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DEBT AS SAFE ASSET

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ABSTRACT

The price of a safe asset reflects not only the expected discounted future cash flows but also future service flows, since retrading allows partial insurance of idiosyncratic risk in an incomplete markets setting. This lowers the issuers' interest burden. As idiosyncratic risk rises during recessions, so does the value of the service flows bestowing the safe asset with a negative b. The resulting exorbitant privilege resolves government debt valuation puzzles and allows the government to run a permanent (primary) deficit without ever paying back its debt, but the government faces a "Debt Laffer Curve".

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1 Introduction

What is a safe asset? What are its features? Why does it have a negative β? Why is government debt of advanced countries like the US and Japan a safe asset? Why does it enjoy an exorbitant privilege? How much of it can the market absorb? Is there a debt valuation puzzle? When can governments run a permanent (primary) deficit without ever paying back its debt, like a Ponzi scheme, and is there a limit, a "Debt Laffer Curve"? When does one lose the safe asset status? How do we have to modify representative agent asset pricing and the government debt valuation equation to account for safe asset features? This paper presents a theory of safe assets that sheds light on these questions.

We define a safe asset by its key characteristic, the *Good Friend Analogy*. A safe asset is like a good friend, it is around; that is, it is valuable and can be traded when one needs it.[1] We illustrate this within a setting in which citizens face uninsurable idiosyncratic risks and save for precautionary reasons. Each citizen adjusts her portfolio consisting of risky physical capital and the safe asset, a government bond. Idiosyncratic shocks that cannot be diversified away (as well as aggregate shocks) make capital risky. This makes the safe asset attractive since it can be sold after an adverse shock. From an individual citizen's perspective it is this ability to retrade which makes the government bond a desirable hedging instrument. Her planned dynamic trading strategy generates a payoff stream that is a good hedge.

Since a safe asset generates this extra service flow in the form of self-insurance, it is attractive even at a low real interest rate, r, its cash flow return. It is instructive to consider a new asset pricing formula which nicely separates the two benefits of the safe asset: cash flows, possibly negative, and a service flow that results from the ability to self-insure through retrading. The real value of a safe asset (or any tradable asset) is

$$price_t = \mathbb{E}_t[PV_{r^{**}}[cash\ flows]] + \mathbb{E}_t[PV_{r^{**}}[service\ flows]].$$

While the traditional asset pricing formula prices the cash flow of a buy and hold strategy of the safe asset, the above stated pricing formula prices the cash flow of a dynamic (equilibrium) strategy whose cash flow is positive when the asset is sold (after

[1] In this paper we focus on risk and abstract from liquidity considerations by assuming that all assets are liquid in our model. But, conceptually, liquidity is also an important aspect of this definition.

a negative shock) and negative when additional safe assets are bought (after a positive shock). Valuing individual dynamic trading cash flow streams and aggregating them leads to the above pricing equation, where a different discount rate, r^{**}, arises naturally.[2] Interestingly, r^{**} can be viewed as the "representative agent interest rate" in an incomplete market setting. It is the risk-free rate that excludes the component that is due to precautionary demand driven by the exposure to uninsurable idiosyncratic risk.

A safe asset is really like a good friend if it not only allows citizens to self-insure against adverse idiosyncratic shocks, but also serves as a safe haven after adverse aggregate shock. That is, it appreciates in recessions due to flight-to-safety capital flows. To see why a safe asset has a non-positive β consider an economy in which idiosyncratic risk rises and aggregate output declines in recessions. A drop in output reduces payoffs and increases the marginal utility, leading to the traditional positive β in the asset pricing equation for the cash flow term. The second term, the service flow term, behaves very differently. As idiosyncratic risk rises in recessions, citizens prefer to shift their portfolio away from capital towards the safe asset, resulting in a force that pushes up the real value of safe assets. It is due to the second term capturing the discounted stream of service flows that the safe asset has a non-positive β.

Our model has also interesting stock market asset pricing implications due to "flight-to safety" phenomena. During recessions, idiosyncratic risk is assumed to rise. While for outside equity, idiosyncratic risk can be diversified away, the residual claimant to each firm is an insider who remains exposed to the idiosyncratic risk via her inside equity holdings. During recessions, these insiders demand a higher insider risk premium which depresses payouts to outside equity holders. As a consequence, the (outside) equity stock index depreciates relative to the safe asset. That is, outside equity has a positive β, even when held in a diversified mutual fund.

For the safe asset the required cash flow return r is low due to its service flow and its negative β. Any entity that issues a safe asset, be it the government or private corporations, benefits from the low required cash flow return. Since similar service flows can be derived from all safe assets, whether issued by the government or by corporations, traditional measures of convenience yields, like the BAA-Treasury interest rate spread used in Krishnamurthy and Vissing-Jorgensen (2012), do not capture the full service

[2] While it is here applied to the safe asset, this "dynamic trading perspective" is a general valuation approach and can be used in any incomplete markets setting to isolate the benefits from equilibrium trades.

flow from trading safe assets.

However, not any entity that issues a safe asset with service flows can run a Ponzi scheme. When precautionary savings due to idiosyncratic risk depress the required real cash flow return r below the economic growth rate g, sustainable Ponzi schemes become feasible. One can pay off the maturing bonds with newly issued debt and issue more to fund additional expenditures. Viewed differently, in this case one can issue a "bubbly" safe asset.[3] In our model with uninsurable idiosyncratic risk, bubbles are possible even though individual citizens' transversality conditions hold. Strictly speaking, which entity can run a Ponzi scheme depends on which equilibrium is selected. In other words, the selected "bubble equilibrium" determines who is subject to a no-Ponzi constraint. Brunnermeier et al. (2021a) argue that the government's ability to tax and impose regulations on the private sector puts it in a unique position to defend a bubble on its debt. According to this view, which we follow in this paper, the government enjoys an exorbitant privilege as a safe asset issuer that sets it apart from private entities. While the latter may also be able to issue a safe asset with service flows, they can – unlike governments – not run a Ponzi scheme.

Note that safe assets do not need to be bubbly. Safe assets can also arise in the absence of bubbles. However, a bubble component can make an asset "safer", since the value of the service flow is proportional to the market value of the (bubbly) asset – and the service flow is highly priced, not least because it carries a negative β. Indeed, under certain circumstances, the same asset without a bubble can have a positive β (driven by discounted cash flows), while when the bubble is associated with the asset, its β becomes negative. In that case, the asset is only a safe asset if the bubble is attached to the asset. In other words, bubble and safe asset characteristics are complementary. This leads us to a second concept, the *Safe Asset Tautology* that holds for bubbly safe assets: A safe asset is safe because it is perceived to be safe. In this case, the safe asset status can be lost, when the bubble pops.

Finally, the government can generate seigniorage revenues by "mining the bubble" (expanding the Ponzi scheme to fund deficits). One contribution of this paper is to document quantitatively that "bubble mining" only raises sizable government revenue if safe public debt has a negative β. A government can generate revenue by issuing bonds at a faster pace, create higher inflation, and thereby reduce the real return on

[3] A standard ("buy and hold") asset pricing equation carries then a bubble term.

holding the government bond. "Printing" bonds at a faster rate acts like a tax on bond holdings or, better said, on partial self-insurance through holding and retrading the safe asset. It is a form of "financial repression." Increasing the tax rate increases the "tax revenue", but erodes the "tax base", the value of the bonds. A "*Debt Laffer Curve*" emerges. When the tax exceeds a certain level, overall tax revenue from bubble mining declines. Our calibration quantifies the Laffer Curve and estabilishes that the negative β is crucial to generate quantitatively significant revenue from "bubble mining".

Literature. This paper touches upon many strands of classic and recent economic literature. It connects to the large literature on the risk-sharing benefits of money or government debt in environments with uninsured idiosyncratic risk. The classic reference is Bewley (1980). Aiyagari and McGrattan (1998) calibrate the optimal debt level in an Aiyagari (1994)-type model without aggregate risk. In Woodford (1990) trading of government debt improves consumption smoothing and crowds in investment. In Kiyotaki and Moore (2008) citizens use money as liquidity to exploit investment opportunity shocks. Our model environment is closest to Angeletos (2007), who studies idiosyncratic investment risks, and Brunnermeier and Sannikov (2016a,b) who add a 'bubbly' safe asset in the form of government debt or money into that framework and allow for aggregate risk. Di Tella (2020) and Merkel (2020) study similar environments in which the safe asset is money and has a convenience yield instead of being bubbly. In this paper, we emphasize that government debt generates a service flow from retrading even in the absence of bubbles and convenience yields. We provide a formal characterization of these service flows by developing a new asset valuation approach in incomplete markets settings, the "dynamic trading perspective", which explicitly highlights the gains from trading. Government debt retains its safe asset role possibly with a negative CAPM β even in a setting with publicly traded equity mutual fund that is also fully liquid and free of idiosyncratic risk.

Our discussion of public debt bubbles is also related to the literature on rational bubbles and debt sustainability when the interest rate r is lower than the economic growth rate g. Papers generating $r < g$ with Overlapping Generations (OLG) include Samuelson (1958), Diamond (1965) with capital, Tirole (1985) with a bubble and, most recently, Blanchard (2019). In the above cited Bewley (1980)-type models, as well as ours, interest rates are depressed due to a precautionary savings motive. Bassetto and Cui (2018), Brunnermeier et al. (2021a) and, more recently, Kaplan et al. (2023) study the fiscal theory of the price level (FTPL) when $r < g$. Brunnermeier et al. (2021a) show how the government can ensure that the bubble is associated with government debt and does not jump to other assets, e.g. crypto assets. They also point out that a bubble gives rise to a "Debt Laffer Curve", but consider an environment without aggregate risk. In this paper, we quantify the Laffer curve and show that aggregate risk and the negative β are quantitatively important. Reis (2021) studies fiscal debt capacity in a related framework with a bubble on government debt. To avoid an opposite infinity problem in the debt valuation equation when $r < g$, Reis (2021) discounts at the higher marginal product of capital $m > g$. In this paper, we propose instead r^{**}-discounting, which also avoids the opposite infinity problem but has a very simple economic interpretation. Aguiar et al. (2023) show that bubbly government debt can lead to a Pareto improvement. Mian et al. (2022) study debt issuance at the zero lower bound in a setting in which households derive utility from holding government bonds. There is also an extensive literature on rational bubbles outside of the government debt context. Survey papers include Miao (2014) and Martin and Ventura (2018).

This paper resolves the "Public Debt Valuation Puzzle" proposed in Jiang et al. (2019), which argues that government debt appears overvalued not least because primary surpluses, the total payments to all bond holders, are procyclical and should thus be discounted at a high rate. In our setting, the price of debt is countercyclical since the value of service flows rises in bad times, resulting in a negative β asset. Second, it also resolves the "Government Debt Risk Premium Puzzle" (Jiang et al., 2020), the puzzle that government debt appears to insure simultaneously bond holders and taxpayers whereas in standard models, it can insure only one of the two groups. Our analysis shows that sufficiently large service flows can make the bond a negative β-asset, a good hedge for bond holders, while primary surpluses are procyclical at the same time, thus providing insurance for taxpayers.

One strand of the safe asset literature points to high market liquidity as the key feature of safe assets. In contrast, we focus on the risk properties of safe assets and fully abstract from any trading costs or asymmetric information frictions. Dang et al. (2015) emphasize the information insensitivity of safe assets. In Gorton and Pennachi (1990), Dang et al. (2017), and Greenwood et al. (2016) intermediaries create information insensitive assets. Trading frictions are also central in the New Monetarist literature and arise there from decentralized exchange (see Lagos et al. (2017) for a survey). Closest to the previously mentioned papers is Rocheteau (2011), wherein the risk-free asset is informationally insensitive and used first in bilateral trades.[4]

Several other papers are loosely related in that they are also about safe assets, but they make different conceptual points and study distinct model environments. He et al. (2019) model a safe asset tautology within a generalized global games setting. Caballero et al. (2017) and Caballero and Farhi (2017) stress the importance of safe asset shortage in zero lower bound episodes. Brunnermeier et al. (2017, 2016) point out that an asymmetric supply of safe assets across countries can lead to eruptive flight-to-safety cross-border capital flows. Brunnermeier et al. (2021b) discuss the loss of safe asset status in the context of an international framework for emerging market economies.

Like us, Constantinides and Duffie (1996) show how variation in idiosyncratic risk exposures in an incomplete markets setting can resolve several asset pricing puzzles. Unlike our paper, they abstract from asset trading and focus exclusively on the no-bubble equilibrium, while in our theory re-trading of the safe asset is key and can generate bubbles. Krueger and Lustig (2010) provide conditions under which aggregate risk premia are unaffected by uninsurable idiosyncratic risk in a model in which bonds are not traded. Heaton and Lucas (1996, 2000a,b) are prominent papers quantifying incomplete market models that include trading costs and the importance of entrepreneurial risk. In our analysis traded equity exhibits excess volatility and predictability as, in recessions, idiosyncratic risk rises, which drives up the risk compensation for inside equity and leaves fewer cash flows for outside (publicly traded) equity holders.

2 Model

2.1 Model Setup

Environment. The model is set in continuous time with an infinite horizon.

There is a continuum of households indexed by $i \in [0, 1]$. All households have identical logarithmic preferences

$$V_0^i := \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right]$$

[4]While not about safe assets, Vayanos and Vila (1999) represents an important early contribution that studies asset pricing in general equilibrium with (reduced-form) trading costs.

with discount rate $\rho > 0$.[5]

Each agent operates one firm that produces an output flow $a_t k_t^i dt$, where k_t^i is the capital input chosen by the firm and a_t is an exogenous productivity process that is common for all agents. Capital of firm i evolves according to

$$\frac{dk_t^i}{k_t^i} = \left(\Phi\left(\iota_t^i\right) - \delta\right) dt + \tilde{\sigma}_t d\tilde{Z}_t^i + d\Delta_t^{k,i}, \tag{1}$$

where Φ is an increasing concave function that captures adjustment costs in capital accumulation, ι_t^i the investment rate (in output goods) per unit of capital, δ is the depreciation rate, $\tilde{\sigma}_t d\tilde{Z}_t^i$ represents idiosyncratic Brownian shocks, and $d\Delta_t^{k,i}$ represents firm i's market transactions in physical capital. Brownian motions $\tilde{Z}^i$ are agent-specific and i.i.d. across agents. The levels of idiosyncratic risk $\tilde{\sigma}_t$ and productivity a_t are exogenous processes.

To obtain simple closed-form expressions, we choose the functional form $\Phi(\iota) = \frac{1}{\phi}\log\left(1 + \phi\iota\right)$ with adjustment cost parameter $\phi \geq 0$ for the investment technology.[6]

Each agent i can reduce idiosyncratic risk exposure by selling equity to other agents. Outside equity claims on i's capital k^i have the same aggregate and idiosyncratic risk as capital itself, but may pay a lower expected return, reflecting an insider premium that i earns for managing the capital stock. Agents can hold a diversified equity portfolio and thereby eliminate idiosyncratic risk.

The key friction in the model is that agents are unable to share idiosyncratic risk perfectly. Specifically, we assume that agents face a skin-in-the-game constraint and must retain at least a fraction $\bar{\chi} \in (0, 1]$ of their capital in undiversified form. As a consequence, agents have to bear the residual idiosyncratic risk of at least $\bar{\chi}\tilde{\sigma}_t d\tilde{Z}_t^i$ per unit of capital in their firms.

Besides this limit on idiosyncratic risk sharing, there are no further financial frictions. Agents are allowed to trade physical capital and contingent claims on aggregate risk subject to standard no Ponzi conditions.

In addition to households, there is a government that issues nominal government bonds, funds government spending, and imposes taxes on firms. Outstanding nominal

[5]In Appendix A.7, we present a generalization with Duffie and Epstein (1992) preferences (continuous time Epstein and Zin (1989) preferences).

[6]This function is defined for all $\iota \geq \underline{\iota} := -1/\phi$. We set $\Phi(\iota) := -\infty$ for $\iota < \underline{\iota}$.

government debt has a face value of $\mathcal{B}_t$ and pays nominal interest i_t. The face value follows a continuous process $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$ with growth rate $\mu_t^{\mathcal{B}}$. There is an exogenous need for real spending $\mathfrak{g} K_t dt$, where $K_t := \int k_t^i di$ is the aggregate capital stock and $\mathfrak{g}$ is a model parameter. The government can finance this spending by setting a proportional tax τ_t (subsidy if negative) on firms' output and by adjusting the bond issuance rate $\mu_t^{\mathcal{B}}$ (repurchasing bonds if negative). We assume that $\mu_t^{\mathcal{B}}$ and i_t are exogenous processes while taxes τ_t adjust to satisfy the nominal government budget constraint

$$i_t \mathcal{B}_t + \mathcal{P}_t \mathfrak{g} K_t = \mu_t^{\mathcal{B}} \mathcal{B}_t + \mathcal{P}_t \tau_t a_t K_t, \tag{2}$$

where $\mathcal{P}_t$ denotes the price level.

We assume that the exogenous processes a_t, $\tilde{\sigma}_t$, $\mu_t^{\mathcal{B}}$, and i_t follow a joint Markov diffusion process that is driven by some Brownian motion Z_t, which captures aggregate risk and is independent of all idiosyncratic Brownian motions $\tilde{Z}_t^i$.

Finally, the aggregate resource constraint is

$$C_t + \mathfrak{g} K_t + \iota_t K_t = a_t K_t, \tag{3}$$

where $C_t := \int c_t^i di$ is aggregate consumption and $\iota_t = \int \iota_t^i k_t^i / K_t di$ is the average investment rate.[7]

Household Problem. We formulate the household problem as a standard consumption-portfolio-choice problem that does not make explicit reference to the capital trading process $d\Delta_t^{k,i}$ as a choice variable. For this purpose, denote by n_t^i the net worth of household i and let $\theta_t^{K,i}$, $\theta_t^{E,i}$, $\theta_t^{\bar{E},i}$ be the fraction of net worth invested into capital, own outside equity, and the diversified portfolio of equity, respectively. The own outside equity share $\theta_t^{E,i}$ is typically negative as this asset is issued by the household. The

[7] To ensure that this constraint can be satisfied given our assumptions on government spending and investment, we assume that the productivity process satisfies $a_t > \mathfrak{g} + \underline{\iota}$ at all times. Here, $\underline{\iota}$ is as in footnote 6.

remaining fraction of net worth is invested in bonds. Net worth evolves according to

$$\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + \underbrace{dr_t^{\mathcal{B}} + \theta_t^{K,i}\left(dr_t^{K,i}\left(\iota_t^i\right) - dr_t^{\mathcal{B}}\right) + \theta_t^{E,i}\left(dr_t^{E,i} - dr_t^{\mathcal{B}}\right) + \theta_t^{\bar{E},i}\left(d\bar{r}_t^{E} - dr_t^{\mathcal{B}}\right)}_{=:dr_t^{n,i}}. \quad (4)$$

where $dr_t^{\mathcal{B}}$, $dr_t^{K,i}(\cdot)$, $dr_t^{E,i}$ and $d\bar{r}_t^{E}$ denote the returns on bonds, capital, own outside equity and the diversified equity portfolio, respectively. These returns depend on the evolution of market prices, which individuals take as given.[8] We provide explicit expressions for them below.

The household chooses consumption c_t^i, real investment ι_t^i, and the portfolio shares $\theta_t^{k,i}$, $\theta_t^{E,i}$, and $\theta_t^{\bar{E},i}$ to maximize utility V_0^i subject to (4), the skin-in-the-game constraint

$$-\theta_t^{E,i} \leq (1-\bar{\chi})\theta_t^{K,i}, \quad (5)$$

and a solvency constraint $n_t^i \geq 0$ that rules out Ponzi schemes.

Prices and Returns. Here we formalize how market prices determine returns in (4). We denote by q_t^K the market price of a single unit of physical capital. Recall that $\mathcal{P}_t$ denotes the nominal price level, so that the real value of a single unit of bonds is $1/\mathcal{P}_t$. It is more convenient to work with $q_t^B := \frac{\mathcal{B}_t/\mathcal{P}_t}{K_t}$, which is the ratio of the real value of government debt to total capital in the economy.[9]

With these definitions, the return on bonds is[10]

$$dr_t^{\mathcal{B}} = i_t dt + \frac{d\left(1/\mathcal{P}_t\right)}{1/\mathcal{P}_t} = i_t dt + \frac{d\left(q_t^B K_t/\mathcal{B}_t\right)}{q_t^B K_t/\mathcal{B}_t} = \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} - \underbrace{\left(\mu_t^{\mathcal{B}} - i_t\right)}_{=:\check{\mu}_t^{\mathcal{B}}} dt \quad (6)$$

[8]For the capital return, individuals take the function $\iota^i \mapsto dr_t^{K,i}(\iota^i)$ as given but do internalize how their own investment choice affects the capital return.

[9]This is a normalized version of the inverse price level $1/\mathcal{P}_t$. While the latter depends on the scale of the economy and the nominal quantity of outstanding bonds in equilibrium, q_t^B turns out to be stationary.

[10]The last equality uses $d\mathcal{B}_t = \mu_t^{\mathcal{B}}\mathcal{B}_t dt$.

and the return on agent i's capital is

$$dr_t^{K,i}\left(\iota_t^i\right) = \frac{(1-\tau_t)\,a_t - \iota_t^i}{q_t^K}dt + \frac{d(q_t^K\tilde{k}_t^i)}{q_t^K\tilde{k}_t^i},$$

where $\tilde{k}_t^i$ follows the same evolution (1) but with trades $d\Delta_t^{k,i}$ set to 0.

In addition to bonds and capital, there are outside equity claims in zero net supply. Outside equity claims issued by household i have the same risk characteristics as the capital return $dr_t^{K,i}$ but may have a different expected return. A return differential may exists because the inside equity holder i requires a compensation for bearing idiosyncratic risk whereas outside equity holders can eliminate this risk by diversification. Specifically,

$$dr_t^{E,i} = \mathbb{E}_t[dr_t^{E,i}] + \left(dr_t^{K,i} - \mathbb{E}_t[dr_t^{K,i}]\right),$$

where the expected return component $\mathbb{E}_t[dr_t^{E,i}]$ is determined in equilibrium. In equilibrium, all agents optimally hold a perfectly diversified equity portfolio. The return on that portfolio is

$$d\bar{r}_t^E = \int dr_t^{E,i}di.$$

Equilibrium. A competitive equilibrium, given an exogenous government policy and initial conditions, is defined in the usual way as a set of allocations and prices such that all households maximize utility and all markets clear. Here, prices and aggregate variables may depend only on aggregate exogenous histories, that is histories of the aggregate exogenous shocks dZ_t. In contrast, individual outcomes for household i can depend on both aggregate and individual idiosyncratic histories, that is joint histories of the shocks dZ_t and $d\tilde{Z}_t^i$.

Formally, we define equilibrium as follows. For ease of exposition, we restrict attention here to a *symmetric* equilibrium in which the expected return $\mathbb{E}_t[dr_t^{E,i}]$ of all outside equity claims is the same and in which all agents make identical choices for scaled consumption $\hat{c}_t^i := c_t^i/n_t^i$, the investment rate ι_t^i, and portfolio weights $\theta_t^{K,i}$, $\theta_t^{E,i}$, $\theta_t^{\bar{E},i}$.[11] This does not mean that all agents are identical because they can differ with regard to the level of their net worth n_t^i.

Definition 1. *Let $K_0 > 0$ be the initial level of capital and let a_t, $\tilde{\sigma}_t$, $\breve{\mu}_t^{\mathcal{B}}$ be exogenous processes*

[11] The restriction to symmetric equilibria is without loss of generality. In our environment, any equilibrium must be symmetric because agents face identical investment opportunities and utility is isoelastic.

adapted to the filtration generated by Z. A symmetric competitive equilibrium *consists of processes for prices* $\left(q_t^B, q_t^K, \mathbb{E}_t[dr_t^E]\right)_{t\geq 0}$*, scaled consumption* $(\hat{c}_t)_{t\geq 0}$*, investment rates* $(\iota_t)_{t\geq 0}$*, portfolio weights* $(\theta_t^K, \theta_t^E, \theta_t^{\bar{E}})_{t\geq 0}$*, taxes* $(\tau_t)_{t\geq 0}$*, and aggregate capital* $(K_t)_{t\geq 0}$*, all adapted to the filtration generated by Z, such that*

1. *Aggregate capital is consistent with the initial condition and satisfies*[12]

$$dK_t = \left(\Phi\left(\iota_t\right) - \delta\right) K_t dt$$

2. *Taxes satisfy the government budget constraint*[13]

$$\tau_t a_t K_t + \breve{\mu}_t^{\mathcal{B}} q_t^B K_t = \mathfrak{g} K_t$$

3. *For each household* $i \in [0,1]$, $c_t^i = \hat{c}_t n_t^i$, $\iota_t^i = \iota_t$, $\theta_t^{K,i} = \theta_t^K$, $\theta_t^{E,i} = \theta_t^E$, $\theta_t^{\bar{E},i} = \theta_t^{\bar{E}}$ *solves i's optimization problem (described previously) for arbitrary* n_0^i *under the assumption that* $\mathbb{E}[dr_t^{E,j}] = \mathbb{E}[dr_t^E]$ *for all* $j \in [0,1]$.

4. *All markets clear:*

   - *goods market clearing: equation* (3) *holds;*
   - *asset market clearing:*[14]

$$\theta_t^K = \frac{q_t^K K_t}{(q_t^K + q_t^B) K_t}, \qquad \theta_t^E + \theta_t^{\bar{E}} = 0.$$

We remark here that we have eliminated nominal quantities from this equilibrium definition by not explicitly adding $\mathcal{B}_0 > 0$ to the initial conditions, by using the scaled bond value q_t^B instead of the nominal price level $\mathcal{P}_t$, and by exogenously specifying government policy in terms of the differential $\breve{\mu}_t^{\mathcal{B}} = \mu_t^{\mathcal{B}} - i_t$. This is convenient because $\mathcal{P}_t$ and $\mathcal{B}_t$ do not enter any decision problem directly. If one is interested in nominal variables, one can easily recover them ex post from any given equilibrium together with a specification for $\mathcal{B}_0$ and for either $\mu_t^{\mathcal{B}}$ or i_t.

[12]This equation follows immediately from the individual capital evolutions (1) and the fact that idiosyncratic shocks and trading average out.

[13]This equation follows from (2) by dividing by $\mathcal{P}_t$ and combining $\mu_t^{\mathcal{B}}$ and i_t.

[14]These are for the capital and equity markets. The bond market clears by Walras' law.

We also remark that we have only defined the equilibrium in terms aggregate variables. However, for any given equilibrium and any given initial net worth distribution $(n_0^i)_{i\in[0,1]}$ consistent with it, i.e. $\int n_0^i di = (q_0^B + q_0^K)K_0$, we can recover individual variables from condition 3 in Definition 1 as this condition requires that choices are optimal for arbitrary initial net worth.[15]

2.2 Model Solution

The key equilibrium quantity is the share of total wealth invested in bonds,

$$\vartheta_t = \frac{\mathcal{B}_t/\mathcal{P}_t}{\mathcal{B}_t/\mathcal{P}_t + q_t^K K_t} = \frac{q_t^B}{q_t^B + q_t^K}.$$

Ultimately, results in this paper are driven by individual demand for safe assets, which is directly related to ϑ_t in equilibrium. In this section, we first establish that all equilibrium quantities of interest can be written in terms of ϑ_t and exogenous variables. In a second step, we characterize the evolution of ϑ_t.

The first step only relies on the optimal consumption and investment choices of households and on goods market clearing. We solve the household problem in Appendix A.1 using the stochastic maximum principle. Here, we merely report the main conclusions. The optimal consumption and investment choice are determined by the two conditions

$$c_t^i = \rho n_t^i,$$
$$q_t^K = \frac{1}{\Phi'(\iota_t^i)} = 1 + \phi \iota_t^i.$$

The first line is the familiar permanent income consumption equation for log preferences. The second line is a Tobin's q condition for physical investment in the presence of capital adjustment costs. The second equality in that line follows from the functional form assumption $\Phi(\iota) = \frac{1}{\phi}\log(1 + \phi\iota)$. Because all households face the same capital price q_t^K, they all choose the same investment rate ι_t^i, so that we may drop the i superscript from now on.

Aggregating the first condition across all agents i and combining the two equations

[15] The individual variables are, of course, dependent on histories of the idiosyncratic shocks $d\tilde{Z}_t^i$ as well, that is these processes for agent i are adapted to the filtration generated by the pair $(Z, \tilde{Z}^i)$.

with goods market clearing (3) and the definition of ϑ_t immediately implies the following lemma.

Lemma 1. *In any equilibrium, the investment rate, (scaled) value of government bond, and price of physical capital are given by*

$$\iota_t = \frac{(1-\vartheta_t)(a_t - \mathfrak{g}) - \rho}{1 - \vartheta_t + \phi\rho}, \tag{7}$$

$$q_t^B = \vartheta_t \frac{1 + \phi (a_t - \mathfrak{g})}{1 - \vartheta_t + \phi\rho}, \tag{8}$$

$$q_t^K = (1-\vartheta_t) \frac{1 + \phi (a_t - \mathfrak{g})}{1 - \vartheta_t + \phi\rho}. \tag{9}$$

These equations determine the equilibrium uniquely as a function of the exogenous process a_t and the (endogenous) bond wealth share ϑ_t. To fully characterize the equilibrium, we thus only need to determine ϑ_t.[16]

ϑ_t can be thought of as a relative price between capital assets (including equity which is a claim to capital) and government bonds. It is determined by households' portfolio choice and asset market clearing. The portfolio choice conditions for $\theta_t^{K,i}$, $\theta_t^{E,i}$, and $\theta_t^{\bar{E},i}$ take the form of standard Merton-type portfolio conditions that ensure that for the optimal portfolio, each asset's expected excess return over $\frac{\mathbb{E}_t[dr_t^{\mathcal{B}}]}{dt}$ is equalized to the risk premium that the agent requires to be willing to hold the aggregate and idiosyncratic risk associated with the asset.[17] We present the formal equations in Appendix A.1.

There, we also show that by combining these portfolio choice conditions and using asset market clearing to eliminate the individual portfolio weights $\theta_t^{K,i}$, $\theta_t^{E,i}$, $\theta_t^{\bar{E},i}$, we can reduce them to a single equation for the expected change in ϑ_t:

Proposition 1. *In any equilibrium, ϑ must satisfy the equation*

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - (1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2\right) \vartheta_t dt. \tag{10}$$

[16] Equilibrium objects from Definition 1 not covered by this lemma are $\hat{c}$, $\mathbb{E}[dr^K]$, τ, K and the portfolio weights. However, $\hat{c}$ follows from the optimal consumption rule, τ from the government budget constraint and K from the capital evolution given ι. We show in Appendix A.1 how $\mathbb{E}[dr^K]$ and the portfolio weights depend on ϑ and exogenous quantities.

[17] In the case of capital and outside equity, there is also a Lagrange multiplier term related to the skin-in-the-game constraint (5). In the problem considered here, the constraint is always binding, so that households issue the maximum amount of outside equity consistent with it.

Conversely, any $[0,1]$-valued solution ϑ to this equation is associated with a competitive equilibrium.

Equation (10) is a backward stochastic differential equation (BSDE) for ϑ_t. It characterizes all possible stochastic processes for the bond wealth share ϑ_t ($\approx$ relative price between bonds and capital) that are consistent with household portfolio choice and market clearing. Together with a specification for the evolution of the exogenous states $\tilde{\sigma}_t$ and a_t and for policy $\breve{\mu}_t^{\mathcal{B}}$, equation (10) determines the equilibrium process for ϑ_t. Equations (7), (8), (9) and goods market clearing (3) can then be used to back out the remaining quantities of interest.

2.3 Uniqueness of Stationary Monetary Equilibria

In our model, equilibria as defined in Definition 1 may not be unique because equation (10) for ϑ is a fixed-point equation that can have multiple solutions. However, we establish here that if we make suitable Markov assumptions on exogenous processes, then there is always at most one solution to equation (10) that is both *non-degenerate*, i.e. different from $\vartheta \equiv 0$, and *stationary*. This solution corresponds to a unique stationary "monetary" equilibrium in which government bonds have always a positive value. This equilibrium has a particularly appealing property: it is the only equilibrium consistent with any belief that, with positive probability, bonds retain a positive value bounded away from zero in the arbitrarily distant future. In this paper, we always focus on this equilibrium with the exception of Section 5, where we briefly discuss how alternative equilibria may be associated with a loss of safe asset status.[18]

We next formulate our uniqueness result. To do so, we first need to provide a precise definition of the notion of stationarity we are requiring.

