrast, $P(b_t^\Delta \to b_t) = 0$ for any Δ if $\overline{R} > G^* - \overline{\varphi}$.

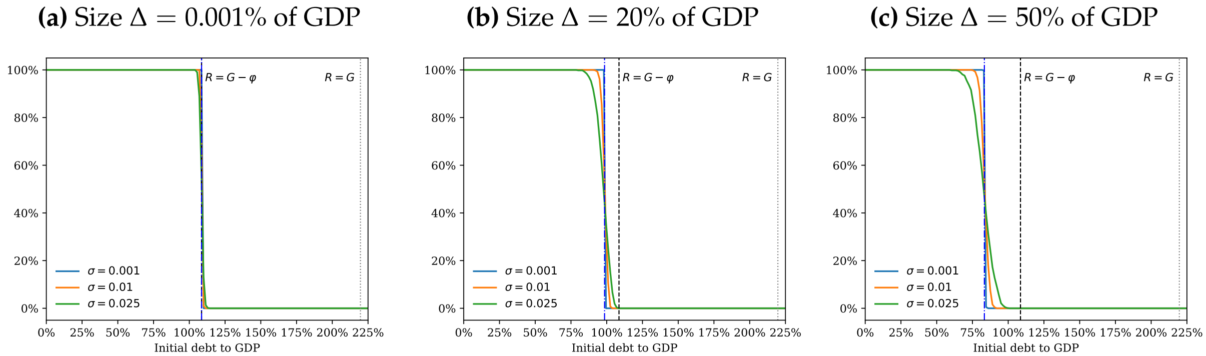
Proof. See Appendix A.2. □

The result generalizes our condition for a free lunch policy from Section 3. Instead of $R < G^* - \varphi$ being the relevant condition, evaluated at a given initial level of debt, it is the average of the condition that matters. Precisely when $\overline{R} < G^* - \overline{\varphi}$, a free lunch success probability arbitrarily close to 1 can be ensured by choosing a suitable Δ.

Figure 8 illustrates the result in Proposition 5 for the U.S. calibration from Section 5. It plots free lunch success probabilities as a function of the initial debt level b_0. For each b_0, we choose a simple fiscal rule $\mathcal{Z}(b_0) = 2\rho\,(b_0 - b)$. We vary the standard deviation of aggregate risk σ from a small value of 0.001 to 0.025, the standard deviation of post-WWII U.S. GDP growth.

The panels vary the size of the one-time deficit Δ. Panel (a) shows success probabilities for a very small value of Δ. As can be seen, success probabilities are essentially a step function: 0% to the left of the threshold $R < G - \varphi$, and 100% to the right. This is a numerical confirmation of Proposition 5. With greater Δ, there is no longer a clean step

Figure 8: Success probabilities of running a free lunch with aggregate risk



Note. The probabilities are computed by simulating 1,000 sample paths for each b_0. Convergence criterion: $|b_t^\Delta - b_t| < 0.01\%$ at any point $t < 10,000$.

function. However, across σ, the success probabilities still line up closely with the vertical dash-dotted blue line, which is the deterministic free lunch threshold for that Δ from Section 3.[25]

This shows that, both theoretically and numerically, the condition $R < G - \varphi$ even applies with aggregate risk.

6.3 Free lunch in the Blanchard [2019] model

We can similarly compute the probability of a free lunch succeeding in the Blanchard [2019] model. We use the recalibrated model that matches the same U.S. calibration targets in 2019 as our baseline model (see Section 5.5 and Appendix C.5 for details). Figure 9 plots the success probabilities of a free lunch policy that raises initial debt by 1% across different levels of initial debt. As before, we keep the path of deficits unchanged.

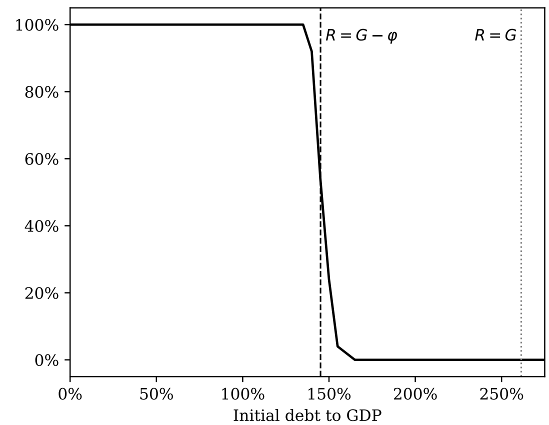
Just like in the previous section, the threshold $R = G - \varphi$ also matters for the viability of a free lunch policy in the Blanchard [2019] model. Here, $R = G - \varphi$ corresponds to the peak of the "risky steady state" locus $z(b)$ shown in Figure 7 (see Appendix C.5 for details).

7 Maturity structure and crowding out of capital

We present two additional insights in this section, building on the basic model in Section 2. First, we study how maturity structure interacts with fiscal space. Second, we extend the

[25] This threshold is simply computed as the value of b_0 for which $b_0 + \Delta$ just converges back to b_0 holding the deficit constant at $z(b_0)$. In other words, b_0 satisfies $z(b_0) = z(b_0 + \Delta)$.

Figure 9: Success probabilities of running a free lunch of 1% of GDP in the Blanchard [2019] model



Note. The probabilities are computed by simulating 50 sample paths for each b_0. Convergence criterion: $|b_t^{\Delta} - b_t| < 0.01\%$ at any point $t < 1,000$.

model in Section 2 to allow for physical capital.

7.1 Maturity structure and fiscal space

We begin by introducing long-term debt into the model. In particular, denote the stock of long-term debt relative to potential GDP by b_t^{LT}. We assume that long-term debt also carriers convenience benefits for savers, albeit less than short-term debt. Thus, we assume a convenience utility of

$$v\left(b_t + \alpha b_t^{LT}\right)$$

where $\alpha \in (0,1)$. This specification implies that the natural interest rate on short-term debt, which we continue to denote by R_t^*, is given by

$$R_t^* = \rho + G^* - v'(b_t + \alpha b_t^{LT})\,(1 - \mu - x)$$

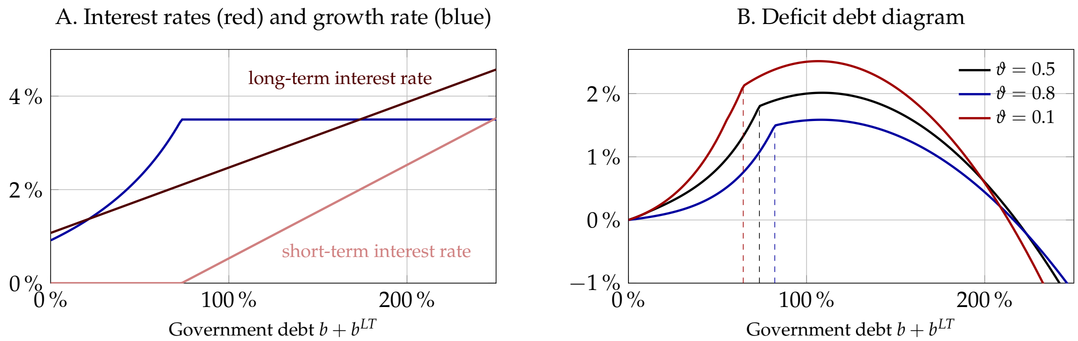
with $R_t = \max\{R_t^*, 0\}$ as before. The interest rate on long-term debt, which we denote by R_t^{LT}, is then

$$R_t^{LT} = R_t + (1-\alpha)\, v'(b_t + \alpha b_t^{LT})\,(1 - \mu - x)\,.$$

In particular, R_t^{LT} is strictly greater than R_t, and the spread between the two shrinks in $b_t + \alpha b_t^{LT}$.

To see how this affects the deficit-debt diagram, we denote the share of LT debt issued

Figure 10: Fiscal space with various shares ϑ of long-term debt



Note. Plot uses $\alpha = 0.7$. Left panel: $\vartheta = 0.5$. Right panel: $\vartheta \in \{0.1, 0.5, 0.8\}$. This figure is only illustrative and uses a slightly lower κ of $\kappa = 0.075$.

by the government by ϑ. The government budget constraint is then

$$\frac{d}{dt}\left(b_t + b_t^{LT}\right) = \left(\overline{R}_t - G_t\right)\left(b_t + b_t^{LT}\right) + z_t$$

where $\overline{R}_t = (1-\vartheta) R_t + \vartheta R_t^{LT}$ and G_t is equal to G^* outside the ZLB and (20) at the ZLB, as before.

Figure 10(A) plots R_t, R_t^{LT}, and G_t as function of total debt $b + b^{LT}$, illustrating the positive spread between R_t and R_t^{LT}, which shrinks at higher debt levels. Figure 10(B) plots the deficit debt locus $z\left(b + b^{LT}\right)$, as function of total debt $b + b^{LT}$, for various shares of long-term debt ϑ. Two observations are noteworthy. First, with greater shares of long-term debt ϑ, there is less fiscal space at small debt levels; the ZLB region is greater; and the boundary of the free lunch region b^* generally shifts to the left. Second, with greater ϑ, there is generally *more* fiscal space at higher debt levels. This is a direct consequence of the fact that long-term debt has smaller convenience benefits, so both interest rates R_t and R_t^{LT} increase less rapidly in long-term debt.

A stylized way to think of large scale purchases of long-term government debt (one type of quantitative easing, QE) is that it changes the maturity composition of government liabilities towards short-term debt, effectively lowering ϑ. As Figure 10(B) shows, this can help an economy escape the ZLB (as in Gertler and Karadi 2018, Caballero and Farhi 2018a, and Cui and Sterk forthcoming), and gives it greater fiscal space at low debt levels. However, it also highlights that QE may reduce fiscal space at higher debt levels.

7.2 Crowding out of capital

Our baseline model does not involve any capital. In this extension, we include capital. To keep things simple, we assume there is no ZLB constraint, as in Section 3. We let (potential) output now be a Cobb-Douglas aggregate of capital k and labor n, which is still equal to $n = 1$ without the ZLB. Thus, $y_t = k_t^\alpha$ after de-trending, letting $\alpha \in [0, 1]$ be the capital share. We let $\delta_k \geq 0$ denote the depreciation rate of capital and assume that government spending is a share x of potential output as y_t may now differ from 1. Following Ball and Mankiw [2021], we allow for an exogenous markup $m \geq 1$; pure profits are earned by savers.

Whether capital is affected by the debt level in our model is not obvious. If capital does not carry a convenience yield, it is entirely unaffected by the debt level.[26] In the literature, capital is often influenced by the debt level as both are treated as substitutable (e.g. in OLG or Bewley models, see appendix C). In our model, this can be captured by including capital in the convenience utility,

$$\max_{\{c_t, b_t\}} \int_0^\infty e^{-\rho t} \left\{ \log c_t + v\left(\frac{b_t + k_t}{y_t} \right) \right\} dt. \tag{35}$$

We now also divide $b_t + k_t$ by (potential) output y_t explicitly. Before, this was unnecessary as potential output was equal to 1. The budget constraint of savers now includes capital and pure profits,

$$c_t + \dot{b}_t + \dot{k}_t \leq (R_t - G_t)\, b_t + \left(r_t^k - \gamma \right) k_t + (1 - \mu)\, w_t n_t - \tau_t + \left(1 - m^{-1} \right) y_t$$

where $r_t^k \equiv \alpha k_t^{\alpha - 1} - \delta_k$ denotes the real net return on capital. By no arbitrage, $R_t = r_t^k + \pi^*$. Two first order conditions jointly pin down the capital stock k and interest rate R^* as a function of debt,[27]

$$R^* = \rho + G^* - (1 - \mu - x)\, v'\left(\frac{b + k}{y} \right) \qquad R^* - G^* = m^{-1} \alpha k^{\alpha - 1} - \delta_k - \gamma. \tag{36}$$

[26] This is for instance the case in the microfoundation proposed in Appendix D.

