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THE CASE OF TWO REGIONS

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ABSTRACT

In this paper we characterize the set of equilibria in a generalized version of the canonical two-region economic geography model that nests the class of models in Allen and Arkolakis (2014) as well as Krugman (1991). We show that the set of (regular) equilibria corresponds to the set of zeros of a function $V(x)$, where x is the relative price of manufacturing goods produced in the two regions (adjusted by the trade elasticity). Using this approach, we provide sufficient conditions for uniqueness of equilibria that — in contrast to the well-know result in Allen and Arkolakis (2014) — allow for positive agglomeration externalities even in the absence of congestion effects, and highlight the key role played by three additional parameters: the trade elasticity, which regulates the strength of the dispersion force associated with the decline in the terms of trade caused by migration into a region; trade costs, which weaken this dispersion force by limiting trade across regions; and the importance of the agricultural sector, which pushes against agglomeration forces in manufacturing. We also discuss how asymmetries between the two regions tend to push against multiplicity.

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1 Introduction

Recently there has been a surge of research on quantitative models of the spatial distribution of economic activity that are tractable yet sophisticated enough to capture first-order features of the data, such as heterogeneous geography, productivity, and amenities, along with trade and migration across regions — see Redding (2020) for a recent survey of the literature. Given that economies of agglomeration constitute a central feature in economic geography, a challenge in this literature is the possibility of multiple equilibria and its implications for counterfactual analysis.

In their seminal paper, Allen and Arkolakis (2014, henceforth AA) provided a sufficient condition for uniqueness of spatial equilibria for an important class of economic geography models featuring trade and agglomeration forces in a single productive sector with regionally differentiated varieties, as in Armington (1969), and congestion forces, as in Helpman (1998). This sufficient condition hinges on the balance between the elasticity of localized external economies of scale, $\psi \geq 0$, and the elasticity of congestion externalities, $\delta \geq 0$, requiring that the latter be weakly stronger than the former, $\psi \leq \delta$. A benefit of this condition is that, if it holds, it ensures uniqueness regardless of the values of all other parameters, including trade costs. However, it is a strong condition that, for example, rules out agglomeration externalities if there are no congestion externalities. Moreover, this sufficiency result does not tell us anything about the set of equilibria if the condition is violated.

In this paper we characterize the set of equilibria in a generalized version of the canonical two-region economic geography model that nests the class of models in Allen and Arkolakis (2014) as well as Krugman (1991). Section 2 presents the model. There are two sectors, manufacturing and agriculture. Manufacturing is modeled as in Armington (1969), with elasticity of substitution $\sigma > 1$ across the goods produced by different regions — so that the trade elasticity is $\varepsilon = \sigma - 1 > 0$, and features external economies of scale with elasticity $\psi \geq 0$. Agriculture consists of a single homogenous good that is subject to frictionless trade. There are two types of labor: manufacturing workers are perfectly mobile across regions and face congestion externalities with elasticity $\delta \geq 0$, while agricultural workers are exogenously located. Preferences are Cobb-Douglas with a share β of income spent on manufacturing. The model is isomorphic to Krugman (1991) when the product agglomeration forces net of congestion effects and the trade elasticity is exactly one, or more precisely $\alpha \equiv [(\psi - \delta) \ / \ (1 + \delta)]\, \varepsilon = 1$, and so we say that $\alpha = 1$ corresponds to the "Krugman case".[1] The AA model obtains when there is no agriculture, $\beta = 1$, and

[1] In Krugman (1991) there is no congestion, $\delta = 0$, while economies of agglomeration arise by way of

AA's sufficient condition for uniqueness corresponds to $\alpha \leq 0$.

Section 3 considers the case of two regions and presents a series of results that characterize the set of equilibria for given values of α, ε, β, iceberg trade costs and the distribution of exogenous productivity, amenities, and the number of agricultural workers across the two regions. After showing that *irregular equilibria* (i.e., equilibria in which all manufacturing workers locate in a single region) are possible only when $\alpha \geq 1$, we show that a *regular equilibrium* (i.e., an equilibrium in which manufacturing workers are located in both regions) is fully characterized by the relative price of manufacturing goods produced in the two regions (adjusted by the trade elasticity), which we denote by x, and that a regular equilibrium satisfies $V(x) = 0$ for an explicit function $V(\cdot)$. Thus, characterizing the set of regular equilibria corresponds to characterizing the set of zeros of the function $V(x)$. Using this approach we derive a set of conditions that ensure uniqueness of equilibrium as long as $\alpha \in (0, 1)$ and then show how these results are affected when $\alpha = 1$, as in the Krugman case.

Broadly speaking, the results of Section 3 establish sufficient conditions for uniqueness of equilibria that — in contrast to the well-know result in Allen and Arkolakis (2014) — allow for positive agglomeration externalities even in the absence of congestion effects, and highlight the key role played by three additional parameters: the *trade elasticity*, which regulates the strength of the dispersion force associated with the decline in the terms of trade caused by migration into a region; *trade costs*, which weaken this dispersion force by limiting trade across regions; and the importance of the *agricultural sector*, which pushes against agglomeration forces in manufacturing. To shed light on these different forces it proves convenient to consider two special cases: the case with no agriculture, as in AA, and the case with symmetric regions, as in Krugman (1991).

Section 4 studies the case with no agriculture, $\beta = 1$, and derives a full characterization of the set of equilibria, including *necessary* and *sufficient* conditions for uniqueness. These conditions show that if $0 < \alpha < 1$ then the equilibrium is unique if trade costs are low while there are multiple equilibria if trade costs are high.[2] Moreover, a lower trade elasticity expands the range of trade costs under which the equilibrium is unique. These results reveal how the terms-of-trade dispersion force can lead to uniqueness even if agglomeration dominates congestion, $\psi > \delta$, so that $\alpha > 0$. Intuitively, terms of trade worsen for a region experiencing a rise in population, and this effect is more severe with a lower trade elasticity and when trade costs are low so that there is more trade and

free entry and love of variety, implying that $\psi\varepsilon = 1$ — see AA for a details.

[2] In Section 3 we show that if $\alpha < 1$ then the equilibrium is unique if trade costs are low enough even with $\beta < 1$. In contrast, the result that there are multiple equilibria for high enough trade costs requires β to be high enough.

terms-of-trade changes have a larger impact on real wages and migration.[3]

The presence of agriculture changes these results in intuitive ways, as shown in Section 5 for the case with symmetric regions. In this case and for $\alpha \in (0, 1)$ the number of equilibria depends on the relative importance manufacturing and agriculture captured by β and trade costs for any given value of the trade elasticity ε. Not surprisingly, the results described above for the case with no agriculture continue to hold when β is close to one. At the other extreme, when β is low enough then the weight of agriculture is so strong that there is a unique equilibrium for all trade costs. For intermediate values of β uniqueness holds if trade costs are either low or high: when both agriculture and manufacturing are important, uniqueness requires trade costs either be low so that terms-of-trade dispersion force is strong enough or high so that dispersion effects associated with the agricultural sector are strong enough.

Finally, Section 6 presents some examples to illustrate the role of asymmetries between the two regions on the number of equilibria. While the results of Section 5 suggest that decreasing α or β would always make it more likely that the equilibrium is unique, in Section 6 we see that this is not always true under asymmetry: depending on other parameters, increasing alpha or beta might result in intermittent regions of uniqueness and multiplicity. In this section we also show how while increasing the asymmetry between the two regions tends to enlarge the set of parameters under which the equilibrium is unique, this is also not universally true.

The completely symmetric case of the Krugman (1991) model was analyzed in Robert-Nicoud (2005), while the asymmetric case was analyzed in Sidorov (2011). Sidorov (2011) considers the Krugman (1991) model with unequal distribution of agricultural labor across two regions, but with symmetric iceberg trade costs, no congestion externalities ($\delta = 0$), and equal productivities and amenities across regions. In this paper we analyze the Krugman model with an unequal distribution of agricultural labor across two regions, asymmetric trade costs, allow for congestion externalities, and unequal productivities and amenities across regions. For the AA model, we analyze a general case with all the same asymmetries as in the Krugman model, but — as in the original AA paper — without the agricultural sector. Importantly, we provide a comprehensive characterization of equilibria for a generalized version of the canonical two-region model that integrates all

[3] If there are no congestion externalities on local amenities, $\delta = 0$, then $\alpha < 1$ is equivalent to $\psi < 1/\varepsilon$. Thus, if the terms-of-trade dispersion force — whose strength is regulated by $1/\varepsilon$ — dominates agglomeration effects — whose strength is regulated by ψ — then uniqueness is guaranteed under frictionless trade even in the absence of congestion effects. In the context of a multi-sector gravity model of international trade with sector-level external economies of scale, Kucheryavyy, Lyn and Rodríguez-Clare (2018) also show that if the scale elasticity is lower than the inverse of the trade elasticity in every sector then there is a unique equilibrium under frictionless trade.

the forces in the AA model as well as those in Krugman (1991).

2 The Model

Here we consider a multi-region spatial model as in AA with an additional agricultural sector as in Krugman (1991). There are N regions indexed by i, j and n, and two sectors: a manufacturing sector with a differentiated good associated with each region, as in the Armington approach, and an agricultural sector producing a homogeneous good. Production of either of these goods uses labor specific to the corresponding sector.

Each region i has $\bar{L}_i^A \geq 0$ agricultural workers who cannot move across regions and inelastically supply their labor to the agricultural sector. Production technology in agriculture is the same across all regions: the agricultural good is produced one-to-one from agricultural labor. Agricultural goods produced by different regions are perfect substitutes in consumption and can be costlessly traded across regions. Thus, the price of the agricultural good is equal to the wage of agricultural workers and is the same across all regions.

In addition to the agricultural workers, the economy has $\bar{L}$ manufacturing workers who can move freely across regions. Manufacturing workers residing in region i inelastically supply their labor to the manufacturing sector in that region. Each manufacturing worker in region i produces A_i units of the manufactured good, with the local productivity given by $A_i \equiv \bar{A}_i L_i^{\psi}$, where $\bar{A}_i$ is an exogenous component of productivity in region i, L_i is the number of manufacturing workers employed in region i, and $\psi \geq 0$ regulates the strength of agglomeration externalities affecting manufacturing production. Trade in manufactured goods between regions is subject to iceberg trade costs: delivering a unit of a manufactured good from region i to region n requires shipping $\tau_{ni} \geq 1$ units of the good, with $\tau_{ii} = 1$ for all i and $\tau_{nl} \leq \tau_{ni}\tau_{il}$ for all n, l, and i (triangular inequality).

Both agricultural and manufacturing workers residing in region i derive utility from a local amenity as well as from consumption of the agricultural good and an aggregate of manufactured goods produced by all regions. Region-specific manufactured goods are aggregated by a constant-elasticity-of-substitution (CES) function with elasticity of substitution $\sigma > 1$. Utility from consumption of the agricultural good and the manufactured aggregate is Cobb-Douglas with share $\beta > 0$ of expenditure devoted to manufactures.

Letting w_i and P_i denote wage and price index in manufacturing in region i and w^A the wage in agriculture, the welfare of manufacturing workers residing in region i is given by

$$U_i \equiv \frac{w_i}{P_i^{\beta}\left[w^A\right]^{1-\beta}} u_i, \tag{1}$$

with the term u_i denoting the amenity in region i given by

$$u_i \equiv \bar{u}_i L_i^{-\delta}. \tag{2}$$

Here $\bar{u}_i$ is an exogenous utility component of amenity in i and $\delta \geq 0$ governs the strength of congestion externalities affecting utility.

All markets are perfectly competitive. The equilibrium conditions are that: (i) the market for the manufactured good from each region clears; (ii) the global market for the agricultural good clears; (iii) welfare of manufacturing workers is equalized across all inhabited regions; and (iv) the global market for manufacturing labor clears. Formally, the set of equilibrium conditions is given by

$$w_i L_i = \sum_{n=1}^{N} \lambda_{ni} \beta \left(w_n L_n + w^A \bar{L}_n^A \right), \quad \text{for all } i; \tag{3}$$

$$\sum_{i=1}^{N} w^A \bar{L}_i^A = (1-\beta) \sum_{i=1}^{N} \left(w_i L_i + w^A \bar{L}_i^A \right); \tag{4}$$

$$L_i \geq 0, \quad \bar{U} - U_i \geq 0, \quad L_i \left(\bar{U} - U_i\right) = 0, \quad \text{for all } i; \tag{5}$$

$$\sum_{i=1}^{N} L_i = \bar{L}; \tag{6}$$

where $\bar{U} > 0$ is the utility level of manufacturing workers, $\lambda_{ni} \equiv \bar{A}_i^{\varepsilon} L_i^{\varepsilon\psi} \left(w_i \tau_{ni}\right)^{-\varepsilon} P_n^{\varepsilon}$ denotes the share of expenditure on manufacturing that region n devotes to imports from region i,

$$P_n = \left[\sum_{j=1}^{N} \bar{A}_j^{\varepsilon} L_j^{\varepsilon\psi} \left(w_j \tau_{nj}\right)^{-\varepsilon} \right]^{-\frac{1}{\varepsilon}} \tag{7}$$

is the price index of manufacturing in region n, and $\varepsilon \equiv \sigma - 1$ is the trade elasticity. We assume that $\sum_{i=1}^{N} \bar{L}_i^A > 0$ for the versions of the model with $\beta < 1$, and that $\sum_{i=1}^{N} \bar{L}_i^A = 0$ for the versions of the model with $\beta = 1$ (no agricultural sector).

Following AA, we call an equilibrium *regular*, if all regions are inhabited, and we call an equilibrium *irregular* otherwise. For future purposes, it is also convenient to introduce an additional parameter,

$$\alpha \equiv \frac{(\psi - \delta)\,\varepsilon}{1+\delta}.$$

This will be key parameter in characterizing equilibria of the model.

Our model above nests various different cases that we consider below. What we label the "Krugman case" obtains with $\alpha = 1$, while the AA case obtains with $\beta = 1$ and $\sum_{i=1}^{N} \bar{L}_i^A = 0$ (no agriculture). To see why $\alpha = 1$ captures the Krugman model, imagine that $\delta = 0$, and note that then $\alpha = 1$ entails $\psi = 1/\varepsilon = 1/(\sigma - 1)$: this is the standard result that in a model with monopolistic competition with free entry and CES preferences, there are economies of scale arising from love of variety with a scale elasticity given by $1/(\sigma-1)$ (see for example Allen and Arkolakis (2014)). As we show below, $\delta \neq 0$ does not have an independent effect on the set of equilibria given α.

3 Spatial Equilibria with Two Regions

In this section we characterize equilibria of the economy described in Section 2 in the general case with two regions ($N = 2$). In Section 4 we provide a sharper equilibrium analysis for the AA economy ($\beta = 1$). In Section 5 we use a fully symmetric two-region case to provide intuition behind the results in the general case. Finally, in Section 6 we discuss the role of asymmetries for multiplicity of equilibria in the general case.

We first introduce the following definitions:

$$G \equiv \left(\frac{\bar{u}_1}{\bar{u}_2}\right)^{\alpha+\varepsilon}\left(\frac{\bar{A}_1}{\bar{A}_2}\right)^{\varepsilon}, \quad \gamma \equiv \frac{\bar{L}_1^A}{\bar{L}_1^A + \bar{L}_2^A},$$

and

$$\overline{\gamma} \equiv 1-\gamma, \quad \overline{\beta} \equiv 1-\beta, \quad \mu \equiv (\alpha/\varepsilon + 1)\,\beta.$$

Also, for some of the results, it is more convenient to use trade freeness parameters, $\phi_1 \equiv \tau_{12}^{-\varepsilon}$ and $\phi_2 \equiv \tau_{21}^{-\varepsilon}$, instead of trade costs.

Parameters G and γ capture asymmetries between the two regions (in Section 5 we consider the case of symmetric regions, which entails $G = 1$ and $\gamma = 1/2$). Parameter μ can be written as $\mu = \frac{(1+\psi)\beta}{1+\delta}$ and captures the interaction of the economies of scale in manufacturing, ψ, with congestion in amenities, δ, adjusted by the size of the manufacturing sector, β. In the Krugman case ($\alpha = 1$), condition $\mu \geq 1$ is the familiar "black-hole condition" from Krugman (1991). Observe that our restrictions that $\psi \geq 0$, $\delta \geq 0$, and $\beta > 0$ imply that $\mu > 0$.

We start by characterizing irregular equilibria of the economy of Section 2 with $N = 2$.

Proposition 1 (Irregular equilibria).

(i) If $\alpha < 1$ then all equilibria are regular.

(ii) If $\alpha = 1$ then:

(ii.a) Allocation $L_1 = \bar{L}$ and $L_2 = 0$ is an equilibrium if and only if

$$G \geq \overline{\gamma}\overline{\beta}\phi_2^{\mu-1} + \left(1 - \overline{\gamma}\overline{\beta}\right)\phi_1\phi_2^{\mu}. \tag{8}$$

(ii.b) Allocation $L_1 = 0$ and $L_2 = \bar{L}$ is an equilibrium if and only if

$$G \leq \left[\gamma\overline{\beta}\phi_1^{\mu-1} + \left(1 - \gamma\overline{\beta}\right)\phi_1^{\mu}\phi_2\right]^{-1}. \tag{9}$$

(iii) If $\alpha > 1$ then both patterns of irregular allocations — (1) $L_1 = \bar{L}$ and $L_2 = 0$, and (2) $L_1 = 0$ and $L_2 = \bar{L}$ — constitute equilibria.

The proof of Proposition 1 is provided in Appendix A.1. Parts (i) and (iii) of Proposition 1 can easily be extended to more than two regions, and their proofs are very similar to the corresponding proofs in Allen and Arkolakis (2014) (which do not have an agricultural sector).

The rest of this section is devoted to the analysis of *regular equilibria* of the two-region economy of Section 2. We emphasize that when we refer to "a unique regular equilibrium", as for example in Propositions 2 and 5 below, it means that there is *one* regular equilibrium, but there may also be irregular equilibria. To arrive at overall uniqueness results we will couple these results on regular equilibria with those in Propositions 1.

The key result that facilitates analysis of regular equilibria is the fact that the equilibrium system of equations (3)-(6) for regular equilibria can be summarized by just one equation.

Lemma 1. *The analysis of multiplicity of regular equilibria of the economy of Section 2 is equivalent to the analysis of multiplicity of positive solutions of the following equation in x:*

$$x^{\alpha-1} = G \cdot \left(\frac{\phi_1}{\phi_2}\right)^{-\alpha} \left[g_{\phi}(x)\right]^{\mu} \cdot \left[g_d(x)\right]^{\alpha}, \tag{10}$$

where

$$g_{\phi}(x) \equiv \frac{1 + \phi_1 x}{\phi_2 + x} \quad \text{and} \quad g_d(x) \equiv \frac{1 + d_1 x}{d_2 + x}, \tag{11}$$

$$d_1 \equiv \frac{\gamma\overline{\beta} + \left(1 - \gamma\overline{\beta}\right)\phi_1\phi_2}{\phi_2} \quad \text{and} \quad d_2 \equiv \frac{\overline{\gamma}\overline{\beta} + \left(1 - \overline{\gamma}\overline{\beta}\right)\phi_1\phi_2}{\phi_1}. \tag{12}$$

The proof Lemma 1 is provided in Appendix A.2. This result implies that we can focus on characterizing the set of solutions of a single equation in one unknown, x, which is the price of the manufacturing good produced in region 1 relative to region 2 adjusted by the trade elasticity,

$$x = \left(\frac{w_1 / \left(\bar{A}_1 L_1^{\psi} \right)}{w_2 / \left(\bar{A}_2 L_2^{\psi} \right)} \right)^{\varepsilon}.$$

In the trivial case in which trade is frictionless and there are no agglomeration or congestion externalities, equation (10) boils down to $x^{1/\varepsilon} = \bar{u}_2 \bar{A}_2 / \bar{u}_1 \bar{A}_1$, which simply says that the relative price of manufacturing has to equal the inverse ratio for the productivities adjusted by amenities. In the general case, equation (10) shows how we need to adjust this equality by trade costs (ϕ_1, ϕ_2), the importance of agriculture in consumption, $\overline{\beta}$, and the relative size of agriculture in both regions, γ. Importantly, once we make the change of variables to focus only on solving for x, agglomeration and congestion elasticities, ψ and δ, matter only through *one* parameter α. Finally, note that parameter α can take any value, and the behavior of equation (10) is qualitatively different depending on whether $\alpha \leq 0$, $0 < \alpha < 1$, $\alpha = 1$, or $\alpha > 1$.

Lemma 1 allows us to immediately see what happens in the case of frictionless trade. In this case, $\phi_1 = \phi_2 = 1$ and the definitions (12) of d_1 and d_2 in Lemma 1 imply that $d_1 = 1$ and $d_2 = 1$. Thus, equation (10) collapses to $x^{\alpha-1} = G$. If $\alpha \neq 1$ then we can explicitly find $x = G^{\frac{1}{\alpha-1}}$, which means that (10) has only one positive solution. If $\alpha = 1$ and $G = 1$ then any positive x is a solution of (10), while if $\alpha = 1$ and $G \neq 1$, then (10) does not have positive solutions. This is summarized in the following proposition.

Proposition 2 (Regular equilibria: Frictionless trade). *Assume that trade is frictionless, $\phi_1 = \phi_2 = 1$.*

(i) If $\alpha \neq 1$ then the economy has a unique regular equilibrium.

(ii) If $\alpha = 1$ and $G = 1$ then any regular allocation satisfying $L_1 + L_2 = \bar{L}$ is an equilibrium. If $\alpha = 1$ and $G \neq 1$ then the economy of Section 2 does not have regular equilibria.

It is interesting to note that Part (i) of Propositions 1 and 2 imply that if $\alpha < 1$ and trade is frictionless then there is a unique equilibrium and this equilibrium is regular.

In the rest of this section we focus on the case of costly trade: $\tau_{12}\tau_{21} > 1$ or, equivalently, $\phi_1\phi_2 < 1$. Lemma 1 can readily be applied to establish existence of a regular equilibrium of the economy of Section 2 in the case with $\alpha \neq 1$ under costly trade. Indeed, taking logarithms on both sides of function (10) in Lemma 1 and bringing all terms to one side, we get equation $V(x) = 0$, where

$$V(x) \equiv (\alpha - 1) \ln x - \ln G + \alpha \ln \left(\frac{\phi_1}{\phi_2} \right) - \mu \ln g_{\phi}(x) - \alpha \ln g_d(x). \tag{13}$$

Existence of a regular equilibrium in the case with $\alpha \neq 1$ then simply follows from the fact that function $V(x)$ is continuous and takes values at the opposite sides of 0 as $x \to 0$ and $x \to \infty$.[4] This result is summarized in the following proposition.

Proposition 3 (Regular equilibria: Existence under costly trade and $\alpha \neq 1$). *If trade is costly ($\phi_1 \phi_2 < 1$) and $\alpha \neq 1$ then the economy has a regular equilibrium.*

Analysis of existence of regular equilibria in the case with $\alpha = 1$ is substantially more complicated and is described in detail in Appendix C.

Lemma 1 can also be used to get results on the maximum number of regular equilibria of the economy of Section 2. One can easily verify that in the case with $\alpha \neq 1$ solving equation $V'(x) = 0$ for $x > 0$ is equivalent to finding roots of a fourth-degree polynomial, and thus function $V(x)$ can have at most four extrema. This implies that equation $V(x) = 0$ can have at most five solutions. Similarly, in the case with $\alpha = 1$ solving equation $V'(x) = 0$ for $x > 0$ is equivalent to finding roots of a quadratic polynomial, and thus function $V(x)$ can have at most two extrema, implying that equation $V(x) = 0$ can have at most three solutions. These results are summarized in the following proposition with its formal proof provided in Appendix A.3.

Proposition 4 (Maximum number of regular equilibria under costly trade). *Assume that trade is costly ($\phi_1 \phi_2 < 1$).*

(i) If $\alpha \neq 1$ then the economy has at most five regular equilibria.

(ii) If $\alpha = 1$ then the economy has at most three regular equilibria.

The fact that for $\alpha \neq 1$ solving equation $V'(x) = 0$ for $x > 0$ is equivalent to finding roots of a fourth-degree polynomial has one more important implication for our analysis. Suppose that $0 < x_1^* < \cdots < x_M^*$ with $0 \leq M \leq 4$ are distinct positive real solutions to $V'(x) = 0$. Since roots of a fourth-degree polynomial can be found analytically, we can obtain closed-form (although, very complicated) expressions for $x_1^*, \ldots, x_M^*$. These expressions depend only on the exogenous parameters of the model. Each of the points $x_1^*, \ldots, x_M^*$ can be either a local extremum of $V(\cdot)$ or an inflection point. Checking the signs of $V(x_1^*), \ldots, V(x_M^*)$ as well as $\lim_{x \to 0} V(x)$ and $\lim_{x \to \infty} V(\infty)$, we can unambiguously determine how many times function $V(\cdot)$ intersects the horizontal axis, which would give us the number of regular equilibria in the original economy.[5] For example, if $M = 0$ then function $V(\cdot)$ is monotonic and so there could be only one regular

[4] If $\alpha < 1$ then $\lim_{x \to 0} V(x) = \infty$ and $\lim_{x \to \infty} V(x) = -G\phi_1^{\mu-\alpha}\phi_2^{\alpha} d_1^{\alpha}$. If $\alpha > 1$ then $\lim_{x \to 0} V(x) = -G\phi_1^{-\alpha}\phi_2^{\alpha-\mu} d_2^{-\alpha}$ and $\lim_{x \to \infty} V(x) = \infty$.

[5] Our exhaustive analysis of regular equilibria in the Krugman case — provided in Appendix C — is based on this logic. The same logic is also used in our analysis of multiplicity of regular equilibria for the AA economy ($\beta = 1$) in the case of $0 < \alpha < 1$, which is summarized in part (ii) of Proposition 7 below.

equilibrium. As another example, suppose that $M = 2$ and $V(x_1^*) < 0$, $V(x_2^*) > 0$, $\lim_{x\to 0} V(x) = \infty$ and $\lim_{x\to\infty} V(x) = -\infty$. In this case x_1^* is a local minimum of $V(\cdot)$, x_2^* is a local maximum of $V(\cdot)$, and $V(\cdot)$ necessarily intersects the horizontal axis three times, implying that there exist three regular equilibria.

Following this approach, we could derive analytically necessary and sufficient conditions for uniqueness of regular equilibria for the case with $\alpha \neq 1$, but the expressions would be complicated and devoid of insights. Instead, we will formulate several parsimonious sufficient conditions for uniqueness of regular equilibria. The proofs of these sufficiency results presented below are based on the fact that under these conditions $V(\cdot)$ is either monotonic or a global contraction mapping.

For the purposes of the next proposition, let us define

$$\overline{\mu} \equiv \frac{\phi_1\phi_2(d_1d_2 - 1)}{1-\phi_1\phi_2} \quad \text{and} \quad \overline{\overline{\mu}} \equiv \frac{d_1d_2 - 1}{d_1d_2(1-\phi_1\phi_2)}. \tag{14}$$

It is straightforward to check that $\overline{\mu} < \overline{\overline{\mu}}$ as long as $d_1d_2 > 1$.[6] It is also straightforward to check that $d_1d_2 > 1$ if and only if

$$\phi_1\phi_2 < \frac{\gamma\overline{\gamma}\overline{\beta}^2}{(1-\overline{\gamma}\overline{\beta})(1-\gamma\overline{\beta})}.$$

That is, $d_1d_2 > 1$ if and only if trade costs are high enough.