Definition 2. *The exogenous processes $\tilde{\sigma}_t$, a_t, and $\breve{\mu}_t^{\mathcal{B}}$ are* stationary *if there exists an ergodic Markov state process X_t on a compact domain $\mathbb{X} \subset \mathbb{R}^n$ and continuous functions $\tilde{\sigma}, a, \breve{\mu}^{\mathcal{B}} : \mathbb{X} \to \mathbb{R}$ such that*

$$\tilde{\sigma}_t = \tilde{\sigma}(X_t), \qquad a_t = a(X_t), \qquad \breve{\mu}_t^{\mathcal{B}} = \breve{\mu}^{\mathcal{B}}(X_t)$$

[18] We make two further remarks with regard to this equilibrium selection. First, the selection of the unique stationary monetary equilibrium is in complete analogy to the standard choice in models with monetary frictions. Second, under the fiscal policy arrangements discussed in Brunnermeier et al. (2021a), this equilibrium would emerge as the unique equilibrium in the sense of Definition 1.

for all $t \geq 0$.[19]

Given stationary exogenous process, we say that a solution ϑ_t *to BSDE* (10) *is* stationary *if there is a continuous function* $\vartheta : \mathbb{X} \to [0,1]$ *such that* $\vartheta_t = \vartheta(X_t)$ *for all* t.

Proposition 2 (Uniqueness of stationary non-degenerate solutions)**.** *Suppose the exogenous processes are stationary and* $\rho + \breve{\mu}^{\mathcal{B}}(X) > 0$ *for all* $X \in \mathbb{X}$*. Then, equation* (10) *has at most one stationary nondegenerate (i.e. not identically 0) solution.*

We prove Proposition 2 in Appendix A.2. The key idea behind the proof is to investigate the finite-horizon version of BSDE (10) and show two key properties. First, the mapping from terminal conditions $\vartheta_T > 0$ to the (always unique) finite-horizon solution over $[0, T]$ represents a contraction in a suitable sense. Second, conditional on a fixed state X_t, the finite-horizon solutions are monotonic in time t. We establish these properties with the help of the comparison theorem for BSDEs (e.g. Pham (2009, Theorem 6.2.2)).

In fact, these properties do not only allow us to establish uniqueness of the non-degenerate stationary solution but also yield an additional limit result: for any given terminal condition $\vartheta_T > 0$, the solution to the finite-horizon equation converges to the unique non-degenerate stationary solution as $T \to \infty$, provided the latter exists. Conceptually, this additional result is important because it implies that all nonstationary solutions must converge to 0 in the distant future. Practically, the additional result matters because it ensures that the solution procedure that we employ for the numerical illustration in Section 4 converges to the desired solution.

2.4 Closed-Form Steady State

We provide a brief characterization of the model's steady state.[20] To do so, we assume that productivity a, idiosyncratic risk $\tilde{\sigma}$, and policy $\breve{\mu}^{\mathcal{B}}$ are constant and look for a constant solution $\vartheta > 0$ to equation (10). By Lemma 1, also q^B, q^K, and ι must be constant in a steady state.

Imposing $d\vartheta_t = 0$ in equation (10) leads to a third-order polynomial equation which-has three (mathematical) solutions: $\vartheta = 0$, $\vartheta = \frac{\tilde{\chi}\tilde{\sigma} + \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\tilde{\chi}\tilde{\sigma}}$, and $\vartheta = \frac{\tilde{\chi}\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\tilde{\chi}\tilde{\sigma}}$. Among

[19]Informally, an ergodic Markov process can travel from any state to any other state.

[20]The "steady state" is in fact a balanced growth path. In our AK-type model, there is always a growth trend in the capital stock K_t.

these solutions, at most the third can satisfy the additional restriction $\vartheta \in (0,1)$. It satisfies this restriction, and is then consistent with a monetary stationary equilibrium, if and only if the condition

$$\bar{\chi}\tilde{\sigma} \geq \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}$$

is satisfied. Effectively, this inequality imposes a constraint on bond growth in excess of interest payments $\breve{\mu}^{\mathcal{B}}$, a measure of dilution of existing bonds. Dilution $\breve{\mu}^{\mathcal{B}}$ cannot be too large for the private sector to remain willing to hold government bonds. The higher is the residual idiosyncratic risk $\bar{\chi}\tilde{\sigma}$ that agents have to bear after optimal outside equity issuance, the less restrictive is this constraint.

If this condition is satisfied, investment is

$$\iota = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(a - \mathfrak{g}) - \rho\bar{\chi}\tilde{\sigma}}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\bar{\chi}\tilde{\sigma}}$$

and the (scaled) real asset values are

$$q^{B} = \frac{\left(\bar{\chi}\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right)\left(1 + \phi\,(a - \mathfrak{g})\right)}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\bar{\chi}\tilde{\sigma}}, \qquad q^{K} = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\left(1 + \phi\,(a - \mathfrak{g})\right)}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\bar{\chi}\tilde{\sigma}}.$$

These closed-form solutions yield straightforward conclusions regarding the impact of parameter changes on equilibrium outcomes. We emphasize here explicitly that capital valuations and investment are strictly decreasing while bond valuation are strictly increasing in idiosyncratic risk $\tilde{\sigma}$. This is because an increase in idiosyncratic risk leads to a portfolio reallocation from capital assets to government bonds as can be readily seen from equation (10). This same force also plays an important role in the flight-to-safety dynamics that we emphasize in Section 4.

2.5 Safe Asset Definition

Individuals hold a safe asset for precautionary reasons, which they can "liquify" at an above average return when they face an idiosyncratic and/or aggregate shock and they attach a high marginal value to extra resources. This marginal value for individual i is measured by that individual's stochastic discount factor (SDF) process, which we denote by ξ^i_t. This process satisfies $\xi^i_0 = 1$ and $d\xi^i_t/\xi^i_t = -r^f_t\,dt - \varsigma_t dZ_t - \tilde{\varsigma}^i_t d\tilde{Z}^i_t$, with a negative drift term equal to the risk-free rate and aggregate and idiosyncratic prices of risk, ς_t, $\tilde{\varsigma}^i_t$ respectively.[21] The return of citizen i's net worth, $r^{n,i}$ is given by equation (4).

The following definition makes the "Good Friend Analogy" of a safe asset precise.

Definition 3. *An asset j is in equilibrium a* safe asset *for individual i at time t if the conditional covariance between her SDF and return of the asset in excess to her net worth return, $dr^j_t - dr^{n,i}_t$, is positive, i.e.* $\mathrm{Cov}_t[d\xi^i_t/\xi^i_t, dr^j_t - dr^{n,i}_t] > 0$.

We make several remarks. First, the safe asset concept is an equilibrium concept. The same asset with the same cash flows can be a safe asset in one equilibrium and not a safe asset in another equilibrium. The returns r^j_t and $r^{n,i}_t$ as well as the SDF ξ^i_t depend on the equilibrium the economy is in.

Second, an asset is safe for individual i relative to her own net worth n^i. A safe asset provides (on average) better payoffs in high marginal utility states than the total net worth portfolio.[22]

Third, the definition is contingent on the economy's state. An asset can be safe at a specific point in time t but may lose its "safe-asset status" at a different time.

Fourth, the definition focuses exclusively on the risk properties of the asset. This is sufficient because, in our model, we have assumed that all assets enjoy perfect market liquidity (all the time). However, as we explain in the next section, safety derives from retrading. Therefore, liquidity is also an important feature of a safe asset. Beyond our model environment, a definition of a safe asset should also include that the asset can be traded easily without any trading frictions, e.g. due to asymmetric information.

The following proposition shows that government bonds are indeed a safe asset in the steady-state solution derived in Section 2.4.

Proposition 3. *If $\tilde{\sigma}_t > 0$, a_t, and $\check{\mu}^{\mathcal{B}}_t$ are constant over time, the government bond is a safe asset for all agents at all times.*

[21] In integral form the individual SDF is

$$\xi^i_t = \underbrace{\exp\left(-\int_0^t r^f_\tau d\tau\right)}_{\text{time discounting}} \cdot \underbrace{\exp\left(-\int_0^t \varsigma_t dZ_\tau - \frac{1}{2}\int_0^t \varsigma^2_\tau d\tau\right)}_{\text{aggregate risk}} \cdot \underbrace{\exp\left(-\int_0^t \tilde{\varsigma}_\tau d\tilde{Z}^i_\tau - \frac{1}{2}\int_0^t \tilde{\varsigma}^2_\tau d\tau\right)}_{\text{idiosyncratic risk}},$$

where the second and third factors are martingales.

[22] Alternatively, one could define an asset as (absolutely) safe by $\mathrm{Cov}_t[d\xi^i_t/\xi^i_t, dr^j_t] \geq 0$.

In equilibrium each citizen's net worth return is positively correlated with her consumption growth rate and hence negatively correlated with her SDF, as the incomplete markets frictions prevents her from hedging her idiosyncratic risk. Since the government bond return is risk-free, $\mathrm{Cov}_t[d\xi_t^i/\xi_t^i, r^f - dr_t^{n,i}] > 0$ and consequently the government bond is a safe asset. Capital is not a safe asset as $\mathrm{Cov}_t[d\xi_t^i/\xi_t^i, dr_t^K] \leq \mathrm{Cov}_t[d\xi_t^i/\xi_t^i, dr_t^{n,i}]$.

More generally, with stochastic idiosyncratic risk $\tilde{\sigma}_t$, the real return of the government bond is not risk-free. However, as we will see in Section 4 in the context of our calibrated model, it depreciates in volatile times less than citizen's net worth. Indeed it even appreciates, and hence more than satisfies the safe asset criterion.

3 Two Perspectives on Asset Valuation Equations

The value of government debt has to satisfy a debt valuation equation that relates the real value of debt to the present value of future primary surpluses. In this section, we contrast the standard asset pricing approach of deriving such an equation with an alternative approach that emphasizes and makes explicit the benefits from retrading of the asset. Both approaches imply an identical valuation formula with complete markets, but lead to two distinct equations when markets are incomplete. These equations provide two different perspectives for asset pricing in incomplete market environments. Here, we apply these perspectives to government debt valuation.[23]

The standard approach to asset valuation is based on a buy-and-hold fiction. An asset is priced as if it was held forever, so that the value of the asset must equal the present value of all future cash flows derived from the asset's payouts.[24] We call this the *"buy and hold perspective"* of asset pricing. Applied to the total government debt stock, the cash flows in the present value formula are precisely the primary surpluses.[25]

We propose an alternative approach that recognized that, in equilibrium, individ-

[23]For concreteness, we present the equations only for government debt in the context of our model. But we remark that our alternative valuation approach, the "dynamic trading perspective", is general and can be applied in any incomplete markets setting to any asset.

[24]To be fully precise, the buy-and-hold fiction does assume an eventual liquidation of the asset which results in a single terminal resale cash flow. The liquidation time is then sent to infinity, so that the present value of this cash flow only matters if there is a bubble.

[25]Despite the label, this may require the agent to trade, but only directly with the issuer, the government, in order to absorb new debt issuance, not with other agents.

ual agents may not intend to buy and hold an asset, but plan to retrade it whenever they face a shock. They raise cash flow by selling the asset and face a cash outflow when buying more of the asset. The aggregate stock of the asset can also be priced by first valuing the cash flows from agents' optimal dynamic trading strategy in equilibrium and then aggregating across all agents. This approach leads to a *"dynamic trading perspective"* of asset pricing. Importantly, the aggregated present value of individual trades may be different from zero, even though trades among private agents wash out in the aggregate. Hence, the dynamic trading perspective incorporates an additional term that makes explicit the aggregate value of equilibrium trades. Applied to government bonds in our model, this term is positive because bonds allow agents to self-insure against idiosyncratic shocks. We refer to this term as the "service flow" term from retrading.

The distinction between the two perspectives is particularly illuminating when there can be rational bubbles. Dynamic programming implies that a transversality condition has to hold only from the dynamic trading perspective, for each individual agent. Optimality does not imply a transversality condition from the buy and hold perspective. For that reason, a gap may appear between the value of debt and the present value of surpluses from the latter perspective. This gap is closed by an additional bubble term.

Unfortunately, it can even happen that both the bubble term and the present value of primary surpluses are infinite with opposite sign, yet their sum still converges as the time horizon approaches infinity. In contrast, the terms in the dynamic trading perspective are always well-defined and finite.

Buy and Hold Perspective. We denote by $s_t := \tau_t a_t - \mathfrak{g}$ the primary surplus per unit of aggregate capital. Recall that ξ_t^i is agent i's SDF process. From the buy and hold perspective, individual uninsurable risk does not enter the valuation equation directly, so that only the aggregate component $\bar{\xi}_t$ of the processes ξ_t^i matters, i.e. $d\bar{\xi}_t/\bar{\xi}_t = -r_t^f dt - \varsigma_t dZ_t$.[26] Absent aggregate shocks (including inflation shocks), the government bond is a risk-free asset and the relevant discount factor is simply $\bar{\xi}_t = \exp(-\int_0^t r_\tau^f d\tau)$.

[26]The aggregate discount factor is the projection of any individual citizen's SDF onto a common filtration generated by the aggregate Brownian Z. Put differently, $\bar{\xi}_t := \mathbb{E}\left[\xi_t^i \mid Z_\tau : \tau \le t\right]$, takes conditional expectations with respect to the history of aggregate shocks dZ_τ up to time t but without any knowledge of idiosyncratic shocks. Equivalently, $\bar{\xi}_t = \int \xi_t^i di$ is the unweighted average of individual SDFs.

Proposition 4 (Buy and Hold Perspective)**.** *The value of government debt at $t = 0$ satisfies*

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T\to\infty}\left(\mathbb{E}\left[\int_0^T \bar{\xi}_t s_t K_t dt\right] + \mathbb{E}\left[\bar{\xi}_T \frac{\mathcal{B}_T}{\mathcal{P}_T}\right]\right). \tag{11}$$

This equation consists of two terms: a discounted stream of primary surpluses plus (the limit of) a discounted terminal value. The latter can be positive even in the limit, giving rise to a possible bubble on government debt.[27] The reason is that individual transversality conditions do not necessarily imply $\mathbb{E}\left[\bar{\xi}_T \frac{\mathcal{B}_T}{\mathcal{P}_T}\right] \to 0$ because agents do not buy and hold a fixed fraction of the government debt stock but constantly trade bonds. If the terminal condition does converge to zero, then we obtain the traditional debt valuation equation that says that the value of debt must equal the present value of primary surpluses.

The derivation of equation (11) is standard. Essentially, one multiplies the government's flow budget constraint by the SDF, iterates forward the resulting equation, and then takes expectations and the limit $T \to \infty$. We relegate the formal details to Appendix A.4.

Dynamic Trading Perspective. Let $\eta_t^i := n_t^i / N_t$ be agent i's net worth share and denote again i's SDF process by ξ_t^i. In our model, η_t^i also represents the share of total bonds held by i because all agents hold identical portfolios (up to scale). Pricing individual bond portfolios and aggregating over agents i yields our main valuation equation from the dynamic trading perspective,

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \int \left(\mathbb{E}\left[\int_0^\infty \xi_t^i \cdot \eta_t^i s_t K_t dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^i \cdot \eta_t^i \left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right]\right) di. \tag{12}$$

The real value of all outstanding public debt $\mathcal{B}_0/\mathcal{P}_0$ is the integral of the valuations of individual debt holdings. Each of these valuations consists of two terms, the discounted value of the share of future primary surpluses, $\eta_t^i s_t K_t := \eta_t^i(\tau_t a - \mathfrak{g})K_t$, paid out to agent i plus the discounted value of future service flows, $\eta_t^i \left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t}$, that agent i derives from trading bonds. The safe asset service flow is due to partial insurance. It increases in the value of public debt and the amount of idiosyncratic risk the citizen is exposed to. The latter depends on her portfolio share on physical capital

[27] The bubble term on government debt is discussed in detail in Brunnermeier et al. (2021a).

$(1-\vartheta_t)$ and undiversified risk $\bar{\chi}\tilde{\sigma}_t$. Government bonds provide a positive service flow because the agent sells bonds precisely when she experiences a negative idiosyncratic shock, so that the bond portfolio generates a positive payout in times of high marginal utility ξ_t^i.

Equation (12) emphasizes that the total value is obtained by aggregating individual portfolio valuations. Mathematically, it is more convenient to interchange the order of integration (Fubini's Theorem). This yield the following proposition.

Proposition 5 (Dynamic Trading Perspective)**.** *The value of government debt at $t = 0$ satisfies*

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \underbrace{\left(\int \xi_t^i \eta_t^i di\right)}_{=:\xi_t^{**}} s_t K_t dt\right] + \mathbb{E}\left[\int_0^\infty \underbrace{\left(\int \xi_t^i \eta_t^i di\right)}_{=:\xi_t^{**}} (1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right]. \tag{13}$$

This equation discounts aggregate cash flows (surpluses and service flows) free of idiosyncratic risk like equation (11) obtained from the buy and hold perspective. But importantly, the "stochastic discount factor" ξ_t^{**} in this equation is a net-worth-weighted average of individual stochastic discount factors. Since a single agent's individual net worth weight η_t^i co-moves negatively with her SDF ξ_t^i, the discount factor is lower (discount rate is higher) than the usual unweighted average discount factor (used in the buy and hold perspective). It turns out this weighted average SDF is not a mere mathematical artifact from swapping integrals but has an economic interpretation. It is the correct marginal rate of intertemporal substitution of aggregate cash flows for a pseudo-representative agent who is forced to distribute aggregate consumption to individuals according to the equilibrium consumption shares c_t^i/C_t in our model. We discuss this interpretation in more detail below.

To derive valuation equations (12) and (13), we start by valuing agent i's bond portfolio at time $t = 0$. Denote by $b_t^i := (1-\theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i})n_t^i$ the value of agent i's bond portfolio at time t and let $b_t^i d\Delta_t^{b,i}$ be the stochastic bond trading process, where

$$d\Delta_t^{b,i} = \mu_t^{\Delta,i} dt + \sigma_t^{\Delta,i} dZ_t + \tilde{\sigma}_t^{\Delta,i} d\tilde{Z}_t^i$$

denotes the proportional appreciation of b_t^i due to trading and payouts between t and $t + dt$. Under the optimal trading policy, the initial bond wealth b_0^i must equal the discounted value of future payouts (=outflows) from the bond portfolio,[28]

$$b_0^i = -\mathbb{E}\left[\int_0^\infty \xi_t^i b_t^i \left(\mu_t^{\Delta,i} - \varsigma_t \sigma_t^{\Delta,i} - \tilde{\varsigma}_t^i \tilde{\sigma}_t^{\Delta,i}\right) dt\right]. \tag{14}$$

As all agents hold the same constant fraction of their net worth in bonds ($\theta_t^i = \vartheta_t$), the value of the individual bond portfolio is simply the product of the agent's net worth share and aggregate bond wealth, $b_t^i = \eta_t^i q_t^B K_t$. In Appendix A.5, we show that the bond trading process satisfies

$$\mu_t^{\Delta,i} = -s_t / q_t^B, \qquad \sigma_t^{\Delta,i} = 0, \qquad \tilde{\sigma}_t^{\Delta,i} = (1 - \vartheta_t)\bar{\chi}\tilde{\sigma}_t. \tag{15}$$

The first equation says that the proportional reduction in the value of all agents' bond portfolios due to trading with the government equals the surplus-debt ratio s_t / q_t^B. This term captures cash flows from payouts, not from retrading among private agents. The second and third term capture such retrading in response to aggregate and idiosyncratic shocks, respectively. Here, agents do not trade in response to aggregate shocks as they are all exposed symmetrically, but agents do trade in response to idiosyncratic shocks: they sell capital and buy bonds when they receive a positive shock and vice versa. We also show in the appendix that the price of idiosyncratic risk satisfies $\tilde{\varsigma}_t^i = (1 - \vartheta_t)\bar{\chi}\tilde{\sigma}_t$, where the right-hand expression is the residual (proportional) idiosyncratic wealth risk that agents have to bear in equilibrium.

Combining the previous equations and using $q_t^B K_t = \mathcal{B}_t / \mathcal{P}_t$ leads to the individual valuation equation

$$\eta_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \xi_t^i \eta_t^i s_t K_t dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^i \eta_t^i \left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right]. \tag{16}$$

Finally, integrating over individuals i yields equation (12).

Comparison of the SDFs $\bar{\xi}_t$ and ξ_t^{**} The SDFs used in equations (11) and (13) are both free of idiosyncratic risk and imply the same aggregate risk premium, but they differ with respect to their average rate of decay, the "risk-free rate" they imply. The average SDF $\bar{\xi}$ decays at the equilibrium risk-free rate r_t^f. It is thus a proper SDF in this model that prices all assets free of idiosyncratic risk. The same is not true for the weighted

[28]A transversality condition always ensures that there is no additional nonvanishing terminal wealth term. We provide a formal derivation of this equation in Appendix A.5.

average SDF ξ_t^{**}. The latter decays at a rate $r_t^f + \tilde{\varsigma}_t \tilde{\sigma}_t^n$, where $\tilde{\sigma}_t^n$ is the idiosyncratic net worth volatility of agents (which is identical for all agents in equilibrium). The weighted average SDF ξ_t^{**} therefore discounts safe cash flows at a higher rate than the risk-free rate that contains a risk premium for idiosyncratic wealth risk. The reason for this is apparent from equation (12) which inverts the order of integration: while aggregate cash flows from bonds are free of idiosyncratic risk, each agent holds a stochastic share η_t^i of the aggregate bond portfolio so that individual bond portfolios do contain priced idiosyncratic risk.

These considerations imply that only equation (11) is a standard asset pricing formula, a discounted present value formula using a SDF that prices all assets (at least those free of idiosyncratic risk). But equation (11) can have a bubble and infinities with opposite sign. It can be more informative to work with equation (13) instead, as this equation makes the source of trading gains transparent. However, we need to keep in mind that this equation uses a SDF that does not (in general) price the assets in the economy correctly without additional service flow terms.

Relating the Dynamic Trading Perspective to a Representative Agent. The weighted-average SDF may not be a proper SDF that prices assets in the competitive equilibrium of our incomplete markets economy. Yet, it turns out to be the correct SDF of a hypothetical representative agent who is forced to distribute aggregate consumption according to the consumption shares that arise in our incomplete markets economy.

More precisely, consider a representative agent whose preferences are described by a weighted welfare function $\mathcal{W}_0 = \int \lambda^i V_0^i di$ with some (positive) welfare weights $(\lambda^i)_{i\in[0,1]}$. If we denote by $\eta_t^i := c_t^i / C_t$ the consumption share of agent i, we can write utility of this representative agent as

$$\mathcal{W}_0 = \mathbb{E}\left[\int_0^\infty e^{-\rho t} \int \lambda^i \log\left(\eta_t^i C_t\right) di dt\right]. \tag{17}$$

For any fixed stochastic processes for the consumption shares η_t^i, $\mathcal{W}_0$ describes standard time-separable preferences in *aggregate consumption* C_t with period utility function $C_t \mapsto \int \lambda^i u(\eta_t^i C_t) di$ (here, $u = \log$, but the following applies more generally).

Using the usual identification between an agent's SDF process and the marginal rate of substitution for consumption in different time periods, the SDF process Ξ_t of this hypothetical representative agent can be defined as

$$\Xi_t = e^{-\rho t} \frac{\int \lambda^i \eta_t^i u'(\eta_t^i C_t) di}{\int \lambda^i \eta_0^i u'(\eta_0^i C_0) di} = \frac{\int \lambda^i u'(c_0^i) \xi_t^i \eta_t^i di}{\int \lambda^i u'(c_0^i) \eta_0^i di},$$

where the second equality uses the definition $\xi_t^i = e^{-\rho t} \frac{u'(c_t^i)}{u'(c_0^i)}$ of the SDF process for agent i.

We are interested in the special case that the η_t^i-processes are the ones that arise in equilibrium in our incomplete markets economy. In this case, $d\eta_t^i = \tilde{\sigma}_t^\eta d\tilde{Z}_t^i$ with a common volatility process $\tilde{\sigma}_t^\eta$. As we show in Appendix A.6, Ξ_t is then independent of welfare weights λ^i, so that we can assume w.l.o.g. that $\lambda^i u'(c_0^i)$ is a constant independent of i.[29] This implies the following proposition.

Proposition 6. *The representative agent's SDF equals the weighted-average SDF,*

$$\Xi_t = \int \xi_t^i \eta_t^i di = \xi_t^{**}.$$

In Appendix A.6 we elaborate more on the connection to a representative agent. Specifically, we note that representative agent utility (17) can also be written in the form

$$\mathcal{W}_0 = w_0 + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^\eta\right)^2\right) dt\right] \tag{18}$$

with some constant w_0. Equation (18) eliminates the direct dependence on i and gives us the alternative interpretation that the representative agent forms preferences over two "goods", aggregate consumption and volatility. In Appendix A.6 we embed a continuum of agents with this utility function into a Lucas (1978) tree economy with two trees. Both trees produce bundles of consumption goods and volatility that resemble the aggregate and cross-sectional cash flows arising in our incomplete markets economy from trading capital and bonds, respectively. We show that asset prices and the consumption allocation in this hypothetical representative agent economy are the same as in our incomplete markets economy. In particular, the valuation equation for the "bond tree" for the representative agent resembles our valuation equation (13) from the dynamic trading perspective.

[29]The independence of the welfare weights holds trivially in our model with log utility, but we show that it holds also if u is a general CRRA utility function.

4 Counter-cyclical Safe Asset and Negative Beta

4.1 Setup for Numerical Illustration

In this section, we illustrate the dynamics of our model with aggregate risk in the context of a numerical example. To ensure that this example captures a quantitatively plausible situation, in particular with regard to the implications for government debt valuation, the example is based on a calibration. However, because the most important takeaways from this section are qualitative, not quantitative, we defer a description and justification of our calibration choices to Section 7.

We introduce aggregate risk as shocks to idiosyncratic risk $\tilde{\sigma}_t$. We specify a Heston (1993) model of stochastic volatility, i.e. we assume that the idiosyncratic variance $\tilde{\sigma}_t^2$ follows a Cox–Ingersoll–Ross process (Cox et al., 1985) process,

$$d\tilde{\sigma}_t^2 = -\psi \left(\tilde{\sigma}_t^2 - \left(\tilde{\sigma}^0 \right)^2 \right) dt - \sigma \tilde{\sigma}_t dZ_t \tag{19}$$

with parameters $\psi, \sigma, \tilde{\sigma}^0 > 0$.

We interpret periods of high idiosyncratic risk as recessions and want them to be associated with lower consumption and higher marginal utility. Rather than microfounding this relationship explicitly, we simply impose an exogenous relationship $a_t = a(\tilde{\sigma}_t)$ with $a'(\cdot) < 0$ that generates the desired correlation structure.[30]

For government policy, summarized by debt growth net of interest payments, $\breve{\mu}_t^{\mathcal{B}}$,[31] we similarly impose a functional relationship $\breve{\mu}_t^{\mathcal{B}} = \breve{\mu}^{\mathcal{B}}(\tilde{\sigma}_t)$ with $(\breve{\mu}^{\mathcal{B}})'(\cdot) > 0$. For sufficiently large $(\breve{\mu}^{\mathcal{B}})'$, this ensures that primary surpluses $s_t = -\breve{\mu}_t^{\mathcal{B}} q_t^B$ are positive for low idiosyncratic risk (in expansions) and negative for high idiosyncratic risk (in recessions). Primary surpluses therefore correlate negatively with marginal utility and any agent in the economy would require a positive risk premium for holding a (hypothetical) claim to primary surpluses, a feature that is empirically plausible (see, e.g., Jiang et al. (2019) for the US).

[30] For models similar to ours in which output and consumption naturally react negatively to risk shocks, see DiTella and Hall (2020) and Li and Merkel (2020).

[31] To be precise, the government also chooses the nominal interest rate i_t. However, in our flexible price model, this policy choice merely affects the equilibrium inflation rate but not real allocations or asset prices.

Imposing tight functional relationships between $\tilde{\sigma}_t$ and the other exogenous variables a_t and $\breve{\mu}_t^{\mathcal{B}}$ is somewhat stylized but allows us to keep the state space of our model one-dimensional. This is helpful to illustrate global dynamics.

We also remark here that, for our numerical example, we replace logarithmic preferences of households with stochastic differential utility (Duffie and Epstein, 1992) with relative risk aversion $\gamma > 1$, but we continue to assume a unit elasticity of intertemporal substitution (EIS). We elaborate more on the details in Section 7. This modification does not matter for the qualitative behavior of the model, but it allows us to generate quantitatively realistic aggregate risk premia. This is important to capture the full extent to which pro-cyclical primary surpluses reduce the value of the cash flow component.

4.2 Equilibrium Dynamics of Bond and Capital Values

Figure 1 illustrates the equilibrium dynamics of the value of the government bond stock q^B (blue line) and the value of the capital stock q^K (red line) per unit of capital in the economy by plotting these valuations as a function of the state variable $\tilde{\sigma}$. The gray shaded area depicts the stationary distribution of $\tilde{\sigma}$. q^B is strictly increasing in idiosyncratic risk whereas q^K is strictly decreasing. We can interpret this observation as flight to safety from capital to bonds in times of elevated idiosyncratic risk. We discuss implications for the pricing of (diversified) equity from flight to safety in Section 4.5 below.

Because output comoves negatively with $\tilde{\sigma}$ by construction, the monotonicity patterns of $q^B(\tilde{\sigma})$ and $q^K(\tilde{\sigma})$ imply that bond valuations are counter-cyclical whereas capital valuations are pro-cyclical. It is this counter-cyclical valuation that makes government bonds a good safe asset in the presence of aggregate risk. We analyze the source of the counter-cyclicality in the following subsection.

4.3 Analyzing the Two Bond Asset Pricing Terms Separately

We now consider the two terms in the government debt valuation equation derived from the dynamic trading perspective, equation (13). Figure 2 plots the two present

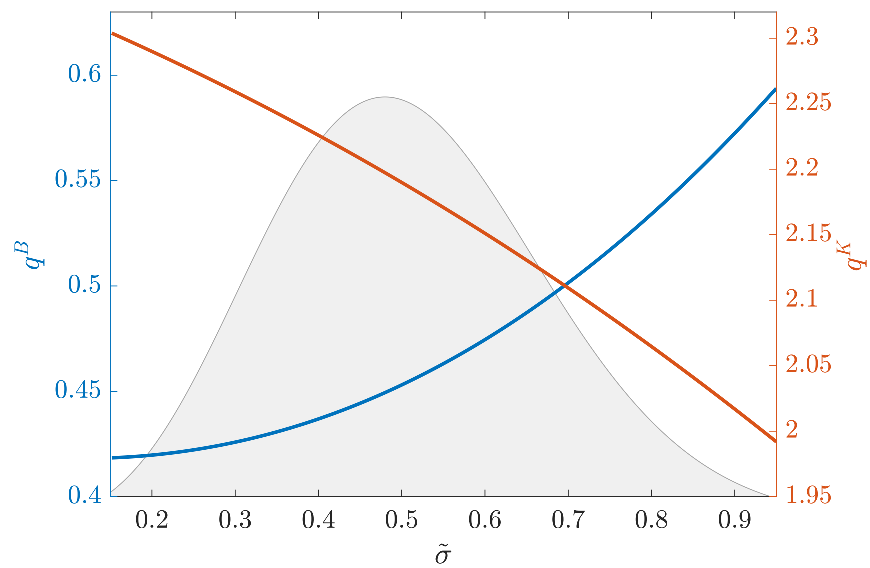

Figure 1: Equilibrium asset valuations q^B (blue line, left scale) and q^K (red line, right scale) as a function of idiosyncratic risk $\tilde{\sigma}$. The gray shaded area in the background depicts the (rescaled) ergodic density of the state variable $\tilde{\sigma}$.