[27] Observe that even if $R^* < G^*$, we may have dynamic efficiency here, that is, $\alpha k^{\alpha - 1} > \delta_k + \gamma$, for m sufficiently above 1.

Expanding both conditions to first order, we find the sensitivity of capital to debt

$$\frac{d\left(k/y\right)}{d\left(b/y\right)}=-\frac{k/y\frac{\varphi}{r^{k}+\delta_{k}}}{b/y+k/y\left(1+\frac{\varphi}{r^{k}+\delta_{k}}\right)} \tag{37}$$

and the sensitivity $R-G$ to debt

$$\frac{d\left(R-G\right)}{d\log\left(b/y\right)}=\varphi\times\frac{b/y}{b/y+k/y\left(1+\frac{\varphi}{R-\pi+\delta_{k}}\right)}. \tag{38}$$

Equation (37) gives us the extent of "crowding out" of capital. Crowding out happens when (a) there is positive capital, $k/y>0$, requiring that $\alpha>0$; and (b) the interest rate is sensitive to wealth, $\varphi>0$. Equation (38) shows that the sensitivity of $R-G$ to b/y is unambiguously smaller with positive capital. This implies an extended region in which a free lunch is available.

Proposition 6. *Crowding out of capital unambiguously increases the free lunch region. The condition for a free lunch is now given by*

$$R<G-\varphi\times\frac{b/y}{b/y+k/y\left(1+\frac{\varphi}{R-\pi+\delta_{k}}\right)} \tag{39}$$

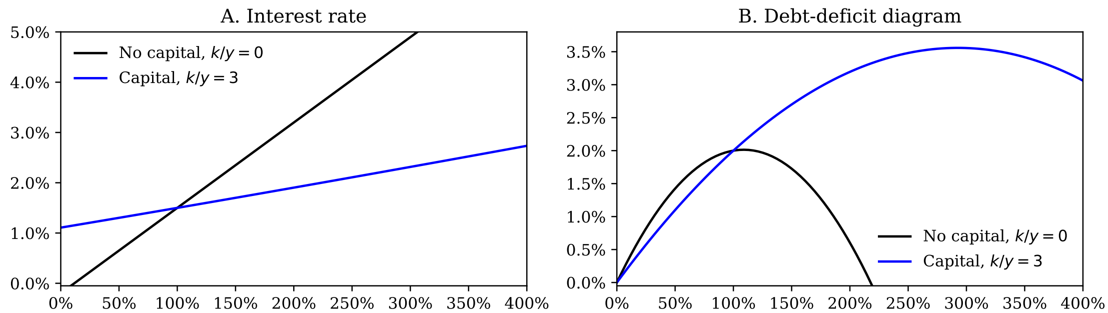
Proof. See Appendix A.3. □

This result may seem surprising at first: Isn't it the case that crowding out of capital increases the marginal product of capital, and hence the interest rate more quickly?

The answer is no. More crowding out due to a higher capital stock k/y, by definition, implies that household wealth $b/y+k/y$ increases *less* quickly with government debt. This, by (36), leads to a weaker interest rate response. This is why k/y reduces the elasticity of $R-G$ to debt in (38). We illustrate this in Figure 11.

Note also that the modified free lunch condition (39) is independent of the markup m. While m matters for welfare, as it determines whether the economy is dynamically inefficient or not when $R<G$, m is irrelevant for whether there exists a free lunch or not (conditional on k/y).

Figure 11: Fiscal space with crowding out of capital



Note. Black line = baseline model without capital. Blue line = calibration of model with capital such that $k/y = 3$ in the initial steady state, with $\delta_k = 0.06$.

8 Conclusion

The textbook view of debt and deficits is that raising deficits lead to an explosive path for government debt unless, at some point, deficits are reduced below their original level. In this paper, we argued that debt may not explode if $R < G - \varphi$ and the increase in deficits is modest ("free lunch"); and that debt may not even rise at all if the economy is at the ZLB and the nominal growth rate is sufficiently responsive to increased deficits. We further illustrated how inequality increases fiscal space outside the ZLB, but may reduce it at the ZLB. For the United States we found very little room for free lunch policies in 2019; but significant room for free lunch policies for Japan.

We have mostly focused on characterizing long-run dynamics in our paper. Our modeling approach, however, is very much amenable to being integrated in richer dynamic models, including models with capital adjustment costs, inertial inflation, and sticky prices. We believe that such models can usefully connect short-run effects of fiscal deficits to the long-run effects we characterize here.

References

Andrew B Abel, N Gregory Mankiw, Lawrence H Summers, and Richard J Zeckhauser. Assessing dynamic efficiency: Theory and evidence. *The Review of Economic Studies*, 56 (1):1–19, 1989.

Mark Aguiar, Manuel Amador, and Cristina Arellano. Micro risks and pareto improving policies. 2021.

S Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 109(3):659–684, 1994.

S Rao Aiyagari and Ellen R McGrattan. The optimum quantity of debt. *Journal of Monetary Economics*, 42(3):447–469, 1998.

George-Marios Angeletos, Fabrice Collard, and Harris Dellas. Public Debt as Private Liquidity: Optimal Policy. Working Paper 22794, National Bureau of Economic Research, November 2020. URL http://www.nber.org/papers/w22794. 00000.

Marina Azzimonti and Pierre Yared. The optimal public and private provision of safe assets. *Journal of Monetary Economics*, 102:126–144, 2019.

Laurence Ball, Douglas W Elmendorf, and N Gregory Mankiw. The deficit gamble. *Journal of Money, Credit and Banking*, pages 699–720, 1998.

Laurence M Ball and N Gregory Mankiw. Market power in neoclassical growth models. Technical report, National Bureau of Economic Research, 2021.

Robert J Barro. Are government bonds net wealth? *Journal of political economy*, 82(6): 1095–1117, 1974.

Robert J Barro. On the determination of the public debt. *Journal of political Economy*, 87(5, Part 1):940–971, 1979.

Robert J Barro. r minus g. Technical report, National Bureau of Economic Research, 2020.

Robert J Barro and Herschel I Grossman. A general disequilibrium model of income and employment. *The American Economic Review*, 61(1):82–93, 1971.

Marco Bassetto and Wei Cui. The fiscal theory of the price level in a world of low interest rates. *Journal of Economic Dynamics and Control*, 89:5–22, 2018.

Marco Bassetto and Thomas J Sargent. Shotgun wedding: Fiscal and monetary policy. *Annual Review of Economics*, 12:659–690, 2020.

Christian Bayer, Benjamin Born, and Ralph Luetticke. The liquidity channel of fiscal policy. 2021.

Jean-Pascal Benassy. *Macroeconomics: an introduction to the non-Walrasian approach*. Academic Press, 1986.

Jess Benhabib, Stephanie Schmitt-Grohé, and Martin Uribe. Monetary policy and multiple equilibria. *American Economic Review*, 91(1):167–186, 2001.

Anmol Bhandari, David Evans, Mikhail Golosov, and Thomas J Sargent. Public debt in economies with heterogeneous agents. *Journal of Monetary Economics*, 91:39–51, 2017.

Francesco Bianchi and Leonardo Melosi. The dire effects of the lack of monetary and fiscal coordination. *Journal of Monetary Economics*, 104:1–22, 2019.

Florin O Bilbiie. Limited asset markets participation, monetary policy and (inverted) aggregate demand logic. *Journal of economic theory*, 140(1):162–196, 2008.

Alberto Bisin. Review of "the deficit myth" by stephanie kelton. *Journal of Economic Literature*, LIII, 2020.

Olivier Blanchard. Public debt and low interest rates. *American Economic Review*, 109(4): 1197–1229, 2019.

Olivier Blanchard and Takeshi Tashiro. Fiscal policy options for japan. Technical report, 2019.

Olivier Blanchard and Philippe Weil. Dynamic efficiency, the riskless rate, and debt ponzi games under uncertainty. *Advances in Macroeconomics*, 1(2), 2001.

Henning Bohn. On testing sustainability of government deficits in a stochastic environment. Technical report, Wharton School-Weiss Center for International Financial Research, 1991.

Henning Bohn. The sustainability of budget deficits in a stochastic economy. *Journal of Money, Credit and Banking*, 27(1):257–271, 1995.

Henning Bohn. The behavior of us public debt and deficits. *the Quarterly Journal of economics*, 113(3):949–963, 1998.

Johannes Brumm, Xiangyu Feng, Laurence J Kotlikoff, and Felix Kubler. Are deficits free? Technical report, Accepted at Journal of Public Economics, 2021a.

Johannes Brumm, Xiangyu Feng, Laurence J Kotlikoff, and Felix Kubler. When interest rates go low, should public debt go high? Technical report, National Bureau of Economic Research, 2021b.

Markus K Brunnermeier, Sebastian A Merkel, and Yuliy Sannikov. Debt as safe asset: Mining the bubble. *Princeton University work in progress*, 2020a.

Markus K Brunnermeier, Sebastian A Merkel, and Yuliy Sannikov. The fiscal theory of price level with a bubble. Technical report, National Bureau of Economic Research, 2020b.

Ricardo Caballero and Emmanuel Farhi. The safety trap. *The Review of Economic Studies*, 85 (1):223–274, 2018a.

Ricardo Caballero and Emmanuel Farhi. The safety trap. *The Review of Economic Studies*, 85 (1):223–274, 2018b.

Ricardo J Caballero, Emmanuel Farhi, and Pierre-Olivier Gourinchas. An equilibrium model of" global imbalances" and low interest rates. *American economic review*, 98(1): 358–93, 2008.

Ricardo J Caballero, Emmanuel Farhi, and Pierre-Olivier Gourinchas. Global imbalances and currency wars at the zlb. 2016.

Ricardo J Caballero, Emmanuel Farhi, and Pierre-Olivier Gourinchas. The safe assets shortage conundrum. *Journal of Economic Perspectives*, 31(3):29–46, 2017.

John Y Campbell and N Gregory Mankiw. Consumption, income, and interest rates: Reinterpreting the time series evidence. *NBER macroeconomics annual*, 4:185–216, 1989.

Matthew Canzoneri, Robert Cumby, and Behzad Diba. Optimal money and debt management: Liquidity provision vs tax smoothing. *Journal of Monetary Economics*, 83:39–53, 2016.

Varadarajan V Chari and Patrick J Kehoe. Optimal fiscal and monetary policy. *Handbook of macroeconomics*, 1:1671–1745, 1999.

John H Cochrane. The fiscal theory of the price level. *Unpublished: Retrieved*, 3(10):2019, 2019.

Nicolas Coeurdacier, Helene Rey, and Pablo Winant. The risky steady state. *American Economic Review*, 101(3):398–401, 2011.

Wei Cui and Vincent Sterk. Quantitative easing with heterogenous agents. *Journal of Monetary Economics*, forthcoming.

Michael R Darby. Some pleasant monetarist arithmetic, 1984.

Peter A Diamond. National debt in a neoclassical growth model. *The American Economic Review*, 55(5):1126–1150, 1965.

David Domeij and Tore Ellingsen. Rational bubbles and public debt policy: A quantitative analysis. *Journal of Monetary Economics*, 96:109–123, 2018.

Eric M Engen and R Glenn Hubbard. Federal government debt and interest rates. *NBER macroeconomics annual*, 19:83–138, 2004.

