NBER WORKING PAPER SERIES

INTERMEDIATION VIA CREDIT CHAINS

Zhiguo He
Jian Li

Working Paper 29632
http://www.nber.org/papers/w29632

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
January 2022

We thank Mikhail Chernov, Will Diamond, Jason Donaldson, Andrea Eisfeldt, Vincent Glode, Valentin Haddad, Francis Longstaff, Konstantin Milbradt, Giorgia Piacentino, and seminar participants at UCLA Anderson, University of Notre Dame, and Wharton for helpful comments. Zhiguo He acknowledges financial support from the John E. Jeuck Endowment at the University of Chicago Booth School of Business. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

Intermediation via Credit Chains
Zhiguo He and Jian Li
NBER Working Paper No. 29632
January 2022
JEL No. D85,E44,E51,G21,G23,G33

ABSTRACT

The modern financial system features complicated financial intermediation chains, with each layer performing a certain degree of credit/maturity transformation. We develop a dynamic model in which an entrepreneur borrows from overlapping-generation households via layers of funds, forming a credit chain. Each intermediary fund in the chain faces rollover risks from its lenders, and the optimal debt contracts among layers are time invariant and layer independent. The model delivers new insights regarding the benefits of intermediation via layers: the chain structure insulates interim negative fundamental shocks and protects the underlying cash flows from being discounted heavily during bad times, resulting in a greater borrowing capacity. We show that the equilibrium chain length minimizes the run risk for any given contract and find that restricting credit chain length can improve total welfare once the available funding from households has been endogenized.

Zhiguo He
University of Chicago
Booth School of Business
5807 S. Woodlawn Avenue
Chicago, IL 60637
and NBER
zhiguo.he@chicagobooth.edu

Jian Li
Columbia University
New York, NY 10025
United States
jl5964@columbia.edu

1 Introduction

Since the mid-1980, the nature of financial intermediation has been changed in a dramatic way by the emergence of securitization and secured lending techniques, giving rise to a more market-based financial system. Shadow banking can be viewed as the product of this market-based financial system; to take one of the most salient examples, it is widely acknowledged that maturity and credit transformation in the shadow banking system contributed to the asset price appreciation in U.S. real estate markets prior to the 2007–09 financial crisis.

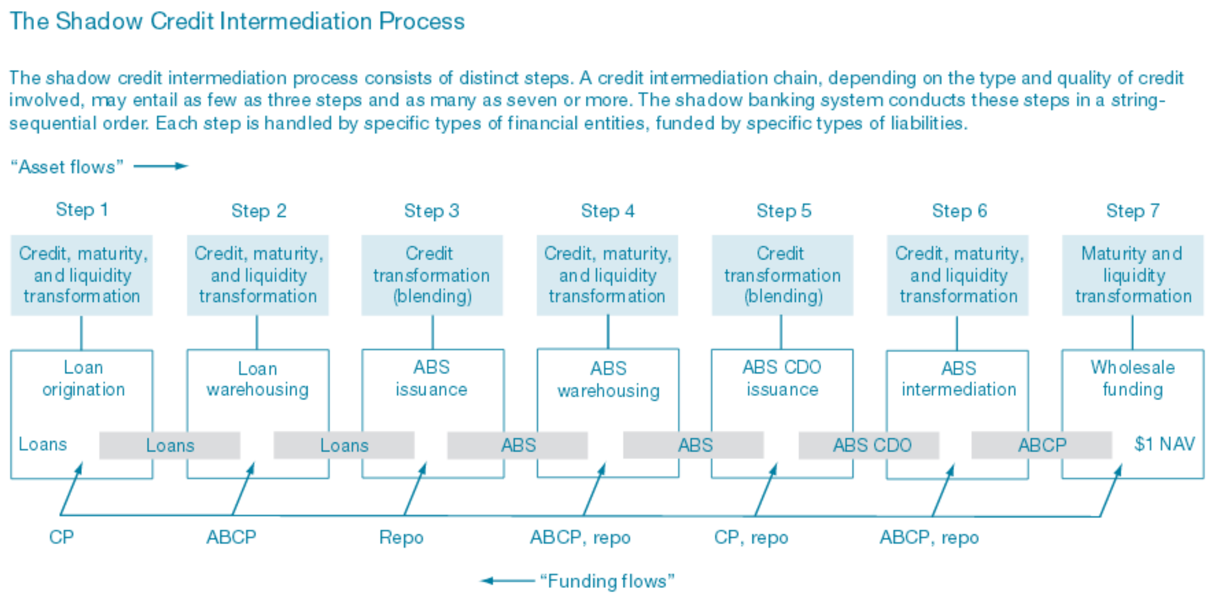
Although the underlying economic mechanism of shadow banking has been well studied by many leading scholars (Adrian and Shin, 2009, 2013; Gennaioli et al., 2013; Duffie, 2019) since the onset of the 2007–09 financial crisis, our paper focuses on one missing piece in the literature on shadow banking. Adrian et al. (2012) explain it vividly:

> Like the traditional banking system, the shadow banking system conducts credit intermediation. However, unlike the traditional banking system, where credit intermediation is performed "under one roof"—that of a bank—in the shadow banking system, it is performed through a daisy-chain of non-bank financial intermediaries in a multi step process. . . . The shadow banking system performs these steps of shadow credit intermediation in a strict, sequential order with each step performed by a specific type of shadow bank and through a specific funding technique. . . . The intermediation chain always starts with origination and ends with wholesale funding, and each shadow bank appears only once in the process.

The thrust of the above description is the concept of a "chain." The common theme in the various shadow banking businesses anatomized by Adrian et al. (2012) is the step-by-step maturity/liquidity and credit transformation, often initiated by loan origination. This is then followed by so-called "loan warehousing," which refers to the act of collecting a significant volume of eligible loans in a special purpose vehicle (SPV), which then issues asset-backed commercial papers (ABCP) to the public, as well as issues loans to the next layer of asset-backed securities (ABS) warehousing. As shown in Figure 1, which we take from Adrian et al. (2012), this process might further involve an ABS collateralized-debt-obligation (CDO), but eventually reaches the wholesale funding markets that are populated by money market investors as well as long-term fixed income investors (say pension funds and insurance companies).

We emphasize that the intermediation credit chain is more general than the stark example of the shadow banking system prior-to the 2007–09 financial crisis. In most modern financial systems, money market mutual funds (MMMFs) issue daily "debt" to households, but hold commercial papers with maturity of one to six months. These commercial papers are issued by banks and other nonbank financial institutions that fund even longer-term and riskier projects. They form the most basic intermediation credit chain. Regulators have increasingly expressed concerns over these nonbank financial intermediaries, which have grown significantly since the global financial crisis (Aramonte et al., 2021).

Figure 1: Illustration of the Credit Intermediation Chain

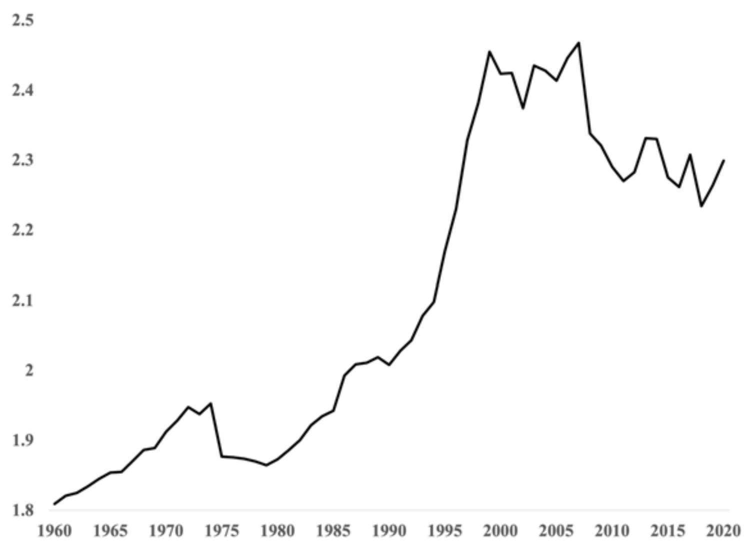


This figure is from Adrian et al. (2012)

Figure 2 plots the credit intermediation index over time, which is the ratio of total liabilities of all sectors in the economy over the total end-user liability. Similar to the "money multiplier" idea, Greenwood and Scharfstein (2013) argue that the credit intermediation index approximates the average credit chain length in the economy, where the total end-user liability is approximated by domestic nonfinancial sector liabilities and the total liabilities of all sectors are measured by the sum of financial and nonfinancial sector liabilities. This ratio grew significantly during the 1990s when structured finance and securitization became popular, consistent with the view in Adrian et al. (2012) mentioned above. It decreased slightly after the global financial crisis, but remains at a high level from a historical perspective. During the last decade, each dollar from investors flows through about 2.2 layers of financial intermediaries on average before reaching the final borrower with potentially wide variation among the types of financing.

Despite the extensive literature on shadow banking and its policy implications, it still remains an open question why market participants rely on *layers of intermediaries* instead of just one (layer of) intermediary to take funding from households and lend it out directly to firms. It is possible that a long credit chain could lure unsophisticated households investors into being the ultimate funding provider; but remember that professional money market funds often invest on behalf of these households. Another often-mentioned explanation is regularity arbitrage; under this view, a long financing chain is intentionally created to obscure certain financial activities conducted by financial institutions. The great body of empirical studies (Acharya et al., 2013; Karolyi and Taboada, 2015; Demyanyk and Loutskina, 2016) on regulatory arbitrage certainly lends support to this view, but it does not explain the rapid growth of the securitization market in the first place

Figure 2: Credit Intermediation Index, 1960–2020



This figure plots the credit intermediation index, following the definition in Greenwood and Scharfstein (2013). It is calculated as the ratio of the total liability of all domestic sectors to the total liability of domestic nonfinancial sectors. Both series are obtained from the Flow of Funds at the annual level.

around the mid-1980's. In fact, there is evidence that securitization is best explained as contracting innovation instead of pure regulatory arbitrage (Calomiris and Mason, 2004).

We shed new light on the economics of credit chains by considering a dynamic model, in which a long-lived entrepreneur borrows from overlapping generations (OLG) of households. The entrepreneur has a time-discount rate $\alpha \in (0, 1)$, and is endowed with a project that matures with certain probability each period and produces cash flows upon maturity. Households, on the other hand, are born with endowments e and live for two periods, but do not receive any discount over their consumption across the two periods.

The relative impatience wedge built into our model implies that the (impatient) entrepreneur would like to pledge out future cash flows and borrows from (patient) households to consume early. However, households are OLG, and their trading in the secondary market needs to be facilitated by financial intermediaries. We hence introduce a third group, "experts"; they are financial intermediaries who can manage funds and facilitate liquidation and trading in the secondary market. All experts also have the same discount rate as the entrepreneur.

The entrepreneur can borrow via layers of funds that are managed by the experts, or directly from OLG households. We assume an exogenous contract maturity rate; each layer of funds optimally designs its debt contract (e.g., debt face value) taking as given other layers' contracts and households' strategies. When contracts mature, the borrower—whether the entrepreneur or an intermediary fund—needs to rollover its debt. Rollover fails when the cash-flow realization falls below an endogenous threshold, in which case the borrower defaults. Creditors liquidate this borrower's assets in the secondary market, where experts serve as buyers who then resell to the next cohort of households. In addition, households pay a dead-weight bankruptcy cost per layer.

Because secondary trading of any long-term securities across cohorts of households involves impatient intermediary experts who demand compensation, the entrepreneur can borrow more from OLG households by using short-term (debt) contracts. This is because short-term contracts minimize the maturity mismatch between the OLG households and the long-term project. Essentially, our model captures the growing appetite for money-like assets in recent decades, as has been well documented in Greenwood et al. (2015) and Carlson et al. (2016).

Interestingly, in our model, a credit chain can increase the entrepreneur's borrowing capacity even further; put differently, a credit chain can supply more money-like securities. Section 2 illustrates the key mechanism, which is new to the literature, by a simple numerical example. When the entrepreneur directly borrow using short-term debt, a negative interim shock forces the entrepreneur's project to be liquidated. In contrast, in the credit chain structure where the entrepreneur borrows (using long-term debt) from a fund who then borrows (using short-term debt) from households, following an interim negative fundamental shock, it is the fund's asset—which is the debt issued by the entrepreneur—that is liquidated. This preserves the subsequent short-term debt claims over the entrepreneur's project, and hence avoids inefficient secondary market trading in the continuation game if future rollover is successful.

As explained above, our model features a stylized trade-off: The impatient entrepreneur would like to pledge out as many cash flows as possible, including future ones; but the associated secondary market liquidation losses will be high. By comparing the borrowing capacities induced by these two cases, our example highlights that the two-layered credit chain structure helps insulate interim negative fundamental shocks and protect the underlying real project (held by the entrepreneur firm) cash flows from heavy discounts in liquidations. In this way, the credit chain structure reduces the tension between maximizing the cash flow pledged out and minimizing liquidation losses, just like what special purpose vehicles (SPVs) achieve in practice.

Section 4 characterizes the equilibrium credit chain in our model. For relatively low funding available e, the equilibrium contracts are shown to be time invariant and layer independent. The time invariant feature is mainly due to the fact that the fundamental is i.i.d., while the layer independence is more subtle. When choosing the optimal contract, each fund trades the proceeds received today against the probability of future rollover failures. Funds closer to households have fewer rollover concerns and would like to borrow more, but are constrained by securities (contracts) they acquire from layers above. Competitive intermediary funds then imply all layers have the same contract in equilibrium. Both contract features (stationarity and layer independence) are important for tractability, which allows us to study the equilibrium chain length.

We show that the equilibrium chain length minimizes households' run threshold. The benefit of borrowing via layers is best illustrated by considering the extreme case without exogenous dead-weight bankruptcy cost; in such a case, the equilibrium chain length is infinity. Households with liquidity needs value short-term debt, but issuing short-term debt against long-term illiquid assets is risky and involves severe liquidation loss when rollover fails. Borrowing via credit chains eases the tension. The intermediate layers preserve subsequent short-term debt claims. So when rollover fails, future cash flows are not discounted as heavily as otherwise. Intermediating via credit chains meets the liquidity needs of the households and simultaneously reduces liquidation losses. This intuition echoes that in Section 2's example, where the credit chain insulates some part of the project's future cash flows from the heavy liquidation discount.

Similar to Samuelson (1958), "money" (debt) in our model serves the important role of storing value and transferring wealth. Along this direction, we endogenize the available funding from households e in Section 5 and investigate whether the decentralized credit chain is excessively long compared to the constrained efficient benchmark. The answer is yes, implying that restricting the chain length can improve total welfare. This is mainly due to the coordination issue between the entrepreneur, who determines the contract but takes the chain length as given, and the funds in the last layer, which determine the chain length but take the contract as given. A shorter credit chain limits rollover risks and, as a result, increases borrowing capacity in every period, which further reduces rollover risks.

Though examining a similar economic phenomenon, our paper differs fundamentally from the literature of asset trading chains. Oftentimes, these papers focus on certain specific market frictions that prevent the asset seller (with a relatively low valuation) from directly selling to the first-best buyer (with the highest valuation); there is, thus, an intermediary who holds the asset temporarily. In this literature, these financial frictions could be either information asymmetry in Glode and Opp (2016), or over-the-counter search frictions in Shen et al. (2021).[1] Our focus is on intermediation credit chains where one agent's liability is another agent's asset, a feature that we often see in the shadow banking system.

Literature Review

Our paper belongs to a recent literature that studies the role and frictions of credit chains, motivated by the growing intermediation chain in the U.S. financial system, particularly in the shadow banking sector (Adrian and Shin, 2010; Adrian et al., 2012). Glode and Opp (2021) focus on strategic debt renegotiation when agents are connected through liabilities in an exogenously given debt chain. They show that the chain structure gives rise to externalities in renegotiation because even though bargaining is bilateral, it affects and depends on renegotiation outcomes in other parts of the chain. In Donaldson and Micheler (2018), credit chains arise when banks rely more on non-resaleable debt, such as repos. The repo borrowing in their framework is similar to borrowing via layers in our setting. In both papers, liquidation losses are smaller in defaults when the borrowing is done via layers; the difference is that, instead of assuming exemption of automatic stay, we start with a common type of frictions and show that having a layer in the middle endogenously results in smaller default losses. Our theory mainly applies to structured investment vehicles and commercial

[1]With a slightly broader interpretation, our model also sheds light on "rehypothecation," i.e., the reuse of collateral in secured financing transactions, which is also called "collateral chains" and is a widespread practice to enhance market functioning between banks and nonbanks (Infante and Saravay, 2020). Because most repo transactions in the U.S. are conducted on an "outright" basis with complete ownership transfer at each leg, rehypothecation in a collateral chain is closer to asset trading chains in our opinion.

paper markets, rather than repos. Di Maggio and Tahbaz-Salehi (2017) study how the distribution of collateral along the credit chain matters for the intermediation capacity and systemic stability. Different from the existing literature, we highlight the asset insulation benefit of intermediating through credit chains.

There is a long literature on the theory of financial intermediation. We focus on the benefit of having multiple layers of intermediaries instead of just one, which is the robust prediction in leading models of financial intermediation. Leland and Pyle (1977) suggest that intermediaries can help resolve information asymmetries between borrowers and lenders, an economic force that is absent in our model. Diamond (1984) shows that financial intermediaries reduce monitoring cost through diversifying projects' idiosyncratic risks; but we do not have idiosyncratic risks in our model and the insulation role of layers in our model is separate from diversification.

Conceptually our paper is closer to Diamond and Rajan (2001). There, an intermediary is necessary—again, a single layer is enough—because it has specific skill in collecting repayments from the firms and can also commit to repaying its creditors by offering demand deposits. Like our paper, Diamond and Rajan (2001) micro-founds the continuation game after asset liquidation, and show that intermediaries increase recovery value if default happens. But inalienable human capital (of entrepreneurs/bankers), which is the backbone of Hart and Moore (1998) and Diamond and Rajan (2001), plays no role in our mechanism; instead, we rely on the households' short-term liquidity needs combined with secondary market trading frictions.

We built upon the literature on bank runs and instability of short-term debt (Diamond and Dybvig, 1983; Calomiris and Kahn, 1991; Goldstein and Pauzner, 2005). We adopt a dynamic debt run setting akin to He and Xiong (2012), but focusing on how runs interact with the endogenous multi-layer structure. The runs between layers in our model capture the repo market and commercial paper runs by institutional investors during the global financial crisis, which has been well studied in the literature (Gorton and Metrick, 2012; Copeland et al., 2014; Krishnamurthy et al., 2014; He and Manela, 2016; Schmidt et al., 2016).

Our work is also related to the network and contagion literature. Allen and Gale (2000) and Elliott et al. (2014) show how financial networks provide diversification and insurance against liquidity shocks, but on the other hand, leads to fragility and cascades of failures. Acemoglu et al. (2015) also demonstrates how small shocks can spread through the network and become systemic risks. Instead of considering general network structures, we focus on a simple form of network, i.e. chains, and endogenize both the contracts among layers as well as the length of the credit chain. Similar to us, Allen et al. (2012) also consider rollover risks of short-term debt in clustered structures. However, we emphasize that having multiple layers can actually reduce the overall rollover risks.

In addition to credit chains, recent literature has also investigated asset trading chains, where an asset is bought and re-sold by a sequence of dealers before it reaches the final buyer. Glode and Opp (2016) show trading via a sequence of moderately informed intermediaries can reduce allocation inefficiency caused by asymmetric information. A sufficient long intermediation chain can also eliminate trading inefficiencies caused by agents with monopoly power screening counterparties (Glode et al., 2019). The literature has also examined the length and price dispersion of intermediation chains in an over-the-counter (OTC) market with search frictions (Atkeson et al., 2015; Hugonnier et al., 2019; Sambalaibat, 2021; Shen et al., 2021). Trading chains arise in equilibrium because of search frictions and/or heterogeneous asset valuation among investors. Our focus is on credit chains where one agent's liability is another agent's asset.

2 An Example: Model Mechanism and Intuition

This section provides a simplified example to illustrate the key intuition of our paper.

2.1 Set-up

Consider a four-date-three-period setting $t = 0, 1, 2, 3$, with timeline given in Figure 3. All agents are risk neutral.

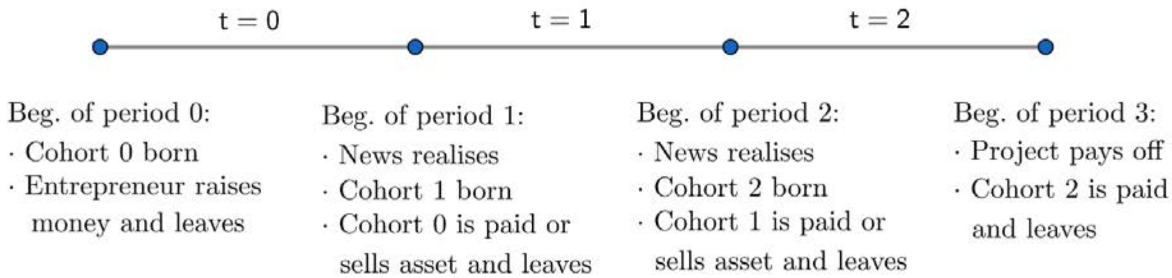
Households. Households are one-period overlapping generations (OLG). Cohort t is born at the beginning of period t, endowed with 1 unit of consumption goods, and has access to a storage technology with zero net return. This cohort can consume $c_t > 0$ or invest in financial market, but leaves the economy at the beginning of period $t+1$ and consume $c_{t+1} > 0$. Households utility is $c_t + c_{t+1}$, so that there is no discount between periods.

Entrepreneur, project, and financial contracts. There is a long-term project that produces cash flows $y \geq 0$ at the end of period $t = 3$. Good news could arrive with probability $p \in (0, 1)$ in period $t = 1, 2$. If good news arrives in either period, then $y = 1$; otherwise, $y = 0$. The arrival of good news is independent across periods.

The project is owned by an entrepreneur who leaves the economy at the end of period 0. Therefore the entrepreneur maximizes the payment of cohort-0 households, by pledging out as much as cash flows to households in different generations. We consider thee financial contracts: three-period debt, two-period debt, and one-period debt. Project only pays cash flow at the end of period 3.

Debt refinance/rollover and secondary market. Toward the end of period t, if the contract (say short-term debt) has matured, then the firm will refinance the debt payment to cohort $t-1$ households from cohort t households. We will call this event "rollover the debt," and throughout the paper we use the word "refinance" and "rollover" interchangeably. If refinance/rollover fails, then the firm has to liquidate its asset at a discount, which is α fraction of next cohort's valuation of the asset. The micro-foundation is that the firm has to sell the asset first to distress experts, who have discount rate α. The discount experts then sell the asset to the cohort t. Hence the proceeds received by the firm is α fraction of cohort t's valuation, which is endogenously determined in equilibrium. If instead the contract has not matured yet, the existing households (the $t-1$ cohort) can sell the securities to a specialized financial intermediary sector, who then sells the securities to

Figure 3: Timing



This figure illustrates the timing of the example in Section 2.

cohort t at the end of period t. The intermediary again has a discount rate of α. Same as in the liquidation case, if cohort t is willing to pay 1 unit for the security, cohort $t-1$ can only receive α units. In other words, liquidation and secondary market trading are treated exactly the same.

Two-layer financing structure with intermediary fund. We departure from the existing literature by studying a two-layer financing structure. Other than issuing debts directly to households, the firm can also adopt a two-layer financing structure where the firm issues short-term debt to an intermediary fund, who then finances itself by issuing its own debt to OLG households. When rollover fails, either at the fund layer or the firm layer, the corresponding creditors liquidates their debt holdings issued by one layer above.