Proposition 5 (Regular equilibria: Sufficient conditions for uniqueness under costly trade and $\boldsymbol{\alpha \neq 1}$). *Assume that $\phi_1\phi_2 < 1$. The economy has a unique regular equilibrium in the following cases:*

(i) $\alpha \leq 0$;

(ii) $0 < \alpha < 1$ *and either of the following three conditions hold:*

- *(ii.a)* $d_1d_2 \leq 1$ *and* $\mu \leq \frac{1+\sqrt{\phi_1\phi_2}}{1-\sqrt{\phi_1\phi_2}} \cdot \frac{1-2\alpha+\sqrt{d_1d_2}}{1+\sqrt{d_1d_2}}$;
- *(ii.b)* $d_1d_2 > 1$ *and* $\mu \leq \alpha\overline{\mu}$;
- *(ii.c)* $d_1d_2 > 1$ *and* $\mu > \alpha\overline{\mu}$ *and* $\max\{\mu - \alpha\overline{\mu},\ \alpha\overline{\overline{\mu}} - \mu\} \leq (1-\alpha)\frac{1+\sqrt{\phi_1\phi_2}}{1-\sqrt{\phi_1\phi_2}}$;

(iii) $\alpha > 1$ *and either of the following three conditions hold:*

- *(iii.a)* $d_1d_2 \leq 1$;
- *(iii.b)* $d_1d_2 > 1$ *and* $\mu \geq \alpha\overline{\overline{\mu}}$;
- *(iii.c)* $d_1d_2 > 1$ *and* $\mu < \alpha\overline{\overline{\mu}}$ *and* $\max\{\mu/\overline{\mu} - \alpha,\ \alpha - \mu/\overline{\overline{\mu}}\} \leq (\alpha-1)\frac{\sqrt{d_1d_2}+1}{\sqrt{d_1d_2}-1}$.

[6] If $d_1d_2 > 1$ then inequality $\overline{\mu} < \overline{\overline{\mu}}$ is equivalent to $\phi_1\phi_2d_1d_2 < 1$, which holds.

The proof of Proposition 5 is provided in Appendix A.4. Recall that the sufficient condition for uniqueness in Allen and Arkolakis (2014) is $\alpha \leq 0$. This is a strong condition that, for example, rules out agglomeration externalities if there are no congestion externalities. Part (i) of Proposition 5 extends this result to the economy with the agricultural sector. At the same time, Propositions 1 and 2 imply that the equilibrium is unique under frictionless trade even with $0 < \alpha < 1$. Proposition 5 allows to extend this result to the case of low enough but strictly positive trade costs, which is established in Corollary 1 below. Also, in the same corollary we establish that there is a unique equilibrium for low enough α and β.

Corollary 1. *Assume that $\phi_1\phi_2 < 1$ and $0 < \alpha < 1$. The economy has a unique regular equilibrium if $\phi_1\phi_2$ is close enough to 1, or α is low enough, or β is low enough.*

The proof of Corollary 1 is provided in Appendix A.5.

The next proposition deals with the case $\alpha = 1$. For the purposes of this proposition, let us define the following constants,

$$\tilde{\phi}_1 \equiv \frac{1-\mu}{1+\mu} \cdot \frac{\overline{\gamma}\overline{\beta}}{1-\overline{\gamma}\overline{\beta}} \left(\frac{\gamma}{\overline{\gamma}}\right)^{\frac{1}{1-\mu}} G^{\frac{1}{1-\mu}}, \tag{15}$$

$$\tilde{\phi}_2 \equiv \frac{1-\mu}{1+\mu} \cdot \frac{\gamma\overline{\beta}}{1-\gamma\overline{\beta}} \left(\frac{\gamma}{\overline{\gamma}}\right)^{-\frac{1}{1-\mu}} G^{-\frac{1}{1-\mu}}, \tag{16}$$

and

$$\tilde{c} \equiv \frac{1-\gamma\overline{\beta}}{1-\overline{\gamma}\overline{\beta}} \left(\frac{\gamma}{\overline{\gamma}}\right)^{\frac{1+\mu}{1-\mu}} G^{\frac{2}{1-\mu}}. \tag{17}$$

Also, let us define the following condition,

$$\overline{\gamma}\overline{\beta}\phi_2^{\mu-1} + \left(1-\overline{\gamma}\overline{\beta}\right)\phi_1\phi_2^{\mu} < G < \left[\gamma\overline{\beta}\phi_1^{\mu-1} + \left(1-\gamma\overline{\beta}\right)\phi_1^{\mu}\phi_2\right]^{-1}. \tag{18}$$

Proposition 6 (Regular equilibria under costly trade and $\alpha = 1$). *Assume that $\alpha = 1$ and $\phi_1\phi_2 < 1$.*

(i) Suppose that one of the following conditions holds:

(i.a) $\overline{\beta} = 0$, or $\gamma = 0$, or $\gamma = 1$;

(i.b) $\mu \geq 1$;

(i.c) $\mu < 1$ and, at the same time, either $\phi_1 \geq \tilde{\phi}_1$ or $\phi_2 \geq \tilde{\phi}_2$, where $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are defined in (15)-(16).

If, in addition to that, condition (18) holds, then the economy of Section 2 has a unique regular equilibrium. If, on the other hand, condition (18) does not hold, then the economy of

Section 2 does not have regular equilibria.

(ii) Suppose that none of the conditions (i.a)-(i.c) holds. Then for any fixed $c = \phi_1/\phi_2$ there exist $\tilde{\tilde{\phi}}_2(c) \in (0, \tilde{\phi}_2]$ and $\overline{\phi}_2(c) \in \left(0, \tilde{\tilde{\phi}}_2(c)\right]$ such that:

(ii.a) If $\phi_2 \leq \overline{\phi}_2(c)$ then the economy of Section 2 has a unique regular equilibrium.

(ii.b) If $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ and condition (18) holds then the economy has a unique regular equilibrium. If $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ but condition (18) does not hold, then the economy of Section 2 does not have regular equilibria.

(ii.c) If $\overline{\phi}_2(c) < \phi_2 < \tilde{\tilde{\phi}}_2(c)$ then the economy has at most three regular equilibria.

(ii.d) If $\phi_2 = \tilde{\tilde{\phi}}_2(c)$ then the economy generically has at most two regular equilibria except for the special case with $c = \tilde{c}$ with $\tilde{c}$ given by (17), in which case $\tilde{\tilde{\phi}}_2(\tilde{c}) = \tilde{\phi}_2$ and the economy of Section 2 has at most one regular equilibrium.

The proof of Proposition 6 is provided in Appendix A.6.

4 No Agricultural Sector

In Section 3 we provided sufficient conditions for uniqueness of the economy of Section 2 in the general case. In this section we provide a *complete* characterization of equilibria for the case in which there is no agriculture, $\beta = 1$. This corresponds to a two-country version of the setup in Allen and Arkolakis (2014).

For the purposes of the next proposition, let us denote

$$\Gamma_1 \equiv \Xi \cdot \left[\phi_1 \phi_2^{\mu}\right]^{-1} (\tilde{\mu} - \phi_1\phi_2 - D)^{1-\alpha} (\tilde{\mu} + \phi_1\phi_2 + D)^{\mu+\alpha},$$
$$\Gamma_2 \equiv \Xi \cdot \left[\phi_1 \phi_2^{\mu}\right]^{-1} (\tilde{\mu} - \phi_1\phi_2 + D)^{1-\alpha} (\tilde{\mu} + \phi_1\phi_2 - D)^{\mu+\alpha},$$

with $\tilde{\mu} \equiv \dfrac{\mu + 2\alpha - 1}{\mu + 1}$, $D \equiv \sqrt{(1 - \phi_1\phi_2)(\tilde{\mu}^2 - \phi_1\phi_2)}$, and

$$\Xi \equiv \frac{G \cdot (\mu+1)^{\mu+1}}{2^{\mu+1} (1-\alpha)^{1-\alpha} (\mu+\alpha)^{\mu+\alpha}}.$$

Proposition 7 (AA economy: Complete Characterization of Equilibria). *Assume that $\beta = 1$ and $\phi_1 \leq 1$ and $\phi_2 \leq 1$.*

(i) If $\alpha \leq 0$ then there is a unique equilibrium, and this equilibrium is regular.

(ii) If $0 < \alpha < 1$ then any equilibrium is regular.

(ii.a) Equilibrium is unique in the following cases:

- *$\phi_1\phi_2 \geq \tilde{\mu}$;*

- $\phi_1\phi_2 < \tilde{\mu}$ *and* $\Gamma_1 < 1$;
- $\phi_1\phi_2 < \tilde{\mu}$ *and* $\Gamma_2 > 1$.

(ii.b) There are two equilibria in the following cases:
- $\phi_1\phi_2 < \tilde{\mu}$ *and* $\Gamma_1 = 1$;
- $\phi_1\phi_2 < \tilde{\mu}$ *and* $\Gamma_2 = 1$.

(ii.c) There are three equilibria if $\phi_1\phi_2 < \tilde{\mu}$ *and* $\Gamma_1 > 1$ *and* $\Gamma_2 < 1$.

(iii) If $\alpha = 1$ *then*
- *Allocation* $L_1 = \bar{L}$ *and* $L_2 = 0$ *is an equilibrium if and only if* $\phi_1\phi_2^{\mu} \leq G$;
- *Allocation* $L_1 = 0$ *and* $L_2 = \bar{L}$ *is an equilibrium if and only if* $\left(\phi_1^{\mu}\phi_2\right)^{-1} \geq G$;
- *If* $\phi_1\phi_2 < 1$ *and* $\phi_1\phi_2^{\mu} < G < \left(\phi_1^{\mu}\phi_2\right)^{-1}$, *then there is a single regular equilibrium;*
- *If* $\phi_1 = \phi_2 = 1$ *and* $G = 1$, *then any regular allocation satisfying* $L_1 + L_2 = \bar{L}$ *is an equilibrium;*
- *There are no regular equilibria in all other cases.*

(iv) If $\alpha > 1$ *then there are two irregular equilibria and one regular equilibrium.*

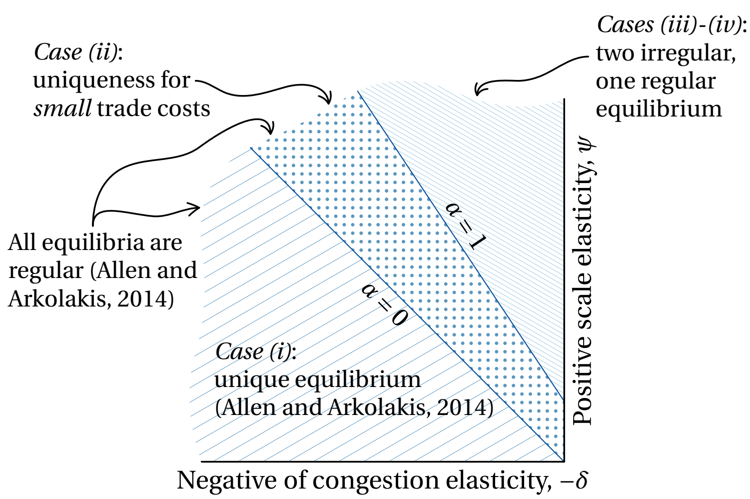


Figure 1: Illustration of Proposition 7 for AA economy.

The cases (i)-(iv) from Proposition 7 are illustrated in Figure 1. In this figure we show the areas with a unique regular equilibrium as well as areas with irregular equilibria in the space of (δ, ψ).[7] In addition to Figure 1, we illustrate cases (ii) and (iii) of Proposition 7 in Figures 2-6.

In part (i) of Proposition 5, we established uniqueness of a regular equilibrium of the economy of Section 2 for $\alpha \leq 0$ and any $\beta > 0$. And part (i) of Proposition 1 implies that all equilibria are regular if $\alpha \leq 0$. This gives us get part (i) of Proposition 7.

[7] Recall that our parameter restriction are $\delta \geq 0$ and $\psi \geq 0$ and our definition of α is $\alpha = \dfrac{(\psi - \delta)\,\varepsilon}{1 + \delta}$.

Part (ii) of Proposition 5 and part (i.a) of Proposition 6 directly imply part (iii) of Proposition 7.

Part (iii.a) of Proposition 5 implies uniqueness of a regular equilibrium of the economy of Section 2 for $\beta = 1$ and $\alpha > 1$, because with $\beta = 1$ we have $d_1 = \phi_1$ and $d_2 = \phi_2$. Combining this with part (iii) of Proposition 1, we get part (iv) of Proposition 7.

Given the above, we only need to prove part (ii) of Proposition 7, which we do in Appendix B.1.

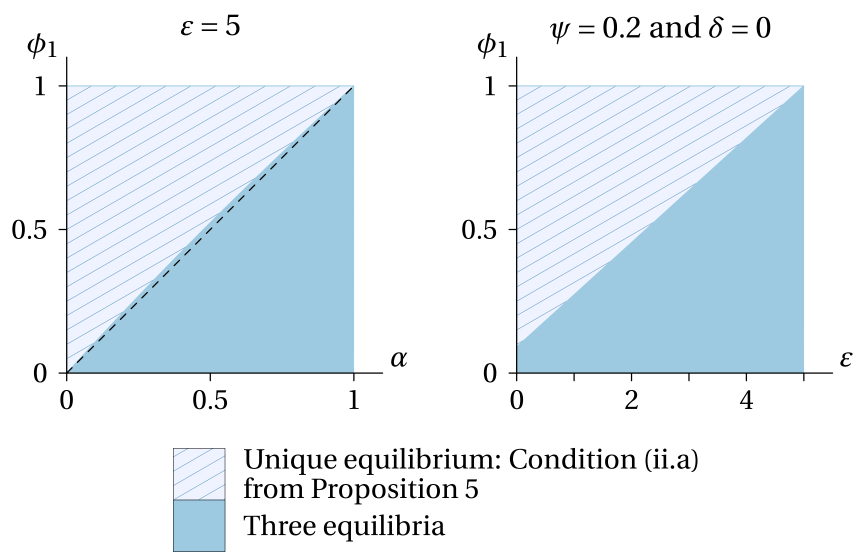


Figure 2: Uniqueness/multiplicity areas in case (ii) of Proposition 7 for AA economy: $\phi_2 = \phi_1$ and $G = 1$. Left picture: varying α and ϕ_1 for $\varepsilon = 5$. Right picture: varying ε and ϕ_1 for $\psi = 0.2$ and $\delta = 0$. Dashed line in the left picture is a 45° line. The boundary between the sets with unique and three equilibria has two equilibria.

Figure 2 shows areas with unique and three equilibria in (α, ϕ_1) and (ε, ϕ_1) spaces for the symmetric economy with $G = 1$ and $\phi_2 = \phi_1$. The maximum value ε on the right picture in Figure 2 is chosen so that $\alpha = \frac{(\psi-\delta)\varepsilon}{(1+\delta)} \in (0, 1)$ for all ε. We see from the left picture in Figure 2 that for each $\alpha \in (0, 1)$ there is a range of trade costs for which the economy has a unique equilibrium. Similarly, we see from the right picture in Figure 2 that for each $\varepsilon \in (0, 5)$ there is a range of trade costs for which the economy has a unique equilibrium. Another message that comes out from Figure 2 is that for each $\alpha \in (0, 1)$ or $\varepsilon \in (0, 5)$ there is a threshold trade freeness value such that there is a unique equilibrium for all trade freeness values above the threshold and there are three equilibria for all trade freeness values below the threshold. We formally establish this fact in Proposition 8 below.

Figure 2 also shows that the range of trade costs resulting in uniqueness tends to be smaller with larger values of α or ε. A key point here is the role of the terms-of-trade dispersion force in leading to uniqueness. Given $\alpha = \frac{\psi-\delta}{(1+\delta)/\varepsilon}$, we have that $0 < \alpha < 1$ implies $0 < \frac{\psi-\delta}{1+\delta} < \varepsilon$. The net agglomeration force captured by $\frac{\psi-\delta}{1+\delta}$ is counteracted by a terms-of-trade dispersion force, here captured by the inverse of the trade elasticity, $1/\varepsilon$. A high value of $1/\varepsilon$ means that as consumers move to a region they suffer a bigger terms of trade loss. With low trade costs, thanks to the terms-of-trade dispersion force, consumers do not tend to concentrate in one region, which results in uniqueness of an equilibrium. High trade costs weaken this dispersion force by limiting trade, thereby making terms of trade changes less relevant, and multiple equilibria more likely.

In part (ii) of Proposition 5 we provided sufficient conditions for uniqueness in the case with $0 < \alpha < 1$. Since $\beta = 1$ implies $d_1 = \phi_1$ and $d_2 = \phi_2$, the condition in part (ii) of Proposition 5 relevant for the case with $\beta = 1$ is the condition in part (ii.a) assuming that $d_1 d_2 \leq 1$. As we can see from Figure 2, condition in part (ii.a) from Proposition 5 happens to cover the entire uniqueness area in the symmetric case with $\beta = 1$. However, as we show in Section 6 below, in the general case, any of the conditions from part (ii) from Proposition 5 can be relevant for different parameter combinations. Moreover, for some parameter combinations, none of the conditions from Proposition 5 might cover some cases of uniqueness. This last point is also illustrated in Figure 3, to which we turn next.

In Figure 3 we show how asymmetries in trade costs and in productivities/amenities of regions impact uniqueness/multiplicity. In addition to this, Figure 3 illustrates the point that we made earlier that larger values of α are more likely to result in multiplicity. We see from panel (b) of Figure 3 that asymmetries in productivities/amenities of regions tend to result in uniqueness. This can be seen directly by looking at conditions for uniqueness in part (ii) of Proposition 7: condition $\Gamma_1 < 1$ is more likely to hold as G becomes lower, and condition $\Gamma_2 > 1$ is more likely to hold as G becomes larger.

Intuitively, multiplicity of equilibria in the symmetric setup arises exactly because all fundamental parameters of regions are the same, which — due to the economies of scale — creates indeterminacy of the outcome: manufacturing production can happen to be concentrated in one region or the other. As we make regions dissimilar in their productivities/amenities, the region with the more favorable characteristic is more likely to become an industrial hub with concentrated manufacturing production, which reduces indeterminacy of the outcome and leads to uniqueness of equilibrium.

Let us now consider asymmetries in trade costs. In order to isolate the effect of asymmetries in trade costs from the effect of the size of trade costs, we focus on combinations of ϕ_1 and ϕ_2 such that the geometric mean of of trade freeness is constant, $(\phi_1 \phi_2)^{1/2} = c$, or more simply $\phi_1 \phi_2 = c$. Given $\phi_1 \phi_2 = c$, the more the ratio ϕ_1/ϕ_2 is different from 1,

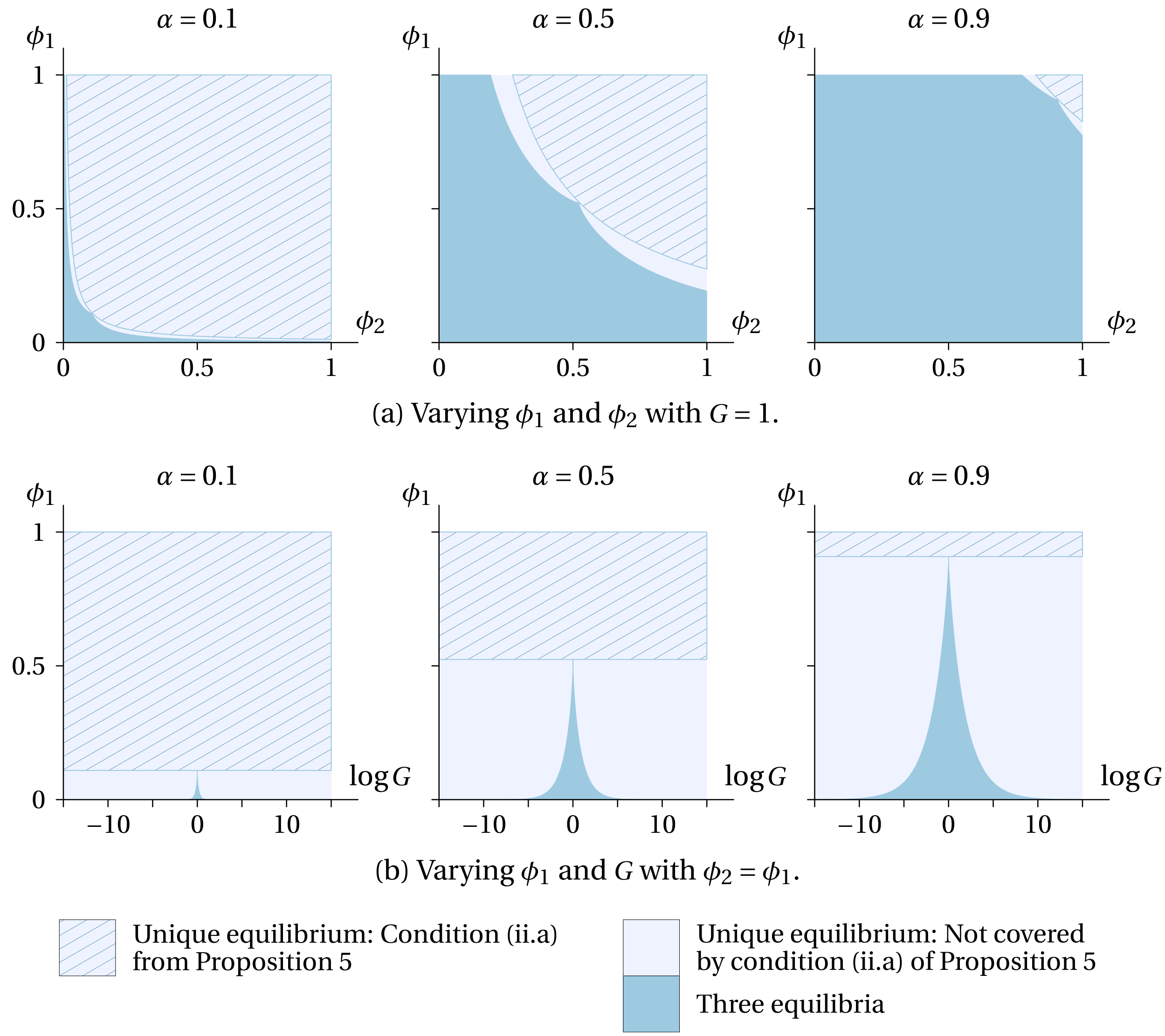


Figure 3: Uniqueness/multiplicity areas in case (ii) of Proposition 7 for AA economy: $0 < \alpha < 1$ and $\varepsilon = 5$.

the larger asymmetries in trade costs are. In Figure 4 we show areas with one and three equilibria in the (ϕ_1, ϕ_2) space — just as in panel (a) of Figure 3. On top of that, we depict curves $\phi_1\phi_2 = c$ for three levels of c. Also, for illustration purposes, we drop the assumption that $\phi_1 \leq 1$ and $\phi_2 \leq 1$. Figure 4 shows that increasing asymmetry in trade costs tends to result in uniqueness. Intuitively, for the same geometric mean of trade costs, asymmetry in trade costs creates incentives to concentrate manufacturing production in the region with high cost of importing and low cost of exporting, which leads to uniqueness. The same outcome is also evident from panel (a) of Figure 3. However, as Figure 3 shows, imposing the restriction $\phi_1 \leq 1$ and $\phi_2 \leq 1$ limits the effect of asymmetries on uniqueness: for example, with $\alpha = 0.9$ the model has three equilibria for most

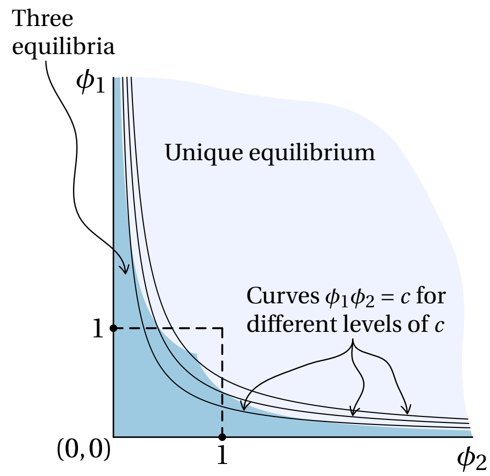


Figure 4: Illustration to Proposition 7 for AA economy: $0 < \alpha < 1$ and $G = 1$. Asymmetries in trade costs.

combinations of trade costs.

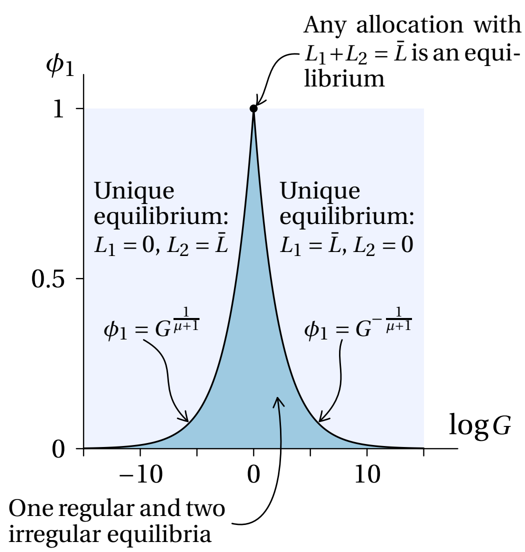


Figure 5: Case (iii) in Proposition 7 for AA economy: $\alpha = 1$, $\varepsilon = 5$, and $\phi_2 = \phi_1$.

Returning to different cases in Proposition 7, observe that conditions for uniqueness in part (ii.a) of Proposition 7 turn into conditions for irregular equilibria in part (iii) of Proposition 7 as $\alpha \to 1$. We can see this visually by comparing the pictures in panel (b) of Figure 3 with Figure 5.

In the next proposition we formally establish the outcome that we observe in Figures 2-4: that in the case with $0 < \alpha < 1$ equilibrium is unique if and only if trade costs are low enough. This proposition gives an alternative perspective on conditions in part (ii) of Proposition 7.

Proposition 8 (**AA economy: Uniqueness boundary for $0 < \alpha < 1$**). *Assume that $0 < \alpha < 1$, $\beta = 1$, and $\phi_1, \phi_2 \in (0, \infty)$. Then for any fixed $c \equiv \phi_1/\phi_2$:*

(a) If $\phi_2 \geq \tilde{\phi}_2(c)$, where

$$\tilde{\phi}_2(c) \equiv \tilde{\mu} \cdot \min\left\{\left[cG^{-1}\right]^{-\frac{1}{\mu+1}}, \left[c^{\mu}G\right]^{-\frac{1}{\mu+1}}\right\}, \tag{19}$$

then the equilibrium is unique.

(b) If $\phi_2 < \tilde{\phi}_2(c)$ then there exists $\tilde{\tilde{\phi}}_2(c) \in (0, \tilde{\phi}_2(c)]$ such that the equilibrium is unique if $\phi_2 > \tilde{\tilde{\phi}}_2(c)$, whereas there are three equilibria if $\phi_2 < \tilde{\tilde{\phi}}_2(c)$. If $\phi_2 = \tilde{\tilde{\phi}}_2(c)$, then generically there are two equilibria except for the special case when $G = c^{\frac{1-\mu}{2}}$, in which case there is a unique equilibrium. Moreover, if $G = c^{\frac{1-\mu}{2}}$ then $\tilde{\tilde{\phi}}_2(c) = \tilde{\phi}_2(c)$.

The proof of Proposition 8 is provided in Appendix B.2. Observe that in Proposition 8 we drop the assumption $\phi_1 \leq 1$ and $\phi_2 \leq 1$ and allow for any positive ϕ_1 and ϕ_2, which means that we allow the iceberg trade costs to be negative. This simplifies the formulation and proof of Proposition 8.

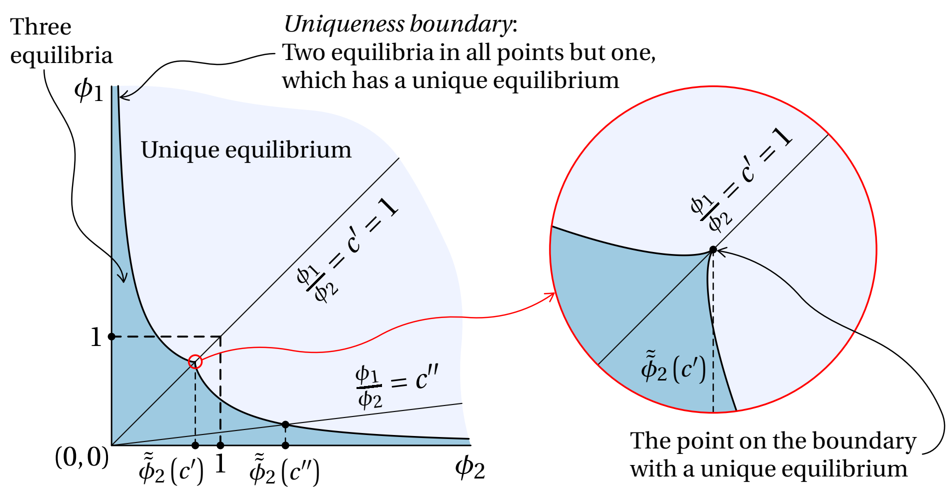


Figure 6: Illustration to Propositions 7 and 8 for AA economy: $0 < \alpha < 1$ and $G = 1$. Asymmetries in trade costs and uniqueness boundary.