Emmanuel Farhi and Matteo Maggiori. A model of the international monetary system. *The Quarterly Journal of Economics*, 133(1):295–355, 2018.

Emmanuel Farhi and Jean Tirole. Bubbly liquidity. *The Review of Economic Studies*, 79(2): 678–706, 2012.

Martin Feldstein. Perceived wealth in bonds and social security: A comment. *Journal of Political Economy*, 84(2):331–336, 1976.

Jason Furman and Lawrence Summers. A reconsideration of fiscal policy in the era of low interest rates. *Unpublished manuscript, Harvard University and Peterson Institute for International Economics*, 2020.

Jordi Galí, J David López-Salido, and Javier Vallés. Understanding the effects of government spending on consumption. *Journal of the european economic association*, 5(1):227–270, 2007.

Mark Gertler and Peter Karadi. Qe 1 vs. 2 vs. 3...: A framework for analyzing large-scale asset purchases as a monetary policy tool. *29th issue (January 2013) of the International Journal of Central Banking*, 2018.

Robin Greenwood, Samuel G Hanson, and Jeremy C Stein. A comparative-advantage approach to government debt maturity. *The Journal of Finance*, 70(4):1683–1722, 2015.

Veronica Guerrieri, Guido Lorenzoni, and Matthew Rognlie. Preserving fiscal space. 2021.

Refet S Gürkaynak, Brian P Sack, and Eric T Swanson. Market-based measures of monetary policy expectations. *Journal of Business & Economic Statistics*, 25(2):201–212, 2007.

Jonathon Hazell, Juan Herreno, Emi Nakamura, and Jón Steinsson. The slope of the phillips curve: evidence from us states. Technical report, National Bureau of Economic Research, 2020.

Tomohiro Hirano and Noriyuki Yanagawa. Asset bubbles, endogenous growth, and financial frictions. *The Review of Economic Studies*, 84(1):406–443, 2016.

Zhengyang Jiang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z Xiaolan. The us public debt valuation puzzle. Technical report, National Bureau of Economic Research, 2019.

Zhengyang Jiang, Hanno N Lustig, Stijn Van Nieuwerburgh, and Mindy Z Xiaolan. Bond convenience yields in the eurozone currency union. *Available at SSRN*, 2020.

Narayana Kocherlakota. Bursting bubbles: Consequences and cures. *Unpublished manuscript, Federal Reserve Bank of Minneapolis*, 2009.

Narayana Kocherlakota. Public debt bubbles in heterogeneous agent models with tail risk. *Unpublished manuscript*, 2021.

Ralph SJ Koijen and Motohiro Yogo. Exchange rates and asset prices in a global demand system. Technical report, National Bureau of Economic Research, 2020.

Arvind Krishnamurthy and Annette Vissing-Jorgensen. The aggregate demand for treasury debt. *Journal of Political Economy*, 120(2):233–267, 2012.

Ricardo Lagos and Randall Wright. A Unified Framework for Monetary Theory and Policy Analysis. *Journal of Political Economy*, 113(3):463–484, June 2005. ISSN 0022-3808. doi: 10.1086/429804. URL http://www.jstor.org/stable/10.1086/429804. ArticleType: research-article / Full publication date: June 2005 / Copyright © 2005 The University of Chicago Press.

Thomas Laubach. New evidence on the interest rate effects of budget deficits and debt. *Journal of the European Economic Association*, 7(4):858–885, 2009.

Eric M Leeper. Equilibria under ?active?and ?passive?monetary and fiscal policies. *Journal of monetary Economics*, 27(1):129–147, 1991.

Abba P. Lerner. Functional finance and the federal debt. *Social Research*, 10(1):38–51, 1943. ISSN 0037783X. URL http://www.jstor.org/stable/40981939.

Edmond Malinvaud. The theory of unemployment reconsidered. 1977.

N Gregory Mankiw. The savers-spenders theory of fiscal policy. *American economic review*, 90(2):120–125, 2000.

Alberto Martin and Jaume Ventura. The macroeconomics of rational bubbles: a user's guide. *Annual Review of Economics*, 10:505–539, 2018.

Neil Mehrotra and Dmitriy Sergeyev. Debt sustainability in a low interest rate world. 2020.

Atif Mian, Ludwig Straub, and Amir Sufi. The saving glut of the rich. 2020.

Atif Mian, Ludwig Straub, and Amir Sufi. Indebted demand. *The Quarterly Journal of Economics*, 136(4):2243–2307, 2021.

Pascal Michaillat and Emmanuel Saez. Resolving new keynesian anomalies with wealth in the utility function. *Review of Economics and Statistics*, pages 1–46, 2019.

Jean-Baptiste Michau. Fiscal policy under secular stagnation: An optimal pump priming strategy. Technical report, Center for Research in Economics and Statistics, 2020.

Preston J Miller and Thomas Sargent. A reply to darby. *Quarterly Review*, 8(Spr), 1984.

Lira Mota. The corporate supply of (quasi) safe assets. *Available at SSRN*, 2020.

Stefan Nagel. The liquidity premium of near-money assets. *The Quarterly Journal of Economics*, 131(4):1927–1971, 2016.

Thomas Piketty, Emmanuel Saez, and Gabriel Zucman. Distributional national accounts: methods and estimates for the united states. *The Quarterly Journal of Economics*, 133(2): 553–609, 2018.

Andrea F Presbitero and Ursula Wiriadinata. Public debt and r-g at risk. Technical report, International Monetary Fund, 2020.

Ricardo Reis. The constraint on public debt when r¡ g but g¡ m. 2021.

Paul A Samuelson. An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of political economy*, 66(6):467–482, 1958.

Stephanie Schmitt-Grohé and Martin Uribe. Downward nominal wage rigidity, currency pegs, and involuntary unemployment. *Journal of Political Economy*, 124(5):1466–1514, 2016.

Miguel Sidrauski. Rational choice and patterns of growth in a monetary economy. *The American Economic Review*, 57(2):534–544, 1967.

Christopher Sims. *Optimal fiscal and monetary policy with distorting taxes*. Benjamin H. Griswold III, Class of 1933, Center for Economic Policy Studies, 2019.

Lawrence H Summers. Us economic prospects: Secular stagnation, hysteresis, and the zero lower bound. *Business economics*, 49(2):65–73, 2014.

Sumiko Takaoka. Convenience yield on government bonds and unconventional monetary policy in japanese corporate bond spreads. 2018.

Jean Tirole. Asset bubbles and overlapping generations. *Econometrica: Journal of the Econometric Society*, pages 1499–1528, 1985.

Quentin Vandeweyer. Treasury debt and the pricing of short-term assets. *European Central Bank and Sciences Po*, 2019.

Michael Woodford. Public debt as private liquidity. *The American Economic Review*, 80(2): 382–388, 1990.

Appendix

A Proofs and model details

A.1 Equilibrium multiplicity and uniqueness

In this section, we prove two results: First, that every steady state of the model is indeed the unique equilibrium outcome and globally stable for some fiscal rule $\mathcal{Z}(b)$. And second, without ZLB constraint, we can never have multiplicity with simple constant z policies.

A.1.1 Unique steady state for some fiscal rule

We proceed in two steps. First, assume $\kappa < \hat{\kappa}$, where $\hat{\kappa}$ is defined in Section 4. In that case, the deficit-debt locus is not "backward bending". Fix a steady state b_0 with deficit z_0 and R_0, G_0. Define $r(b) \equiv R(b) - G(b)$. Let $\tilde{r}(b)$ be a continuously differentiable function that lies strictly below $r(b)$ for $b < b_0$ and strictly above $r(b)$ for $b > b_0$, with $\tilde{r}(b_0) = r(b_0)$.

We claim that a fiscal rule defined as $\mathcal{Z}(b) \equiv -\tilde{r}(b)b$ establishes the steady state at b_0 as unique equilibrium. To show this, we need to show that the steady state is locally saddle-path stable. Then, global uniqueness follows the global version of the Picard–Lindelöf theorem.

When is the steady state locally saddle path stable? If the economy is not at the ZLB in the steady state b_0, there is nothing to show, as the behavior of b_t in that case follows from a simple one-dimensional ODE, see (13). If the economy is at the ZLB, there is locally a two-dimensional ODE system that characterizes the joint dynamics of saver consumption and debt, c_t, b_t,

$$\begin{aligned}
\frac{\dot{c}_t}{c_t} &= 0 - (G^* - \kappa(1 - y_t)) - \rho + v'(b_t)c_t \\
y_t &= c_t + x + \mu y_t \\
\dot{b}_t &= \mathcal{Z}(b_t) + (0 - (G^* - \kappa(1 - y_t)))\, b_t
\end{aligned}$$

or simplified in terms of just c_t and b_t,

$$\begin{aligned}
\frac{\dot{c}_t}{c_t} &= \kappa\left(1 - \frac{x + c_t}{1 - \mu}\right) - G^* - \rho + v'(b_t)c_t \\
\dot{b}_t &= \mathcal{Z}(b_t) - \left(G^* - \kappa\left(1 - \frac{x + c_t}{1 - \mu}\right)\right) b_t
\end{aligned}$$

The linearized homogeneous ODEs around the steady state, stated in terms of linear deviations $\left(\hat{c}_t, \hat{b}_t\right)$ are given by

$$\dot{\hat{c}}_t = -\frac{\kappa}{1-\mu}\hat{c}_t + v'(b_0)\hat{c}_t + v''(b_0)c_0\hat{b}_t$$
$$\dot{\hat{b}}_t = \left(\mathcal{Z}'(b_0) - G_0\right)\hat{b}_t - \frac{\kappa}{1-\mu}b_0\hat{c}$$

or stacked in vector form,

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{b}}_t \end{pmatrix} = \begin{pmatrix} -\frac{\kappa}{1-\mu} + v'(b_0) & v''(b_0)c_0 \\ -\frac{\kappa}{1-\mu}b_0 & \mathcal{Z}'(b_0) - G_0 \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{b}_t \end{pmatrix}$$

By construction of $\mathcal{Z}(b)$, we have that

$$\mathcal{Z}'(b_0) = \tilde{r}(b_0) + \tilde{r}'(b_0)b_0 < G(b_0) + G'(b_0)b_0$$

Define the matrix as $\mathbf{A}$. Its characteristic polynomial is given by

$$P(\lambda) \equiv \det\left(\mathbf{A} - \lambda\mathbf{I}\right) =$$
$$\lambda^2 - \lambda\left(\mathcal{Z}'(b_0) - G_0 - \frac{\kappa}{1-\mu} + v'(b_0)\right) - \left(\left(G_0 - \mathcal{Z}'(b_0)\right)\left(v'(b_0) - \frac{\kappa}{1-\mu}\right) + \left(-v''(b_0)\right)c_0\kappa b_0\right)$$

From $\kappa < \hat{\kappa}$ and $b_0 < b^{ZLB}$, it follows that $v'(b_0) > v'(b^{ZLB}) > \frac{\kappa}{1-\mu}$. Moreover, since $\mathcal{Z}'(b_0) < G_0$, $P(0) < 0$. Together with the fact that $P(\lambda)$ has a positive coefficient in front of λ^2, this leads us to conclude that $P(\lambda)$ has exactly one positive and one negative root. Thus, b_0 is locally saddle path stable, giving us local (and global) uniqueness.

The case with $\kappa > \hat{\kappa}$ follows analogously.