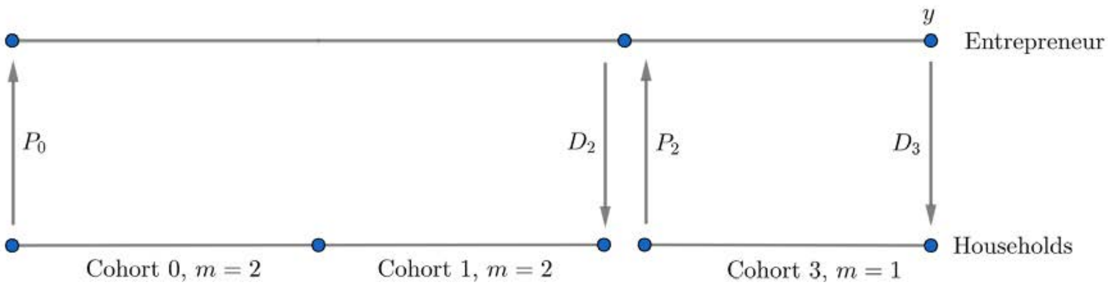
2.2 A Numeric Example

To illustrate the model mechanism, we provide a numeric example with $\alpha = 0.5$ and $p = 0.6$. For simplicity we consider four financing structures: three-period debt, two-period debt, one-period debt, and a two-layer credit chain. The three-period debt case serves as benchmark; it has the longest maturity. Our discussion focuses on why the two-layer financial intermediation can increase the entrepreneur's borrowing capacity, though the comparison between long-term contracts and short-term debt is also useful in delivering the intuition.

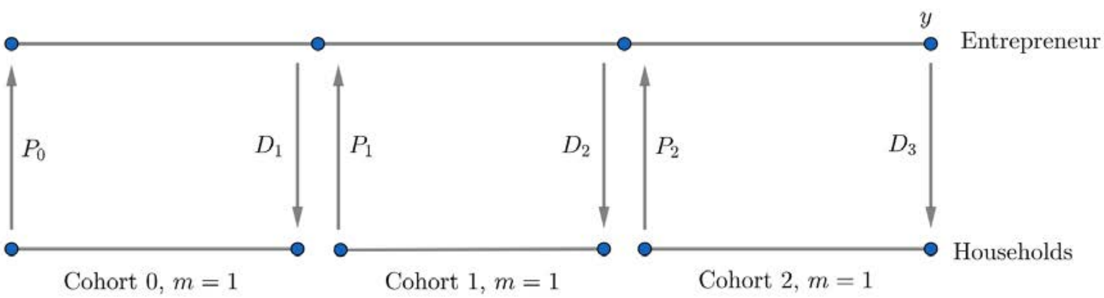
In the following calculation, we take the contract (including face value) as given; Appendix A verifies that they are optimal given the financing structure in each case, thanks to the binary distribution of cash-flows and that entrepreneur maximizes period 0 proceeds received.

Case 0: Long-Term Three-Period Debt The entrepreneur directly issues long-term claims (i.e., three-period debt) against the entire cash-flows to cohort 0, who will then sell it to cohort 2 and 3 later. Since each sale on the market incurs a discount $\alpha = 0.5$, the entrepreneur is able to

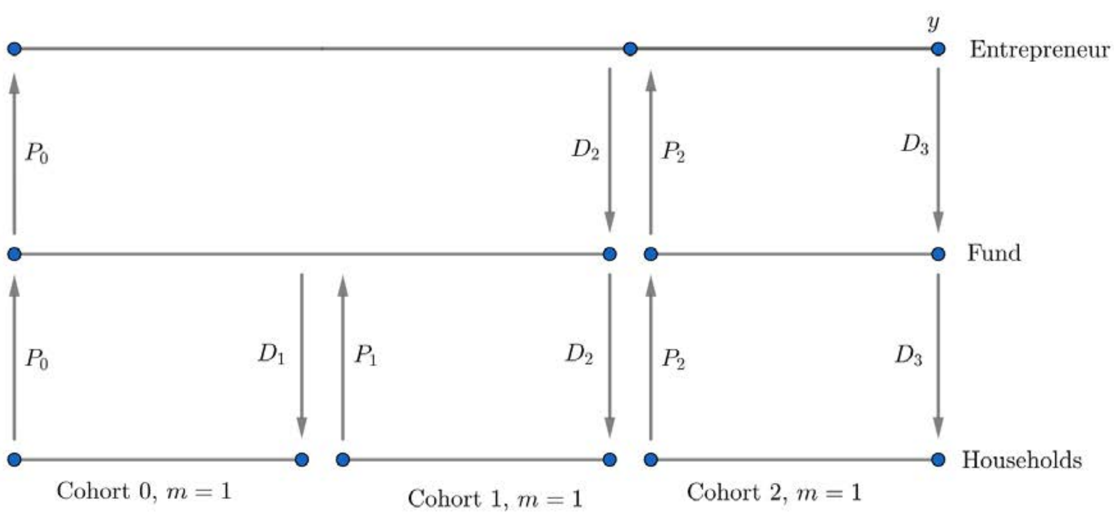
Figure 4: Illustration of the Example



(a) Direct Financing via Two-Period Debt



(b) Direct Financing via One-Period Debt



(c) Two-Layer Credit Chain

This figure illustrates different financing structures in the example of Section 2. Panel (a) illustrates the flow of money in case 1: direct financing using a two-period debt followed by a one-period debt. Panel (b) illustrates the flow of money in case 2: direct financing using only one-period debt. Panel (c) illustrates the flow of money in case 3: financing via a fund. The funding structure between the entrepreneur and the intermediate fund is the same as that in panel (a) and the funding structure between the intermediate fund the households is the same as that in panel (b).

raise (at most)

$$P_0\,(\text{three-period debt}) = 0.6 \times 0.5^2 + 0.4 \times 0.6 \times 0.5 = 0.21 \tag{1}$$

Case 1: Long-Term Two-Period Debt The entrepreneur first issues two-period debt to households that matures in period 2, with face value D_2. The entrepreneur then issues another one-period debt from period 2 to 3, with face value D_3 and issuance proceeds P_2. We are interested in the amount of proceeds P_0 that the entrepreneur can raise at period 0.

At $t = 2$, the entrepreneur can raise $P_2 = 1$ if good news has arrived, otherwise $P_2 = 0$. To maximize the payout to cohort-1 households, the entrepreneur will set the period 2 debt face value to be $D_2 = 1$.[2] This implies that rollover is only successful when good news has been realised, in which cohort-1 creditors receive 1. (There is no discount applied in the good state, which contributes to Case 1's advantage over Case 0.) If no good news has arrived, the entrepreneur is forced into liquidation with liquidation value equals to 0.

In period 1, if good news arrives, then cohort-1 creditors can for sure receive $D_2 = 1$ in the next period; otherwise, cohort-1 can receive $D_2 = 1$ with probability $p = 0.6$ in the next period. Hence cohort-1's valuation for debt is $0.6 \times 1 + 0.4 \times 0.6 \times 1 = 0.84$.

At the end of period 1, cohort 0 can sell the debt contract to cohort 1, with a discount rate α, receiving $0.5 \times 0.84 = 0.42$

$$P_0(\text{two-period debt}) = 0.42 \tag{2}$$

This is larger than the 0.21 that he can raise by issuing three-period debt. Unlike the three-period debt case, not all cash flows here are discounted by α: when rollover is successful in period 2, the debt payment flowing to the cohort 2 involves no discount.

Case 2: Short-Term One-Period Debt The entrepreneur issues one period debt contract with face value D_t that matures in period t. The proceeds from issuing D_t is P_{t-1}. The structure is illustrated in Figure 4b.

The calculation of $t = 2$ is the same as before: we have $P_2 = D_2 = 1$ if good news has arrived. At $t = 1$, P_1 can be calculated as the expected payment at $t = 2$. If good news arrives in period 1, then $P_1 = 1$, otherwise $P_1 = 0.6$. As shown in Appendix A, rollover again is only successful in the good state, in which case without any discount the creditors receive the full face value $D_1 = 1$. In the bad state, the entrepreneur is forced into liquidation, with a liquidation value of α times the expected future cash-flows: $0.5^2 \times 0.6 = 0.15$.

We then calculate the $t = 0$ price of D_1 to be:

$$P_0\,(\text{one-period debt}) = 0.6 \times 1 + 0.4 \times 0.15 = 0.66. \tag{3}$$

By issuing short-term debt, the entrepreneur can (at most) raise 0.66 in period 0. This is larger than the 0.42 raised in the two-period debt case.

[2]This result is due to the binary structure in our example. Setting a significantly lower face value could help avoid liquidation in the bad state, but the payout of this riskless debt is too small.

Case 3: Two-Layer Credit Chain Consider the following two-layer structure with financial intermediaries. The entrepreneur issues a two-period debt to the intermediary from period 0 to period 2, with a face value D_2; the $t = 0$ price is P_0. The intermediary then issues one period debt to the households, with face value D_1 in period 1 and D_2 in period 2. This structure, which combines Case 1 and Case 2, is illustrated in Figure 4c. In this example, the intermediary layer also features a maturity transformation, i.e., the debt contract between the entrepreneur and the fund is longer than the one between the fund and households.

The calculation of $t = 2$ is the same as before: we have $P_2 = D_2 = 1$ if good news has arrived, otherwise $P_2 = 0$. Same as before, $D_1 = 1$, so rollover is only successful if good news arrives in period 1.

When rollover fails in period 1, the fund's asset, which is a debt claim with face value D_2 over the project, is liquidated at the secondary market. (In case 2, it is as if the entire project gets liquidated in this scenario; there we rule out the possibility that the entrepreneur sells its liquidated asset to funds, which we consider in the formal model). The value of that claim is $0.6 \times 1 = 0.6$, hence the proceeds from the liquidation is $0.5 \times 0.6 = 0.3$.

We now calculate P_0, which equals the expected payment to be received in period 1

$$P_0\,(\text{two-layer}) = 0.6 \times 1 + 0.4 \times 0.3 = 0.72. \tag{4}$$

The entrepreneur is able to raise 0.72 via a two-layer structure with an intermediary fund, which is even larger than the proceeds from one-period debt direct financing in case 2.

2.3 Intuition

Three-period contract has the longest maturity, followed by two-period contract in case 1, and the one-period contract in case 2. Compared with long-term contracts (either three-period debt in case 0 or two-period debt in case 1), the benefit of issuing short-term debt in case 2 comes from the fact that successfully rolling over debt avoids transaction cost in the secondary market. In Appendix A, we show the difference between the case 2 and 1 is:

$$P_0\,(\text{one-period}) - P_0\,(\text{two-period}) = \underbrace{p}_{g}(1-\alpha) - \underbrace{(1-p)p}_{bg}\alpha(1-\alpha) \tag{5}$$

The first term captures the fact that if rollover is successful in period 1 (good news arrives), then there is no discount applied to $y = 1$. The second term captures the cost of short-term debt: if rollover fails in period 1 (no good news arrives), then the entrepreneur's asset has to be liquidated, even if good news eventually arrives (with probability p). In this case, the α discount is applied twice. However, in the two-period debt case, if rollover is successful in period 2, then the second discount can be avoided. On net, the benefit of short-term debt is larger than the cost. The benefit of short-debt over equity is even larger, since equity has the longest effective maturity. The mechanism is similar.

The difference between the short-debt case and two-layer case comes from the fact that liquidating the project–which is the entrepreneur's asset–is more costly than liquidating fund's asset.

We show that

$$P_0\,(\text{two-layer}) - P_0\,(\text{one-period debt}) = \underbrace{(1-p)p}_{bg}\,\alpha(1-\alpha). \tag{6}$$

To see the intuition, we know that rollover fails in the bad state of $t = 1$ for both cases. In the short-debt case, the entrepreneur fails to rollover, and the project is liquidated; this implies that y will be discounted twice no matter what happens in the subsequent period $t = 2$. In contrast, in the two-layer case, the intermediary fails to rollover, and it is fund's asset — which is a one-period debt backed by the firm — is liquidated. There, if $t = 2$ is in good state, then entrepreneur can still successfully rollover debt and y is only discounted once. In other words, in the case when period 1 rollover fails, the two-layer structure preserves subsequent short-term debt claims, instead of repeatedly discounting future cash flow. Compared with the previous cases, two-layer financing structure provides the benefit of short-term debt yet avoids the additional liquidation losses in the one-period debt.

We make two remarks before moving to analyze the full dynamic model.

Remark 1 *Our example highlights a key trade-off that is new to the literature. The impatient entrepreneur would like to pledging out as much cash flows as possible at $t = 0$, including future ones; but the associated secondary market liquidation losses will be high. We show that the credit chain structure reduces the tension between maximizing cash flow pledged out and minimizing liquidation losses, because the two-layered credit chain structure, just like special purpose vehicles (SPVs) that we observe in the practice, helps insulate interim negative fundamental shocks and protect the underlying real firms from heavy discounts.*

Remark 2 *Our main dynamic model will feature credit chains with general L layers. If we generalize the numerical example in this section to a project that matures in L periods, the logic behind the benefit of credit chains illustrated in Section 2.3 implies that an $(L-1)$-layer credit chain should be formed, where the layer ℓ holds debt with maturity $L-\ell$ and issue debt with maturity $L-\ell-1$. Though intuitive, this leads to intractability in a dynamic setting. In our main model, instead of deterministic debt maturity as in the example, we assume that each layer's debt contract matures with some random probability, and also matures if above-layers' debts mature. As we will show, this random maturity setup is much more tractable, while generating similar maturity structure and same economic mechanisms as in the example.*[3]

3 The Model

In this section we first present each ingredient on our dynamic model. We then write down the optimization problem for each fund in different layers in the credit chain, before we define the equilibrium formally in this economy.

[3] Suppose that each debt will mature in each period with an exogenous probability λ_d, for all layers. Then the effective maturity of debt at layer-ℓ is $1-(1-\lambda_d)^\ell$, which means layer-ℓ effectively hold debt with maturity $1-(1-\lambda_d)^\ell$ and issue debt with maturity $1-(1-\lambda_d)^{\ell+1}$.

3.1 The Setting

We consider a discrete-time economy, in which there are three types of risk-neutral agents: OLG households, a long lived entrepreneur, and a group of long lived experts.

Endowment, agents, and timing. A long-lived entrepreneur with a discount rate $\alpha \in (0, 1)$ (hereafter he) has a long term project at $t = 0$ that matures with a constant probability λ_y in each of following periods; the project produces nothing before maturity and the game ends. More specifically, within each period $t > 0$, the public "news" on the cash-flow $y_t \geq 0$ arrives at the beginning of the period, where y_t is i.i.d. across periods. The entrepreneur operates the project during the period; if the project matures during the period with probability λ_y, it delivers y_t units of consumption good at the end of period. (We will explain the timing in more detail soon.) Denote by $H(\cdot)$ the cumulative distribution function (CDF) and by $h(\cdot)$ the corresponding probability density function (PDF).

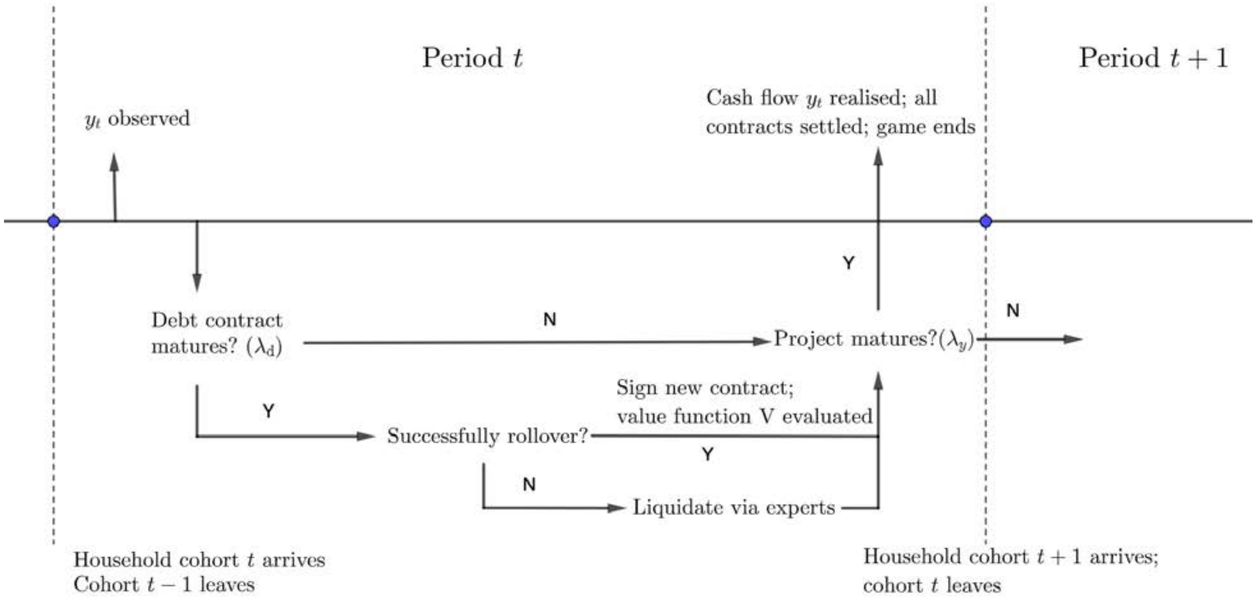
There are OLG households in this economy. Cohort t is born in period t and leaves the economy at the end of period $t+1$. Each cohort consists of a measure 1 of representative households, who are endowed with e units of consumption good when born. They can choose to consume c_t^t in period t or invest in the securities issued by the firm or funds, and consume c_{t+1}^t in period $t + 1$ (and then leave the market). Household's utility is $c_t^t + c_{t+1}^t$. In Section 5 we will consider a richer setting where the endowment e is endogenized.

There is another financial intermediary sector which consists of a group of "experts." In contrast to OLG households, each expert (hereafter she) is long lived, and with a discount rate $\alpha \in (0, 1)$; for simplicity we take the experts' discount rate to be the same as that of the entrepreneur's. In our model, expert can serve different roles in the financial market; they can operate some funds who raise financing from households and in turn provide credit to the firm; or they can run distress funds who purchase liquidated assets in the secondary market. There are many interpretations for the their discount rate α besides their opportunity costs of time; for instance, following He and Krishnamurthy (2012) and He and Krishnamurthy (2013), experts needs to commit certain equity capital to operate the distressed funds, which is costly.

Note that we have set both the entrepreneur and experts to have the same time-discount rate α, while (each) household cohort is more patient with a discount rate 1. This implies that in our model the gain of trade comes from financing from households. Just as illustrated in our simple example in Section 2, the key issue is how to sell the project's cash flows from the hands of relatively impatient entrepreneur to the patient but OLG households.

We now explain the timing of the model. As shown in Figure 5, at the beginning of each period, everyone learns the value of y_t first; then whether debt contracts mature or not. Cohort-t households are then born, and after that, cohort-$t - 1$ households (who receive the debt payment or liquidation value) leave the economy. At the end of each period, whether the project matures or not is realized. We denote the information set at the end of period t by $\mathcal{F}_t$.

Figure 5: Timing



This figure illustrates the timing of events in each period for Section 3.

Debt contracts. Financing contracts in our model are restricted to the class of "debt"-like contracts. More specifically, let T be the contract termination time (either project or debt matures, which is a stopping time measurable to $\mathcal{F}_t$). Denote by π_t a generic debt contract; we assume that it takes the form of $\pi_t = \{\tilde{F}_{y,s}, F_{d,s+1}\}_{s=t}^{T}$, with an exogenously given debt maturity parameter λ_d.[4] More specifically, this contract specifies that the future promised payments from the debtor to the creditor are

$$\tilde{F}_{y,s} \cdot \mathbf{1}_{\text{project matures at period } s, \text{ w.p. } \lambda_y} + F_{d,s+1} \cdot \mathbf{1}_{\text{debt matures at period } s+1, \text{ w.p. } \lambda_d}, \tag{7}$$

where both $\{\tilde{F}_{y,s}\}$ and $\{F_{d,s+1}\}$ are $\mathcal{F}_s$-measurable for any $s \geq t$. Note, the information set $\mathcal{F}_s$ includes the realisation of y_t as well as whether debt from previous period has matured. The time indexes for $\tilde{F}_{y,s}$ and $F_{d,s+1}$ reflect the fact that a new debt contract is signed after the existing debt matures with y's information in hand, but before knowing whether project matures or not; see Figure 5.

We impose limited liability throughout the paper, so that $\tilde{F}_{y,s}(y_s)$ has to be bounded by the project payoff: $\tilde{F}_{y,s}(y_s) \leq y_s$. We assume that $\tilde{F}_{y,s}$ takes the form of a debt contract, i.e. $\tilde{F}_{y,s}(y_s) = \min(F_{y,s}, y_s)$ for some optimally chosen face value $F_{y,s}$. (In the optimal contract, $F_{y,s}$ equals some endogenous constant F_y^*.) If debt matures in period s but y_s is sufficiently low, then

[4] We focus on credit chain length and therefore leave endogenous debt maturity choice to future research. As we explain in Section 4.4, adding layers to the credit chain has certain advantage over maturity shortening. For models with endogenous debt maturity structure, see He and Milbradt (2016) and Hu et al. (2021).

$\tilde{F}_{y,s} \leq y_s$ is constrained to be low and the entrepreneur/fund may not be able to raise enough funding from the market to rollover its debt. In contrast, the "promised" payment at the debt maturity $\{F_{d,s+1}\}$ cannot depend on tomorrow's fundamental y_{s+1}. We will later show that in the optimal contract $F_{d,s+1}$ is constant over time (i.e., $F_{d,s+1} = F_d^*$). For simplicity we focus on debt contracts that are issued at par, so that F_d^* is also the value of debt when rollover is successful.

We emphasize that it is the "debtness" of $\{F_{d,s+1}\}$, not the "debtness" of $\tilde{F}_{y,s}(y_s)$, that drives our result. As we will see, inefficient liquidation caused by rigid debt payment only occurs after a debt contract matures (and when the y_t is sufficiently low), while the game ends without inefficient liquidation after the project matures.[5]

From now on we denote by the contract π_t the sequence of face values $\{F_{y,s}, F_{d,s+1}\}_{s=t}^{T}$. Denote the space of debt contracts by $\Pi \equiv \mathbb{R}_+^{T-t+1} \times \mathbb{R}_+^{T-t}$, so that each period t all funds (and entrepreneur) can choose $\pi_t \in \Pi$ if their previous debt contracts mature. For simplicity, to rule out dilution concerns, we assume that any debt contract is with a covenant so that issuers (the firm or funds) cannot raise new debt before their existing debt matures.

We further allow creditors, after knowing the realization of y_t, to renegotiate by "prepaying" the debt contract. Effectively, in our model creditors have the option of unilaterally triggering the debt to "mature," so that they pay the lender F_d and eliminate all future obligations. Without loss of generality we focus on renegotiation proof contracts; in other words this renegotiation never occurs along the equilibrium path. Shortly we will show that, this renders our model to be "stationary," so that the optimal debt contract chosen at any period along the equilibrium path is independent of history.[6] We therefore will suppress the time t index in the following model description, unless necessary.

Credit chain and prepayment clauses along the chain. The model starts with the entrepreneur who owns the project issues debt at period 0 to household creditors via a credit chain. See Figure 6 for an illustration.