Proposition 8 implies that we can trace the "uniqueness boundary" in the space $\phi_1 > 0$ and $\phi_2 > 0$ by varying the ratio $c = \phi_1/\phi_2$ between 0 and $+\infty$. A typical uniqueness boundary for the case with $G = 1$ is illustrated in Figure 6. In this figure, we depict two threshold values for ϕ_2, $\tilde{\tilde{\phi}}_2(c')$ and $\tilde{\tilde{\phi}}_2(c'')$, that correspond to two fixed levels of the ratio ϕ_1/ϕ_2, $c' = 1$ and $c'' < 1$. Proposition 8 says that any ray corresponding to a fixed ratio ϕ_1/ϕ_2 intersects the uniqueness boundary only once. At the same time, as the right part of Figure 6 shows, lines with fixed levels of one of the trade costs can intersect the uniqueness boundary several times. In particular, the vertical line at $\phi_2 = \tilde{\tilde{\phi}}_2(c')$ in Figure 6 intersects the uniqueness boundary two times.

In Figure 6 we also depict the lines $\phi_1 = 1$ and $\phi_2 = 1$ to give the reader an idea of the role of the usual restriction $\phi_1 \leq 1$ and $\phi_2 \leq 1$. As one can see, accommodating the restriction $\phi_1 \leq 1$ and $\phi_2 \leq 1$ is straightforward: we just need to intersect the region given by this restriction with the regions with unique equilibrium and three equilibria outlined by the uniqueness boundary.

According to Proposition 8, we generically have two equilibria — an even number — on the uniqueness boundary. At the same time, the Index Theorem (Kehoe, 1980) implies that an economy generically has an odd number of equilibria. The reason for having an even number of equilibria on the uniqueness boundary is that the Jacobian of the equilibrium system (3)-(6) is singular in one of the equilibrium points with parameters on the uniqueness boundary. Thus, conditions of the Index Theorem are not satisfied in this case. Intuitively, as we change trade costs to go from the uniqueness area to the multiplicity area, we get a new equilibrium point on the uniqueness boundary. This new equilibrium point is associated with a singular Jacobian of the equilibrium system of equations. After we cross the uniqueness boundary, this new equilibrium point is split into two different equilibrium points each of which is associated with non-singular Jacobians of the equilibrium system. In the special case with $G = c^{\frac{1-\mu}{2}}$, the two equilibrium points on the uniqueness boundary happen to coincide, which gives us a unique equilibrium. In this case, as we change trade costs to go to the multiplicity area, the point on the uniqueness boundary is split into three equilibrium points.

Finally, observe that the threshold $\tilde{\phi}_2(c)$ defined in (19) in Proposition 8 gives us an important result in the case of symmetric trade costs. In this case $c = 1$ and it is easy to check that $\tilde{\phi}_2(1) < 1$. Hence, if trade costs are symmetric, there is always a range of trade costs with a unique equilibrium and there is always a range of trade costs with three equilibria. Note that in the general case the threshold $\tilde{\phi}_2(c)$ gives us only a sufficient condition for uniqueness. That is, there can be a unique equilibrium for $\phi_2 < \tilde{\phi}_2(c)$. This is true even in the case of symmetric trade costs. At the same time, if $G = c^{\frac{1-\mu}{2}}$ then, according to Proposition 8, $\tilde{\tilde{\phi}}_2(c) = \tilde{\phi}_2(c)$ and so the threshold $\tilde{\phi}_2(c)$ gives us both a necessarily and sufficient condition for uniqueness. Condition $G = c^{\frac{1-\mu}{2}}$ holds, for example, in the fully symmetric case with $\phi_1 = \phi_2$ and $G = 1$.

5 Symmetric Regions

In Section 4 we provided intuition on how the interaction of the terms-of-trade dispersion force with trade costs leads to uniqueness or multiplicity when $\beta = 1$. In the current section we provide further intuition for the results of Section 3 by focusing on the case with symmetric regions, but allowing $\beta \neq 1$. The case with symmetric regions is formally given by parameter restrictions $\gamma = 0.5$, $G = 1$, and $\phi_1 = \phi_2 \equiv \phi$ (or, alternatively, $\tau_{12} = \tau_{21} = \tau$). Observe that in this case $x = 1$ is always a solution to equation (10) defined in Lemma 1. This solution is the symmetric outcome where each region gets half of the manufacturing labor force.

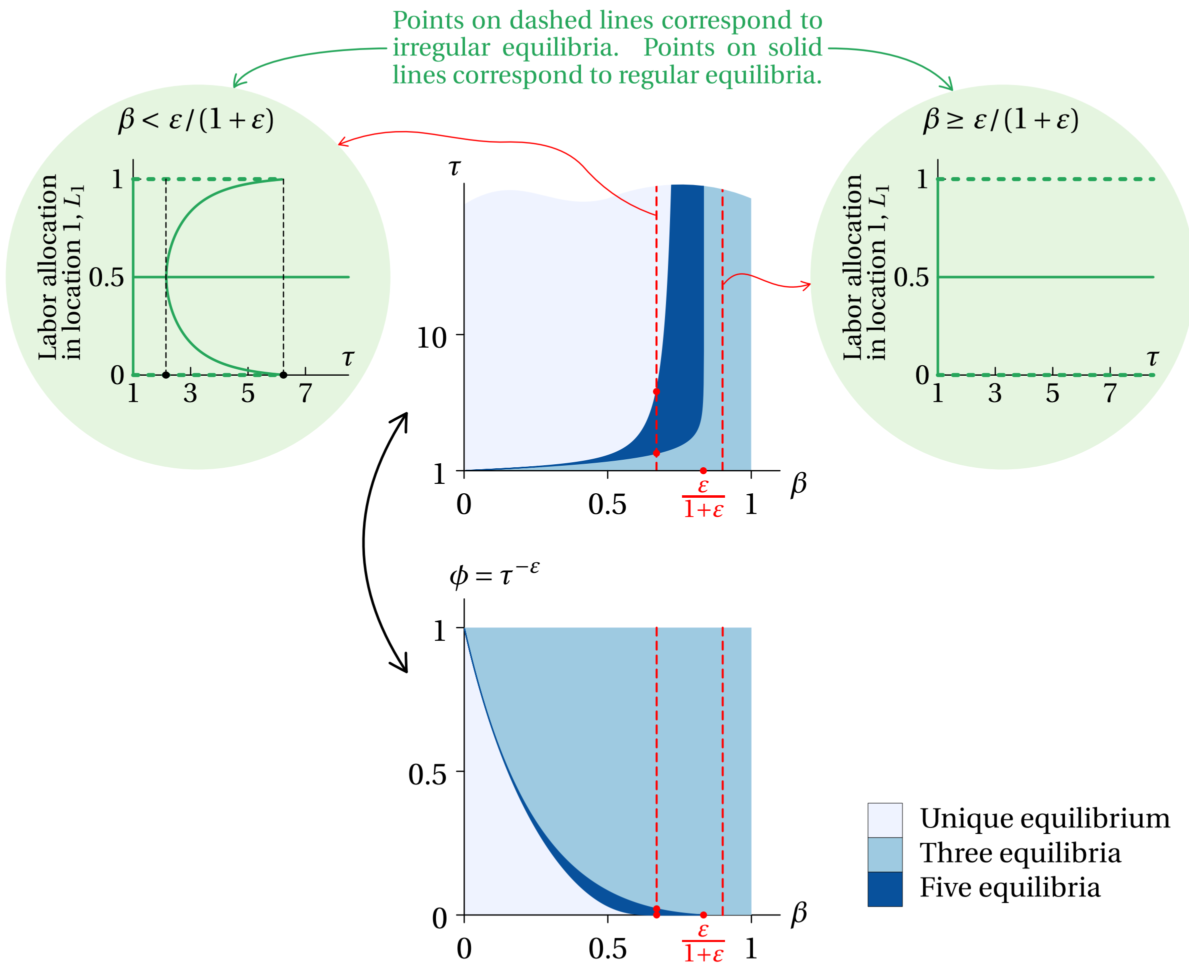


Figure 7: Symmetric Krugman case, $\varepsilon = 5$, $\alpha = 1$, $\bar{L} = 1$, and $\tau_{12} = \tau_{21} \equiv \tau$. Both regular and irregular equilibria are counted in the figures.

To set the stage, let us first consider the relatively well-studied Krugman case ($\alpha = 1$), which is illustrated in Figure 7.[8] In the symmetric case, if $\mu \geq 1$ then both inequalities (8) and (9) hold for all levels of trade costs, implying that both irregular equilibria exist for all levels of trade costs. Similarly, if $\mu \geq 1$ then condition (18) holds under costly trade, and, thus, part (i.b) of Proposition 6 implies that the economy of Section 2 has a unique regular equilibrium under costly trade.

Now consider the case with $\mu < 1$ and focus on regular equilibria first. We have $\tilde{\phi}_1 = \tilde{\phi}_2 < 1$, where $\tilde{\phi}_1$ and $\tilde{\phi}_2$ were defined in (15) and (16). Therefore, part (ii.b) of Proposition 6 implies that, for low enough trade costs, the economy of Section 2 has at most one regular equilibrium. This equilibrium is the symmetric allocation, which is always an equilibrium in the symmetric case. Also, part (ii.a) of Proposition 6 implies that there exists a unique regular equilibrium for high enough trade costs (low $\phi_1 = \phi_2$) when $\mu < 1$. Finally, part (ii.c) of Proposition 6 implies that for intermediate levels of trade costs the economy of Section 2 has at most three regular equilibria.

Turning to irregular equilibria in the case with $\mu < 1$, observe that inequalities (8) and (9) are weak versions of inequalities in (18). Thus, Proposition 1 implies that both irregular equilibria exist if and only if regular equilibria in parts (ii.a)-(ii.b) of Proposition 6 exist. In other words, both irregular equilibria exist for low and intermediate levels of trade costs if $\mu < 1$.

Given the definition of μ, condition $\mu \geq 1$ combined with $\alpha = 1$ is equivalent to $\beta \geq \frac{\varepsilon}{\varepsilon+1}$, which is the so-called "black-hole condition" from Krugman (1991). Thus, the results described above are exactly what we know about the original Krugman (1991) model (see, for example, Combes *et al.*, 2008): under the black-hole condition, the core-periphery structure is always a possibility, and the symmetric equilibrium is the third possible equilibrium. If, on the other hand, the black-hole condition does not hold, then the picture that shows labor allocation patterns for each value of trade costs is "the tomahawk diagram". In this case there is a unique regular equilibrium if trade costs are low or high, while intermediate levels of trade costs result in three regular equilibria. At the same time, irregular equilibria exist only for low and intermediate levels of trade costs.

Relaxing the assumption $\alpha = 1$ of the Krugman case — by having $\alpha \in (0, 1)$ — significantly changes the set of equilibrium labor allocations. First, as established in Proposition 1, all equilibria are regular. Thus, as shown in Figure 8, there are no longer dashed lines associated with equilibria with $L_1 = 0$ or $L_1 = \bar{L}$. Figure 8 also shows that with

[8] In Figure 7, we show equilibria maps in two alternative spaces: (β, τ) and (β, ϕ). The focus of this figure is the (β, τ) space, for which we illustrate two points in the (τ, L_1) coordinates. The figure in the (β, ϕ) space allows us to zoom-in on the area close to the frictionless trade. This proves to be especially useful when $\alpha < 1$, as is made evident in Figure 8. In other sections, we show equilibria maps only for the trade freeness parameters ϕ_i.

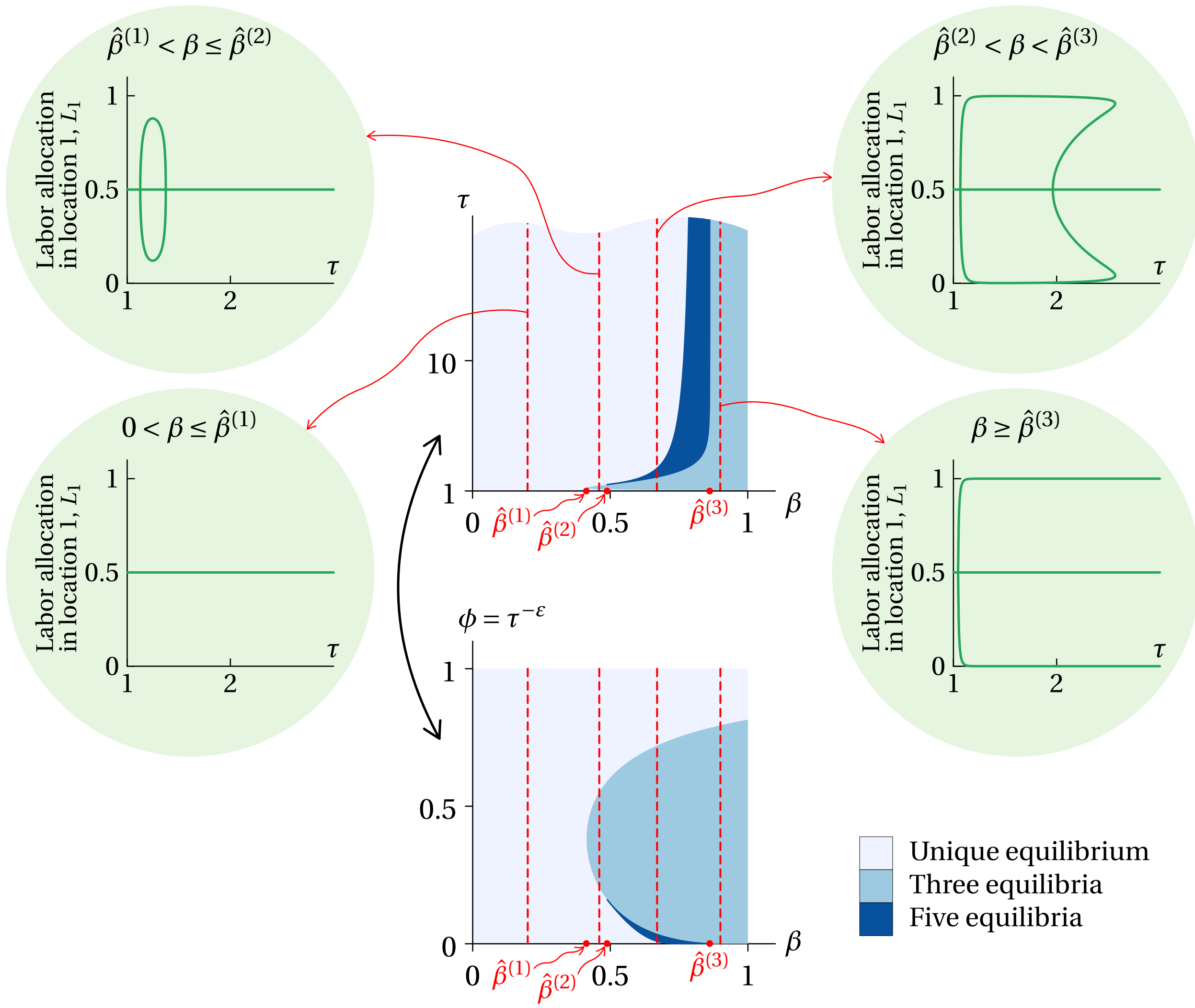


Figure 8: Symmetric case with $\varepsilon = 5$, $\alpha = 0.8$, and $\overline{L} = 1$. All equilibria are regular.

$\alpha \in (0, 1)$ there are four different patterns of labor allocations depending on the value of β. For low values of β the equilibrium is unique. For lower to medium values of β the pattern of labor allocations is a displaced ellipse. For medium to higher values of β the pattern of labor allocations is a displaced tomahawk. And for high values of β the pattern of labor allocations is a pitchfork. As we established in Corollary 1 in Section 3, with $0 < \alpha < 1$ the equilibrium is always unique for low enough trade costs. This creates the "displaced" patterns in Figure 8.

As we discussed in Section 4, parameter α captures the relative strength of the agglomeration force and the terms-of-trade dispersion force. For lower values of α the terms-of-trade dispersion force dominates, which tends to result in uniqueness. The opposite happens for larger values of α: the agglomeration force dominates, which tends to result in multiplicity.

The agricultural sector is yet another important dispersion force. Agricultural workers-consumers are stuck in their regions and need to be served by the manufactured good. If trade costs are low, this can be done through trade. However, if trade costs are high, it might be cheaper to serve agricultural workers-consumers by the manufactured good through local production. Thus, in the presence of the agricultural sector, high trade costs go against concentration of production, which potentially leads to uniqueness of equilibrium.

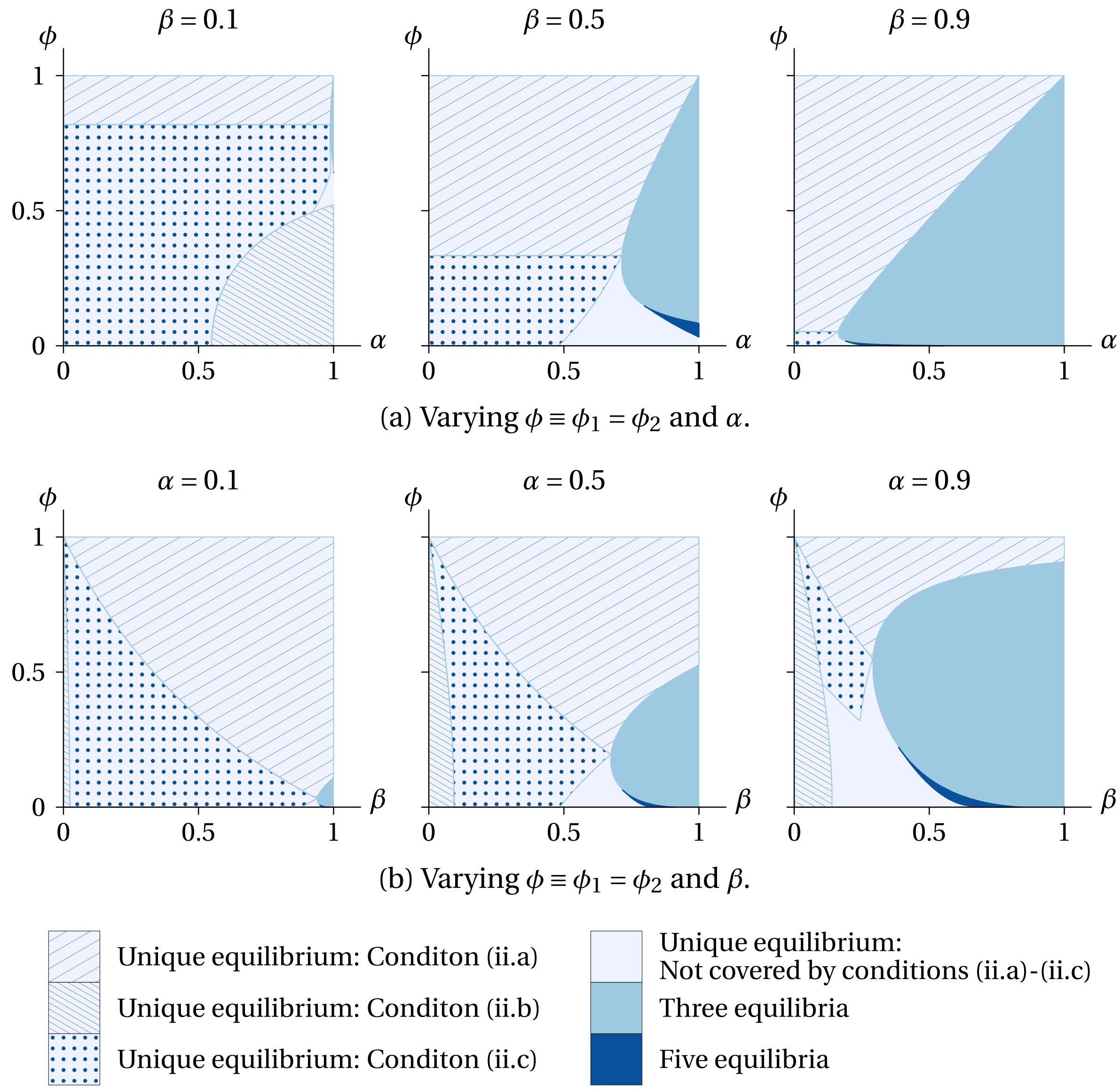
Low trade costs can lead to uniqueness because of the terms-of-trade dispersion force, while high trade costs can lead to uniqueness because of the presence of the agricultural workers-consumers. If the share of the agricultural good in consumption is high enough (β is low), then we have a unique equilibrium for all levels of trade costs. For intermediate values of β we have a region of multiplicity for intermediate level of trade costs. And for high values of β we have uniqueness only for low trade costs. And overall there tends to be a smaller range of trade costs resulting in multiplicity for lower values of β.

6 Role of Asymmetries in the Case with $0 < \alpha < 1$

In this section we focus on the role of asymmetries for the number of equilibria in the case with $0 < \alpha < 1$. The cases (ii.a)-(ii.c) of Proposition 5 as well as areas with one, three, and five equilibria are illustrated in Figures 9-13 for various model parameter values. In all of these figures, the vertical axis measures ϕ_1, while the horizontal axis measures one of the other parameters: α, β, ϕ_2, γ or G. For all of these figures we set $\varepsilon = 5$.[9] Figure 9 corresponds to the symmetric case with $\phi_1 = \phi_2 \equiv \phi$, $\gamma = 0.5$, and $G = 1$, while Figures 10-13 illustrate the consequences of asymmetries in trade costs and parameters γ and G.

As discussed in Section 3, the number of equilibria for a particular set of parameters is unambiguously determined by the number of critical points of $V(x)$ — given by condition $V'(x) = 0$ — and the sign pattern of $V(x)$ evaluated at these critical points, where $V(x)$ is the equilibrium function defined in (13). Following this approach, we find that parameters strictly inside the sets of one, three, and five equilibria in Figures 9-13 yield solutions to $V(x) = 0$ that satisfy $V'(x) \neq 0$, while parameters on the boundaries between these sets yield solutions to $V(x) = 0$ that satisfy $V'(x) = 0$. Thus, the boundary between sets with five equilibria and sets with three or one equilibria generically has four equilibria, while the boundary between sets with three equilibria and sets with one

[9] Figures for other values of ε are very similar.



(a) Varying $\phi \equiv \phi_1 = \phi_2$ and α.

(b) Varying $\phi \equiv \phi_1 = \phi_2$ and β.

Figure 9: Uniqueness/multiplicity areas, $\varepsilon = 5$. Interaction of trade costs and parameters α and β in the symmetric case with $\phi_1 = \phi_2 \equiv \phi$, $\gamma = 0.5$, $G = 1$. Points on the boundaries between sets of one and three equilibria have a unique equilibrium. Points on the boundaries between sets of one and five equilibria have three equilibria. Points on the boundaries between sets of three and five equilibria have three equilibria.

equilibrium generically has two equilibria.[10]

[10] Exceptions are the cases when several critical points of $V(x)$ simultaneously satisfy $V(x) = 0$. In the general asymmetric case, this can happen only at a finite number of points. In such cases, the boundary between sets with five and sets with three or one equilibria can have three equilibria, while the boundary between sets with three and one equilibria can have one equilibrium. Since Figure 9 corresponds to the symmetric case with $\phi_1 = \phi_2 \equiv \phi$, $\gamma = 0.5$, and $G = 1$, the boundaries between sets of one, three, and five equilibria feature an odd number of equilibria, while in Figures 10-13 the boundaries generically feature an even number of equilibria. An odd number of equilibria on the boundaries in the symmetric case follows

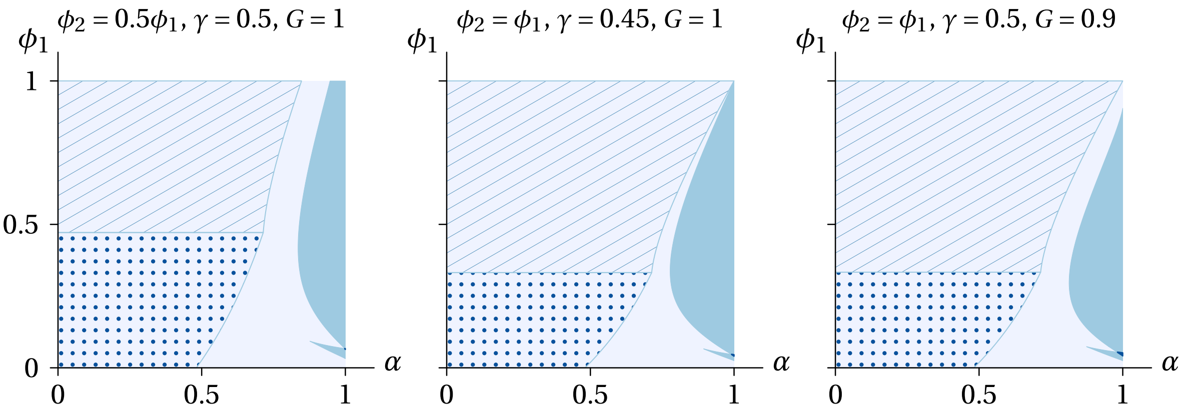
Let us start with Figure 9 devoted to the symmetric case. This figure summarizes some of the things that we have learned so far. Namely, uniqueness of equilibria depends, among other things, on the interaction of trade costs and parameters α and β, so that the sufficient condition for uniqueness $\alpha \leq 0$ used — following Allen and Arkolakis (2014) — in the recent economic geography literature is particularly strong and somewhat conceals the complexity of the outcomes. Another message of Figure 9 — familiar from Sections 5 and 4 — is that lower values of α and/or β tend to result in a bigger set of trade costs for which the economy has a unique equilibrium. A fresh insight coming out of Figure 9 is that each of the sufficient conditions (ii.a)-(ii.c) of Proposition 5 is relevant for some parameter values, and there are areas of uniqueness not covered by any of these conditions. In other words, the sufficient conditions in Proposition 5 are not necessary.

Figure 9 seems to suggest that lower values of α and/or β are more likely to result in uniqueness. We see from panel (a) of Figure 9 that, for each level of trade costs and each value of parameter β, either the equilibrium is unique for all $\alpha \in (0, 1)$, or there is a threshold value $\hat{\alpha}$ such that the equilibrium is unique if and only if $\alpha < \hat{\alpha}$. That is, by increasing α and keeping everything else fixed, we can only go from the region of uniqueness to the region of multiplicity, and never in reverse. Panel (b) of Figure 9 shows a similar behavior in terms of β. Such monotonic behavior, however, is, if anything, a consequence of the symmetry between regions and is generally not true. This is demonstrated in Figure 10.