A.1.2 Without ZLB, simple deficit rule gives uniqueness

Without a ZLB constraint and with a simple deficit rule, $z_t = z = const$, the dynamics of debt are governed entirely by (13). This is a simple ordinary differential equation (ODE), which admits a unique solution of the right hand side is Lipschitz continuous in b. For any initial debt $b_0 > 0$ and $z > 0$, we can pick a lower bound $\underline{b}$ below which the solution won't go since $\dot{b}_t > 0$ for any $b < b_0$. Above any such $\underline{b}$, the right hand side of (13) is Lipschitz continuous in b, as v' is weakly convex with domain $(\underline{b}, \infty)$ and thus has a bounded derivative. The Picard-Lindelöf theorem then establishes uniqueness.

A.2 Details on the model with aggregate risk

A.2.1 Derivation of $R(b_t)$ in (32) with aggregate risk.

The saver solves (29) subject to (30). Denote by λ_t the costate of B_t. This implies a first order condition of

$$\lambda_t = C_t^{-\nu}$$

and a law of motion of λ_t of

$$\mathbb{E}_t [d\lambda_t] = \lambda_t (\rho + G^* - R_t) dt - y_t^{-\gamma} v'(b_t) dt \tag{40}$$

Denote $c_t \equiv C_t / y_t$. In equilibrium, $c_t = 1 - x - \mu$ is a constant by goods market clearing. Thus,

$$\lambda_t = y_t^{-\nu} c^{-\nu}$$

and so

$$d\lambda_t = \frac{1}{2} (\nu\sigma)^2 \lambda_t dt - \nu\sigma\lambda_t dZ_t \tag{41}$$

Combining (40) with (41) we find

$$\frac{1}{2} (\nu\sigma)^2 \lambda_t = \lambda_t (\rho + G^* - R_t) - y_t^{-\gamma} v'(b_t)$$

which simplifies to (32),

$$R_t = R(b_t) = G^* + \rho - \frac{1}{2}\nu^2\sigma^2 - c^\nu v'(b_t)$$

A.2.2 Derivation of normalized government budget constraint (33).

To derive (33), observe that the usual budget constraint still holds,

$$dB_t = (R(b_t) - G^*) B_t dt + (x y_t - T_t) dt \tag{42}$$

Therefore, the evolution of $b_t \equiv B_t / y_t$ is given by

$$db_t = z_t dt + \left(R(b_t) - G^* + \frac{\sigma^2}{2} \right) b_t dt - b_t \sigma dZ_t$$

where the $\frac{\sigma^2}{2}$ term is the Ito correction coming from the volatility of y_t.

A.2.3 Derivation of sufficiency of (34) for the transversality condition.

Define $h_t \equiv e^{-\rho t} C_t^{-\nu} B_t$. To derive (34), observe that $h_t = e^{-\rho t} C_t^{-\nu} B_t = e^{-\rho t} \lambda_t B_t$. Therefore, building on (41) and (42),

$$dh_t = -\rho h_t dt + \frac{1}{2} (\nu\sigma)^2 h_t dt - \nu\sigma h_t dZ_t + (R(b_t) - G^*) h_t dt + \frac{z_t}{b_t} h_t dt$$

If (34) holds, we can bound h_t above by process $\tilde{h}_t$, which evolves as

$$d\tilde{h}_t = -\epsilon \tilde{h}_t dt - \nu\sigma \tilde{h}_t dZ_t$$

at all times $t > T$ where T is chosen such that (34) holds. $\tilde{h}_t$ is a standard geometric Brownian motion whose expectation $\mathbb{E}_0 \tilde{h}_t$ converges to zero. Thus, $\mathbb{E}_0 h_t \le \mathbb{E}_0 \tilde{h}_t$ must converge to zero as well.

A.2.4 Proof of Proposition 5

The evolution of debt b_t without the increase in debt by Δ is given by

$$db_t = z_t dt + \left(R(b_t) - G^* + \frac{\sigma^2}{2} \right) b_t dt - b_t \sigma dZ_t$$

The evolution of debt b_t^Δ after increasing debt by $\Delta > 0$ at date 0 is

$$db_t^\Delta = z_t dt + \left(R(b_t^\Delta) - G^* + \frac{\sigma^2}{2} \right) b_t^\Delta dt - b_t^\Delta \sigma dZ_t$$

Given the convenience yield is affine-linear, as in (24), the interest rate schedule $R(b)$ has a constant slope $\phi \equiv R'(b)$. In the notation of (24), $\phi = \varphi / b_0$. Here, we use $\varphi(b)$ to denote the local semi-elasticity of R to debt around an arbitrary debt level b, $\varphi(b) = \frac{\partial R(b)}{\partial \log b} = \phi b$. Of course, around b_0, $\varphi(b_0)$ is exactly equal to the φ in (24).

We denote the difference between the two by $\Delta b_t \equiv b_t^\Delta - b_t$. It satisfies the SDE

$$d(\Delta b_t) = \left[R(b_t) - G^* + \varphi(b_t) + \frac{\sigma^2}{2} + \phi \Delta b_t \right] \Delta b_t dt - \Delta b_t \cdot \sigma dZ_t \qquad (43)$$

Our goal is to show that $\lim_{\Delta \to 0} P(\Delta b_t \to 0) = 1$.We do so by first analyzing a simpler process, $\widetilde{\Delta b_t}$, defined by

$$d\left(\widetilde{\Delta b_t}\right) = \left[R(b_t) - G^* + \varphi(b_t) + \frac{\sigma^2}{2}\right]\widetilde{\Delta b_t}dt - \widetilde{\Delta b_t}\cdot\sigma dZ_t \tag{44}$$

with same initial condition $\widetilde{\Delta b_0} = \Delta b_0 = \Delta$.

Characterizing the process $\widetilde{\Delta b_t}$. The SDE for $\log\widetilde{\Delta b_t}$ is given by

$$d\log\widetilde{\Delta b_t} = [R(b_t) - G^* + \varphi(b_t)]\,dt - \sigma dZ_t \tag{45}$$

We can integrate (45),

$$\log\widetilde{\Delta b_T} - \log\Delta = \int_0^T (R(b_t) - G^* + \varphi(b_t))\,dt - \sigma Z_T$$

We note here that $\widetilde{\Delta b_T}$ scales with Δ.

Since b_t follows a stationary Markov process, the strong law of large numbers (Ergodic Theorem) holds,

$$\frac{1}{T}\int_0^T b_t dt \to \int b\mathcal{F}(db) \qquad \text{a.s.}$$

By linearity of R, φ, it follows that

$$\frac{1}{T}\int_0^T (R(b_t) - G^* + \varphi(b_t))\,dt \to \overline{R} - G^* + \overline{\varphi} \qquad \text{a.s.} \tag{46}$$

with $\overline{R}, \overline{\varphi}$ as defined in the text of Proposition 5. Moreover, another application of the strong law of large numbers gives[28]

$$\frac{1}{T}Z_T \to 0 \qquad \text{a.s.} \tag{47}$$

Together, (46) and (47) imply that

$$\frac{1}{T}\left(\log\widetilde{\Delta b_T} - \log\Delta\right) \to \overline{R} - G^* + \overline{\varphi} \qquad \text{a.s.}$$

We now distinguish two cases, depending on the sign of $\overline{R} - G^* + \overline{\varphi}$. Suppose first that $\overline{R} - G^* + \overline{\varphi} < 0$. Pick some $\delta > 0$ such that $\overline{R} - G^* + \overline{\varphi} + \delta < 0$. For almost any sample

[28] See e.g. https://www.stat.berkeley.edu/~pitman/s205s03/lecture15.pdf, Example 15.6 for a proof.

path of $\widetilde{\Delta b_T}$, we can find a time $\underline{T}$, such that for any $T > \underline{T}$,

$$\frac{1}{T}\left(\log \widetilde{\Delta b_T} - \log \Delta\right) \leq \overline{R} - G^* + \overline{\varphi} + \delta$$

This implies

$$\widetilde{\Delta b_T} \leq \Delta \cdot \exp\left\{\left(\overline{R} - G^* + \overline{\varphi} + \delta\right) T\right\}$$

Given $\overline{R} - G^* + \overline{\varphi} + \delta < 0$, this establishes that $\widetilde{\Delta b_T} \to 0$ along almost any sample path, and hence $\widetilde{\Delta b_T} \to 0$ almost surely. In addition, it establishes that $\widetilde{\Delta b_T}$ is integrable along almost any sample path, that is,

$$\int_0^\infty \widetilde{\Delta b_T} dT < \infty \qquad \text{a.s.}$$

Now consider the case $\overline{R} - G^* + \overline{\varphi} > 0$ and chose δ such that $\overline{R} - G^* + \overline{\varphi} - \delta > 0$. Then, for almost any sample path of $\widetilde{\Delta b_T}$, we can find a time $\underline{T}$, such that for any $T > \underline{T}$,

$$\frac{1}{T}\left(\log \widetilde{\Delta b_T} - \log \Delta\right) \geq \overline{R} - G^* + \overline{\varphi} - \delta$$

and therefore

$$\widetilde{\Delta b_T} \geq \Delta \cdot \exp\left\{\left(\overline{R} - G^* + \overline{\varphi} + \delta\right) T\right\}$$

Given $\overline{R} - G^* + \overline{\varphi} - \delta > 0$, this establishes that in this case, $\widetilde{\Delta b_T} \to \infty$ along almost any sample path, and hence $\widetilde{\Delta b_T} \to \infty$ almost surely.

Having investigated the properties of $\widetilde{\Delta b_t}$, we now return to Δb_t.

Characterizing the process Δb_t. Δb_t differs from $\widetilde{\Delta b_t}$ as the former has an additional nonlinear term, $\phi\left(\Delta b_t\right)^2$, in its SDE (43). We therefore clearly have that $\Delta b_t \geq \widetilde{\Delta b_t}$. This already gives us our first result, namely that $\overline{R} - G^* + \overline{\varphi} > 0$ implies almost sure divergence of Δb_t, or in other words, $P(b_t^\Delta \to b_t) = 0$.

Accordingly, we focus on the case $\overline{R} - G^* + \overline{\varphi} < 0$ hereafter. We can formally study the difference between Δb_t and $\widetilde{\Delta b_t}$ by characterizing the SDE of $\widetilde{\Delta b_t}/\Delta b_t$ (which must lie between 0 and 1). After some algebra combining (43) and (44), we find

$$d\left(\frac{\widetilde{\Delta b_t}}{\Delta b_t}\right) = -\phi \widetilde{\Delta b_t} dt$$

This SDE has the solution

$$\frac{\widetilde{\Delta b_T}}{\Delta b_T} - 1 = -\int_0^T \phi \widetilde{\Delta b_t} dt$$

or, equivalently,

$$\Delta b_T = \frac{\widetilde{\Delta b_T}}{1 - \int_0^\infty \phi \widetilde{\Delta b_t} dt}$$

which is well defined for any sample path with $\int_0^T \phi \widetilde{\Delta b_t} dt < 1$. Since we showed above that (a) for $T \to \infty$, $\int_0^T \widetilde{\Delta b_t} dt$ is finite almost surely and (b) $\widetilde{\Delta b_t}$ scales in Δ, it follows that for any Δ for which $\int_0^\infty \phi \widetilde{\Delta b_t} dt < 1$,

$$\lim_{T \to \infty} \Delta b_T = \frac{\lim_{T \to \infty} \widetilde{\Delta b_T}}{1 - \int_0^\infty \phi \widetilde{\Delta b_t} dt} = 0$$

Thus,

$$P\left(\Delta b_T \to 0\right) \geq P\left(\int_0^\infty \phi \widetilde{\Delta b_t} dt < 1\right)$$

but the latter probability approaches 1 as we take $\Delta \to 0$, since $\widetilde{\Delta b_t}$ scales in Δ. Therefore,

$$\lim_{\Delta \to 0} P\left(\Delta b_T \to 0\right) = 1$$

which is what we set out to prove.