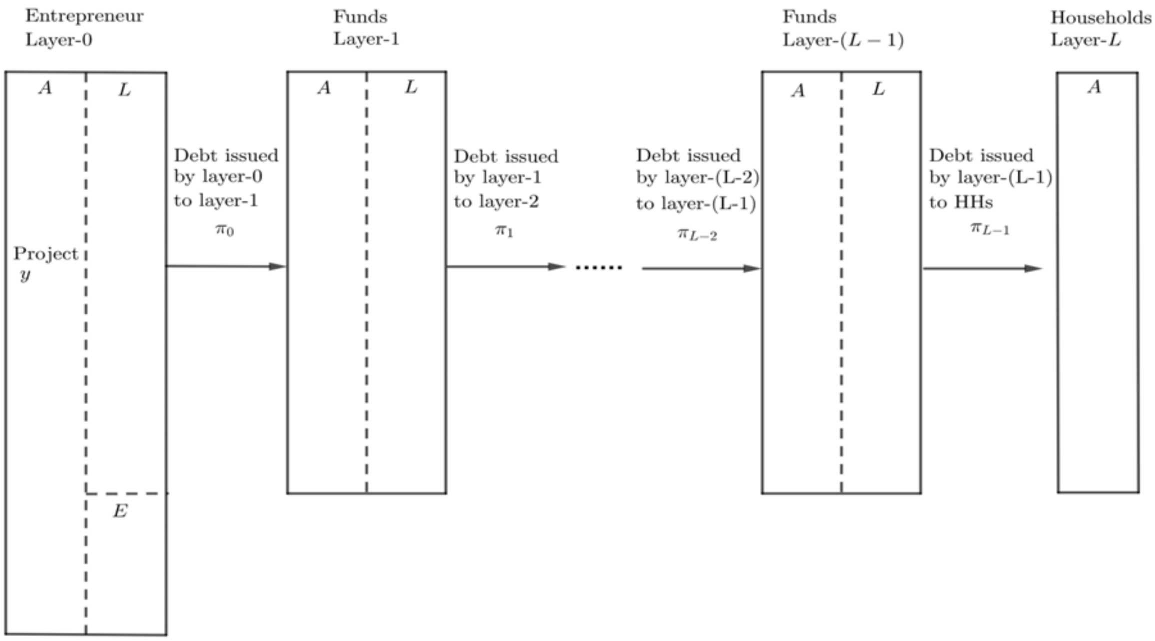
Consider a credit chain with length L, and a fund in the chain is indexed by its position l, where $0 \leq l \leq L$. A fund in layer l borrows from layer $l+1$ using a debt contract $\pi_l = \{F_{y,l}, F_{d,l}\}$. We refer to 0-layer fund of a credit chain as the firm with real project—the ultimate borrower, and L-layer as the households—the ultimate lenders. And, we call funds that sit at layer $i < l$ ($i > l$) to be the upper (lower) layers of fund l. With slight abuse of notation, we use $F_{y,l}$ (or $F_{d,l}$) to denote the payment when the project (or debt) matures at the corresponding period.

The debt contracts in the credit chain needs to have some other "prepayment" clauses if other debt contracts (or the project) in the chain mature. We assume the following. First, when the

[5] We follow the corporate finance literature that inefficient liquidation cost is a fraction of continuation fundamental value, because "experts" who are managing the distressed funds are less patient.

[6] Because of the stationary structure of the fundamental (i.e., y_t's are i.i.d.), the optimal debt contracts would have been stationary if we assume debt contracts to be short-term. Essentially, the prepayment option (of the lenders) is the minimum element to guarantee the stationarity of optimal contracting in our model. It is also worth emphasizing that this prepayment option, which is about the debt itself, differs from "the prepayment clauses" introduced shortly, which are regarding prepayments triggered by events along the credit chain.

Figure 6: Credit Chains



This figure illustrates the structure of the credit chain. Layer-0 is the entrepreneur, holding the project on the asset side and issuing debt contract π_0 to layer-1 funds. Funds in layer-1 hold the debt issued by layer-0 on the asset side, and issue debt contract π_1 to layer-2. The households hold debt contract π_{L-1} issued by the last layer of funds, layer-$(L-1)$.

real project matures, the creditors of layer l get paid by $F_{y,l}$ and the game ends. Due to limited liability, we have

$$F_{y,l} \leq F_{y,l-1} \leq y_t \qquad \text{for} \quad \forall l, \tag{8}$$

and hence this payment trickles down to the households. (In equilibrium $F_{y,l} = F_{y,l-1}$ for $l > 1$.) We can define $F_{y,-1} \equiv y_t$.

Second, when $l+1$'s debt claim issued by l matures, all the debts issued by lower layers $i \geq l+1$ mature, and the payment from $l+1$—whether l makes it full or gets liquidated—will trickle down to the ultimate household creditors who will then leave the economy avoiding the secondary market transaction costs. Our analysis takes this "prepayment" clause as given; however, we conjecture that this will be the outcome of optimal contracting, as it facilitates the payment directly to departing households as soon as possible, avoiding secondary market transaction costs (to be introduced shortly). Finally, it follows from these prepayment clauses that if multiple contracts in different layers mature, only the one with the highest layer (the smallest layer number) matters.

Without loss of generality we focus on the class of issue-at-par debt contracts, i.e., their market values at issuance equal their face value, so that $F_{d,l}$ is also the value of debt issued by layer l. Because the layer-l fund is essentially using its asset holding with a market value of $F_{d,l-1}$ to back its debt issuance with a market value of $F_{d,l}$, and fund managers have no initial wealth, we impose the following condition throughout the paper:

$$F_{d,l} \leq F_{d,l-1} \leq e \text{ for } \forall l. \tag{9}$$

The first part of the condition (9) essentially rules out the "Ponzi" scheme by any fund in which a fund maintains a debt that is underfunded relative to its asset holdings but keeps rolling over this debt from OLG households. A side benefit of this assumption is that it simplifies the prepayment process, as the cash-flows trickle down to the bottom. The second part $F_{d,l} \leq e$ in condition (9) captures the fact that households can only afford to pay e.

To simplify expression, we refer to the scenario that "either the project matures, or any debt contract issued by any fund $i \in \{1, \cdots, l-1\}$ matures" simply as that "layers above l mature."

Credit chain, debt rollover, and secondary market. We have explained the payment flow along the credit chain following a debt maturing event in a fund l. Now consider a borrower fund l who needs to refinance/rollover its debt contract (so that contractual payments can ensue as described above).

Suppose that rollover is successful, i.e., the fund l is able to raise enough money in the market to pay back $F_{d,l}$ to fund $l+1$, which occurs when y exceeds above certain endogenous threshold $\hat{y}$ in equilibrium. (We will show shortly that $\hat{y} = F_y$.) Due to prepayment clauses, all debt between layer l and the households matures. The fund l can use the proceeds raised from new-born households to pay back $F_{d,l}$, so that all funds between layer l and the households are paid in full with the common face value, as well as the departing households. Since the optimal chain length does not change, they can renegotiate and form a new credit chain with the optimal length of L.[7]

Otherwise, when $y < \hat{y}$, rollover fails. Creditors take over and liquidate the asset held by fund l, which could be the real project of the firm, or the debt issued by some intermediary fund $l-1$. The liquidation occurs on the secondary market where the buyers are experts (who run distressed funds), who then sell this asset to the next cohort of households at a price $B_l(y, L)$, where subscript l refers to the layer that fails to rollover.

We assume that with probability $\beta \in [0, 1]$, the chain is restored immediately, in which case the next cohort values the debt at $V_L(L)$. With probability $1 - \beta$, the households need to hold the debt issued by layer-l directly for one period and the chain is restored in the following period absent another run. We essentially need some bankruptcy cost, and a probabilistic delay of chain length restoration is perhaps the simplest way to capture this inefficiency.[8] Layer $l-1$ and the new funds brought in via restoration can (re)design new contracts given to their creditors. We derive

[7] There are many different ways to implement the same outcome, as essentially in this arrangement departing households receive the payment $F_{d,l}$ financed by new-born households. For instance, all funds can simply ask their corresponding lender funds for rollover. In the final layer, the new-born households simply replace departing households. The credit chain stays exactly the same going forward.

[8] For simplicity, we have assumed that restoration (to the optimal credit chain length) occurs for sure after one period. Our mechanism goes through in another stationary setting where restoration occurs with a constant probability β each period.

$B_l(y, L)$ in Section 3.2, and show that the liquidation value $B_l(y, L)$ is higher for a greater l. As we explain shortly, that the liquidation value increases with the chain position l is the key feature that drives the benefit of a longer credit chain in the market solution.

In the case when layer-0 (the entrepreneur) fails to rollover its debt, bankruptcy occurs, but we assume the expert finds the original entrepreneur to continue running the project (so the original chain is restored). The rationale is that the original entrepreneur has the most project-specific human capital and skills.[9] We specify the entrepreneur's exact payoff in Section 3.2.2.

We further assume that there is a restructuring/legal cost $c \geq 0$ for each layer that is experiencing this bankruptcy, which is paid by households. We will study the special case of $c = 0$. To summarize, the direct creditor fund l recovers $\min(\alpha B_l(y, L), F_{d,l})$ from the liquidation of fund l's asset (intermediated by experts), where the liquidation value $B_l(y, L)$ is endogenously determined in equilibrium. This payment then trickles down to the ultimate creditors and departing households hence receive

$$\min(\alpha B_l(y, L), F_{d,l}) - c \cdot (L - l).$$

3.2 Value Functions and Bellman Equation

Denote period t value function of layer l fund by $V_{l,t}(y_t, \pi_{l,t}; \pi_{l-1,t}, L)$, which is evaluated after debt maturity is realised and before the project maturity is realised; see Figure 5. Fund l takes as given the debt contract from its preceding layer $\pi_{l-1,t}$ and the credit chain length L, which will be determined endogenously in equilibrium. From now on we will suppress the time subscript $V_l(y, \pi; \pi_{l-1}, L)$ thanks to stationarity in our model. For the entrepreneur with $l = 0$, we have

$$\pi_{-1} \equiv \emptyset, \text{ and } F_{y,-1} \equiv y;$$

while for households with $l = L$, we have

$$\pi_L \equiv \emptyset, \text{ and } F_{y,l} \equiv 0.$$

Throughout the paper, subscripts indicate positions in the chain. Denote the market price of the debt issued by layer-l under contract $\pi_{l,t}$ by $P_l(\pi_l, y; \pi_{l-1}, L)$. It is a function of the contract set by layer-l (π_l) and the project fundamental (y), taking as given the total chain length (L) and the contract from the layer above (π_{l-1}). We may write $P_l(\pi_l, y; \pi_{l-1}, L)$ simply as $P_l(y)$ whenever there is no risk of confusion.

3.2.1 Fund managers

Layer-l's ($0 < l < L$) payoff in period 0 is then

$$P_l(\pi_l, y; \pi_{l-1}, L) - P_{l-1}(y) + V_l(y, \pi_l; \pi_{l-1}, L). \tag{10}$$

[9] This assumption ensures that the private loss in a bankruptcy is the same as the social loss.

Here, layer-l issues its debt π_l for a proceed of P_l, and then purchases the debt from layer-$(l-1)$ at a price of P_{l-1}, where P_l and P_{l-1} are the market prices of the underlying debt. The last term captures its continuation payoff.

In subsequent periods, if the debt issued by layer-l ($l < L$) matures, then layer-l needs to refinance the debt. If $P_l - F_{d,l} \geq 0$, rollover is successful, and layer-l's value is

$$P_l(\pi_l, y; \pi_{l-1}, L) - F_{d,l} + V_l(y, \pi_l; \pi_{l-1}, L). \tag{11}$$

If rollover fails ($P_l < F_{d,l}$), the layer-l fund asset gets liquidated and the manager recovers nothing.

We can write $V(y, \pi_l; \pi_{l-1}, l, L)$ for $0 < l < L$ recursively as (where we have followed the convention to use prime to indicate variables in the next period),

$$V_l(y, \pi_l; \pi_{l-1}, L) = \lambda_y \underbrace{(\tilde{F}_{y,l-1} - \tilde{F}_{y,l})}_{\text{Project matures}} \tag{12}$$

$$+ (1-\lambda_y)\alpha \Bigg\{ (1-\lambda_d)^{l+1}\mathbb{E}\big[\underbrace{V_l(y', \pi_l; \pi_{l-1}, L)}_{\text{Neither debt issued by nor held by layer } l \text{ matures}} \big] \tag{13}$$

$$+ \sum_{i=0}^{l-1}(1-\lambda_d)^i \lambda_d \mathbb{E}\big[\underbrace{\mathbf{1}^i_{rollover}(-F_{d,l} + F_{d,l-1} - P'_{l-1} + \max_{\pi'_l}(P'_l + V_l(y', \pi'_l; \pi'_{l-1}, L)))}_{\text{Debt held by layer } l \text{ matures}} \big] \tag{14}$$

$$+ (1-\lambda_d)^l \lambda_d \mathbb{E}\big[\underbrace{\mathbf{1}^l_{rollover}(-F_{d,l} + \max_{\pi'_l}(P'_l + V_l(y', \pi'_l; \pi_{l-1}, L)))}_{\text{Debt held by layer } l \text{ does not mature but debt issued by layer } l \text{ matures}} \big] \Bigg\} \tag{15}$$

where we denote $\mathbf{1}^i_{rollover} = 1$ if and only if layer-i successfully rolls-over its debt. The first part captures the payoff to layer-l when the project matures with probability λ_y; otherwise with probability $1 - \lambda_y$, we have the next three terms in the curly brackets.

The first term (13) in the curly bracket captures the continuation value of layer-l when neither its asset side nor liability side matures, which occurs with probability $(1-\lambda_d)^{l+1}$. Here the fund manager as an expert discount her future by α, and y' is the next period project cash flow realization.

The second term (14) in the curly bracket captures the payoff if layer-l's asset side matures; this happens whenever debt issued by any layer-i ($i < l$) matures. In this case, if rollover is not successful, layer-l's payoff is simply 0. When rollover is successful, the layer-l receives $F_{d,l-1}$ from its debtors, and pays $F_{d,l}$ to its creditors. In the refinancing stage, it receives P'_l from its new creditors and gives P'_{l-1} to its debtors. Going forward, layer l's valuation is $V(y', \pi'_l; \pi'_{l-1}, l, L)$, where π'_l is the new contract issued by layer-l and π'_{l-1} is a new contract given to layer-l. We highlight that fund l is optimally choosing a new contract π'_l to maximize the sum of new debt proceeds and its continuation payoff $P'_l + V_l(y', \pi'_l; \pi_{l-1}, L)$.

Finally, the last term in (15), which occurs with probability $(1-\lambda_d)^l \lambda_d$, considers the expected payoff to layer-l if debt issued by layer-l matures but layer-l's asset has not matured yet. In this case, if rollover is successful, layer-l raises P_l', pays off $F_{d,l}$ to existing creditors and chooses a new contract π_l'. Otherwise layer-l's payoff is 0.

3.2.2 Entrepreneur

Recall that entrepreneur is labeled as layer 0. Just like fund managers, the entrepreneur's value is given by:

$$V_0(y,\pi_0;L) = \underbrace{\lambda_y(y-\tilde{F}_{y,l})}_{\text{Project matures}} + (1-\lambda_y)\alpha\Bigg\{(1-\lambda_d)\mathbb{E}\Big[\underbrace{V_0(y',\pi_0;L)}_{\text{Debt issued by layer-0 does not mature}}\Big] \tag{16}$$

$$+\lambda_d\mathbb{E}\Big[\underbrace{\mathbf{1}^0_{rollover}(-F_{d,0}+\max_{\pi_0'}(P_0'+V_0(y',\pi_0';L)))}_{\text{Debt issued by layer-0 matures and rollover succeeds}}+ \tag{17}$$

$$\underbrace{(1-\mathbf{1}^0_{rollover})[(\beta+(1-\beta)(1-\lambda_y)\alpha)(-P_{-1}'+\max_{\pi_0'}(P_0'+V_0(y',\pi_0';L)))]\Big]}_{\text{Debt issued by layer-0 matures and rollover fails}}\Bigg\}. \tag{18}$$

Similar to the value function of fund managers, the second term of Eq. (16) captures the continuation value when debt does not mature, and (17) captures the value when debt matures and rollover is successful. The main difference between the entrepreneur's payoff and intermediary funds' payoffs is reflected in the last term in Eq. (18), when debt matures but rollover fails.

Because of the entrepreneur's unique human capital in the project, he is re-hired back after the bankruptcy if the chain is restored.[10] Essentially, the expert in the distress fund sells the project back to the entrepreneur at price P_{-1}' (one can view the distress fund as layer -1). The entrepreneur takes price P_{-1}' as given, chooses a new contract π_0' (and hence initializes a new chain) to maximize the sum of proceeds from issuing debt (P_0') and his continuation value (V_0). This is crucial for keeping the contract stationary over time. Since the entrepreneur has no savings when he is rehired,[11] the price charged by the distress fund P_{-1}' cannot be larger than the debt proceeds that the entrepreneur can raise P_0'. We assume the distress fund has all the bargaining power so that $P_{-1}' = P_0'$.[12]

3.2.3 Households

Now consider the value function of households. Regardless of whether debt matures or whether rollover is successful, the new-born households are paying P_{L-1} for the debt. So their payoff is

$$e - P_{L-1}(y) + V_L(y;\pi_{L-1},L). \tag{19}$$

[10] The chain is restored with probability β this period, and $(1-\beta)(1-\lambda_y)$ in the next period. Discount rate α is applied to the continuation value if restoration happens in the next period.

[11] Due to the discount rate α, the entrepreneur would not save the debt proceeds from before.

[12] This assumption implies that the liquidation value equals the fair value of the debt, a property that is consistent with how liquidation value is determined when other layers are broken.

In equilibrium, households are paying the competitive price, $P_{L-1}(y) = V_L(y;\pi_{L-1}, L)$, which is defined recursively as below:

$$V_L(y;\pi_{L-1},L) = \quad \lambda_y \quad \underbrace{\tilde{F}_{y,L-1}}_{\text{Project matures}} \tag{20}$$

$$+ (1-\lambda_y)\Bigg\{(1-\lambda_d)^L \mathbb{E}[\underbrace{\alpha V_L(y';\pi_{L-1},L,L)}_{\text{Debt does not mature}}] \tag{21}$$

$$+ \sum_{l=0}^{L-1}(1-\lambda_d)^l \lambda_d \mathbb{E}[\underbrace{\mathbf{1}^l_{rollover} F_{d,l-1} + (1-\mathbf{1}^l_{rollover})(\alpha B_l(y,L) - c(L-l))}_{\text{Debt matures}}]\Bigg\}. \tag{22}$$

Similar to before, $\tilde{F}_{y,L-1}$ is the payoff to households if the project matures. Given $\tilde{F}_{y,L-1} = \min(F_{y,L-1}, y)$, a sufficiently low $y < \hat{y}$—hence a sufficiently low $\min(F_{y,L-1}, y)$—implies that V_L is too low to convince the new cohort of households to rollover the debt. Therefore $F_{y,L-1}$ is closely tied with the run threshold $\hat{y}$. If neither the project nor the debt matures, then the departing households resell their debt on the secondary market, at a discount α. If the project does not mature but debt issued by layer-l matures, then the households get paid by $F_{d,l-1}$ if rollover is successful (the first part of (22) inside the expectation). Otherwise, layer-l's asset (debt issued by layer–$(l-1)$) is liquidated, and the households only receive the liquidation proceeds $\alpha B_l(y,L)$ net of the legal cost $c(L-l)$ (the second part of Eq. (22) inside the expectation), where $B_l(y,L)$ is the price of liquidated asset at which the experts sell to the market at the beginning of the next period.

We now determine $B_l(y,L)$ from the perspective of the buyer (i.e., new households), with the following valuation equation:

$$B_l(y,L) = \beta \underbrace{V_L(L)}_{\text{If chain is restored}} + (1-\beta)\Big\{\underbrace{\lambda_y \tilde{F}_{y,l-1}}_{\text{Project matures}} + (1-\lambda_y)\Big[(1-\lambda_d)^l \underbrace{\mathbb{E}[\alpha V_L(y';L)]}_{\text{Debt does not mature}} \tag{23}$$

$$+ \sum_{i=0}^{l-1}\lambda_d(1-\lambda_d)^i \mathbb{E}\underbrace{\left[\mathbf{1}^i_{rollover}F_{d,i-1} + (1-\mathbf{1}^i_{rollover})(\alpha B_i(y,L) - c(l-i))\right]}_{\text{Debt matures}}\Big]\Big\}. \tag{24}$$

With probability β, the chain is restored to length L immediately, in which case the households' valuation for the debt is V_L. With probability $1-\beta$, households hold the liquidated asset (debt issued by layer $i-1$) directly for one period, and the chain is restored to L in the following period. If the project matures during this period, then households get paid $\tilde{F}_{y,l-1}$ (define $\tilde{F}_{y,-1} \equiv y$); if neither the project nor the debt matures, then it is sold to the next cohort of households at discount α. Since the next cohort of households will hold debt issued by the restored chain, their valuation of debt is V_L. This is the same term as in Eq. (21). Lastly, if the project does not mature but debt matures, then the households either get paid by $F_{d,i-1}$ if rollover is successful or receive the liquidation proceeds $\alpha B_i(y,L) - c(l-i)$ if rollover fails. The liquidation loss is different depending on where the chain breaks. We will show soon that $B_l(y,L)$ is increasing in l.

3.3 Equilibrium Definition

Define $\hat{\Pi}$ as the set of feasible contracts that are renegotiation proof and subject to the resource constraint (imposed by limited endowment from OLG households):

$$\hat{\Pi} \equiv \{\pi \in \Pi : V_L(\{F_{y,s}, F_{d,s+1}\}_{s=t}^{T}, L) \leq F_{d,t} \leq e \quad \text{for} \quad \forall t\}. \tag{25}$$

Definition 1 *The equilibrium credit chain is a set of contracts* $\{\pi_{l,t}\}_{0 \leq l \leq L-1}$ *and credit chain length* L^* *such that*

1. *When layer-l's liability matures,*[13]

$$\pi_l = \underset{\pi \in \hat{\Pi}}{\arg\max} \quad \mathbf{1}_{rollover}^{l}(P_l(y, \pi; \pi_{l-1}, L^*) + V_l(y, \pi; \pi_{l-1}, L^*)), \tag{26}$$

$$s.t. \quad F_{y,l} \leq F_{y,l-1} \leq y \text{ in } (8) \qquad F_{d,l} \leq F_{d,l-1} \leq e \text{ in } (9). \tag{27}$$

2. *The equilibrium* L^* *is such that the final layer of fund manager* $(L^* - 1)$ *prefers to borrow directly from households than to borrow via other fund managers:*

$$P_{L^*-1}(L^*) + V_{L^*-1}(L^*) \geq P_{L^*-1}(L^* + l) + V_{L^*-1}(L^* + l) \quad \text{for} \quad l \geq 1. \tag{28}$$

   Furthermore, for all other funds $0 < l < L^* - 1$,

$$P_l(L^*) + V_l(L^*) \geq P_l(l+1) + V_l(l+1). \tag{29}$$

   In other words, the funds in intermediary layers prefer to borrow via other funds than to borrow from the households.

3. *Due to perfect competition,*

$$P_l - P_{l-1} + V_l = 0. \tag{30}$$

4 Equilibrium Credit Chain

We analyze the equilibrium credit chain in this section. We first show the stationarity of optimal contract under certain parameterization assumption, and further establish that the optimal contract is independent of the position in the chain. We then characterize and analyze the equilibrium credit chain length L^* in equilibrium.

4.1 Optimal Contract

Layer-l chooses a new contract for its creditors when either the debt issued by himself or the debt held by himself matures, i.e., the event $\mathbf{1}_{rollover}^{l}$ in Eq. (14) and (15) occurs. There, we can see

[13] When $t = 0$, $\mathbf{1}_{rollover}^{l} = 1$ for all l.

layer-l's $(0 < l < L)$ problem is equivalent to:

$$\max_{\pi_l} \quad P_l + V_l(y, \pi_l; \pi_{l-1}, L) \tag{31}$$

$$s.t. \quad P_l = P_{l+1} + V_{l+1}(y, \pi_{l+1}; \pi_l, L) \tag{32}$$

$$F_{d,l} \leq F_{d,l-1} \qquad F_{y,l} \leq F_{y,l-1}. \tag{33}$$

Optimal contracting: stationarity and layer independence. Throughout the paper we impose the following assumption on our parameterization.