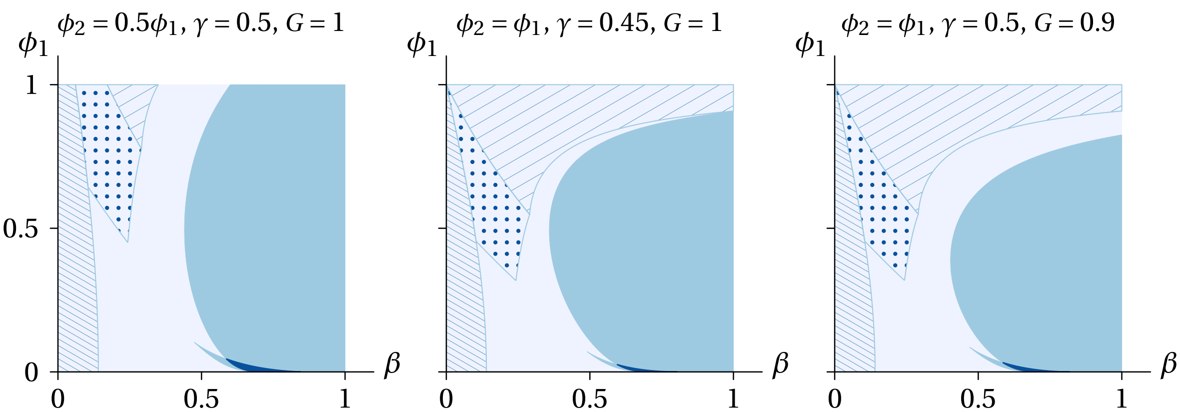
Panel (a) of Figure 10 shows the number of equilibria in (α, ϕ_1) coordinates for $\beta = 0.5$ (and $\varepsilon = 5$) and for three types of asymmetries: the left picture of panel (a) corresponds to asymmetries in trade costs between regions given by $\phi_2/\phi_1 = 0.5$ (implying that it is costlier to export from region 1 to region 2 than in the opposite direction); the central picture of panel (a) corresponds to asymmetries in agricultural labor endowments given by $\gamma = 0.45$ (implying more agricultural labor in region 2); and the right picture of panel (a) corresponds to asymmetries in productivities and/or amenity endowments of regions given by $G = 0.9$ (implying higher productivity/amenities in region 2). Panel (b) of Figure 10 is similar to panel (a) with the difference that panel (b) shows the number of equilibria in (β, ϕ_1) coordinates for $\alpha = 0.5$. As we can see from Figure 10, increasing α or β, and keeping everything else fixed, generally can result in intermitting regions of uniqueness and multiplicity.

Comparing pictures in panel (a) of Figure 10 with the picture corresponding to $\beta = 0.5$ in panel (a) of Figure 9, we see that introduction of asymmetries in just one characteristic

from the fact that in the symmetric case, for each x^* satisfying $V(x) = 0$ and $V'(x) = 0$, we have that $[x^*]^{-1}$ also satisfies the same conditions. In other words, in the symmetric case, conditions $V(x) = 0$ and $V'(x) = 0$ are always satisfied for a pair of critical points.



(a) Varying ϕ_1 and α with $\beta = 0.5$.



(b) Varying ϕ_1 and β with $\alpha = 0.9$.

Figure 10: Uniqueness/multiplicity areas, $\varepsilon = 5$. Interaction of trade costs and parameters α and β in the case of asymmetries in only one characteristic of regions. Legend is the same as in Figure 9. Points on the boundaries between sets of one and three equilibria generically have two equilibria. Points on the boundaries between sets of one and five equilibria generically have four equilibria. Points on the boundaries between sets of three and five equilibria generically have four equilibria.

of regions tends to increase uniqueness areas in the (α, ϕ_1) space. A similar observation can be made with regard to the (β, ϕ_1) space — compare pictures in panel (b) of Figure 10 with the picture corresponding to $\alpha = 0.9$ in panel (b) of Figure 9.

Figure 11 further reinforces the two points made above: lower values of α as well as asymmetries in one characteristic of regions tend to result in a larger set of uniqueness outcomes. In this figure, we set $\varepsilon = 5$, $\beta = 0.5$, and $\alpha = 0.5$, 0.9 and 0.99, and show the uniqueness/multiplicity areas as we vary the trade freeness parameter ϕ_1 and one of the three other parameters: ϕ_2, γ or G.

As we can see from Figure 11, all equilibria are unique in all pictures corresponding to $\alpha = 0.5$, while the set of unique equilibria is the smallest in pictures corresponding to

$\alpha = 0.99$. However, the dependence of the size of the uniqueness area on parameter α is not necessarily monotonic. We also see from panel (a) of Figure 11 that making trade costs asymmetric (while keeping the mean trade freeness $(\phi_1\phi_2)^{1/2}$ fixed) tends to result in uniqueness, but this behavior is not always monotonic. Observe the splitting "tail" of the region of uniqueness for low values of ϕ_1 and ϕ_2. Finally, panels (b) and (c) of Figure 11 show that making the regions asymmetric by taking parameters γ or G to their extreme values tends to result in uniqueness for a given value of ϕ_1. However, again, this behavior is not always monotonic.

Figure 12 is similar to Figure 11 and shows the uniqueness/multiplicity areas for $\varepsilon = 5$, $\alpha = 0.9$, and $\beta = 0.1$, 0.5 and 0.9. The message is the same as that of Figure 11: lower values of β as well as asymmetries in one characteristic of regions tend to result in a larger set of uniqueness outcomes.

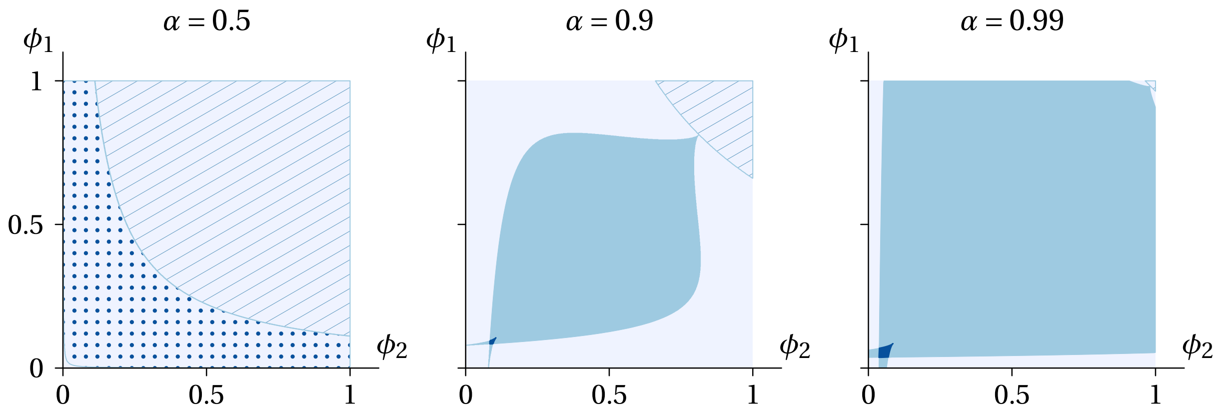
Figure 13 shows the consequences of introducing asymmetries in two or all three region characteristics. In this figure, we fix $\varepsilon = 5$, $\alpha = 0.9$, and $\beta = 0.5$. Consider panel (a) of this figure, which shows uniqueness/multiplicity areas in the (ϕ_1, ϕ_2) space for three sets of values of parameters γ and G: $\gamma = 0.2$ and $G = 1$ (asymmetries only in the agricultural labor endowments); $\gamma = 0.5$ and $G = 1.5$ (asymmetries only in productivities/amenities); and $\gamma = 0.2$ and $G = 1.5$ (asymmetries both in agricultural labor endowments and productivities/amenities). Observe how in the picture corresponding to $\gamma = 0.2$ and $G = 1$ the multiplicity region shifts to the bottom and to the right relative to the multiplicity region in the case with $\gamma = 0.5$ and $G = 1$ shown in the central picture in panel (a) of Figure 11. This outcome is intuitive. Having less agricultural labor in region 1 ($\gamma = 0.2 < 1$) combined with relatively high costs of shipping manufactured goods from region 1 to region 2 (low values of ϕ_2/ϕ_1) makes region 2 a more attractive place to concentrate manufacturing production, which reduces indeterminacy of outcomes and, thus, tends to result in uniqueness. At the same time, when costs of shipping goods from region 1 to region 2 are relatively low, region 1 can be a potentially attractive location for concentration of manufacturing production despite low agricultural labor endowment. This creates indeterminacy of outcomes and potentially leads to multiplicity.

Next, consider the case with $\gamma = 0.5$ and $G = 1.5$ in panel (a) of Figure 13. In this case region 1 has larger productivity/amenities. Relative to the case with $\gamma = 0.5$ and $G = 1$, the multiplicity region shifts up and to the left in the (ϕ_1, ϕ_2) space, which is, again, intuitive — the explanation is similar to the one for the case with $\gamma = 0.2$ and $G = 1$.

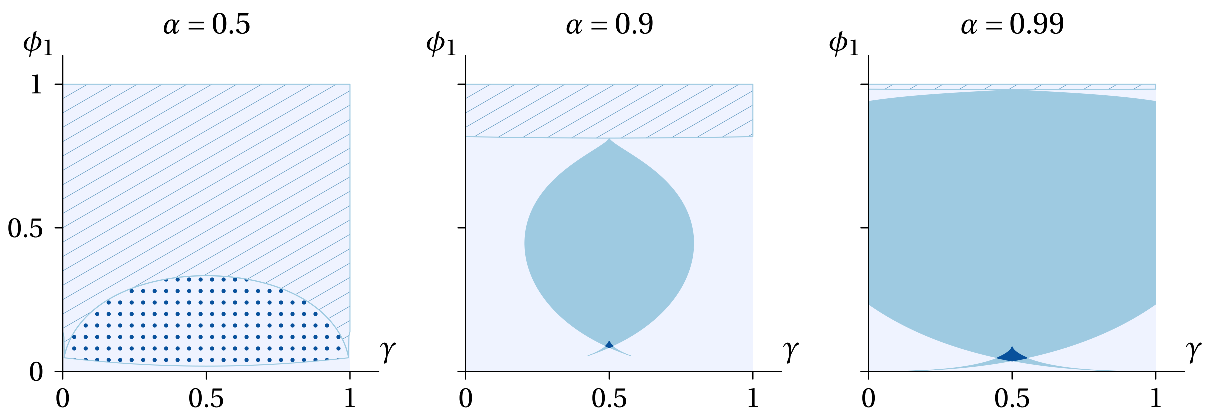
More interesting is the case with $\gamma = 0.2$ and $G = 1.5$, where asymmetries in the agricultural labor endowments and productivities/amenities work against each other:

region 1 is more attractive due to its productivity/amenities, while region 2 is more attractive due to its agricultural labor abundance. Introduction of asymmetries in trade costs on top of this reinforces attractiveness of one region and diminishes attractiveness of the other region. This creates indeterminacy of outcomes for relatively small asymmetries in trade costs, but reduces indeterminacy of outcomes for large asymmetries in trade costs. This explains why the multiplicity region expands to both sides of the line $\phi_1 = \phi_2$ relative to the case with $\gamma = 0.5$ and $G = 1$.

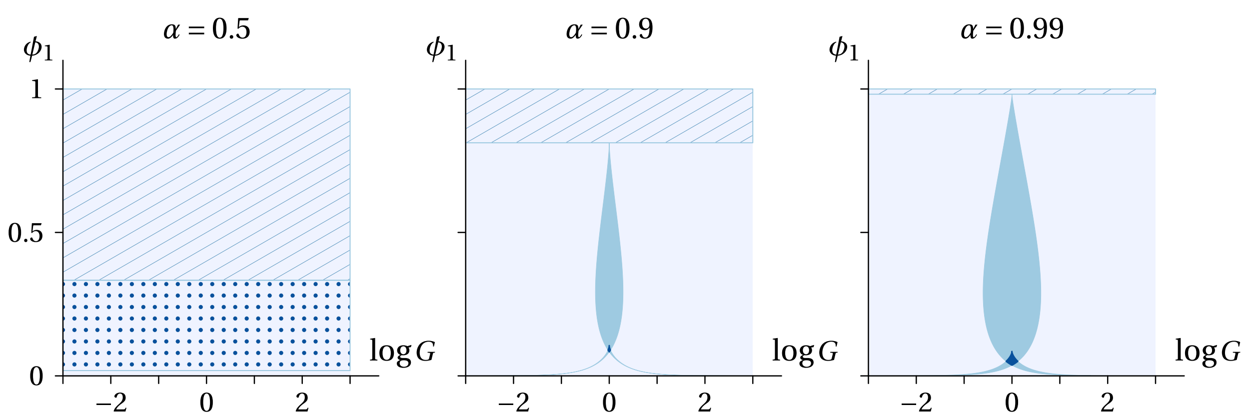
The pictures in panels (b) and (c) of Figure 13 show how the uniqueness/multiplicity areas shift in the (ϕ_1, γ) and $(\phi_1, \log G)$ spaces. The explanation is conceptually the same as for the panel (a). The upshot of this analysis is that making regions more asymmetric in several characteristics can make a uniqueness outcome more or less likely depending on whether asymmetries in characteristics favor one of the regions or make these regions similarly attractive for concentration of manufacturing production.



(a) Varying ϕ_1 and ϕ_2 with $\gamma = 0.5$ and $G = 1$.

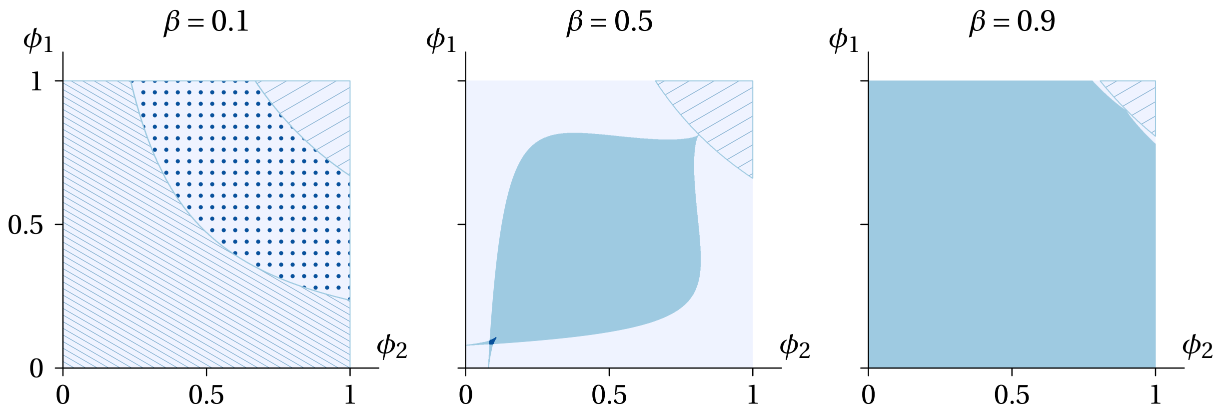


(b) Varying ϕ_1 and γ with $\phi_2 = \phi_1$ and $G = 1$.

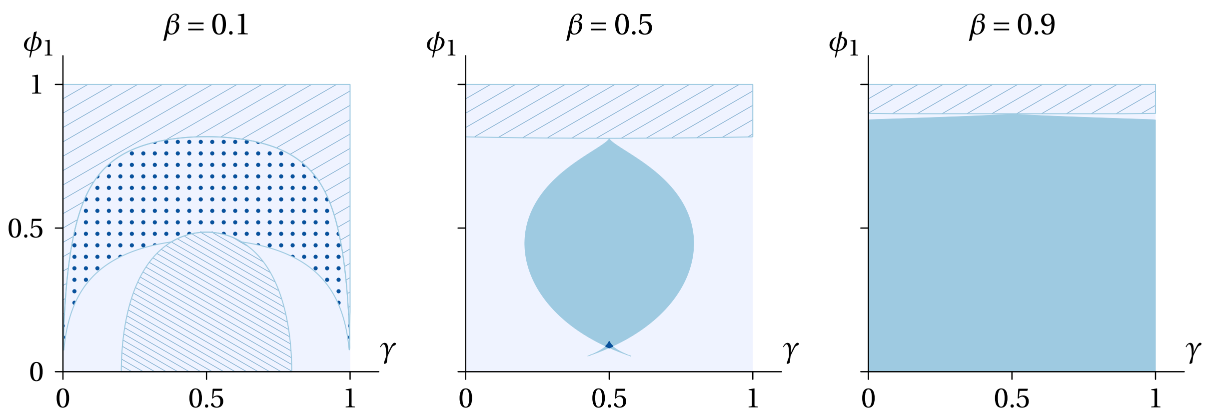


(c) Varying ϕ_1 and G with $\phi_2 = \phi_1$ and $\gamma = 0.5$.

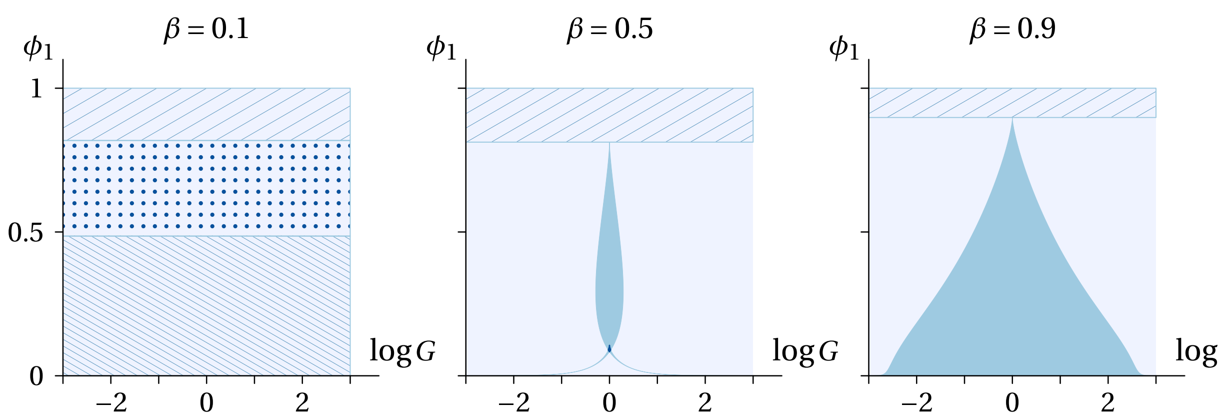
Figure 11: Uniqueness/multiplicity areas, $\varepsilon = 5$ and $\beta = 0.5$. Interaction of trade costs and characteristics of regions: allowing asymmetry in only one characteristic at a time. Legend is the same as in Figure 9. Points on the boundaries between sets of one and three equilibria generically have two equilibria. Points on the boundaries between sets of one and five equilibria generically have four equilibria. Points on the boundaries between sets of three and five equilibria generically have four equilibria.



(a) Varying ϕ_1 and ϕ_2 with $\gamma = 0.5$ and $G = 1$.



(b) Varying ϕ_1 and γ with $\phi_2 = \phi_1$ and $G = 1$.



(c) Varying ϕ_1 and G with $\phi_2 = \phi_1$ and $\gamma = 0.5$.

Figure 12: Uniqueness/multiplicity areas, $\varepsilon = 5$ and $\alpha = 0.9$. Interaction of trade costs and characteristics of regions: allowing asymmetry in only one characteristic at a time. Legend is the same as in Figure 9. Points on the boundaries between sets of one and three equilibria generically have two equilibria. Points on the boundaries between sets of one and five equilibria generically have four equilibria. Points on the boundaries between sets of three and five equilibria generically have four equilibria.

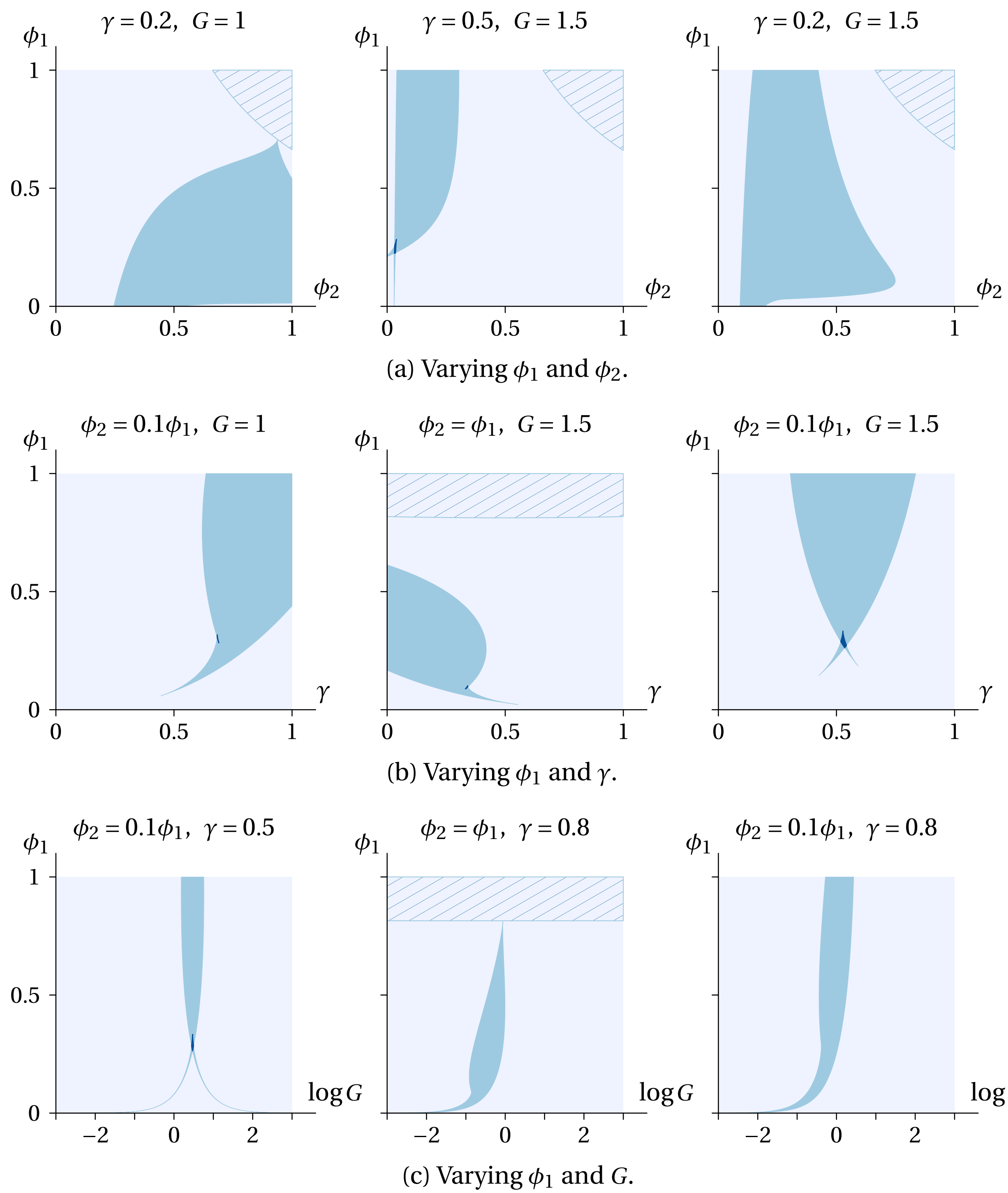


Figure 13: Uniqueness/multiplicity areas, $\varepsilon = 5$, $\alpha = 0.9$, and $\beta = 0.5$. Interaction of trade costs and characteristics of regions: allowing asymmetries in two or all three characteristics. Legend is the same as in Figure 9. Points on the boundaries between sets of one and three equilibria generically have two equilibria. Points on the boundaries between sets of one and five equilibria generically have four equilibria. Points on the boundaries between sets of three and five equilibria generically have four equilibria.

7 Concluding Remarks

In this paper, we characterize the set of equilibria in a generalized version of the canonical two-region spatial equilibrium model. We show how the set of equilibria is affected by parameters governing the importance of agriculture, agglomeration economies in manufacturing, and congestion forces affecting migration, as well as trade costs and the distribution of exogenous productivity, amenities and agriculture labor in the two regions. The critical parameters are: α, which captures the net effect of forces of agglomeration, terms-of-trade in manufacturing and congestion externalities on local amenities; β, which captures the importance of agriculture; and $\{\tau_{12}, \tau_{21}\}$, which capture trade costs. Allen and Arkolakis (2014) have already established that the equilibrium is regular if $\alpha < 1$ and unique if $\alpha < 0$. Our main contribution is to derive a set of sufficient conditions for uniqueness when $\alpha \in (0, 1]$. Most importantly, the equilibrium can be unique even if $\alpha > 0$, for example with positive agglomeration externalities even in the absence of congestion externalities. This is possible thanks to the role played by the dispersion force associated with a finite trade elasticity, which implies that a region's terms of trade worsen as it gets larger. Since the volume of trade magnifies the aggregate relevance of this force, trade costs combined with the trade elasticity now play an important role in ensuring uniqueness. In the extreme, the equilibrium is unique under frictionless trade if $\alpha < 1$. If the two regions are symmetric then lower values of α and β enlarge the set of other parameters under which the equilibrium is unique, but this result does not always hold if the two regions are asymmetric. Similarly, making the two regions more asymmetric tends to enlarge the set of parameters under which the equilibrium is unique, but again this is not always the case.

The main limitation of our analysis, of course, is that we restrict it to two regions. With more than two regions, it is straightforward to establish that AA's sufficient condition for uniqueness, $\alpha < 0$, is valid even when there is agriculture, $\beta < 1$. It should also be possible to generalize to more than two regions our result in Proposition (5) that if $\alpha < 1$ then there is a unique equilibrium if trade costs or β are low enough. Broadly speaking, we see our results as suggestive of the importance of trade costs, the trade elasticity, and the importance of fixed factors (here agriculture) in determining whether the equilibrium is unique in more general environments. Beyond that, providing tight conditions on parameters that ensure that the equilibrium is unique with more than two regions seems very challenging.

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Appendices

A Proofs for Section 3

A.1 Proof of Proposition 1

In equilibrium, at least one region is inhabited. Hence, $L_i^{\varepsilon\psi} w_i^{-\varepsilon} > 0$ for at least one i — otherwise w_i needs to be infinity for all i with $L_i > 0$, which cannot be an equilibrium. This implies that $P_i^{-\varepsilon} > 0$ for all i. Next, if $L_i^{\varepsilon\psi} w_i^{-\varepsilon}$ is infinite for at least one i, then $P_i^{-\varepsilon}$ is infinite for all i, which is impossible. Hence, in any equilibrium, $L_i^{\varepsilon\psi} w_i^{-\varepsilon}$ is finite for all i and P_i is positive and finite for all i.

For any region i with $L_i > 0$, the manufactured goods market clearing condition (3) implies

$$w_i = L_i^{\frac{\varepsilon\psi - 1}{1+\varepsilon}} \bar{A}_i^{\frac{\varepsilon}{1+\varepsilon}} \left[\sum_{n=1}^{N} \tau_{ni}^{-\varepsilon} P_n^{\varepsilon} \beta \left(w_n L_n + w^A \bar{L}_n^A \right) \right]^{\frac{1}{1+\varepsilon}}.$$

Substituting this into (1) and combining with (2), we get

$$U_i = \left[\sum_{n=1}^{N} \tau_{ni}^{-\varepsilon} P_n^{\varepsilon} \beta \left(w_n L_n + w^A \bar{L}_n^A \right) \right]^{\frac{1}{1+\varepsilon}} P_i^{-\beta} \left[w^A \right]^{-(1-\beta)} \bar{A}_i^{\frac{\varepsilon}{1+\varepsilon}} \bar{u}_i L_i^{\frac{(1+\delta)(\alpha-1)}{1+\varepsilon}}. \tag{20}$$

Taking the limit $L_i \to 0$, we see that $U_i \to \infty$ if $\alpha < 1$ (as we have argued above, P_i has to be finite for all regions, including regions with $L_i = 0$). Hence, in the case with $\alpha < 1$ irregular equilibria are not possible. This proves part (i) of Proposition 1.

Next, consider the case with $\alpha > 1$ and suppose that $L_1 = \bar{L}$ and $L_2 = 0$. Then (20) implies that U_1 is a finite positive number while $U_2 = 0$, and thus condition (5) is satisfied with $\bar{U} = U_1$. Therefore, the allocation $L_1 = \bar{L}$ and $L_2 = 0$ is an equilibrium. Clearly, the mirror image allocation $L_1 = 0$ and $L_2 = \bar{L}$ is also an equilibrium. This proves part (iii) of Proposition 1.