A.3 Details on the model with capital

We begin with the household optimization problem (35). The Euler equation for bonds is given by

$$\frac{\dot{c}_t}{c_t} = R_t - G_t - \rho + \frac{c_t}{y_t} v'\left(\frac{b_t + k_t}{y_t}\right) \tag{48}$$

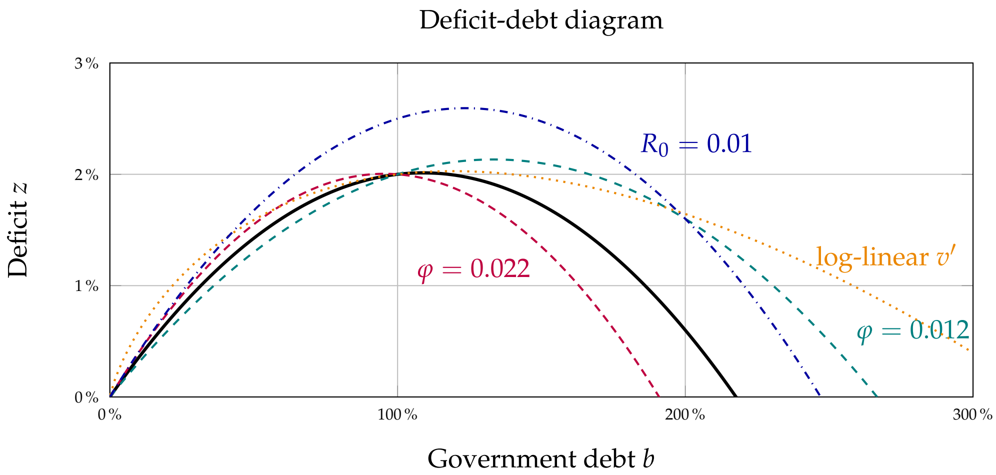
The Euler equation for capital is given by

$$\frac{\dot{c}_t}{c_t} = r_t^k - \gamma - \rho + \frac{c_t}{y_t} v'\left(\frac{b_t + k_t}{y_t}\right) \tag{49}$$

We characterize the steady state. Then, subtracting (48) from (49), we find

$$R^* - G^* = r^k - \gamma = m^{-1} \alpha k^{\alpha - 1} - \delta_k - \gamma = m^{-1} \alpha \frac{y}{k} - \delta_k - \gamma$$

Figure 12: Robustness in deficit-debt diagram



Moreover, in a steady state, (48) looks as usual

$$R^* = \rho + G^* - (1 - \mu - x)\, v' \left(\frac{b + k}{y} \right).$$

This completes our derivation of (36). Linearizing the two equations in (36), we obtain

$$dR = \varphi \left(\frac{b + k}{y} \right) \left(d\frac{b}{y} + d\frac{k}{y} \right) \qquad \text{and} \qquad dR = -\left(k/y\right)^{-1} \frac{dR}{R - G^* + \delta_k + \gamma}$$

Combining these two equations, yields (37), (38), and Proposition 6.

B Robustness of the U.S. deficit-debt diagram

The implied numbers by our calibration in Section 5 should not be taken as gospel. They instead illustrate how our stylized model can be put to work to parse through recent U.S. and Japanese data. We show robustness across alternative calibrations in Figure 12. For simplicity, we do this only for the U.S. economy, and focus on the case without binding ZLB. In particular, we show deficit debt loci for smaller or greater elasticities φ; for the log-linear functional form (25); and for a reduced pre-Covid natural interest rate $R_0 = 1\%$. Shifting G^* and R_0 in parallel (e.g. $G^* = 4\%$, $R_0 = 2\%$) does not affect the deficit-debt schedule above the ZLB. Also, neither ρ nor the level of government spending x, nor the spenders' income share μ affects the deficit-debt schedule conditional on calibrating R_0. Across the alternatives shown in Figure 12, the maximum deficit z^* remains relatively robust around

2-2.5%. Among alternatives with the linear functional form (24), the debt level b^* at which z^* is attained varies in the range 100% to 130%. The debt level at which R crosses G is most uncertain, with estimates varying between just below 200% to just above 300%.

C The deficit-debt diagram in other models

In this section, we derive the interest rate and growth rate schedules $R(b)$ and $G(b)$ in a variety of models, and compute the deficit-debt locus $z(b) \equiv (G(b) - R(b))\, b$.

C.1 Model inspired by Reis [2021]

Here, we sketch a version of the two-agent model in Reis [2021] and use it to derive the corresponding functions for $R(b)$ and $G(b)$.

There are two types of agents, entrepreneurs E and financiers F. Each instant t, an agent i is randomly allocated to be either E or F, with probabilities α and $1-\alpha$ for E and F. Agents solve

$$\max \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log c_t^i dt \right]$$

subject to the budget constraint

$$da_t^i = \left(R_t b_t^i + r_t^l l_t^i + r_t^i k_t^i - \tau_t - c_t^i \right) dt \tag{50}$$

where $a_t^i = b_t^i + l_t^i + k_t^i$ is agent i's total wealth, and subject to the constraint

$$b_t^i \geq 0, k_t^i \geq 0.$$

Here, b_t^i is agent i's holdings of bonds, l_t^i agent i's lending (or if negative, borrowing), and k_t^i agent i's holding of capital. τ_t is a lump-sum tax. Thus, each agent can invest in three different assets each instant: government bonds b_t^i paying rate R_t, loans l_t^i paying rate r_t^l and capital paying rate r^i.

The return on capital r^i crucially differs by type. If i is type E, then r_t^i is constant, and equal to $r_t^i = A - \delta \equiv m > 0$. If i is type F, r_t^i is subject to idiosyncratic investment risk and given by

$$r_t^i = \eta(A - \delta) - \sigma dz_t^i$$

where $\eta \in (0,1)$ captures reduced capital quality in the hands of type F agents. We simplify the model here and set $\eta \to 0$. This essentially assumes that type F agents do not invest in capital.[29]

To avoid too much investment on the side of type E agents, we also impose a borrowing constraint

$$-r_t^l l_t^i \leq \gamma r_t^i k_t^i$$

for some $\gamma > 0$. For type F agents, the borrowing constraint is simply assumed to be $l_t^i \geq 0$. In equilibrium, aggregate bonds outstanding B_t have to equal the sum of all individual positions,

$$B_t = \int b_t^i di$$

and the market for loans has to clear,

$$0 = \int l_t^i di.$$

Our goal is to use this description of the household side to solve for both the steady state interest rate R and the steady state growth rate G as a function of the overall supply of steady state bonds B.

Given the iid type switching, we can split total wealth a_t into wealth held by E's, $a_t^E = \alpha a_t$ and wealth held by F's, $a_t^F = (1-\alpha)\, a_t$. E's always borrow to their maximum. Further, we assume that γ is sufficiently high so that E's do not hold any government bonds. Then, from (50) and the fact that agents always consume $c_t^i = \rho a_t^i$, E's wealth evolves as

$$\dot{a}_t^E = \frac{(1-\gamma) m r_t^l}{r_t^l - \gamma m} a_t^E - \rho a_t^E$$

with positions in capital and lending markets given by

$$a_t^E = k_t^E - \gamma \frac{m k^E}{R}.$$

Given capital k_t^E, output is simply

$$y_t = A k_t^E. \tag{51}$$

F's hold all government bonds, and lend, so that $r_t^l = R_t$. Their wealth then evolves as

$$\dot{a}_t^F = (R_t - \rho)\, a_t^E$$

[29] The case with $\eta > 0$ is similar, it just requires a case distinction.

and is given by

$$a_t^F = \gamma \frac{mk_t^E}{R_t} + B_t. \tag{52}$$

In a steady state, total wealth evolves according to

$$\frac{\dot{a}_t}{a_t} = \alpha \frac{(1-\gamma)mR_t}{R_t - \gamma m} + (1-\alpha)R_t - \rho \tag{53}$$

and is given by

$$a_t = k_t^E + B_t. \tag{54}$$

We denote by $b_t \equiv B_t/y_t$ government debt relative to GDP.

This gives us all the equations we need. Assuming that b is constant, we combine (51), (53) and (54) to find a steady state growth rate G of the economy of

$$G = \alpha \frac{(1-\gamma)mR}{R - \gamma m} + (1-\alpha)R - \rho. \tag{55}$$

The interest rate R is itself determined by the amount of lending in equilibrium, using (52), (54) and the fact that $a_t^F = (1-\alpha)\, a_t$,

$$\gamma \frac{mk^E}{R} + B = (1-\alpha)\left(k^E + B\right).$$

Solving for R we find

$$R(b) = \frac{\gamma m}{1 - \alpha - \alpha A b}. \tag{56}$$

Together with (55), we can solve for G as function of b as well,

$$G(b) = \frac{(1-\gamma)m}{1 + Ab} + (1-\alpha)\frac{\gamma m}{1 - \alpha - \alpha A b} - \rho.$$

We sketch the two schedules in Figure 13 and the implied deficit-debt diagram.[30]

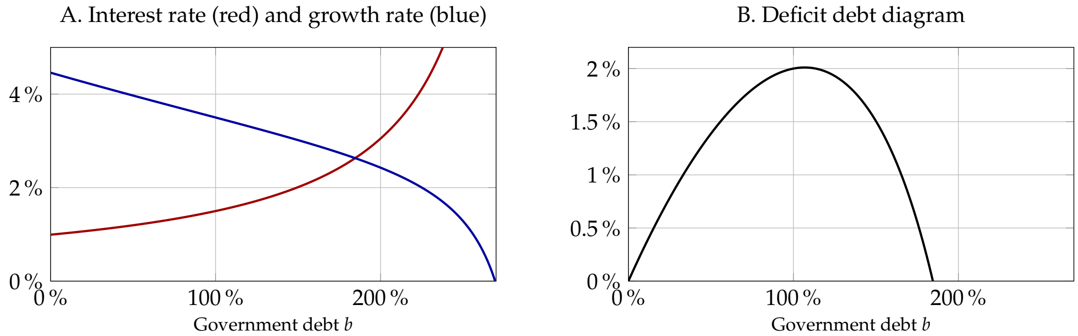
C.2 Model inspired by Brunnermeier et al. [2020a]

Next, we study a model that is inspired by recent models studying $R < G$ with the fiscal theory of the price level (Brunnermeier et al. 2020a,b, Bassetto and Cui [2018], Sims 2019).[31]

[30] We calibrate the model exactly as above, matching $R_0 = 1.5\%$, $G_0 = 3.5\%$, $\varphi = 1.7\%$, $b_0 = 1$. This yields $\delta = 0.04$, $\rho = 0.03$, $\gamma = 0.033$, $A = 0.12$, $\alpha = 0.74$.

[31] For a recent book on the fiscal theory, see Cochrane [2019]. For a classic reference, see Leeper [1991].

Figure 13: $R(b), G(b)$ and deficits in the Reis [2021] model



We do so using a version of our model in Section 2. In particular, we assume away the zero lower bound, assume flexible prices with price level P_t, $\mu = 0$, and nominal bonds B_t (now measured relative to *real* GDP), so that the budget constraint of the government is given by

$$P_t z_t + (R_t - \gamma) B_t \leq \dot{B}_t \tag{57}$$

Since B_t is measured relative to real GDP, it is no longer G_t that is subtracted from R_t, but instead the real growth rate γ. Preferences continue to be those in (1) with the real value of government bonds still denoted by b_t, only that here, b_t is endogenous and given by $b_t = B_t / P_t$.