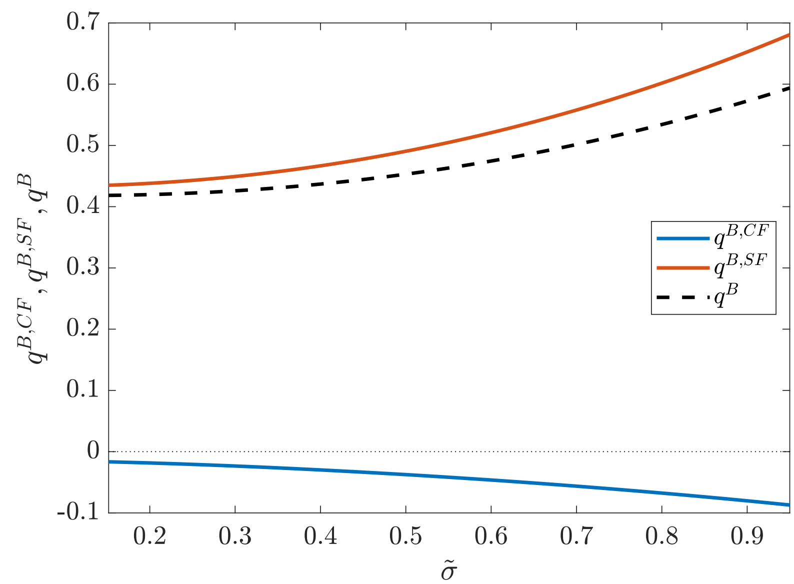

Figure 2: Decomposition of the value of government debt as a function of idiosyncratic risk $\tilde{\sigma}$. The blue solid line shows the present value of primary surpluses ($q^{B,CF}$), the red solid line the present value of service flows ($q^{B,SF}$) and the black dashed line the total value of government debt (q^B), all normalized by the capital stock.

values[32]

$$q^{B,CF}(\tilde{\sigma}) := \mathbb{E}\left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di\right) s_t K_t dt \mid \tilde{\sigma}_0 = \tilde{\sigma}, K_0\right] / K_0,$$
$$q^{B,SF}(\tilde{\sigma}) := \mathbb{E}\left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di\right) (1-\vartheta_t)^2 \gamma \tilde{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \mid \tilde{\sigma}_0 = \tilde{\sigma}, K_0\right] / K_0.$$

The blue solid line shows the present value of future primary surpluses (cash flows) $q^{B,CF}$ as a function of the single state variable $\tilde{\sigma}$. This value is strictly decreasing in idiosyncratic risk and has a low – in fact negative – value. Comparing the present value of surpluses $q^{B,CF}K$ in our model to the market value of government debt $q^B K$, which is represented by the black dashed line in Figure 2, reveals a large gap $(q^B - q^{B,CF})K$, a "debt valuation puzzle". In addition, when compared with the present value of surpluses $q^{B,CF}K$, the total value of government debt $q^B K$ has also the opposite correlation with the aggregate state. Yet, there is no puzzle from the perspective of our model: government debt is a safe asset valued for its service flow from re-trading which is represented by the component $q^{B,SF}(\tilde{\sigma})$. As the red solid line in Figure 2 shows, this value is positive, large and positively correlated with $\tilde{\sigma}_t$. This additional component dominates the overall dynamics of the value of government debt and is the reason that q^B appreciates in bad times despite the simultaneous drop in $q^{B,CF}$. That $q^{B,SF}$ must be positively correlated with $\tilde{\sigma}$ can also be seen from the present value equation: for our policy specification, residual net worth risk $(1-\vartheta_t)\tilde{\chi}\tilde{\sigma}_t$ is increasing in $\tilde{\sigma}_t$, so that an increase in idiosyncratic risk increases the value of insurance service flows from re-trading.[33]

The correlation structure apparent in Figure 2 implies that, if the two claims $q^{B,CF}$ and $q^{B,SF}$ could be traded separately, the cash flow claim would be a high-β asset, while the service flow claim would be a negative-β asset. The presence of this second, negative-β component makes government debt as a whole a negative β asset. Government debt emerges as a "good friend" also with respect to aggregate shocks.

[32] Relative to equation (13), here an additional factor γ appears because we no longer assume logarithmic preferences.

[33] This is not an entirely rigorous argument as it ignores changes in the discount rate. The effective discount rate in the weighted-average SDF $\int \xi_t^i \eta_t^i di$ can both increase or decrease with the aggregate state $\tilde{\sigma}_t$ depending on whether the *aggregate* risk premium increases or decreases. Note however, that the level of idiosyncratic risk does not directly matter for the effective discount rate because the risk premium on idiosyncratic risk exactly offsets the lower risk-free rate due to a precautionary motive.

We remark that, while the exact numbers in Figure 2 depend on our calibration, the broad qualitative pattern described in this section is fairly robust, so long as the calibration is consistent with the following two stylized facts about US primary surpluses: (1) the average primary surplus is close to zero or even slightly negative and (2) primary surpluses are pro-cyclical. Fact (1) implies that even a risk-free claim to the cash flows would have a non-positive value and, together with fact (2), the most likely outcome is a negative cash flow component that has a positive β. The model can then only generate a large positive total value of government debt if the service flow component dominates.

4.4 The Possibility of Insuring Bond Holders and Tax Payers at the Same Time

In our simple setting, households are both capital owners and bond holders. In this section, we conceptually separate each household into two sub-units, a capital owner and a government debt holder. Surprisingly, it is possible to follow a government policy that provides insurance against negative aggregate shocks for both tax payers and bond holders at the same time. By cutting taxes (or even granting subsidizes) for capital owners in recessions, their tax burden is positively correlated with their income providing insurance to tax payers. At the same time, the safe asset service flow rises in recessions, which provides insurance to government bond holders. This finding in our incomplete market setting with a safe asset is in sharp contrast to traditional asset pricing in which either tax payers or government bond holders can be insured, but not both, as pointed out by Jiang et al. (2020).

We remark that the government nevertheless faces a trade-off also in our setting. By making debt issuance more counter-cyclical, tax payers become better insured whereas insurance to bond holders is reduced. This is the case because bond holders are also exposed to the cash flow component, which captures the conventional logic emphasized by Jiang et al. (2020). However, the trade-off is considerably more favorable because the cash flow component represents only a small fraction of the total value of debt.

4.5 Volatile, Flight-to-Safety Prone Equity Markets

The presence of idiosyncratic risk and government debt as a safe asset has also implications for equity markets. We explain in this section why the diversified equity portfolio does not emerge as a safe asset and how flight to safety can generate quantitatively large additional equity return volatility.

Why Stocks Are not Safe Assets. In our model, agents can hold a diversified stock portfolio. Like government bonds, this stock portfolio is free of idiosyncratic risk and thus allows agents to self-insure against idiosyncratic consumption fluctuations. However, unlike government bonds, stocks are poor aggregate risk hedges as they are ultimately claims to capital, which looses in value in recessions. This implies that stocks are positive-β assets in our model.

To understand why stock prices fall in times of high idiosyncratic risk, even though idiosyncratic equity risk can be diversified away, note that the marginal holder of capital in our model is always an insider who has to bear the increased idiosyncratic risk. As a consequence, when idiosyncratic risk goes up, so does the insider premium earned by the managing households, which is achieved by a reduction in the dividend that is paid to outside equity holders.[34] This makes stock dividends more procyclical than production cash flows, so that stocks lose value precisely when idiosyncratic risk goes up.

When evaluating the diversified stock portfolio with regard to the key characteristic of safe assets, the Good Friend Analogy, stocks fail to qualify as safe assets in the same way as government debt does. Stocks have the good friend property only partially: stocks are valuable when an agent experiences a negative idiosyncratic shock, but due to their positive β, they are not in bad aggregate times.

Flight-to-safety Volatility. While the focus of this paper is on government bonds, our model can also match the empirical mean and volatility of the excess return on the stock market in excess of government bonds as we discuss in Section 7 below. The realistic Sharpe ratio is clearly a feature of recursive preferences with a high risk aversion, but the ability of our simple model to generate large return volatility in the presence of

[34]Formally, the Lagrange multiplier λ_t^i on the skin-in-the-game constraint (5) that governs the spread $\mathbb{E}_t[dr_t^{K,i}] - \mathbb{E}_t[dr_t^{E,i}]$ increases, compare Appendix A.1.

realistic levels of output variation is noteworthy[35] and directly related to the existence of safe government bonds.

To gain intuition, let's abstract from the distinction between capital and outside equity[36] and consider the following equation, which aggregates the intertemporal budget constraints of all households:

$$q_t^K K_t + q_t^B K_t = \mathbb{E}_t \left[\int_t^\infty \frac{\int \xi_s^i \eta_s^i di}{\int \xi_t^i \eta_t^i di} C_s ds \right]. \tag{20}$$

In many macro asset pricing models, government debt does not represent positive net wealth, $q_t^B = 0$, and thus equation (20) implies for such models that the value of the capital stock equals the present value of future consumption. In other words, in these models, pricing the aggregate equity claim is equivalent to pricing the aggregate consumption claim.[37] In the presence of realistic consumption volatility, large volatility in capital valuations $q_t^K K_t$ is then hard to generate (and requires substantial time variation in the SDF $\int \xi_s^i \eta_s^i di$).

Our model with $q^B \neq 0$ suggests an additional explanation for the high observed stock market volatility. When idiosyncratic risk $\tilde{\sigma}_t$ rises, there is a flight to safety that increases the value of bonds (q_t^B) and lowers the value of capital (q_t^K). Even in the absence of changes in the present value on the right-hand side of equation (20), this portfolio reallocation generates *flight-to-safety volatility* in capital valuations and thus in the stock market.

To understand how much flight-to-safety volatility matters quantitatively, we compare the excess stock return volatility in our model to the one generated by a version of the model without government debt. In that alternative version, $q_t^B = 0$ at all times and thus flight-to-safety volatility disappears.[38] We find that the average (annualized) excess return volatility in the alternative model would be 2.4% as opposed to 11.7% in

[35]This is so because we work with preferences that feature a unit EIS. It is well-known from the long-run risk literature that recursive preferences can also generate large return volatility, but *only* if the EIS is sufficiently larger than 1. In contrast, the mechanism we describe here works even for EIS ≤ 1.

[36]As discussed previously, a state-dependent insider premium will ensure that equity values and capital values move in lockstep despite the fact that idiosyncratic equity risk can be diversified away.

[37]Because the equation results from aggregating individual intertemporal budget constraints, the SDF used in this pricing equation is again the weighted-average SDF as in the dynamic trading perspective.

[38]Formally, we selection the "non-monetary" equilibrium in our model (degenerate solution $\vartheta \equiv 0$ to equation (10)). We keep all parameters as in our baseline model.

our baseline model.[39] We can therefore conclude that flight-to-safety volatility accounts for more than three quarters of the overall excess return volatility in our framework.

5 Safe Asset Tautology and Loss of Safe Asset Status

In this section, we clarify the relationship between safe assets and bubbles as well as the fragility of the safe asset status. We offer two key takeaways: First, while safe assets and bubbles are two distinct concepts, they complement each other. If an asset is associated with a bubble, it is more likely to be a safe asset. Second, safe-asset status is fragile and may be lost when the bubble pops. The same asset with the same payoffs might be a safe asset in one equilibrium, but not a different equilibrium. In this sense, a safe asset is safe because it is perceived to be safe, a tautology. In contrast to our standard equilibrium selection (compare Section 2.3), in this section – and in this section only – we do not restrict attention to stationary equilibria in which government bonds have a positive value. Then Proposition 2 does not apply and multiple equilibria may possibly arise.

Bubbles and Safe Assets. While bubbles and safe assets are distinct concepts, there is a complementarity between bubbles and the negative β property of safe assets:

First, low-β assets can sustain bubbles more easily that high-β assets. A bubble on an asset is possible if, in the buy and hold perspective, the discounted terminal value does not necessarily vanish in the limit. For example, in the case of government debt, the terminal value may not vanish if $\mathbb{E}\left[\bar{\xi}_T \mathcal{B}_T / \mathcal{P}_T\right]$ does not necessarily converge to zero as $T \to \infty$ (compare equation (11)). In the long-run, the value of government debt $\mathcal{B}_T / \mathcal{P}_T$ grows on average at the same rate as the aggregate economy, so that the discounted terminal value grows on average at the rate

$$g - r^f - \text{risk premium on gov. debt}$$

where g is the average growth rate of the economy and r^f is the average risk-free rate. The (average) risk premium on government debt scales linearly with its (average) β, at least approximately. Hence, the smaller is the β, the larger is the growth rate of the

[39] As the benchmark asset in the alternative model, we choose a zero net supply risk-free bond, the most common choice in the literature.

discounted terminal value and the easier it is to sustain a bubble on government debt. The same argument applies of course also to other assets than government debt.

Second, a bubble component can lower an asset's β and thereby make it safer. To understand this point, take again government debt as an example but now consider the dynamic trading perspective, equation (13). Suppose the situation is as depicted in Figure 2 with a cash flow component that has a positive β and a service flow component that has a negative β.[40] The relative contribution of the two components to the overall value of government debt determines the size and sign of the asset's β. The service flow component is proportional to the market value of government debt whereas the cash flow component does not directly depend on it. A bubble component raises the market value of the debt and thus increases the relative contribution of the service flow component, which lowers the asset's β.

The previous discussion implies that the safe asset status can be bubbly. An asset whose cash flow component has a sufficiently large β may be safe in some equilibria but not safe in others. Specifically, in an equilibrium in which the asset has a (sufficiently large) bubble component, the service flow component dominates, it has a negative total β and its required risk-adjusted rate of return is low, such that the asset can easily sustain the bubble. In a different equilibrium without a bubble component, the cash flow component dominates, the total β is positive and, as a result, the risk-adjusted rate of return so large that the asset does not appear to be able to sustain bubbles.

Bubbly safe assets give rise to the *Safe Asset Tautology*: the asset is safe in a given equilibrium because it is perceived to be safe, but there are alternative equilibria in which the asset is not safe. In these alternative equilibria, the asset has a positive β and is therefore not a good friend after negative aggregate shocks. When government debt is merely a bubbly safe asset, the safe asset status can be fragile. If markets coordinate on a different equilibrium, the bubble bursts and the government loses the safe asset status together with the fiscal space it implies.

Bubbly versus Fundamentally Safe Government Debt. In our model environment, a precautionary savings motive arising from uninsured idiosyncratic risk can depress discount rates sufficiently to make bubbles possible. But importantly, it depends on government policy how strong this precautionary savings motive is. If the government

[40] As we have explained previously, this is the relevant case to make sense of the empirical facts about primary surpluses and debt from the perspective of our theory.

makes bonds more attractive by raising higher primary surpluses (equivalently, by lowering $\breve{\mu}_t^{\mathcal{B}}$), the bond wealth share ϑ_t increases, idiosyncratic risk sharing improves and the precautionary motive is dampened.

A government that makes its debt very attractive can therefore raise discount rates sufficiently to eliminate any space for rational bubbles. Under such a policy, government debt can still be a safe asset, however. In this case, government debt is a fundamentally safe asset whose safe asset status does not require the continued belief of market participants in its safety, unlike for bubbly safe assets.

The simplest way to retain a safe asset status in the absence of a bubble is for the government to give up insurance of tax payers in recessions and make the β of the cash-flow component negative. Alternatively, there may still be a sufficient amount of residual idiosyncratic risk for the counter-cyclical service flow component to be substantial even though the precautionary savings motive is not large enough to sustain bubbles. In this case, also a mildly pro-cyclical surplus process may be consistent with safe government debt in the absence of bubbles.

Selecting the Public Debt Bubble. When the safe asset status is bubbly, a government with access to a richer set of policy tools than considered in this paper may still be able to select a unique equilibrium. Brunnermeier et al. (2021a) analyze how fiscal policy and asset regulation can affect the set of possible equilibria in environments with public debt bubbles. Their results also apply to our model:

The government can impose a number of specific policy measures that target bubbles on alternative assets or give an advantage to government bonds: it can eliminate private Ponzi schemes by imposing no Ponzi conditions on private agents through insolvency laws, it can tax competing bubbly assets, and it can use financial repression tools such as reserve and liquidity requirements to generate additional demand for its own liabilities. But most importantly, if a government can credibly commit to create a fundamentally safe asset off equilibrium by raising surpluses, it can eliminate all other equilibria. Hence, a government with sufficient backup fiscal capacity does not need to fear a loss of the safe asset status of its debt.

6 Privately Issued Safe Assets and Convenience Yields

So far, we have emphasized government debt as a safe asset. In this section, we discuss safe asset issuance by private agents. We also elaborate on the difference between service flows derived from retrading and convenience yields.[41]

Privately Issued Safe Assets. We consider a model extension with privately issued safe assets. We discuss here merely the economic conclusions and relegate the formal details to Appendix A.8.

We assume that each agent can issue nominally risk-free bonds, just like the government. Our safe asset definition, based on the Good Friend Analogy, applies equally also to such debt instruments issued by private citizens. For any individual asset holder, government bonds and privately issued safe bonds are perfect substitutes. As a consequence, the equilibrium interest rate i_t^p that private agents have to pay on their bonds equals the government's, $i_t^p = i_t$.

In equilibrium, agents are then indifferent as to how many bonds to issue and how many privately issued bonds of other agents to hold. In Appendix A.8, we consider a simple example in which the quantity of outstanding private bonds is always proportional to the quantity of government bonds and all agents keep the relative allocation to private and government bonds in their portfolios constant, so that they must trade them in constant proportions. Just like government bonds, we can value bonds issued by some agent j ("j-bonds") from the dynamic trading perspective by pricing the cash flows of the portfolios of j-bonds held by all other agents $i \neq j$. The resulting equation is in complete analogy to equation (13) for government bonds. In particular, the service flow that agents derive per real unit of j-bonds outstanding is the same as for government bonds.

Overall, the model extension in Appendix A.8 highlights that privately issued bonds are equally suitable as safe assets for their holders. However, private bond issuance also comes with a short position in the bond for the issuing agent. In the same spirit as before, we can value the short position by determining the present value of net payouts that an issuer makes to all bond holders. That valuation exercise reveals that the short position generates a negative service flow for the issuing agent. This negative service flow results from the fact that the agent repays outstanding debt after negative idiosyn-

[41] For the equations presented in this section, we revert back to the logarithmic preference specification.

cratic shocks and issues additional debt after positive idiosyncratic shocks. While the cash flows generated from this contingent bond issuance are zero on average, they are systematically correlated with marginal utility and thus tend to increase the overall riskiness of the agent's portfolio.

Once we aggregate all long and short positions of all privately issued safe bonds, the service flows "earned" by bond holders and the service flows "paid" by bond issuers exactly cancel out.[42] Therefore, unlike government debt, private safe asset creation does not generate net service flows for the economy as a whole.

Convenience Yields. A conclusion from the previous model extension is that $\Delta i_t := i_t^p - i_t = 0$, the yield spread between privately issued and government debt is zero. Government debt is not special. In the presence of idiosyncratic risk, a precautionary motive depresses all asset returns symmetrically. Equivalently, a service flow from re-trading can be derived from all assets that are both free of idiosyncratic risk and tradeable on liquid markets.

Such a service flow is conceptually different from a convenience yield. A convenience yield on government debt captures the special role that government bonds play in certain transactions. It can be measured by a positive yield spread $\Delta i_t > 0$ between government debt and safe corporate debt of equal maturity. In contrast, the service flow from retrading we emphasize in this paper affects also safe corporate debt. It is therefore unrelated to the spread Δi_t.

To illustrate this difference further, we augment our model so that government debt has a convenience yield. We model the source of the convenience yield by simply putting government bond holdings in agents' utility functions. Other mechanisms like collateral constraints require richer environments but would lead to the same conclusions. We present the formal model equations in Appendix A.9.

In the augmented model, we can once again price government debt according to our two valuation perspectives:

Proposition 7. *In the model with convenience yields, the value of government debt at $t = 0$*

[42]This conclusion is generally true, not just in the specific example analyzed in Appendix A.8.

satisfies from the buy and hold perspective:

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T \to \infty} \left(\mathbb{E}\left[\int_0^T \bar{\bar{\xi}}_t s_t K_t dt \right] + \mathbb{E}\left[\int_0^T \bar{\bar{\xi}}_t \Delta i_t \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right] + \mathbb{E}\left[\bar{\bar{\xi}}_T \frac{\mathcal{B}_T}{\mathcal{P}_T} \right] \right),$$

and from the dynamic trading perspective:

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \xi_t^{**} s_t K_t dt \right] + \mathbb{E}\left[\int_0^\infty \xi_t^{**} \left(\Delta i_t + (1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \right) \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right].$$

From the latter, dynamic trading perspective, the service flows from bonds in the utility function (captured by Δi_t) and from self-insurance through retrading (captured by $(1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2$) appear symmetrically. However, the buy and hold perspective reveals an asymmetry. The convenience yield still enters the valuation explicitly as a service flow term. In contrast, the service flow from retrading is absent in this perspective. Instead, it is implicitly contained in the stochastic discount factor $\bar{\bar{\xi}}_t$ and results in a lower discount rate due to precautionary savings as well as – potentially – a bubble term.

The terms arising from the buy and hold perspective are the ones that are typically measured in empirical asset pricing. The best an empirical researcher can do when estimating a SDF based on aggregate asset price data is to identify $\bar{\bar{\xi}}_t$. When looking at yield differences between safe corporate and government bonds, the empirical researcher identifies an estimate of Δi_t. The importance of self-insurance service flows can only be determined indirectly, e.g. by finding a bubble component.[43]

We interpret the empirical findings of Jiang et al. (2019) as evidence in support of such a bubble component. They conclude that an empirical estimate of the present value of surpluses and convenience yields falls short of the market value of government debt. As their empirical analysis is conducted from the buy and hold perspective, the gap must be explained by a bubble component.

[43] The presence of a bubble component in the buy and hold perspective means that even at the low discount rates implied by $\bar{\bar{\xi}}_t$, cash flows $s_t K_t$ and convenience yield service flows $\Delta i_t \mathcal{B}_t / \mathcal{P}_t$ are insufficient to explain the total value of government debt. The same always remains true if we discount at the higher rates implied by ξ_t^{**}, so that the self-insurance service flow must explain the gap.

7 Calibration and Quantitative Model Fit

In this section we describe the calibration underlying the numerical illustration presented in Section 4 and argue that it leads to predictions for asset prices and macro aggregates that are broadly realistic.

Details on the Model Setup We remind the reader that $\tilde{\sigma}_t$ is assumed to follows a Heston (1993) model of stochastic volatility and that we impose functional relationships $a_t = a(\tilde{\sigma}_t)$ and $\breve{\mu}^{\mathcal{B}}_t = \breve{\mu}^{\mathcal{B}}(\tilde{\sigma}_t)$ (compare Section 4, in particular equation (19)).

For the functional relationships for a_t and $\breve{\mu}^{\mathcal{B}}_t$, we choose parsimonious linear specifications

$$a(\tilde{\sigma}_t) = a^0 - \alpha^a(\tilde{\sigma}_t - \tilde{\sigma}^0), \tag{21}$$

$$\breve{\mu}^{\mathcal{B}}_t = \breve{\mu}^{\mathcal{B},0} + \alpha^{\mathcal{B}}(\tilde{\sigma}_t - \tilde{\sigma}^0) \tag{22}$$

with parameters a^0, $\breve{\mu}^{\mathcal{B},0}$, α^a, and $\alpha^{\mathcal{B}}$. Sufficiently large coefficients $\alpha^a, \alpha^{\mathcal{B}} > 0$ ensure that output, consumption, and primary surpluses all fall when idiosyncratic risk rises.

In order to match certain aspects of the data better, we also make two small modifications to the model itself. First, for our model to generate a quantitatively realistic price of aggregate risk, we replace logarithmic preferences with stochastic differential utility with unit EIS and arbitrary relative risk aversion $\gamma > 0$: household i maximizes V^i_0, where V^i_t is recursively defined by

$$V^i_t = \mathbb{E}_t\left[\int_t^\infty (1-\gamma)\rho V^i_s\left(\log(c^i_s) - \frac{1}{1-\gamma}\log\left((1-\gamma)\,V^i_s\right)\right)ds\right].$$

In the special case $\gamma = 1$, this specification collapses to our baseline specification with logarithmic utility. Second, to separate the level of investment from the adjustment cost parameter ϕ, which governs fluctuations in investment and capital prices, we consider the slightly more general capital adjustment cost function

$$\hat{\Phi}(\iota) = \iota^0 + \Phi(\iota - \iota^0)$$

with the additional parameter ι^0. All solution formulas for the baseline model remain valid for this more general specification if we replace a_t with $a_t - \iota^0$ and ι_t with $\iota_t - \iota^0$.

Table 1: Parameter Choices

parameter	description	value	parameter	description	value
$\tilde{\sigma}^0$	$\tilde{\sigma}_t^2$ stoch. steady state	0.54	$\mathfrak{g}$	gov. expenditures	0.138
ψ	$\tilde{\sigma}_t^2$ mean reversion	0.67	$\breve{\mu}^{\mathcal{B},0}$	$\breve{\mu}_t^{\mathcal{B}}$ stoch. steady state	0.0026
σ	$\tilde{\sigma}_t^2$ volatility	0.4	α^a	a_t slope	0.072
$\bar{\chi}$	undiversifiable risk	0.3	$\alpha^{\mathcal{B}}$	$\breve{\mu}_t^{\mathcal{B}}$ slope	0.12
γ	risk aversion	6	ϕ	capital adj. cost	8.1
ρ	time preference	0.138	ι^0	capital adj. intercept	-0.022
a^0	a_t stoch. steady state	0.625	δ	depreciation rate	0.055

We present the model solution for this generalized model in Appendix A.7. There, we also describe our numerical solution algorithm.

Calibration Strategy We calibrate our model such that, when we feed in a quantitatively realistic process for idiosyncratic risk, the model generates variation in macro aggregates and aggregate risk premia that are broadly consistent with US data. We briefly outline our calibration strategy here. The resulting parameter choices are summarized in Table 1. Additional details as well as a description of the underlying data sources can be found in Appendix A.10.

We normalize the time period in our model to one year. Because ours is a continuous-time model, this is merely a choice of units that does not affect any results.

With regard to the parameters $\tilde{\sigma}^0$, ψ, σ of the exogenous risk process $\tilde{\sigma}_t$, we tie our hands by estimating them externally. Specifically, we choose these parameters such that $\tilde{\sigma}_t^2$ closely matches, in a maximum likelihood sense, the common idiosyncratic volatility (CIV) factor proposed by Herskovic et al. (2016). As that paper shows, CIV is a priced risk factor that is correlated with idiosyncratic risk exposures of both firms and households. In Appendix A.10.2, we argue that CIV is a model-consistent data counterpart of $\tilde{\sigma}_t^2$.[44]

For the share of undiversifiable idiosyncratic risk, $\bar{\chi}$, the previous literature provides some imperfect guidance. Angeletos (2007) reports that, in the aggregate, private and public equity firms in the US are of approximately equal importance for production, employment, and wealth and uses this as the basis to calibrate a model with id-

[44]We have also considered alternative measures for idiosyncratic uncertainty (Bloom et al., 2018; Hassan et al., 2019) but ultimately chosen CIV both due to the quality and length of the available data series and because of the theoretical link between $\tilde{\sigma}_t^2$ in the model and CIV.

iosyncratic return risk similar to ours that features both private and public equity. His calibration suggests $\bar{\chi} \approx 0.5$ in our framework. Heaton and Lucas (2000a,b) provide evidence on the portfolio composition of the wealth of US stockholders. Broadly speaking, two fifths of wealth are composed of liquid assets, and three fifths are composed of real estate and business wealth. Depending on whether one treats real estate wealth, which has no direct counterpart in our model, as diversifiable or non-diversifiable capital, their evidence suggests values of $\bar{\chi}$ between 0.2 and 0.6.[45] Based on this broad evidence, we choose an intermediate value of $\bar{\chi} = 0.3$.

The calibration of the parameters $\tilde{\sigma}^0$ and $\bar{\chi}$ jointly affects the (average) level of non-diversifiable idiosyncratic risk faced by agents in our model. To further assess our calibration choices of these parameters, we check how the predictions of our model for the volatility of wealth growth compare with estimates in the data. The ergodic mean of the total wealth growth volatility ($\sqrt{(\sigma^n)^2 + (\tilde{\sigma}^n)^2}$) in our model is 0.13. Most of this is due to idiosyncratic volatility: the ergodic mean of idiosyncratic wealth growth volatility alone ($\tilde{\sigma}^n$) is 0.12. While these numbers are homogeneous in our model, it is less clear what the right comparison group is in the data. Compared to empirical estimates for households at the top of the wealth distribution, who are arguably most relevant for asset pricing, risk exposures in our model appear to be on the low end. For the US, Gomez (2023, Table II) reports wealth growth standard deviations between 0.25 and 0.31 but his data only include very wealthy individuals in the Forbes 400. Based on a richer administrative dataset from Sweden, Bach et al. (2020, Table II) report wealth growth standard deviations for different groups in the top-10% of the wealth distribution that range from 0.11 to 0.33 and are monotonically increasing in wealth.[46] However, when assessed against the numbers reported by Bach et al. (2020) for wealth groups in the bottom-90% (their Table II), risk exposures in our model appear to be too high.

To discipline the comparison with the empirical estimates better, we set up in Appendix A.11 an extension of our model with two types to generate heterogeneity in the volatility of wealth growth also in the model. We show that, under certain con-

[45] More recent work based on administrative data from Sweden and Norway (Bach et al., 2020; Fagereng et al., 2020) provides corroborating evidence that private business wealth of households in the top percentile of the wealth distribution – those most relevant for asset pricing – makes up well above 20% of their total portfolio holdings.

[46] These authors also attempt to isolate the idiosyncratic component based on a factor model and find that, in the top-10% group, the share of idiosyncratic risk ranges from one third to two thirds, again monotonically increasing in wealth Bach et al. (2020, Table I).

ditions, this model generates the exact same predictions as our baseline model with homogeneous wealth growth volatility, provided the dynamics for the (cross-sectional) *wealth-weighted average* of the variance of (idiosyncratic) net worth growth are identical. Motivated by this observation, we conclude that such a wealth-weighted average of the empirical estimates is the correct comparison statistic to our model-implied volatilities.[47] To compute this comparison statistic, we combine data from Bach et al. (2020) for wealth growth volatility across the wealth distribution (in Sweden) with data reported by Smith et al. (2023) for wealth shares estimated for the US. We provide additional details of the procedure in Appendix A.10.4. Depending on the precise choices, we find numbers between 0.16 and 0.19 for total wealth growth volatility and between 0.09 and 0.1 for the idiosyncratic portion. We conclude from this exercise that our calibration implies broadly realistic amounts of risk exposures in total portfolios, albeit with a slightly overstated idiosyncratic component.[48]

We choose the nine parameters γ, ρ, a^0, $\mathfrak{g}$, $\breve{\mu}^{\mathcal{B},0}$, α^a, $\alpha^{\mathcal{B}}$, ϕ, ι^0 such that the model generates values for a number of moments that are broadly in line with the empirical evidence.[49] These moments are the average ratios of consumption, government expenditures, primary surpluses, capital, and debt to output, the average investment rate, the volatilities of output, consumption, investment, and the surplus-output ratio, and the equity premium and equity sharpe ratio.[50] Our empirical moments are based on a sample from 1970 to 2019 just prior to the start of the covid pandemic with two exceptions. The first is the debt-output ratio. Over the largest part of our sample period, this ratio has exhibited a clear upward trend. For this reason, we target the average over the last decade in the sample (0.71) instead of the average over the full sample (0.37). The second exception is the average investment rate, $\mathbb{E}[I/K]$, which we do not compute ourselves but take directly from Cooper and Haltiwanger (2006), who report a value estimated from micro data.

As we explain in Appendix A.10.3, matching the average ratios is directly infor-

[47] To be consistent with the result from the two-type model, we take a wealth-weighted average of the variances, not the standard deviations.

[48] Our calibration may not even overstate idiosyncratic risk if we adopt a broader interpretation that includes idiosyncratic human capital risk, which is absent from our model, but arguably an important source of idiosyncratic risk outside the top wealth group.

[49] We do not employ a formal simulated method of moments estimation but merely adjust parameters manually to achieve a good visual fit.

[50] Volatilities of macro aggregates are at the quarterly frequency. Equity moments refer to annualized quantities but are measured at a monthly frequency.

mative for the average value of the endogenous variable ϑ_t and the parameters ρ, a^0, $\mathfrak{g}$, $\breve{\mu}^{\mathcal{B},0}$, and ι^0. Requiring the model to match the macro volatilities is standard in the business cycle literature and also ensures that the model generates a broadly realistic amount of aggregate macro risk.[51] While including the surplus volatility is less standard, this moment is important to discipline the (cyclical) variation in primary surpluses, a key ingredient into the valuation of government debt.[52] Finally, requiring the model to match the equity premium and equity sharpe ratio ensures that this aggregate macro volatility is realistically priced in capital markets.

The remaining parameter δ does not affect anything of interest for the purpose of this paper.[53] We set it to 0.055, a value slightly smaller than but broadly in line with typical calibrations. With this choice, the average growth rate of our model economy is 2.0%, close to the empirical counterpart of 2.1% in our sample.

Model Fit. Table 2 summarizes the quantitative model fit. In addition to our target moments, we report in Table 2 also a number of untargeted moments: the correlations of consumption, investment, and primary surpluses with output, the standard deviation of the debt-output ratio, and the average and standard deviation of the risk-free rate.

Section (1) of Table 2 reveals that our model achieves overall a very good fit to the targeted moments. As we have varied only nine of our parameters to match twelve moments, this is by no means a trivial observation. Most importantly, our model is consistent with the observed large equity premium and price of risk (Sharpe ratio) while at the same time matching the volatility and comovement of macro aggregates. This verifies that our model is capable of generating realistic aggregate risk premia without requiring excessive real volatility. Notably, our model achieves quantitatively plausible aggregate risk pricing with a moderate risk aversion parameter (γ) of just 6.