Finally, consider the case with $\alpha = 1$. Consider the allocation $L_1 = \bar{L}$ and $L_2 = 0$. We have $P_1 = w_1 / \left(A_1 \bar{L}^{\psi} \right)$ and $P_2 = \tau_{21} P_1$. Also, $\lambda_{11} = \lambda_{21} = 1$ and $\lambda_{12} = \lambda_{22} = 0$. From (20) we find

$$U_1 = \beta^{\frac{1}{1+\varepsilon}} \left[w_1 \bar{L} + w^A \bar{L}_1^A + w^A \bar{L}_2^A \right]^{\frac{1}{1+\varepsilon}} w_1^{\frac{\varepsilon}{1+\varepsilon} - \beta} \bar{A}_1^{\beta} \left[w^A \right]^{-(1-\beta)} \bar{u}_1 \bar{L}^{-\psi\left(\frac{\varepsilon}{1+\varepsilon} - \beta\right)},$$

$$U_2 = \beta^{\frac{1}{1+\varepsilon}} \left[\tau_{12}^{-\varepsilon} \left(w_1 \bar{L} + w^A \bar{L}_1^A \right) + \tau_{21}^{\varepsilon} w^A \bar{L}_2^A \right]^{\frac{1}{1+\varepsilon}}$$
$$\times \tau_{21}^{-\beta} w_1^{\frac{\varepsilon}{1+\varepsilon} - \beta} \left(\frac{\bar{A}_2}{\bar{A}_1} \right)^{\frac{\varepsilon}{1+\varepsilon}} \bar{A}_1^{\beta} \left[w^A \right]^{-(1-\beta)} \bar{u}_2 \bar{L}^{-\psi\left(\frac{\varepsilon}{1+\varepsilon} - \beta\right)}.$$

The equilibrium condition that $U_1 \geq U_2$ gives

$$w_1 \bar{L} + w^A \bar{L}_1^A + w^A \bar{L}_2^A \geq \left[\phi_1 \left(w_1 \bar{L} + w^A \bar{L}_1^A \right) + \phi_2^{-1} w^A \bar{L}_2^A \right] \phi_2^{\mu} G^{-1}, \tag{21}$$

where $G \equiv (\bar{u}_1/\bar{u}_2)^{1+\varepsilon} \left(\bar{A}_1/\bar{A}_2\right)^{\varepsilon}$, $\mu \equiv \beta\,(1/\varepsilon + 1)$, and $\phi_1 \equiv \tau_{12}^{-\varepsilon}$ and $\phi_2 \equiv \tau_{21}^{-\varepsilon}$.

Consider the case with $\beta = 1$ and $\bar{L}_1^A = \bar{L}_2^A = 0$ (no agricultural sector in both regions). In this case, inequality (21) becomes $G \geq \phi_1 \phi_2^{\mu}$. Thus, we get that the allocation $L_1 = \bar{L}$ and $L_2 = 0$ is an equilibrium if and only if $G \geq \phi_1 \phi_2^{\mu}$, which is the condition in part (ii.a) of Proposition 1 for $\beta = 1$ (or, equivalently, for $\overline{\beta} = 0$).

Now consider the case with $0 < \beta < 1$. In this case, the agricultural goods market clearing condition (4) gives

$$\frac{w_1}{w^A} = \frac{\beta}{1-\beta} \cdot \frac{\bar{L}_1^A + \bar{L}_2^A}{\bar{L}}.$$

Substituting this into (21) and after doing some algebra, we get the condition from part (ii.a) of Proposition 1

$$G \geq \overline{\gamma}\overline{\beta}\phi_2^{\mu-1} + \left(1 - \overline{\gamma}\overline{\beta}\right) \phi_1 \phi_2^{\mu}.$$

Proving part (ii.b) of Proposition 1 can be done in an analogous manner.

A.2 Proof of Lemma 1

Denote $l \equiv L_2/L_1$ and $w \equiv w_2/w_1$. Also, denote $a \equiv (\bar{u}_2/\bar{u}_1)^{\varepsilon}$, $b \equiv (\bar{A}_2/\bar{A}_1)^{\varepsilon}$, and $\phi_1 \equiv \tau_{12}^{-\varepsilon}$ and $\phi_2 \equiv \tau_{21}^{-\varepsilon}$. In the case of regular equilibria, the complementary slackness condition (5) implies that both regions have the same welfare $\bar{U}$, and we can combine (1)-(2) for the two regions and get

$$l^{\delta} = a^{\frac{1}{\varepsilon}} w \left(\frac{P_1}{P_2} \right)^{\beta}. \tag{22}$$

Dividing onto each other price indices for the two countries given by expression (7), we get

$$\frac{P_1}{P_2} = \left[\frac{1+\phi_1 b l^{\varepsilon\psi} w^{-\varepsilon}}{\phi_2 + b l^{\varepsilon\psi} w^{-\varepsilon}}\right]^{-\frac{1}{\varepsilon}}.$$

After substituting this expression into (22) and doing some algebra, we get

$$a^{\frac{1}{\beta}} w^{\frac{\varepsilon}{\beta}} l^{-\frac{\delta\varepsilon}{\beta}} = \frac{1+\phi_1 b l^{\varepsilon\psi} w^{-\varepsilon}}{\phi_2 + b l^{\varepsilon\psi} w^{-\varepsilon}}. \tag{23}$$

Next, the manufactured and agricultural goods market clearing conditions (3) and (4) for the first region can be written as

$$1 = \frac{1}{1+\phi_1 b l^{\varepsilon\psi} w^{-\varepsilon}} \beta \left(1 + \frac{\bar{L}_1^A}{w_1 L_1}\right) + \frac{\phi_2}{\phi_2 + b l^{\varepsilon\psi} w^{-\varepsilon}} \beta \left(wl + \frac{w^A \bar{L}_2^A}{w_1 L_1}\right),$$
$$\beta \left(\frac{w^A \bar{L}_1^A + w^A \bar{L}_2^A}{w_1 L_1}\right) = (1-\beta)(1+wl).$$

Solving for $w_1 L_1$ from the second equation and substituting into the first, gives

$$1 = \frac{1}{1+\phi_1 b l^{\varepsilon\psi} w^{-\varepsilon}} \left(\beta + \gamma\overline{\beta} + \gamma\overline{\beta} wl\right) + \frac{\phi_2}{\phi_2 + b l^{\varepsilon\psi} w^{-\varepsilon}} \left(\overline{\gamma}\overline{\beta} + \left(\beta + \overline{\gamma}\overline{\beta}\right) wl\right), \tag{24}$$

where $\overline{\beta} \equiv 1-\beta$, $\gamma \equiv \bar{L}_1^A \big/ \left(\bar{L}_1^A + \bar{L}_2^A\right)$, and $\overline{\gamma} \equiv 1-\gamma$.

The equilibrium system of equations is given by (23) and (24). Let us introduce the change of variables $x_1 \equiv b l^{\varepsilon\psi} w^{-\varepsilon}$ and $x_2 \equiv lw$. Then $l = b^{-\frac{1}{\varepsilon(\psi+1)}} x_1^{\frac{1}{\varepsilon(\psi+1)}} x_2^{\frac{1}{\psi+1}}$ and $w = b^{\frac{1}{\varepsilon(\psi+1)}} x_1^{-\frac{1}{\varepsilon(\psi+1)}} x_2^{\frac{\psi}{\psi+1}}$, and equations (23) and (24) become

$$a^{\frac{1}{\beta}} b^{\frac{(1+\delta)}{\beta(1+\psi)}} \left(x_1^{-\delta-1} x_2^{\varepsilon(\psi+\delta)}\right)^{\frac{1}{\beta(\psi+1)}} = \frac{1+\phi_1 x_1}{\phi_2 + x_1},$$
$$x_2 = \frac{\left[\left(\overline{\gamma}\overline{\beta} + \left(1-\overline{\gamma}\overline{\beta}\right)\phi_1\phi_2\right) + \phi_1 x_1\right] x_1}{\phi_2 + \left(\gamma\overline{\beta} + \left(1-\gamma\overline{\beta}\right)\phi_1\phi_2\right) x_1}.$$

Substituting x_2 from the second equation into the first, we get

$$x_1^{\alpha-1} = a^{-\left(\frac{\alpha}{\varepsilon}+1\right)} b^{-1} \left(\frac{\phi_1}{\phi_2}\right)^{-\alpha} \left(\frac{1+\phi_1 x_1}{\phi_2 + x_1}\right)^{\mu} \left(\frac{1+d_1 x_1}{d_2 + x_1}\right)^{\alpha},$$

where $\alpha \equiv \frac{(\psi-\delta)\varepsilon}{1+\delta}$, $\mu \equiv \frac{\beta(\psi+1)}{1+\delta} = \beta\left(\frac{\alpha}{\varepsilon}+1\right)$, and

$$d_1 \equiv \frac{\gamma\overline{\beta} + \left(1-\gamma\overline{\beta}\right)\phi_1\phi_2}{\phi_2} \quad \text{and} \quad d_2 \equiv \frac{\overline{\gamma}\overline{\beta} + \left(1-\overline{\gamma}\overline{\beta}\right)\phi_1\phi_2}{\phi_1}.$$

A.3 Proof of Proposition 4

Take logarithms from both sides of equation (10) and write the resulting equation as $V(x) = 0$, where

$$V(x) \equiv (\alpha-1)\ln x - \ln G + \alpha \ln\left(\frac{\phi_1}{\phi_2}\right) - \mu \ln g_\phi(x) - \alpha \ln g_d(x).$$

Taking the first derivative of $V(x)$, we get $V'(x) = x^{-1}\, W_2(x)/W_1(x)$, where

$$\begin{aligned} W_1(x) &\equiv (1+\phi_1 x)(\phi_2 + x)(1+d_1 x)(d_2+x), \\ W_2(x) &\equiv (\alpha-1)W_1(x) + \mu(1-\phi_1\phi_2)(1+d_1x)(d_2+x)x \\ &\qquad + \alpha(1-d_1d_2)(1+\phi_1x)(\phi_2+x)x. \end{aligned}$$

We see that $V'(x) = 0$ if and only if $W_2(x) = 0$. Expanding the terms of $W_2(x)$, we can see that in the case with $\alpha \neq 1$ it is a 4th degree polynomial, while in the case with $\alpha = 1$ function $x^{-1}W_2(x)$ is a quadratic polynomial. Since a nth degree polynomial has at most n real roots, function $V(x)$ has at most 4 extrema if $\alpha \neq 1$ and it has at most 2 extrema if $\alpha = 1$. Since a continuous function with n extrema can intersect the horizontal axis at most $n+1$ times, we get that equation $V(x) = 0$ can have at most 5 solutions if $\alpha \neq 1$ and it can have at most 3 solutions if $\alpha = 1$.

A.4 Proof of Proposition 5

Taking both sides of (10) to the power $1/(\alpha-1)$ (which is well-defined since $\alpha \neq 1$), then taking logarithms from the both sides and denoting $y \equiv \ln x$, we get equation $y = F(y)$, where

$$F(y) \equiv \ln G^{\frac{1}{\alpha-1}} - \frac{\alpha}{\alpha-1}\ln\left(\frac{\phi_1}{\phi_2}\right) + \frac{\mu}{\alpha-1}\ln\left(\frac{1+\phi_1 e^y}{\phi_2+e^y}\right) + \frac{\alpha}{\alpha-1}\ln\left(\frac{1+d_1e^y}{d_2+e^y}\right). \quad (25)$$

Differentiating $F(y)$, we get

$$F'(y) = -\frac{\mu}{\alpha-1}(1-\phi_1\phi_2)f_\phi(y) - \frac{\alpha}{\alpha-1}(1-d_1d_2)f_d(y), \quad (26)$$

where

$$f_{\phi}(y) \equiv \frac{e^{y}}{(1+\phi_1 e^{y})(\phi_2+e^{y})} \quad \text{and} \quad f_{d}(y) \equiv \frac{e^{y}}{(1+d_1 e^{y})(d_2+e^{y})}. \tag{27}$$

Consider $f_{\phi}(y)$. We have

$$f_{\phi}'(y) = \frac{\phi_1 e^{y}\left(e^{2y_{\phi}^*} - e^{2y}\right)}{(1+\phi_1 e^{y})^2 (\phi_2+e^{y})^2},$$

where $y_{\phi}^* \equiv \ln\left(\phi_1^{-\frac{1}{2}}\phi_2^{\frac{1}{2}}\right)$. We see that $f_{\phi}'\left(y_{\phi}^*\right) = 0$ and $f_{\phi}'(y) > 0$ for $y < y_{\phi}^*$ and $f_{\phi}'(y) < 0$ for $y > y_{\phi}^*$, which means that y_{ϕ}^* is the global maximum of $f_{\phi}(\cdot)$. Evaluating $f_{\phi}\left(y_{\phi}^*\right)$ gives $f_{\phi}\left(y_{\phi}^*\right) = (1+\sqrt{\phi_1\phi_2})^{-2}$. Similarly, $f_d(y)$ achieves its global maximum at $y_d^* \equiv \ln\left(d_1^{-\frac{1}{2}}d_2^{\frac{1}{2}}\right)$ with $f_d(y_d^*) = \left(1+\sqrt{d_1 d_2}\right)^{-2}$. Clearly, $f_{\phi}(y) > 0$ and $f_d(y) > 0$ for all y, and so we have the following bounds for $f_{\phi}(y)$ and $f_d(y)$,

$$0 < f_{\phi}(y) \le \frac{1}{(1+\sqrt{\phi_1\phi_2})^2} \quad \text{and} \quad 0 < f_d(y) \le \frac{1}{(1+\sqrt{d_1 d_2})^2}. \tag{28}$$

Case (i): $\boldsymbol{\alpha \le 0}$. If $\alpha \le 0$ then $\alpha/(\alpha-1) \ge 0$. Also, since $\mu > 0$ we have $\mu/(\alpha-1) < 0$. Then, using (28) and our proposition assumption that $\phi_1\phi_2 < 1$, we get

$$\left|F'(y)\right| \le \frac{\mu}{1-\alpha}\cdot\frac{1-\sqrt{\phi_1\phi_2}}{1+\sqrt{\phi_1\phi_2}} + \frac{-\alpha}{1-\alpha}\cdot\frac{\left|1-\sqrt{d_1 d_2}\right|}{1+\sqrt{d_1 d_2}} \le \rho,$$

where

$$\rho \equiv \frac{\mu-\alpha}{1-\alpha}\cdot\max\left\{\frac{1-\sqrt{\phi_1\phi_2}}{1+\sqrt{\phi_1\phi_2}}, \frac{\left|1-\sqrt{d_1 d_2}\right|}{1+\sqrt{d_1 d_2}}\right\}.$$

Finally, having $\alpha \le 0$ together with $0 < \beta \le 1$ implies that $\mu \le 1$, and, therefore, $0 < (\mu-\alpha)/(1-\alpha) \le 1$. Also, obviously,

$$\frac{1-\sqrt{\phi_1\phi_2}}{1+\sqrt{\phi_1\phi_2}} < 1 \quad \text{and} \quad \frac{\left|1-\sqrt{d_1 d_2}\right|}{1+\sqrt{d_1 d_2}} < 1.$$

Hence, we get that $\rho < 1$.

The usual argument involving the mean value theorem then implies that $F(y)$ is a contraction mapping on any closed interval of $\mathbb{R}$.[11] Therefore, there exists at most one

[11]Consider any $y', y'' \in \mathbb{R}$ with $y' \neq y''$. By the mean value theorem, there exists $\bar{y}$ between y_1 and

fixed point of $F(y)$ on any closed interval of $\mathbb{R}$. Invoking the result of Proposition 3, we conclude that there is exactly one fixed point of $F(y)$ on $\mathbb{R}$. In other words, there is a unique solution to $y = F(y)$ for $y \in \mathbb{R}$, and, thus, there is a unique positive solution to (10).

Case (ii): $0 < \alpha < 1$. Consider equation $y = F(y)$ and write it as $V(y) \equiv y - F(y) = 0$. Suppose that $d_1 d_2 \leq 1$. Combining expression (26) for $F'(y)$ with the upper bounds (28) on $f_\phi(x)$ and $f_d(x)$ and the restriction $0 < \alpha < 1$, we get

$$V'(y) \geq 1 - \frac{\mu}{1-\alpha} \cdot \frac{1-\sqrt{\phi_1\phi_2}}{1+\sqrt{\phi_1\phi_2}} - \frac{\alpha}{1-\alpha} \cdot \frac{1-\sqrt{d_1 d_2}}{1+\sqrt{d_1 d_2}}.$$

From here we see that if

$$\mu \leq \frac{1+\sqrt{\phi_1\phi_2}}{1-\sqrt{\phi_1\phi_2}} \cdot \frac{1-2\alpha+\sqrt{d_1 d_2}}{1+\sqrt{d_1 d_2}}, \tag{29}$$

then $V'(y) > 0$ for all y except for, perhaps, one point y^* with $V'(y^*) = 0$. This point y^* exists only if $f_\phi(y)$ and $f_d(y)$ achieve their corresponding global maxima at the same point and, in addition to that, inequality (29) holds with equality. Thus, if inequality (29) holds, then $V(y)$ is increasing in all points except for, maybe, one. This implies that $V(y)$ can intersect the horizontal axis only once, which proves part (ii.a) of Proposition 5.

Next, suppose that $d_1 d_2 > 1$. To proceed, we need the following auxiliary result:

Lemma 2. *If $\phi_1\phi_2 < 1$, then there exist $\bar{\epsilon}_1, \bar{\epsilon}_2 \in (0,1)$ such that for for any $\epsilon_1 \in [0, \bar{\epsilon}_1]$ and $\epsilon_2 \in [0, \bar{\epsilon}_2]$ and any y we have*

$$(1-\epsilon_1)^{-1}(\phi_1\phi_2) f_\phi(y) \leq f_d(y) \leq (1-\epsilon_2)(d_1 d_2)^{-1} f_\phi(y), \tag{30}$$

where

$$f_\phi(y) \equiv \frac{e^y}{(1+\phi_1 e^y)(\phi_2 + e^y)} \quad \textit{and} \quad f_d(y) \equiv \frac{e^y}{(1+d_1 e^y)(d_2+e^y)}.$$

Proof. Given the definitions (12) of d_1 and d_2 in Lemma 1,

$$d_1 \equiv \frac{\gamma\bar{\beta} + (1-\gamma\bar{\beta})\,\phi_1\phi_2}{\phi_2} \quad \text{and} \quad d_2 \equiv \frac{\bar{\gamma}\bar{\beta} + (1-\bar{\gamma}\bar{\beta})\,\phi_1\phi_2}{\phi_1},$$

we can write $d_1 = u_1 \phi_2^{-1}$ and $d_2 = u_2 \phi_1^{-1}$, where

$$u_1 \equiv \gamma\bar{\beta} + (1-\gamma\bar{\beta})\,\phi_1\phi_2 \quad \text{and} \quad u_2 \equiv \bar{\gamma}\bar{\beta} + (1-\bar{\gamma}\bar{\beta})\,\phi_1\phi_2.$$

y_2 such that $|F(y') - F(y'')| = |F'(\bar{y})| \cdot |y' - y''|$. Then, since $|F'(y)| \leq \rho < 1$ for all $y \in \mathbb{R}$, we get $|F(y') - F(y'')| \leq \rho \cdot |y' - y''|$.

Then the definitions of $f_\phi(y)$ and $f_d(y)$ imply

$$f_\phi(y) = [\phi_1\phi_2]^{-1} h_d(y) f_d(y) \quad \text{and} \quad f_d(y) = [d_1 d_2]^{-1} h_\phi(y) f_\phi(y), \tag{31}$$

where

$$h_\phi(y) \equiv \frac{1+\phi_1 e^y}{1+u_2^{-1}\phi_1 e^y} \cdot \frac{\phi_2 + e^y}{u_1^{-1}\phi_2 + e^y} \quad \text{and} \quad h_d(y) \equiv \frac{1+d_1 e^y}{1+u_1^{-1} d_1 e^y} \cdot \frac{d_2 + e^y}{u_2^{-1} d_2 + e^y}.$$

Clearly, $u_1 < 1$ and $u_2 < 1$ as long as $\phi_1\phi_2 < 1$. This implies that $h_\phi(y) < 1$ and $h_d(y) < 1$ for all y. Let us focus on $h_\phi(y)$ and show that a stronger result holds: there exists some $0 < \epsilon_\phi < 1$ such that $h_\phi(y) \leq 1 - \epsilon_\phi$. Proving a similar result for $h_d(y)$ can be done in an analogous manner.

Consider the following function

$$\tilde{h}_\phi(y) \equiv \frac{u_1^{-1}\phi_2 + e^y}{\phi_2 + e^y} - \frac{1+\phi_1 e^y}{1+u_2^{-1}\phi_1 e^y}.$$

We have

$$\tilde{h}'_\phi(y) = \frac{\left[\phi_2 - B + \left(1 - u_2^{-1}\phi_1 B\right) e^y\right] \cdot \left[\phi_2 + B + \left(1 + u_2^{-1}\phi_1 B\right) e^y\right] \left(u_2^{-1} - 1\right) \phi_1 e^y}{\left(1 + u_2^{-1}\phi_1 e^y\right)^2 (\phi_2 + e^y)^2},$$

where

$$B \equiv \sqrt{\frac{\left(u_1^{-1} - 1\right)\phi_2}{\left(u_2^{-1} - 1\right)\phi_1}}. \tag{32}$$

Let us go through different cases for the signs of $(\phi_2 - B)$ and $\left(1 - u_2^{-1}\phi_1 B\right)$.

(i) $\phi_2 - B = 0$ and $1 - u_2^{-1}\phi_1 B = 0$. Then $\tilde{h}_\phi(y) = 1 - u_2 > 0$ for all y.

(ii) $\phi_2 - B < 0$ and $1 - u_2^{-1}\phi_1 B \leq 0$; or $\phi_2 - B = 0$ and $1 - u_2^{-1}\phi_1 B < 0$. In these cases $\tilde{h}'_\phi(y) < 0$ for all y and, thus, $\tilde{h}_\phi(y)$ is a decreasing function and $\tilde{h}_\phi(y) \geq \lim_{y\to\infty} \tilde{h}_\phi(y) = 1 - u_2 > 0$ for all y.

(iii) $\phi_2 - B > 0$ and $1 - u_2^{-1}\phi_1 B \geq 0$; or $\phi_2 - B = 0$ and $1 - u_2^{-1}\phi_1 B > 0$. In these cases $\tilde{h}'_\phi(y) > 0$ and, thus, $\tilde{h}_\phi(y)$ is an increasing function and $\tilde{h}_\phi(y) \geq \lim_{y\to-\infty} \tilde{h}_\phi(y) = u_1^{-1} - 1 > 0$ for all y.

(iv) $\phi_2 - B > 0$ and $1 - u_2^{-1}\phi_1 B < 0$. In this case $\tilde{h}_\phi(y)$ is a concave function that is bounded below by the minimum of $\lim_{y\to-\infty} \tilde{h}_\phi(y)$ and $\lim_{y\to\infty} \tilde{h}_\phi(y)$, which are both positive.

(v) $\phi_2 - B < 0$ and $1 - u_2^{-1}\phi_1 B > 0$. This is the most involved case, in which $\tilde{h}_\phi(y)$ is a convex function that achieves its global minimum at $y^* = \ln\left[\frac{B-\phi_2}{1-u_2^{-1}\phi_1 B}\right]$. We have

$$\tilde{h}_\phi(y^*) = \frac{u_1^{-1}\phi_2\left(1 - u_2^{-1}\phi_1 B\right) + \left(u_2^{-1} - 1\right)\phi_1 B^2 + (\phi_1 B - 1)\phi_2}{\left(1 - u_2^{-1}\phi_1\phi_2\right) B}.$$

Substituting B^2 from (32) into this expression and after doing some algebra, we get

$$\tilde{h}_\phi(y^*) = \frac{[2(1-u_1)u_2 - (1-u_1u_2)\phi_1 B]\phi_2}{\left(1 - u_2^{-1}\phi_1\phi_2\right)u_1u_2B}. \tag{33}$$

Given our case-(v) assumptions that $\phi_2 < B$ and $1 - u_2^{-1}\phi_1 B > 0$, we get $1 - u_2^{-1}\phi_1\phi_2 > 1 - u_2^{-1}\phi_1 B > 0$. Thus, we have $1 - u_2^{-1}\phi_1\phi_2 > 0$ and thus the denominator of (33) is positive.

Next, suppose that $1 - 2u_1 + u_1u_2 \geq 0$. This is equivalent to $1 - u_1u_2 \leq 2(1-u_1)$. Using this inequality in the numerator of (33), we get

$$2(1-u_1)u_2 - (1-u_1u_2)\phi_1 B \geq 2(1-u_1)(u_2 - \phi_1 B) > 0,$$

where the second (strict) inequality above follows from our case-(v) assumption that $\phi_1 B < u_2$. This allows us to conclude that $\tilde{h}_\phi(y^*) > 0$ if $1 - 2u_1 + u_1u_2 \geq 0$.

Now suppose that $1 - 2u_1 + u_1u_2 < 0$. Our case-(v) assumption that $B > \phi_2$ is equivalent to

$$(1-u_1)u_2 > (1-u_2)u_1\phi_1\phi_2. \tag{34}$$

Substituting B from (32) into the numerator of (33) and using inequality (34), we get

$$\begin{aligned}
&2(1-u_1)u_2 - (1-u_1u_2)\phi_1 B \\
&\quad = \left[2\sqrt{(1-u_1)(1-u_2)u_1u_2(\phi_1\phi_2)^{-1}} - (1-u_1u_2)\right]\phi_1 B \\
&\quad > -(1-2u_1+u_1u_2)\phi_1 B,
\end{aligned}$$

where the expression in the last line is positive due to our supposition that $1 - 2u_1 + u_1u_2 < 0$. This implies that $\tilde{h}_\phi(y^*) > 0$.

Thus, we get that in case (v), regardless of whether $1 - 2u_1 + u_1u_2 \geq 0$ or $1 - 2u_1 + u_1u_2 < 0$, we have $\tilde{h}_\phi(y^*) > 0$. Then, given that in case (v) function $\tilde{h}_\phi(y)$ is convex and y^* is its global minimum, we get that $\tilde{h}_\phi(y) \geq \tilde{h}_\phi(y^*) > 0$ for all y.

Cases (i)-(v) imply that there always exists a $\underline{h}_\phi > 0$ such that $\tilde{h}_\phi(y) \geq \underline{h}_\phi$ for all y. The definitions of $\tilde{h}_\phi(y)$ and $\tilde{h}_\phi(y)$ then imply that

$$h_\phi(y) = \left(1 + \frac{1 + u_2^{-1}\phi_1 e^y}{1 + \phi_1 e^y}\tilde{h}_\phi(y)\right)^{-1} \leq \left(1 + \frac{1 + u_2^{-1}\phi_1 e^y}{1 + \phi_1 e^y} \cdot \underline{h}_\phi\right)^{-1}.$$

Since

$$\frac{1 + u_2^{-1}\phi_1 e^y}{1 + \phi_1 e^y} \cdot \underline{h}_\phi \geq \underline{h}_\phi$$

for any y, we get that $h_\phi(y) \leq 1 - \epsilon_\phi$, where $\epsilon_\phi \equiv \underline{h}_\phi / \left(1 + \underline{h}_\phi\right)$. Clearly, $0 < \epsilon_\phi < 1$.

Repeating the same proof for $h_d(y)$, we can show that there exists $0 < \epsilon_d < 1$ such that $h_d(y) \leq 1 - \epsilon_d$. Going back to expressions (31), we get

$$f_\phi(y) \leq \left(1 - \epsilon_\phi\right)[\phi_1\phi_2]^{-1} f_d(y) \quad \text{and} \quad f_d(y) \leq (1 - \epsilon_d)[d_1 d_2]^{-1} f_\phi(y).$$

Clearly, if the above inequalities are satisfied for $\epsilon_\phi > 0$ and $\epsilon_d > 0$, they are also satisfied for any $\epsilon_1 \in \left[0, \epsilon_\phi\right]$ and $\epsilon_2 \in [0, \epsilon_d]$ in place of ϵ_ϕ and ϵ_d, respectively. This completes the proof of Lemma 2. □

Applying inequalities (30) from Lemma 2 to $F'(y)$, we get that for any $\epsilon_1 \in [0, \bar{\epsilon}_1]$ and $\epsilon_2 \in [0, \bar{\epsilon}_2]$,

$$\left(\mu - (1 - \epsilon_2)\,\alpha\overline{\overline{\mu}}\right)\frac{1 - \phi_1\phi_2}{1 - \alpha} f_\phi(y) \leq F'(y) \leq \left(\mu - (1 - \epsilon_1)^{-1}\alpha\overline{\mu}\right)\frac{1 - \phi_1\phi_2}{1 - \alpha} f_\phi(y), \tag{35}$$

where

$$\overline{\mu} \equiv \frac{\phi_1\phi_2(d_1 d_2 - 1)}{1 - \phi_1\phi_2} \quad \text{and} \quad \overline{\overline{\mu}} \equiv \frac{d_1 d_2 - 1}{d_1 d_2(1 - \phi_1\phi_2)}.$$

From here we see that if $\mu \leq \alpha\overline{\mu}$ then $F'(y) < 0$ for all y. This implies that the right-hand side of equation $y = F(y)$ is a decreasing function, while the left-hand side of this equation is an increasing function. Therefore, in this case, there is at most one solution to this equation. This proves part (ii.b) of Proposition 5.