We follow the literature and assume that B_t grows at an exogenous rate $\mu_B > 0$ and pays an exogenous nominal interest rate R_t. We also assume a fixed level of the primary deficit $z \in \mathbb{R}$. In equilibrium, as we continue to normalize relative to potential, goods market clearing implies $c_t + x = 1$ at all dates. This means that along any transition, the Euler equation still implies a version of (9),

$$R_t = \rho + \gamma + \pi_t - v'(b_t) \cdot (1 - x). \tag{58}$$

Moreover, (57), together with $\dot{B}_t / B_t = \mu_B$, implies that

$$z + (R_t - \gamma - \mu_B) b_t = 0. \tag{59}$$

Finally, for any positive t, the real value of debt b_t changes according to

$$\dot{b}_t = (\mu_B - \pi_t) b_t. \tag{60}$$

We guess and verify that, irrespective of the initial level of nominal government debt

B_0, this economy always exhibits a constant real value of debt b_t. Thus, guessing that $b_t = b = const$, we find that inflation is pinned down by growth in nominal debt due to (60), $\pi_t = \mu_B > 0$. The level of debt is pinned down by (58),

$$v'(b) = \frac{\rho + \gamma + \mu_B - R}{1 - x}$$

and the primary surplus that can be financed follows from (59),

$$z = (\gamma + \mu_B - R)\, b. \tag{61}$$

This illustrates the key differences between our approach and an approach based on the fiscal theory of the price level. In the latter, the price level flexibly adjusts to achieve a given real value of debt, for an exogenously chosen nominal rate R and nominal growth rate $G = \gamma + \mu_B$. This happens because the monetary authority sets a (passive) fixed nominal interest rate R here while the fiscal authority sets an exogenous path for nominal debt. This leads to an expression for the primary deficit (61) that is to be read like the revenue from seignorage: setting $R = 0$ to capture money, we can rewrite this as

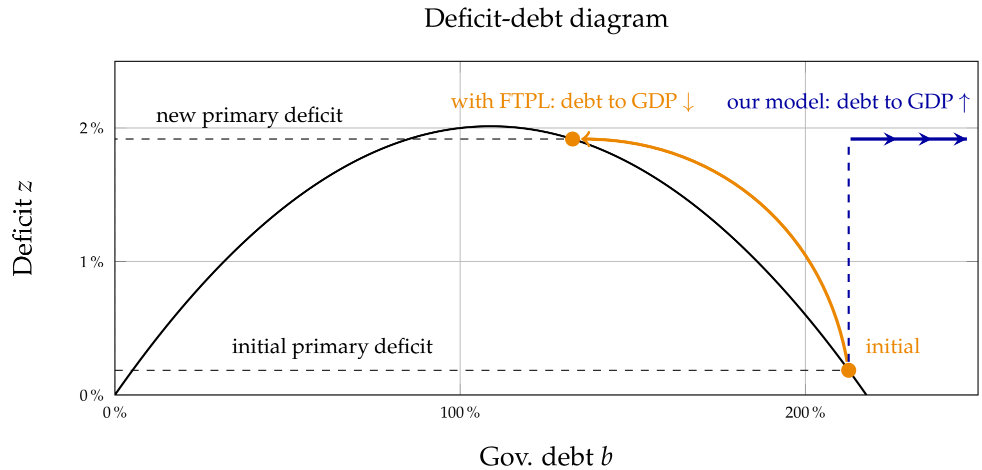
$$z = G \cdot (v')^{-1} \left(\frac{\rho + G}{1 - x} \right) \tag{62}$$

That is, by choosing inflation π, and thus nominal growth G, the fiscal authority can trace out a Laffer curve for seignorage revenue.

By contrast, the monetary authority in our model follows an active Taylor rule to implement the inflation target $\pi_t = \pi^*$, while the fiscal authority chooses primary deficits z_t. Thus, unless the economy is at the ZLB, nominal growth G is *entirely unaffected* by fiscal policy. In that sense, it cannot simply maximize the Laffer curve (62). Instead, debt b_t is a backward looking state variable that is controlled by primary deficits z_t.

The difference is not just semantics. To illustrate, imagine an economy that starts with a given level of debt b_0 that is to the right of the peak, $b_0 > b^*$. What happens when policymakers would like to increase the deficit z? They can do so by raising $\mu_B - R$, that is, by either increasing the rate of nominal debt growth μ_B or reducing the nominal interest rate paid on debt. Crucially, and very differently from the dynamics in Section 3.3, the increase in the deficit z leads to a reduction in levels of debt to GDP. In our model, the same experiment would lead to an increase in debt to GDP. We view this as an important distinction between our model and models based on the fiscal theory of the price level (or

Figure 14: Deficit debt diagram and response to sudden increase in primary deficit with fiscal theory of the price level



models based on seignorage).

C.3 Model inspired by Aiyagari and McGrattan [1998]

For this model, we move to discrete time. Compared to the model in Section 2, however, the main difference is that agents no longer enjoy any convenience utility from holding government debt; instead, they are hit by idiosyncratic income shocks and value government debts for their liquidity, which allows them to partially self insure against the income risk. Recent papers in this vein are Domeij and Ellingsen [2018] and Bayer et al. [2021]. We set $\mu = 0$ for this section.

Specifically, households solve

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} e^{-\rho t} u(c_{it}) \tag{63}$$

subject to the budget constraint

$$c_{it} + b_{it} \leq \frac{1+R_t}{1+G_t} b_{it-1} + (1-\tau_t)\, e_{it} w_t n_t \tag{64}$$

and a borrowing constraint $b_{it} \geq 0$. Here, e_{it} follows a Markov chain with a mean of the stationary distribution of 1. τ_t is a proportional labor income tax. We assume all agents have a labor endowment of 1, as before, and that, at the ZLB, all endowments are equally rationed, and equal to n_t. Inflation is downwardly rigid, just as before, with

$1+\pi_t \geq (1+\pi^*)(1-\kappa(1-y_t))$. We continue to work with the same linear aggregate production function, so that the real wage w_t is still equal to one. Aggregating across households, we find the aggregate demand for bonds

$$b_t \equiv \int b_{it} di.$$

The government budget constraint in discrete time is given by

$$x + \frac{1+R_t}{1+G_t} b_{t-1} \leq b_t + \tau_t w_t n_t$$

where $z_t \equiv x - \tau_t w_t n_t$ continues to denote the primary deficit relative to GDP.

As is well known from Aiyagari [1994], and more recently studied in a two-asset context in Bayer et al. [2021], the household problem above implies a steady state schedule

$$1+\mathcal{R}(b,y)$$

so that if $\frac{1+R_t}{1+G_t} = 1+\mathcal{R}(b,y)$ and $(1-\tau_t) w_t n_t = y$, for all t, in (64), the steady state demand for bonds is equal to b. Observe that $\mathcal{R}(b,y)$ is homogeneous of degree zero. This defines an increasing function for the natural interest rate $R^*(b)$,

$$1+R^*(b) = (1+G^*)(1+\mathcal{R}(b,1-\tau)).$$

What happens when the natural rate is negative, $R^*(b) < 0$? In that case, the ZLB is binding. Output y and tax rate τ are pinned down jointly by

$$1 = (1+G^*)(1-\kappa(1-y))(1+\mathcal{R}(b,(1-\tau)y))$$

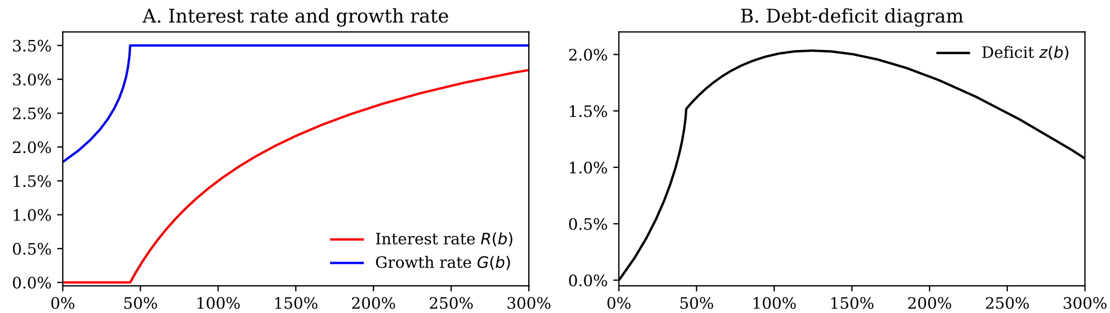
and the government budget constraint

$$\tau y = \mathcal{R}(b,(1-\tau)y)b + x.$$

This then determines $1+G(b) = (1+G^*)(1-\kappa(1-y))$.

To show that this procedure, while more involved, can predicts behavior similar to our more reduced form model in Section 2, we simulate this model with a standard AR(1) process for $\log e_{it}$ (annual persistence 0.90, standard deviation of $\log e_{it}$ of 0.70). We use the risk aversion in u to match the slope $\varphi = \frac{\partial \mathcal{R}(b,1)}{\partial \log b}$ at some initial debt level b_0. The other

Figure 15: $R(b), G(b)$ and deficits in a model based on liquidity



parameters are similar to those in Section 5.3.[32]

The left panel of Figure 15 shows the interest rate $R(b)$ and growth rate $G(b)$ schedules as a function of the debt level. The right panel of Figure 15 shows the deficit-debt diagram $(G(b) - R(b))\, b$. The plots look very similar to those in Section 5.4, and the robustness plots in Appendix B (specifically those with the log-linear functional form for $v'(b)$).

C.4 Model inspired by Diamond [1965]

We sketch the well-known Cobb-Douglas version of the Diamond [1965] model and show that it implies a simple closed-form deficit schedule $z(b)$, and derive the conditions under which there is a free lunch (which in the Diamond [1965] model coincides with the region of dynamic inefficiency).

The model operates in discrete time and consists of two-period-lived overlapping generations. The generation born at date t has G^t members, where $G > 1$. Each maximizes preferences

$$(1-\beta)\log c_{yt} + \beta \log c_{ot+1}$$

over consumption when young c_{yt} and when old c_{ot+1}, subject to the budget constraints

$$c_{yt} + a_t \le w_t\,(1-\tau_t) \qquad c_{ot+1} = R_{t+1} a_t.$$

We have $\beta \in (0,1)$, τ_t is an income tax. The policy function is then

$$a_t = \beta w_t\,(1-\tau_t)\,. \tag{65}$$

[32] We choose ρ, G^* to match the same initial interest rate and growth rate at the same initial debt level b_0, we use the same steady state x and τ. The parameters we find are: $x = 0.14$, $\tau = 0.12$, EIS = 1.37, $\beta = 0.99$, $G^* = 3.5\%$. We choose $\kappa = 0.075$ here.

The per capita saving a_t of generation t finances capital for $t+1$ and bonds maturing in $t+1$. Normalizing the latter two in terms of the population size at $t+1$, we have an asset market clearing condition

$$G^{-1}a_t = k_{t+1} + B_{t+1}. \tag{66}$$

Production in period t is neoclassical with aggregate output per capita

$$y_t = k_t^{\alpha} l_t^{1-\alpha}$$

where l_t is the labor endowment of each member of generation t, which we normalize to 1. Thus, the wage is $w_t = (1-\alpha)\, k_t^{\alpha}$ and, with a depreciation rate of 1, the return is $R_{t+1} = \alpha k_{t+1}^{\alpha-1} + 1 - \delta$. With (65) and (66), the law of motion for capital is then

$$k_{t+1} = G^{-1}\beta\,(1-\alpha)\,(1-\tau)\,k_t^{\alpha} - B_{t+1} \tag{67}$$

The government's budget constraint is simply given by

$$GB_{t+1} = R_t B_t - \tau_t w_t + X_t$$

where X_t denotes government spending per capita.