However, Section (2) of Table 2, which reports untargeted moments, reveals some quantitative shortcomings of our model with regard to the risk-free rate and the debt-output ratio.[54] In particular, the debt-output ratio is too volatile and the risk-free rate

[51]These moments also discipline the model parameter α^a and ϕ.

[52]That moment also disciplines the parameter $\alpha^{\mathcal{B}}$.

[53]This is due to the combination of the AK structure of our economy with a unit EIS. The former implies that δ merely affects the growth rate of the economy and the latter that income and substitution effects from permanent variations in growth rates cancel out.

[54]While three correlations reported also appear too high relative to the data, particularly the correlation between output and surpluses, we do not view this as a major issue. In a model with one state variable, all non-zero correlations are close to 1 in absolute value, such that only the sign is truly informative, not

Table 2: Quantitative Model Fit

moment symbol	moment description	model	data
	(1) targeted moments		
$\sigma(Y)$	output volatility	1.3%	1.3%
$\sigma(C)/\sigma(Y)$	relative consumption volatility	0.61	0.64
$\sigma(I)/\sigma(Y)$	relative investment volatility	3.35	3.38
$\sigma(S/Y)$	surplus volatility	1.1%	1.1%
$\mathbb{E}[C/Y]$	average consumption-output ratio	0.58	0.56
$\mathbb{E}[G/Y]$	average government expenditures-output ratio	0.22	0.22
$\mathbb{E}[S/Y]$	average surplus-output ratio	-0.0005	-0.0005
$\mathbb{E}[I/K]$	average investment rate	0.12	0.12
$\mathbb{E}[q^K K/Y]$	average capital-output ratio	3.48	3.73
$\mathbb{E}[q^B K/Y]$	average debt-output ratio	0.74	0.71
$\mathbb{E}[d\bar{r}^E - dr^{\mathcal{B}}]$	average (unlevered) equity premium	3.59%	3.40%
$\frac{\mathbb{E}[dr^E - dr^{\mathcal{B}}]}{\sigma(dr^E - dr^{\mathcal{B}})}$	equity sharpe ratio	0.31	0.31
	(2) untargeted moments		
$\rho(Y,C)$	correlation of output and consumption	0.98	0.92
$\rho(Y,I)$	correlation of output and investment	0.99	0.94
$\rho(Y,S/Y)$	correlation of output and surpluses	0.98	0.60
$\sigma(q^B K/Y)$	volatility of debt-output ratio	4.8%	2.0%
$\mathbb{E}[r^f]$	average risk-free rate	5.18%	0.64%
$\sigma(r^f)$	volatility of risk-free rate	5.47%	2.25%

Notes: For $x \notin \{dr^E - dr^{\mathcal{B}}, r^f\}$, $\sigma(x)$ denotes the standard deviation of x and $\rho(x,y)$ denotes the correlation of x and y, both at a quarterly frequency. Inputs x and y are HP-filtered with smoothing parameter 1600. For $x, y \in \{Y, C, I, G\}$, we take logarithms before filtering. $\mathbb{E}[x]$ denotes expectations over the ergodic model distribution, inputs x are *not* HP-filtered. $x \in \{dr^E - dr^{\mathcal{B}}, r^f\}$ refer to annualized returns measured at monthly frequency and are also *not* HP-filtered. Y: (aggregate) output, C: consumption, I: investment, G: government expenditures, S: primary surplus, r^f: risk-free rate; K, q^K, q^B, dr^B, $d\bar{r}^E$ are defined as in Section 2.

is too high on average and also too volatile.

The first two observations are, in fact, closely related. As Figure 1 reveals, a large fraction of the price adjustments in a flight-to-safety episode are due to (deflationary) adjustments in the value of bonds q_t^B as opposed to adjustments in the capital price q_t^K. When prices are flexible, large fluctuations in q_t^B due to (instantaneous) deflationary and inflationary adjustments in response to shocks are a robust feature of this class of models. If, instead, the relative price adjustment between the two asset was the same but all the impact adjustment was due to movements in capital values, then the volatility of the debt-output ratio would be reduced and, at the same time, the risk-free rate would fall and be closer to the average return on bonds, which is approximately correct in our model (2.63% vs. 2.40% in the data).

One way to shift all instantaneous volatility from inflationary or deflationary bond value adjustments into capital prices is to introduce sticky prices. Li and Merkel (2020) show, in the context of a related model, that introducing even an arbitrarily small degree of price stickiness leads to sluggish movements in q_t^B and "overshooting" in q_t^K, so that all relative price volatility between the two assets is shifted into capital price adjustments.

A bigger challenge is the volatility of the risk-free rate. It is unclear how much this moment matters because the risk-free rate does not perform a direct allocational function in our model. Still, it is difficult to match it well without additional ingredients. The reason is that the risk-free rate can be written as

$$r_t^f = \rho + \mu_t^n - \varsigma_t \sigma_t^n - \tilde{\varsigma}_t \tilde{\sigma}_t^n,$$

where μ_t^n, σ_t^n, and $\tilde{\sigma}_t^n$ are the (common) geometric drift and (aggregate and idiosyncratic) volatility loadings of net worth n_t^i. The mechanism we emphasize in this paper requires that the idiosyncratic risk premium $\tilde{\varsigma}_t \tilde{\sigma}_t^n$ shoots up in recessions and, with the exogenous process (19), also the aggregate wealth risk premium $\tilde{\varsigma}_t \sigma_t^n$ increases in recessions. Both dynamics appear plausible, but they require the risk-free rate to fall in recessions unless expected net worth growth μ_t^n offsets the increase in risk premia. Expected net worth growth μ_t^n, in turn, cannot vary too much without generating counterfactually large aggregate consumption volatility. We leave the question how to resolve this tension for future research.

the precise numerical value.

8 Quantifying the Bubble Mining Laffer Curve

When idiosyncratic risk is large, safe asset demand may be sufficient to sustain a public debt bubble. This is indeed the case for our calibration. As Brunnermeier et al. (2021a) point out, such public debt bubbles represent a fiscal resource that can be "mined" for revenue as a substitute for taxation. However, the ability to mine a bubble does not imply that a government can expand spending without limits. Bubble mining affects the sustainability of bubbles and thereby creates a "debt Laffer curve".

Here, we briefly revisit the Laffer curve logic and then use our calibrated model to quantify the Laffer curve for the US. The main takeaway is that the negative β property of government debt matters considerably. The (average) maximum permanent deficit is above 2% of GDP in our dynamic model but merely 0.1% if we hold idiosyncratic risk constant over time.

The Laffer curve logic follows from the following simple formula for primary deficits per unit of capital[55]

$$-s_t = \breve{\mu}_t^{\mathcal{B}} q_t^B.$$

The first factor, $\breve{\mu}_t^{\mathcal{B}}$, measures revenue raised by bond issuance that is not distributed to bond holders in the form of interest payments. If it is positive, the claim of old bond holders is diluted by the issuance of new bonds, i.e., a higher $\breve{\mu}_t^{\mathcal{B}}$ represents a tax on existing bond holders. The second factor, q_t^B, is the tax base, the real value of existing debt (per unit of capital). Permanent deficits are possible if this tax base remains positive even for (permanently) negative primary surpluses. That this is a possibility can be seen from both perspectives to debt valuation discussed in Section 3: the value of debt remains positive despite negative surpluses if, in the buy and hold perspective (equation (13)), a positive bubble term offsets the negative surplus term, or, equivalently, in the dynamic trading perspective (equation (13)), the service flow term is sufficiently large. In this case, the tax base is positive, but it nevertheless reacts negatively to an increase in the rate of bubble mining $\breve{\mu}_t^{\mathcal{B}}$.[56] This negative reaction creates a Laffer curve.

[55] This equation, in turn, follows immediately from the government budget constraint.

[56] We can see analytically that higher $\breve{\mu}^{\mathcal{B}}$ lowers the equilibrium value of $q^{\mathcal{B}}$ in steady state, compare the formulas in Section 2.4. Outside of the steady state, equation (8) tells us that there is a negative relationship if an upward shift in $\breve{\mu}_t^{\mathcal{B}}$ at all dates decreases ϑ_t. Equation (10) suggests that this is indeed the case, but additional technical considerations are required to make this a fully rigorous argument.

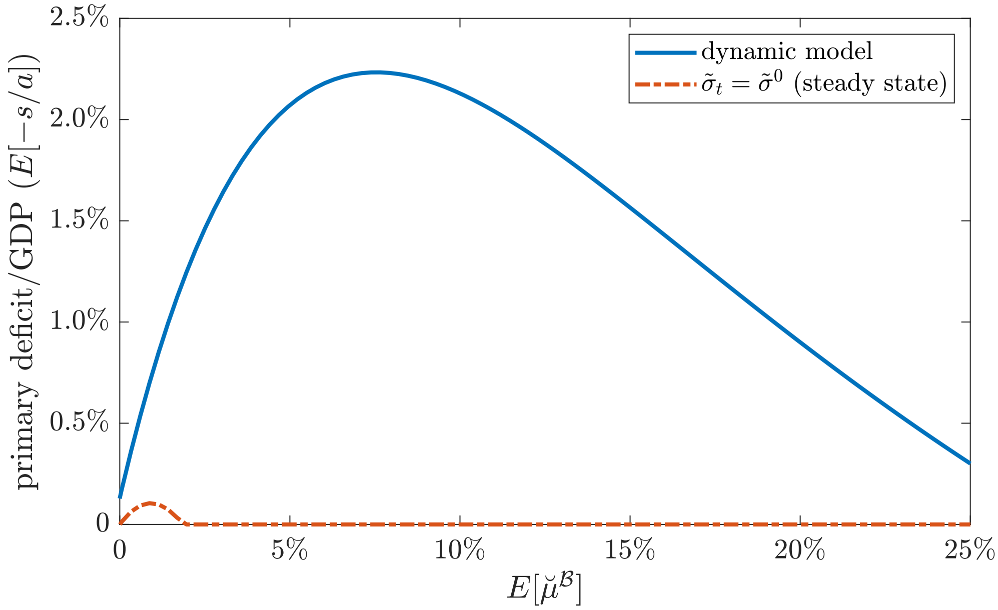

Figure 3: Debt Laffer curve for dynamic model and in steady state (constant $\tilde{\sigma}_t$) when there is a bubble on government debt. $\mathbb{E}[\breve{\mu}^{\mathcal{B}}]$ is varied by varying parameter $\breve{\mu}^{\mathcal{B},0}$ while keeping all other parameters as in Table 1.

The blue line in Figure 3 depicts the debt Laffer curve for our calibrated model. Specifically, the figure plots the average deficit-GDP ratio that can be sustained for different debt growth policies of the form (22) with identical $\alpha^{\mathcal{B}}$ (identical cyclicality of debt growth and surpluses) but varying $\breve{\mu}^{\mathcal{B},0}$, i.e. the average level of (interest-adjusted) debt growth varies across different policies on the x-axis. The implicit assumption in Figure 3 is that $\mathfrak{g}$ remains unchanged, so that larger deficits imply smaller output taxes.[57]

In Figure 3, if the bubble is mined too aggressively so that the average $\breve{\mu}^{\mathcal{B}}$ exceeds 7.3%, the government fails to raise additional real revenues. In particular, there is a limit to bubble mining and the government still faces a constraint on real spending. Our calibrated model suggests that the average primary deficit that can be sustained by bubble mining is bounded above by 2.25% of GDP.

It turns out that the negative β property is very important for the qualitative and quantitative shape of the Laffer curve depicted in Figure 3. If we abstract from countercyclical idiosyncratic risk and consider a constant level of $\tilde{\sigma}_t = \tilde{\sigma}^0$ instead, the resulting

[57]If instead the increased revenues from bubble mining are used to fund additional government expenditures (higher $\mathfrak{g}$), the slope of the Laffer curve is uniformly smaller.

Laffer curve is as depicted by the red dashed line in Figure 3. This steady-state Laffer curve reveals two differences compared to the dynamic model. First, the Laffer curve is quantitatively tiny. The maximum (average) permanent deficit is merely 0.1% (and it is reached at a much lower average value of $\breve{\mu}^{\mathcal{B}}$). Second, the steady-state Laffer curve quickly decays to zero, so that the tax base is more quickly eroded as the government dilutes the claims of existing bond holders at a faster rate. Instead, in the dynamic model, agents hold on to some bonds even at very large levels of average (interest-adjusted) debt growth rates of more than 10% despite the high inflation rates that they imply. The reason is that the insurance against adverse aggregate events makes bonds attractive for agents even if they pay negative rates of return on average.

How plausible and robust are these conclusions? First, the quantitative importance of the negative β property appears to be robust. We have found that it does not hinge on our specific calibration choices, except for the assumption that there is counter-cyclical idiosyncratic risk and the insistence that the model generate a realistic (i.e., large) price of aggregate risk. Second, the exact numbers such as the 2.3% maximum permanent deficit are more sensitive to calibration choices, in particular to the target level for the average debt-output ratio.[58] As we show in Appendix A.12, a calibration that reduces the target for the debt-output ratio by one third generates a much smaller Laffer curve that peaks already at a permanent deficit of 0.9% of output. In contrast, the other alternative calibration choices in Appendix A.12 that leave the target for the debt-output ratio unaltered have only minor effects on the maximum sustainable deficit.[59]

There are now also several other recent papers that quantify the debt Laffer curve in different model environments. Aguiar et al. (2023) and Kaplan et al. (2023) contain quantitative evaluations of Aiyagari-Huggett models. Aguiar et al. (2023, Appendix D.3) plot only a portion of the Laffer curve, as their paper has a different focus. Still, the information they provide is sufficient to conclude that the peak must be above 2% and likely between 2% and 2.5% of GDP. Kaplan et al. (2023) report estimates for the maximum sustainable deficits of 4.6%–4.8% of GDP.[60] Mian et al. (2022, Figure 7)

[58] Furthermore, the comparison with the steady-state Laffer curve reveals that also the calibration of the exogenous process for $\tilde{\sigma}_t$ must be important for the quantitative predictions. However, in a previous version of this paper we calibrated our model based on the alternative measures for idiosyncratic risk mentioned in footnote 44. The resulting Laffer curve was quantitatively similar to Figure 3.

[59] The observation that the calibration target for the debt-output ratio is quantitatively important is also shared by Mian et al. (2022) in the context of a different model.

[60] However, their calibration targets are not fully comparable. They target a steady-state debt-output ratio of 1.1 (0.74 in our model) and a steady-state primary deficit-output ratio of 0.033 (0.0004 in our

plot the Laffer curve across four different models, all calibrated to the same targets, and show that they all imply a similar peak estimate of about 2%–2.5%, albeit at different levels of total debt. In light of these complementary findings, we conclude that our own estimates appear plausible – and possibly even slightly conservative.

9 Conclusion

In this paper we have developed a safe asset theory of government debt based on time-varying idiosyncratic insurance service flows generated by trading government bonds. Our model matches properties of US government debt qualitatively and quantitatively and can resolve the empirical puzzles emphasized by Jiang et al. (2019, 2020). The theory also features a novel explanation for the large equity return volatility based on flight to safety into government bonds.

Throughout this paper we have assumed that government bonds are traded on liquid markets. The (bubbly) safe asset status rests on this assumption because the service flow that citizens derive from government debt is directly tied to their ability to trade it as they experience adverse shocks. The government through its central bank can engage as market maker of last resort so that citizens can trade the asset facing only small bid-ask spreads. This helps ensuring that government debt retains the safe asset status. Private assets do not enjoy this privilege.

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model). Therefore, their numbers must be larger than ours by construction.

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A Appendix

A.1 Omitted Steps in the Derivation of the Model Solution

Rewriting the Return Processes. We first write the return processes stated in Section 2 explicitly as diffusion processes using Ito's lemma. To do so, we first postulate that q_t^B and q_t^K follow a generic Ito evolution of the form

$$dq_t^B = \mu_t^{q,B} q_t^B dt + \sigma_t^{q,B} q_t^B dZ_t, \qquad dq_t^K = \mu_t^{q,K} q_t^K dt + \sigma_t^{q,K} q_t^K dZ_t.$$

Whenever $q_t^B, q_t^K \neq 0$, the unknown (geometric) drifts $\mu_t^{q,B}, \mu_t^{q,K}$ and volatilities $\sigma_t^{q,B}, \sigma_t^{q,K}$ are uniquely determined by the local behavior of q_t^B and q_t^K, respectively. In the following, we also use the notation μ_t^ϑ and σ_t^ϑ for the (geometric) drift and volatility of ϑ_t.[61]

Using Ito's lemma, the bond return can be written as[62]

$$\begin{aligned} dr_t^{\mathcal{B}} &= \breve{\mu}_t^{\mathcal{B}} dt + \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} \\ &= \left(\Phi(\iota_t) - \delta + \mu_t^{q,B} - \breve{\mu}_t^{\mathcal{B}}\right) dt + \sigma_t^{q,B} dZ_t, \end{aligned}$$

Similarly, the return on agent i's capital can be written as

$$\begin{aligned} dr_t^{K,i}\left(\iota_t^i\right) &= \frac{(1-\tau_t)\, a_t - \iota_t^i}{q_t^K} + \frac{d(q_t^K \tilde{k}_t^i)}{q_t^K \tilde{k}_t^i} \\ &= \left(\frac{(1-\tau_t)\, a_t - \iota_t^i}{q_t^K} + \Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}\right) dt + \sigma_t^{q,K} dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^i. \end{aligned}$$

Using the government budget constraint (2) allows us to eliminate taxes $\tau_t a_t$ and yields

$$dr_t^{K,i}\left(\iota_t^i\right) = \left(\frac{a_t - \mathfrak{g} - \iota_t^i}{q_t^K} + \frac{q_t^B}{q_t^K}\breve{\mu}_t^{\mathcal{B}} + \Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}\right) dt + \sigma_t^{q,K} dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^i.$$

[61] This means, $d\vartheta_t = \mu_t^\vartheta \vartheta_t dt + \sigma_t^\vartheta \vartheta_t dZ_t$.

[62] In line with our symmetric equilibrium definition and the discussion in the main text, we have assumed here $\iota_t^i = \iota_t$ for all i so that $\int \Phi(\iota_t^i) di = \Phi(\iota_t)$. This assumption is also verified below from the optimal choice conditions of households.

Given the capital return, the return on outside equity issued by agent i can be written as

$$dr_t^{E,i} = \mathbb{E}_t[dr_t^{E,i}] + \sigma_t^{q,K} dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^i.$$

Finally, the return on the equity portfolio is then

$$d\bar{r}_t^E = \int dr_t^{E,i} di = \mathbb{E}_t[d\bar{r}_t^E] + \sigma_t^{q,K} dZ_t,$$

because idiosyncratic risk $\tilde{\sigma}_t d\tilde{Z}_t^i$ of individual equity securities averages out in the portfolio.

Hamiltonian of the Household Problem. We use the stochastic maximum principle to derive optimal choice conditions for the household problem. After substituting the return expressions stated previously into the net worth evolution (4), we see that the Hamiltonian for the household problem is

$$\begin{aligned} H_t^i = e^{-\rho t} \log c_t^i + \xi_t^i \Bigg[-c_t^i + n_t^i \Bigg(\frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} + \theta_t^{K,i} \left(\frac{\mathbb{E}_t\left[dr_t^{K,i}(\iota_t^i)\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} \right) \\ + \theta_t^{E,i} \left(\frac{\mathbb{E}_t[dr_t^{E,i}]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} \right) + \theta_t^{\bar{E},i} \left(\frac{\mathbb{E}_t[d\bar{r}_t^{E}]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} \right) \Bigg) \Bigg] \\ - \varsigma_t^i \xi_t^i n_t^i \left(\sigma_t^{q,B} - \left(\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i} \right) \frac{\sigma_t^{\vartheta}}{1 - \vartheta_t} \right) - \tilde{\varsigma}_t^i \xi_t^i n_t^i \left(\theta_t^{K,i} + \theta_t^{E,i} \right) \tilde{\sigma}_t, \end{aligned}$$

where we have used $\sigma_t^{q,K} - \sigma_t^{q,B} = \frac{\sigma_t^{\vartheta}}{1-\vartheta_t}$. Here ξ_t^i denotes the costate ("Lagrange multiplier") for the net worth evolution (4), and we write the loadings of ξ_t^i with respect to the Brownian motions dZ_t and $d\tilde{Z}_t^i$ as $-\varsigma_t^i \xi_t^i$ and $-\tilde{\varsigma}_t^i \xi_t^i$, respectively.[63]

As this is a standard portfolio choice problem, we conjecture that the value function of the problem inherits the functional form of the utility function, i.e. $V_t\left(n^i\right) = v_t + \frac{1}{\rho} \log n^i$, where v_t depends on (aggregate) investment opportunities, but not on individual net worth n^i.[64] The usual relationship between the value function and the costate, $\xi_t^i = e^{-\rho t} V_t'(n_t^i)$ then implies $\xi_t^i = e^{-\rho t}/(\rho n_t^i)$, which we can use to eliminate ξ_t^i from the Hamiltonian H_t^i.

By the stochastic maximum principle, the optimal choice $(c_t^i, \iota_t^i, \theta_t^{K,i}, \theta_t^{E,i}, \theta_t^{\bar{E},i})$ must

[63] We use the same notation ξ_t^i for the costate in the household problem and the SDF in Sections 2.5 and 3 because the two in fact coincide.

[64] The verification argument for this conjecture is entirely standard, see e.g. Brunnermeier et al. (2021a), Appendix A.2.

maximize the Hamiltonian subject to the skin-in-the-game constraint (5).

Optimal Consumption and Investment Choice. Taking first-order conditions with respect to c_t^i and ι_t^i yields the two equations

$$c_t^i = \rho n_t^i, \tag{23}$$
$$\frac{d}{d\iota}\mathbb{E}_t\left[dr_t^{K,i}(\iota)\right]\Big|_{\iota=\iota_t^i} = 0.$$

The first equation is precisely the optimal consumption condition stated in the main text. The second equation implies the Tobin's q condition stated in the main text once we take the derivative in the explicit formula for $dr_t^{K,i}(\iota)$ stated in Section 2.2. We reproduce the Tobin's q condition here for the convenience of the reader:[65]

$$q_t^K = 1 + \phi\iota_t. \tag{24}$$

Derivation of Equations (7), (8), and (9). Integrating the optimal consumption condition (23) across all households i yields

$$C_t = \int c_t^i di = \rho \int n_t^i di = \rho(q_t^B + q_t^K)K_t,$$

where the last equality follows from the fact that aggregate net worth consists precisely of all capital and bond wealth combined.[66]

We next use $q_t^B + q_t^K = q_t^K/(1-\vartheta_t)$ by the definition of ϑ_t to replace the right-hand side of the previous equation.

$$C_t = \frac{\rho}{1-\vartheta_t} q_t^K K_t.$$

Substituting this into goods market clearing (3), canceling K_t, and using equation (24) to eliminate q_t^K yields the equation

$$\frac{\rho}{1-\vartheta_t}\left(1+\phi\iota_t\right) + \mathfrak{g} + \iota_t = a_t.$$

[65] We have already dropped the i superscript on ι because all households choose the same investment rate as argued in the main text.

[66] Note that, in our formulation, taxes are effectively imposed on capital holdings such that the present value of tax liabilities of households is implicitly capitalized in capital valuations. Also note that outside equity claims are in zero net supply and thus do not contribute to aggregate net worth.

This is a simple linear equation for ι_t. Solving it implies equation (7) as stated in the main text. Equation (9) can then be recovered by substituting the resulting expression for ι back into equation (24). Finally, equation (8) follows by exploiting the relationship $q_t^B = \frac{\vartheta_t}{1-\vartheta_t} q_t^K$, which is a direct consequence of the definition of ϑ_t.

This also completes the proof of Lemma 1.

Optimal Portfolio Choice. The first-order conditions for maximizing the Hamiltonian with respect to the portfolio shares $\theta_t^{K,i}$, $\theta_t^{E,i}$, and $\theta_t^{\bar{E},i}$ yields three Merton portfolio choice equations

$$\frac{\mathbb{E}_t\left[dr_t^{K,i}\left(\iota_t^i\right)\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} = -\varsigma_t^i \frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \tilde{\varsigma}_t^i \tilde{\sigma}_t - \lambda_t^i \left(1-\bar{\chi}\right), \tag{25}$$

$$\frac{\mathbb{E}_t\left[dr_t^{E,i}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} = -\varsigma_t^i \frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \tilde{\varsigma}_t^i \tilde{\sigma}_t - \lambda_t^i, \tag{26}$$

$$\frac{\mathbb{E}_t\left[d\bar{r}_t^{E}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} = -\varsigma_t^i \frac{\sigma_t^{\vartheta}}{1-\vartheta_t}. \tag{27}$$

Here, λ_t^i is a scaled Lagrange multiplier on the constraint (5) (skin-in-the-game constraint). Combining the last two equations and using $\frac{\mathbb{E}_t\left[d\bar{r}_t^{E}\right]}{dt} = \frac{\mathbb{E}_t\left[dr_t^{E,i}\right]}{dt}$ in equilibrium, we obtain a simple characterization of λ_t^i:

$$\lambda_t^i = \tilde{\varsigma}_t^i \tilde{\sigma}_t.$$

As we will show below, $\tilde{\varsigma}_t^i$ is always positive and so the constraint (5) must always be binding – households issue the maximum possible amount of outside equity. In particular,

$$\theta_t^{K,i} + \theta_t^{E,i} = \theta_t^{K,i} \bar{\chi}. \tag{28}$$

We now perform two substitutions in the first portfolio choice condition stated above. First, we eliminate λ_t^i on the right-hand side using the previously derived equation. Second, we plug in the expected return expressions implied by the return equations stated in Section 2.2. The condition then becomes

$$\frac{a_t - \mathfrak{g}_t - \iota_t}{q_t^K} - \frac{\mu_t^{\vartheta} - \breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t} - \frac{\left(\sigma_t^{q,B} - \sigma_t^{\vartheta}\right)\sigma_t^{\vartheta}}{1-\vartheta_t} = -\varsigma_t^i \frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \tilde{\varsigma}_t^i \bar{\chi} \tilde{\sigma}_t. \tag{29}$$

Characterizing the Costate Volatility Loadings ς_t^i and $\tilde{\varsigma}_t^i$. To determine the values of ϑ_t^i and $\tilde{\varsigma}_t^i$ in the previous equations, recall that, by definition, $-\varsigma_t^i \xi_t^i$ and $-\tilde{\varsigma}_t^i \xi_t^i$ are the loadings of $d\xi_t^i$ with respect to dZ_t and $d\tilde{Z}_t^i$, respectively. We can use $\xi_t^i = e^{-\rho t}/(\rho n_t^i)$ to conclude that

$$\varsigma_t^i = \sigma_t^{n,i}, \qquad \tilde{\varsigma}_t^i = \tilde{\sigma}_t^{n,i},$$

where $\sigma_t^{n,i}$ and $\tilde{\sigma}_t^{n,i}$ are the (geometric) volatility loading of net worth n_t^i for aggregate and idiosyncratic risk, respectively. The net worth evolution (4) combined with the return expressions stated in the beginning of this appendix furthermore implies[67]

$$\sigma_t^{n,i} = \sigma_t^{q,B} - \left(\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i}\right) \frac{\sigma_t^{\vartheta}}{1 - \vartheta_t}, \qquad \tilde{\sigma}_t^{n,i} = \left(\theta_t^{K,i} + \theta_t^{E,i}\right) \tilde{\sigma}_t.$$

We now eliminate the equity portfolio weights. For idiosyncratic volatility, we can use equation (28) to eliminate $\theta_t^{E,i}$. For aggregate volatility, we use that in equilibrium all agents face the same portfolio conditions (28) and (29) and thus optimally choose the same portfolio allocation $\theta_t^{K,i}$, $\theta_t^{E,i}$, $\theta_t^{\bar{E},i}$ (i.e., these quantities do not depend on i). Market clearing in the outside equity market then implies $\theta_t^{E,i} = -\theta_t^{\bar{E},i}$, which allows us to eliminate the sum $\theta_t^{E,i} + \theta_t^{\bar{E},i}$ in the aggregate risk loading.

By combining all equations, we obtain

$$\varsigma_t^i = \sigma_t^{q,B} - \theta_t^{K,i} \frac{\sigma_t^{\vartheta}}{1 - \vartheta_t}, \qquad \tilde{\varsigma}_t^i = \theta_t^{K,i} \bar{\chi} \tilde{\sigma}_t. \tag{30}$$

Derivation of Equation (10). We start from the portfolio choice condition (29), substitute in the costate voatility loadings as stated in equation (30), use that all households choose identical portfolios together with capital market clearing $\theta_t^{K,i} = 1 - \vartheta_t$ as well as the fact that $q_t^K = (1 - \vartheta_t)(q_t^B + q_t^K)$, and rearrange:

$$\frac{1}{1 - \vartheta_t} \frac{a_t - \mathfrak{g} - \iota_t}{q_t^B + q_t^K} - \frac{\mu_t^{\vartheta} - \breve{\mu}_t^{\mathcal{B}}}{1 - \vartheta_t} = (1 - \vartheta_t) \bar{\chi}^2 \tilde{\sigma}_t^2.$$

[67]Compare also the Hamiltonian stated earlier. There, the same expressions enter because n_t^i is a controlled state variable.

By goods market clearing, the second factor in the first term on the left equals ρ. Solving the resulting equation for μ_t^{ϑ} yields

$$\mu_t^{\vartheta} = \rho + \check{\mu}_t^{\mathcal{B}} - (1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2.$$

Equation (10) follows from this equation by multiplying both sides by ϑ_t and using $\mathbb{E}_t[d\vartheta_t] = \mu_t^{\vartheta} \vartheta_t$.

This also proves the first part of Proposition 1. We comment on the proof of the second part below.

Remark on Determination of Equilibrium Quantities in Definition 1. We have claimed in the main text after Lemma 1 that all equilibrium quantities in Definition 1 are functions of ϑ_t and exogenous variables. We still have to justify this claim for the portfolio weights and the expected equity return.

Regarding the portfolio weights, first note that $\theta_t^E = -\theta_t^{\bar{E}}$ by equity market clearing, so that it is sufficient to determine only θ_t^E. This weight, in turn, can be expressed in terms of θ_t^K by equation (28). Finally, capital market clearing, $\theta_t^K = 1 - \vartheta_t$, expresses θ_t^K in terms of ϑ_t.

Regarding the expected equity return, note that the portfolio choice first-order conditions and the characterization of the Lagrange multiplier λ_t^i imply

$$\mathbb{E}[dr_t^E] = \mathbb{E}_t[dr_t^K] - \tilde{\varsigma}_t \tilde{\sigma}_t + \lambda_t^i (1 - \bar{\chi}) = -\bar{\chi} \tilde{\varsigma}_t \tilde{\sigma}_t$$

and then equation (30) together with capital market clearing yield

$$\mathbb{E}[dr_t^E] = -(1 - \vartheta_t) \bar{\chi}^2 \tilde{\sigma}_t^2.$$

The right-hand side is a function of ϑ_t and the exogenous variable $\tilde{\sigma}_t$.

This also concludes the proof of the second part of Proposition 1, as we have constructed all variables in Definition 1 as functions of ϑ_t. By following the previous derivation steps backwards, one verifies that then indeed households maximize utility and all markets clear.

A.2 Uniqueness of Stationary Monetary Equilibria

BSDE (10) is a fixed-point condition for the key equilibrium process ϑ. In this appendix, we show that the BSDE is well-behaved on the domain $(0,1)$ and represents a contraction in a suitable sense to be made precise. The contraction property implies that the equation has at most one nondegenerate stationary solution on this domain.

Let us first consider the finite-horizon version of the BSDE (10) for a fixed terminal condition ϑ_T. In integral form, this BSDE can be written as

$$\forall t \in [0,T]: \; \vartheta_t = \mathbb{E}_t \left[\vartheta_T + \int_t^T \left((1-\vartheta_s)^2 \bar{\chi}^2 \tilde{\sigma}_s^2 - \rho - \breve{\mu}_s^{\mathcal{B}} \right) \vartheta_s ds \right]. \tag{31}$$

Standard results from BSDE theory imply that, under suitable conditions on $\breve{\mu}^{\mathcal{B}}$ and $\tilde{\sigma}$ (boundedness is sufficient), there is a unique solution to the BSDE for any bounded terminal condition (see, e.g., Pham (2009, Theorem 6.2.2)).

The following auxiliary lemma shows that solutions to the finite-horizon BSDE satisfy a type of monotonicity property with respect to the terminal condition. It also implies that, if the terminal condition is in $(0,1]$, then so is the full solution.