Assumption 1 *The primitives of our model satisfies:*

$$e \leq \bar{e} \tag{34}$$

where $\bar{e}$ is defined by

$$\bar{e} = \max_{y \in (e, \frac{e}{\lambda_y})} \quad \lambda_y(1-\alpha)\frac{1-H(y)}{h(y)} - c\left[\frac{log_{(1-\lambda_d)}\frac{cH(y)}{cH(y)(1-\lambda_d)+(1-H(y))(1-\alpha)\lambda_d\bar{e}}}{1-\frac{cH(y)}{cH(y)(1-\lambda_d)+(1-H(y))(1-\alpha)\lambda_d\bar{e}}} + 1 - \frac{1}{\lambda_d}\right] \tag{35}$$

Under Assumption 1, the optimal contract in our economy is independent of history. This stationarity feature is convenient for our analysis. In essence, Assumption 1 guarantees that inequality (9) always binds (so that in the optimal contract $F_{d,l,t} = e$), and it is more likely to be true when e is relatively small. We put back the time subscript only in this subsection; and later we will omit $*$ when we refer to the optimal contract.

Assumption 2 *The following inequality holds for all y,*

$$\frac{\lambda_y}{(1-\lambda_y)e}\frac{1-(1-\lambda_y)\alpha H(y)}{h(y)} - 1 \geq 0 \tag{36}$$

We make assumption 2 to ensure the uniqueness of the equilibrium rollover threshold F_y^*.

For later analysis, we denote by m_l the probability that layer l's asset does not mature

$$m_l \equiv (1-\lambda_d)^l, \tag{37}$$

which satisfies $1 - m_{l+1} = 1 - m_l + m_l\lambda_d$. We present the main result on the equilibrium contract in Proposition 1.

Proposition 1 *Under Assumption 1, the optimal debt contract is stationary and independent of fund position l, so that $\tilde{F}_{y,l,t} = \min(y_t, F_y^*)$, and $F_{d,l,t} = e$. Under Assumption 2, the equilibrium rollover threshold F_y^* is the unique solution to the following equation*

$$e = \lambda_y F_y + \underbrace{\underbrace{\begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix}}_{1\times(L+1)}(\Psi(F_y)^{-1}\eta(F_y))}_{=v_L(L)} \tag{38}$$

where Ψ is a $(L+1) \times (L+1)$ matrix and η is a $(L+1) \times 1$ vector, with both being functions of F_y. The exact expressions for Ψ and η are in Appendix B.1.

The formal proof is in Appendix B. Household's valuation for the debt $V_L(L)$, together with all the liquidation values $B_l(L)$ $(0 \leq l \leq L-1)$, forms a system of linear equations with dimension $L+1$. We solve this system of linear equations and take the last entry which is the value of $v_L(L)$, to be last part in Eq. (38). Matrix Ψ and vector η only depend on F_y and exogenous parameters.

We start explaining the intuition of stationarity. First, we point out that $F_{y,l,t+s} = \hat{y}_{l,t+s}$, which is the rollover threshold of fund l in the credit chain. When designing the contract in period t, y_t is observed but whether project matures or not is still uncertain. The fund l chooses $\tilde{F}_{y,l,t} = \min(F_{y,l,t}, y)$ in order to convince the new-cohort households to refinance the maturing debt, and he has no incentive to promise more than what is needed—of course, unless it cannot afford. This reasoning implies that $F_{y,l,t} = \hat{y}_{l,t}$ is exactly the minimal threshold level for successful rollover; and whenever $y_t < F_{y,l,t}$ the fund l's fundamental is falling short of this threshold, leading to a rollover failure. This logic applies to future periods as well. Importantly, because the entrepreneur and funds can always renegotiate $F_{y,l,t+s}$ $(s \geq 1)$ down to the minimum value at which they can refinance the debt in period $t+s$, in a renegotiation-proof contract $F_{y,l,t+s} = \hat{y}_{l,t+s}$ equals the run threshold for all periods.

We first explain why $F_{d,l,t+s}(s \geq 0)$ is independent of both t and s. Thanks to the i.i.d. nature of fundamental shocks y, without rollover concerns $F_{d,l,t+s}$ should be constant over time (both t and s). However, when y is small, borrowers who face rollover difficulties may try to increase future promised payments $F_{d,l,t+s}$ in order to refinance today. This possibility is ruled out by Assumption 1, which guarantees that $F_{d,l,t+s} \leq e$ binds for all $t+s$. That $F_{d,l,t+s} = F_{d,l}$ is constant over time immediately implies that the endogenous rollover threshold $F_{y,l,t+s}$ is also constant over time, i.e. $F_{y,l,t+s} = F_{y,l}$.

We next explain why $F_{d,l}$ is independent of layer position l. The main concern for setting a high $F_{d,l}$ is that it increases the probability of rollover failures. Because the market is competitive, via debt prices top layers (layers with small l) internalize the rollover risks faced by all layers below. This implies that layers further away from households tend to set smaller $F_{d,l}$. As a result, the first part of inequality (9) binds and all layers have the same F_d.

Lastly, given that the optimal $F_{d,l} = F_d$ is the same across layers, $F_{y,l}$ has to be the same as well. To see this, thanks to market competition $V_l(\{F_{y,l}, F_{d,l}\}; \{F_{y,l-1}, F_{d,l-1}\}, L) = 0$. When $F_{d,l-1} = F_{d,l} = F_d$, the above equation is satisfied if and only if $F_{y,l} = F_{y,l-1}$. Intuitively, if $F_{y,l-1}$ is smaller than $F_{y,l}$, then layer-l earns positive spread when the project matures, implying strictly positive profit in expectation. This cannot be true under perfect competition.

Characterizing the optimal contracts. Given the contract is stationary and layer independent, the run thresholds for all layers are the same and constant over time. We can simplify the households' value function by taking advantage of the fact that $F_{d,l} = F_d$ and $\mathbb{E}[\mathbf{1}^l_{rollover}] = H(F_y)$ (recall $H(\cdot)$ is the cumulative distribution function of y):

$$V_L(\{F_y, F_d\}, L; y) = \lambda_y \min(F_y, y) \tag{39}$$

$$+ \underbrace{(1-\lambda_y)[m_L \alpha V_L + (1-m_L)(1-H(F_y))F_d + H(F_y)\sum_{l=0}^{L-1} m_l \lambda_d(\alpha \mathbb{E}[B_l(y', L)|y' \leq F_y] - c(L-l))]}_{v_L(\{F_y, F_d\}, L)}. \tag{40}$$

We further define $v_L(L)$ to be the part of $V_L(L)$ that is independent of the current realization of y:

$$v_L(\{F_y, F_d\}, L) \equiv V_L(L) - \lambda_y \min(F_y, y) \tag{41}$$

Because $v_L(\{F_y, F_d\}, L)$ only depends on total chain length L and contract parameters (F_y, F_d), it is constant over time.

Conditional on rollover being successful ($y \geq F_y$), the households' valuation of the debt $V_L(L)$ should equal F_d. Therefore the following equation pins down F_y as a function of F_d and L,

$$\lambda_y F_y + v_L(\{F_y, F_d\}, L) = F_d. \tag{42}$$

Under Assumption 1, $F_d \leq e$ is binding; we have explained that this is crucial for the optimal contracting being stationary. After plugging in $F_d = e$, solving for the debt value ($V_L(L)$) and liquidation values ($B_l(L)$), we get Eq. (38) which determines the equilibrium F_y. Lastly, Assumption 2 guarantees that the equilibrium F_y^* is unique.

4.2 Credit Chain Length

For any given F_d and L, define $F_y(F_d, L)$ as the solution to Eq. (42). The next proposition characterizes the equilibrium credit chain length L^*.

Proposition 2 *The equilibrium chain length L^* is characterized by*

$$L^* = \arg\min_L F_y(e, L), \tag{43}$$

which is characterized by the following equation uniquely:

$$0 = \lambda_d(1-\lambda_d)^{L^*}(1-H(F_y))(1-\alpha)e - cH(F_y)(1-(1-\lambda_d)^{L^*+1}). \tag{44}$$

Proof: See Appendix C.

As explained in the previous section, F_y corresponds to the run threshold. Given $F_d = e$, payoff of funds in all layers is decreasing in F_y. Funds in layer $L-1$ will only borrow via another layer of funds if extending the credit chain reduces F_y. Otherwise, they will borrow directly from the households. Hence, the equilibrium L effectively minimizes F_y. Since all layers have the same F_y, deviating by borrowing from the households directly would lead to a chain length with higher F_y, and lower payoff. Hence no layer has incentive to deviate.

Eq. (44) is the first order condition that determines the equilibrium chain length L^*. The first term gives the marginal benefit of longer chains, which comes from the wedge of time discount factors between households (1) and the entrepreneur/fund managers $(1-\alpha)$. A longer chain facilitates maturity transformation, insulates asset from large liquidation losses and hence generates higher value from the lending relationship. We will isolate and explain in detail the different benefits in Section 4.4. On the cost side, when rollover fails, the bankruptcy cost is increasing in the number of layers disrupted. Hence the second term in Eq. (44), capturing marginal cost of more layers, is proportional to bankruptcy cost c and probability of rollover failure $H(F_y)$.

4.3 Liquidation Value

As illustrated by the simple example in Section 2, part of the goal for financial intermediaries to form credit chains is to increase the liquidation value $B_l(y, L)$ toward departing households. Consistent with the intuition illustrated in the simple example, the next proposition formally gives two key properties of $B_l(y, L)$ that drive the benefit of a long-chain (in a decentralized market). These two properties will be critical in understanding the result in Section 4.4.

Proposition 3 *The following features of liquidation value $B_l(y, L)$ hold*

1. *Liquidation value $B_l(y, L)$ is increasing in l for $L \leq L^*$ and any $l \leq L$.*
2. *Liquidation value $B_{L-j}(y, L)$ is increasing in L for $L \leq L^*$ and any $j \leq L$.*

Proof: See Appendix D.

Proposition 3 shows that the liquidation value $B_l(y, L)$ depends on l, the position where the chain breaks. As l increases, the chain's breaking point becomes further away from the entrepreneur and in the same time closer to the households (i.e., $L - l$). The second part in Proposition 3, by fixing the distance to households (j in the second part), highlights that the key is being further away from the entrepreneur. This is important for why layer-structures emerge in equilibrium. If the key reason for higher liquidation value is for the bankruptcy layer to be closer to households, then the equilibrium chain length should be as short as possible. In contrast, establishing more layers is the only way to take advantage of the benefit if a higher liquidation value is driven by a greater distance from from the entrepreneur.

To see the mechanism behind this result, consider the asset being liquidated when the breaking point is further away from the entrepreneur. This asset in liquidation is the debt directly issued by one layer above, but essentially can be considered as a collection of debt contracts issued by all layers above. Evaluating this asset in liquidation, investors understand that there are possible future (before the project matures) favorable fundamental y realizations under which debt payments flow toward departing households in a frictionless way (i.e., without the discount factor α). Because this possibility is greater if the layer is further away from the entrepreneur, the liquidation value is increasing in its distance to the entrepreneur.

We highlight that the above intuition is exactly the same as in our example in Section 2, which shows that two-layer structure dominates that of short-debt. Essentially, compared to the short-term debt structure, the two-layer structure insulates interim negative shocks and protects underlying long term cash flows from being discounted repeatedly. Proposition 3 confirms that the force in our simple example also exists in the model: The more the layers between the point of bankruptcy and the underlying project, the greater the protection.

4.4 Special Case: $c = 0$

The special case of no restructuring cost, i.e. $c = 0$, helps illustrate the benefit of setting up long chains. In essence, as explained shortly, a longer chain effectively insulates the project and reduces liquidation loss during rollover failures. Because there is zero physical cost when rollover fails given $c = 0$, the optimal chain length becomes infinity.

Corollary 1 *When $c = 0$, the equilibrium length of credit chain is infinity.*

Proof: See Appendix E.

To see the benefit of long chains, consider the difference in households value when the chain length is L versus $L + 1$. Using equation (40), one can show that

$$v_{L+1}(L+1) - v_L(L) = \frac{(1-\lambda_y)(1-H(F_y))\lambda_d m_L(F_d - \alpha \overbrace{\mathbb{E}[V_L(L)|y \geq F_y]}^{=F_d})}{1-(1-\lambda_y)\alpha(m_{L+1} + H(F_y)K_{L+1})} > 0. \tag{45}$$

where the constant $K_l \leq 1 - m_l$ for any $l \geq 0$ (see Appendix E for detailed derivation and expression for K_l.). This implies that the last layer of funds always prefer to keep extending the credit chain.

To better understand the benefit of adding more layers, consider a hypothetical structure, where there are only L layers, but the maturity rate between households (layer L) and the last fund (layer $L-1$) is $1-(1-\lambda_d)^2$ instead of λ_d; this way, households hold debt contracts with an aggregate maturity rate of $1-(1-\lambda_d)^{L+1}$.[14] In principle, the hypothetical structure alters the debt maturity in the last layer without changing the number of layers, which helps isolate the debt maturity effect only.

Denote by $\tilde{V}_L$ the households' value function from this hypothetical structure, and correspondingly $\tilde{B}_l(L)$ the liquidation value when layer l fails to rollover its debt. We have

$$\tilde{V}_L(\{F_y, F_d\}, L) = \lambda_y \min(F_y, y) + (1-\lambda_y)\mathbb{E}[m_{L+1}\alpha\tilde{V}_L + (1-m_{L+1})\mathbf{1}_{y_{t+1} \geq F_y} F_d \tag{46}$$

$$+ \mathbf{1}_{y_{t+1} < F_y} \sum_{i=0}^{L-2} m_i \lambda_d \alpha \tilde{B}_i(L) + \mathbf{1}_{y_{t+1} < F_y} m_{L-1}(1-(1-\lambda_d)^2)\alpha\tilde{B}_{L-1}(L)], \tag{47}$$

and let us compare the difference between $\tilde{V}_L$ and V_L:

$$\tilde{V}_L - V_L = \underbrace{\frac{(1-\lambda_y)\lambda_d m_L(1-H(F_y))(F_d - \alpha\mathbb{E}[V_L(L)|y \geq F_y])}{1-(1-\lambda_y)\alpha(m_{L+1} + H(F_y)K_{L+1})}}_{=v_{L+1}(L+1)-v_L(L) \text{ in Eq.(45)}} - \underbrace{\frac{(1-\lambda_y)\lambda_d m_L H(F_y)(\tilde{B}_L(L) - \tilde{B}_{L-1}(L))}{1-(1-\lambda_y)\alpha(m_{L+1} + H(F_y)K_{L+1})}}_{\text{net benefit of one more layer}}. \tag{48}$$

[14]Because of prepayment clauses, household's debt matures as long as one of the layer's debt matures. With $L+1$ layers, the probability of household's debt maturing is $1 - m(L+1) = 1-(1-\lambda_d)^{L+1}$. In the hypothetical structure, the probability of household's debt maturing is $1-(1-\lambda_d)^{L-1}(1-\lambda_d)^2 = 1-(1-\lambda_d)^{L+1}$.

By design, the difference between $\tilde{V}_L$ and V_L comes from the fact that debt held by households effectively has shorter maturity in the hypothetical structure. Short-term debt is preferable because households are short-lived and reselling debt involves a discount. Consider the situation where the debt issued by layer $L-1$ matures in the hypothetical structure but does not in our L-layer structure, and rollover is successful. In the hypothetical structure, households who hold the matured debt leave the economy with a full debt payment F_d. In contrast, in our L-layer case, it is as if there is no debt matures on the entire chain, and households who need to resell debt at a discount receive $\alpha V_L(L)$ only. When rollover is successful ($y \geq F_y$), we know $V_L(L) = F_d$. Hence $\alpha \mathbb{E}[V_L(L)|y \geq F_y] = \alpha F_d < F_d$, first term on the right hand side of Eq. (48) is positive. Mapping back to our example in Section 2, this reflects the difference between short-term debt and long-term contract.

More importantly, there is a downside to shortening the effective debt maturity. Because debt maturity shortening raises the probability of rollover failure, second term in Eq. (48) is negative and captures the additional liquidation loss $\tilde{B}_L - \tilde{B}_{L-1}$ when rollover fails for the hypothetical structure. This additional term is a loss ($\tilde{B}_L - \tilde{B}_{L-1} > 0$) due to Proposition 3. We show that, however, adding a layer (without changing the maturity faced by households) removes this cost, as evident from Eq. (48) which compares $\tilde{V}_L - V_L$ in with $v_{L+1} - v_L$ in Eq. (45). This decomposition highlights that the unique benefit of having multiple layers is to increase the liquidation value (reduce liquidation loss) by insulating the final project from interim negative shocks, as illustrated by the comparison between two-layer structure and short-term debt in our example.

4.5 Comparative Statics

The previous subsection illustrates that intermediaries in the market would like to extend the credit chains. When $c > 0$, additional cost in the case of rollover failure increases with L, leading the optimal chain length to be finite.

Proposition 4 *Under Assumption 1, the equilibrium credit chain length is decreasing in bankruptcy cost c and increasing in project maturity rate λ_y.*

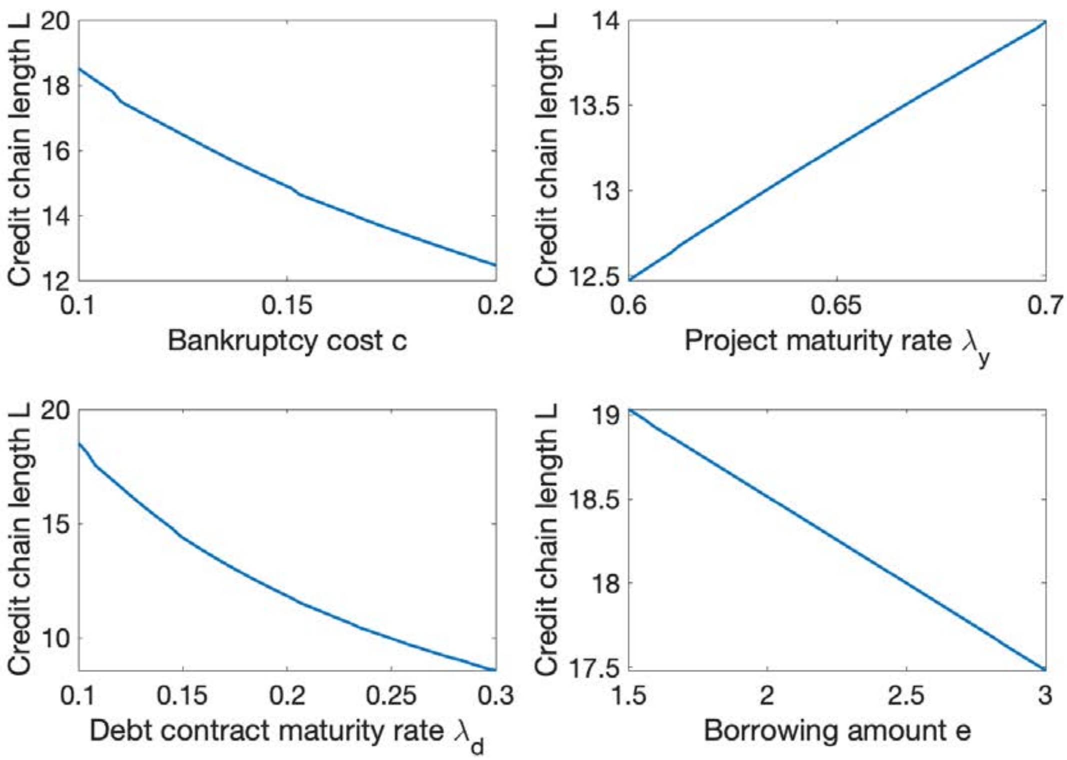
$$\frac{\partial L^*}{\partial c} \leq 0, \quad \frac{\partial L^*}{\partial \beta} > 0 \quad \textit{and} \quad \frac{\partial L^*}{\partial \lambda_y} \geq 0. \tag{49}$$

Proof: See Appendix F.

Figure 7 plots several numerical illustrations of comparative statics with respect to c and λ_y, together with other two parameters (λ_d and e). Intuitively, when the liquidation cost c is higher, it is more costly to add layers, hence in equilibrium chain length is shorter. Related, when β is larger, the liquidation loss is smaller, which motivates more maturity transformation and a longer chain. On the other hand, the more likely the project matures (a higher λ_y), the less severe the rollover risk, and hence the longer the equilibrium credit chain.

Recall under Assumption 1 the outstanding debt market value, which is also the face par-value, is binding at the endogenously given households endowment e. A lager e has the following

Figure 7: Comparative Statics of Credit Chain Length



Numerical illustration of comparative statics related to chain length L. Parameter values (unless specified in the x-axis): $\beta = 0.5$, $\lambda_d = 0.1$, $c = 0.2$, $\alpha = 0.2$, $\lambda_y = 0.6$, $g(y) = \gamma \exp(-\gamma y)$, $\gamma = 0.1$, $e = 2$.

two opposing forces. On one hand, the benefit of maturity transformation, which is proportional to the market value of debt, is larger. On the other hand, a larger debt face value implies a greater probability of a run conditional on debt maturing, which favors a shorter credit chain. The trade-off of these two forces depends on how binding the constraint $F_d \leq e$ is. When e is close to the unconstrained optimal F_d, the second force dominates, i.e., a larger e leads to a shorter credit chain length in order to limit the rollover risk. This is demonstrated in Figure 7.

Regarding the contract maturity rate, the larger the λ_d, the smaller the marginal benefit of increasing maturity rate of debt held by households (proportional to $(1 - \lambda_y)\lambda_d m_L$). This force pushes towards shorter credit chain via the maturity channel. Moreover, the marginal benefit via the asset insulation channel is also proportional to $(1 - \lambda_y)\lambda_d m_L$ (the second term in Eq. (48)), which decreases in λ_d. Both forces imply a shorter equilibrium chain length when λ_d is larger.

Finally, one may ask how the discount rate of experts in our economy, i.e., α, impacts the equilibrium credit chain length. The direct effect of a greater α reduces chain length. This is because a greater α implies a smaller wedge in relative impatience between the entrepreneur/managers and households, therefore a smaller benefit of maturity transformation. On the other hand, a greater

α also implies a smaller liquidation loss, so the cost of maintaining long chains is smaller as well. In general the net effect of α is ambiguous.

5 Welfare Analysis

In this section, we focus on whether the decentralized equilibrium is constrained-efficient from the social planner's perspective. Specifically, we ask the following question: Can the social planner improve welfare by restricting the credit chain length, say via a regulation which caps the credit chain length? Throughout the analysis, we assume that the only tool that the social planner has is to adjust the credit chain length, which affects the resulting market equilibrium (i.e., the debt contracts and allocations).

The answer to the above question is yes—welfare is larger if the social planner limits the chain length. The key source of inefficiency comes from the fact that it is the bottom part of the chain (layer $L-1$) who determines the total chain length. But the bottom layer does not take into account its impact on top layer's contract design and profit.

5.1 Model Modification

We modify the setup slightly for a richer equilibrium outcome while maintaining the stationarity feature over time. The key difference from the model in Section 3 is that we now allow households, before entering the financial market, to choose endogenously how much to set aside for purchasing debt offered by funds.