Suppose that $\alpha\overline{\mu} < \mu < \alpha\overline{\overline{\mu}}$. Then for all small enough $\epsilon_1 \geq 0$ and $\epsilon_2 \geq 0$ (including a nonzero measure of $\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$) we have $\mu - (1 - \epsilon_2)\,\alpha\overline{\overline{\mu}} < 0$ and $\mu - (1 - \epsilon_1)^{-1}\alpha\overline{\mu} > 0$. This implies that for all such ϵ_1 and ϵ_2 we have

$$\left|F'(y)\right| \leq \rho_{\epsilon_1,\epsilon_2} \equiv \frac{1}{1 - \alpha}\max\left\{\mu - (1 - \epsilon_1)^{-1}\alpha\overline{\mu},\ (1 - \epsilon_2)\,\alpha\overline{\overline{\mu}} - \mu\right\}\frac{1 - \sqrt{\phi_1\phi_2}}{1 + \sqrt{\phi_1\phi_2}},$$

where we used the upper bound on $f_\phi(y)$ given by (28). Denote

$$\rho_{0,0} \equiv \frac{1}{1-\alpha} \max\left\{\mu - \alpha\overline{\mu},\ \alpha\overline{\overline{\mu}} - \mu\right\} \frac{1-\sqrt{\phi_1\phi_2}}{1+\sqrt{\phi_1\phi_2}}.$$

Observe that if $\rho_{0,0} \le 1$ then $\rho_{\epsilon_1,\epsilon_2} < 1$ for small enough $\epsilon_1 > 0$ and $\epsilon_2 > 0$. This implies that if $\rho_{0,0} \le 1$, then $F(y)$ is a contraction mapping in $\mathbb{R}$, which means that there is at most one solution to $y = F(y)$.

Now suppose that $\mu \ge \alpha\overline{\overline{\mu}}$. Then, since $\overline{\mu} < \overline{\overline{\mu}}$, we have $0 \le \mu - (1-\epsilon_2)\alpha\overline{\overline{\mu}} < \mu - (1-\epsilon_1)^{-1}\alpha\overline{\mu}$ for all small enough $\epsilon_1 \ge 0$ and $\epsilon_2 \ge 0$. This implies that

$$0 \le F'(y) \le \tilde{\rho}_{\epsilon_1} \equiv \frac{1}{1-\alpha}\left(\mu - (1-\epsilon_1)^{-1}\alpha\overline{\mu}\right)\frac{1-\sqrt{\phi_1\phi_2}}{1+\sqrt{\phi_1\phi_2}},$$

for all small enough $\epsilon_1 \ge 0$. Observe that $\tilde{\rho}_{\epsilon_1} = \rho_{\epsilon_1,\epsilon_2}$ when $\mu \ge (1-\epsilon_2)\alpha\overline{\overline{\mu}}$ because $\mu - (1-\epsilon_1)^{-1}\alpha\overline{\mu}$ is positive while $(1-\epsilon_2)\alpha\overline{\overline{\mu}} - \mu$ is nonpositive and so the maximum of these two expressions is $\mu - (1-\epsilon_1)^{-1}\alpha\overline{\mu}$. Thus, again, if $\rho_{0,0} \le 1$ then $\tilde{\rho}_{\epsilon_1} < 1$ and so $F(y)$ is a contraction mapping in $\mathbb{R}$. This proves part (ii.c) of Proposition 5.

Case (iii): $\boldsymbol{\alpha > 1}$. The proof for this case is very similar to the proof for case (ii).

Since we have $\mu > 0$ and $\alpha > 1$, expression (26) for $F'(y)$ immediately implies that if $d_1d_2 \le 1$ then $F'(y) < 0$ for all y. This means that the right-hand side of equation $y = F(y)$ is a decreasing function of y, while the left-hand side of this equation is an increasing function of y. Therefore, if $d_1d_2 \le 1$ then there can be at most one intersection of the functions on the right- and left-hand sides of $y = F(y)$, which proves part (iii.a) of Proposition 5.

Suppose now that $d_1d_2 > 1$. Applying inequalities (30) from Lemma 2 to $F'(y)$, we get

$$\left(\alpha - (1-\epsilon_1)\,\mu\big/\overline{\mu}\right)\left(\frac{d_1d_2-1}{\alpha-1}\right) f_d(y) \le F'(y) \le \left(\alpha - (1-\epsilon_2)^{-1}\mu\big/\overline{\overline{\mu}}\right)\frac{(d_1d_2-1)}{\alpha-1} f_d(y), \tag{36}$$

From here we see that if $\mu \ge \alpha\overline{\overline{\mu}}$ then $F'(y) < 0$ for all y. This implies that the right-hand side of equation $y = F(y)$ is a decreasing function, while the left-hand side of this equation is an increasing function. Therefore, in this case, there is at most one solution to this equation. This proves part (iii.b) of Proposition 5.

Suppose that $\alpha\overline{\mu} < \mu < \alpha\overline{\overline{\mu}}$. Then for all small enough $\epsilon_1 \ge 0$ and $\epsilon_2 \ge 0$ (including a nonzero measure of $\epsilon_1 \ne 0$ and $\epsilon_2 \ne 0$) we have $\alpha - (1-\epsilon_1)\,\mu\big/\overline{\mu} < 0$ and

$\alpha - (1-\epsilon_2)^{-1}\mu\Big/\overline{\overline{\mu}} > 0$. This implies that for all such ϵ_1 and ϵ_2 we have

$$\left|F'(y)\right| \leq \rho_{\epsilon_1,\epsilon_2} \equiv \frac{1}{\alpha-1}\max\left\{(1-\epsilon_1)\,\mu\Big/\overline{\mu} - \alpha,\ \alpha - (1-\epsilon_2)^{-1}\mu\Big/\overline{\overline{\mu}}\right\}\frac{\sqrt{d_1d_2}-1}{\sqrt{d_1d_2}+1},$$

where we used the upper bound on $f_d(y)$ given by (28). Denote

$$\rho_{0,0} \equiv \frac{1}{\alpha-1}\max\left\{\mu\Big/\overline{\mu} - \alpha,\ \alpha - \mu\Big/\overline{\overline{\mu}}\right\}\frac{\sqrt{d_1d_2}-1}{\sqrt{d_1d_2}+1}.$$

Observe that if $\rho_{0,0} \leq 1$ then $\rho_{\epsilon_1,\epsilon_2} < 1$ for small enough $\epsilon_1 > 0$ and $\epsilon_2 > 0$. This implies that if $\rho_{0,0} \leq 1$, then $F(y)$ is a contraction mapping in $\mathbb{R}$, which means that there is at most one solution to $y = F(y)$.

Now suppose that $\mu \leq \alpha\overline{\mu}$. Then, since $\overline{\mu} < \overline{\overline{\mu}}$, we have $0 \leq \alpha - (1-\epsilon_1)\,\mu\Big/\overline{\mu} < \alpha - (1-\epsilon_2)^{-1}\mu\Big/\overline{\overline{\mu}}$ for all small enough $\epsilon_1 \geq 0$ and $\epsilon_2 \geq 0$. This implies that

$$0 \leq F'(y) \leq \tilde{\rho}_{\epsilon_2} \equiv \frac{1}{\alpha-1}\left(\alpha - (1-\epsilon_2)^{-1}\mu\Big/\overline{\overline{\mu}}\right)\frac{\sqrt{d_1d_2}-1}{\sqrt{d_1d_2}+1},$$

for all small enough $\epsilon_2 \geq 0$. Observe that $\tilde{\rho}_{\epsilon_2} = \rho_{\epsilon_1,\epsilon_2}$ when $(1-\epsilon_1)\,\mu \leq \alpha\overline{\mu}$ because $\alpha - (1-\epsilon_2)^{-1}\mu\Big/\overline{\overline{\mu}}$ is positive while $\alpha - (1-\epsilon_1)\,\mu\Big/\overline{\mu}$ is nonpositive and so the maximum of these two expressions is $\alpha - (1-\epsilon_2)^{-1}\mu\Big/\overline{\overline{\mu}}$. Thus, again, if $\rho_{0,0} \leq 1$ then $\tilde{\rho}_{\epsilon_2} < 1$ and so $F(y)$ is a contraction mapping in $\mathbb{R}$. This proves part (iii.c) of Proposition 5.

A.5 Proof of Corollary 1

Suppose that $\phi_1\phi_2 \lessapprox 1$, where "$\lessapprox$" means approximately equal but less than. Then $d_1d_2 \lessapprox 1$ and so to establish uniqueness we need to show that the condition $\mu \leq \frac{1+\sqrt{\phi_1\phi_2}}{1-\sqrt{\phi_1\phi_2}} \cdot \frac{1-2\alpha+\sqrt{d_1d_2}}{1+\sqrt{d_1d_2}}$ in part (ii.a) of Proposition 5 holds. With $\phi_1\phi_2 \lessapprox 1$ we have that $\frac{1+\sqrt{\phi_1\phi_2}}{1-\sqrt{\phi_1\phi_2}}$ is positive and large, and so it is enough to show that $1 - 2\alpha + \sqrt{d_1d_2}$ is positive. This is equivalent to $\frac{1+\sqrt{d_1d_2}}{2} > \alpha$. For any α there are always trade costs that are low enough so that d_1d_2 is close to 1 and, consequently, $\frac{1+\sqrt{d_1d_2}}{2}$ is close enough to 1. Since $\alpha < 1$, then the inequality follows.

Consider now the case with $\beta \gtrapprox 0$ combined with $0 < \alpha < 1$ and $\phi_1\phi_2 < 1$. Also, assume for simplicity that $0 < \gamma < 1$ (the cases with $\gamma = 0$ or $\gamma = 1$ can be analyzed in a similar manner). Having $\beta \gtrapprox 0$ implies that $d_1d_2 - 1 \approx \frac{(1-\gamma)\gamma(1-\phi_1\phi_2)^2}{\phi_1\phi_2}$, which is positive for $0 < \gamma < 1$. In other words, $d_1d_2 > 1$ for low enough β and $0 < \gamma < 1$. Then we get $\mu \approx 0$ and $\overline{\mu} \approx \gamma(1-\gamma)(1-\phi_1\phi_2) > 0$ for $\beta \gtrapprox 0$ and $0 < \gamma < 1$. This implies that $\mu < \alpha\overline{\mu}$ for $\beta \gtrapprox 0$ and $0 < \gamma < 1$, and thus we can invoke part (ii.b) of Proposition 5 to get uniqueness.

Finally, consider the case with $\alpha \gtrapprox 0$ and $0 < \beta < 1$. Suppose first that $d_1 d_2 \leq 1$. We have that $\mu \approx \beta$ and $\frac{1+\sqrt{\phi_1\phi_2}}{1-\sqrt{\phi_1\phi_2}} \cdot \frac{1-2\alpha+\sqrt{d_1 d_2}}{1+\sqrt{d_1 d_2}} \approx \frac{1+\sqrt{\phi_1\phi_2}}{1-\sqrt{\phi_1\phi_2}} > 1$, and thus $\mu < \frac{1+\sqrt{\phi_1\phi_2}}{1-\sqrt{\phi_1\phi_2}} \cdot \frac{1-2\alpha+\sqrt{d_1 d_2}}{1+\sqrt{d_1 d_2}}$ for low enough α. Part (ii.a) of Proposition 5 then implies uniqueness for $d_1 d_2 \leq 1$ and $\alpha \gtrapprox 0$. Now suppose that $d_1 d_2 > 1$. Then, $\alpha\overline{\mu} \approx 0$ for $\alpha \gtrapprox 0$, while we still have $\mu \approx \beta$. Therefore, $\mu > \alpha\overline{\mu}$ and $\mu - \alpha\overline{\mu} \approx \beta$ for $\alpha \gtrapprox 0$. Also, according to its definition (14), $\overline{\overline{\mu}}$ does not depend on α and so $\alpha\overline{\overline{\mu}} \approx 0$ and $\alpha\overline{\overline{\mu}} - \mu \approx -\beta$ for $\alpha \gtrapprox 0$. Therefore, $\max\left\{\mu - \alpha\overline{\mu}, \alpha\overline{\overline{\mu}} - \mu\right\} \approx \beta$ for $\alpha \gtrapprox 0$. At the same time, $(1-\alpha)\frac{1+\sqrt{\phi_1\phi_2}}{1-\sqrt{\phi_1\phi_2}} > 1$ for $\alpha \gtrapprox 0$. Part (ii.c) of Proposition 5 then implies uniqueness for $d_1 d_2 > 1$ and $\alpha \gtrapprox 0$.

A.6 Proof of Proposition 6

We prove Proposition 6 in a sequence of lemmas.

Lemma 3 (Regular equilibria: Sufficient conditions for uniqueness under costly trade and $\alpha = 1$). *Suppose that $\alpha = 1$ and $\phi_1\phi_2 < 1$. If, in addition to that, either*

(a) $\overline{\beta} = 0$, or $\gamma = 0$, or $\gamma = 1$; or

(b) $\mu \geq 1$; or

(c) $\mu < 1$ and, at the same time, either $\phi_1 \geq \tilde{\phi}_1$ or $\phi_2 \geq \tilde{\phi}_2$;

then the economy of Section 2 has at most one regular equilibrium.

Proof. Consider equation (10) for $\alpha = 1$

$$1 = G \cdot \left(\frac{\phi_1}{\phi_2}\right)^{-1} \left(\frac{1+\phi_1 x}{\phi_2 + x}\right)^{\mu} \left(\frac{1+d_1 x}{d_2 + x}\right), \tag{37}$$

where

$$d_1 \equiv \frac{\gamma\overline{\beta} + \left(1-\gamma\overline{\beta}\right)\phi_1\phi_2}{\phi_2} \quad \text{and} \quad d_2 \equiv \frac{\overline{\gamma}\overline{\beta} + \left(1-\overline{\gamma}\overline{\beta}\right)\phi_1\phi_2}{\phi_1}.$$

Introduce the change of variables

$$z = \left(\frac{1+\phi_1 x}{\phi_2 + x}\right)^{1+\mu},$$

which implies that

$$x = \frac{1 - \phi_2 z^{\frac{1}{1+\mu}}}{z^{\frac{1}{1+\mu}} - \phi_1}.$$

Observe that z ranges from $\phi_2^{-(1+\mu)}$ to $\phi_1^{1+\mu}$ as x ranges from 0 to ∞. After doing some algebra, we can write (37) as $V(z) = 0$, where

$$V(z) \equiv \overline{\gamma}\overline{\beta} z^{\frac{1}{1+\mu}} - \gamma\overline{\beta} G \cdot z^{\frac{\mu}{1+\mu}} - \left(1 - \gamma\overline{\beta}\right)\phi_2 G z + \left(1 - \overline{\gamma}\overline{\beta}\right)\phi_1. \tag{38}$$

Formally, equation $V(z) = 0$ is equivalent to equation (37) only for $z \in \left(\phi_1^{1+\mu}, \phi_2^{-(1+\mu)}\right)$. At the same time, function $V(z)$ is defined for any $z \geq 0$. Moreover, $V(z) = 0$ has at least one solution for some $z > 0$, because $V(0) > 0$ and $\lim_{z\to\infty} V(z) = -\infty$, while equation (37) might have no positive solutions. In the proof of the current lemma, we analyze multiplicity of solutions of equation $V(z) = 0$ for all $z > 0$ and then in Lemma 5 below impose the condition that $z \in \left(\phi_1^{1+\mu}, \phi_2^{-(1+\mu)}\right)$.

If $\overline{\beta} = 0$ then we can explicitly find that the solution of equation $V(z) = 0$ is $z = \phi_1\phi_2^{-1}G^{-1}$. If $\gamma = 0$ and $\overline{\beta} > 0$ then $V(z)$ is a concave function achieving its maximum at a positive z^*. Then, given that $V(0) > 0$ and $\lim_{z\to\infty} V(z) = -\infty$, we conclude that $V(z)$ intersects the horizontal axis only once. If $\gamma = 1$ and $\overline{\beta} > 0$ then $V(z)$ is a strictly decreasing function with $V(0) > 0$ and $\lim_{z\to\infty} V(z) = -\infty$. Hence, again, $V(z)$ intersects the horizontal axis only once. This proves part (a) of the current lemma.

If $\mu = 1$ then equation $V(z) = 0$ turns into the quadratic equation,

$$\left(1 - \gamma\overline{\beta}\right)\phi_2 G z - \left(\overline{\gamma}\overline{\beta} - \gamma\overline{\beta} G\right) z^{\frac{1}{2}} - \left(1 - \overline{\gamma}\overline{\beta}\right)\phi_1 = 0,$$

which has a unique positive solution. This proves the case with $\mu = 1$ of part (b) of the current lemma.

In what follows, we assume that $\overline{\beta} \neq 0$ and $0 < \gamma < 1$ and consider the cases $\mu < 1$ and $\mu > 1$. We have

$$V'(z) = \frac{1}{1+\mu}\overline{\gamma}\overline{\beta} z^{-\frac{\mu}{1+\mu}} - \frac{\mu}{1+\mu}\gamma\overline{\beta} G \cdot z^{-\frac{1}{1+\mu}} - \left(1 - \gamma\overline{\beta}\right)\phi_2 G, \tag{39}$$

$$V''(z) = \frac{\mu\gamma\overline{\beta} G}{(1+\mu)^2} z^{-\frac{\mu}{1+\mu}-1}\left(z^{-\frac{1-\mu}{1+\mu}} - \overline{z}_0^{-\frac{1-\mu}{1+\mu}}\right), \tag{40}$$

where

$$\overline{z}_0 \equiv \left(\frac{\gamma}{\overline{\gamma}} G\right)^{\frac{1+\mu}{1-\mu}}. \tag{41}$$

Consider the case with $\mu > 1$. In this case, $V''(z) < 0$ for $z < \overline{z}_0$ and $V''(z) > 0$ for

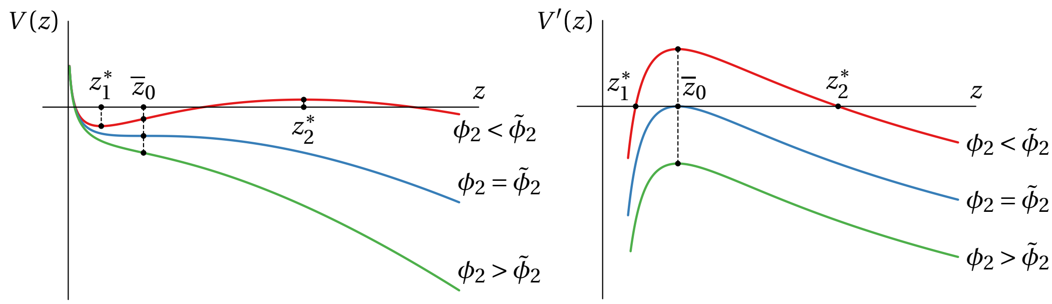


Figure 14: Krugman case, $\mu < 1$: V and V'

$z > \overline{z}_0$. Thus, $V'(z)$ is convex function with $\overline{z}_0$ being its minimum. We have

$$V'(\overline{z}_0) = -\frac{\mu - 1}{\mu + 1}\overline{\beta}\gamma^{\frac{\mu}{\mu-1}}\overline{\gamma}^{-\frac{1}{\mu-1}}G^{\frac{\mu}{\mu-1}} - \left(1 - \gamma\overline{\beta}\right)\phi_2 G.$$

Clearly, $V'(\overline{z}_0) < 0$, which, together with $\lim_{z\to 0} V'(z) = \infty$ and $\lim_{z\to\infty} V'(z) < 0$, implies that $V'(z)$ intersects the horizontal axis only once for some $z_1^* < \overline{z}_0$. Therefore, $V(z)$ is increasing for $z < z_1^*$ and decreasing for $z > z_1^*$. Then, given that $V(0) > 0$ and $\lim_{z\to\infty} V(z) = -\infty$, we have that $V(z)$ intersect the horizontal axis only once for some $\tilde{z} > z_1^*$. This completes the proof of part (b) of the current lemma.

Now consider the case with $\mu < 1$. It helps to refer to Figure 14, which sketches the shapes of $V(z)$ and $V'(z)$ in the case with $\mu < 1$. In this case, $V''(z) > 0$ for $z < \overline{z}_0$ and $V''(z) < 0$ for $z > \overline{z}_0$. Thus, $V'(z)$ is a concave function with $\overline{z}_0$ being its maximum. It is straightforward to check that if $\phi_2 > \tilde{\phi}_2$ — where $\tilde{\phi}_2$ is defined in (16) — then $V'(\overline{z}_0) < 0$, which implies that $V'(z) < 0$ for all z. This means that in this case $V(z)$ is a decreasing function that intersects the horizontal only once.

The case with $\phi_2 = \tilde{\phi}_2$ (and $\mu < 1$) is similar to the case with $\phi_2 > \tilde{\phi}_2$ with the only difference that if $\phi_2 = \tilde{\phi}_2$ then $V'(\overline{z}_0) = 0$ and $V(z)$ is decreasing for all $z \neq \overline{z}_0$. Clearly, equation $V(z) = 0$ has a unique solution in this case as well. This proves the case with $\phi_2 \geq \tilde{\phi}_2$ of part (c) of the current lemma.

To prove the case with $\phi_1 \geq \tilde{\phi}_1$ of part (c) of the current lemma, let us rewrite function $V(z)$ given by (38) as $V(z) = -Gz\widetilde{V}\left(z^{-1}\right)$,where

$$\widetilde{V}(z) \equiv -G^{-1}zV\left(z^{-1}\right) = \gamma\overline{\beta}z^{\frac{1}{1+\mu}} - \overline{\gamma}\overline{\beta}G^{-1}z^{\frac{\mu}{1+\mu}} - \left(1 - \overline{\gamma}\overline{\beta}\right)\phi_1 G^{-1}z + \left(1 - \gamma\overline{\beta}\right)\phi_2.$$

Clearly, $V(z)$ and $\widetilde{V}(z)$ have the same number of positive solutions. Repeating the above analysis for $\widetilde{V}(z)$ instead of $V(z)$, we get that $\widetilde{V}(z)$ intersects the horizontal axis once and only once if $\phi_1 \geq \tilde{\phi}_1$. This completes part (c) of the current lemma. $\square$

We next proceed with an analysis of the case with $\mu < 1$ and $\phi_1 < \tilde{\phi}_1$ and $\phi_2 < \tilde{\phi}_2$, where $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are defined in (15)-(16). As we show below, in this case, the two-dimensional space (ϕ_1, ϕ_2) can be divided into two areas: one area where equation (38) has at most one positive solution and one area where equation (38) has at most three positive solutions. A natural and insightful way to describe these areas is to use parameters ϕ_2 and $c \equiv \phi_1/\phi_2$ (rather than ϕ_1 and ϕ_2) to trace the "uniqueness boundary" that separates these areas. A benefit of this approach is that it naturally covers the case of symmetric trade costs with $c = 1$.

Lemma 4 (Regular equilibria: Uniqueness and multiplicity under costly trade and $\boldsymbol{\alpha = 1}$). *Suppose that $\mu < 1$, $\phi_1\phi_2 < 1$, and $\phi_1 < \tilde{\phi}_1$ and $\phi_2 < \tilde{\phi}_2$, where $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are defined in* (15)-(16)*. Then for any fixed $c = \phi_1/\phi_2$ there exists $\tilde{\tilde{\phi}}_2(c) \in (0, \tilde{\phi}_2]$ such that for any $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ the economy of Section 2 has at most one regular equilibrium, and for any $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ the economy of Section 2 has at most three regular equilibria. For $\phi_2 = \tilde{\tilde{\phi}}_2(c)$, the economy of Section 2 generally has at most two regular equilibria except for the special case with $c = \tilde{c}$ given by* (17)*, in which case $\tilde{\tilde{\phi}}_2(\tilde{c}) = \tilde{\phi}_2$ and the economy of Section 2 at most one regular equilibrium.*

Proof. Consider function V defined in (38) in Lemma 3. Replace ϕ_1 in the definition of V by $c\phi_2$ and introduce arguments ϕ_2 and c into the notation of V by writing this function as

$$V(z, \phi_2, c) \equiv \overline{\gamma}\overline{\beta} z^{\frac{1}{1+\mu}} - \gamma\overline{\beta} G z^{\frac{\mu}{1+\mu}} - \left(1 - \gamma\overline{\beta}\right)\phi_2 G z + \left(1 - \overline{\gamma}\overline{\beta}\right) c\phi_2.$$

Denote the first and the second derivatives of $V(z, \phi_2, c)$ with respect to z as $V_1'(z, \phi_2, c)$ and $V_1''(z, \phi_2, c)$ These derivatives are given by (39) and (40).

We break the proof of the current lemma into two steps.

STEP 1. We first prove that for each $c > 0$ the system of equations $V(z, \phi_2, c) = 0$ and $V_1'(z, \phi_2, c) = 0$ has a unique solution $\tilde{\tilde{z}}(c)$ and $\tilde{\tilde{\phi}}_2(c)$ with $\tilde{\tilde{z}}(c) > 0$ and $0 < \tilde{\tilde{\phi}}_2(c) \leq \tilde{\phi}_2$.

Solving for ϕ_2 from equation $V_1'(z, \phi_2, c) = 0$, substituting the result into equation $V(z, \phi_2, c) = 0$, and after doing some algebra, we get equation $H(z, c) = 0$ in z, where

$$H(z, c) \equiv \mu\overline{\gamma} z^{\frac{2}{1+\mu}} + \frac{1 - \overline{\gamma}\overline{\beta}}{1 - \gamma\overline{\beta}}\overline{\gamma} G^{-1} c z^{\frac{1-\mu}{1+\mu}} - \gamma G z - \frac{1 - \overline{\gamma}\overline{\beta}}{1 - \gamma\overline{\beta}}\mu\gamma c.$$

We are going to show that for any $c > 0$ there is a unique $\tilde{\tilde{z}}(c) > 0$ such that $H(\tilde{\tilde{z}}(c), c) = 0$.

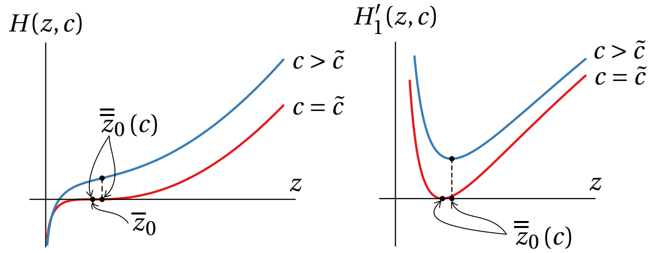


Figure 15: Krugman case, $\mu < 1$: H and H'

The first and the second derivatives of $H(z,c)$ with respect to z are given by

$$H_1'(z,c) = \frac{2\mu}{1+\mu}\overline{\gamma}z^{\frac{1-\mu}{1+\mu}} + \frac{1-\mu}{1+\mu}\cdot\frac{1-\overline{\gamma}\overline{\beta}}{1-\gamma\overline{\beta}}\overline{\gamma}G^{-1}cz^{-\frac{2\mu}{1+\mu}} - \gamma G,$$
$$H_2'(z,c) = \frac{2\mu(1-\mu)}{(1+\mu)^2}\overline{\gamma}z^{-\frac{2\mu}{1+\mu}-1}\left(z-\bar{\bar{z}}_0(c)\right),$$

where

$$\bar{\bar{z}}_0(c) \equiv \frac{1-\overline{\gamma}\overline{\beta}}{1-\gamma\overline{\beta}}G^{-1}c. \tag{42}$$

From here we see that $H_1''(z,c) < 0$ if and only if $z < \bar{\bar{z}}_0(c)$. Thus, for any $c > 0$, $H_1'(z,c)$ is a convex function in z, and $\bar{\bar{z}}_0(c)$ is its minimum (see Figure 15, which sketches the shapes of $H(z,c)$ and $H_1'(z,c)$).