Next, we focus on steady states, at which all prices and per capita quantities are constant. Moreover, we normalize government debt and spending by output $y = k^{\alpha}$. We denote $b \equiv B/y$ as before and $x = X/y$. Then, (67) becomes

$$k^{1-\alpha} = G^{-1}\beta\,(1-\alpha)\,(1-\tau) - b$$

and we can rearrange it to obtain an expression for the interest rate

$$R(b) = \frac{\alpha G}{\beta\,(1-\alpha)\,(1-\tau) - Gb}.$$

The normalize government budget constraint can be written as usual

$$z(b) = (G - R(b))\,b$$

where we defined the primary deficit relative to GDP as $z(b) \equiv x - \tau\,(1-\alpha)$. Different from our model in Section 2, it turns out that for this analysis, it is somewhat more tractable to fix the tax rate τ and instead vary government spending x if $z(b)$ changes.

We can analyze the deficit schedule $z(b)$ just like before. In particular, we can ask when higher debt levels allow for a greater primary deficit $z(b)$, which in this model is equivalent to dynamic inefficiency. The condition for this is

$$R(b) < G - b \cdot R'(b) \tag{68}$$

where $\varphi = b \cdot R'(b)$. Observe that the standard condition for dynamic inefficiency that is usually taught in this model is $R < G$, or in terms of primitives, $\frac{\alpha}{1-\alpha} < \beta(1-\tau)$. Yet, as (68) highlights this condition is only accurate for levels of government debt around zero, where $\varphi = 0$. When $b > 0$, $\varphi > 0$, and the relevant condition becomes $R < G - \varphi$. In terms of primitives, this corresponds to

$$b < \frac{\beta(1-\alpha)(1-\tau)}{G} - \frac{1}{G}\sqrt{\alpha \cdot \beta(1-\alpha)(1-\tau)} \equiv b^*$$

where b^* is, as before, the deficit-maximizing level of debt. The deficit associated with b^* is given by

$$z^* = \left(\sqrt{\beta(1-\alpha)(1-\tau)} - \sqrt{\alpha}\right)^2.$$

We thus find that OLG models based on Diamond [1965] admit a similar interest rate schedule as the one we derived in Section 3, and the relevant condition for a free lunch (here equivalent to dynamic inefficiency) is given by $R < G - \varphi$, which only in the case without debt reduces to $R < G$.

C.5 The Blanchard model

Here, we compute the deficit-debt locus of the (Cobb-Douglas) model in Blanchard [2019]. The model is a stochastic version of the model in the previous section, which we briefly recap here. The model operates in discrete time and consists of two-period-lived overlapping generations. Each period corresponds to $N = 25$ years. There is no population growth, $G = 0$, so all returns have to be considered detrended. Households solve

$$\max \log c_{y,t} + \beta \frac{1}{1-\gamma} \log \mathbb{E}_t \left[c_{o,t+1}^{1-\gamma}\right]$$

over consumption when young c_{yt} and when old c_{ot+1}, subject to the budget constraints

$$c_{yt} + k_t + b_t \leq w_t(1-\tau_t) \qquad c_{ot+1} = R_{t+1}k_t + R^f_{t+1}b_t.$$

Here, agents can choose between a risk-free bond, paying the risk free rate R^f_{t+1}, and risky capital, paying R_{t+1}. As before, production is Cobb-Douglas per head of generation t

$$y_t = A_t k_{t-1}^{\alpha} l_t^{1-\alpha}$$

where l_t is the labor endowment of each member of generation t, which we normalize to 1. Thus, the wage is $w_t = (1-\alpha) A_t k_{t-1}^{\alpha}$ and the return on capital is $R_{t+1} = \alpha A_{t+1} k_t^{\alpha-1}$. $\log A_t \sim \mathcal{N}(\mu, \sigma^2)$ is iid stochastic technology.

The government's budget constraint is still given by

$$b_t = R^f_t b_{t-1} - \tau_t w_t + x_t.$$

We look for a "risky steady state", characterizing the steady state of the path along which $\log A_t$ continues to realize at its mean μ. For this exercise, we set government spending to zero (as done by Blanchard 2019), $x_t = 0$. For a given (end of period) debt per capita b, the risky steady state is described by the following four equations: The two budget constraints

$$c_y = (1-\alpha) e^{\mu} k^{\alpha} + \left(1 - R^f\right) b \qquad c_o(A) = \alpha A k^{\alpha} + R^f b$$

and two Euler equations

$$\frac{1}{c_y} = \frac{\beta}{1-\gamma} \frac{1}{\mathbb{E}_A\left[c_o(A)^{1-\gamma}\right]} \mathbb{E}_A\left[c_o(A)^{-\gamma} \alpha A k^{\alpha-1}\right]$$

$$\mathbb{E}_A\left[c_o(A)^{-\gamma} \alpha A k^{\alpha-1}\right] = R^f \mathbb{E}_A\left[c_o(A)^{-\gamma}\right]$$

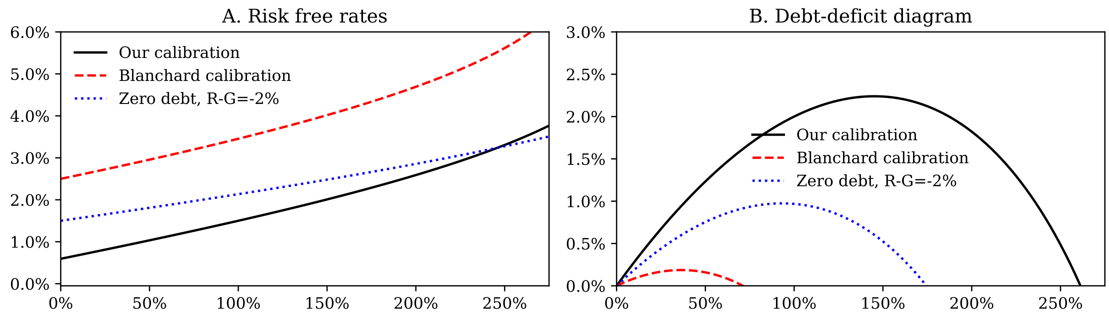
Together, the equations pin down $k, R^f, c_y, c_o(A)$ for any given b. By normalizing by output $y = e^{\mu} k^{\alpha}$ (all normalized variables are denoted with a hat), we can simplify this:

$$\hat{c}_y = 1 - \alpha + \left(1 - R^f\right) \hat{b} \qquad \hat{c}_o(A) = \alpha A e^{-\mu} + R^f \hat{b}$$

The risk free rate $R^f = R^f(\hat{b})$ then solves the risk-free Euler equation

$$\frac{1}{\hat{c}_y} = \beta \frac{1}{\mathbb{E}_A\left[\hat{c}_o(A)^{1-\gamma}\right]} R^f \mathbb{E}_A\left[\hat{c}_o(A)^{-\gamma}\right]$$

Figure 16: Interest rates and deficit-debt schedule in the risky steady state of the Blanchard [2019] model



and the capital output ratio can then be computed from

$$\frac{k}{y} = \frac{\alpha}{R^f} \frac{\mathbb{E}_A \left[\hat{c}_o(A)^{-\gamma} A e^{-\mu} \right]}{\mathbb{E}_A \left[\hat{c}_o(A)^{-\gamma} \right]}$$

The expected return on capital is then equal to

$$\mathbb{E}R = \frac{\alpha}{k/y} \mathbb{E}_A \left[A e^{-\mu} \right]$$

We calibrate the model as in the Cobb-Douglas calibration of Blanchard [2019]. With zero initial debt, we choose $\alpha = 1/3$, a period length of 25 years, $\sigma = 0.2$, and calibrate γ and β to match jointly a riskless rate R^f equal to -1% annual (i.e. one percent below G^*) and an expected return on capital $\mathbb{E}R$ equal to $+2\%$ annual, (i.e. two percent above G^*). This yields $\gamma = 18.7$ and $\beta = 0.31$ (not annualized).

Figure 16(A) shows the annual risk-free rate and the annual expected return on capital as we vary the debt-to-GDP ratio $\hat{b}$. Without loss, we add $G^* = 3.5\%$ to both to ease comparability with our own analysis. Both interest rates increase with debt, with a roughly constant risk premium separating them. Figure 16(B) shows the annualized deficit-debt schedule implied by this calibration, constructed as

$$z(\hat{b}) = \left(1 - \left(R^f(\hat{b}) \right)^{1/N} \right) \cdot N\hat{b}.$$

In Figure 7 we also present two additional calibrations. First, a calibration which reduces both R^f and $\mathbb{E}R$ annually by 1%. This gives $\gamma = 18.8$, $\beta = 0.40$. Second, a calibration that matches those same interest rate targets, but at an initial level of debt of 100% of GDP. This gives $\gamma = 21.7$, $\beta = 0.45$.

Simulating the model. To simulate the Blanchard [2019] model forward, we begin with initial values k_{-1}, b_{-1}, R^f_{-1}, a sequence of deficits z_t, and draw a random sequence of productivity shocks $\{A_t\}$.

At each step t, we compute output as

$$y_t = A_t k_{t-1}^{\alpha}$$

We evolve debt forward with

$$b_t = R^f_{t-1} b_{t-1} + z_t$$

We use this to write consumption of the currently young generation as

$$c_{yt} = (1-\alpha)\, y_t + z_t \qquad\qquad c_{ot+1}(A', R^f_t) = \alpha A' k_t^{\alpha} + R^f_t b_t$$

We solve for the unknown k_t and R^f_t by solving

$$\frac{1}{c_{yt}} = \beta \frac{1}{\mathbb{E}_{A'}\left[c_{ot+1}(A', R^f_t)^{1-\gamma}\right]} \mathbb{E}_{A'}\left[c_{ot+1}(A', R^f_t)^{-\gamma} \alpha A' {k_t}^{\alpha-1}\right]$$

$$\mathbb{E}_{A'}\left[c_{ot+1}(A', R^f_t)^{-\gamma} \alpha A' k'^{\alpha-1}\right] = R^f_t \mathbb{E}_{A'}\left[c_{ot+1}(A', R^f_t)^{-\gamma}\right]$$

To construct Figure 9, we first simulate the economy for a steady state $k_{-1} = k_{ss}, b_{-1} = b_{ss}$ associated with a given initial debt-to-GDP level $\hat{b}_{ss}$. We assume a deficit rule that avoids explosive debt levels,

$$z_t = z_{ss} - 0.08 \cdot N \cdot \left(R^f_{t-1} b_{t-1} - R^f_{ss} b_{ss}\right)$$

Then, for every $\hat{b}_{ss}$, we simulate the same economy again, for the same shocks $\{A_t\}$, except that (a) the initial debt level b_{-1} is now increased by 1% of initial GDP; and (b) the deficit path z_t is unchanged, taken directly from the economy *without* the initial 1% debt shift.

C.6 Model with indebted demand and convenience yield

The model in Section 2 can easily be extended to allow for "indebted demand" as in Mian et al. [2021].[33] To do so, we include a term in savers' preferences that captures their average

[33] We ignore the ZLB in this section.

saving motive, not specific to bonds,

$$\max_{\{c_t,b_t\}} \int_0^\infty e^{-\rho t} \{\log c_t + v(b_t) + \hat{v}(b_t + d_t + h_t)\} dt \tag{69}$$

Here, $\hat{v}(b+d+h)$ is a utility over total wealth, bonds b as well as private debt d. For tractability, we include human capital h in total wealth. We denote the return on assets other than government bonds by $\hat{R}_t$. R_t continuous to denote the return on government debt.