Lemma 2. *Let ϑ_t solve the BSDE (31) with terminal condition ϑ_T taking values in $(0,1]$. If ϑ'_t is another solution with terminal condition $\vartheta'_T < \vartheta_T$, then $\vartheta'_t < \vartheta_t$ for all $t \in [0,T]$. If $\rho + \breve{\mu}_t^{\mathcal{B}} > 0$ for all $t \in [0,T]$, then $\vartheta_t \in (0,1)$ for all $t < T$.*

Proof. First, observe that whenever $\vartheta_T \geq \vartheta'_T$, the comparison principle for BSDEs implies that $\vartheta_t \geq \vartheta'_t$ (see, for example, Pham (2009, Theorem 6.2.2)). Furthermore, $\vartheta_t > \vartheta'_t$ if $\vartheta_T > \vartheta'_T$ with positive probability.

Second, let us compare the solution ϑ_t of BSDE (31), which we write in integrated form as

$$-d\vartheta_t = \underbrace{\left((1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 - \rho - \breve{\mu}_t^{\mathcal{B}} \right) \vartheta_t}_{=f_t^1(\vartheta_t)} dt - \nu_t dZ_t$$

with the solution $\bar{\vartheta}_t = 1$ of BSDE

$$-d\bar{\vartheta}_t = \underbrace{0}_{=f_t^2(\bar{\vartheta}_t)} dt - \nu_t dZ_t$$

with terminal condition $\bar{\vartheta}_T = 1$.

Since terminal conditions satisfy $\bar{\vartheta}_T \geq \vartheta_T$, and generators satisfy

$$0 = f_t^2(\bar{\vartheta}_t) > f_t^1(\bar{\vartheta}_t) = -\left(\rho + \breve{\mu}_t^{\mathcal{B}}\right),$$

the comparison principle implies that $\vartheta_t < \bar{\vartheta}_t = 1$ for all $t \in [0, T)$. □

One could now attempt to solve the infinite-horizon BSDE (10) by starting at some terminal guess ϑ_T of the finite-horizon BSDE and considering longer and longer time horizons ($T \to \infty$). It is, however, a priori unclear whether this procedure converges and, if so, whether the limit is independent of the assumed terminal guess.

The following technical lemma is key in establishing that this strategy succeeds (under certain conditions).

Lemma 3. *Suppose $\breve{\mu}_t^{\mathcal{B}} + \rho > 0$ for all t. Then the finite-horizon BSDE (31) is a contraction on logarithmic scale:*
Consider any two distinct terminal conditions ϑ_T and ϑ'_T with values in $(0, 1)$. Let ϑ_t and ϑ'_t be the corresponding solutions. Then for all $t < T$, ϑ_t and ϑ'_t have values in $(0, 1)$ and satisfy[68]

$$\log \vartheta_t - \log \vartheta'_t \in \left(\operatorname{ess\,inf}\left(\log \vartheta_T - \log \vartheta'_T\right), \operatorname{ess\,sup}\left(\log \vartheta_T - \log \vartheta'_T\right)\right). \qquad (32)$$

Proof. The statement of the lemma is equivalent to

$$\frac{\vartheta_t}{\vartheta'_t} \in \left(\operatorname{ess\,inf}\frac{\vartheta_T}{\vartheta'_T}, \operatorname{ess\,inf}\frac{\vartheta_T}{\vartheta'_T}\right).$$

Let us prove that

$$\frac{\vartheta_t}{\vartheta'_t} > x := \operatorname{ess\,inf}\frac{\vartheta_T}{\vartheta'_T}$$

as the other bound is symmetric. Without loss of generality, let us assume that $\vartheta_T \leq \vartheta'_T$, because replacing ϑ_T with $\min(\vartheta_T, \vartheta'_T)$ only weakly lowers ϑ_t by Lemma 2 and makes the bound harder to prove.

[68] Here ess inf and ess sup denote the essential infimum and essential supremum, respectively, and are taken over all outcomes of the underlying probability space.

Equation (31) implies that ϑ_t satisfies

$$\vartheta_t = \mathbb{E}_t\left[\exp\left(\int_t^T \left((1-\vartheta_s)^2\bar{\chi}^2\tilde{\sigma}_s^2 - \rho - \breve{\mu}_s^{\mathcal{B}}\right) ds\right)\vartheta_T\right] \tag{33}$$

and an analogous expression holds for ϑ'_t. Since $\vartheta_s \leq \vartheta'_s < 1$ for all $s \in [t, T]$ we have $(1-\vartheta_s)^2 \leq (1-\vartheta'_s)^2$ and so

$$\begin{aligned}\vartheta_t &\geq \mathbb{E}_t\left[\exp\left(\int_t^T \left((1-\vartheta'_s)^2\bar{\chi}^2\tilde{\sigma}_s^2 - \rho - \breve{\mu}_s^{\mathcal{B}}\right) ds\right)\vartheta_T\right] \\ &\geq \mathbb{E}_t\left[\exp\left(\int_t^T \left((1-\vartheta'_s)^2\bar{\chi}^2\tilde{\sigma}_s^2 - \rho - \breve{\mu}_s^{\mathcal{B}}\right) ds\right) x\vartheta'_T\right] = x\vartheta'_t,\end{aligned}$$

Hence, $\vartheta_t/\vartheta'_t \geq x$. If ϑ_T and ϑ'_T are distinct, then the inequality must be strict. □

Lemma 3 has important implications for the infinite-horizon BSDE (10) if the economy is *stationary* (compare Definition 2).

Proposition 8. *Suppose the exogenous processes are stationary and $\rho + \breve{\mu}^{\mathcal{B}}(X) > 0$ for all $X \in \mathbb{X}$. Then, equation (10) has at most one stationary nondegenerate (i.e. not identically 0) solution.*

If this solution exists, $\vartheta_t = \vartheta^*(X_t)$, then

- *for all $X \in \mathbb{X}$, $\vartheta^*(X) > 0$;*
- *for any function $\vartheta' : \mathbb{X} \to (0,1)$, the solution to the finite-horizon equation (31) with terminal condition $\vartheta_T = \vartheta'(X_T)$ converges to $\vartheta_t = \vartheta^*(X_t)$ as $T \to \infty$.*

If equation (10) has no stationary nondegenerate solution, then for any terminal condition $\vartheta_T = \vartheta'(X_T)$, the solution to the finite-horizon equation converges to $\vartheta_t = 0$ as $T \to \infty$.

We note that Proposition 8 encompasses all statements in Proposition 2 stated in the main text. Proving the former therefore also implies the latter. Before we present the proof, we first establish another small technical lemma.

Lemma 4. *Suppose that economy is stationary and let $\vartheta_t = \vartheta(t, X_t)$ solve (31) with terminal condition $\vartheta_T = \vartheta(T, X_T)$, with values in $(0, 1]$. If $\vartheta(t, X) > \vartheta(T, X)$ for all X and $t < T$, then $\vartheta(t, X)$ increases as t declines. If $\vartheta(t, X) < \vartheta(T, X)$, then $\vartheta(t, X)$ declines as t declines.*

Proof. The two statements are symmetric, so let us prove the first one. We would like to show that $\vartheta(t-s, X) > \vartheta(t, X)$. These are solutions to (31) with time horizon $T-t$ and terminal conditions $\vartheta(T-s, X) > \vartheta(T, X)$. By Lemma 2, $\vartheta(t-s, X) > \vartheta(t, X)$. $\square$

Proof of Proposition 8. First, let us show that any stationary nondegenerate solution $\vartheta(X)$ must be strictly positive. If $\vartheta(X') = 0$ for some $X' \in \mathbb{X}$, then $\vartheta_t = 0$ when $X_t = X$, hence equation (33) can hold only if $\vartheta_T = 0$ almost surely for all future T. Since state process X_t is ergodic, it follows that $\vartheta(X) = 0$ almost surely, but then (33) implies that $\vartheta(X) = 0$ for all X. Therefore, ϑ cannot degenerate to 0 at any single point.

Let us prove that there is at most one stationary nondegenerate solution. Suppose $\vartheta_1(X)$ and $\vartheta_2(X)$ are two distinct solutions, with $\vartheta_1(X) < \vartheta_2(X)$ for some $X \in \mathbb{X}$. Then $x := \inf_{X \in \mathbb{X}} \frac{\vartheta_1(X)}{\vartheta_2(X)} < 1$, and by the compactness of the domain $\mathbb{X}$, the infimum is attained at some point $\underline{X}$ (as ϑ_1, ϑ_2 are assumed to be continuous).

Now, suppose $X_t = \underline{X}$ and consider solutions ϑ and ϑ' of equation (31) with terminal conditions $\vartheta_T = \vartheta_1(X_T)$ and $\vartheta'_T = \vartheta_2(X_T)$. Then, by uniqueness of solutions to the BSDE (31), we have $\vartheta_t = \vartheta_1(\underline{X})$ and $\vartheta'_t = \vartheta_2(\underline{X})$. But then

$$\frac{\vartheta_t}{\vartheta'_t} = \frac{\vartheta_1(\underline{X})}{\vartheta_2(\underline{X})} \leq \inf \frac{\vartheta_T}{\vartheta'_T},$$

a contradiction to Lemma 3.

Now, suppose (10) has no stationary nondegenerate solution. Consider the solution to equation (31) with terminal condition $\vartheta_T = 1$. Then by Lemma 2, $\vartheta_{T-s} < 1$ for all $s > 0$, and by Lemma 4, ϑ_t declines for each X as the horizon T increases. Hence, ϑ_t must converge to some function $\vartheta^*(X)$. By continuity $\vartheta^*(X)$ is a solution to (10), and because there are no stationary nondegenerate solutions, the limit must be $\vartheta^*(X) = 0$. Now, if ϑ'_t a solution with a different terminal condition $\vartheta'_T < 1$, then $\vartheta'_t < \vartheta_t$ by the comparison principle (Lemma 2), hence ϑ'_t must also converge to 0.

Finally, suppose (10) does have a stationary nondegenerate solution. Then the solution ϑ from the terminal condition $\vartheta_T = 1$ is likewise declining as we go backwards in time and converges to a solution. Since ϑ stays above the stationary nondegenerate solution ϑ^* by Lemma 2, it must converge to ϑ^*. Likewise, the solution ϑ' from the terminal condition $\vartheta'_T = \epsilon\hat{\vartheta}$ increases as we go backwards in time (by Lemmas 4 and 3), and converges to ϑ^*. By the comparison principle, any other solution ϑ''_t with terminal condition $\vartheta''_T(X) \in [\epsilon\hat{\vartheta}(X), 1]$ will also be squeezed between ϑ'_t and ϑ_t, hence will converge to $\hat{\vartheta}$. □

Proposition 8 implies that to solve (10), we do not need a good guess of the terminal condition. Any nonzero guess will converge to a stationary solution and, if it exists, the nondegenerate one.

We remark that when the standard solution is nondegenerate, then equation (10) does have many other nonstationary solutions (i.e. the uniqueness result applies only to the stationary solution). However, Proposition 8 implies that all nonstationary solutions converge to 0 in the distant future.

A.3 Proof of Proposition 3

Note that, in general, because $\xi_t^i = e^{-\rho t}/(\rho n_t^i)$ and $dr_t^{n,i}$ has the same risk loadings as dn_t^i/n_t^i (compare equation (4)),

$$\mathrm{Cov}_t\left(\frac{d\xi_t^i}{\xi_t^i}, dr_t^{n,i}\right) = \mathrm{Cov}_t\left(\frac{d(1/n_t^i)}{1/n_t^i}, \frac{dn_t^i}{n_t^i}\right) = -\left(\left(\sigma_t^{n,i}\right)^2 + \left(\tilde{\sigma}_t^{n,i}\right)^2\right)$$

Under the assumptions of the proposition, ϑ_t and thus prices q_t^K, q_t^B do not load on the aggregate shock dZ_t, $\sigma_t^{\vartheta} = \sigma_t^{q,B} = \sigma_t^{q,K} = 0$. In particular, $\sigma_t^{n,i} = 0$ (compare equation (30) and recall that prices of risk and net worth loadings coincide). Thus

$$\mathrm{Cov}_t\left(\frac{d\xi_t^i}{\xi_t^i}, dr_t^{n,i}\right) = \mathrm{Cov}_t\left(\frac{d(1/n_t^i)}{1/n_t^i}, \frac{dn_t^i}{n_t^i}\right) = -\left(\tilde{\sigma}_t^{n,i}\right)^2 = -(1-\vartheta)^2\bar{\chi}^2\tilde{\sigma}^2 < 0,$$

where the last equation follows from equation (30) and market clearing for $\theta_t^{K,i}$.

In contrast, $\sigma_t^{q,B} = 0$ implies that $dr_t^{\mathcal{B}}$ is locally deterministic (does not load on Brownian shocks), so that

$$\mathrm{Cov}_t\left(\frac{d\xi_t^i}{\xi_t^i}, dr_t^{\mathcal{B}}\right) = 0.$$

Comparing the two covariances reveals that the former is always strictly smaller. Thus the bond is a safe asset for agent i at all times t.

A.4 Derivation of Equation (11)

We derive here equation (11), the debt valuation equation from the buy and hold perspective. We start by using $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$ to rewrite the government flow budget constraint (2) as

$$-\left(d\mathcal{B}_t - i_t \mathcal{B}_t dt\right) = \mathcal{P}_t \underbrace{\left(\tau a_t - \mathfrak{g}\right)}_{=s_t} K_t dt.$$

Multiplying both sides by the nominal SDF $\xi_t^i / \mathcal{P}_t$ of agent i and using Ito's product rule to replace $\xi_t^i / \mathcal{P}_t d\mathcal{B}_t$ with $d\left(\xi_t^i / \mathcal{P}_t \mathcal{B}_t\right) - \mathcal{B}_t d(\xi_t^i / \mathcal{P}_t)$[69] yields

$$-d\left(\xi_t^i \mathcal{B}_t / \mathcal{P}_t\right) + \mathcal{B}_t \left(d\left(\xi_t^i / \mathcal{P}_t\right) + i_t \xi_t^i / \mathcal{P}_t dt\right) = \xi_t^i s_t K_t dt.$$

Integrating this equation from $t = 0$ to $t = T$, taking expectations, and solving for $\xi_0^i \mathcal{B}_0 / \mathcal{P}_0$ implies

$$\xi_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^T \xi_t^i s_t K_t dt\right] - \mathbb{E}\left[\int_0^T \mathcal{B}_t \left(d\left(\xi_t^i / \mathcal{P}_t\right) + i_t \xi_t^i / \mathcal{P}_t dt\right)\right] + \mathbb{E}\left[\xi_T^i \frac{\mathcal{B}_T}{\mathcal{P}_T}\right]. \quad (34)$$

Equation (34) is simply an accounting identity, an integrated version of the government flow budget constraint (2). We now note that the individual SDF ξ_t^i must price the bond because agent i is marginal in the bond market. This implies that the associated nominal SDF $\xi_t^i / \mathcal{P}_t$ must decay on average at the nominal market interest rate, so that the second term in equation (34) vanishes. In addition, we can replace the individual SDF ξ_t^i with the average SDF $\bar{\xi}_t$ because equation (34) holds for all individuals i and $s_t K_t$ and $\mathcal{B}_T / \mathcal{P}_T$ are free of idiosyncratic risk. When taking the limit $T \to \infty$, we obtain equation (11).

A.5 Omitted Steps in the Derivation of Equation (13)

We present here the missing steps in the derivation of equation (13) left out in the main text. There, we have used without proof equation (14), which expresses the value of the bond portfolio as the present value of trading cash flows, and equation (15), which characterizes the bond trading process. In addition, we have used that the price

[69] There is no quadratic covariation term because $d\mathcal{B}_t$ is absolutely continuous.

of idiosyncratic risk is given by

$$\tilde{\varsigma}_t^i = (1 - \vartheta_t)\bar{\chi}\tilde{\sigma}_t.$$

This equation has already been derived in Appendix A.1 (compare equation (30) and the market clearing equation for $\theta_t^{K,i}$ stated in the subsequent paragraph).[70] We thus only derive equations (14) and (15) here.

Derivation of equation (14). We can write the evolution of the bond portfolio as

$$\frac{db_t^i}{b_t^i} = dr_t^{\mathcal{B}} + d\Delta_t^{b,i}. \tag{35}$$

Absent trading and payouts, the bond portfolio grows at the (stochastic) bond return $dr_t^{\mathcal{B}}$, but the actual portfolio value has to be adjusted for cash inflows $b_t^i d\Delta_t^{b,i}$ due to trading and payouts. Writing

$$dr_t^{\mathcal{B}} = \mu_t^{r^{\mathcal{B}}} dt + \sigma_t^{r^{\mathcal{B}}} dZ_t, \qquad d\Delta_t^{b,i} = \mu_t^{\Delta,i} dt + \sigma_t^{\Delta,i} dZ_t + \tilde{\sigma}_t^{\Delta,i} d\tilde{Z}_t^i$$

and using Ito's product rule, we obtain for the discounted bond wealth

$$\begin{aligned} \frac{d(\xi_t^i b_t^i)}{\xi_t^i b_t^i} = &\left(\underbrace{\mu_t^{r^{\mathcal{B}}} - r_t^f - \varsigma_t \sigma_t^{r^{\mathcal{B}}}}_{=0} + \mu_t^{\Delta,i} - \varsigma_t \sigma_t^{\Delta,i} - \tilde{\varsigma}_t^i \tilde{\sigma}_t^{\Delta,i} \right) dt \\ &+ \left(\sigma_t^{r^{\mathcal{B}}} + \sigma_t^{\Delta,i} - \varsigma_t \right) dZ_t + \left(\tilde{\sigma}_t^{\Delta,i} - \tilde{\varsigma}_t \right) d\tilde{Z}_t^i. \end{aligned} \tag{36}$$

Here, the first part of the drift is zero by standard asset pricing logic because agent i is marginal in the market for government bonds. Integrating over $t \in [0, T]$, taking expectations, and rearranging yields

$$\xi_0^i b_0^i = -\mathbb{E}_0 \left[\int_0^T \xi_t^i b_t^i \left(\mu_t^{\Delta,i} - \varsigma_t \sigma_t^{\Delta,i} - \tilde{\varsigma}_t^i \tilde{\sigma}_t^{\Delta,i} \right) dt \right] + \mathbb{E}_0 \left[\xi_T^i b_T^i \right].$$

Optimal behavior implies a transversality condition $\lim_{T\to\infty} \mathbb{E}\left[\xi_T^i n_T^i\right] = 0$ on total wealth n_T^i of agent i as a necessary choice condition. Because total wealth consists of

[70] Note that we have defined ξ_t^i in Appendix A.1 as the costate in the household's problem whereas we use the same notation for the individual SDF in Section 3. The two are the same (up to scaling) because both measure the marginal utility of an additional unit of wealth at time t in a given state.

bond wealth and capital wealth and the latter cannot become negative, a transversality condition for bond wealth b_T^i immediately follows. Consequently, the second term converges to zero as $T \to \infty$ and we obtain equation (14) in the limit.

Derivation of equation (15). To characterize the trading process $d\Delta_t^{b,i}$, start from (35):

$$d\Delta_t^{b,i} = \frac{db_t^i}{b_t^i} - dr_t^{\mathcal{B}}. \tag{37}$$

Because all agents hold the same fraction $\theta_t^i = \vartheta_t$ of their net worth in bonds, we have $b_t^i = \eta_t^i q_t^B K_t$. As η_t^i loads only on the idiosyncratic Brownian and $q_t^B K_t$ only on the aggregate Brownian, their quadratic covariation vanishes and thus Ito's product rule simply implies

$$\frac{db_t^i}{b_t^i} = \frac{d(q_t^B K_t)}{q_t^B K_t} + \frac{d\eta_t^i}{\eta_t^i}.$$

Furthermore, the return on bonds can be written as (compare equation (6))

$$dr_t^{\mathcal{B}} = \frac{d(q_t^B K_t)}{q_t^B K_t} - \breve{\mu}_t^{\mathcal{B}} dt.$$

Substituting the previous two equations into (37) yields

$$d\Delta_t^{b,i} = \breve{\mu}_t^{\mathcal{B}} dt + \frac{d\eta_t^i}{\eta_t^i} = \breve{\mu}_t^{\mathcal{B}} dt + \sigma_t^{\eta,i} d\tilde{Z}_t^i,$$

which implies

$$\mu_t^{\Delta,i} = \breve{\mu}_t^{\mathcal{B}}, \qquad \sigma_t^{\Delta,i} = 0, \qquad \tilde{\sigma}_t^{\Delta,i} = \sigma_t^{\eta,i}.$$

The equations in formula (15) follow, if we can show that

$$\breve{\mu}_t^{\mathcal{B}} = -s_t / q_t^B, \tag{38}$$

$$\sigma_t^{\eta,i} = (1 - \vartheta_t)\,\bar{\chi}\tilde{\sigma}_t. \tag{39}$$

Equation (38) follows immediately from the government budget constraint (2) and the definition of s_t. For the proof of equation (39), note that individual net worth n_t^i and total net worth $N_t := (q_t^K + q_t^B)K_t$ have identical drifts and volatility loadings on the aggregate Brownian dZ_t, so that simply

$$\frac{d\eta_t^i}{\eta_t^i} = \frac{d(n_t^i/N_t)}{n_t^i/N_t} = \tilde{\sigma}_t^{n,i} d\tilde{Z}_t^i$$

because n_t^i loads on the idiosyncratic Brownian $d\tilde{Z}_t^i$, but N_t does not. Combining the net worth evolution (4) with the equilibrium portfolio weights, we obtain

$$\tilde{\sigma}_t^{n,i} = (1 - \vartheta_t)\bar{\chi}\tilde{\sigma}_t \tag{40}$$

which completes the proof of (39).

A.6 Representative Agent Formulation

In this Appendix we present additional details on the representative agent formulation summarized in Section 3. In Part A.6.1, we outline the setup of the hypothetical representative agent tree economy that generates the same asset prices and allocations as our incomplete markets economy and discuss substantive economic takeaways. Additional technical derivation details, including the omitted steps in the arguments that lead to Proposition 6 in the main text, can be found in Part A.6.2.

A.6.1 The Representative Agent Economy

We present a Lucas (1978)-type asset pricing economy that generates the same allocation as in the competitive equilibrium of our incomplete markets economy. In this economy, we interpret aggregate capital and aggregate bonds as two "trees" and we show that equation (13) is precisely the valuation equation for the "bond tree" from the perspective of the representative agent. The dynamic trading perspective is therefore equivalent to the perspective of a hypothetical representative agent.

As stated in the main text, we consider a representative agent whose preferences are represented by a weighted welfare function $\mathcal{W}_0 = \int \lambda^i V_0^i di$. We denote by $\eta_t^i := c_t^i/C_t$ the consumption share of agent i and assume that $d\eta_t^i = \tilde{\sigma}_t^\eta d\tilde{Z}_t^i$ with volatility process $\tilde{\sigma}_t^\eta$ specified below in equation (42). As shown in the main text, utility $\mathcal{W}_0$ satisfies equation (17), which expresses utility in terms of aggregate consumption C_t and consumption shares η_t^i. We show below (Part A.6.2) that utility can also be represented in the form (this is equation (18) in the main text)

$$\mathcal{W}_0 = w_0 + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^\eta\right)^2\right)dt\right] \tag{41}$$

with some constant w_0. This equation eliminates the direct dependence on i and gives us the alternative interpretation that two "goods" enter the representative agent's utility function, the aggregate consumption good and a "volatility good" (which generates disutility).[71]

We assume that the representative agent has access to two assets, capital K_t, which produces a certain bundle of the aggregate consumption good and volatility $\tilde{\sigma}_t^\eta$, and "derivatives" X_t, which mimic the cash flows to individuals i generated by bond trades in our incomplete markets model and thereby reduce volatility. Capital grows at rate $g_t := \Phi(\iota_t) - \delta$ over time and generates consumption goods at rate $\left((1-\tau_t)a_t - \iota_t\right) K_t dt$. For the purpose of this representative agent economy, g_t, τ_t, a_t, ι_t are exogenous processes. But, of course, we choose for them the stochastic processes implied by the competitive equilibrium of our incomplete markets model.[72] The same remark holds for other lower-case variables q_t^B, q_t^K, $\breve{\mu}_t^{\mathcal{B}}$ used below. The face value X_t of derivatives evolves according to

$$dX_t/X_t = \left(g_t + \mu_t^{q,B}\right)dt + \sigma_t^{q,B} dZ_t,$$

where $\mu_t^{q,B}$, $\sigma_t^{q,B}$ are the drift and volatility processes of q_t^B implied by the competitive equilibrium of the incomplete markets model. Derivatives generate a cash flow $-\breve{\mu}_t^{\mathcal{B}} X_t$ and reduce fluctuations in consumption shares η_t^i. Specifically, the volatility loading $\tilde{\sigma}_t^\eta$ satisfies the equation

$$\left(q_t^K K_t + X_t\right)\tilde{\sigma}_t^\eta = q_t^K K_t \bar{\chi} \tilde{\sigma}_t, \tag{42}$$

where q_t^K is the capital price process from the incomplete markets economy. We can interpret the product $X_t \tilde{\sigma}_t^\eta$ as a measure of the aggregate gross trading cash flows from bond trades in response to idiosyncratic shocks in the incomplete markets economy.[73]

[71]The representative agent's objective is akin to a money in utility (MIU) model. Holding the derivative asset introduced below reduces volatility $\tilde{\sigma}_t^\eta$ in a similar way as holding money in a MIU model generates utility services.

[72]We could also endogenize the real investment decision by letting the representative agent choose ι_t. The representative agent would choose precisely the rate ι_t we are taking here as exogenous.

[73]$q_t^K k_t^i \bar{\chi} \tilde{\sigma}_t$ is sensitivity of an agent i's capital wealth to shocks $d\tilde{Z}_t^i$ before portfolio rebalancing and

Let Q_t^K be the capital price that the representative agent faces, P_t^X the price per unit (face value) of derivatives, and let $N_t := Q_t^K K_t + P_t^X X_t$ be the representative agent's total net worth. The budget constraint of the representative agent is

$$dN_t = -C_t dt + Q_t^K K_t dr_t^K + P_t^X X_t dr_t^X \tag{43}$$

with return processes

$$dr_t^K = \left(\frac{(1-\tau_t)a_t - \iota_t}{Q_t^K} + \mu_t^{Q,K} + g_t \right) dt + \sigma_t^{Q,K} dZ_t,$$
$$dr_t^X = \left(\mu_t^{P,X} + g_t - \breve{\mu}_t^{\mathcal{B}} + \sigma_t^{q,B}\sigma_t^{P,X} \right) dt + \left(\sigma_t^{q,B} + \sigma_t^{P,X} \right) dZ_t.$$

The representative agent chooses $C_t, \tilde{\sigma}_t^{\eta}, K_t, X_t$ to maximize utility $\mathcal{W}_0$ subject to the budget constraint (43) and the risk constraint (42) taking the prices Q_t^K, P_t^X and the return processes as given. The representative agent model is closed by time-zero supplies of capital (K_0) and derivatives (X_0). We impose the additional relationship $X_0 = q_0^B K_0$, where q_0^B is the initial value of q_t^B in the incomplete markets model. While this supply restriction for X_0 may appear ad hoc, it can be micro-founded in an environment with information frictions in which idiosyncratic shocks are private information and agents have access to hidden trade and savings.[74] In such an environment, incentive compatibility requires that any insurance transfer to an agent must be precisely offset by a reduction in the present value of that agent's future consumption. Otherwise, the agent would have incentives to misreport the size of the shock and secretly trade capital. Incentive compatibility thus limits the amount of insurance that can be provided, i.e. the quantity X of derivatives.

We show below that the competitive equilibrium of this representative agent economy features prices $Q_t^K = q_t^K$ and $P_t^X = 1$ (and thus $P_t^X X_t = q_t^B K_t$), so that asset prices are the same as in the incomplete markets economy.[75] Also, as we have already stated in the main text, the representative agent's SDF process satisfies $\Xi_t = \xi_t^{**}$ (compare Proposition 6).

$q_t^K k_t^i \tilde{\sigma}_t^{\eta}$ is the shock sensitivity after rebalancing. The difference, $q_t^K k_t^i \left(\bar{\chi}\tilde{\sigma}_t - \tilde{\sigma}_t^{\eta} \right)$ measures trading cash flows per unit of $d\tilde{Z}_t^i$ and aggregating over all agents yields $X_t \tilde{\sigma}_t^{\eta}$.

[74] Details on this micro-foundation can be found in Brunnermeier et al. (2020). This information environment has also been employed by Di Tella (2020) in a closely related model.

[75] Also aggregate consumption C_t and the consumption shares η_t^i are as in the incomplete markets economy. The representative agent economy therefore leads to the same allocation.

The valuation equation for derivatives from the perspective of the representative agent is

$$P_0^X X_0 = \mathbb{E}\left[\int_0^\infty \Xi_t \cdot \left(-\breve{\mu}_t^{\mathcal{B}} X_t\right) dt\right] + \mathbb{E}\left[\int_0^\infty \Xi_t \cdot (1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 X_t dt\right]. \tag{44}$$

Here, the first term represents the discounted present value of cash flows $-\breve{\mu}_t^{\mathcal{B}} X_t$ and the second term represents the discounted volatility reduction service flows that derivatives provide by lowering $\tilde{\sigma}^{\eta}$ in the utility function (18). As derivatives in the representative agent economy play the same role as bonds in the incomplete markets economy, we can make the identification $X_t = q_t^{\mathcal{B}} K_t$ and $-\breve{\mu}_t^{\mathcal{B}} X_t = s_t K_t$. With these replacements (and $P_0^X = 1$), equation (44) becomes equation (13), the debt valuation equation from the dynamic trading perspective.

A.6.2 Additional Derivation Details and Proofs

Missing Step in Proof of Proposition 6: Ξ is Independent of Welfare Weights. For CRRA utility with parameter γ, we have

$$\Xi_t = e^{-\rho t} \frac{\int \lambda^i \eta_t^i u'(\eta_t^i C_t) di}{\int \lambda^i \eta_0^i u'(\eta_0^i C_0) di} = e^{-\rho t} \frac{\int \lambda^i (\eta_0^i)^{1-\gamma} (\eta_t^i/\eta_0^i)^{1-\gamma} di}{\int \lambda^i (\eta_0^i)^{1-\gamma} di} \frac{C_t^{-\gamma}}{C_0^{-\gamma}}.$$

Furthermore, η_t^i/η_0^i is given by

$$(\eta_t^i/\eta_0^i)^{1-\gamma} = \exp\left((1-\gamma)\int_0^t \tilde{\sigma}_\tau^{\eta} d\tilde{Z}_\tau^i - \frac{1-\gamma}{2}\int (\tilde{\sigma}_\tau^{\eta})^2 d\tau\right)$$

and the distribution of this object conditional on aggregate information (i.e. information in Z) does not depend on i. In particular, there is a random variable X_t such that

$$X_t = \mathbb{E}[(\eta_t^i/\eta_0^i)^{1-\gamma} \mid Z_\tau : \tau \le t]$$

for all i. Because Ξ_t is adapted to the filtration generated by the aggregate Brownian motion Z,

$$\Xi_t = \mathbb{E}[\Xi_t \mid Z_\tau : \tau \le t] = e^{-\rho t} \frac{\mathbb{E}[\int \lambda^i (\eta_0^i)^{1-\gamma} (\eta_t^i/\eta_0^i)^{1-\gamma} di \mid Z_\tau : \tau \le t]}{\int \lambda^i (\eta_0^i)^{1-\gamma} di} \frac{C_t^{-\gamma}}{C_0^{-\gamma}}$$

$$= e^{-\rho t} \frac{\int \lambda^i (\eta_0^i)^{1-\gamma} \mathbb{E}[(\eta_t^i / \eta_0^i)^{1-\gamma} \mid Z_\tau : \tau \leq t] di}{\int \lambda^i (\eta_0^i)^{1-\gamma} di} \frac{C_t^{-\gamma}}{C_0^{-\gamma}} = e^{-\rho t} \underbrace{\frac{\int \lambda^i (\eta_0^i)^{1-\gamma} di}{\int \lambda^i (\eta_0^i)^{1-\gamma} di}}_{=1} X_t \frac{C_t^{-\gamma}}{C_0^{-\gamma}}.$$

Hence, Ξ_t does not depend on the choice of the weights λ^i.