Cohort t of households are born with endowment e with utility $c_t + c_{t+1}$, where c_t is the consumption in period t and c_{t+1} is the consumption in $t+1$. Households can either consume immediately (c_t) or save via the credit chain (i.e., purchasing debts issued funds) at an endogenous rate r_t. So far, this is equivalent to our previous setup. We modify the setting slightly as follows.

- When cohort t households are born and before y_t is realised, each household chooses c_t^D to consume. Here, "D" stands for day;
- After y_t realises, households can choose to consume $c_t^N \geq 0$ in addition ("N" stands for night) or save via the credit chain. However, households only receive a utility of $1-\epsilon$ per unit of c_t^N. This implies that i) households period t consumption is $c_t = c_t^D + (1-\epsilon)c_t^N$; and ii) they invest $e - c_t^D - c_t^N \geq 0$ in the credit chain either through buying debt or purchasing asset;
- In period $t+1$, households collect money from the credit chain (or proceeds from liquidation), consume and exit the economy.
- We assume households cannot observe historical contracts; Section 5.2 discusses the role of this assumption.

Essentially, $\epsilon > 0$ leads to an irreversible day consumption decision c_t^D, leaving $e - c_t^D$ at households' hands for their investment in credit chains.

In this modified setting, households take the credit chain and its equilibrium return r_t as given, and solve the following problem,

$$\max_{c_t^D \geq 0, c_t^N \geq 0} \quad m_l \mathbb{E}[c_t^D + (1-\epsilon)c_t^N(y;\mathrm{NM})\mathbf{1}_{c_t^N(y;\mathrm{NM})\geq 0} + r_t(e - c_t^D - c_t^N(y;\mathrm{NM}))] \tag{50}$$

$$(1-m_l)\mathbb{E}[c_t^D + (1-\epsilon)c_t^N(y;\mathrm{M})\mathbf{1}_{c_t^N(y;\mathrm{M})\geq 0} + r_t(e - c_t^D - c_t^N(y;\mathrm{M}))] \tag{51}$$

$$s.t. \quad c_t^D + c_t^N \leq e. \tag{52}$$

where $c_t^N(y;\mathrm{M})$ ($c_t^N(y;\mathrm{NM})$) is night consumption when the debt held by previous cohort of households matures (does not mature). In the case when debt has not matured or rollover fails, $c_t^N(y;\mathrm{NM})$ so households will consume in the night. When debt matures and rollover is successful, we have $c_t^N(y;\mathrm{M}) = 0$. As before, contracts $\{F_y, F_d\}$ are layer independent.

Throughout our analysis we focus on the case where $\epsilon \to 0$; we will show that this limiting case maps to our baseline model exactly, except with an endogenous day consumption c_t^D. Because households who obtain a linear utility $(1-\epsilon)c_t^N$ are essentially endowed with a constant savings technology with a constant rate $1-\epsilon$, when $\epsilon \to 0$ we have the endogenous return from investing in the credit chain $r_t \to 1$. Furthermore, $\epsilon \to 0$ implies that $c_t^D \to e - F_d$, i.e., given the equilibrium debt (face) value F_d households set aside just enough funds for potential debt purchase (after a sufficiently favorable y realization, if some debt matures along the credit chain.)

We make the above seemingly stark assumptions to ensure that the modified model exactly matches the baseline model solved in Section 4. The key economic mechanism will be the same if we adopt a "smoother" modification, which features endogenous but irreversible financial-skill investment decision for young households. To establish a clean inefficiency result, we only need some endogenous margin in the amount of resource that is available to invest in the project (via credit chains).[15]

5.2 Optimality Condition of F_d

We have emphasized that our modified model features an endogenous households' c_t^D decision and hence an endogenous equilibrium debt value F_d, which will binds in equilibrium at $e - c_t^D$.

Households who do not observe historical contracts will save based on their (rational) expectation of contracts offered by the equilibrium credit chain. Therefore we determine F_d by the entrepreneur's first order condition from maximizing $P_0 + V_0$, as shown in Eq. (53):

[15] More specifically, cohort t households can spend $I \in [0, e]$ before y_t is realized. After the y_t realization, households with remaining endowment $e - I > 0$ can purchase debt issued by funds as in our main model, or save in a linear saving technology whose return is $R(I)$; and they consume the proceeds when they leave the economy. Importantly, $R'(I) > 0$ and $R''(I) < 0$; in words, young households can "invest" I in their financial skills to improve the return of their savings technology $R(I)$. This extended model nests our main model in Section 3 by setting $I = 0$ and $R(0) = 1$. What we really need is the endogenous margin embedded in I (or c^D in the main text) that helps establish the inefficiency.

$$(1-\alpha)\,(1-m_L)(1-H(F_y))-h(F_y)\underbrace{\frac{dF_{y,t}}{dF_{d,t}}\Big|_{F_{d,t}=F_d}}_{=\frac{1}{\lambda_y}}\sum_{l=0}^{L-1}\lambda_d m_l\left(F_d-\alpha B_l(F_y,L)+c(L-l)\right)\geq 0. \tag{53}$$

To understand (53), we note that generally speaking the benefit of a larger F_d comes from the wedge between the interest rate 1 and the entrepreneur/funds time discount rate α, whereas the cost is rooted in rollover failure. Condition (53) is derived from considering a one shot (more precisely, downward) deviation of $F_{d,t}$ at period t from the entrepreneur's perspective. Since the households cannot observe contracts or realisation of y when making daytime consumption decisions, c^D_{t+s} is constant over time and $F_{d,t+s} \leq e - c^D_{t+s}$ for all $s \geq 1$ thanks to households' irreversible consumption choice. The entrepreneur who face this constraint effectively takes all future $F_{d,t+s}$ ($s \geq 1$) as given when evaluating any deviation of $F_{d,t}$ in period t, and it follows that all future $F_{y,t+s}$ ($s \geq 1$) is fixed as well due to the contract's prepayment option.[16] This explains that only the adjustment of $F_{y,t}$ is considered when evaluating the deviation of $F_{d,t}$.

To sum up, given the binding constraint of $F_{d,t} \leq e - c^D_t$, any F_d for which the left hand side of Eq. (53) is weakly positive constitutes an equilibrium (so entrepreneur never wants to lower F_d). Within this class of equilibria, because our goal is to compare the welfare of decentralized equilibria to the one under the planner's constrained-efficient solution, we focus on the equilibrium that yields the highest welfare, which is the one that satisfies Eq. (53) exactly:

$$(1-\alpha)\,(1-m_L)(1-H(F_y))-h(F_y)\frac{1}{\lambda_y}\sum_{l=0}^{L-1}\lambda_d m_l\left(F_d-\alpha B_l(F_y,L)+c(L-l)\right)=0. \tag{54}$$

Before moving on to the next section, we stress that Condition (54) just amounts to one type of equilibrium selection (i.e., the one with the highest welfare). We can motivate this equilibrium selection by the following equivalent setting, in which the key driver is the non-transparency of credit chain. Note that when initiating the contract at time 0, the cohort 0 households naturally observe the proposed contract before making the daytime consumption decision. In contrast, because credit chains are typically obscure due to the complicated layer structure, later cohorts cannot observe the contract history in the credit chain. As a result, in period 0, the entrepreneur picks F_d to satisfy Eq. (53) exactly, taking all future F_d's as given. This alternative setup yields the same equilibrium outcome as characterized by Condition (54).

5.3 Special Case $c = 0$ Revisited

We come back to the special case when $c = 0$. As explained in Section 4.4 Corollary 1, the equilibrium chain length is infinity regardless of β, and F_y is determined by $V_{L=\infty}(\{F_y, F_d\}, L = \infty) = F_d$. As before in Section 4.2, $F_y(F_d, L)$ denotes the solution to $F_d = V_L(\{F_y, F_d\}, L)$ for any given F_d and L.

[16] Otherwise, the entrepreneur can refinance later at lower F_y's and prepay the current contract.

We now consider whether the social planner can improve the welfare by constrain the chain length L to be a finite number, taking into account that the equilibrium contract parameters are affected by the chain length. The total social welfare W is given by:

$$\begin{aligned} W(\{F_y, F_d\}, L) =& e + \lambda_y y + (1-\lambda_y)[(1-m)(1-H(F_y))((1-\alpha)F_d + \alpha\mathbb{E}[W|y > F_y]) \\ &+ \alpha\lambda_d H(F_y)\sum_{l=0}^{L-1} m_l \left(\mathbb{E}[W|y \leq F_y] + b_l - v_L(L) + m_l\mathbb{E}W\right). \end{aligned} \tag{55}$$

In Eq. (55), $(1-m)(1-H)(1-\alpha)F_d$, which captures the benefit of a larger F_d due to the difference in impatience between households and the entrepreneur, is increasing in F_d. But a higher F_d is also costly as it raises run thresholds; and a longer credit chain results in more maturity transformation and increases the chance that a run occurs (fixing the run threshold). It is also useful to point out that F_y only impacts the welfare through the probability of rollover failure $H(F_y)$. When the project matures and F_y is actually paid out, it is merely a transfer from the entrepreneur to the households and hence only has redistribution effect in that case.

Consider the impact of varying credit chain length on the total welfare, evaluated at the decentralised equilibrium $L = \infty$ in the baseline case of $c = 0$:

$$\underbrace{\frac{dW}{dF_d}}_{>0}\Big|_{L=\infty} \underbrace{\frac{dF_d}{dL}}_{<0}\Big|_{L=\infty} + \frac{dW}{dF_y}\Big|_{L=\infty} \underbrace{\frac{dF_y}{dL}}_{=0}\Big|_{L=\infty} + \underbrace{\frac{dW}{dL}}_{=0}\Big|_{L=\infty} < 0, \tag{56}$$

where all the terms are evaluated at the point of the decentralized equilibrium. As shown in Proposition 2, in the decentralized market, the equilibrium L is chosen to minimize $F_y(F_d, L)$. Hence $\frac{dF_y}{dL} = 0$ and the entire second term disappears. Furthermore, the direct effect of chain length on welfare, $\frac{dW}{dL}$, is also 0 at the decentralized equilibrium thanks to the last layer's first order condition. In other words, for any given value of F_d, the equilibrium F_y and L are socially efficient.

This leaves us only the first term in the welfare evaluation. The equilibrium F_d is decreasing in chain length L. When the chain is shorter, the degree of maturity mismatch is reduced and the chance of run given any run threshold is smaller. As a result, the entrepreneur increases the borrowing amount F_d. Moreover, welfare is increasing in F_d, i.e., the first term in Eq. (56) is strictly negative; we will explain this property shortly. As a result, we have the next proposition.

Proposition 5 *Suppose that $c = 0$. For any $\beta \in [0, 1]$, relative to the decentralized equilibrium, the social planner can improve total welfare by reducing the credit chain length L.*

Proof: See Appendix G.

In the decentralized market solution, the last layer in the credit chain decides the equilibrium chain length, taking the borrowing face value F_d as given. The top layer — the entrepreneur — optimally chooses the borrowing amount F_d, taking the credit chain length as given because he is unable to control the borrowing decision of other funds in the chain. Hence the impact of chain length L on F_d is not internalised by any agent. This coordination issue between the top layer and bottom layer of the chain gives rise to inefficiency.

For the impact of L on F_d to matter, it must be the case that the value of F_d itself is suboptimal. The suboptimality of F_d is rooted in the fact that entrepreneur and funds design F_d period by period and cannot commit to future contract values. Just like the key friction in the setting of dynamic debt runs (He and Xiong, 2012) — today's agent take future run decisions as given— today's entrepreneur and funds also take future borrowing amount as given. As a result, the marginal cost (by increasing run probability) from raising borrowing amount only at time t is more severe than if all future borrowing amount is increased.

To see this point clearly, consider the cost of raising $F_{d,t}$ from a fund's or the entrepreneur's perspective. When its debt matures, more financing will need to be raised from the market in order to pay back the previous cohort. Since F_d is taken as given for all future periods, only F_y can be adjusted, which directly leads to higher run probability than otherwise. On the other hand, the social planner understands all F_d's can be adjusted in equilibrium. Therefore, the decentralized equilibrium F_d is too small relative to social planner's optimal solution.

To summarize, reducing credit chain length L improves total welfare. When the chain is shorter, the entrepreneur increases the borrowing amount F_d in equilibrium and the social value generated from the lending relationship is higher.

Connection to the role of money in Samuelson (1958) Our result is related to the seminal work of Samuelson (1958), who illustrates the important role of money as storage of value in OLG models. In our model, the debt issued by the credit chain is essentially money that facilitates the transfer of wealth among generations.[17] Since the money is privately produced, the "trust" in the money is endogenous and the entrepreneur/funds cannot issue unlimited amount. Our result shows that the amount of money produced in the decentralized equilibrium is too low relative to the social solution. This inefficiency arises from the coordination issue among different cohorts and the social planner can partially alleviate this coordination problem by restricting chain length, which reduces the rollover risk and degree of strategic complementarity among different periods. This effect of credit chain length is not internalised by any private agent in the market as explained above.

5.4 General Case of $c > 0$

Proposition 6 shows the inefficiency exists for general cases as well: When $c > 0$, the decentralised equilibrium has finite chain length when $\beta = 1$. The intuition is the same as in the previous section.

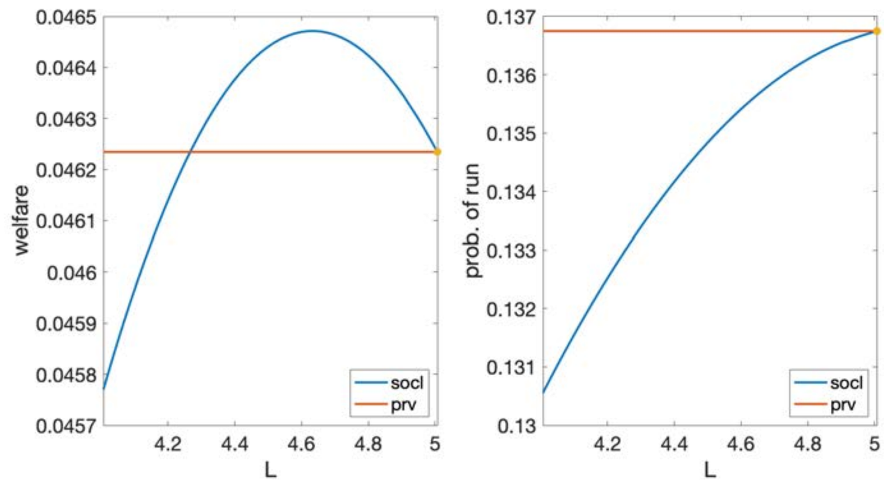
Proposition 6 *For any $1 - \alpha \geq \lambda_y$ and $\beta = 1$, relative to the decentralized equilibrium, the social planner can improve total welfare by reducing the credit chain length L.*

Proof: See Appendix G.

Although we have not been able to prove this result in the most general case of $c > 0$ and $\beta \in (0, 1)$, all numerical solutions so far support this claim. Figure 8 provides such a numerical

[17] The entrepreneur can be thought of as cohort -1.

Figure 8: Social v.s. Private Welfare and Probability of Run



Numerical illustration of what happens when social planner restricts chain length L. Social welfare increases and probability of a run decreases. Parameter values: $\beta = 1$, $\lambda_d = 0.1$, $c = 0.15$, $\alpha = 0.4$, $\lambda_y = 0.6$, $g(y) = \gamma \exp(-\gamma y)$, $\gamma = 0.4$.

illustration. The orange dot shows on the right side of each sub-figure denotes decentralized equilibrium. As L becomes lower, total welfare turns larger. The probability of a run occurring in any period is also lower when the credit chain is shorter.

6 Conclusion

By highlighting a feature that we often see in the modern market-based financial system, we study a new dimension of the credit intermediation where one agent's liability is another agent's asset in the credit chain. We illustrate the trade-off of credit chains in our framework, characterize the equilibrium credit chain, and then study the policy implication of regulating the credit chain from a welfare perspective.

Different from existing research that only looks at systemic risk for each part of the financial system one at a time, our paper tries to provide a holistic view of the financial system when analyzing risks and welfare. This is important because regulations that impact one sector of the financial system will induce changes in the whole sector, affecting other institutions that interact with that sector. Without a model that includes the linkages among different institutions, we cannot properly assess the impact of any individual institution or policy.

References

Acemoglu, Daron, Asuman Ozdaglar, and Alireza Tahbaz-Salehi, "Systemic Risk and Stability in Financial Networks," *American Economic Review*, 2015, *105* (2), 564–608.

Acharya, Viral V., Philipp Schnabl, and Gustavo Suarez, "Securitization without risk transfer," *Journal of Financial Economics*, 2013, *107* (3), 515–536.

Adrian, Tobias, Adam Ashcraft, Hayley Boesky, and Zoltan Pozsar, "Shadow Banking," *Federal Reserve Bank of New York Staff Reports*, 2012, *2* (458), 603–618.

_ **and Hyun Song Shin**, "Money, Liquidity, and Monetary Policy," *The American Economic Review*, 2009, *99* (2), 600–605.

_ **and** _ , "The Changing Nature of Financial Intermediation and the Financial Crisis of 2007–2009," *Annual Review of Economics*, 2010, *2* (1), 603–618.

_ **and** _ , "Procyclical Leverage and Value-at-Risk," *The Review of Financial Studies*, 10 2013, *27* (2), 373–403.

Allen, Franklin, Ana Babus, and Elena Carletti, "Asset commonality, debt maturity and systemic risk," *Journal of Financial Economics*, 2012, *104* (3), 519–534. Market Institutions, Financial Market Risks and Financial Crisis.

_ **and Douglas Gale**, "Financial Contagion," *Journal of Political Economy*, 2000, *108* (1), 1–33.

Aramonte, Sirio, Andreas Schrimpf, and Hyun Song Shin, "Non-bank Financial Intermediaries and Financial Stability," *BIS Working Paper*, October 2021, (972).

Atkeson, Andrew G., Andrea L. Eisfeldt, and Pierre-Olivier Weill, "ENTRY AND EXIT IN OTC DERIVATIVES MARKETS," *Econometrica*, 2015, *83* (6), 2231–2292.

Calomiris, Charles W. and Charles M. Kahn, "The Role of Demandable Debt in Structuring Optimal Baking Arrangements," *American Economic Review*, 1991, *81* (3), 497–513.

_ **and Joseph R. Mason**, "Credit Card Securitization and Regulatory Arbitrage," *Journal of Financial Services Research*, 2004, pp. 5–17.

Carlson, Mark, Burcu Duygan-Bump, Fabio Natalucci, Bill Nelson, Marcelo Ochoa, Jeremy Stein, and Skander Van den Heuvel, "The Demand for Short-Term, Safe Assets and Financial Stability: Some Evidence and Implications for Central Bank Policies," *International Journal of Central Banking*, 2016, *12* (4), 307–333.

Copeland, Adam, Antoine Martin, and Michael Walker, "Repo Runs: Evidence from the Tri-Party Repo Market," *The Journal of Finance*, 2014, *69* (6), 2343–2380.

Demyanyk, Yuliya and Elena Loutskina, "Mortgage companies and regulatory arbitrage," *Journal of Financial Economics*, 2016, *122* (2), 328–351.

Diamond, Douglas W., "Financial Intermediation and Delegated Monitoring," *The Review of Economic Studies*, 1984, *51* (3), 393–414.

_ **and Philip H. Dybvig**, "Bank Runs, Deposit Insurance, and Liquidity," *The Journal of Political Economy*, 1983, *91* (3), 401–419.

_ **and Raghuram G. Rajan**, "Liquidity Risk, Liquidity Creation, and Financial Fragility: A Theory of Banking," *Journal of Political Economy*, 2001, *109* (2), 287–327.

Donaldson, Jason Roderick and Eva Micheler, "Resaleable debt and systemic risk," *Journal of Financial Economics*, 2018, *127* (3), 485–504.

Duffie, Darrell, "Prone to Fail: The Pre-crisis Financial System," *Journal of Economic Perspectives*, February 2019, *33* (1), 81–106.

Elliott, Matthew, Benjamin Golub, and Matthew O. Jackson, "Financial Networks and Contagion," *American Economic Review*, October 2014, *104* (10), 3115–53.

Gennaioli, Nicola, Andrei Shleifer, and Robert W. Vishny, "A Model of Shadow Banking," *The Journal of Finance*, 2013, *68* (4), 1331–1363.

Glode, Vincent and Christian C. Opp, "Private Renegotiations and Government Interventions in Debt Chains," *Working Paper*, 2021.

_ **and Christian Opp**, "Asymmetric Information and Intermediation Chains," *American Economic Review*, 2016, *106* (9), 2699–2721.

_ **, Christian C. Opp, and Xingtan Zhang**, "On the efficiency of long intermediation chains," *Journal of Financial Intermediation*, 2019, *38*, 11–18.

Goldstein, Itay and Ady Pauzner, "Demand-Deposit Contracts and the Probability of Bank Runs," *The Journal of Finance*, 2005, *16* (3).

Gorton, Gary and Andrew Metrick, "Securitized Banking and the Run on Repo," *Journal of Financial Economics*, 2012, *104* (3).

Greenwood, Robin and David Scharfstein, "The Growth of Finance," *Journal of Economic Perspectives*, May 2013, *27* (2), 3–28.

_ **, Samuel G. Hanson, and Jeremy C. Stein**, "A Comparative-Advantage Approach to Government Debt Maturity," *Journal of Finance*, 2015, *LXX* (4), 1683–1722.

Hart, Oliver and John Moore, "Default and Renegotiation: A Dynamic Model of Debt," *The Quarterly Journal of Economics*, 1998, *113* (1), 1–41.

He, Zhiguo and Arvind Krishnamurthy, "A Model of Capital and Crises," *The Review of Economic Studies*, 09 2012, *79* (2), 735–777.

_ **and** _ , "Intermediary Asset Pricing," *American Economic Review*, April 2013, *103* (2), 732–70.

— **and Asaf Manela**, “Information acquisition in rumor-based bank runs,” *The Journal of Finance*, 2016, *71* (3), 1113–1158.

— **and Konstantin Milbradt**, “Dynamic Debt Maturity,” *The Review of Financial Studies*, 08 2016, *29* (10), 2677–2736.

— **and Wei Xiong**, “Dynamic Debt Run,” *The Review of Financial Studies*, 2012, *25* (6), 1799–1843.

Hu, Yunzhi, Felipe Varas, and Chao Ying, “Debt Maturity Management,” *Working Paper*, 2021.

Hugonnier, Julien, Benjamin Lester, and Pierre-Olivier Weill, “Frictional Intermediation in Over-the-Counter Markets,” *The Review of Economic Studies*, 07 2019, *87* (3), 1432–1469.

Infante, Sebastian and Zack Saravay, “What Drives U.S. Treasury Re-use?,” Technical Report 2020.

Karolyi, G. Andrew and Alvaro G. Taboada, “Regulatory Arbitrage and Cross-Border Bank Acquisitions,” *The Journal of Finance*, 2015, *70* (6), 2395–2450.

Krishnamurthy, Arvind, Stefan Nagel, and Dmitry Orlov, “Sizing up Repo,” *The Journal of Finance*, 2014, *69* (6), 2381–2417.

Leland, Hayne E. and David H. Pyle, “Informational Asymmetries, Financial Structure, and Financial Intermediation,” *The Journal of Finance*, 1977, *32* (2), 371–387.

Maggio, Marco Di and Alireza Tahbaz-Salehi, “Collateral Shortages and Intermediation Networks,” *Working Paper*, 2017.

Sambalaibat, Batchimeg, “Endogenous Specialization and Dealer Networks,” *Working Paper*, July 2021.

Samuelson, Paul A., “An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money,” *Journal of Political Economy*, 1958, *66* (6), 467–482.