Then, human wealth is equal to

$$\left(\hat{R}_t - G^*\right) h_t = (1-\mu) w_t n_t - \tau_t + \dot{h}_t$$

and the budget constraint can be written as

$$c_t + \dot{b}_t + \dot{d}_t \leq (R_t - G^*) b_t + \left(\hat{R}_t - G^*\right) d_t + (1-\mu) w_t n_t - \tau_t. \tag{70}$$

At the steady state, the first order conditions for bonds and other assets pin down R and $\hat{R}$,

$$\hat{R} = G^* + \frac{\rho}{1 + \hat{v}'(b+d+h)(b+d+h)}$$

$$R = \hat{R} - v'(b)\left(1 - \mu - x + \left(\hat{R} - G^*\right) d\right)$$

The first equation is like the one in Mian et al. [2021]: Increased total wealth of savers means a reduced return on wealth $\hat{R}$. Despite this, interest rates on government debt behave in a more nuanced way. R unambiguously falls when other wealth (e.g. d) increases, since that increases savers incomes and their desired saving. But R can rise when savers' holdings of government debt increase, as it diminishes the convenience yield $\hat{R} - R$ on government debt, in line with the analysis in this paper.

D Microfoundations for the convenience utility

So far, we have taken the convenience yield $v(b)$ as given, deriving implications for fiscal space. We next propose a microfoundation for the convenience yield $v(b)$ that is fully consistent with our previous analysis. This also allows us to study implications of debt with multiple maturities.

Convenience benefits are typically thought of as either liquidity or safety premia. Many microfoundations exist for liquidity (e.g. Lagos and Wright 2005), and some have been shown to reduce to a $v(b)$ function (Angeletos et al. 2020). In this section, instead, we propose a model of safety premia, interpreting bonds as being safe if they are likely to pay out even after a big disaster.[34]

Consider an economy like the one in Section 2, with two changes. First, there is no ad-hoc convenience utility function v. Second, there is a flow probability $\lambda > 0$ with which a disaster occurs. Conditional on the disaster occurring, it reduces potential output y^* from 1, our normalized pre-disaster value, to $\delta \in (0,1)$, with probability $f(\delta)$, where $\int_0^1 f(\delta)d\delta = 1$. The only friction that we assume in this model is that the government can only raise tax revenue τ_t from savers up to some fraction $\overline{\tau} + x$ of output.[35] For simplicity, it cannot tax spenders, $\tilde{\tau} = 0$. If debt service requires greater taxes, we assume that the government defaults. We assume that default entails default costs (in the form of transfers to households, not resource costs) that are sufficiently large so that all bond wealth is lost.[36]

We analyze this model in two steps. First, we focus on the economy after a disaster of size δ happened. Then, we study the economy before the disaster shock, and argue that it is isomorphic to our model in Section 2.[37]

When a disaster of size δ materializes, the interest rate rises to $R = G^* + \rho$, as bonds lose their "specialness". This requires the economy to run a primary surplus of $\rho b/\delta$ relative to GDP. Given the upper bound on taxes of $\overline{\tau} + x$, default occurs when output after the shock δ falls below $\underline{\delta} \equiv \rho b/\overline{\tau}$. We denote by $\tilde{V}_t(b;\delta)$ the utility of an individual saver with bond position b after shock δ realizes.

Before the disaster occurs, savers now maximize utility

$$\rho V_t(b) \equiv \max_c \log c + \lambda \int_0^1 f(\delta) \left(\tilde{V}_t(b;\delta) - V_t(b)\right) d\delta + \dot{V}_t(b) + V_t'(b)\dot{b}_t \tag{71}$$

where $\dot{b}_t$ is given by the budget constraint (2). Combining the first order condition for c and the Envelope theorem for $V_t(b)$, this formulation can be shown to imply a natural rate

[34] We describe in Appendix C a number of alternative models and show that they numerically have similar implications to our reduced-form convenience-yield model of Section 2.

[35] We include the share of government spending x here so that the government can always finance its spending. This is equivalent to a cap on the primary surplus of $\overline{\tau}$.

[36] This is equivalent to the government defaulting on all its debt. The case with partial default is very similar to the analysis below.

[37] This is again similar to the "risky steady state" in Coeurdacier et al. [2011], and to the Poisson shocks in Caballero and Farhi [2018a] and Caballero, Farhi, and Gourinchas [2016].

before the disaster that depends on b and is given by

$$R^*(b) = \rho + G^* - \lambda F\left(\frac{\rho b}{\overline{\tau}}\right) \tag{72}$$

where we defined

$$F(\underline{\delta}) \equiv \int_{\underline{\delta}}^{1} f(\delta)\delta^{-1}d\delta - 1. \tag{73}$$

$F(\underline{\delta})$ determines the insurance value of a bond that pays off whenever the shock is better than $\underline{\delta}$. If $F(\underline{\delta}) < 0$, this implies that the bond, on net, is risky, which will be the case for $\underline{\delta}$ close to 1. If $F(\underline{\delta}) > 0$, which will be the case for $\underline{\delta}$ closer to zero, the bond is, on net, safe. In that case, $\lambda F\left(\frac{\rho b}{\overline{\tau}}\right)$ corresponds to the convenience yield, analogous to $v'(b)(1-x)$ in (9). As before, the convenience yield falls in b.

The definition of F in (73) illustrates exactly the premium for "safety": bonds that pay out in very adverse states of the world, with low δ, carry a higher convenience yield. In the special case where the density is equal to $f(\delta) = 2\delta$ and $F(\underline{\delta}) = 1 - 2\underline{\delta}$, we find that the convenience yield is given by

$$\lambda F\left(\frac{\rho b}{\overline{\tau}}\right) = \lambda - 2\frac{\rho\lambda}{\overline{\tau}}b$$

microfounding our affine-linear specification (24).

E Details on estimation of φ

E.1 Further discussion of estimates from the literature

The estimates of $\frac{\partial(\rho+G-R)}{\partial \log b}$ from Krishnamurthy and Vissing-Jorgensen [2012] reported in Table 1above come from their Table 1, columns 4 and 5. The measure of the spread is the Baa corporate yield minus the Treasury bond yield, which they prefer because "Aaa bonds offer some convenience services of Treasuries and thus the Baa-Treasury spread is more appropriate for capturing the full effect of Treasury supply on the Treasury convenience yield." For the estimates of $b_0\frac{\partial(\rho+G-R)}{\partial b}$, we collected the same data as in Krishnamurthy and Vissing-Jorgensen [2012] and regressed the Baa minus Treasury spread on the level of the debt to GDP ratio. We multiply this coefficient $\frac{\partial(\rho+G-R)}{\partial b}$ (which is -0.027 and -0.048 for the long and short time periods, respectively) by the average level of the debt to GDP ratio b_0 (which is 0.42 and 0.36 for the long and short timer periods, respectively) to get the estimate.

The Greenwood et al. [2015] estimate is from column 1 of Panel B of their Table 1. The measure of the spread is the difference between the actual yield on an 2-week Treasury bill and the 2-week fitted yield, based on the fitted Treasury yield curve in Gürkaynak, Sack, and Swanson [2007]. The derivative is with respect to the amount of Treasury bills outstanding scaled by GDP. The implied estimate of $\frac{\partial(\rho+G-R)}{\partial b}$ is -0.167, which we then multiply by the average Treasury bill to GDP ratio b_0 (which is 0.084) to get the estimate. We use the estimate from Panel B which goes only through 2007 because of the endogeneity issues discussed by Greenwood et al. [2015] surrounding the Great Recession and financial crisis (see the last full paragraph on page 1689 of their article). The Vandeweyer [2019] regression estimate comes from column 2 of Table 4 of his study. The measure of the spread is the 3-month T-bill rate minus the 3-month General Collateral Repo rate, and this is regressed on the ratio of outstanding T-bills to GDP. The implied estimate of $\frac{\partial(\rho+G-R)}{\partial b}$ is -0.040, which we then multiply by the average Treasury bill to GDP ratio b_0 (which is 0.010) to get the estimate. We use column 2 of Table 4, as this regression controls for the Federal Funds rate as suggested by Nagel [2016]. The Vandeweyer [2019] natural experiment involves the 2016 money market reform which led to a large rise in demand for T-bills by money market funds. Money market funds increased their holdings of T-bills by $400 billion, which was about 20% of the stock outstanding. Vandeweyer [2019] uses a model-based counter-factual to show that this shock led to an 18 basis point reduction in yields on government debt, which gives $\frac{\partial(\rho+G-R)}{\partial \log b} = 0.009$. The estimate from Takaoka [2018] comes from Table 4, and the estimate from Jiang et al. [2020] comes from Table 5, panel A, column 2. For the Jiang et al. [2020] estimate of -0.01, we multiply by the average government debt to GDP ratio in their sample to get the final estimate of -0.008.

The estimates of $b_0\frac{\partial(G-R)}{\partial b}$ come from Presbitero and Wiriadinata [2020], Table A3, column 1. The coefficients $\frac{\partial(G-R)}{\partial b}$come from that table (-0.027 for advanced economies, -0.024 for the full sample), and then these are multiplied by the average government debt to GDP ratiob_0 for the respective samples, which are 0.53 and 0.56 for the advanced economies and the full sample, respectively.

E.2 Regressions based on Presbitero and Wiriadinata [2020]

The other estimates from Table 1 come from our own data analysis using a data set constructed exactly as the one used by Presbitero and Wiriadinata [2020]. The associated regression table is Table 3.

Table 3: Results from regressions on Presbitero and Wiriadinata [2020] data

	Left hand side: G - R					
	(1)	(2)	(3)	(4)	(5)	(6)
Log(Gov Debt/GDP)	-0.024***	-0.031***	-0.015**	-0.025**	-0.028**	-0.020**
	(0.006)	(0.005)	(0.004)	(0.007)	(0.006)	(0.003)
Observations	1184	1184	1184	490	490	490
R^2	0.103	0.179	0.553	0.162	0.209	0.698
FE		Country	Country and Year		Country	Country and Year
Sample						

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

Standard errors are heteroskedasticity-robust, clustered by country.

Note. This table presents coefficient estimates of $G - R$ on government debt to GDP ratios. The sample for the first three columns are the 17 advanced economies covered by the JST Macrohistory data base. The sample for columns 4 through 6 is G7 countries (Canada, France, Germany, Italy, Japan, United Kingdom, United States). The time period covered is 1950 to 2019. Please see Presbitero and Wiriadinata [2020] for more details.

E.3 Georgia Senate election

The Georgia Senate election of January 5, 2021 offers a unique opportunity to assess how markets perceive a sudden rise in expected government debt. On the eve of the election, trading at Electionbettingodds.com implied a 50.8% probability of the Republicans controlling the Senate, and a 49.1% probability of the Democrats controlling the Senate. It was widely reported in the press that President-Elect Biden's administration would propose a $1.9 trillion "American Rescue Plan" once the President-Elect took office. Our assumption in the calculation below is that the win by the two Democrats in the Georgia Senate election of January 5, 2021 increased the expected government debt by $2 trillion, which at the time was about 7.4% of total debt outstanding.

Figure 17 shows the effect on the 10 year nominal interest rate, the 10 year TIPS interest rate, and expected inflation. As it shows the victory by the Democrats in the Georgia Senate election led to a 15 basis point immediate reaction which then declined to an 8 basis point reaction after a week. Taken together, these numbers imply that a 3.7% rise in total government debt outstanding relative to prior expectations led to an 8 basis point decline in $G - R$, which gives an estimate of $\frac{\partial(G-R)}{\partial \log b}$ of -0.022. The data for these calculations come from Bloomberg.

Figure 17: The change in real interest rates around the January 5th, 2021 Georgia run-off election.