Derivation of Utility Representation (18). By Ito's formula,

$$\log \eta_t^i = \log \eta_0^i - \frac{1}{2} \int_0^t \left(\tilde{\sigma}_s^\eta \right)^2 ds + \int_0^t \tilde{\sigma}_s^\eta d\tilde{Z}_s^i$$

and thus

$$\begin{aligned} \int_0^\infty e^{-\rho t} \int \lambda^i \mathbb{E} \left[\log \eta_t^i \right] didt &= \int_0^\infty e^{-\rho t} \int \lambda^i \log \eta_0^i didt - \frac{1}{2} \int_0^\infty e^{-\rho t} \int \lambda^i \int_0^t \left(\tilde{\sigma}_s^\eta \right)^2 dsdidt \\ &= \frac{1}{\rho} \int \lambda^i \log \eta_0^i di - \frac{1}{2} \int \lambda^i di \int_0^\infty e^{-\rho t} \int_0^t \left(\tilde{\sigma}_s^\eta \right)^2 dsdt \\ &= \frac{1}{\rho} \int \lambda^i \log \eta_0^i di - \frac{1}{2\rho} \int_0^\infty e^{-\rho t} \left(\tilde{\sigma}_t^\eta \right)^2 dt, \end{aligned}$$

where the last line uses that $\int \lambda^i di = 1$. Substituting this into equation (17) (with interchanged order of integration where necessary) implies

$$\mathcal{W}_0 = \frac{1}{\rho} \int \lambda^i \log \eta_0^i di + \mathbb{E} \left[\int_0^\infty e^{-\rho t} \left(\log C_t - \frac{1}{2\rho} \left(\tilde{\sigma}_t^\eta \right)^2 \right) dt \right]$$

With the definition $w_0 := \frac{1}{\rho} \int \lambda^i \log \eta_0^i di$, this is precisely equation (18).

Competitive Equilibrium in Representative Agent Economy. As this is a representative agent economy, we can fully characterize the allocation by determining goods and asset supplies. The problem of the representative agent only needs to be considered to determine asset prices.

The assumed growth rate process for capital K_t is the same as in the equilibrium of the incomplete markets model, so that K_t must follow precisely the same process as in that equilibrium if we start from the same initial K_0 (which we can assume w.l.o.g. as this only scales the overall size of the economy). Because $dX_t/X_t = d(q_t^B K_t)/(q_t^B K_t)$ and $X_0 = q_0^B X_0$ by the condition on initial supply, we then also have $X_t = q_t^B K_t$ for all

t. Total consumption goods produced by the two "trees" in period t are

$$\begin{aligned} C_t &= \left((1-\tau_t)\, a_t - \iota_t\right) K_t - \breve{\mu}_t^{\mathcal{B}} X_t \\ &= \left(a_t - \iota_t + \tau_t a_t - \breve{\mu}_t^{\mathcal{B}} q_t^B\right) K_t \\ &= (a_t - \mathfrak{g} - \iota_t)\, K_t, \end{aligned}$$

where the last line follows from the government budget constraint (2) (in the incomplete markets model). The aggregate consumption goods supply is thus the same as the (endogenous) aggregate consumption process in the incomplete markets economy.

We now turn to the remaining "good", volatility reduction. Total volatility "supply" is determined by equation (42),

$$\tilde{\sigma}_t^{\eta} = \frac{q_t^K K_t}{q_t^K K_t + X_t} \bar{\chi} \tilde{\sigma}_t = \frac{q_t^K K_t}{q_t^K K_t + q_t^B K_t} \bar{\chi} \tilde{\sigma}_t = (1-\vartheta_t)\, \bar{\chi} \tilde{\sigma}_t.$$

This is also the same as the (endogenous) volatility of consumption shares η_t^i in the incomplete markets economy. The representative agent economy therefore generates the same allocation as the equilibrium in our incomplete markets model.

We now turn to asset prices. As this is the decision problem of a consumer with logarithmic utility, the optimal consumption rule is $C_t = \rho N_t$, exactly as for the agents in our incomplete markets economy.[76] This fact can be derived using the stochastic maximum principle in precisely the same way as in Appendix A.1, so that we skip the details here. Using the definition $N_t = Q_t^K K_t + P_t^X X_t$ and the supplies $X_t = q_t^B K_t$, $C_t = (q_t^B + q_t^K) K_t$ derived previously, we obtain

$$(q_t^B + q_t^K) K_t = \frac{C_t}{\rho} = Q_t^K K_t + P_t^X X_t = (Q_t^K + P_t^X q_t^B) K_t.$$

Therefore, if we can show $P_t^X = 1$, $Q_t^K = q_t^K$ is automatically implied. $P_t^X = 1$, in turn, follows from equation (44) and the remarks following it in the main text. Consequently, we only need to derive equation (44) to complete the equilibrium characterization.

Valuation Formula (44) for "Derivatives". We can use standard asset pricing logic. From the perspective of the representative agent, this is an entirely standard complete

[76]In the utility function here, there is also a second term ($\tilde{\sigma}_t^{\eta}$). But because it is additively separated, it does not affect the optimal consumption rule.

markets economy with two consumption goods. The price of a single unit of an asset measured in time-zero consumption units must thus equal the sum of the present discounted value of its future marginal consumption flow dividends and the present discounted future consumption value of its marginal volatility flow dividends, both discounted with the SDF Ξ_t, the marginal rate of substitution between consumption at time t and consumption at time 0.

The consumption flow term is straightforward. One unit of derivatives at time 0 turns into X_t/X_0 units of derivatives at time t and each of them produces a consumption flow $-\breve{\mu}_t^{\mathcal{B}} dt$. The present discounted value of these future consumption flows is therefore

$$\mathbb{E}\left[\int_0^\infty \Xi_t \left(-\breve{\mu}_t^{\mathcal{B}} \frac{X_t}{X_0}\right) dt\right].$$

For the volatility flow term, note that the "marginal volatility product of derivatives" at time t is

$$\frac{\partial \tilde{\sigma}_t^\eta}{\partial X_t} = -\frac{q_t^K K_t}{\left(q_t^K K_t + X_t\right)^2} \bar{\chi} \tilde{\sigma}_t = -\frac{\tilde{\sigma}_t^\eta}{\left(q_t^K + q_t^B\right) K_t} = -\frac{\tilde{\sigma}_t^\eta}{N_t}$$

and the marginal rate of substitution between time-t consumption and time-t volatility is

$$\frac{\partial \left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^\eta\right)^2\right) / \partial \tilde{\sigma}_t^\eta}{\partial \left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^\eta\right)^2\right) / \partial C_t} = \frac{-\tilde{\sigma}_t^\eta / \rho}{1/C_t} = -\frac{C_t}{\rho} \tilde{\sigma}_t^\eta.$$

The consumption value of the marginal volatility reduction of X_t/X_0 derivatives at time t is therefore

$$\frac{\partial \left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^\eta\right)^2\right) / \partial \tilde{\sigma}_t^\eta}{\partial \left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^\eta\right)^2\right) / \partial C_t} \cdot \frac{\partial \tilde{\sigma}_t^\eta}{\partial X_t} \cdot \frac{X_t}{X_0} = \frac{C_t}{\rho N_t} \left(\tilde{\sigma}_t^\eta\right)^2 \frac{X_t}{X_0} = \left(\tilde{\sigma}_t^\eta\right)^2 \frac{X_t}{X_0},$$

here the last equation follows from $C_t = \rho N_t$. Consequently, the discounted value of volatility flows generates by one unit of derivatives is

$$\mathbb{E}\left[\int_0^\infty \Xi_t \left(\tilde{\sigma}_t^\eta\right)^2 \frac{X_t}{X_0} ds\right].$$

Combining the two present values and using $\tilde{\sigma}^{\eta}_t = (1-\vartheta_t)\bar{\chi}\tilde{\sigma}_t$ (derived previously) yields

$$P_0^X = \mathbb{E}\left[\int_0^\infty \Xi_t \left(-\breve{\mu}_t^{\mathcal{B}} \frac{X_t}{X_0}\right) dt\right] + \mathbb{E}\left[\int_0^\infty \Xi_t \left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{X_t}{X_0} ds\right].$$

After multiplying both sides by X_0, we obtain equation (44).

A.7 Model Solution with Stochastic Differential Utility

The model setup is identical to the one described in Section 2, except that logarithmic preferences are replaced with the utility recursion

$$V_t^i = \mathbb{E}_t\left[\int_t^\infty f(c_s^i, V_s^i) ds\right],$$

where the aggregator f is defined by

$$f(c,V) = (1-\gamma)\rho V \left(\log(c) - \frac{1}{1-\gamma}\log\left((1-\gamma)V\right)\right)$$

We can solve this augmented model as we have solved the baseline model in Section 2.2 (compare also Appendix A.1). The Hamiltonian of the household problem is precisely as stated in Appendix A.1, except that the very first term $e^{-\rho t}\log c_t^i$ must be replaced with $f(c^i, V_t(n^i))$.[77]

We use again a standard guess for the value function to eliminate the costate variable from the Hamiltonian. The guess here is $V_t(n^i) = v_t \frac{(n^i)^{1-\gamma}}{1-\gamma}$, where v_t is, again, a variable that does not depend on individual net worth. The relationship between the value function and the costate requires $\xi_t^i = V_t'(n_t^i) = v_t (n_t^i)^{-\gamma}$.[78] We write μ_t^v and σ_t^v for the (geometric) drift and aggregate volatility of v_t. Note that v_t does not load on the idiosyncratic Brownian because it merely depends on aggregate conditions.

The model solution procedure follows the same steps as for the baseline model. Here, we merely highlight the differences that occur on the way.

[77] On a small technical note, the resulting Hamiltonian here is a "current value Hamiltonian" whereas the one used in Appendix A.1 is a "present value Hamiltonian". The costate must thus be discounted differently here. Otherwise, this does not affect the solution procedure.

[78] It is here that the difference between "present value" and "current value" matters. For this reason, there is no time discounting term (such as $e^{-\rho t}$) in this equation, unlike in Appendix A.1.

The first difference is that the first-order condition for optimal consumption is not immediately equation (23), but instead of the more complicated form

$$v_t(n_t^i)^{-\gamma} = \partial_c f(c_t, V_t) = (1-\gamma)\rho \frac{V_t}{c_t}.$$

However, once the value function $V_t = v_t \frac{(n_t^i)^{1-\gamma}}{1-\gamma}$ is plugged in, the condition reduces again to the familiar form of equation (23).

The second difference is in the characterization of the costate volatility loadings ς_t^i and $\tilde{\varsigma}_t^i$. Because the costate is now $\xi_t^i = v_t(n_t^i)^{-\gamma}$, Ito's lemma implies

$$\varsigma_t^i = \gamma\sigma_t^{n,i} - \sigma_t^v, \qquad \tilde{\varsigma}_t^i = \gamma\tilde{\sigma}_t^{n,i}. \tag{45}$$

The net worth volatilities $\sigma_t^{n,i}$ and $\tilde{\sigma}_t^{n,i}$ take the same form as before such that we simply need to replace the final equation (30) with the slightly more complicated form

$$\varsigma_t^i = \gamma \left(\sigma_t^{q,B} - \theta_t^{K,i} \frac{\sigma_t^{\vartheta}}{1-\vartheta_t} \right) - \sigma_t^v, \qquad \tilde{\varsigma}_t^i = \gamma\theta_t^{K,i}\bar{\chi}\tilde{\sigma}_t.$$

The third difference is that the modified expressions for ς_t^i and $\tilde{\varsigma}_t^i$ affect the derivation and final result of equation (10). Following the same steps as in Appendix A.1, we obtain the slightly modified equation

$$\mathbb{E}_t\left[d\vartheta_t\right] = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \left(\sigma_t^v - (\gamma-1)\,\sigma_t^{\bar{q}} \right) \sigma_t^{\vartheta} - \gamma\left(1-\vartheta_t\right)^2 \bar{\chi}^2\tilde{\sigma}_t^2 \right) \vartheta_t dt,$$

where $\sigma_t^{\bar{q}}$ is the volatility of $\bar{q}_t := q_t^B + q_t^K$.

The fourth and final difference is that we now also have to characterize the process v_t as it affects the BSDE for ϑ_t through the term σ_t^v.[79] To characterize v_t, we start from the costate equation (a necessary optimality condition by the stochastic maximum principle), which is here given by

$$\mathbb{E}_t[d\xi_t^i] = -\left(\partial_V f(c_t^i, V_t^i)\xi_t^i + \frac{\partial H_t^i}{\partial n_t^i} \right) dt$$

[79]There is no need to solve for v_t in the baseline model because there it enters the value function additively and thus only impacts total utility but not optimal choices.

$$= -\left((1-\gamma)\rho \log(c_t^i/n_t^i) - \rho \log v_t - \rho + \mu_t^{n,i} + \frac{c_t^i}{n_t^i} - \varsigma_t^i \sigma_t^{n,i} - \tilde{\varsigma}_t^i \tilde{\sigma}_t^{n,i}\right) \xi_t^i dt$$

$$= -\left((1-\gamma)\rho \log \rho - \rho \log v_t + \mu_t^{n,i} - \left(\gamma \sigma_t^{n,i} - \sigma_t^v\right) \sigma_t^{n,i} - \gamma \left(\tilde{\sigma}_t^{n,i}\right)^2\right) \xi_t^i dt, \tag{46}$$

where the last line uses $c_t^i/n_t^i = \rho$ and the price of risk formulas (45). We also know $\xi_t^i = v_t(n_t^i)^{-\gamma}$ and applying Ito's lemma to this equation yields for the drift term

$$\mathbb{E}_t[d\xi_t^i] = \left(\mu_t^v - \gamma \mu_t^{n,i} + \frac{\gamma(\gamma+1)}{2}\left(\left(\sigma_t^{n,i}\right)^2 + \left(\tilde{\sigma}_t^{n,i}\right)^2\right) - \gamma \sigma_t^v \sigma_t^{n,i}\right) \xi_t^i dt \tag{47}$$

Combining equations (46) and (47) and solving for μ_t^v yields

$$\begin{aligned}
\mu_t^v &= \gamma \mu_t^{n,i} - \frac{\gamma(\gamma+1)}{2}\left(\left(\sigma_t^{n,i}\right)^2 + \left(\tilde{\sigma}_t^{n,i}\right)^2\right) + \gamma \sigma_t^v \sigma_t^{n,i} \\
&\quad - \left((1-\gamma)\rho \log \rho - \rho \log v_t^i + \mu_t^{n,i} - \left(\gamma \sigma_t^{n,i} - \sigma_t^v\right) \sigma_t^{n,i} - \gamma \left(\tilde{\sigma}_t^{n,i}\right)^2\right) \\
&= \rho \log v_t + (\gamma - 1)\left(\rho \log \rho + \mu_t^{n,i} - \frac{\gamma}{2}\left(\left(\sigma_t^{n,i}\right)^2 + \left(\tilde{\sigma}_t^{n,i}\right)^2\right) + \sigma_t^v \sigma_t^{n,i}\right) \\
&= \rho \log v_t + (\gamma - 1)\left(\rho \log \rho + \mu_t^{\bar{q}} + \Phi(\iota_t) - \delta - \frac{\gamma}{2}\left(\left(\sigma_t^{\bar{q}}\right)^2 + (1-\vartheta)^2 \bar{\chi}^2 \tilde{\sigma}_t^2\right) + \sigma_t^v \sigma_t^{\bar{q}}\right),
\end{aligned}$$

where in the last line we use that individual net worth has the same drift and aggregate volatility as aggregate net worth $\bar{q}_t K_t$, while its idiosyncratic volatility is $\tilde{\sigma}_t^{n,i}$, as determined previously. The previous equation for μ_t^v leads to a second BSDE

$$\mathbb{E}_t[dv_t] = \mu_t^v v_t dt$$

that has to be solved numerically jointly with the BSDE for ϑ_t stated previously.

Numerical Model Solution. We solve the model numerically using a finite difference method. This is a standard approach employed in the literature to solve models of this type. Here, we only briefly outline the procedure. A more comprehensive description of the method can be found, e.g., in Brunnermeier et al. (2020), Chapter 3 (specifically Sections 3.2.6 and 3.2.7).

For our numerical solution, we impose the functional relationships $\vartheta_t = \vartheta(t, \tilde{\sigma}_t)$,

$v_t = v(t, \tilde{\sigma}_t)$ and use the known forward equation for the state variable $\tilde{\sigma}_t$ to transform the two BSDEs into partial differential equations in time t and the state $\tilde{\sigma}_t$. We choose suitable terminal guesses for the functions ϑ and v[80] at a finite terminal time T and solve the two PDEs backward in time using a finite difference method. We choose T sufficiently large such that an increase in T no longer changes the solutions at $t = 0$, $\vartheta(0, \cdot)$ and $v(0, \cdot)$, noticeably. These solution functions $\vartheta(0, \cdot)$ and $v(0, \cdot)$ represent our numerical approximation to the stationary (Markov) equilibrium functions $\tilde{\sigma} \mapsto \vartheta(\tilde{\sigma}), v(\tilde{\sigma})$.[81]

A.8 Model Extension with Privately Issued Safe Assets

In this appendix, we present the formal details for the model extension with privately issued safe assets. We restrict attention to the baseline model from Section 2 with logarithmic preferences.

Setup and Model Solution. Each agent i issues nominally risk-free bonds ("i-bonds") of total real value $B_t(i) \geq 0$ and holds a real quantity $b_t^i(j) \geq 0$ of j-bonds issued by other agents $j \neq i$. The clearing conditions at all times t and for all varieties j are

$$B_t(j) = \int b_t^i(j) di.$$

We denote by i_t^p the nominal interest a household has to pay in equilibrium on its privately issued debt[82] and by $B_t^p := \int B_t(j) dj$ the aggregate quantity of privately issued bonds outstanding. Because privately issued debt is nominally risk-free, its return is

$$dr_t^b = \left(i_t^p - i_t \right) dt + dr_t^{\mathcal{B}},$$

where, as before, $dr_t^{\mathcal{B}}$ is the return on government bonds (compare equation (6)). By no arbitrage, in equilibrium $i_t^p = i_t$. Thus, the yields on privately issued bonds and government bonds are identical.

[80]Specifically, we use the functions implied by the steady state equilibrium with $\tilde{\sigma}_t = \tilde{\sigma}^0$ forever.

[81]Note that our results in Appendix A.2 imply that this solution procedure always selects the unique nondegenerate stationary solution the BSDE for ϑ.

[82]Theoretically, i_t^p could depend on the issuing household j. However, as all privately issued bonds are required to be nominally risk-free, it is obvious that they all have to pay the same nominal rate in equilibrium.

We can solve household i's problem as in the baseline model. Denote by $\theta_t^{B,i} := -B_t(i)/n_t^i \leq 0$ the negative of bond issuance as a share of net worth and by $\theta_t^{b,i}(j) := b_t^i(j)/n_t^i \geq 0$ holdings of j-bonds as a fraction of net worth. Relative to the baseline model, the household has the additional choice variables $\theta_t^{B,i}$ and $(\theta_t^{b,i}(j))_{j\in[0,1]}$ subject to the nonnegativity constraints. However, the Hamiltonian of the household's problem does not change relative to Appendix A.1: due to $dr_t^b = dr_t^{\mathcal{B}}$, choices of $\theta_t^{B,i}$ and $(\theta_t^{b,i}(j))_{j\in[0,1]}$ do not affect either the expected return or the risk characteristics of the household's portfolio, such that the additional terms in the Hamiltonian cancel out.

We can draw two immediate conclusions from the previous observation. First, because the Hamiltonian remains unaffected, the model solution steps outlined in Appendix A.1 remain valid in this extended model. Consequently, all equilibria with private bond issuance must feature the same real allocation and the same prices of government bond (q_t^B) and capital (q_t^K) as in the baseline model. Second, all households are indifferent between any choice of private bond issuance and holdings of bonds issued by other agents as long as these holdings do not interfere with the optimal plans for capital holdings ($\theta_t^{K,i}$), outside equity issuance ($\theta_t^{E,i}$), and diversified equity holdings ($\theta_t^{\bar{E},i}$).

There are thus many different equilibria that all feature the same consumption allocation and valuation of government bonds, equity, and capital, but differ with regard to the quantities $B_t(j)$ of private bonds in circulation.

A Simple Example. To illustrate how privately issued bonds can serve as safe assets in precisely the same way as government bonds, we consider an example in which all agents trade private and government bonds in equal proportions.[83] Specifically, we make the following choices: (a) the aggregate real value of privately issued bonds is proportional to the value of government bonds, $B_t^p \propto q_t^B K_t$, (b) the total bonds issued by each agent j is proportional to the agent's net worth share, $B_t(j) = \eta_t^j B_t^p$, and (c) all agents hold a portfolio of j-bonds for $j \neq i$ and government bonds according to market capitalization weights.

We now discuss the debt valuation equations verbally referenced in the main text. We defer a derivation of the following equations to the end of this appendix.

[83]While valuation equations for individual bond types depend on what we assume about trading of individual bonds (which is indeterminate due to indifference), none of the economic conclusions from the example crucially depend on this choice.

For each agent i, the value of the long position $b_t^i(j)$ in j-bonds must equal the present value of future cash inflows from the portfolio of j-bonds, either due to payments made by agent j or due to trading of j-bonds. This insight leads to an equation in full analogy to equations (14) and (16) for government bonds that we have derived in the context of the dynamic trading perspective:

$$b_0^i(j) = \mathbb{E}\left[\int_0^\infty \xi_t^i x_t b_t^i(j) dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^i \left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 b_t^i(j) dt\right]. \tag{48}$$

Here, x_t denotes the expected net payouts made by agent j to all holders of j-bonds per real unit of j-bonds outstanding. Total expected net payouts $x_t B_t(j)$ made by agent j are the private debt counterparts of primary surpluses $s_t K_t$, which represent the net payouts made by the government to public debt holders.

Equation (48) emphasizes that the valuation of j-bonds for agent i depends on a cash flow component resulting from payouts made by agent j and a service flow component resulting from the fact that i trades j-bonds with agents other than j. When aggregating these equations for all $i \neq j$, we obtain a debt valuation equation from the dynamic trading perspective for the aggregate long position in j-bonds:[84]

$$B_0(j) = \mathbb{E}\left[\int_0^\infty \xi_t^{**} x_t B_t(j) dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^{**} \left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 B_t(j) dt\right]. \tag{49}$$

The key takeaway is that this equation looks precisely like equation (13) for government bonds. In particular, the service flow component is identical.

Equation (49) emphasizes the similarity between government bonds and privately issued bonds for their holders. However, private bond issuance also comes with a short position in the bond for the issuer j. In the same spirit as before, we can value that short position by determining the present value of all net payouts that j makes to holders of j-bonds,

$$-B_0(j) = \mathbb{E}\left[\int_0^\infty \xi_t^j \left(-x_t\right) B_t(j) dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^j \left(-\left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2\right) B_t(j) dt\right]. \tag{50}$$

This equation illustrates that issuing bonds according to the specified issuance strategy effectively exposes the agent to negative service flows. Because $B_t(j) = \eta_t^j B_t^p$ is proportional to η_t^j, cash flows from debt issuance and repayments are systematically

[84] Relative to equation (48), the following equation also interchanges integrals and uses $b_t^i(j) = \eta_t^i B_t(j)$.

correlated with marginal utility in a way that increases the riskiness of j's portfolio.

Once we integrate equations (49) and (50) over all bond types j, the integrated service flow terms on the right-hand side become identical in absolute value but have opposite sign. In other words, in the aggregate the positive service flows derived from privately issued bonds by their holders exactly cancel with the negative service flows generated for their issuers. Private safe asset creation does not generate additional net service flows for the economy.

Derivation of Equations (48), (49), and (50). In precisely the same way as in Appendix A.5, we can derive equations in analogy to equation (14) for the portfolios of j-bonds held by agents i and j:

$$b_0^i(j) = -\mathbb{E}_0\left[\int_0^\infty \xi_t^i b_t^i(j)\left(\mu_t^{\Delta,i}(j) - \varsigma_t\sigma_t^{\Delta,i}(j) - \tilde{\varsigma}_t\tilde{\sigma}_t^{\Delta,i,i}(j)\right)\right], \tag{51}$$

$$-B_0^j = -\mathbb{E}_0\left[\int_0^\infty \xi_t^i(-B_t^j)\left(\mu_t^{\Delta,j}(j) - \varsigma_t\sigma_t^{\Delta,j}(j) - \tilde{\varsigma}_t\tilde{\sigma}_t^{\Delta,j}(j)\right)\right]. \tag{52}$$

Here, $d\Delta^{b,i}(j)_t$ and $d\Delta_t^{B,j}$ are the trading processes for j-bonds of agents i and j, respectively:

$$\begin{aligned} d\Delta_t^{b,i}(j) &= \mu_t^{\Delta,i}(j)dt + \sigma_t^{\Delta,i}(j)dZ_t + \tilde{\sigma}_t^{\Delta,i,i}(j)d\tilde{Z}_t^i + \tilde{\sigma}_t^{\Delta,i,j}(j)d\tilde{Z}_t^j, \\ d\Delta_t^{B,j} &= \mu_t^{\Delta,j}(j)dt + \sigma_t^{\Delta,j}(j)dZ_t + \tilde{\sigma}_t^{\Delta,j}(j)d\tilde{Z}_t^i. \end{aligned}$$

As in Section 3 and Appendix A.5, $b_t^i(j)d\Delta^{b,i}(j)_t$ represents the real value of new j-bonds purchased by agent i at time t (net of payouts made by agent j on existing bonds). Similarly, but with opposite sign due to the short position, $-B_t^j d\Delta_t^{B,j}$ represents the real value of new j-bonds (re-)purchased by agent j. In other words, $-d\Delta_t^{B,j}$ corresponds to the payouts that the issuer j makes to bond holders.

To derive equations (48) and (50), we have to characterize the trading processes. In full analogy to Appendix A.5, these processes must satisfy

$$d\Delta_t^{b,i}(j) = \frac{db_t^i(j)}{b_t^i(j)} - dr_t^b, \tag{53}$$

$$d\Delta_t^{B,j} = \frac{dB_t^j}{B_t^j} - dr_t^b. \tag{54}$$

We first characterize the second process. By definition, $\mu_t^{\Delta,j}(j) = -x_t$ corresponds to the negative of the expected net payouts made by agent j to holders of j-bonds per real unit of bonds outstanding. To determine the volatility loadings of the trading process, we use $B_t^j = \eta_t^j B_t^p \propto \eta_t^j q_t^B K_t$, so that

$$\frac{dB_t^j}{B_t^j} = \frac{d\eta_t^j}{\eta_t^j} + \frac{d(q_t^B K_t)}{q_t^B K_t}.$$

The volatility loadings of $dr_t^b = dr_t^{\mathcal{B}}$ coincide with the ones of $d(q_t^B K_t)/(q_t^B K_t)$, compare equation (6). Thus,

$$d\Delta_t^{B,j} = \text{drift terms} + \tilde{\sigma}_t^\eta d\tilde{Z}_t^j.$$

In total, we get

$$\mu_t^{\Delta,j}(j) = -x_t, \qquad \sigma_t^{\Delta,j}(j) = 0, \qquad \tilde{\sigma}_t^{\Delta,j}(j) = \tilde{\sigma}_t^\eta.$$

Substituting this into equation (52) and using $\tilde{\varsigma}_t = \tilde{\sigma}_t^\eta = \bar{\chi}(1-\vartheta_t)\tilde{\sigma}_t$ implies equation (50).

The previous discussion also implies (using equation (54))

$$dr_t^b = \frac{dB_t(j)}{B_t(j)} - d\Delta_t^{B,j} = x_t dt + \frac{d\left(q_t^B K_t\right)}{q_t^B K_t}$$

and substituting this into equation (53) and using $b_t^i(j) = \eta_t^i B_t^j = \eta_t^i \eta_t^j B_t^p$ implies

$$\begin{aligned} d\Delta_t^{b,i}(j) &= \frac{d\eta_t^i}{\eta_t^i} + \frac{d\eta_t^j}{\eta_t^j} + \frac{dB_t^p}{B_t^p} - \left(x_t dt + \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} \right) \\ &= \tilde{\sigma}_t^\eta d\tilde{Z}_t^i + \tilde{\sigma}_t^\eta d\tilde{Z}_t^j + \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} - x_t dt - \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} \\ &= -x_t dt + \tilde{\sigma}_t^\eta d\tilde{Z}_t^i + \tilde{\sigma}_t^\eta d\tilde{Z}_t^j. \end{aligned}$$

In other words,

$$\mu_t^{\Delta,i}(j) = -x_t, \qquad \sigma_t^{\Delta,i}(j) = 0, \qquad \tilde{\sigma}_t^{\Delta,i,i}(j) = \tilde{\sigma}_t^{\Delta,i,j}(j) = \tilde{\sigma}_t^{\eta}.$$

Substituting these equations into equation (51) implies equation (48).

It is left to derive equation (49). This equation easily follows from the previously derived equation (48) by integrating over all holders i:

$$\begin{aligned}
B_0(j) &= \int b_t^i(j)di \\
&= \int \left(\mathbb{E}\left[\int_0^\infty \xi_t^i x_t b_t^i(j)dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^i \left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 b_t^i(j)dt\right]\right) di \\
&= \mathbb{E}\left[\int_0^\infty \int \xi_t^i x_t \eta_t^i B_t(j)didt\right] + \mathbb{E}\left[\int_0^\infty \int \xi_t^i \left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \eta_t^i B_t(j)didt\right] \\
&= \mathbb{E}\left[\int_0^\infty \int \xi_t^i \eta_t^i di \cdot x_t B_t(j)dt\right] + \mathbb{E}\left[\int_0^\infty \int \xi_t^i \eta_t^i di \cdot \left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 B_t(j)dt\right] \\
&= \mathbb{E}\left[\int_0^\infty \xi_t^{**} x_t B_t(j)dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^{**} \left(1-\vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 B_t(j)dt\right].
\end{aligned}$$

A.9 Model Extension with Convenience Yields

.In this appendix, we present the model extension with bonds in the utility function to generate a convenience yield and derive the two debt valuation equations stated in Section 6

Setup and Equilibrium Characterization. To keep equations as simple as possible, we only consider the case of logarithmic consumption preferences and introduce separable logarithmic bond utility as in Di Tella (2020). Each agent i maximizes

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left((1-v)\log c_t^i + v \log b_t^i\right) dt\right],$$

where

$$b_t^i = (1 - \theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i}) n_t^i$$

are real government bond holdings of the agent as in Section 3. v measures the utility share derived from bond holdings. For $v = 0$, the model collapses to the baseline model. As in the main text, but unlike in Appendix A.8, we assume here that the gross holdings or privately issued nominal debt are zero, so that all bonds are government bonds. So long as privately issued bonds do not provide utility, this assumption is without loss of generality.

However, as in Appendix A.8, we use the notation i_t^p to denote the (shadow) nominal short rate on such privately issued bonds. As these bonds do not enter utility, the spread $\Delta i_t := i_t^p - i_t$ can be positive in this augmented model. It captures the convenience yield on government bonds.

The augmented model has almost the same equilibrium solution as our baseline model. ι, q^B, and q^K are given by the equations

$$\iota_t = \frac{(1-\vartheta_t)(a_t - \mathfrak{g}) - (1-v)\rho}{1-\vartheta_t + \phi(1-v)\rho}, \tag{55}$$

$$q_t^B = \vartheta_t \frac{1+\phi(a_t - \mathfrak{g})}{1-\vartheta_t + \phi(1-v)\rho}, \tag{56}$$

$$q_t^K = (1-\vartheta_t)\frac{1+\phi(a_t - \mathfrak{g})}{1-\vartheta_t + \phi(1-v)\rho} \tag{57}$$

as a function of the bond wealth share ϑ_t. The latter is determined by the dynamic equation

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \Delta i_t - (1-\vartheta_t)^2 \tilde{\chi}^2 \tilde{\sigma}_t^2\right)\vartheta_t dt, \tag{58}$$

where $\Delta i_t = v\rho/\vartheta_t$ is the equilibrium convenience yield on government bonds. This equation differs from equation (10) only by the presence of the convenience yield term Δi_t, which raises the equilibrium level of ϑ_t.

We present a proof of equations (55)–(57) and (58) at the end of this appendix.

Debt Valuation Equations (Proposition 7). We next sketch the derivations of the two debt valuation equations stated in the main text. The derivation steps are in complete analogy to the ones presented in Section 3 for the baseline model.

The valuation from the buy and hold perspective starts again from the government flow budget constraint (2) and follows precisely the same steps as in Section 3 up to the derivation of equation (34) stated in the main text and restated here for convenience:

$$\xi_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^T \xi_t^i s_t K_t dt\right] - \mathbb{E}\left[\int_0^T \mathcal{B}_t \left(d\left(\xi_t^i/\mathcal{P}_t\right) + i_t \xi_t^i/\mathcal{P}_t dt\right)\right] + \mathbb{E}\left[\xi_T^i \frac{\mathcal{B}_T}{\mathcal{P}_T}\right].$$

From here on, the derivation departs slightly. Because the nominal SDF $\xi_t^i / \mathcal{P}_t$ in this model does *not* price nominal government debt but nominal private debt, it decays on average at rate $i_t^p = i_t + \Delta i_t$, and the second term does not vanish. Instead, we obtain

$$-\mathbb{E}\left[\int_0^T \mathcal{B}_t \left(d\left(\xi_t^i / \mathcal{P}_t\right) + i_t \xi_t^i / \mathcal{P}_t dt\right)\right] = \mathbb{E}\left[\int_0^T \xi_t^i \Delta i_t \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right],$$

which is the present value of convenience yield service flows derived from government debt between $t = 0$ and $t = T$. From here on, the derivation is again analogous to the one in Section 3. Once we replace ξ_t^i with $\bar{\xi}_t$ and take the limit $T \to \infty$, we arrive at the equation stated in Section 6.