Schmidt, Lawrence, Allan Timmermann, and Russ Wermers, “Runs on Money Market Mutual Funds,” *American Economic Review*, September 2016, *106* (9), 2625–57.

Shen, Ji, Bin Wei, and Hongjun Yan, “Financial Intermediation Chains in an Over-the-Counter Market,” *Management Science*, 2021, *67* (7), 4623–4642.

Appendix A Equilibrium in the Example

As a benchmark, the entrepreneur can issue three-period debt to cohort 0, who will then sell the debt to cohort 1 and 2 later. The price of the equity contract is

$$P_0(\text{three-period debt}) = [p + (1-p)p]\alpha^2 \tag{57}$$

The entrepreneur's payoff is P_0(three-period debt).

A.1 Direct Financing Using Two-Period Contract

Here we consider the general case where the debt face value is D_2. We show that in equilibrium D_2 is such that rollover is only successful in the good state.

We solve the problem backward. In period 2, the firm can at most raise $P_2 = 1$ from cohort 2. This happens when good news has arrived, otherwise, $P_2 = 0$. If $D_2 = 0$, then the firm is never in liquidation; if $1 \leq D_2 > 0$, the firm is only liquidated if no good news has arrived; finally, if $D_2 > 1$, the firm will always be liquidated.

$$D_2 \begin{cases} \leq 0 & \text{never liquidate} \\ \in (0,1] & \quad \text{only liquidate when no good news arrives} \\ > 1 & \text{always liquidate.} \end{cases} \tag{58}$$

The amount of money that can be raised in period 0 is,

$$P_0(D_2) = \mathbf{1}_{D_2 \leq 0}\alpha D_2 + \mathbf{1}_{0<D_2\leq 1}[p + (1-p)p]\alpha D_2 + \mathbf{1}_{D_2>1}[p + (1-p)p]\alpha^2 \tag{59}$$

The entrepreneur chooses D_2 to maximize P_0,

$$D_2 = 1 \tag{60}$$

$$P_0\,(\text{two-period debt}) = [p + (1-p)p]\alpha \tag{61}$$

Liquidation only happens in period 2 and when no good news arrives. Compared with the three-period debt case,

$$P_0\,(\text{two-period debt}) - P_0(\text{three-period debt}) = [p + (1-p)p]\alpha(1-\alpha) \tag{62}$$

In the three-period debt case, discount is always applied twice on the final cash flow. In the two-period debt case, α is always applied once due to the trading in period 1. However, if debt is rolled over successful in period 2, then final cash flow need not be discounted again. That situation happens with probability $p + (1-p)p$ and the saving is $1 - \alpha$.

A.2 Direct Financing Using One-Period Contract

The problem in period 2 is the same as in the two-period contract case

$$D_2 \begin{cases} \leq 0 & \text{never liquidate} \\ \in (0,1] & \quad \text{only liquidate when no good news arrives} \\ > 1 & \text{always liquidate.} \end{cases} \tag{63}$$

So the proceeds of issuing debt in period 1 if good news arrives is,

$$P_1(g; D_{2,g}) = \mathbf{1}_{D_{2,g}\leq 0} D_{2,g} + \mathbf{1}_{0<D_{2,g}\leq 1} D_{2,g} + \mathbf{1}_{D_{2,g}>1}\alpha \tag{64}$$

if good news has not arrived in period 1, then

$$P_1(b; D_{2,b}) = \mathbf{1}_{D_{2,b}\leq 0} D_{2,b} + \mathbf{1}_{0<D_{2,b}\leq 1} p D_{2,b} + \mathbf{1}_{D_{2,b}>1} p\alpha \tag{65}$$

Given the amount of money that can be raised in period 1 (P_1),

$$D_1 \begin{cases} \leq P_{1,b} & \text{never liquidate} \\ \in (P_{1,b}, P_{1,g}] & \text{liquidate if no good news} > P_{1,g} \quad \text{always liquidate} \end{cases} \tag{66}$$

The amount of money that can be raised in period 0 is

$$P_0(D_1, D_2) = \mathbf{1}_{D_1\leq P_{1,b}} D_1 + \mathbf{1}_{P_{1,b}<D_1\leq P_{1,g}} [pD_1 + (1-p)p\alpha^2] + \mathbf{1}_{D_1>P_{1,g}} \alpha^2[p + (1-p)p] \tag{67}$$

The entrepreneur's problem is to

$$\max_{D_1, D_2} \quad P_0(D_1, D_2) \tag{68}$$

Solution is the following

$$D_{2,g} = D_{2,b} = 1 \tag{69}$$

$$D_1 = 1 \tag{70}$$

$$P_0(\text{one-period debt}) = p + (1-p)p\alpha^2 \tag{71}$$

So liquidation in period 2 happens when no good news arrives and liquidation in period 1 happens when no good news arrives in period 1.

Comparing the funds raised via one-period debt with the funds raised via two-period debt,

$$P_0(\text{one-period debt}) - P_0(\text{two-period debt}) = \underbrace{p(1-\alpha)}_{\text{Rollover succeeds}} \underbrace{-(1-p)\alpha p(1-\alpha)}_{\text{Rollover fails}} \tag{72}$$

The benefit comes from avoiding transaction cost in the secondary market when short-term debt can be successfully rolled-over.

A.3 Financing via Intermediary Funds

Similar to before, we solve the problem backward. The problem at period 2 is exactly the same as the previous subsection,

$$D_2 \begin{cases} \leq 0 & \text{never liquidate} \\ \in (0, 1] & \text{only liquidate when no good news arrives} \\ > 1 & \text{always liquidate.} \end{cases} \tag{73}$$

Same as before, the proceeds of issuing debt in period 1 if good news arrives is,

$$P_1(g; D_{2,g}) = \mathbf{1}_{D_{2,g}\le 0} D_{2,g} + \mathbf{1}_{0<D_{2,g}\le 1} D_{2,g} + \mathbf{1}_{D_{2,g}>1}\alpha \tag{74}$$

if good news has not arrived in period 1, then

$$P_1(b; D_{2,b}) = \mathbf{1}_{D_{2,b}\le 0} D_{2,b} + \mathbf{1}_{0<D_{2,b}\le 1} pD_{2,b} + \mathbf{1}_{D_{2,b}>1} p\alpha \tag{75}$$

Next, we consider the issuance of debt in period 0. Given the amount of money that can be raised in period 1 (P_1),

$$D_1 \begin{cases} \le P_{1,b} & \text{fund never liquidates} \\ \in (P_{1,b}, P_{1,g}] & \text{fund only liquidates when no good news} \\ > P_{1,g} & \text{fund always liquidates} \end{cases} \tag{76}$$

Notice the liquidation in period 1 is at the fund level, i.e. the asset being sold on the market is the debt contract between the entrepreneur and the fund.

The amount of money that can be raised in period 0 is

$$P_0(D_1, D_2) = \mathbf{1}_{D_1\le P_{1,b}} D_1 + \mathbf{1}_{P_{1,b}<D_1\le P_{1,g}} [pD_1 + (1-p)\alpha p] \tag{77}$$

$$+ \mathbf{1}_{D_1>P_{1,g}} (p + (1-p)p)\alpha \tag{78}$$

The entrepreneur's problem is $\max_{D_1,D_2} P_0$, this gives us

$$D_{2,g} = D_{2,b} = 1 \tag{79}$$

$$D_1 = 1 \tag{80}$$

$$P_0(\text{two-layer}) = p + (1-p)p\alpha \tag{81}$$

Comparing this with the one-period direct financing case

$$P_0(\text{two-layer}) - P_0(\text{one-period debt}) = (1-p)p\alpha(1-\alpha) \tag{82}$$

The difference occurs in the case when no good news arrive in period 1, so rollover fails in the first period. In the one-period debt financing case, the entrepreneur's asset is being liquidated, where as in the two-period financing case, only the fund's asset is being liquidated. Since short-term asset incurs lower discount on the secondary market, the indirect financing method is able to raise more funds.

Appendix B Proof for Proposition 1

We first show that $F_{d,l,t} = F_{d,l}$, i.e. the optimal F_d for each layer is constant over time if the managers do not face rollover issues in this period. We start from the problem between layer $(L-1)$ and the households. Layer $(L-1)$ is given a contract π_{L-2} by layer $(L-2)$; the contract specifies a sequence of payments if debt matures $\{F_{d,L-2,t}\}_{t=0}^{T}$ and a payment if project matures

$F_{y,L-2}$. T is the stopping time, either when the contract or when the project matures. Plugging in P_{L-1}, layer $L-1$ maximizes the following,

$$\max_{F_{d,l-1}} -P_{L-2} + \lambda_y F_{y,L-2} + (1-\lambda_y)\mathbb{E}\Big[(1-\lambda_d)^L \alpha(V_{L-1}(y', \pi_{L-1}; \pi_{L-2}, L) + V_L(y'; \pi_{L-1}, L)) \tag{83}$$

$$+ \sum_{i=0}^{L-2} (1-\lambda_d)^i [\lambda_d \mathbf{1}^i_{rollover}(\alpha V_{L-1}(y, \pi'_{L-1}; \pi'_{L-2}, L) + \alpha F_{d,L-2} + (1-\alpha)F_{d,l-1}) + (1-\mathbf{1}^i_{rollover})(\alpha B_i(y,L) - c(L-i-1))] \tag{84}$$

$$+ (1-\lambda_d)^{L-1}\lambda_d \mathbf{1}^{L-1}_{rollover}(\alpha V_{L-1}(y, \pi'_{L-1}; \pi_{L-2}, L) + (1-\alpha)F_{d,l-1}) + (1-\mathbf{1}^{L-1}_{rollover})(\alpha B_{L-1}(y,L) - c)\Big] \tag{85}$$

$$s.t. \quad F_{d,l-1} \le F_{d,L-2} \tag{86}$$

The first order condition with respect to $F_{d,l-1,t}$ is

$$0 = -\mu^{\lambda_d}_{L-1,t} + \mathbb{E}[(1-\alpha)\sum_{i=0}^{L-1}(1-\lambda_d)^i \lambda_d \mathbf{1}^i_{rollover}] \tag{87}$$

$$+ (1-\lambda_d)^{L-1}\lambda_d \frac{d\Pr(\text{rollover at layer } L-1)}{dF_{d,l-1,t}}(F_{d,l-1,t} - \alpha B_{L-1}(y,L) + c) \tag{88}$$

where $\mu^{\lambda_d}_{L-1,t}$ is the Lagrangian Multiplier in front of $F_{d,L-2,t} - F_{d,L-1,t} \ge 0$.

If $\pi_{L-2} = \pi^*_{L-2}$ is stationary and $F_{d,L-2,t}$ is constant over time, then $F^*_{d,l-1,t} = F_{d,l-1}$.

The same logic applies to $F^*_{d,l,t} = F_{d,l}$ for all $0 \le l \le L-1$. For $0 \le l < L-1$, its objective can be written as

$$\max_{F_{d,l}} -P_{l-1} + \lambda_y F_{y,l-1} + (1-\lambda_y)\alpha\Big\{(1-\lambda_d)^{l+2}\mathbb{E}V_l(y', \pi_l; \pi_{l-1}, L) + (1-\lambda_d)^{l+1}\lambda_d \mathbb{E}(1-\mathbf{1}^{l+1}_{rollover})V_l(y', \pi'_l; \pi_{l-1}, L) \tag{89}$$

$$+ \sum_{i=0}^{l-1}(1-\lambda_d)^i \lambda_d \mathbb{E}[\mathbf{1}^i_{rollover}(F_{d,l-1} - F_{d,l+1} - P'_{l-1} - P'_l + \max_{\pi'_l}(P'_l + V_l(y', \pi'_l; \pi'_{l-1}, L)) + \max_{\pi'_{l+1}}(P'_{l+1} + V_{l+1}(y', \pi'_{l+1}; \pi_l, L)))] \tag{90}$$

$$+ (1-\lambda_d)^l \lambda_d \mathbb{E}[\mathbf{1}^l_{rollover}(-F_{d,l+1} - P'_l + \max_{\pi'_l}(P'_l + V_l(y', \pi'_l; \pi_{l-1}, L))) + \max_{\pi_{l+1}'}(P'_{l+1} + V_{l+1}(y', \pi'_{l+1}; \pi_l, L))] \tag{91}$$

$$+ (1-\lambda_d)^{l+2}\mathbb{E}V_{l+1}(y', \pi_{l+1}; \pi_l, L) + (1-\lambda_d)^{l+1}\lambda_d \mathbb{E}[\mathbf{1}^{l+1}_{rollover}(-F_{d,l+1} + \max_{\pi'_{l+1}} V_{l+1}(y', \pi'_{l+1}; \pi_l, L))]\Big\} + P_{l+1} \tag{92}$$

we know in equilibrium $P'_{l-1} = \max_{\pi'_l}(P'_l + V_l(y', \pi'_l; \pi'_{l-1}, L))$ and $P'_l = \max_{\pi'_{l+1}}(P'_{l+1} + V_{l+1}(y', \pi'_{l+1}; \pi'_l, L))$, so the above can be simplified as

$$\max_{F_{d,l}} -P_{l-1} + \lambda_y F_{y,l-1} + (1-\lambda_y)\alpha\Big\{(1-\lambda_d)^{l+2}\mathbb{E}V_l(y', \pi_l; \pi_{l-1}, L) + (1-\lambda_d)^{l+1}\lambda_d \mathbb{E}(1-\mathbf{1}^{l+1}_{rollover})V_l(y', \pi'_l; \pi_{l-1}, L) \tag{93}$$

$$+ \sum_{i=0}^{l-1}(1-\lambda_d)^i \lambda_d \mathbb{E}[\mathbf{1}^i_{rollover}(F_{d,l-1} - F_{d,l+1})] + (1-\lambda_d)^l \lambda_d \mathbb{E}[\mathbf{1}^l_{rollover}(-F_{d,l+1} + V_l(y', \pi'_l; \pi_{l-1}, L) + V_{l+1}(y', \pi'_{l+1}; \pi'_l, L) + P'_{l+1})] \tag{94}$$

$$+ (1-\lambda_d)^{l+2}\mathbb{E}V_{l+1}(y', \pi_{l+1}; \pi_l, L) + (1-\lambda_d)^{l+1}\lambda_d \mathbb{E}[\mathbf{1}^{l+1}_{rollover}(-F_{d,l+1} + P'_{l+1} + V_{l+1}(y', \pi'_{l+1}; \pi_l, L))]\Big\} + P_{l+1} \tag{95}$$

subject to $F_{d,l,t} \le F_{d,l-1,t}$. Denote the Lagrangian multiplier as $\mu^{\lambda_d}_{l,t}$. The first order condition of $F_{d,l,1}$ is

$$0 = -\mu^{\lambda_d}_{l,1} + \mu^{\lambda_d}_{l+1,1} + \frac{dP_{l+1}}{dF_{d,l,1}} \tag{96}$$

$$= -\mu^{\lambda_d}_{l,t} + \mu^{\lambda_d}_{l+1,t} + (1-\lambda_d)^l \lambda_d \frac{d\Pr(\text{rollover at layer } l)}{dF_{d,l,1}}(F_{d,l-1} - \alpha B_{L-1}(y,L) + c) \tag{97}$$

The first order condition with respect to $F_{d,l,t}$ is

$$0 = -\mu_{l,t}^{\lambda_d} + \mu_{l+1,t}^{\lambda_d} + \frac{dP_{l+1}}{dF_{d,l,t}} \tag{98}$$

$$= -\mu_{l,t}^{\lambda_d} + \mu_{l+1,t}^{\lambda_d} + (1-\lambda_d)^l \lambda_d \frac{d\mathrm{Pr}(\text{rollover at layer } l)}{dF_{d,l,t}}(F_{d,l-1} - \alpha B_{l-1}(y,L) + c) \tag{99}$$

If π_{l-1}^* does not depend on history and is stationary, then it is straightforward that $F_{d,l,t}^* = F_{d,l}$.

Next, we show that $F_{d,l} = F_d$ across layers. Since the problem is identical over time, we loose the time subscript. The first order condition with respect to $F_{d,l-1}$ in equilibrium is

$$0 = -\mu_{L-1}^{\lambda_d} + (1-\alpha)\sum_{l=0}^{L-1}(1-\lambda_d)^l \lambda_d \mathrm{Pr}(\text{rollover at layer } l) + \tag{100}$$

$$(1-\lambda_d)^{L-1}\lambda_d \frac{d\mathrm{Pr}(\text{rollover at layer } L-1)}{dF_{d,l-1}}[F_{d,l-1} - \alpha B_{l-1}(y,L) + c] \tag{101}$$

The first order condition with respect to $F_{d,l}$ for $0 < l < L-1$ is,

$$0 = -\mu_l^{\lambda_d} + \mu_{l+1}^{\lambda_d} + (1-\lambda_d)^l \lambda_d \frac{d\mathrm{Pr}(\text{rollover at layer l})}{dF_{d,l}}(F_{d,l-1} - \alpha B_l(y,L) + c(L-l)) \tag{102}$$

For $l = 0$, the first order condition is

$$0 = -\mu_0^{\lambda_d} + \mu_1^{\lambda_d} + \lambda_d \frac{d\mathrm{Pr}(\text{rollover at layer l})}{dF_{d,0}}(F_{d,l-1} - \alpha B_l(y,L) + c(L-l)) \tag{103}$$

Substituting in all the Lagrangian multipliers.

$$\begin{aligned} 0 = &-\mu_0^{\lambda_d} + (1-\alpha)\sum_{l=0}^{L-1}(1-\lambda_d)^l \lambda_d \mathrm{Pr}(\text{rollover at layer } l) \\ &+ \sum_{l=0}^{L-1}(1-\lambda_d)^l \lambda_d \frac{d\mathrm{Pr}(\text{rollover at layer } l)}{dF_{d,l}}(F_{d,l-1} - \alpha B_l(y,L) + c(L-l)) \end{aligned} \tag{104}$$

Denote layer-0's choice as $F_{d,0} = F_d$, satisfying equation (104). If $\mu_0^{\lambda_d} > 0$, then $F_d = e$, and since

$$\mu_{L-1}^{\lambda_d} \geq \mu_{L-2}^{\lambda_d} \geq ... \geq \mu_0^{\lambda_d} > 0 \tag{105}$$

so all the constraints are binding, i.e. $F_{d,l-1} = F_{d,L-2} = ... = F_d$.

If $\mu_0^{\lambda_d} = 0$, then $F_d < e$, it must be the case that $\frac{d\mathrm{Pr}(\text{rollover at layer } l)}{dF_{d,l}} < 0$ holds for at least one l. Denote $\hat{l}$ as the smallest l such that $\frac{d\mathrm{Pr}(\text{rollover at layer } l)}{dF_{d,l}} < 0$. This implies that for $l < \hat{l}$, $\frac{d\mathrm{Pr}(\text{rollover at layer l})}{dF_{d,l}} = 0$, so the first order conditions for $F_{d,l}$ ($l \geq \hat{l}$) are the same as that for $F_{d,0}$. In other words, $F_{d,l} = F_d$. For $l < \hat{l}$, we have $\mu_l^{\lambda_d} > 0$, so the constraint is binding, i.e. $F_{d,l-1} = F_{d,L-2} = ... = F_{d,\hat{l}-1} = F_d$.

So far we have shown that when there is no rollover concerns, we have $F_{d,l,t} = F_d$ being constant over time and across layers. Now we just to show when y is small, and when the money raised from the unconstrained optimal contract is smaller than the amount owed, the managers cannot deviate and set higher F_d. For managers in layer 1 to layer $L-1$, because $F_{d,l} \leq F_{d,l-1}$ is binding, they cannot set higher F_d. For layer 0, as we will show in Appendix B, Assumption 1 ensures that $F_{d,0} \leq e$ is binding. Hence the entrepreneur at layer 0 cannot deviate and set higher F_d either. As a result, $F_{d,l,t} = F_d$ for all layer l and time t.

We next proceed to show that $F_{y,l,t} = \min(F_{y,l-1,t}, F_{y,l})$, where $F_{y,l}$ is a function of $F_{d,l}$.

At time t, for a given sequence of future payments $\{F_{y,L-1,t+j}\}_{j=1}^{\infty}$, there exists $F_{y,L-1,t}$ such that

$$P_{L-1} = V_L(\{F_{y,L-1,t}, \{F_{y,L-1,t+j}\}_{j=1}^{\infty}, F_{d,l-1}\}, L) = \hat{R} \tag{106}$$

Because y_t is i.i.d. across periods, $F_{y,L-1,t}$ does not depend on the history of y.

Since the fund manager can always renegotiate with the households, it must be the case that

$$F_{y,l-1,t} \leq \min\{F_{y,L-1,t}, F_{y,L-2,t}\} \tag{107}$$

If $F_{y,l-1,t} < \min(F_{y,L-1,t}, F_{y,L-2,t})$, then by setting $\tilde{F}_{y,l-1,t} = \min(F_{y,L-1,t}, F_{y,L-2,t})$ and setting $\tilde{F}_{y,L-1,t+1} = F_{y,L-1,t+1} - \alpha(\min(F_{y,L-1,t}, F_{y,L-2,t}) - F_{y,l-1,t})$, both the households and the fund manager remain indifferent. So without loss of generality, we can assume

$$F_{y,l-1,t} = \min(F_{y,L-1,t}, F_{y,L-2,t}) \tag{108}$$

Next, we proceed to show $F_{y,L-1,t}$ must be a constant.

Given our hypothesized form of $F_{y,L-2}$ and the fact that y_t is i.i.d., the distribution of $F_{y,L-2,t}$ is stationary. If $F_{\lambda_y,t} < F_{\lambda_y,t+1}$, then it must exist j, such that

$$\mathbb{E}_t[F_{y,L-1,t+j}] > \mathbb{E}_{t+1}[F_{y,L-1,t+j+1}] \tag{109}$$

$$\Rightarrow \mathbb{E}_t[\min(F_{y,L-1,t+j}, F_{y,L-2,t+j})] > \mathbb{E}_{t+1}[(F_{y,L-1,t+j+1}, F_{y,L-2,t+j+1})] \tag{110}$$

$$\Rightarrow F_{y,L-1,t+j} > F_{y,L-1,t+j+1} \tag{111}$$

However, at time $t+j$, the problem faced by the fund is exactly the same as at time t because of stationarity: at both point t and $t+j$, the manager is trying to find the best subsequent of payment such the debt is worth $F_{d,l-1}$ to households. The two problems are identical. Hence it must be the case that

$$F_{y,L-1,t+j} < F_{y,L-1,t+j+1} \tag{112}$$

This is a contradiction. So $F_{y,L-1,t} = F_{y,l-1}$, i.e. it must be a constant over time.

Next, we consider the lending relationship between layer $l-1$ and layer l $(1 \leq l < L)$,

Similar to before, because creditors can always renegotiate

$$F_{y,l-1,t} \leq \min\{F_{y,L-1,t}, F_{y,L-2,t}\} \tag{113}$$

where $F_{y,L-1,t}$ is defined as

$$P_{l-1}(\{F_{y,L-1,t}, \{F_{y,l-1,t+j}\}_{j=0}^{\infty}, F_{d,l-1}\}) = F_{d,l-1} \tag{114}$$

If $F_{y,l-1,t} < \min\{F_{y,L-1,t}, F_{y,L-2,t}\}$, we show then there is a (weakly) better contract for layer $l-1$ manager. Denote $\Delta = \min(F_{y,L-1,t}, F_{y,L-2,t}) - F_{y,L-2,t}$. Set $\tilde{F}_{y,l-1,t} = \min(F_{y,L-1,t}, F_{y,L-2,t})$, and

$$\tilde{F}_{y,L-1,t+1} = F_{y,L-1,t+1} - \frac{\lambda_y \Delta}{\alpha(1-\lambda_y)[(1-\lambda_d)^{l+1}\lambda_y + (1-\lambda_d)^l \lambda_d \Pr(\text{rollover at layer } l)\lambda_y + (1-\lambda_d)^l \lambda_d \frac{d\Pr(\text{rollover at layer } l)}{dF_{y,l-1}} V_l]} \tag{115}$$

layer l's payoff then remains unchanged. For layer $l-1$, the change in payoff is also 0. Hence without loss of generality, we can assume

$$F_{y,l-1,t} = \min(F_{y,L-1,t}, F_{y,L-2,t}) \tag{116}$$

The proof for $F_{y,L-1,t} = F_{y,l-1}$ carries directly over for a general l $F_{y,L-1,t} = F_{y,l-1}$. Hence

$$F_{y,l-1,t} = \min(F_{y,l-1}, F_{y,L-2,t}) \tag{117}$$

For layer 0 (the entrepreneur),

$$F_{y,0,t} = \min(F_y, y_t) \tag{118}$$

We have now established stationarity. We move on to show $F_{d,l} = F_d$, and $F_{y,l} = F_y$, i.e. they are the same constant across layers.