Evaluating $H_1'\left(\bar{\bar{z}}_0(c), c\right)$, we get

$$H_1'\left(\bar{\bar{z}}_0(c), c\right) = \overline{\gamma}\left(\frac{1-\overline{\gamma}\overline{\beta}}{1-\gamma\overline{\beta}}G^{-1}c\right)^{\frac{1-\mu}{1+\mu}} - \gamma G,$$

which allows us to see that $H_1'\left(\bar{\bar{z}}_0(c), c\right) > 0$ if and only if $c > \tilde{c}$, where $\tilde{c}$ was defined in (17). From here we immediately see that if $c > \tilde{c}$, then $H_1'(z,c) > 0$ for all z, and so $H(z,c)$ increases in z. Then, given that $H(0,c) < 0$ and $\lim_{z\to\infty} H(z,c) = \infty$, we conclude that, for any fixed $c > \tilde{c}$, $H(z,c)$ intersects the horizontal axis $z = 0$ only once at some $\tilde{\tilde{z}}(c) > 0$.

The case with $c = \tilde{c}$ is similar to the case with $c > \tilde{c}$ with the only difference that $H(z,c)$ is an increasing function for all $z \neq \bar{\bar{z}}_0(\tilde{c})$ and $H\left(\bar{\bar{z}}_0(\tilde{c}), \tilde{c}\right) = H_1'\left(\bar{\bar{z}}_0(\tilde{c}), \tilde{c}\right) = 0$ (and so $\tilde{\tilde{z}}(\tilde{c}) = \bar{\bar{z}}_0(\tilde{c})$).

Now consider the case with $c < \tilde{c}$. We can write $H(z,c) = -\frac{1-\overline{\gamma}\overline{\beta}}{1-\gamma\overline{\beta}}cz^{\frac{2\alpha\mu}{\alpha\mu+1}}\tilde{H}\left(z^{-1}, c\right)$, where

$$\tilde{H}(z,c) \equiv -\frac{1-\gamma\overline{\beta}}{1-\overline{\gamma}\overline{\overline{\beta}}}c^{-1}z^{\frac{2}{1+\mu}}H\left(z^{-1},c\right)$$
$$= \mu\gamma z^{\frac{2}{1+\mu}} + \frac{1-\gamma\overline{\beta}}{1-\overline{\gamma}\overline{\overline{\beta}}}\gamma G c^{-1} z^{\frac{1-\mu}{1+\mu}} - \overline{\gamma}G^{-1}z - \frac{1-\gamma\overline{\beta}}{1-\overline{\gamma}\overline{\overline{\beta}}}\mu\overline{\gamma}c^{-1}.$$

Obviously, functions $H(z,c)$ and $\tilde{H}(z,c)$ have the same number of zeros for $z > 0$. Observe that function $\tilde{H}(z,c)$ is similar to function $H(z,c)$ with the difference that γ is swapped with $\overline{\gamma}$; G is swapped with G^{-1}; and c is swapped with c^{-1}. Applying the same analysis to function $\tilde{H}(z,c)$ as to function $H(z,c)$, we get that $\tilde{H}(z,c)$ increases in z if $c < \tilde{c}$. Then, given that $\tilde{H}(0,c) < 0$ and $\lim_{z\to\infty}\tilde{H}(z,c) = \infty$, we conclude that $\tilde{H}(z,c)$ intersects the horizontal axis $z = 0$ once and only once for some $\tilde{\tilde{z}}(c) > 0$. The corresponding unique solution to equation $H(z,c) = 0$ is $[\tilde{\tilde{z}}(c)]^{-1}$.

At this point, we have established that for any $c > 0$ there is a unique $\tilde{\tilde{z}}(c) > 0$ such that $H(\tilde{\tilde{z}}(c),c) = 0$. This, of course, means that our original system $V(z,\phi_2,c) = 0$ and $V_1'(z,\phi_2,c) = 0$ has a unique solution $\left(\tilde{\tilde{z}}(c),\tilde{\tilde{\phi}}_2(c)\right)$ with $\tilde{\tilde{z}}(c) > 0$, where we use equation $V_1'(z,\phi_2,c) = 0$ to find $\tilde{\tilde{\phi}}_2(c)$ corresponding to $\tilde{\tilde{z}}(c)$, which gives

$$\tilde{\tilde{\phi}}_2(c) = \frac{1}{1+\mu}\cdot\frac{\overline{\gamma}\overline{\beta}}{1-\gamma\overline{\overline{\beta}}}G^{-1}\left[\tilde{\tilde{z}}(c)\right]^{-\frac{\mu}{1+\mu}} - \frac{\mu}{1+\mu}\cdot\frac{\gamma\overline{\beta}}{1-\gamma\overline{\overline{\beta}}}\left[\tilde{\tilde{z}}(c)\right]^{-\frac{1}{1+\mu}}.$$

We need to verify that $0 < \tilde{\tilde{\phi}}_2(c) \leq \tilde{\phi}_2$. The upper bound $\tilde{\tilde{\phi}}_2(c) \leq \tilde{\phi}_2$ simply follows from the fact that $V_1'\left(\tilde{\tilde{z}}(c),\tilde{\tilde{\phi}}_2(c),c\right) = 0$, and we know from Lemma 3 that $V_1'(z,\phi_2,c) < 0$ for all $z > 0$ and $\phi_2 > \tilde{\phi}_2$. Verifying positivity of $\tilde{\tilde{\phi}}_2(c)$ is equivalent to verifying that $\tilde{\tilde{z}}(c) > \tilde{\mu}\overline{z}_0$, where $\tilde{\mu} \equiv \mu^{\frac{1+\mu}{1-\mu}}$ and $\overline{z}_0$ was defined in (41) in Lemma 3. Simple algebra reveals that

$$H(\tilde{\mu}\overline{z}_0,c) = -\left(\mu^{-2}-1\right)\mu^{1+\frac{2}{1-\mu}}\overline{\gamma}\left(\gamma\overline{\gamma}^{-1}G\right)^{\frac{2}{1-\mu}} < 0.$$

Also, our analysis above implies that for any $c > 0$, regardless of the shape of $H(z,c)$, we have $H(z,c) < 0$ for $z < \tilde{\tilde{z}}(c)$ and $H(z,c) > 0$ for $z > \tilde{\tilde{z}}(c)$. This allows us to conclude that $\tilde{\mu}\overline{z}_0 < \tilde{\tilde{z}}(c)$ and, thus, $\tilde{\tilde{\phi}}_2(c) > 0$.

STEP 2. We are now going to prove the statement of the current lemma about the existence of a unique $\tilde{\tilde{\phi}}_2(c) > 0$ that traces the uniqueness boundary.

Consider $V_1'(z,\phi_2,c)$ for any $c > 0$ and $\phi_2 < \tilde{\phi}_2$ (this case is illustrated in Figure 14 by the curve labeled "$\phi_2 < \tilde{\phi}_2$" as well as in Figure 16). In Lemma 3, we have established that $V_1'(\overline{z}_0,\phi_2,c) > 0$ for $\phi_2 < \tilde{\phi}_2$, where $\overline{z}_0$ was defined in (41) and is a maximum of

$V_1'(z, \phi_2, c)$. This implies that $V_1'(z, \phi_2, c)$ intersects the horizontal axis $z = 0$ at exactly two points: one lower than $\bar{z}_0$ and one larger than $\bar{z}_0$. This fact allows us to define functions

$$z_1^*(\phi_2, c) \equiv \{z | z \le \bar{z}_0 \text{ and } V_1'(z, \phi_2, c) = 0\},$$
$$z_2^*(\phi_2, c) \equiv \{z | z \ge \bar{z}_0 \text{ and } V_1'(z, \phi_2, c) = 0\},$$

both with domain $\phi_2 < \tilde{\phi}_2$ and $c > 0$. Definitions of $z_1^*(\phi_2, c)$ and $z_2^*(\phi_2, c)$ imply that $z_1^*(\phi_2, c) < \bar{z}_0 < z_2^*(\phi_2, c)$. Moreover, we have that $V_1'(z, \phi_2, c) < 0$ for $z \in (0, z_1^*(\phi_2, c)) \cup (z_2^*(\phi_2, c), \infty)$ and $V_1'(z, \phi_2, c) > 0$ for $z \in (z_1^*(\phi_2, c), z_2^*(\phi_2, c))$. Therefore, $z_1^*(\phi_2, c)$ is a local minimum of $V(z, \phi_2, c)$ and $z_2^*(\phi_2, c)$ is a local maximum of $V(z, \phi_2, c)$, and $V(z_1^*(\phi_2, c), \phi_2, c) < V(z_2^*(\phi_2, c), \phi_2, c)$.

We have shown in Step 1 that there exists a unique solution $\left(\tilde{\tilde{z}}(c), \tilde{\tilde{\phi}}_2(c)\right)$ to the system of equations $V(z, \phi_2, c) = 0$ and $V_1'(z, \phi_2, c) = 0$. Moreover, as we have argued in Step 1, $H(z, c) < 0$ if and only if $z < \tilde{\tilde{z}}(c)$. Simple algebra reveals that

$$H(\bar{z}_0, c) = (1 - \mu) \frac{\gamma(1 - \bar{\gamma}\bar{\beta})}{1 - \gamma\bar{\beta}} (c - \tilde{c}),$$

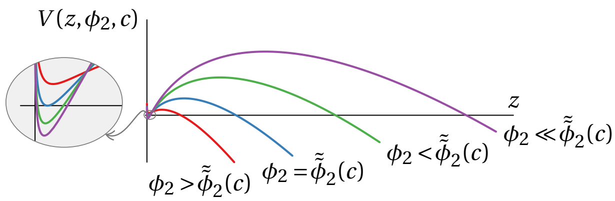
and, thus, $H(\bar{z}_0, c) < 0$ if and only if $c < \tilde{c}$, where $\tilde{c}$ is defined in (41). Therefore, $\tilde{\tilde{z}}(c) > \bar{z}_0$ if and only if $c < \tilde{c}$. This, in turn, implies that $\tilde{\tilde{z}}(c) = z_1^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$ for $c > \tilde{c}$ and $\tilde{\tilde{z}}(c) = z_2^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$ for $c < \tilde{c}$, while $\tilde{\tilde{z}}(\tilde{c}) = \bar{z}_0$ and $\tilde{\tilde{\phi}}_2(\tilde{c}) = \tilde{\phi}_2$.

Next, using the fact that $V_1'\left(z_i^*(\phi_2, c), \phi_2, c\right) = 0$, we find that

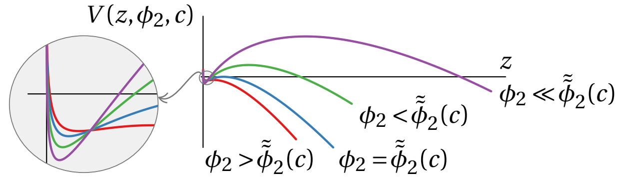
$$\frac{dV\left(z_i^*(\phi_2, c), \phi_2, c\right)}{d\phi_2} = -\left(1 - \gamma\bar{\beta}\right) G z_i^*(\phi_2, c) + \left(1 - \bar{\gamma}\bar{\beta}\right) c,$$

and, thus, $dV\left(z_i^*(\phi_2, c), \phi_2, c\right)/d\phi_2 > 0$ if and only if $z_i^*(\phi_2, c) < \bar{\bar{z}}_0(c)$, where $\bar{\bar{z}}_0(c)$ was defined in (42). Using the definitions of $\bar{z}_0$ and $\bar{\bar{z}}_0(c)$, we get that $\bar{\bar{z}}_0(c) < \bar{z}_0$ if and only if $c < \tilde{c}$. Thus, given the fact that $z_1^*(\phi_2, c) < \bar{z}_0 < z_2^*(\phi_2, c)$ for any $\phi_2 < \tilde{\phi}_2$ and $c > 0$, we get that if $c > \tilde{c}$ then $z_1^*(\phi_2, c) < \bar{\bar{z}}_0(c)$, and if $c < \tilde{c}$ then $z_2^*(\phi_2, c) > \bar{\bar{z}}_0(c)$. This, in turn, implies that if $c > \tilde{c}$ then $V(z_1^*(\phi_2, c), \phi_2, c)$ is increasing in ϕ_2, and if $c < \tilde{c}$ then $V(z_2^*(\phi_2, c), \phi_2, c)$ is decreasing in ϕ_2.

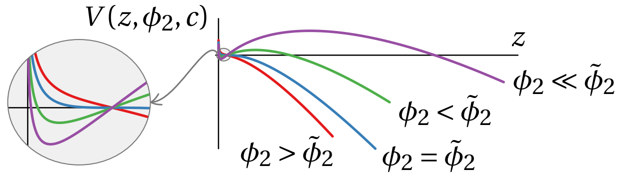
We are now ready to bring all facts together to characterize multiplicity of solutions of equation $V(z, \phi_2, c) = 0$. Fix any $c > \tilde{c}$ and consider function $V(z, \phi_2, c)$ as we change ϕ_2 (see Figure 16a). For $\phi_2 = \tilde{\tilde{\phi}}_2(c)$, the horizontal axis $z = 0$ is tangent to the local minimum of $V(z, \phi_2, c)$ at point $\tilde{\tilde{z}}(c) = z_1^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$ (this case is depicted in Figure 16a by the curve labeled "$\phi_2 = \tilde{\tilde{\phi}}_2(c)$"). Thus, $V\left(z_1^*\left(\tilde{\tilde{\phi}}_2(c), c\right), \tilde{\tilde{\phi}}_2(c), c\right) = 0$ and for all



(a) $c > \tilde{c}$



(b) $c < \tilde{c}$



(c) $c = \tilde{c}$

Figure 16: Krugman case, $\mu < 1$: V for different c

points $z \in \left(0, z_2^*\left(\tilde{\tilde{\phi}}_2(c), c\right)\right)$ different from $z_1^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$ we have $V\left(z, \tilde{\tilde{\phi}}_2(c), c\right) > 0$. For $z > z_2^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$, function $V\left(z, \tilde{\tilde{\phi}}_2(c), c\right)$ monotonically decreases from a positive value to $-\infty$ as $z \to \infty$. This implies that function $V\left(z, \tilde{\tilde{\phi}}_2(c), c\right)$ crosses the horizontal axis $z = 0$ only once for some $\tilde{z} > z_2^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$. Thus, for $\phi_2 = \tilde{\tilde{\phi}}_2(c)$ there are two solutions to equation $V(z, \phi_2, c) = 0$: $z_1^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$ and $\tilde{z}$.

Next, as we have argued above, $V(z_1^*(\phi_2, c), \phi_2, c)$ is increasing in ϕ_2 for $c > \tilde{c}$. Therefore, for $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ we have $V(z_1^*(\phi_2, c), \phi_2, c) > 0$, which implies that $V(z, \phi_2, c) > 0$ for all $z \in (0, z_2^*(\phi_2, c))$ (this case is depicted in Figure 16a by the curve labeled "$\phi_2 > \tilde{\tilde{\phi}}_2(c)$"). And for $z > z_2^*(\phi_2, c)$, again, function $V(z, \phi_2, c)$ intersects the horizontal axis $z = 0$ once and only once. Thus, for $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ there is a unique solution to equation $V(z, \phi_2, c) = 0$.

Finally, for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ we have $V(z_1^*(\phi_2, c), \phi_2, c) < 0$ (this case is depicted in Figure 16a by the curves labeled "$\phi_2 < \tilde{\tilde{\phi}}_2(c)$" and "$\phi_2 \ll \tilde{\tilde{\phi}}_2(c)$" with the latter curve corresponding to a lower value of ϕ_2 than the former curve). At the same time, we necessarily have $V(z_2^*(\phi_2, c), \phi_2, c) > 0$ for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$. To see this, observe that $V(z_2^*(\phi_2, c), \phi_2, c) > 0$ for $\phi_2 \in \left[\tilde{\tilde{\phi}}_2(c), \tilde{\phi}_2\right)$, because $V(z_1^*(\phi_2, c), \phi_2, c) \geq 0$ for $\phi_2 \in \left[\tilde{\tilde{\phi}}_2(c), \tilde{\phi}_2\right)$ and $V(z_1^*(\phi_2, c), \phi_2, c) < V(z_2^*(\phi_2, c), \phi_2, c)$ for all $\phi_2 < \tilde{\phi}_2$. If there is some $\phi_2' \in \left(0, \tilde{\tilde{\phi}}_2(c)\right)$ such that $V(z_2^*(\phi_2', c), \phi_2, c) \leq 0$ then continuity of $V(z_2^*(\phi_2, c), \phi_2, c)$ in ϕ_2 implies that there is also some $\phi_2'' \in \left[\phi_2', \tilde{\tilde{\phi}}_2(c)\right)$ such that $V(z_2^*(\phi_2'', c), \phi_2'', c) = 0$. Then the pair $z_2^*(\phi_2'', c)$ and ϕ_2'' is a solution of the system of equations $V(z, \phi_2, c) = 0$ and $V_1'(z, \phi_2, c) = 0$. Moreover, $\phi_2'' \neq \tilde{\tilde{\phi}}_2(c)$, while the pair $z_1^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$ and $\tilde{\tilde{\phi}}_2(c)$ is another solution of the same system of equations. This contradicts to the fact established at Step 1 that the system of equations $V(z, \phi_2, c) = 0$ and $V_1'(z, \phi_2, c) = 0$ has a unique solution. Thus, indeed, $V\left(z_2^*\left(\phi_2,c\right),\phi_2,c\right)>0$ for all $\phi_2<\tilde{\tilde{\phi}}_2\left(c\right)$.

The facts that $V\left(0,\phi_2,c\right)>0$ and that for any $\phi_2<\tilde{\tilde{\phi}}_2\left(c\right)$ we have $V\left(z_1^*\left(\phi_2,c\right),\phi_2,c\right)<0$ and $V\left(z_2^*\left(\phi_2,c\right),\phi_2,c\right)>0$ imply that for $z\in\left(0,z_2^*\left(\phi_2,c\right)\right)$ function $V\left(z,\phi_2,c\right)$ intersects the horizontal axis $z=0$ exactly two times. In addition to that — similarly to the cases with $\phi_2>\tilde{\tilde{\phi}}_2\left(c\right)$ and $\phi_2=\tilde{\tilde{\phi}}_2\left(c\right)$ — function $V\left(z,\phi_2,c\right)$ intersects the horizontal axis $z=0$ one more time for some $\tilde{z}>z_2^*\left(\phi_2,c\right)$. Thus, for $\phi_2<\tilde{\tilde{\phi}}_2\left(c\right)$ equation $V\left(z,\phi_2,c\right)=0$ has three solutions.

Analysis of multiplicity of solutions of $V\left(z,\phi_2,c\right)$ for $c<\tilde{c}$ is similar to the above analysis with $c>\tilde{c}$ (see Figure 16b for illustration). The difference is that for $c<\tilde{c}$ the horizontal axis $z=0$ is tangent to the local maximum of $V\left(z,\tilde{\tilde{\phi}}_2\left(c\right),c\right)$ at point $\tilde{\tilde{z}}\left(c\right)=z_2^*\left(\tilde{\tilde{\phi}}_2\left(c\right),c\right)$, and $V\left(z_2^*\left(\phi_2,c\right),\phi_2,c\right)$ is a decreasing function of ϕ_2.

The case with $c=\tilde{c}$ is special (see Figure 16c for illustration). In this case, $V\left(\overline{z}_0,\phi_2,\tilde{c}\right)=0$ for any ϕ_2. We know from Lemma 3 that for any $\phi_2\geq\tilde{\phi}_2$ equation $V\left(z,\phi_2,c\right)=0$ has a unique solution. Thus, for all $\phi_2\geq\tilde{\phi}_2$ the unique solution to $V\left(z,\phi_2,\tilde{c}\right)=0$ is $\overline{z}_0$. For $\phi_2<\tilde{\phi}_2$, we have that $z_1^*\left(\phi_2,\tilde{c}\right)<\overline{z}_0<z_2^*\left(\phi_2,\tilde{c}\right)$ and that $V_1'\left(z,\phi_2,\tilde{c}\right)<0$ for $z\in\left(0,z_1^*\left(\phi_2,\tilde{c}\right)\right)\cup\left(z_2^*\left(\phi_2,\tilde{c}\right),\infty\right)$ and $V_1'\left(z,\phi_2,\tilde{c}\right)>0$ for $z\in\left(z_1^*\left(\phi_2,\tilde{c}\right),z_2^*\left(\phi_2,\tilde{c}\right)\right)$. Therefore, $V\left(z_1^*\left(\phi_2,\tilde{c}\right),\phi_2,\tilde{c}\right)<0<V\left(z_2^*\left(\phi_2,\tilde{c}\right),\phi_2,\tilde{c}\right)$. Then, given that $V\left(0,\phi_2,\tilde{c}\right)>0$ and $\lim_{z\to\infty}V\left(z,\phi_2,\tilde{c}\right)=-\infty$, we conclude that $V\left(z,\phi_2,\tilde{c}\right)$ intersects the horizontal axis $z=0$ once for $z<z_1^*\left(\phi_2,\tilde{c}\right)$ and once for $z>z_2^*\left(\phi_2,\tilde{c}\right)$. Thus, overall, for $\phi_2<\tilde{\phi}_2$, equation $V\left(z,\phi_2,\tilde{c}\right)=0$ has three solutions (one of which is $\overline{z}_0$). □

Lemma 5 (Regular equilibria: Existence and uniqueness under costly trade and $\alpha=1$.). *Suppose that $\alpha=1$ and $\phi_1\phi_2<1$.*

(i) In any of the cases of Lemmas 3 and 4, in which the economy of Section 2 is guaranteed to have at most one regular equilibrium, this equilibrium exists only if condition (18) holds.

(ii) Suppose that $\mu<1$ and $\overline{\beta}\neq0$ and $0<\gamma<1$. Then for any fixed $c=\phi_1/\phi_2$ there exists $\overline{\phi}_2(c)\in\left(0,\tilde{\tilde{\phi}}_2\left(c\right)\right]$ such that for all $\phi_2<\overline{\phi}_2(c)$ the economy of Section 2 has a unique regular equilibrium.

Proof. In the proof of this proposition we are going to use notation introduced in Lemmas 3 and 4.

Part (i). As argued in Lemma 3, any solution $\tilde{z}$ to equation $V\left(z\right)=0$ will be a solution to the original equation (10) only if it falls within the interval $\left(\phi_1^{1+\mu},\phi_2^{-(1+\mu)}\right)$. The proofs in Lemmas 3 and 4 imply that in all cases when there is a unique solution $\tilde{z}$ to equation $V\left(z\right)=0$, we necessarily have that $V\left(z\right)>0$ for all $z<\tilde{z}$ and $V\left(z\right)<0$ for all $z>\tilde{z}$. Therefore, in all cases with the unique solution $\tilde{z}$ to $V\left(z\right)=0$, we have

$\tilde{z} \in \left(\phi_1^{1+\mu}, \phi_2^{-(1+\mu)}\right)$ if and only if $V\left(\phi_1^{1+\mu}\right) > 0 > V\left(\phi_2^{-(1+\mu)}\right)$, which, after some algebra, gives condition (18).

Part (ii). Suppose that $\mu < 1$ and $0 < \overline{\beta} < 1$ and $0 < \gamma < 1$. From Lemma 4 we know that for any fixed $c = \phi_1/\phi_2$ there exists $\tilde{\tilde{\phi}}_2(c) > 0$ such that for all $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ equation $V(z) = 0$ has three solutions. Denote these solutions as $\tilde{z}_1(\phi_2, c) < \tilde{z}_2(\phi_2, c) < \tilde{z}_3(\phi_2, c)$. Let us verify that for all low enough $\phi_2 > 0$ we have $\tilde{z}_1(\phi_2, c) < c^{1+\mu}\phi_2^{1+\mu} < \tilde{z}_2(\phi_2, c)$ and $\tilde{z}_2(\phi_2, c) < \phi_2^{-(1+\mu)} < \tilde{z}_3(\phi_2, c)$, which implies that $\tilde{z}_2(\phi_2, c)$ — and only $\tilde{z}_2(\phi_2, c)$ — is the solution of the original equation (10).

The shape of $V(z)$ for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ implies that inequalities $\tilde{z}_1(\phi_2, c) < c^{1+\mu}\phi_2^{1+\mu} < \tilde{z}_2(\phi_2, c)$ hold if $V\left(c^{1+\mu}\phi_2^{1+\mu}\right) < 0$ and $c^{1+\mu}\phi_2^{1+\mu} < \overline{z}_0$, where $\overline{z}_0$ was defined in (41). These conditions, in turn, hold if

$$G > \max\left\{c^{1-\mu}\left[\gamma\overline{\beta}\phi_2^{\mu-1} + \left(1-\gamma\overline{\beta}\right)c\phi_2^{\mu+1}\right]^{-1}, \frac{\overline{\gamma}}{\gamma}c^{1-\mu}\phi_2^{1-\mu}\right\}. \tag{43}$$

Similarly, the shape of $V(z)$ for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ implies that inequalities $\tilde{z}_2(\phi_2, c) < \phi_2^{-(1+\mu)} < \tilde{z}_3(\phi_2, c)$ hold if $V\left(\phi_2^{-(1+\mu)}\right) > 0$ and $\phi_2^{-(1+\mu)} > \overline{z}_0$. These conditions, in turn, hold if

$$G < \min\left\{\overline{\gamma}\overline{\beta}\phi_2^{\mu-1} + \left(1-\overline{\gamma}\overline{\beta}\right)c\phi_2^{\mu+1}, \frac{\overline{\gamma}}{\gamma}\phi_2^{\mu-1}\right\}. \tag{44}$$

Observe that as $\phi_2 \to 0$, the right-hand side of inequality (43) goes to 0, while the right-hand side of inequality (44) goes to ∞. Thus, for all low enough ϕ_2 inequalities (43) and (44) hold. □

Now we can combine Lemmas 3-5 to get a proof of Proposition 6. Lemma 3 and part (i) of Lemma 5 together immediately imply part (i) of Proposition 6, while Lemma 4 and part (ii) of Lemma 5 together imply part (ii) of Proposition 6.

B Proofs for Section 4

B.1 Proof Part (ii) of Proposition 7

In the case with $\beta = 1$, expressions (12) for d_1 and d_2 collapse to $d_1 = \phi_1$ and $d_2 = \phi_2$, and equation (10) becomes $x^{\alpha-1} = G \cdot (\phi_1/\phi_2)^{-\alpha}\left[g_\phi(x)\right]^{\alpha+\mu}$. Taking logarithm from both sides and bringing all terms to one side, we get equation $V(x) = 0$, where

$$V(x) \equiv (1-\alpha)\ln x + \ln G - \alpha \ln \frac{\phi_1}{\phi_2} + (\alpha+\mu)\ln \frac{1+\phi_1 x}{\phi_2 + x}.$$

We have

$$V'(x) = \frac{(1-\alpha)(\phi_2+x)(1+\phi_1 x) - (\alpha+\mu)(1-\phi_1\phi_2)x}{(\phi_2+x)(1+\phi_1 x)x}.$$

Thus, $V'(x) \geq 0$ if and only if $W(x) \geq 0$, where

$$W(x) \equiv (1-\alpha)\phi_1 x^2 + [(\mu+1)\phi_1\phi_2 - (\mu+2\alpha-1)]x + (1-\alpha)\phi_2.$$

Observe that $W(x)$ is a quadratic polynomial in x. Denoting by $\widetilde{D}$ the discriminant of equation $W(x) = 0$, we can write

$$\widetilde{D} \equiv (\mu+1)^2(1-\phi_1\phi_2)\left(\tilde{\mu}^2 - \phi_1\phi_2\right),$$

where

$$\tilde{\mu} \equiv \frac{\mu+2\alpha-1}{\mu+1}.$$

Observe that $\tilde{\mu} < 1$ for $0 < \alpha < 1$. If $\tilde{\mu} < \phi_1\phi_2 < 1$, then $W(x) > 0$ for all x and so $V(x)$ is an increasing function of x for all $x > 0$. Then, given that $\lim_{x\to 0} V(x) = -\infty$ and $\lim_{x\to\infty} V(x) = \infty$, we get that for $\tilde{\mu} < \phi_1\phi_2 < 1$ function $V(x)$ intersects the horizontal axis $x = 0$ once and only once.