The valuation from the dynamic trading perspective proceeds precisely as in Section 3. The only difference is that the derivation no longer results in the intermediate equation (14) but in the slightly modified version

$$b_0^i = -\mathbb{E}\left[\int_0^\infty \xi_t^i b_t^i \left(-\Delta i_t + \mu_t^{\Delta,i} - \varsigma_t \sigma_t^{\Delta,i} - \tilde{\varsigma}_t^i \tilde{\sigma}_t^{\Delta,i}\right) dt\right],$$

where the term $-\Delta i_t$ is new. After replacing equation (14) with this variant and otherwise following the steps outlined in Section 3, we obtain the valuation equation from the dynamic trading perspective stated in Section 6.

To understand where the additional term $-\Delta i_t$ comes from, note that also the derivation steps for equation (14) in Appendix A.5 remain unchanged except for one detail: in that appendix, we have used in equation (36) that

$$\mu_t^{r^\mathcal{B}} - r_t^f - \varsigma_t \sigma_t^{r^\mathcal{B}} = 0$$

by standard asset pricing logic. That argument is valid if the SDF ξ_t^i prices the government bond, so that the expected return $\mu_t^{r^\mathcal{B}}$ equals the risk-adjusted required return $r_t^f + \varsigma_t \sigma_t^{r^\mathcal{B}}$. Due to the presence of utility services from government bonds, this is not true anymore in the augmented model. The expected return on a privately issued bond $\mu_t^{r^\mathcal{B}} + \Delta i_t$ still equals the required return, but the expected return on the government bond is lower by Δi_t. Consequently, we must use the modified relationship

$$\mu_t^{r^\mathcal{B}} - r_t^f - \varsigma_t \sigma_t^{r^\mathcal{B}} = -\Delta i_t$$

in equation (36). This explains the additional term $-\Delta i_t$ above.

Model Solution Details. The model solution follows the same steps as in Appendix A.1. The difference here is that the term $\log c_t^i$ in the Hamiltonian must be replaced with

$$(1-v)\log c_t^i + v\log\left(1-\theta_t^{K,i}-\theta_t^{E,i}-\theta_t^{\bar{E},i}\right) + v\log n_t^i.$$

We only discuss how this affects the solution without repeating all steps from Appendix A.1 explicitly.

With the same conjecture for the value function (and thus for ξ_t^i) as in Appendix A.1, the first-order condition for optimal consumption becomes

$$c_t^i = (1-v)\,\rho n_t^i$$

while the first-order condition for the optimal investment choice remains unaffected. Following the aggregation steps in Appendix A.1, we obtain again equations (7), (8), and (9) for ι_t, q_t^B, and q_t^K from the maintext with the difference that ρ in these equations, which represents the consumption-wealth ratio, must be replaced with $(1-v)\,\rho$. With this replacement, these equations take the form equations (55), (56), and (57).

The first-order conditions for the portfolio shares $\theta_t^{K,i}$, $\theta_t^{E,i}$, and $\theta_t^{\bar{E},i}$ are the same as in Appendix A.1 except that there is an additional term[85]

$$\frac{\rho v}{1-\theta_t^{K,i}-\theta_t^{E,i}-\theta_t^{\bar{E},i}} = \rho v\frac{n_t^i}{b_t^i} = \Delta i_t$$

on the right-hand side of all three conditions that is due to the marginal utility of bond holdings:

$$\begin{aligned}
\frac{\mathbb{E}_t\left[dr_t^{K,i}(\iota_t^i)\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} &= -\varsigma_t^i\frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \tilde{\varsigma}_t^i\tilde{\sigma}_t - \lambda_t^i\left(1-\bar{\chi}\right) + \Delta i_t,\\
\frac{\mathbb{E}_t\left[dr_t^{E,i}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} &= -\varsigma_t^i\frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \tilde{\varsigma}_t^i\tilde{\sigma}_t - \lambda_t^i + \Delta i_t,\\
\frac{\mathbb{E}_t\left[d\bar{r}_t^{E}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} &= -\varsigma_t^i\frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \Delta i_t.
\end{aligned}$$

[85]The last equality follows from the fact that a hypothetical zero net supply nominal bond not entering the utility function but with otherwise identical risk profile would only have this term in the first-order condition for its excess return.

From here, we can follow the same steps as in Appendix A.1, which yield again equation (28), but a modified version of equation (29):

$$\frac{a_t - \mathfrak{g} - \iota_t}{q_t^K} - \frac{\mu_t^\vartheta - \breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t} - \frac{\left(\sigma_t^{q,B} - \sigma_t^\vartheta\right)\sigma_t^\vartheta}{1-\vartheta_t} = -\varsigma_t^i \frac{\sigma_t^\vartheta}{1-\vartheta_t} + \tilde{\varsigma}_t^i \bar{\chi}\tilde{\sigma}_t + \Delta i_t.$$

Replacing equation (29) with the previous one but following otherwise the steps in Appendix A.1 yields for μ_t^ϑ

$$\mu_t^\vartheta = (1-v)\rho + \breve{\mu}_t^{\mathcal{B}} - (1-\vartheta_t)^2\bar{\chi}^2\tilde{\sigma}_t^2 - (1-\vartheta_t)\Delta i_t.$$

To bring this into the form (58), note that $\Delta i_t = \rho v \frac{n_t^i}{b_t^i} = \frac{\rho v}{\vartheta_t}$ in equilibrium and hence

$$(1-\vartheta_t)\Delta i_t + v\rho = \frac{\rho v}{\vartheta_t} - \rho v + v\rho = \frac{v\rho}{\vartheta_t} = \Delta i_t.$$

The previous equation therefore simplifies to

$$\mu_t^\vartheta = \rho + \breve{\mu}_t^{\mathcal{B}} - (1-\vartheta)^2\bar{\chi}^2\tilde{\sigma}_t^2 - \Delta i_t.$$

Multiplying both sides by ϑ_t yields equation (58).

A.10 Calibration Details

A.10.1 Data Sources and Definitions

The data series for the CIV factor (Herskovic et al., 2016) have been retrieved from Bernard Herskovic's website (https://bernardherskovic.com/data/). That series (column "CIV") represents an annualized return variance measure of the common idiosyncratic volatility in stock returns.

All other data used in this paper have been retrieved from the FRED database maintained by the Federal Reserve Bank of St. Louis (https://fred.stlouisfed.org/). We briefly describe next how we map model quantities into FRED data series.

For the macro aggregates Y, C, I, and G, we use quarterly data from 1970Q1 to 2019Q4. Output is defined as $Y = C + I + G$ (in particular, exclusive of net exports) while we define the three series C, I, and G as follows:

- In line with the business cycle literature, we exclude consumption of durable goods from our consumption measure. To compute C, we start from real personal consumption expenditures (FRED code PCECC96) and subtract real expenditures for durable goods. We identify the latter by multiplying total real consumption expenditures by the ratio of nominal expenditures for durable goods (PCDG) and nominal total consumption expenditures (PCEC).

- We define investment I as the sum of two components: (1) real gross private domestic investment (GPDIC1) net of the change in private inventories (CBIC1) and (2) real consumption expenditures for durable goods (measured as described previously). We include durables in investment as we have removed them from consumption but they nevertheless represent an important part of overall private expenditures.[86]

- We government spending G as real government consumption expenditures and gross investment (GCEC1).

The ratios of primary surpluses and government debt to GDP, S/Y and $q^B K/Y$, respectively, are measured from nominal data. We use again quarterly data series from 1970Q1 to 2019Q4. We define the nominal primary surplus as current receipts (FGRECPT) minus current expenditures (FGEXPND) but add back current interest expenditures (A091RC1Q027SBEA) of the federal government. We define nominal debt as the market value of marketable treasury debt (MVMTD027MNFRBDAL). We compute the ratios S/Y and $q^B K/Y$ by dividing both nominal primary surpluses and nominal debt by nominal GDP (GDP).[87]

Data on the capital stock to compute the capital-output ratio is based on the Penn World Tables (Feenstra et al., 2015) and only available annually. We again choose the time period from 1970 to 2019. The capital-output ratio $q^K K/Y$ is defined as capital stock at constant national prices (RKNANPUSA666NRUG) divided by real GDP at constant national prices (RGDPNAUSA666NRUG), both for the US.

[86]Excluding durables altogether from our measures of economic activity does not substantially change our computed data moments: it lowers the volatility of output somewhat but otherwise only marginally affects results.

[87]Unlike for our time series of macro aggregates, we do not correct the GDP measure for components not in the model. Doing so would have only a minor impact on the resulting numbers. Not doing so is also consistent with how we compute the capital-output ratio below.

For returns on bonds and equity, we use monthly data from February 1970 to December 2019.[88] We first construct monthly log returns from these data sources as follows:[89]

- We measure the return on government debt using data on the market yield on treasury securities at 5-year constant maturity (DGS5). We chose the 5-year maturity as this approximately reflects the average duration of federal debt. We convert the yield data into (holding period) returns using the well-known formula

$$r_{t+1}^T = Ty_t^T - (T-1)y_{t+1}^{T-1}$$

  that relates the log holding period return r_{t+1}^T over the period from t to $t+1$ of a bond with time to maturity of T at date t to the log yield y_t^T of a T-period bond at t and the log y_{t+1}^{T-1} of a $T-1$-period bond at $t+1$. To operationalize this formula, we approximate the unknown 59-month yield y_{t+1}^{T-1} with the observed 60-month yield y_{t+1}^T. This procedure generates a series $\hat{r}_t^{\mathcal{B}}$ of monthly log returns for government bonds.

- As a proxy for the total equity market, we take the Wilshire 5000 index. We compute monthly log returns by dividing successive end-of-month values of the total market index (Wilshire 5000 Total Market Index, FRED series WILL5000IND), which includes dividend reinvestments, and then taking natural logarithms. As market returns are based on leveraged equity returns, this procedure yields a series $\hat{r}_t^{E,\text{leverage}}$ of leveraged monthly log returns for equity.

Based on these data series, we construct the sample estimates for $\mathbb{E}\left[d\bar{r}^E - dr^{\mathcal{B}}\right]$ and $\sigma\left(d\bar{r}^E - dr^{\mathcal{B}}\right)$ reported in Table 2 as follows. We first define for leveraged returns:

$$\begin{aligned}
\mathbb{E}\left[d\bar{r}^{E,\text{leverage}}\right] &= 12 \cdot \text{sample mean}\left(\hat{r}^{E,\text{leverage}}\right) + \frac{12}{2} \cdot \text{sample var}\left(\hat{r}^{E,\text{leverage}}\right), \\
\mathbb{E}\left[dr^{\mathcal{B}}\right] &= 12 \cdot \text{sample mean}\left(\hat{r}^{\mathcal{B}}\right) + \frac{12}{2} \cdot \text{sample var}\left(\hat{r}^{\mathcal{B}}\right), \\
\sigma^2\left(d\bar{r}^{E,\text{leverage}} - dr^{\mathcal{B}}\right) &= 12 \cdot \text{sample var}\left(\hat{r}^{E,\text{leverage}} - \hat{r}^{\mathcal{B}}\right).
\end{aligned}$$

[88] February 1970 is the first month at which some of the required series are available on FRED.

[89] To be precise, the following definitions are for nominal returns while the returns in the model are real. However, for the purpose of computing return differentials, as we do, this distinction is irrelevant.

However, the model counterpart $d\bar{r}^E$ of the market equity return is closer to a delevered equity return. The theoretical relationship between the delevered equity return $d\bar{r}^E$ and the leveraged return $d\bar{r}^{E,\text{leverage}}$ is

$$d\bar{r}^E = dr^{\mathcal{B}} + \frac{1}{\ell}\left(d\bar{r}^{E,\text{leverage}} - dr^{\mathcal{B}}\right),$$

where $\ell \geq 1$ is financial leverage as measured by the ratio of total assets to equity. We therefore define:

$$\mathbb{E}\left[d\bar{r}^E - dr^{\mathcal{B}}\right] = \frac{1}{\ell}\left(\mathbb{E}\left[d\bar{r}^{E,\text{leverage}}\right] - \mathbb{E}\left[dr^{\mathcal{B}}\right]\right),$$
$$\sigma\left(d\bar{r}^E - dr^{\mathcal{B}}\right) = \frac{1}{\ell}\sigma\left(d\bar{r}^{E,\text{leverage}} - dr^{\mathcal{B}}\right).$$

We use $\ell = 1.5$ to compute delevered equity returns.

For the real risk-free rate we also use monthly data from February 1970 to December 2019. We approximate the nominal risk-free rate by the (annualized) 3-month Treasury Bill secondary market rate (DTB3). We convert nominal rates to real rates using realized inflation based on the consumer price index for all urban consumers (CPIAUCSL_PC1). We compute $\mathbb{E}[r^f]$ and $\sigma(r^f)$ based on sample means and variance of the logged risk-free rate series in the same way as for other financial returns (but without the factor 12 given that the returns are already annualized).

A.10.2 Calibration of the Exogenous $\tilde{\sigma}_t$ Process

We estimate the coefficients $\tilde{\sigma}^0$, ψ, and σ of the idiosyncratic risk process (19) such that it matches the observed CIV series. Here, we first describe the details of the estimation procedure and then explain why CIV is a suitable data counterpart for idiosyncratic risk $\tilde{\sigma}_t^2$ in the model.

Parameters Estimation. We use a maximum likelihood estimation (MLE) to determine $\tilde{\sigma}^0$, ψ, and σ based on a monthly CIV sample from January 1945 to December 2019. MLE is straightforward here because the conditional density of the CIR process $\tilde{\sigma}_t^2$ has a known closed-form expression (e.g. Aït-Sahalia (1999), equation (20)).

While not directly targeted by MLE, the estimated process generates first and second ergodic moments of $\tilde{\sigma}_t$, 0.5078 and 0.1701, respectively, that closely match their empirical counterparts (based on square roots of the CIV sample), 0.4950 and 0.1817, respectively.

CIV as a Model-consistent Measure of $\tilde{\sigma}_t^2$. We briefly outline why CIV indeed measures $\tilde{\sigma}_t^2$. Herskovic et al. (2016) construct CIV as the cross-sectional mean of the idiosyncratic return variance of individual stocks in their sample. The idiosyncratic return variance of an individual stock, in turn, is defined as the variance of the residual of a factor regression on the market factor.

In our model, this procedure broadly amounts to a (population) regression of the type

$$dr_t^{E,i} - r_t^f dt = \alpha_t^i + \beta_t^i \left(d\bar{r}_t^E - r_t^f dt \right) + \varepsilon_t^i$$

for stocks issued by all agents i. Comparing the return expressions for $dr_t^{E,i}$ and $d\bar{r}_t^E$ stated in Section 2.2, it is clear that this regression yields $\alpha_t^i = 0$, $\beta_t^i = 1$ and $\varepsilon_t^i = \tilde{\sigma}_t d\tilde{Z}_t^i$. The variance of each individual residual ε_t^i therefore exactly equals $\tilde{\sigma}_t^2$, and so does the cross-sectional mean over all residual variances. In other words, if the real-world data was generated by the model, measured CIV at time t would exactly correspond to $\tilde{\sigma}_t^2$.

A.10.3 Calibration of Remaining Model Parameters

The calibration choices for χ and δ are explained in the main text. The remaining nine parameters, γ, ρ, a^0, $\mathfrak{g}$, $\breve{\mu}^{\mathcal{B},0}$, α^a, $\alpha^{\mathcal{B}}$, ϕ, ι^0, are chosen to match twelve moments as described in the main text. We briefly explain here (heuristically) how these moments identify the model parameters.

First, given the estimated $\tilde{\sigma}_t$ process, the capital productivity process

$$a_t = a(\tilde{\sigma}_t) = a^0 - \alpha^a(\tilde{\sigma}_t - \tilde{\sigma}^0)$$

is exogenous and fully determined by the two parameters a^0 and α^a. While output $Y_t = a_t K_t$ still contains an endogenous term K_t, the capital stock is slow-moving such that most of the variation in HP-filtered output is due to variation in a_t. Therefore, the parameter α^a is effectively determined by the target moment $\sigma(Y)$.

Second, because $\mathfrak{g}$ is constant, the variability of output left for private uses, $Y - G$, is also determined by the parameter α^a. By the aggregate resource constraint $Y - G = C + I$, so that the choice of α^a also constrains the variation of the sum of consumption and investment. The parameter ϕ effectively controls how much of that variation is absorbed by the individual components of that sum. While in principle the full details of the model matter for the dynamics of investment opportunities, ϕ controls to which extent changes in investment opportunities change actual physical investment as opposed to simply driving up or down capital valuations. For $\phi \to 0$, investment reacts a lot while for $\phi \to \infty$, investment is fixed and only prices react. Therefore, the two relative volatilities $\sigma(C)/\sigma(Y)$ and $\sigma(I)/\sigma(Y)$ effectively determine ϕ.[90]

Third, the ratio of primary surpluses to output is given by

$$S_t/Y_t = -\breve{\mu}_t^{\mathcal{B}} \frac{q_t^B}{a_t} = -\left(\breve{\mu}^{\mathcal{B},0} + \alpha^{\mathcal{B}}(\tilde{\sigma}_t - \tilde{\sigma}^0)\right) \frac{q_t^B}{a_t}.$$

While the dynamics of this variable depend on the endogenous price q_t^B, the parameter $\alpha^{\mathcal{B}}$ is nevertheless able to control the overall volatility of S_t/Y_t.[91] The parameter $\alpha^{\mathcal{B}}$ is therefore determined by the moment $\sigma(S/Y)$.

Fourth, the six average ratio targets in the calibration effectively determine the five parameters ρ, a^0, $\mathfrak{g}$, $\breve{\mu}^{\mathcal{B},0}$, and ι^0. To see this, we explain how, in the stochastic steady state of the model, the five parameters map directly into functions of target ratios and how this mapping can be inverted to obtain the parameters. While we do not target the stochastic steady state but the ergodic mean when matching moments, the two are quantitatively very close.

The identity $C + I + G = Y$ and the level targets for C/Y and G/Y imply $I/Y = 1 - C/Y - G/Y$. We can thus write for capital productivity a^0 in the stochastic steady state

$$a^0 = \frac{Y}{K} = \frac{I/K}{I/Y} = \frac{I/K}{1 - C/Y - G/Y}.$$

This determines a^0 as a function of targets. Due to $G = \mathfrak{g}K$, we obtain immediately also

$$\mathfrak{g} = G/Y \cdot a^0.$$

Because G/Y is a target and a^0 has already been determined, this equation determines

[90] This does not imply that we are always able to pick ϕ in a way that matches both relative volatilities precisely. It just means that if model dynamics are such that they can be matched at all, then this works only for one value of ϕ.

[91] This is not a rigorous theoretical statement but an empirical one based on observed numerical model solutions.

$\mathfrak{g}$.

Next, ρ represents the ratio of consumption to total wealth in the model, that is

$$\rho = \frac{C}{(q^B + q^K)K} = \frac{C/Y}{q^B K/Y + q^K K/Y}$$

and the right-hand expression is a function of targeted ratios. Hence, the targets also determine ρ.

By the government budget constraint, the policy variable $\breve{\mu}^{\mathcal{B}}$ in the stochastic steady state must satisfy

$$\breve{\mu}^{\mathcal{B},0} = -\frac{s}{q^B} = -\frac{S/Y}{q^B K/Y}$$

and, again, the right-hand expression is a function of targeted ratios.

Finally, the capital price in the stochastic steady state can be related to the capital-output ratio by the equation $q^{K,0} = q^K K/Y \cdot a^0$. Because a^0 is a function of targeted ratios, so is $q^{K,0}$. It is easy to show that the investment rate is $I/K = \iota^0 + \frac{q^{K,0}-1}{\phi}$. This expression only depends on $q^{K,0}$ and the parameters ι^0 and ϕ. For any given parameter ϕ, ι^0 is therefore determined by targets through the equation

$$\iota^0 = I/K - \frac{q^{K,0} - 1}{\phi}.$$

We remark that the six average ratios do not only identify the five parameters ρ, a^0, $\mathfrak{g}$, $\breve{\mu}^{\mathcal{B},0}$, and ι^0 (in the stochastic steady state) but also the average value ϑ^0 of the endogenous variable ϑ_t, namely

$$\vartheta^0 = \frac{q^B}{q^B + q^K} = \frac{q^B K/Y}{q^B K/Y + q^K K/Y}.$$

This generates an implicit target that must be somehow matched by varying parameters other than ρ, a^0, $\mathfrak{g}$, $\breve{\mu}^{\mathcal{B},0}$, and ι^0 in order to match all six average ratios.

Fifth, because ρ, $\bar{\chi}$, and the dynamics of $\breve{\mu}^{\mathcal{B}}$ and $\tilde{\sigma}_t$ are already determined by external calibration choices or the targeted average ratios, the counterpart of equation (10) in Appendix A.7 implies that this implicit target ϑ^0 for the average value of ϑ_t must be matched by a sufficient size of the risk premium terms in that equation. The only "free" variables in these terms are σ_t^v and γ and the former is effectively also determined by γ

(once ρ, $\bar{\chi}$, and the dynamics of $\tilde{\sigma}_t$ are fixed). In fact, the risk premium terms are strictly increasing in γ given the remaining parameter choices. Therefore, the implicit target ϑ^0 is only achieved for a specific value of γ. At the same time, γ affects also the average equity premium $\mathbb{E}[d\bar{r}^E - dr^{\mathcal{B}}]$ and the equity sharpe ratio $\mathbb{E}[d\bar{r}^E - dr^{\mathcal{B}}]/\sigma(d\bar{r}^E - dr^{\mathcal{B}})$. The parameter γ is thus certainly identified by the set of target moments, but it is generally not possible to match all of them.

A.10.4 Calculation of Wealth-weighted Risk Exposures Discussed in the Main Text

In this appendix we explain how we compute empirical counterparts for the wealth-weighted total and idiosyncratic risk exposures discussed in Section 7.

Our data for risk exposures by wealth group are from Bach et al. (2020). These authors report the standard deviation of the excess return on gross wealth (Table I, column (2)) and net wealth (Table II, column (3)) for 16 wealth groups categorized by their relative position in the wealth distribution. For the gross wealth data, the authors also report the fraction that is due to idiosyncratic risk (Table I, column (3)) relative to a factor asset pricing model. In lack of other data, we assume that the same fractions also apply to the net wealth figures. We use these observations to compute for each group both the total and the idiosyncratic variance of the excess return on wealth, both for gross wealth and net wealth. As the observations are based on Swedish administrative data, our implicit assumption is that the mapping from wealth groups to these variances is similar for the the US, where no such data are observable.

To match these variances with wealth shares for the US, we take estimates from Smith et al. (2023) who calculate wealth shares using different methodologies for the following five wealth groups (see their Table I): "Full population", "Top 10%", "Top 1%", "Top 0.1%", "Top 0.01%".[92] We use both their "baseline" and their "equal returns" estimate for wealth shares.

Unfortunately, the wealth groups formed by Smith et al. (2023) are coarser than the ones reported in Bach et al. (2020). Where the Smith et al. (2023) estimates only tell us the combined wealth share of several groups based on the Bach et al. (2020) split, we allocate the wealth equally across the groups formed in the latter paper.

[92]They also have a sixth group "Top 0.001%". However, the Bach et al. (2020) is not sufficiently fine-grained at the very top of the wealth distribution, so that we ignore this additional group throughout.

Table 3 reports the square roots of the resulting wealth-weighted cross-sectional averages for the variances, both for idiosyncratic and total risk exposures. For each type of estimate, we report four values, depending on which wealth share estimate we use and whether we take gross wealth or net wealth figures for risk exposures.

Table 3: Wealth-weighted risk exposures

wealth shares	"baseline"		"equal returns"	
gross/net wealth	gross	net	gross	net
idiosyncratic risk	0.09	0.09	0.1	0.1
total risk	0.16	0.18	0.17	0.19

A.11 A Model with Two Types

In this appendix we present a model variant with two types of agents that have heterogeneous access to the different asset markets in our economy and therefore heterogeneous idiosyncratic and possibly aggregate risk exposures. We derive theoretical results that link the predictions of the two-type model to the predictions of the one-type model presented in the main text.

Setup. The model is the same as the baseline model in the main text, except for the following modification. At each time, each agent i is either an expert ("e") or a household ("h"). Experts can manage capital directly and therefore face precisely the same portfolio (and real investment) choice as all agents in our baseline model. Households, in turn, are restricted to only hold financial assets (equity and bonds). All agents have identical preferences regardless of type. We allow for both logarithmic preferences as in Section 2 ($\gamma = 1$) and more general stochastic differential utility preferences with risk aversion $\gamma > 0$ as considered in Section 4.

Agents receive idiosyncratic (Poisson) type switching shocks. Experts become households with arrival rate $\lambda^e > 0$ and households become experts with arrival rate $\lambda^h > 0$.[93]

Let e_t^i be an indicator that is 1 if agent i is an expert at time t and zero otherwise. In

[93]Without type switching experts would eventually dominate the economy because they earn higher expected returns on average. In the stationary distribution, the model would then reduce to the one-type model studied in the main text.

what follows, we define the shares

$$\eta_t^e := \int \eta_t^i e_t^i di, \qquad \eta_t^h := \int \eta_t^i (1 - e_t^i) di$$

of total wealth that is owned by experts (η_t^e) and households (η_t^h), respectively. One of these variables is a sufficient summary of the cross-sectional wealth distribution for the purposes of solving for the aggregate dynamics and asset prices in this model (the other variable can be backed out form $\eta_t^e + \eta_t^h = 1$). When solving the model, we therefore include η_t^e as an additional state variable.

Sketch of the Model Solution. The model can be solved along the same lines as our baseline model. We briefly sketch the solution procedure here and provide more details on the steps that are new relative to the baseline model.

First, everything that is said in Section 2.2 before Lemma 1 as well as that lemma itself remains valid in the two-type model without any modification. As a consequence, the dynamics of asset prices, aggregate consumption, and aggregate investment are fully determined by the dynamics of the endogenous process ϑ_t and the exogenous process a_t.

Second, the optimal portfolio choice conditions (25), (26), and (27) remain unchanged for those agents i that are experts at time t. For households, instead, the first two conditions do not apply, as households do not hold capital and issue outside equity. Nevertheless, equation (27) remains valid also for households. Therefore, for experts the exact same steps as in Appendix A.1 lead once again to equation (29) stated there. This equation depends on the agent index i only through the prices of risk ς_t^i and $\tilde{\zeta}_t^i$. We argue next that these prices of risk are actually not i-dependent. Specifically, because equation (27) holds for all agents regardless of type, $\varsigma_t^i = \varsigma_t$ is the same for all i.[94] Furthermore, using $\varsigma_t^i = \varsigma_t$ and $\lambda_t^i = \tilde{\zeta}_t^i \tilde{\sigma}_t$ (compare Appendix A.1) in equation (25) for any agent i that is an expert implies that $\tilde{\zeta}_t^i$ is identical for all experts. We call this common value $\tilde{\zeta}_t^e$ from now on. Consequently, the combined portfolio choice condition

[94] This conclusion rests on the implicit assumption $\sigma_t^\vartheta \neq 0$. However, it is easy to verify ex post that $\sigma_t^\vartheta = 0$ if and only if there is no consumption-relevant aggregate risk. But in this case, trivially $\varsigma_t^i = 0 =: \varsigma_t$ for all agents.

equation (29) can be written here as

$$\frac{a_t - \mathfrak{g}_t - \iota_t}{q_t^K} - \frac{\mu_t^{\vartheta} - \breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t} - \frac{\left(\sigma_t^{q,B} - \sigma_t^{\vartheta}\right)\sigma_t^{\vartheta}}{1-\vartheta_t} = -\varsigma_t \frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \tilde{\varsigma}_t^e \bar{\chi}\tilde{\sigma}_t. \tag{59}$$

Third, the factor v_t in the costate ξ_t^i for agent i now becomes type-specific, v_t^e if $e_t^i = 1$ and v_t^h if $e_t^i = 0$. Hence, equations (45) for the prices of risk remain valid. However, in the first equation, σ_t^v has to be interpreted as $\sigma_t^{v,e}$ for experts and and as $\sigma_t^{v,h}$ for households. In this model, it makes sense to determine the aggregate net worth risk loadings $\sigma_t^{n,i}$ slightly differently to before. Specifically, $\eta_t^i = n_t^i / N_t$ implies that (by Ito's lemma) $\sigma_t^{n,i} = \sigma_t^N + \sigma_t^{\eta,i}$. Using $N_t = q_t^B / \vartheta_t K_t$, we have furthermore $\sigma_t^N = \sigma_t^{q,B} - \sigma_t^{\vartheta}$. We can therefore write for the price of aggregate risk

$$\varsigma_t = \gamma\left(\sigma_t^{q,B} - \sigma_t^{\vartheta}\right) + \gamma\sigma_t^{\eta,e} - \sigma_t^{v,e} = \gamma\left(\sigma_t^{q,B} - \sigma_t^{\vartheta}\right) + \gamma\sigma_t^{\eta,h} - \sigma_t^{v,h}.$$

The price of idiosyncratic risk is type-specific and given by

$$\tilde{\varsigma}_t^e = \gamma\frac{1-\vartheta_t}{\eta_t^e}\bar{\chi}\tilde{\sigma}_t, \qquad \tilde{\varsigma}_t^h = 0.$$

Fourth, while the steps in Appendix A.1 that lead to Proposition 1 remain exactly the same, the ultimate dynamic equation for ϑ_t is different from equation (10) because the prices of risk are different. Plugging the prices of risk for experts into the combined portfolio choice equation (59) and otherwise following the same steps as in Appendix A.1 yields the equation

$$\mathbb{E}_t\left[d\vartheta_t\right] = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \left(\eta_t^e\sigma_t^{v,e} + \eta_t^h\sigma_t^{v,h} - (\gamma-1)\,\sigma_t^{\bar{q}}\right)\sigma_t^{\vartheta} - \gamma\frac{(1-\vartheta_t)^2\,\bar{\chi}^2\tilde{\sigma}_t^2}{\eta_t^e}\right)\vartheta_t dt, \tag{60}$$

where, as in Appendix A.7, $\sigma_t^{\bar{q}}$ denotes the volatility of $\bar{q}_t := q_t^B + q_t^K$.

Fifth, an additional law of motion for the endogenous state variable η_t^e needs to be determined. This is relatively straightforward for the volatility $\sigma_t^{\eta,e}$. Using the two expressions for ς_t, from experts' and households' perspective, and the fact that $\eta_t^e\sigma_t^{\eta,e} +$

$\eta_t^h \sigma_t^{\eta,h} = 0$ (by construction), we obtain

$$\sigma_t^{\eta,e} = \frac{(1-\eta_t^e)(\sigma_t^{v,e} - \sigma_t^{v,h})}{\gamma}.$$

In particular, in the special case of log utility, $\sigma_t^{v,e} = \sigma_t^{v,h} = 0$ and, hence, also $\sigma_t^{\eta,e} = 0$, so that the wealth share η_t^e evolves locally deterministically.