Lastly, we show $F_{y,l} = F_y$ follows from $F_{d,l} = F_d$. By the definition of $F_{y,l}$,

$$F_{d,l} = P_{l+1} + V_{l+1}(\{F_{d,l+1}, F_{y,l+1}\}; \{F_{d,l}, F_{y,l}\}) \tag{119}$$

In competition, it must be the case that

$$P_l = P_{l+1} + V_{l+1} \tag{120}$$

$$P_l = F_{d,l} = F_d \tag{121}$$

The same is true for $P_{l+1} = F_d$. This implies

$$V_{l+1}(\{F_d, F_{y,l+1}\}; \{F_d, F_{y,l}\}) = 0 \tag{122}$$

From the HJB of V_{l+1}, we can see that it is proportional to $F_{y,l} - F_{y,l+1}$. Hence for $V_{l+1} = 0$, it must be the case that

$$F_{y,l} = F_{y,l+1} = F_y \tag{123}$$

As mentioned before, in equilibrium, F_y is the minimal payment if project matures such that the new households are willing to rollover debt, for a given F_d. By definition

$$F_d = V_L(\{F_y, F_d\}, L) \qquad \text{for } y \geq F_y \tag{124}$$

$$\Rightarrow F_d = \lambda_y F_y + v_L(\{F_y, F_d\}, L) \tag{125}$$

Since all layers have the same F_y and rollover fails when $y < F_y$, we have

$$\Pr(\text{rollover at layer } l) = 1 - H(F_y) \tag{126}$$

Plug this expression in the first order condition of F_d, we get

$$-\mu_0^{\lambda_d} + (1-\alpha)\sum_{l=0}^{L-1} m_l\lambda_d(1-H(F_y)) - \sum_{l=0}^{L-1} m_l\lambda_d h(F_y)\frac{dF_y}{dF_d}(F_d - \alpha B_l(F_y, L) + c(L-l)) = 0 \tag{127}$$

$$\mu_0^{\lambda_d}(e - F_d) = 0 \tag{128}$$

$$\mu_0^{\lambda_d} \geq 0 \tag{129}$$

When $F_d \leq e$ is binding,

$$\frac{dF_y}{dF_d} = \frac{1}{\lambda_y} \tag{130}$$

hence

$$(1-\alpha)(1-m_L)(1-H(F_y)) - \sum_{l=0}^{L-1} m_l\lambda_d h(F_y)\frac{1}{\lambda_y}(F_d - \alpha B_l(F_y, L) + c(L-l)) \tag{131}$$

$$\geq (1-\alpha)(1-m_L)(1-H(F_y)) - \sum_{l=0}^{L-1} m_l\lambda_d h(F_y)\frac{1}{\lambda_y}(F_d + c(L-l)) \tag{132}$$

Under Assumption 1, the above equation is greater than or equal to 0. Hence $F_d = e$.

B.1 Solving for Value Functions

Similar to the definition of $v_L(L)$, define $b_l(L)$ as the stationary component of $B_l(L)$,

$$b_l(L) \equiv B_l(L) - \lambda_y \min(F_y, y) \tag{133}$$

Using the expressions for $B_l(L)$ and $V_L(L)$, we can write them in matrix form,

$$\Psi \begin{bmatrix} b_0(L) \\ b_1(L) \\ \ldots \\ b_{L-1}(L) \\ v_L(L) \end{bmatrix} = \eta \tag{134}$$

where

$$\Psi = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & -\beta \\ 0 & 1 & 0 & ... & 0 & -\beta \\ ... & & & & & \\ 0 & 0 & 0 & ... & 1 & -\beta \\ 0 & 0 & 0 & ... & 0 & 1 \end{bmatrix} \tag{135}$$

$$- (1-\lambda_y)H(F_y)\alpha\lambda_d \begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 & 0 \\ (1-\beta)m_0 & 0 & 0 & ... & 0 & 0 & 0 \\ (1-\beta)m_0 & (1-\beta)m_1 & 0 & ... & 0 & 0 & 0 \\ (1-\beta)m_0 & (1-\beta)m_1 & (1-\beta)m_2 & ... & 0 & 0 & 0 \\ ... & & & & & & \\ (1-\beta)m_0 & (1-\beta)m_1 & (1-\beta)m_2 & ... & (1-\beta)m_{L-2} & 0 & 0 \\ m_0 & m_1 & m_2 & ... & m_{L-2} & m_{L-1} & 0 \end{bmatrix} \tag{136}$$

and

$$\eta = (1-\lambda_y)[\alpha\lambda_y \int^{F_y} ydH(y) + (1-H(F_y))e - cH(F_y)(1-\frac{1}{\lambda_d})] \begin{bmatrix} 1-\beta \\ 1-\beta \\ ... \\ 1-\beta \\ 1 \end{bmatrix} + \begin{bmatrix} (1-\beta)\lambda_y\mu_y \\ 0 \\ ... \\ 0 \\ 0 \end{bmatrix} \tag{137}$$

$$+ (1-\lambda_y)[\alpha(e-\lambda_y F_y H(F_y)) - (1-H(F_y))e + cH(F_y)(1-\frac{1}{\lambda_d})] \begin{bmatrix} (1-\beta)m_0 \\ (1-\beta)m_1 \\ ... \\ (1-\beta)m_{L-1} \\ m_L \end{bmatrix} \tag{138}$$

$$- (1-\lambda_y)cH(F_y) \begin{bmatrix} 0 \\ (1-\beta) \\ ... \\ (1-\beta)(L-1) \\ L \end{bmatrix} \tag{139}$$

We can further simplify v_L as

$$v_L = (1-\lambda_y)\Big\{(1-\lambda_y)(1-\beta)H(F_y)\alpha\lambda_d M \times \Big(\mathcal{I}_L - (1-\beta)(1-\lambda_y)H(F_y)\alpha\lambda_d\hat{M} - \beta(1-\lambda_y)H(F_y)\alpha\lambda_d \mathbb{1}_L M\Big)^{-1}\eta[1:L]$$
$$+ \frac{\alpha m_L(e-\lambda_y F_y H(F_y)) + \alpha\lambda_y \int^{F_y} ydH(y) + (1-m_L)(1-H(F_y))e - cH(F_y)(L+(1-\frac{1}{\lambda_d})(1-m_L))}{1-\beta(1-\lambda_y)H(F_y)\alpha\lambda_d M(\mathcal{I}_L - (1-\beta)(1-\lambda_y)H(F_y)\alpha\lambda_d\hat{M})^{-1}\mathbb{1}_L}\Big\} \tag{140}$$

where $\mathcal{I}_L$ is the identity matrix of dimension L, $\mathbb{1}_L$ is a $L \times 1$ vector of 1's, M is a $1 \times L$ vector where $M_i = m_{i-1}$, $\eta[1:L]$ is the first L elements of η. And finally,

$$\hat{M} = \begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 \\ m_0 & 0 & 0 & ... & 0 & 0 \\ m_0 & m_1 & 0 & ... & 0 & 0 \\ ... \\ m_0 & m_1 & m_2 & ... & m_{L-2} & 0 \end{bmatrix}$$

B.2 Existence and Uniqueness of F_y

A given cohort of household's strategy (run threshold) is $F_y = \frac{e - v_L(F'_y)}{\lambda_y}$, where F'_y is other cohort's strategy. A symmetric equilibrium is where $F_y = F'_y$. Moreover, $\frac{\frac{e - v_L(F'_y)}{\lambda_y}}{dF'_y} \leq 1$ at the equilibrium point.

Given the following

$$\frac{e - v_L(0)}{\lambda_y} > 0 \qquad \lim_{x \to \infty} \frac{e - v_L(x)}{\lambda_y} - x < 0 \tag{141}$$

there exists at least one intersection of $y = \frac{e - v_L(x)}{\lambda_y}$ with $y = x$ from above. So equilibrium exists.

Next, to argue uniqueness, we just need to show that $\frac{d\frac{e - v_L(F_y)}{\lambda_y}}{dF_y} \leq 1 \Leftrightarrow \lambda_y + \frac{dv_L(F_y)}{dF)y} \geq 0$. As shown before,

$$\frac{dv_L}{dF_y} > \frac{-(1-\lambda_y)[\lambda_y m_L \alpha H + h((1-m_L)(1-\alpha)F_d + c\sum_{i=0}^{L-1}\lambda_d m_i(L-i) + \alpha\sum_{i=0}^{L-1}\lambda_d m_i(v_L - b_i(L)))]}{1-(1-\lambda_y)(1-m_L)\alpha H} \tag{142}$$

$$\lambda_y + \frac{dv_L}{dF_y} > \frac{\lambda_y - (1-\lambda_y)[\lambda_y \alpha H + h((1-m_L)(1-\alpha)F_d + c\sum_{i=0}^{L-1}\lambda_d m_i(L-i) + \alpha\sum_{i=0}^{L-1}\lambda_d m_i(v_L - b_i(L)))]}{1-(1-\lambda_y)(1-m_L)\alpha H} \tag{143}$$

$$> \frac{\lambda_y(1-(1-\lambda_y)\alpha H) - (1-\lambda_y)h(1-m_L)e}{1-(1-\lambda_y)(1-m_L)\alpha H} \tag{144}$$

The last term is positive for all F_y under Assumption 2.

Appendix C Proof for Proposition 2

In equilibrium, $F_y(F_d, L)$ is determined by $F_d = \lambda_y F_y + v_L(L)$ and $F_d = e$

$$v_L = (1-\lambda_y)\Big[m\alpha(e + \lambda_y \int^{F_y} yh(y)dy - \lambda_y H F_y) + (1-m)(1-H)e - cH\sum_{i=0}^{L-1}\lambda_d m_i(L-i) \tag{145}$$

$$\sum_{i=0}^{L-1} m_i \lambda_d \alpha \mathbb{E}[B_i(y, L)|y \leq F_y]\Big] \tag{146}$$

where

$$\sum_{i=0}^{L-1}\lambda_d m_i(L-i) = L + (1 - \frac{1}{\lambda_d})(1-(1-\lambda_d)^L) \tag{147}$$

In equilibrium, $L^* = \arg\max_L v_L(L)$. Taking difference with respect to L ($dv_L(L) = v_L(L) - v_{L-1}(L-1)$),

$$\frac{dv_L(L)}{dL} = (1-\lambda_y)\Big[-\lambda_d(1-\lambda_d)^{L-1}\alpha(e+\lambda_y\int^{F_y} yh(y)dy - \lambda_y HF_y) + \lambda_d(1-\lambda_d)^{L-1}(1-H)e \tag{148}$$

$$+\lambda_d(1-\lambda_d)^{L-1}\alpha\lambda_y\int^{F_y} yh(y)dy + \lambda_d(1-\lambda_d)^{L-1}\alpha Hb_L(L) + H(1-m)\alpha\frac{dv_L(L)}{dL} - cH(1-(1-\lambda_d)^L) \tag{149}$$

$$\sum_{i=0}^{L-2} m_i\lambda_d\alpha\underbrace{(\mathbb{E}[B_i(y,L)|y\le F_y] - \mathbb{E}[B_i(y,L-1)|y\le F_y])}_{=\frac{d\mathbb{E}[B_i(y,L)|y\le F_y]}{dL}}\Big] \tag{150}$$

To examine $\frac{d\mathbb{E}[B_i(y,L)|y\le F_y]}{dL}$,

$$\frac{d\mathbb{E}[B_i(y,L)|y\le F_y]}{dL} = \beta\frac{dv_L(L)}{dL} + (1-\beta)(1-\lambda_y)\Big[(1-\lambda_d)^i\alpha\frac{dv_L(L)}{dL} + \sum_{l=0}^{i-1}\lambda_d m_l\alpha\frac{d\mathbb{E}[B_l(y,L)|y\le F_y]}{dL}\Big] \tag{151}$$

At $\frac{dv_L(L)}{dL} = 0$, $\frac{d\mathbb{E}[B_i(y,L)|y\le F_y]}{dL} = 0$ for all $1 \le i \le L$. And $b_L(L) = v_L(L)$. Hence the first order condition with respect to L is,

$$FOC_{L,prv} = (1-H)(1-\alpha)\lambda_d(1-\lambda_d)^L e - cH + cH(1-\lambda_d)^{L+1} = 0 \tag{152}$$

The second order condition for L holds

$$\frac{\partial FOC_{L,prv}}{\partial L} = (1-H)(1-\alpha)\lambda_d e(1-\lambda_d)^L log(1-\lambda_d) + cH(1-\lambda_d)^{L+1}log(1-\lambda_d) < 0 \tag{153}$$

Appendix D Proof for Proposition 3

As shown in Appendix C, we have

$$(1-H)(1-\alpha)\lambda_d m_L F_d - cH + cH(1-\lambda_d)^{L+1} = 0 \tag{154}$$

For $i = L$,

$$v_L - b_{L-1} = (1-\beta)(1-\lambda_y)\Big[\lambda_d m_L((1-H)(1-\alpha)F_d - \alpha H(v_L - b_{L-1}(L))) - cH(1-m_L)\Big] \tag{155}$$

$$v_L - b_{L-1} = \frac{(1-H)(1-\alpha)\lambda_d m_{L-1}F_d - cH + cH(1-\lambda_d)^L}{1+(1-\beta)(1-\lambda_d)\alpha H} > 0 \tag{156}$$

Suppose for all l where $L \geq l \geq i$, we have $b_l \geq b_{l-1}$, then

$$b_i - b_{i-1} = (1-\beta)(1-\lambda_y)\Big[\lambda_d m_i((1-H)(1-\alpha)F_d - \alpha H(v_L - b_{i-1}(L))) - cH(1-m_i)\Big] \tag{157}$$

$$= (1-\beta)(1-\lambda_y)\Big[\lambda_d m_i((1-H)(1-\alpha)F_d - \alpha H(v_L - b_i(L) + b_i(L) - b_{i-1}(L))) - cH(1-m_i)\Big] \tag{158}$$

$$= (1-\beta)(1-\lambda_y)\frac{(1-H)(1-\alpha)\lambda_d m_i F_d - \lambda_d m_i \alpha H(v_L - b_i) - cH(1-m_i)}{1+(1-\beta)(1-\lambda_y)\lambda_d m_i \alpha H} \tag{159}$$

Since we have $b_{i+1} - b_i > 0$, we have

$$(1-H)(1-\alpha)\lambda_d m_{i+1} F_d - \alpha H \lambda_d m_{i+1}(v_L - b_i) - cH(1-m_{i+1}) > 0 \tag{160}$$

$$(1-H)(1-\alpha)\lambda_d F_d - \alpha H \lambda_d (v_L - b_i) + cH > 0 \tag{161}$$

Hence

$$b_i - b_{i-1} = (1-\beta)(1-\lambda_y)\frac{(1-H)(1-\alpha)\lambda_d m_i F_d - \lambda_d m_i \alpha H(v_L - b_i) - cH(1-m_i)}{1+(1-\beta)(1-\lambda_y)\lambda_d m_i \alpha H} \tag{162}$$

$$> \frac{(1-H)(1-\alpha)\lambda_d m_{i+1} F_d - \alpha H m_{i+1}(v_L - b_i) - cH(1-m_{i+1})}{1+(1-\beta)(1-\lambda_y)\lambda_d m_i \alpha H} \tag{163}$$

$$= \frac{b_{i+1} - b_i}{(1-\beta)(1-\lambda_y)[1+(1-\beta)(1-\lambda_y)\lambda_d m_i \alpha H]} > 0 \tag{164}$$

Hence we have $b_i(L)$ and hence $B_i(y, L)$ increase in i.

Appendix E Proof for Corollary 1

We prove the equilibrium chain length is infinity by showing that the firm manager's payoff is always higher with more layers of financial intermediaries, for a given set of contract parameters.

From the proof of optimal contract, it is straightforward that rollover fails when $y < F_y$. This is true for any layer l. Suppose $V_L(\{F_y, F_d\}, L) = F_d$, we will show that $V_{L+1}(\{F_y, F_d\}, L+1) > F_d$, which implies the equilibrium $F_y(F_d, L+1) < F_y(F_d, L)$. In the following proof, unless specified otherwise, $F_y = F_y(F_d, L)$.

We can re-write households' value function as,

$$V_L(\{F_y, F_d\}, L) = \lambda_y \min(F_y, y) + (1-\lambda_y)\mathbb{E}[m_L \alpha V_L + (1-m_L)\mathbf{1}_{y_{t+1} \geq F_y} F_d \tag{165}$$

$$+ \mathbf{1}_{y_{t+1} < F_y} \sum_{i=0}^{L-1} m_i \lambda_d \alpha B_i(L)] \tag{166}$$

consider adding a layer, households' value function becomes

$$V_{L+1}(\{F_y, F_d\}, L+1) = \lambda_y \min(y, F_y) + (1-\lambda_y)\mathbb{E}[m_{L+1} \alpha V_{L+1} + (1-m_{L+1})\mathbf{1}_{y_{t+1} \geq F_y} F_d \tag{167}$$

$$+ \mathbf{1}_{y_{t+1} < F_y} \sum_{i=0}^{L} m_i \lambda_d \alpha B_i(L+1)] \tag{168}$$

To compare the two,

$$
\begin{aligned}
V_{L+1} - V_L =& v_{L+1} - v_L = (1-\lambda_y)\mathbb{E}\Big[m_{L+1}\alpha[v_{L+1} - v_L] - \lambda m_L \alpha V_L(L) && (169)\\
&+ \lambda_d m_L \mathbf{1}_{y_{t+1}\geq F_y} F_d + m_L \lambda \mathbf{1}_{y_{t+1}<F_y} \alpha B_L(L+1) + \mathbf{1}_{y_{t+1}<F_y} \sum_{i=0}^{L-1} m_i \lambda_d \alpha (B_i(L+1) - B_i(L))\Big] && (170)\\
=&(1-\lambda_y)\mathbb{E}\Big[m_{L+1}\alpha[v_{L+1} - v_L] + \lambda_d m_L \mathbf{1}_{y_{t+1}\geq F_y}(F_d - \alpha V_L) && (171)\\
&+ \mathbf{1}_{y_{t+1}<F_y} \sum_{i=0}^{L} m_i \lambda_d \alpha (B_i(L+1) - B_i(L))\Big] && (172)
\end{aligned}
$$

where $B_L(L) = V_L(L)$. First of all, $F_d > \alpha V_L$. Moreover, we show that $B_i(L+1) - B_i(L) > 0$ as well.

$$
\begin{aligned}
B_i(L+1) - B_i(L) &= b_i(L+1) - b_i(L) && (173)\\
b_i(L+1) - b_i(L) &= \beta(v_{L+1}(L+1) - v_L(L)) + (1-\beta)(1-\lambda_y)\sum_{l=0}^{i-1} \alpha H(F_y)(b_l(L+1) - b_l(L)) && (174)
\end{aligned}
$$

and

$$
b_0(L+1) - b_0(L) = [\beta + (1-\beta)(1-\lambda_y)\alpha](v_{L+1}(L+1) - v_L(L)) \tag{175}
$$

Denote $\sum_{l=0}^{n-1} \lambda_d m_l (b_l(L+1) - b_l(L))$ by $K_n \times (v_{L+1}(L+1) - v_L(L))$, and define $K_0 = \lambda_d[\beta + (1-\beta)(1-\lambda_y)\alpha]$. For $n \geq 1$,

$$
\begin{aligned}
K_n &= \beta\lambda_d(1-\lambda_d)^n + [1 + \lambda_d(1-\lambda_d)^n(1-\beta)(1-\lambda_y)\alpha H]K_{n-1} && (176)\\
K_n + \frac{\beta}{(1-\beta)(1-\lambda_y)\alpha H} &= [1 + \lambda_d(1-\lambda_d)^n(1-\beta)(1-\lambda_y)\alpha H](K_{n-1} + \frac{\beta}{(1-\beta)(1-\lambda_y)\alpha H}) && (177)\\
\Rightarrow K_n &= \Pi_{l=1}^{n}[1 + \lambda_d(1-\lambda_d)^l(1-\beta)(1-\lambda_y)\alpha H]\left(K_0 + \frac{\beta}{(1-\beta)(1-\lambda_y)\alpha H}\right) - \frac{\beta}{(1-\beta)(1-\lambda_y)\alpha H} && (178)
\end{aligned}
$$

Plug in $\sum_{l=0}^{L} \lambda_d m_l (b_l(L+1) - b_l(L)) = K_L \times (v_{L+1}(L+1) - v_L(L))$,

$$
v_{L+1}(L+1) - v_L(L) = \frac{(1-\lambda_y)(1-H(F_y))\lambda_d m_L (F_d - \alpha\mathbb{E}[V_L(L)|y \geq F_y])}{1 - (1-\lambda_y)\alpha(m_{L+1} + HK_{L+1})} \tag{179}
$$

Since

$$
\begin{aligned}
b_0(L+1) - b_0(L) &< v_{L+1}(L+1) - v_L(L) && (180)\\
\Rightarrow b_1(L+1) - b_1(L) &< v_{L+1}(L+1) - v_L(L) && (181)
\end{aligned}
$$

Suppose for any $l < i$, we have $b_l(L+1) - b_l(L) < v_{L+1}(L+1) - v_L(L)$, then from Eq. (174), we can show

$$b_i(L+1) - b_i(L) < v_{L+1}(L+1) - v_L(L) \tag{182}$$

Hence

$$K_n < \sum_{i=0}^{n-1} \lambda_d m_i < 1 - m_n \tag{183}$$

So the denominator $1 - (1-\lambda_y)\alpha(m_{L+1} + HK_{L+1})$ is positive. Hence,

$$v_{L+1}(L+1) - v_L(L) > 0 \tag{184}$$

Which means to make the households break-even, $F_y(L+1) < F_y(L)$.

Layer $L-1$ manager's value is decreasing in $F_y(L+1)$. Hence, in equilibrium, the chain length is infinity.