The drift of η_t^e can be computed using some straightforward but tedious algebra that is omitted here in the interest of space.[95] The final result is

$$\mu_t^{\eta,e} = \left(-\sigma_t^{v,e} + \gamma\sigma_t^{\eta,e} + (\gamma-1)\,\sigma_t^{\bar{q}}\right)\sigma_t^{\eta,e} + \frac{1-\eta_t^e}{\eta_t^e}\gamma\frac{\bar{\chi}^2\,(1-\vartheta_t)^2}{\eta_t^e}\tilde{\sigma}_t^2 + \frac{\lambda^h\,(1-\eta_t^e) - \lambda^e\eta_t^e}{\eta_t^e}. \tag{61}$$

Sixth and finally, there are now two BSDEs for the value function factors $\mathbb{E}_t[dv_t^e]$ and $\mathbb{E}_t[dv_t^h]$. These can be derived in precisely the same way as in Appendix A.7, except that we have to account for type switching. The two counterparts of the costate equation, equation (46), are entirely analogous:

$$\mathbb{E}_t[d\xi_t^{e,i}] = -\left((1-\gamma)\rho\log\rho - \rho\log v_t^e + \mu_t^{n,e,i} - \left(\gamma\sigma_t^{n,e,i} - \sigma_t^{v,e}\right)\sigma_t^{n,e,i} - \gamma\left(\tilde{\sigma}_t^{n,e,i}\right)^2\right)\xi_t^{e,i}dt,$$
$$\mathbb{E}_t[d\xi_t^{h,i}] = -\left((1-\gamma)\rho\log\rho - \rho\log v_t^h + \mu_t^{n,h,i} - \left(\gamma\sigma_t^{n,h,i} - \sigma_t^{v,h}\right)\sigma_t^{n,h,i}\right)\xi_t^{h,i}dt.$$

The counterparts of equation (47) change slightly because, when applying Ito's lemma to $\xi_t^e = v_t^e(n_t^i)^{-\gamma}$ and $\xi_t^h = v_t^h(n_t^i)^{-\gamma}$, additional jump terms appear:

$$\mathbb{E}_t[d\xi_t^{e,i}] = \left(\mu_t^{v,e} - \gamma\mu_t^{n,e,i} + \frac{\gamma\,(\gamma+1)}{2}\left(\left(\sigma_t^{n,e,i}\right)^2 + \left(\tilde{\sigma}_t^{n,e,i}\right)^2\right) - \gamma\sigma_t^{v,e}\sigma_t^{n,e,i} + \lambda^e\frac{\xi_t^{h,i} - \xi_t^{e,i}}{\xi_t^{e,i}}\right)\xi_t^{e,i}dt,$$
$$\mathbb{E}_t[d\xi_t^{h,i}] = \left(\mu_t^{v,h} - \gamma\mu_t^{n,h,i} + \frac{\gamma\,(\gamma+1)}{2}\left(\sigma_t^{n,h,i}\right)^2 - \gamma\sigma_t^{v,h}\sigma_t^{n,h,i} + \lambda^h\frac{\xi_t^{e,i} - \xi_t^{h,i}}{\xi_t^{h,i}}\right)\xi_t^{h,i}dt.$$

Combining the two sets of equations and solving for $\mu_t^{v,e}$ and $\mu_t^{v,h}$, respectively, yields

$$\mu_t^{v,e} = \rho\log v_t^e + (\gamma-1)\left(\rho\log\rho + \mu_t^{\bar{q}} + \Phi(\iota_t) - \delta + \mu_t^{\eta,e} + \frac{\lambda^e\eta_t^e - \lambda^h\eta_t^h}{\eta_t^e}\right)$$

[95] Essentially, one applies Ito's lemma to $\eta_t^e = N_t^e/N_t$, where $N_t^e := \int n_t^i e_t^i di$ and its evolution as an Ito process can be determined from the evolution of n_t^i for all i that are experts.

$$+ (\gamma - 1)\left(-\frac{\gamma}{2}\left(\left(\sigma_t^{\bar{q}} + \sigma_t^{\eta,e}\right)^2 + \frac{(1-\vartheta)^2\bar{\chi}^2}{(\eta_t^e)^2}\tilde{\sigma}_t^2\right) + \sigma_t^{v,e}\left(\sigma_t^{\bar{q}} + \sigma_t^{\eta,e}\right)\right) + \lambda^e\frac{v_t^h - v_t^e}{v_t^e},$$

$$\mu_t^{v,h} = \rho\log v_t^h + (\gamma-1)\left(\rho\log\rho + \mu_t^{\bar{q}} + \Phi(\iota_t) - \delta + \mu_t^{\eta,h} + \frac{\lambda^h\eta_t^h - \lambda^e\eta_t^e}{\eta_t^h}\right)$$

$$+ (\gamma - 1)\left(-\frac{\gamma}{2}\left(\sigma_t^{\bar{q}} + \sigma_t^{\eta,h}\right)^2 + \sigma_t^{v,h}\left(\sigma_t^{\bar{q}} + \sigma_t^{\eta,h}\right)\right) + \lambda^h\frac{v_t^e - v_t^h}{v_t^h}.$$

Relationship with Baseline Model. We next provide two theoretical results that highlight relationships between the dynamics of the two-type model and the dynamics of our baseline model with just one type. These results emphasize that the statistic of the cross-sectional distribution of idiosyncratic risk exposures that matters most for our model's predictions is $\int \eta_t^i(\tilde{\sigma}_t^{n,i})^2 di$, i.e. the wealth-weighted cross-sectional mean of the idiosyncratic net worth variance. We conjecture that a similar conclusion would also hold in more general n-type models. To improve the reading flow, we first summarize here the results and present the proofs at the end of this appendix.

In what follows, we always make the following assumptions and use the following notation:

Let K_0 be an initial condition for the capital stock and a_t, $\breve{\mu}_t^{\mathcal{B}}$, $\tilde{\sigma}_t^1$, $\tilde{\sigma}_t^2$ be exogenous processes, such that both $(a_t, \breve{\mu}_t^{\mathcal{B}}, \tilde{\sigma}_t^1)$ and $(a_t, \breve{\mu}_t^{\mathcal{B}}, \tilde{\sigma}_t^2)$ are functions of some finite-dimensional Markov process. Suppose that stationary monetary equilibria exist both for the one-type model with exogenous processes $(a_t, \breve{\mu}_t^{\mathcal{B}}, \tilde{\sigma}_t^1)$ and for the two-type model with exogenous processes $(a_t, \breve{\mu}_t^{\mathcal{B}}, \tilde{\sigma}_t^2)$ based on the same parameters for ρ, $\mathfrak{g}$, and ϕ (but not necessarily for other model parameters).[96] *For any model variable x, denote by x_t^j the stochastic process for x in the equilibrium for the j-type model ($j \in \{1,2\}$).*

Before establishing the main results, we remark that most interesting predictions of the two models only depend on the stochastic processes for the exogenous variables a_t and $\breve{\mu}_t^{\mathcal{B}}$ and the endogenous variable ϑ_t in equilibrium.

Lemma 5. *If $\vartheta_t^1 = \vartheta_t^2$ for all t (almost surely), then also the following equations hold for all t*

[96] By Proposition 1 and the results in Appendix A.2, these equilibria are then also unique. These results hold analogously also for the two-type model.

(almost surely):[97]

$$K_t^1 = K_t^2, \qquad q_t^{B,1} = q_t^{B,2}, \qquad q_t^{K,1} = q_t^{K,2}, \qquad \iota_t^1 = \iota_t^2,$$
$$C_t^1 = C_t^2, \qquad \tau_t^1 = \tau_t^2, \qquad \mathbb{E}_t[dr_t^{\mathcal{B},1}] = \mathbb{E}_t[dr_t^{\mathcal{B},2}], \qquad \mathbb{E}_t[dr_t^{K,1}] = \mathbb{E}_t[dr_t^{K,2}]$$

The previous lemma establishes that if two equilibria feature the same endogenous process ϑ_t, then they make exactly the same predictions for a large range of variables. We next provide sufficient conditions for $\vartheta_t^1 = \vartheta_t^2$.

We start with the special case of log utility ($\gamma = 1$) as then several equations simplify. First, note that for log utility $v_t^e = v_t^h = 1$, so that the decision-relevant portion of agents' value functions is independent of the agent type. As a consequence, we also obtain $\sigma_t^{\eta,e} = 0$, agents find it optimal to fully share aggregate risk. Equation (60) then simplifies to

$$\begin{aligned}\mathbb{E}_t\left[d\vartheta_t\right] &= \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \frac{(1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2}{\eta_t^e}\right)\vartheta_t dt \\ &= \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \left(\eta_t^e \left(\tilde{\sigma}_t^{n,e}\right)^2 + \eta_t^h \left(\tilde{\sigma}_t^{n,h}\right)^2\right)\right)\vartheta_t dt, \end{aligned} \tag{62}$$

where it has been used that $\tilde{\sigma}_t^{n,e} = \frac{1-\vartheta_t}{\eta_t}\bar{\chi}\tilde{\sigma}_t$ and $\tilde{\sigma}_t^{n,h} = 0$. Similarly, the corresponding equation in the baseline model with just one type is[98]

$$\mathbb{E}_t\left[d\vartheta_t\right] = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - \left(\tilde{\sigma}_t^n\right)^2\right)\vartheta_t dt. \tag{63}$$

Note that, beyond the log utility case, equations (62) (for the two-type model) and (63) (for the one-type model) continue to hold more generally if we shut down aggregate shocks (by setting the dZ_t-loading of the exogenous processes to zero).

In either case, log utility or no aggregate shocks, equations (62) and (63) are identical if (and only if) the stochastic process for $\eta_t^e \left(\tilde{\sigma}_t^{n,e}\right)^2 + \eta_t^h \left(\tilde{\sigma}_t^{n,h}\right)^2$ in the two-type model is the same as the stochastic process for $\left(\tilde{\sigma}_t^n\right)^2$ in the one-type model. If this is the case,

[97] For the last equality, note that in both models, $dr_t^{K,i}$ does not depend on i except for the identity of the idiosyncratic shock $d\tilde{Z}_t^i$ which plays no role for the expectation (and it only makes sense if i is an expert in the two-type model). Therefore, the equation is written without i-superscripts

[98] Compare equation (10) in Proposition 1. Alternatively, simply set $\eta_t^e = 1$, $\eta_t^h = 0$ in the previous equation – the two-type model effectively collapses to the baseline model if η_t^e is held fixed at 1.

then the two models imply the same dynamics for ϑ_t and, hence, the same dynamics for aggregates and asset prices. This reasoning leads to the following proposition.

Proposition 9. *Let either $\gamma = 1$ (log utility) or assume that there are no aggregate shocks. Suppose that the following condition hold (for all t almost surely)*

$$\left(\tilde{\sigma}_t^{n,1}\right)^2 = \eta_t^e \left(\tilde{\sigma}_t^{n,e,2}\right)^2 + \eta_t^h \left(\tilde{\sigma}_t^{n,h,2}\right)^2. \tag{64}$$

Then the equilibrium dynamics for all macro aggregates and the expectations and aggregate volatility loadings of all asset returns are identical in both equilibria.

We now turn to the general case that there are aggregate shocks and, potentially, $\gamma \neq 1$. In this case, similar derivations as before show that equation (62) for the two-type model takes the form

$$\mathbb{E}_t \left[d\vartheta_t\right] = \left(\rho + \check{\mu}_t^{\mathcal{B}} - \left(\eta_t^e \sigma_t^{v,e} + \eta_t^h \sigma_t^{v,h} - (\gamma - 1)\, \sigma_t^{\bar{q}}\right) \sigma_t^{\vartheta} - \gamma \left(\eta_t^e \left(\tilde{\sigma}_t^{n,e}\right)^2 + \eta_t^h \left(\tilde{\sigma}_t^{n,h}\right)^2\right)\right) \vartheta_t dt \tag{65}$$

and equation (63) for the one-type model takesthe form

$$\mathbb{E}_t \left[d\vartheta_t\right] = \left(\rho + \check{\mu}_t^{\mathcal{B}} - \left(\sigma_t^{v} - (\gamma - 1)\, \sigma_t^{\bar{q}}\right) \sigma_t^{\vartheta} - \gamma \left(\tilde{\sigma}_t^{n}\right)^2\right) \vartheta_t dt. \tag{66}$$

In this general case, condition (64) is no longer sufficient to make the two equations identical. In addition, we would need the extra condition $\sigma_t^v = \eta_t^e \sigma_t^{v,e} + \eta_t^h \sigma_t^{v,h}$, which is unlikely to be satisfied in general as an inspection of the BSDEs for v_t in the one-type model and for v_t^e, v_t^h in the two-type model reveals. This is because, for $\gamma \neq 1$ and aggregate shocks, hedging demands induce different types to take on heterogeneous aggregate risk exposures, which generates additional dynamics that are absent from the more stylized one-type model. However, these additional dynamics disappear in the limit case that type switching is infinitely fast as then the value functions of households and experts align. Then, condition (64) is again sufficient for the two models to generate identical predictions:

Proposition 10. *The conclusion of Proposition 9 remains valid even for $\gamma \neq 1$ and with aggregate shocks, if the equilibrium in the two-type model is understood to be the limit as $\lambda^e, \lambda^h \to \infty$ with the ratio $\lambda^h / \lambda^e \in (0, \infty)$ held constant.*

Proofs.

Proof of Lemma 5. Lemma 1 holds for both models. We observe immediately from that lemma that if ϑ_t and a_t are the same in two equilibria, then so are q_t^B, q_t^K, and ι_t, provided the equilibria correspond to models with identical parameters ρ, $\mathfrak{g}$, and ϕ, as we have assumed here. If ι_t is the same across the two equilibria, then so is dK_t/K_t by the law of motion of aggregate capital (compare Definition 1 and note that an identical equation also holds in the two-type model). This completes the proof of the first four equations. We now discuss the remaining four equations.

For equality of aggregate consumption C_t, note that $a_t^1 = a_t^2$, $K_t^1 = K_t^2$, and $\iota_t^1 = \iota_t^2$ imply together with goods market clearing (equation (3))

$$C_t^1 = (a_t^1 - \mathfrak{g} - \iota_t^1)K_t^1 = (a_t^2 - \mathfrak{g} - \iota_t^2)K_t^2 = C_t^2.$$

For equality of taxes, we use similarly the government budget constraint and $a_t^1 = a_t^2$, $\breve{\mu}_t^{\mathcal{B},1} = \breve{\mu}_t^{\mathcal{B},2}$, and $q_t^{B,1} = q_t^{B,2}$:

$$\tau_t^1 = \frac{\mathfrak{g} - \breve{\mu}_t^{\mathcal{B},1} q_t^{B,1}}{a_t^1} = \frac{\mathfrak{g} - \breve{\mu}_t^{\mathcal{B},2} q_t^{B,2}}{a_t^2} = \tau_t^2.$$

For equality of the expected return on bonds consider the equation for the return $dr_t^{\mathcal{B}}$ stated in the first part of Appendix A.1. This equations holds for both the one-type and the two-type model. Observe that the drift of $dr_t^{\mathcal{B}}$ only depends on ι_t, $\mu_t^{q,B}$, and $\breve{\mu}_t^{\mathcal{B}}$, which are identical for $j = 1$ and $j = 2$. Hence, $\mathbb{E}_t[dr_t^{\mathcal{B},1}] = \mathbb{E}_t[dr_t^{\mathcal{B},2}]$.

An analogous argument holds for the last equality. Also the final expression for $dr_t^{K,i}(\iota_t^i)$ in Appendix A.1 holds for both models. This expression depends on a_t, ι_t, q_t^B, q_t^K, $\mu_t^{q,K}$, all of which have been shown to be identical in both equilibria. □

Proof of Proposition 9. Comparing equations (62) and (63), it is apparent that if $\vartheta_t^1 = \vartheta_t^2$ and condition (64) holds, then the right-hand sides of both equations are identical state by state. Uniqueness of the solution (compare Appendix A.2) then implies that, indeed, $\vartheta_t^1 = \vartheta_t^2$ is the only possibility.

From Lemma 5 we can then immediately conclude that K_t, q_t^B, q_t^K, ι_t, C_t, τ_t, $\mathbb{E}_t[dr_t^{\mathcal{B}}]$, and $\mathbb{E}_t[dr_t^K]$ must be identical across the two models. All macro aggregates can be written as functions of the first six variables (and possibly the exogenous processes a_t

and $\breve{\mu}_t^{\mathcal{B}}$, which are the same for both models), so that, indeed, the dynamics of all macro aggregates must be identical in both equilibria.

In addition, if q_t^B and q_t^K are identical, then so are $\sigma_t^{q,B}$ and $\sigma_t^{q,K}$ by Ito's lemma. Hence, the aggregate volatility loadings on all returns $dr_t^{\mathcal{B}}$, dr_t^K, dr_t^E, $d\bar{r}_t^E$ must be identical.

Because Lemma 5 already implies that the expected returns on capital and bonds are identical across the two equilibria, it is only left to show that also $\mathbb{E}_t[dr_t^E] = \mathbb{E}_t[d\bar{r}_t^E]$ is the same in both equilibria. Because of equation (27), which holds in both models, and $\vartheta_t^1 = \vartheta_t^2 \Rightarrow \sigma_t^{\vartheta,1} = \sigma_t^{\vartheta,2}$, the desired equality holds if and only if $\varsigma_t^1 = \varsigma_t^2$. This is trivially satisfied if there are no aggregate shocks so that we can from now on assume that $\gamma = 1$. Then, $\varsigma_t^1 = \varsigma_t^2$ follows from the following considerations:

- In the one-type model, the aggregate price of risk is given by (compare Appendix A.1)

$$\varsigma_t^1 = \sigma_t^n = \sigma_t^{\bar{q}}.$$

- In the two-type model, the aggregate price of risk is given by

$$\varsigma_t^2 = \sigma_t^{\bar{q}} + \sigma_t^{\eta,e} = \sigma_t^{\bar{q}},$$

  because $\sigma_t^{\eta,e} = 0$ in the log utility case.

□

Proof of Proposition 10. We first establish some properties of the limit economy in the two-type model. Let $\eta^* := \frac{\lambda^h/\lambda^e}{1+\lambda^h/\lambda^e}$. By the assumptions that $\lambda^h/\lambda^e \in (0,\infty)$ is held constant, we know that $\eta^* \in (0,1)$ is constant along any limit sequence. Also $\frac{\lambda^e}{1-\eta^*} = \frac{\lambda^h}{\eta^*}$ by definition of η^*.

Consider now the last term in the drift of η_t, equation (61):

$$\begin{aligned}
\frac{\lambda^h\left(1-\eta_t^e\right) - \lambda^e\eta_t^e}{\eta_t^e} &= \frac{1}{\eta_t^e}\left(\frac{\lambda^h}{\eta^*}\eta^*\left(1-\eta_t^e\right) - \frac{\lambda^e}{1-\eta^*}\left(1-\eta^*\right)\eta_t^e\right) \\
&= \frac{\lambda^h}{\eta^*}\frac{1}{\eta_t^e}\left(\eta^* - \eta_t^e\right).
\end{aligned}$$

This term is positive for $\eta_t^e < \eta^*$ and negative for $\eta_t^e > \eta^*$ and as $\lambda^h \to \infty$, it becomes arbitrarily large in absolute value. In contrast, all other terms in equation (61) remain bounded for η_t in a (sufficiently small) neighborhood of η^*. Hence, the last term dominates in the limit and ensures that $\eta_t = \eta^*$ at all times.

Next, consider the equations for $\mu_t^{v,e}$ and $\mu_t^{v,h}$ for the two-type model stated above. We only discuss the equation for $\mu_t^{v,e}$ but note that everything said here applies symmetrically to $\mu_t^{v,h}$. The type switching intensities λ^e and λ^h appear in two places. First, in the first line, there is a term

$$\frac{\lambda^e \eta_t^e - \lambda^h \eta_t^h}{\eta_t^e} = \frac{\lambda^h}{\eta^*} \frac{\eta_t^e - \eta^*}{\eta_t^e},$$

which is, up to the sign, exactly the same term as the last term in the drift of η_t. Because the drift is finite (in fact, zero) in the limit equilibrium, this term must also vanish in the limit $\lambda^h \to \infty$. Second, the last term in the expression for $\mu_t^{v,e}$ also depends on switching intensities,

$$\lambda^e \frac{v_t^h - v_t^e}{v_t^e},$$

and, in the limit $\lambda^e \to \infty$, this term becomes arbitrarily large unless $v_t^e = v_t^h$. Because the term is positive if $v_t^h > v_t^e$, negative if $v_t^h < v_t^e$, and this equation describes a backward equation for v_t^e, it must indeed be the case that $v_t^e = v_t^h$ in the limit.

Furthermore, once we impose $v_t^e = v_t^h =: v_t$ and $\eta_t = \eta^*$, use that either of these two equations implies $\sigma_t^{\eta,e} = 0$, and plugs these equations into the equation for either $\mu_t^{v,h}$ or $\mu_t^{v,e}$ (in the limit as $\lambda^e, \lambda^h \to \infty$), we obtain an equation that is identical to the equation for μ_t^v in the one-type model stated at the end of Appendix A.7.

We use the previous considerations to conclude that if condition (64) is satisfied, then $v_t^1 = v_t^2$ and $\vartheta_t^1 = \vartheta_t^2$ (where, v_t^2 is the common value for $v_t^h = v_t^e$ in the two-type model in the limit economy). First, if the condition is satisfied and these two equations hold, then equations (65) and (66) have identical right-hand sides state by state, so indeed the ϑ-solutions must satisfy $\vartheta_t^1 = \vartheta_t^2$. Similarly, as just observed, then $\mu_t^{v,1}$ and $\mu_t^{v,2}$ must be identical state by state, such that the value function solutions must satisfy $v_t^1 = v_t^2$. While this logical appears somewhat circular, the previous observations are indeed sufficient to establish that under condition (64), there is a solution such that

$v_t^1 = v_t^2$ and $\vartheta_t^1 = \vartheta_t^2$.[99] Uniqueness of the non-degenerate stationary solution then also implies that this is the only possibility.

Having established that $\vartheta_t^1 = \vartheta_t^2$, arguments in full analogy to the proofs of Lemma 5 and Proposition 9 show that the conclusion of Proposition 9 remains valid.

□

A.12 Alternative Calibration Choices and Robustness

In this appendix we report results for three alternative calibration choices and show that our main conclusions are robust to them.

First, one concern with our calibration may be that it overstates the real effects of variation in idiosyncratic risk $\tilde{\sigma}_t$. This concern arises because we impose a perfectly linear relationship between this variable and productivity a_t and choose the sensitivity α^a of productivity to variation in $\tilde{\sigma}_t$ to match total output volatility. However, empirically, the correlation between measures of (total factor) productivity and volatility are not nearly as strong as imposed in our model, so that some of the empirically observed output volatility is likely due to factors unrelated to variation in (idiosyncratic) risk.

Here, we show that this is not an issue. Theoretically, the dynamics of the endogenous variable ϑ_t matter most for the predictions of our model (see Proposition 1), but, at least in the log utility case, a_t-dynamics do not affect the determination of ϑ_t at all (compare equation (10)). While the same is no longer exactly true for the preferences we use in our calibrated model, we can verify numerically that the parameter α^a is not particularly important for any of our results. We do so by showing that lowering α^a to half its value in the baseline calibration lowers output volatility (by construction) but otherwise has only marginal effects on model predictions. We report the parameters and model moments for this alternative specification in the column "lower α^a" of Tables 4 and 5, respectively.

We remark that we still need $\alpha^a > 0$ to be sufficiently large such that aggregate consumption falls in times of high idiosyncratic risk. Otherwise, our model fails to match the correct sign of all aggregate risk premia.[100]

A second concern is that our calibration target for the debt-output ratio is too large,

[99] Simply take the solution for $j = 1$ as given and conjecture that processes defined by $\vartheta_t^2 := \vartheta_t^1, v_t^2 := v_t^1$ represent a valid solution to the equations in the two-type model. The previous logic verifies that this is

Table 4: Alternative Parameter Specifications: Parameters

parameter	baseline	lower α^a	lower debt/GDP target	matching cov$(S/Y, Y)$
$\tilde{\sigma}^0$	0.54	0.54	0.54	0.54
ψ	0.67	0.67	0.67	0.67
σ	0.4	0.4	0.4	0.4
$\bar{\chi}$	0.3	0.3	0.3	0.3
γ	6	6	5.4	5.9
ρ	0.138	0.138	0.138	0.138
a^0	0.63	0.63	0.62	0.63
$\mathfrak{g}$	0.138	0.138	0.136	0.138
$\breve{\mu}^{\mathcal{B},0}$	0.0026	0.0026	0.0042	0.0017
α^a	0.071	0.036	0.071	0.072
$\alpha^{\mathcal{B}}$	0.12	0.12	0.19	0.07
ϕ	8.1	8.1	6.2	8.6
ι^0	-0.022	-0.022	-0.0877	-0.0131
δ	0.055	0.055	0.028	0.057

Table 5: Alternative Parameter Specifications: Moments

moment	baseline	lower α^a	lower debt/GDP target	matching cov$(S/Y, Y)$
$\sigma(Y)$	1.3%	0.7%	1.3%	1.3%
$\sigma(C)/\sigma(Y)$	0.61	0.35	0.60	0.61
$\sigma(I)/\sigma(Y)$	3.35	4.44	3.32	3.37
$\sigma(S/Y)$	1.1%	1.1%	1.1%	0.6%
$\mathbb{E}[C/Y]$	0.58	0.58	0.58	0.58
$\mathbb{E}[G/Y]$	0.22	0.22	0.22	0.22
$\mathbb{E}[S/Y]$	-0.0005	-0.0005	-0.0005	-0.0005
$\mathbb{E}[I/K]$	0.12	0.12	0.12	0.12
$\mathbb{E}[q^K K/Y]$	3.48	3.49	3.72	3.48
$\mathbb{E}[q^B K/Y]$	0.74	0.71	0.48	0.74
$\mathbb{E}[d\tilde{r}^E - dr^{\mathcal{B}}]$	3.59%	3.26%	2.83%	3.78%
$\frac{\mathbb{E}[dr^E - dr^{\mathcal{B}}]}{\sigma(dr^E - dr^{\mathcal{B}})}$	0.31	0.28	0.29	0.29
$\rho(Y, C)$	0.98	0.67	0.99	0.98
$\rho(Y, I)$	0.99	0.97	0.99	0.99
$\rho(Y, S/Y)$	0.98	0.97	0.98	0.98
$\sigma(q^B K/Y)$	4.8%	4.7%	2.9%	5.29%
$\mathbb{E}[r^f]$	5.18%	5.41%	4.74%	5.50%
$\sigma(r^f)$	5.47%	5.97%	5.95%	5.31%

Notes: All variables are defined in precisely the same way as in Table 2 in the main text.

not only because we take the average over the last decade (for the reason explained in the main text) but also because we do not account for the fact that a substantial fraction of US government debt is held abroad and that share has risen over our sample period.

We have chosen not to exclude foreign held debt in our baseline calibration because it is not at all clear that this portion should indeed be excluded. This portion of debt is also relevant for the government budget and for pricing total debt. The implicit assumption in our calibration is, however, that foreign holders of US debt have a qualitatively and quantitatively comparable safe asset demand for this debt as domestic holders (so that one should think of them as also being agents in our model). It is unclear whether this is really the case.

For this reason, we report in Tables 4 and 5 in the column "lower debt/GDP target" an alternative calibration that reduces the target for the debt-output ratio by a third, which is approximately the fraction of US federal debt held abroad over the last decade. The new target is therefore 0.47 instead of 0.71 in the baseline calibration. We follow otherwise precisely the same procedure as outlined in the maintext to choose our parameters[101] We find that this modification affects the ability of our model to match the moments only marginally. Specifically, holding the dynamics of idiosyncratic risk constant due to our calibration choices, we need to lower risk aversion γ to reduce safe asset demand for bonds to match the lower debt-output ratio. This leads to a slight reduction in the equity premium and Sharpe ratio relative to the baseline specification. All other moments can still be matched equally well.

A third concern is that our calibration overstates the procyclicality of primary surpluses and therefore underestimates the value of the cash flow component in Figure 2. This concern arises because we target both the volatilities of output and the surplus-output ratio, but operate within a model environment that presumes a next to perfect correlation between the two variables while the empirical correlation is much weaker, 0.60. An alternative choice would be to ignore the empirical volatility of S/Y and instead target the covariance with output as the covariance is more directly related to pricing. This is equivalent to targeting a volatility $\sigma(S/Y)$ that is lowered by the factor

indeed the case.

[100] If, for example, a_t was constant, then consumption would rise in times of high idiosyncratic risk, so that equity and capital would command a negative aggregate risk premium and government bonds a positive one.

[101] We choose δ, which does not affect anything of interest, to keep the average growth rate in the model the same as in the baseline calibration.

0.60/0.98, the ratio between the empirical and the model-implied correlation between the two variables.

We provide results for this alternative calibration choice in the column "matching $\text{cov}(S/Y, Y)$" of Tables 4 and 5. The resulting moments are largely identical to the ones for the baseline specification.

Beyond the effects on model moments, we also plot the counterparts of our key Figure 2 that decomposes the value of government debt into a cash flow and a service flow component for the alternative calibration choices. Figure 4 depicts the results for the two calibration choices "lower debt/GDP target" (left panel) and "matching $\text{cov}(S/Y, Y)$" (right panel).[102] The qualitative and quantitative takeaways remain the same as in Section 4.3.

Figure 5 depicts the Debt Laffer Curves for the dynamic models arising from the four alternative specifications. It shows that only lowering the target for the debt-output ratio considerably affects the size of the sustainable permanent deficit. The rationale for reducing the target was that a substantial fraction of US debt is held abroad. One way to interpret the difference between the orange line and the blue line in Figure 5 is therefore that the latter depicts the Laffer curve trade-off if non-domestic demand for US debt as a safe asset continues to absorb a significant fraction of debt issuance whereas the former depicts a Laffer curve that the US would face if US treasury debt lost its status as a global safe asset.

Not shown in Figure 5 as the comparison Laffer curves for the steady state models arising under the alternative specifications. However, the main conclusion from Figure 3 that the negative-β property is quantitatively important for the Laffer curve arise here analogously. In fact, for all but one specification the steady-state Laffer curve is almost identical with the one shown in Figure 3. The exception is the specification "lower debt/GDP target". In this case, there is no public debt bubble in the steady state model such that the maximum sustainable deficit is zero. Clearly, the conclusion that the negative-β property matters holds in this case as well.

[102] In the interest of space, the specification "lower α^a" is omitted. It looks almost identical to Figure 2.

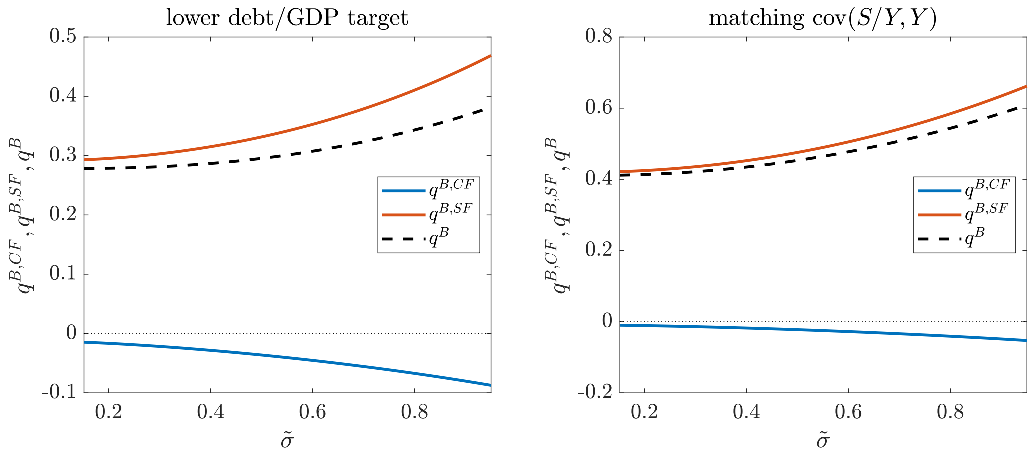

Figure 4: Decomposition of the value of government debt for alternative calibration choices. The description of Figure 2 applies analogously.

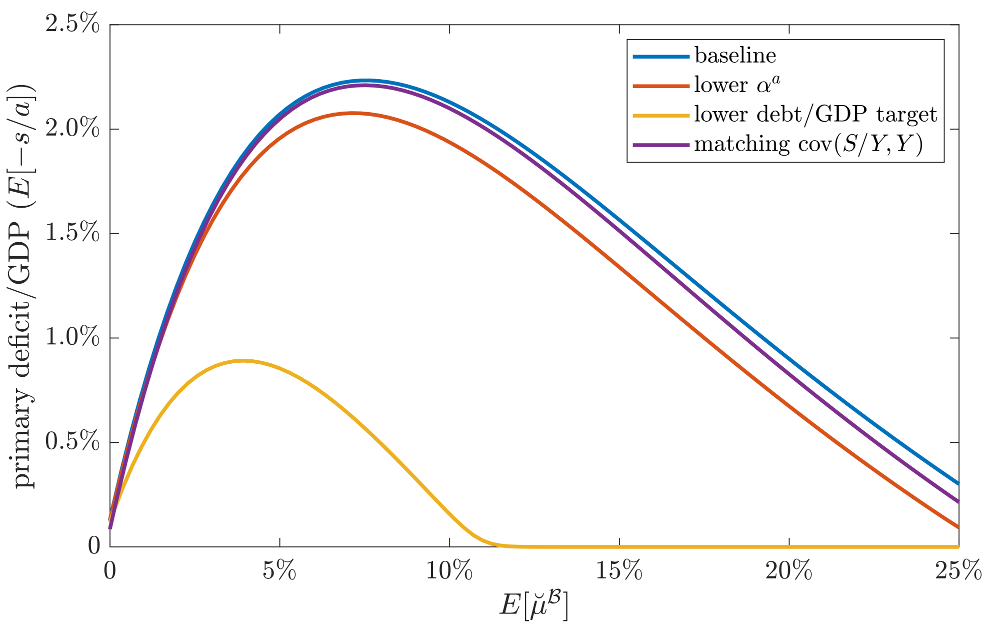

Figure 5: Debt Laffer curves for the alternative parameter specifications. The description of Figure 3 applies analogously to all four lines (for line "dynamic model" in that figure).