Appendix F Proof for Proposition 4

F.1 Comparative statics with respect to c

We first consider the comparative statics with respect to the per layer bankruptcy cost c,

$$\frac{\partial FOC_{L,prv}}{\partial c} = \frac{\partial FOC_{L,prv}}{\partial F_y}\frac{\partial F_y}{\partial c} - H(1-(1-\lambda_d)^{L+1}) \tag{185}$$

$$\frac{\partial F_y}{\partial c} = -\frac{\frac{\partial v_L(L)}{\partial c}}{\lambda_y + \frac{\partial v_L(L)}{\partial F_y}} \tag{186}$$

$$\frac{\partial v_L(L)}{\partial c} < 0 \quad \Rightarrow \quad \frac{\partial F_y}{\partial c} > 0 \tag{187}$$

$$\frac{\partial FOC_{L,prv}}{\partial F_y} = -h[(1-\alpha)\lambda_d(1-\lambda_d)^L e + c(1-(1-\lambda_d)^{L+1})] < 0 \tag{188}$$

$$\Rightarrow \frac{\partial FOC_{L,prv}}{\partial c} < 0 \tag{189}$$

By implicit function theorem,

$$\frac{\partial L^*}{\partial c} < 0 \tag{190}$$

F.2 Comparative statics with respect to λ_y

Lastly, we consider the comparative statics with respect to project maturity rate λ_y.

$$\frac{\partial FOC_{L,prv}}{\partial \lambda_y} = \frac{\partial FOC_{L,prv}}{\partial F_y}\frac{\partial F_y}{\partial \lambda_y} \tag{191}$$

As shown in the previous part,

$$\frac{\partial FOC_{L,prv}}{\partial F_y} < 0 \tag{192}$$

The other part,

$$\frac{\partial F_y}{\partial \lambda_y} = -\frac{\frac{\partial v_L(L)}{\partial \lambda_y}}{\lambda_y + \frac{\partial v_L(L)}{\partial F_y}} \tag{193}$$

$$\frac{\partial v_L(L)}{\partial \lambda_y} = F_y - v_L(L) \tag{194}$$

We know $v_L(L) \leq e$, and $\lambda_y F_y + (1-\lambda_y)v_L(L) = e$, it must be the case that

$$F_y \geq e \geq v_L(L) \tag{195}$$

Hence

$$\frac{\partial v_L(L)}{\partial \lambda_y} \geq 0 \tag{196}$$

$$\frac{\partial F_y}{\partial \lambda_y} \leq 0 \tag{197}$$

$$\Rightarrow \frac{\partial FOC_{L,prv}}{\partial \lambda_y} \geq 0 \tag{198}$$

By implicit function theorem,

$$\frac{\partial L^*}{\partial \lambda_y} \geq 0 \tag{199}$$

Appendix G Proof for Proposition 6

Households take L and r_t as given, and solves the following problem,

$$\max_{c_{1,t},c_{2,t}} \quad m_l \mathbb{E}[c_{1,t} + (1-\epsilon)c_{2,t}(y;\mathrm{NM})\mathbf{1}_{c_{2,t}(y;\mathrm{NM})\geq 0} + r_t(e - c_{1,t} - c_{2,t}(y;\mathrm{NM}))] \tag{200}$$

$$(1-m_l)\mathbb{E}[c_{1,t} + (1-\epsilon)c_{2,t}(y;\mathrm{M})\mathbf{1}_{c_{2,t}(y;\mathrm{M})\geq 0} + r_t(e - c_{1,t} - c_{2,t}(y;\mathrm{M}))] \tag{201}$$

$$s.t. \quad c_{1,t} + c_{2,t} \leq e \tag{202}$$

where r_t is the return from investing in the credit chain.

When $I_1 \leq e - F_d$, first order condition wrt I_1 and I_2 (with $\epsilon \to 0$)

$$[c_{1,t}]: \quad 1 - r \tag{203}$$

$$[c_{2,t}]: \quad (1-\epsilon) - r \tag{204}$$

So $c_{1,t} \geq e - F_d$.
When $c_{1,t} > e - F_d$, it's first order condition is

$$[c_{1,t}]: \quad m\epsilon + (1-m)\epsilon H(F_y) + (1-m)(1-r_t) \tag{205}$$

$$[c_{2,t}]: \quad (1-\epsilon) - r_t \tag{206}$$

as $\epsilon \to 0$, $r_t \to 1$ and $c_{1,t} \to e - F_d$.

G.1 Equilibrium

Households observe nothing when making day-time consumption decisions. We want to show the following conditions characterize a class of equilibria indexed by F_d. Households consume $c_t^D = e - F_d$ during the day, and save the rest in the night. F_d satisfies the following equilibrium

$$(1-\alpha)(1-m_L)(1-H(F_y)) - h(F_y) \underbrace{\frac{dF_{y,t}}{dF_{d,t}}\Big|_{F_{d,t}=F_d}}_{=\frac{1}{\lambda_y}} \sum_{l=0}^{L-1} \lambda_d m_l (F_d - \alpha B_l(F_{y,t}, L) + c(L-l)) \geq 0 \tag{207}$$

where $\frac{dF_{y,t}}{dF_{d,t}}$ is derived from the implicit function theorem using the equation below.

$$F_{d,t} = \lambda_y F_{y,t} + \upsilon_L(\{F_y, F_d\}, L) \tag{208}$$

and F_d is taken as given for all future periods.

To show this is an equilibrium, We consider one-time deviations. First, note that deviating $F_{d,t}$ from F_d to $F_d + \delta$ ($\delta > 0$) is not feasible, because $F_{d,t} \leq F_d = e - c^D$ has to hold. Next, consider deviating downward to $F_d - \delta$ ($\delta > 0$), just for $F_{d,t}$. The cost is $(1-\alpha)(1-m_L)(1-H(F_y))$. Fixing future F_d's, the benefit of this deviation is $h(F_y)\frac{dF_y}{dF_d}\sum_{l=0}^{L-1} \lambda_d m_l (F_d - \alpha B_l(F_y, L) + c(L-l))$. If we allow future F_d's to be adjusted, note that it can only be adjusted downwards. The decrease in run probability, which is the benefit of such deviation, is even less. According to inequality (207), the cost of such deviation is larger than benefit. As a result, we have shown this is an equilibrium, at least when we only consider one-shot deviations.

Denote the F_d at which (207) holds at equality by $\bar{F}_d$. Next, we argue that for $F_d' > \bar{F}_d$, $F_{d,t} = F_d'$ for all t it is not a stationary equilibrium. Because $(1-\alpha)(1-m_L)(1-H(F_y)) - h(F_y)\frac{dF_y}{dF_d}\sum_{l=0}^{L-1} \lambda_d m_l (F_d - \alpha B_l(F_y, L) + c(L-l))$ is decreasing in F_d, at $F_d' > \bar{F}_d$, we have

$$(1-\alpha)(1-m_L)(1-H(F_y)) - h(F_y)\frac{1}{\lambda_y}\sum_{l=0}^{L-1} \lambda_d m_l (F_d - \alpha B_l(F_y, L) + c(L-l))\Big|_{F_d = F_d'} < 0 \tag{209}$$

So the benefit of deviating from F_d' to $F_d' - \delta$ is $h(F_y)\frac{dF_y}{dF_d}\sum_{l=0}^{L-1} \lambda_d m_l (F_d - \alpha B_l(y, L) + c(L-l))$ larger than the cost $(1-\alpha)(1-m_L)(1-H(F_y))$. As a result, this cannot be an equilibrium.

Hence, we have a continuum of equilibria characterized by (207). We focus on the one that has the largest welfare within this class of equilibria, which is when $F_d = \bar{F}_d$.

G.2 Special Case $c = 0$ and $\beta \in [0, 1]$

In the special case when $c = 0$, the equilibrium chain length is infinity.

$$\frac{dW}{dF_d}\frac{dF_d}{dL} + \frac{dW}{dF_y}\frac{dF_y}{dL} + \frac{dW}{dL} \tag{210}$$

where $\frac{dW}{dL}$ is proportional to $FOC_{L,prv}(L)$ plus $\sum_{l=0}^{L-1} \lambda_d \frac{d(b_l(L)-v_L(L))}{dL}$. As we have shown before,

$$\frac{dv_L}{dL} = 0 \Rightarrow \frac{db_l(L)}{dL} = 0 \quad \text{for} \quad 0 \le l \le L-1 \tag{211}$$

Hence when evaluated at the private optimal $L = \infty$, $\frac{dW}{dL}\Big|_{L=\infty} = 0$.

From second order condition, we know

$$\frac{\partial FOC_{F_L,prv}}{\partial L} \le 0 \tag{212}$$

Furthermore,

$$\frac{\partial FOC_{L,prv}}{\partial F_d}\Big|_{L=\infty} = -ch\frac{dF_y}{dF_d} < 0 \tag{213}$$

By implicit function theorem,

$$\frac{dF_d}{dL} < 0 \tag{214}$$

Finally,

$$\frac{dW}{dF_d} =(1-\alpha)(1-H) - h\left(\frac{dF_y}{dF_d}\right)\sum_{i=0}^{\infty}\lambda_d m_i(F_{d,t} - \alpha B_i(L)) \tag{215}$$

$$+ H\alpha\sum_{l=0}^{\infty} m_l\lambda_d\left(\frac{d(b_l(L)-v_L(L))}{dF_d} + \frac{d(b_l(L)-v_L(L))}{dF_y}\left(\frac{dF_y}{dF_d}\right)\right) \tag{216}$$

Given the expression of b_l,

$$b_l - v_L(L) = (1-\beta)(1-\lambda_y)[-(m_l - m_L)(1-\alpha)(1-H)F_d + \sum_{i=l}^{L-1}\lambda_d m_i \alpha H(v_L - b_i)] \tag{217}$$

$$b_0 - v_L(L) = (1-\beta)(1-\lambda_y)[-(1-\alpha)(1-H)F_d + \sum_{i=0}^{L-1}\lambda_d m_i \alpha H(v_L - b_i)] \tag{218}$$

Plug in

$$b_0 = \beta v_L + (1-\beta)(1-\lambda_y)\alpha[(1-H)F_d + \lambda_y\underline{X}(F_y) + Hv_L] \tag{219}$$

Denote $\frac{d\sum_{l=0}^{L}(b_l(L)-v_L(L))}{dF_d}$ by $D(F_d)$ and $\frac{d\sum_{l=0}^{L}(b_l(L)-v_L(L))}{dF_y}$ by $D(F_y)$,

$$\alpha HD(F_d) = -(1-\beta)(1-\lambda_y)(1-H) + [(1-\beta)(1-(1-\lambda_y)\alpha H) + \alpha H]\frac{dv_L}{dF_d} \tag{220}$$

Using the expression for v_L,

$$\frac{dv_L}{dF_d} = (1-\lambda_y)[1 - H + \alpha HD(F_d)] \tag{221}$$

Using equation (220) and (221), we get

$$\alpha HD(F_d) = \frac{\alpha H(1-\lambda_y)(1-H)}{1-(1-\lambda_y)\alpha H} > 0 \tag{222}$$

$$\frac{dv_L}{dF_d} = (1-\lambda_y)\frac{1-H}{1-(1-\lambda_y)\alpha H} \tag{223}$$

Similarly,

$$\alpha HD(F_y) = (1-\beta)(1-\lambda_y)(1-\alpha)hF_d + (1-\beta)(1-(1-\lambda_y)\alpha H)\frac{dv_L}{dF_y} \tag{224}$$

$$\frac{dv_L}{dF_y} = (1-\lambda_y)[-hF_d(1-\alpha) + \alpha h\sum_{l=0}^{\infty} m_l\lambda_d(b_l - v_L) + \alpha HD(F_y)] \tag{225}$$

$$\Rightarrow \alpha HD(F_y) = \frac{(1-\beta)(1-\lambda_y)\alpha HhF_d + (1-\beta)(1-(1-\lambda_y)\alpha H)(1-\lambda_y)\alpha h\sum_{l=0}^{\infty} m_l\lambda_d(b_l - v_L)}{1-(1-\lambda_y)(1-\beta)(1-(1-\lambda_y)\alpha H)} \tag{226}$$

Evaluating $\frac{dW}{dF_d}$ at the private optimal point, we get

$$\frac{dW}{dF_d} = H\alpha D(F_d) + H\alpha D(F_y)\frac{1-\frac{dv_L}{dF_d}}{\lambda_y + \frac{dv_L}{dF_y}} + h\left(\frac{\frac{dv_L}{dF_d} - 1}{\lambda_y + \frac{dv_L}{dF_y}} + \frac{1}{\lambda_y}\right)\sum_{i=0}^{\infty}\lambda_d m_i(F_{d,t} - \alpha B_i(L)) \tag{227}$$

$$\tag{228}$$

As shown previously, $\frac{db_l(L)}{dF_y}$ is decreasing in l. This implies $D(F_y) \geq 0$. It is also straightforward that $\frac{1-\frac{dv_L}{dF_d}}{\lambda_y + \frac{dv_L}{dF_y}} > 0$ and $F_{d,t} \geq \alpha B_i(L)$.

The only term left to consider is $\frac{\frac{dv_L}{dF_d}-1}{\lambda_y+\frac{dv_L}{dF_y}} + \frac{1}{\lambda_y} = \frac{\frac{dv_L}{dF_d} + \frac{1}{\lambda_y}\frac{dv_L}{dF_y}}{\lambda_y + \frac{dv_L}{dF_y}}$

$$\frac{dv_L}{dF_y} = (1-\lambda_y)\Big[\frac{(1-\beta)(1-\lambda_y)\alpha HhF_d + \alpha h\sum_{l=0}^{\infty} m_l\lambda_d(b_i - v_L)}{1-(1-\lambda_y)(1-\beta)(1-(1-\lambda_y)\alpha H)} - hF_d(1-\alpha)\Big] \tag{229}$$

use the first order condition of F_d,

$$\alpha h\sum_{l=0}^{\infty} m_l\lambda_d(b_i - v_L) = -\lambda_y(1-\alpha)(1-H) + h(1-\alpha)F_d \tag{230}$$

Moreover, $\frac{hF_d}{\lambda_y} \geq (1-\alpha)(1-H)$.

$$\frac{dv_L}{dF_d} + \frac{1}{\lambda_y}\frac{dv_L}{dF_y} = \frac{(1-H)(1-\lambda_y)}{[1-(1-\lambda_y)\alpha H][1-(1-\lambda_y)(1-\beta)(1-(1-\lambda_y)\alpha H)]} \tag{231}$$

$$\times\Big\{1-(1-(1-\lambda_y)\alpha H)\Big[-(1-\alpha)(1-\beta)(1-\lambda_y)[\alpha H + (1-\alpha)(1-(1-\lambda_y)\alpha H)] \tag{232}$$

$$+ (1-\alpha) + (1-\lambda_y)(1-\beta)\Big]\Big\} > 0 \tag{233}$$

Hence

$$\underbrace{\frac{dW}{dF_d}}_{>0}\underbrace{\frac{dF_d}{dL}}_{<0} < 0 \tag{234}$$

reducing L increases welfare.

G.3 Special Case with $c > 0$ and $\beta = 1$

For the managers, they take future values of F_d as given, so

$$\frac{dF_y}{dF_d} = \frac{1}{\lambda_y} \tag{235}$$

The equilibrium F_d is determined by

$$FOC_{F_d,prv}(F_d) = (1-\alpha)(1-m_L)(1-H) - h\frac{1}{\lambda_y}\sum_{l=0}^{L-1}\lambda_d m_l(F_{d,t} - \alpha B_l(y,L) + c(L-l)) \tag{236}$$

The equilibrium L is determined by $\frac{dv_L(L)}{dL} = 0$:

$$FOC_{L,prv}(L) = (1-H)(1-\alpha)\lambda_d(1-\lambda_d)^L F_d - c(1-(1-\lambda_d)^{L+1})H = 0 \tag{237}$$

Consider total welfare W and how welfare changes if the social planner regulates L,

$$\frac{dW}{dF_d}\frac{dF_d}{dL} + \frac{dW}{dF_y}\frac{dF_y}{dL} + \frac{dW}{dL} \tag{238}$$

It is straightforward to show that $\frac{dW}{dL}\Big|_{L=L^*} = 0$ and $\frac{dF_y}{dL}\Big|_{L=L^*} = 0$. To evaluate the first term,

$$\frac{dW}{dF_d} =(1-\alpha)(1-m_l)(1-H) - h\left(\frac{dF_y}{dF_d}\right)\sum_{i=0}^{L-1}\lambda_d m_i(F_{d,t} - \alpha B_i(L) + c(L-i)) \tag{239}$$

When evaluated as the private equilibrium, we first show

$$\left(\frac{dF_y}{dF_d}\right)^{socl} = \frac{1-\frac{dv_L}{dF_d}}{\lambda_y + \frac{dv_L}{dF_y}} < \frac{1}{\lambda_y} \tag{240}$$

Consider

$$\left(\frac{dF_y}{dF_d}\right)^{socl} - \frac{1}{\lambda_y} = \frac{1-\frac{dv_L}{dF_d}}{\lambda_y + \frac{dv_L}{dF_y}} - \frac{1}{\lambda_y} = \frac{-\lambda_y\frac{dv_L}{dF_d} - \frac{dv_L}{dF_y}}{\lambda_y(\lambda_y + \frac{dv_L}{dF_y})} \tag{241}$$

$$\frac{dv_L}{dF_d} = \frac{(1-\lambda_y)[m_L\alpha + (1-m_L)(1-H)]}{1-(1-\lambda_y)\alpha H(1-m_L)} \tag{242}$$

$$\frac{dv_L}{dF_y} = \frac{-(1-\lambda_y)[\lambda_y m_L\alpha H + h((1-m_L)(1-\alpha)F_d + c\sum_{l=0}^{L-1}\lambda_d m_l(L-l))]}{1-(1-\lambda_y)\alpha H(1-m_L)} \tag{243}$$

$$\lambda_y\frac{dv_L}{dF_d} + \frac{dv_L}{dF_y} = \frac{(1-\lambda_y)}{1-(1-\lambda_y)\alpha H(1-m_L)}\Big[\lambda_y m_L\alpha(1-H) + \lambda_y(1-m_L)(1-H) \tag{244}$$

$$-\lambda_y(1-\alpha)(1-m_L)(1-H)\Big] > 0 \tag{245}$$

The last equation is using the first order condition of F_d.
So $\frac{dW}{dF_d} > 0$ at the decentralised equilibrium point. Next, we want to evaluate the sign of $\frac{dF_d}{dL}$. From the second order condition of L, we know

$$\frac{\partial FOC_{L,prv}}{\partial L} \leq 0 \tag{246}$$

Furthermore,

$$\frac{\partial FOC_{L,prv}}{\partial F_d} = -m'(1-H)(1-\alpha) + h\left(\frac{dF_y}{dF_d}\right)^{socl}[m'(1-\alpha)F_d - c(1-m)] \tag{247}$$

$$= m'(1-\alpha)\big[F_d \frac{h}{H}\left(\frac{dF_y}{dF_d}\right)^{socl} - (1-H)\big] \tag{248}$$

Using $FOC_{F_d,prv}(F_d) = 0$ and plug in the expression of $\left(\frac{dF_y}{dF_d}\right)^{socl}$ to get

$$F_d h\left(\frac{dF_y}{dF_d}\right)^{socl} - (1-H)H \tag{249}$$

$$= \frac{\frac{F_d h}{\lambda_y}[1-(1-\lambda_y)\alpha H(1-m_L) - (1-\lambda_y)[m_L\alpha + (1-m_L)(1-H)]]}{\lambda_y - \lambda_y(1-\lambda_y)\alpha H - (1-\lambda_y)\lambda_y(1-\alpha)(1-m_L)(1-H))} - (1-H)H \tag{250}$$

From the first order condition of F_d, $h \geq \frac{(1-\alpha)(1-H)\lambda_y}{F_d}$,

$$F_d h\left(\frac{dF_y}{dF_d}\right)^{socl} - (1-H)H \tag{251}$$

$$= \frac{(1-H)}{\lambda_y - \lambda_y(1-\lambda_y)\alpha H - (1-\lambda_y)\lambda_y(1-\alpha)(1-m_L)(1-H))} \tag{252}$$

$$\times \Big[(1-\alpha)(1-(1-\lambda_y)\alpha H - (1-\lambda_y)(\alpha(1-H) + (1-\alpha)(1-H)(1-m_L))) \tag{253}$$

$$- H\lambda_y[1-(1-\lambda_y)\alpha H - (1-\lambda_y)(1-\alpha)(1-H)(1-m_L)]\Big] \tag{254}$$

When $1-\alpha \geq \lambda_y$, we have

$$(1-\alpha)[1-(1-\lambda_y)\alpha H - (1-\lambda_y)(\alpha(1-H) + (1-\alpha)(1-H)(1-m_L))] \tag{255}$$

$$\geq \lambda_y[1-(1-\lambda_y)\alpha H - (1-\lambda_y)(\alpha(1-H) + (1-\alpha)(1-H)(1-m_L))] \tag{256}$$

$$\geq H\lambda_y[1-(1-\lambda_y)\alpha H - (1-\lambda_y)(1-\alpha)(1-H)(1-m_L)] \tag{257}$$

Hence

$$\frac{\partial FOC_{L,prv}}{\partial F_d} \leq 0 \tag{258}$$

By implicit function theorem,

$$\frac{dF_d}{dL} \leq 0 \tag{259}$$

Hence

$$\underbrace{\frac{dW}{dF_d}}_{\geq 0}\underbrace{\frac{dF_d}{dL}}_{\leq 0} + \frac{dW}{dF_y}\underbrace{\frac{dF_y}{dL}}_{=0} + \underbrace{\frac{dW}{dL}}_{=0} \leq 0 \tag{260}$$

So the social planner can increase total welfare by limiting the credit chain length.

How will the fragility change? We define fragility as $(1-m_l)H$, which is the probability of bankruptcy in any period.

$$\frac{d(1-m)H}{dL} = -m'H + (1-m)h\left(\frac{dF_y}{dF_d}\right)^{socl}\frac{dF_d}{dL} \tag{261}$$

To determine $\frac{dF_d}{dL}$,

$$\frac{dF_d}{dL} = -\frac{\frac{\partial FOC_{\lambda_d,prv}}{\partial L}}{\frac{\partial FOC_{\lambda_d,prv}}{\partial F_d}} \tag{262}$$

$$= -\frac{m'(1-\alpha)\left[F_d\frac{h}{H}\left(\frac{dF_y}{dF_d}\right)^{socl} - (1-H)\right]}{-h\left(\frac{dF_y}{dF_d}\right)^{prv}(1-m) - \frac{(1-\alpha)(1-m)(1-H)}{\left(\frac{dF_y}{dF_d}\right)^{prv}}\frac{\partial^2 F_y}{\partial F_d^2}} \tag{263}$$

$$\frac{\partial^2 F_y}{\partial F_d^2} = (1-\lambda_y)(1-m)h\left[\left(\frac{dF_y}{dF_d}\right)^{prv}\right]^2 \tag{264}$$

$$\frac{dF_d}{dL} = \frac{m'(1-\alpha)\left[F_d\frac{h}{H}\left(\frac{dF_y}{dF_d}\right)^{socl} - (1-H)\right]}{h\left(\frac{dF_y}{dF_d}\right)^{prv}(1-m) + (1-\alpha)(1-m)^2(1-H)(1-\lambda_y)h\left(\frac{dF_y}{dF_d}\right)^{prv}} \tag{265}$$

Hence

$$\frac{d(1-m)H}{dL} = -m'H + \frac{m'(1-\alpha)(1-(1-\lambda_y)(m\alpha+(1-m)(1-H)))\left[F_d\frac{h}{H}\left(\frac{dF_y}{dF_d}\right)^{socl} - (1-H)\right]}{1+(1-\alpha)(1-m)(1-H)(1-\lambda_y)} \tag{266}$$

$$F_d h\left(\frac{dF_y}{dF_d}\right)^{socl} \leq (1-H)(1-(1-\lambda_y)(m\alpha+(1-m)(1-H))) \tag{267}$$

$$\frac{d(1-m)H}{dL} \geq -m'\Big[H - \frac{(1-\alpha)[1-(1-\lambda_y)(m\alpha+(1-m)(1-H))]\left[\frac{(1-H)(1-(1-\lambda_y)(m\alpha+(1-m)(1-H)))}{H} - (1-H)\right]}{1+(1-\alpha)(1-m)(1-H)(1-\lambda_y)}\Big] \tag{268}$$

This could be positive or negative. We can work out conditions under which restricting L would result in lower vulnerability.