The case with $\phi_1\phi_2 = \tilde{\mu}$ is similar to the case with $\tilde{\mu} < \phi_1\phi_2 < 1$ with the only difference that $V(x)$ is an increasing function for all $x > 0$ except for one $x = x^*$ such that $V'(x^*) = 0$. In this case $V(x)$ also intersects the horizontal axis once and only once.

Consider the case with $\phi_1\phi_2 < \tilde{\mu}$. In this case equation $W(x) = 0$ has two distinct positive solutions

$$x_1^* = \frac{(\mu+1)(\tilde{\mu}-\phi_1\phi_2) - \sqrt{\widetilde{D}}}{2(1-\alpha)\phi_1} \quad \text{and} \quad x_2^* = \frac{(\mu+1)(\tilde{\mu}-\phi_1\phi_2) + \sqrt{\widetilde{D}}}{2(1-\alpha)\phi_1}.$$

We have that $W(x) > 0$ for $x < x_1^*$ and $x > x_2^*$, and $W(x) < 0$ for $x_1^* < x < x_2^*$. Thus, function $V(x)$ is increasing for $x < x_1^*$ and $x > x_2^*$, and $V(x)$ is decreasing for $x_1^* < x < x_2^*$. Given that $\lim_{x\to 0} V(x) = -\infty$ and $\lim_{x\to\infty} V(x) = \infty$, we have that if $V(x_1^*) < 0$ or $V(x_2^*) > 0$ then function $V(x)$ intersects the horizontal axis only once.

Denote

$$\Gamma_1 \equiv \frac{G(\mu+1)^{\mu+1}\left[\phi_1\phi_2^{\mu}\right]^{-1}}{2^{\mu+1}(1-\alpha)^{1-\alpha}(\mu+\alpha)^{\mu+\alpha}}\left(\tilde{\mu}-\phi_1\phi_2-D\right)^{1-\alpha}\left(\tilde{\mu}+\phi_1\phi_2+D\right)^{\mu+\alpha},$$
$$\Gamma_2 \equiv \frac{G(\mu+1)^{\mu+1}\left[\phi_1\phi_2^{\mu}\right]^{-1}}{2^{\mu+1}(1-\alpha)^{1-\alpha}(\mu+\alpha)^{\mu+\alpha}}\left(\tilde{\mu}-\phi_1\phi_2+D\right)^{1-\alpha}\left(\tilde{\mu}+\phi_1\phi_2-D\right)^{\mu+\alpha},$$

where

$$D \equiv \sqrt{(1-\phi_1\phi_2)\left(\tilde{\mu}^2-\phi_1\phi_2\right)}.$$

Then condition $V(x_1^*) < 0$ can be written as $\Gamma_1 < 1$, while condition $V(x_2^*) > 0$ can be written as $\Gamma_2 > 1$.

If $V(x_1^*) = 0$ then the horizontal axis is tangent to function $V(x)$ at x_1^* and also $V(x)$ intersects the horizontal axis at some point $x > x_1^*$. Thus, in this case equation $V(x) = 0$ has two solutions. Similarly, if $V(x_2^*) = 0$ then the horizontal axis is tangent to function $V(x)$ at x_2^* and also $V(x)$ intersects the horizontal axis at some point $x < x_1^*$, and in this case equation $V(x) = 0$ also has two solutions. Condition $V(x_1^*) = 0$ can be written as $\Gamma_1 = 1$, while condition $V(x_2^*) = 0$ can be written as $\Gamma_2 = 1$.

Finally, if $V(x_1^*) > 0$ and $V(x_2^*) < 0$, which can be written as $\Gamma_1 > 1$ and $\Gamma_2 < 1$, then function $V(x)$ intersects the horizontal axis three times.

B.2 Proof of Proposition 8

In the case with $\beta = 1$, expressions (12) for d_1 and d_2 collapse to $d_1 = \phi_1$ and $d_2 = \phi_2$, and equation (10) becomes $x^{\alpha-1} = G \cdot (\phi_1/\phi_2)^{-\alpha}\left[g_\phi(x)\right]^{\alpha+\mu}$. Introducing the change of variables $z = x^{1-\frac{1-\alpha}{\alpha+\mu}}$ and after doing some algebra, we get equation $V(z,\phi_2,c) = 0$, where

$$V(z,\phi_2,c) \equiv \phi_2 z^{1-\frac{\alpha+\mu}{2\alpha+\mu-1}} - \phi_2 c^{\frac{\mu}{\mu+\alpha}}\tilde{G}z^{\frac{\alpha+\mu}{2\alpha+\mu-1}} + z - \tilde{G}c^{-\frac{\alpha}{\mu+\alpha}},$$

and $\tilde{G} \equiv G^{\frac{1}{\mu+\alpha}}$, and $c \equiv \phi_1/\phi_2$. Observe that $1 - \frac{1-\alpha}{\alpha+\mu} = \frac{2\alpha+\mu-1}{\alpha+\mu} = (2+1/\varepsilon)\frac{\alpha}{\alpha+\mu} > 0$ and so the change of variables $z = x^{1-\frac{\mu-\alpha\mu}{\alpha\mu+1}}$ is well-defined. The first and the second derivatives of $V(z,\phi_2,c)$ with respect to z are given by

$$V_1'(z,\phi_2,c) = -\frac{1-\alpha}{2\alpha+\mu-1}\phi_2 z^{-\frac{\alpha+\mu}{2\alpha+\mu-1}} - \frac{\alpha+\mu}{2\alpha+\mu-1}\phi_2 c^{\frac{\mu}{\mu+\alpha}}\tilde{G}z^{\frac{1-\alpha}{2\alpha+\mu-1}} + 1,$$
$$V_1''(z,\phi_2,c) = \frac{(1-\alpha)(\alpha+\mu)}{(2\alpha+\mu-1)^2}\phi_2 z^{\frac{1-\alpha}{2\alpha+\mu-1}-1}\left(z^{-\frac{\mu+1}{2\alpha+\mu-1}} - \left[\bar{z}_0(c)\right]^{-\frac{\mu+1}{2\alpha+\mu-1}}\right),$$

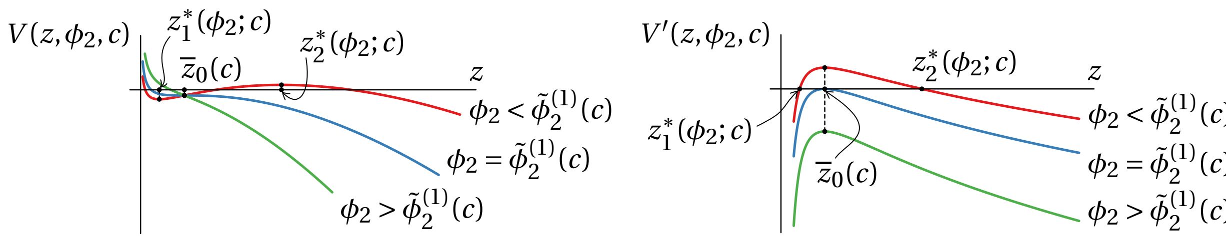


Figure 17: No agricultural sector, $0 < \alpha < 1$: V and V'

where

$$\overline{z}_0(c) \equiv \left(c^{\frac{\mu}{\alpha+\mu}} \tilde{G}\right)^{-\frac{2\alpha+\mu-1}{\mu+1}}. \tag{45}$$

From here we see that $V_1''(z,\phi_2,c) > 0$ if and only if $z < \overline{z}_0(c)$. Thus, $V_1'(z,\phi_2,c)$ is a concave function in z that achieves its maximum at $\overline{z}_0(c)$. Evaluating $V_1'(\overline{z}_0(c),\phi_2,c)$, we get

$$V_1'(\overline{z}_0(c),\phi_2,c) = -\frac{\mu+1}{2\alpha+\mu-1}\phi_2 c^{\frac{\mu}{\mu+1}} \tilde{G}^{\frac{\alpha+\mu}{\mu+1}} + 1,$$

and so $V_1'(\overline{z}_0(c),\phi_2,c) < 0$ if and only if $\phi_2 > \tilde{\phi}_2^{(1)}(c)$, where

$$\tilde{\phi}_2^{(1)}(c) \equiv \frac{2\alpha+\mu-1}{\mu+1} c^{-\frac{\mu}{\mu+1}} \tilde{G}^{-\frac{\alpha+\mu}{\mu+1}}.$$

If $\phi_2 > \tilde{\phi}_2^{(1)}(c)$, then $V_1'(\overline{z}_0(c),\phi_2,c) < 0$ and, hence, $V_1'(z,\phi_2,c) < 0$ for all z, and so function $V(z,\phi_2,c)$ is decreasing in z (see Figure 17 for illustration). Then, given that $\lim_{z\to 0} V(z,\phi_2,c) = \infty$ and $\lim_{z\to\infty} V(z,\phi_2,c) = -\infty$, we conclude that function $V(z,\phi_2,c)$ intersects the horizontal axis $z = 0$ once and only once for some $\tilde{z} > 0$. The case with $\phi_2 = \tilde{\phi}_2^{(1)}(c)$ is similar to the case with $\phi_2 > \tilde{\phi}_2^{(1)}(c)$ with the only difference that function $V\left(z,\tilde{\phi}_2^{(1)}(c),c\right)$ is decreasing for all z except for $z = \overline{z}_0(c)$. In this case as well $V(z,\phi_2,c)$ intersects the horizontal axis $z = 0$ only once for some $\tilde{z} > 0$.

Let us now write $V(z,\phi_2,c)$ as $V(z,\phi_2,c) = -c^{-\frac{\alpha}{\mu+\alpha}} \tilde{G} z \tilde{V}\left(z^{-1},\phi_1,c\right)$, where

$$\tilde{V}(z,\phi_1,c) \equiv \phi_1 z^{1-\frac{\alpha+\mu}{2\alpha+\mu-1}} - \phi_1 c^{-\frac{\mu}{\mu+\alpha}} \tilde{G}^{-1} z^{\frac{\alpha+\mu}{2\alpha+\mu-1}} + z - \tilde{G}^{-1} c^{\frac{\alpha}{\mu+\alpha}}.$$

Clearly, functions $V(z,\phi_2,c)$ and $\tilde{V}(z,\phi_1,c)$ have the same number of zeros for $z > 0$. Moreover, function $\tilde{V}(z,\phi_1,c)$ is similar to function $V(z,\phi_2,c)$ with the difference that ϕ_1 is swapped with ϕ_2, c^{-1} is swapped with c and $\tilde{G}^{-1}$ is swapped with $\tilde{G}$. Repeating for $\tilde{V}(z,\phi_1,c)$ the same analysis as for $V(z,\phi_2,c)$ above, we get that if $\phi_1 \geq \tilde{\phi}_1^{(2)}(c)$, where

$$\tilde{\phi}_1^{(2)}(c) \equiv \frac{2\alpha+\mu-1}{\mu+1} c^{\frac{\mu}{\mu+1}} \tilde{G}^{\frac{\alpha+\mu}{\mu+1}},$$

then $\tilde{V}(z,\phi_1,c)$ has a unique solution. Having $\phi_1 \geq \tilde{\phi}_1^{(2)}(c)$ for a particular c is equivalent to having $\phi_2 \geq \tilde{\phi}_2^{(2)}(c)$ for this c, where

$$\tilde{\phi}_2^{(2)}(c) \equiv c^{-1}\tilde{\phi}_1^{(2)}(c) = \frac{2\alpha+\mu-1}{\mu+1}c^{-\frac{1}{\mu+1}}\tilde{G}^{\frac{\alpha+\mu}{\mu+1}}.$$

Thus, we get that $V(z,\phi_2,c) = 0$ has a unique solution if $\phi_2 \geq \tilde{\phi}_2^{(1)}(c)$ or $\phi_2 \geq \tilde{\phi}_2^{(2)}(c)$. This is equivalent to

$$\phi_2 \geq \tilde{\phi}_2(c) \equiv \min\left\{\tilde{\phi}_2^{(1)}(c), \tilde{\phi}_2^{(2)}(c)\right\} = \frac{2\alpha+\mu-1}{\mu+1}\min\left\{[c^{\mu}G]^{-\frac{1}{\mu+1}}, \left[cG^{-1}\right]^{-\frac{1}{\mu+1}}\right\},$$

where we used the definition of $\tilde{G} = G^{\frac{1}{\alpha+\mu}}$. This proves part (a) of Proposition 8.

For later use, note that $\tilde{V}(z,\phi_1,c)$ is a decreasing function of z if $\phi_1 > \tilde{\phi}_1^{(2)}(c)$ or, equivalently, if $\phi_2 > \tilde{\phi}_2^{(2)}(c)$. This means that $\tilde{V}\left(z^{-1},\phi_1,c\right)$ is an increasing function of z if $\phi_2 > \tilde{\phi}_2^{(2)}(c)$ and, since $V(z,\phi_2,c) = -c^{-\frac{\alpha}{\mu+\alpha}}\tilde{G}z\tilde{V}\left(z^{-1},\phi_1,c\right)$, we have that $V(z,\phi_2,c)$ is a decreasing function of z if $\phi_2 > \tilde{\phi}_2^{(2)}(c)$. Earlier we argued that $V(z,\phi_2,c)$ is a decreasing function of z if $\phi_2 > \tilde{\phi}_2^{(1)}(c)$. This means that $V(z,\phi_2,c)$ is a decreasing function of z if $\phi_2 > \tilde{\phi}_2(c)$.

For the rest of this proof, we focus on the case with $\phi_2 < \tilde{\phi}_2(c)$ and prove part (b) of Proposition 8. We divide this proof into two steps.

STEP 1. We first prove that the system of equations $V(z,\phi_2,c) = 0$ and $V_1'(z,\phi_2,c) = 0$ in z and ϕ_2 has a unique solution $\tilde{\tilde{z}}(c) > 0$ and $0 < \tilde{\tilde{\phi}}_2(c) \leq \tilde{\phi}_2(c)$.

Solving for ϕ_2 from equation $V_1'(z,\phi_2,c) = 0$, substituting the result into equation $V(z,\phi_2,c) = 0$, and after doing some algebra, we get equation $H(z,c) = 0$, where

$$H(z,c) \equiv \tilde{G}^{-1}z^{-\frac{2(1-\alpha)}{2\alpha+\mu-1}} - \frac{1-\alpha}{\alpha+\mu}c^{-\frac{\alpha}{\alpha+\mu}}z^{-\frac{\mu+1}{2\alpha+\mu-1}} + \frac{1-\alpha}{\alpha+\mu}c^{\frac{\mu}{\alpha+\mu}}z - c^{\frac{\mu-\alpha}{\alpha+\mu}}\tilde{G}.$$

We are going to show that for any $c > 0$ there is a unique $\tilde{\tilde{z}}(c) > 0$ such that $H(\tilde{\tilde{z}}(c),c) = 0$.

The first and the second derivatives of $H(z,c)$ with respect to z are given by

$$H_1'(z,c) = \frac{1-\alpha}{2\alpha+\mu-1}\left(-2\tilde{G}^{-1}z^{-\frac{\mu+1}{2\alpha+\mu-1}} + \frac{\mu+1}{\alpha+\mu}c^{-\frac{\alpha}{\alpha+\mu}}z^{-\frac{2(\alpha+\mu)}{2\alpha+\mu-1}}\right) + \frac{1-\alpha}{\alpha+\mu}c^{\frac{\mu}{\alpha+\mu}},$$
$$H_1''(z,c) = \frac{2(1-\alpha)(\mu+1)}{(2\alpha+\mu-1)^2}\tilde{G}^{-1}z^{-\frac{2(\alpha+\mu)}{2\alpha+\mu-1}-1}\left(z-\bar{\bar{z}}_0(c)\right).$$

where $\bar{\bar{z}}_0(c) \equiv \tilde{G}c^{-\frac{\alpha}{\alpha+\mu}}$. From here we see that $H_1''(z,c) > 0$ if and only if $z > \bar{\bar{z}}_0(c)$. Thus,

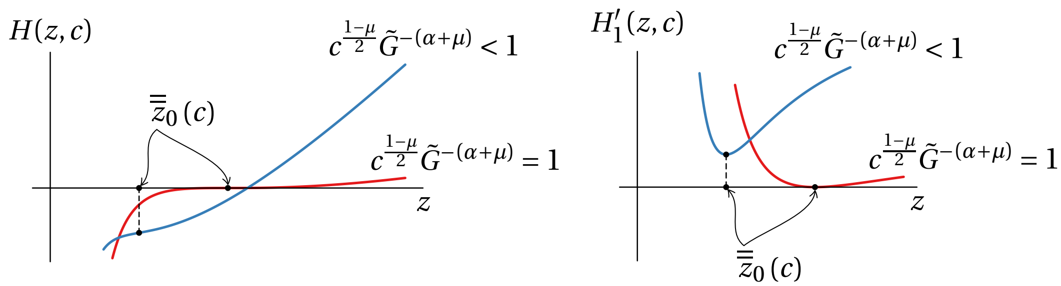


Figure 18: No agricultural sector, $0 < \alpha < 1$: H and H'

for any $c > 0$, $H_1'(z,c)$ is a convex function in z, and $\bar{\bar{z}}_0(c)$ is its minimum (see Figure 18 for illustration).

Evaluating $H_1'\left(\bar{\bar{z}}_0(c), c\right)$, we get

$$H_1'\left(\bar{\bar{z}}_0(c), c\right) = -\frac{1-\alpha}{\alpha+\mu} c^{\frac{\mu}{\alpha+\mu}} \left(\left(c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} \right)^{\frac{2}{2\alpha+\mu-1}} - 1 \right),$$

and so $H_1'\left(\bar{\bar{z}}_0(c), c\right) > 0$ if and only if $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} < 1$. From here we immediately see that if $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} < 1$, then $H_1'(z,c) > 0$ for all z, and so $H(z,c)$ increases in z. Then, given that $\lim_{z\to 0} H(z,c) = -\infty$ and $\lim_{z\to\infty} H(z,c) = \infty$, we conclude that if $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} < 1$ then $H(z,c)$ intersects the horizontal axis $z = 0$ once and only once at some $\tilde{\tilde{z}}(c) > 0$.

The case with $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} = 1$ is similar to the case with $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} < 1$ with the only difference that $H(z,c)$ is an increasing function for all $z \neq \bar{\bar{z}}_0(c)$ and $H\left(\bar{\bar{z}}_0(c), c\right) = 0$, which means that $H(z,c)$ intersects the horizontal axis $z = 0$ only once at $\bar{\bar{z}}_0(c)$.

Now consider the case with $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} > 1$. We can write

$$H(z,c) = -c^{\frac{\mu-\alpha}{\alpha+\mu}} z^{-\frac{2(1-\alpha)}{2\alpha+\mu-1}} \tilde{H}\left(z^{-1}, c\right),$$

where

$$\tilde{H}(z,c) \equiv \tilde{G} z^{-\frac{2(1-\alpha)}{2\alpha+\mu-1}} - \frac{1-\alpha}{\alpha+\mu} c^{\frac{\alpha}{\alpha+\mu}} z^{-\frac{\mu+1}{2\alpha+\mu-1}} + \frac{1-\alpha}{\alpha+\mu} c^{-\frac{\mu}{\alpha+\mu}} z - c^{-\frac{\mu-\alpha}{\alpha+\mu}} \tilde{G}^{-1}.$$

Obviously, functions $H(z,c)$ and $\tilde{H}(z,c)$ have the same number of zeros for $z > 0$. Observe that function $\tilde{H}(z,c)$ is similar to function $H(z,c)$ with the difference that $\tilde{G}$ is swapped with $\tilde{G}^{-1}$, and c is swapped with c^{-1}. Applying the same analysis to function $\tilde{H}(z,c)$ as to function $H(z,c)$, we get that $\tilde{H}(z,c)$ increases in z if $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} > 1$. Then, given that $\lim_{z\to 0} \tilde{H}(z,c) = -\infty$ and $\lim_{z\to\infty} \tilde{H}(z,c) = \infty$, we conclude that $\tilde{H}(z,c)$ intersects the horizontal axis $z = 0$ once and only once for some $\tilde{\tilde{z}}(c) > 0$. The corresponding unique solution to equation $H(z,c) = 0$ is $[\tilde{\tilde{z}}(c)]^{-1}$.

At this point, we have established that for any $c > 0$ there is a unique $\tilde{\tilde{z}}(c) > 0$ such that $H(\tilde{\tilde{z}}(c), c) = 0$. This, of course, means that our original system $V(z,\phi_2,c) = 0$ and $V_1'(z,\phi_2,c) = 0$ has a unique solution $\left(\tilde{\tilde{z}}(c), \tilde{\tilde{\phi}}_2(c)\right)$ with $\tilde{\tilde{z}}(c) > 0$, where we use equation $V_1'(z,\phi_2,c) = 0$ to find $\tilde{\tilde{\phi}}_2(c)$ corresponding to $\tilde{\tilde{z}}(c)$, which gives

$$\tilde{\tilde{\phi}}_2(c) = \left(\frac{1-\alpha}{2\alpha+\mu-1}\left[\tilde{\tilde{z}}(c)\right]^{-\frac{\alpha+\mu}{2\alpha+\mu-1}} + \frac{\alpha+\mu}{2\alpha+\mu-1}c^{\frac{\mu}{\alpha+\mu}}\tilde{G}_2\left[\tilde{\tilde{z}}(c)\right]^{\frac{1-\alpha}{2\alpha+\mu-1}}\right)^{-1}.$$

Obviously, $\tilde{\tilde{\phi}}_2(c) > 0$, while the upper bound $\tilde{\tilde{\phi}}_2(c) \leq \tilde{\phi}_2(c)$ simply follows from the fact that $V_1'\left(\tilde{\tilde{z}}(c), \tilde{\tilde{\phi}}_2(c), c\right) = 0$, and we know that $V_1'(z,\phi_2,c) < 0$ for all $z > 0$ and $\phi_2 > \tilde{\phi}_2(c)$.[12]

STEP 2. We are now going to prove the statement of Proposition 8 about the existence of a unique $\tilde{\tilde{\phi}}_2(c) > 0$ that traces the uniqueness boundary.

Consider $V_1'(z,\phi_2,c)$ for any $c > 0$ and $\phi_2 < \tilde{\phi}_2(c)$. We know that $V_1'(\bar{z}_0,\phi_2,c) > 0$ for $\phi_2 < \tilde{\phi}_2(c)$, where $\bar{z}_0(c)$ was defined in (45) and is a maximum of $V_1'(z,\phi_2,c)$. This implies that $V_1'(z,\phi_2,c)$ intersects the horizontal axis $z = 0$ at exactly two points: one lower than $\bar{z}_0(c)$ and one larger than $\bar{z}_0(c)$. This fact allows us to define functions

$$z_1^*(\phi_2;c) \equiv \left\{z \middle| z \leq \bar{z}_0(c) \text{ and } V_1'(z,\phi_2,c) = 0\right\},$$
$$z_2^*(\phi_2;c) \equiv \left\{z \middle| z \geq \bar{z}_0(c) \text{ and } V_1'(z,\phi_2,c) = 0\right\},$$

both with domain $\phi_2 < \tilde{\phi}_2(c)$ and parameterized by $c > 0$. Definitions of $z_1^*(\phi_2;c)$ and $z_2^*(\phi_2;c)$ imply that $z_1^*(\phi_2;c) < \bar{z}_0(c) < z_2^*(\phi_2;c)$. Moreover, we have that $V_1'(z,\phi_2,c) < 0$ for $z \in (0, z_1^*(\phi_2;c)) \cup (z_2^*(\phi_2;c), \infty)$ and $V_1'(z,\phi_2,c) > 0$ for $z \in (z_1^*(\phi_2;c), z_2^*(\phi_2;c))$. Therefore, $z_1^*(\phi_2;c)$ is a local minimum of $V(z,\phi_2,c)$ and $z_2^*(\phi_2;c)$ is a local maximum of $V(z,\phi_2,c)$, and $V(z_1^*(\phi_2;c),\phi_2,c) < V(z_2^*(\phi_2;c),\phi_2,c)$.

We have shown in Step 1 that there exists a unique solution $\left(\tilde{\tilde{z}}(c), \tilde{\tilde{\phi}}_2(c)\right)$ to the system of equations $V(z,\phi_2,c) = 0$ and $V_1'(z,\phi_2,c) = 0$. Moreover, the argument in Step 1 implies that $H(z,c) < 0$ if and only if $z < \tilde{\tilde{z}}(c)$. Simple algebra reveals that

$$H(\bar{z}_0(c), c) = \frac{\mu+1}{\alpha+\mu}c^{\frac{\mu-\alpha}{\alpha+\mu}}\tilde{G}\left(\left(c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)}\right)^{\frac{2}{\mu+1}} - 1\right),$$

and, thus, $H(\bar{z}_0(c), c) < 0$ if and only if $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} < 1$. Therefore, $\tilde{\tilde{z}}(c) > \bar{z}_0(c)$ if and

[12] As we have argued above, $V(z,\phi_2,c)$ is a decreasing function of z if $\phi_2 > \tilde{\phi}_2(c)$.

only if $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} < 1$. This, in turn, implies that $\tilde{\tilde{z}}(c) = z_1^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$ if $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} > 1$ and $\tilde{\tilde{z}}(c) = z_2^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$ if $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} < 1$, while if $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} = 1$ then $\tilde{\tilde{z}}(c) = \overline{z}_0(c)$ and $\tilde{\tilde{\phi}}_2(c) = \tilde{\phi}_2(c)$.

Next, using the fact that $V_1'\left(z_i^*(\phi_2;c),\phi_2,c\right) = 0$, we find that

$$\frac{dV\left(z_i^*(\phi_2;c),\phi_2,c\right)}{d\phi_2} = \left[z_i^*(\phi_2;c)\right]^{\frac{\alpha+\mu}{2\alpha+\mu-1}}\left(\left[z_i^*(\phi_2;c)\right]^{-\frac{\mu+1}{2\alpha+\mu-1}} - \left[\overline{z}_0(c)\right]^{-\frac{\mu+1}{2\alpha+\mu-1}}\right),$$

and, thus, $dV\left(z_i^*(\phi_2;c),\phi_2,c\right)/d\phi_2 > 0$ if and only if $z_i^*(\phi_2;c) < \overline{z}_0(c)$. This implies that $V(z_1^*(\phi_2;c),\phi_2,c)$ is increasing in ϕ_2, and $V(z_2^*(\phi_2;c),\phi_2,c)$ is decreasing in ϕ_2.

We are now ready to bring all facts together to characterize multiplicity of solutions of equation $V(z,\phi_2,c) = 0$. Fix any $c > 0$ such that $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} > 1$ and consider function $V(z,\phi_2,c)$ as we change ϕ_2. For $\phi_2 = \tilde{\tilde{\phi}}_2(c)$, the horizontal axis $z = 0$ is tangent to the local minimum of $V(z,\phi_2,c)$ at point $\tilde{\tilde{z}}(c) = z_1^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$. Thus, $V\left(z_1^*\left(\tilde{\tilde{\phi}}_2(c);c\right),\tilde{\tilde{\phi}}_2(c),c\right) = 0$ and for all points $z \in \left(0,z_2^*\left(\tilde{\tilde{\phi}}_2(c);c\right)\right)$ different from $z_1^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$ we have $V\left(z,\tilde{\tilde{\phi}}_2(c),c\right) > 0$. For $z > z_2^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$, function $V\left(z,\tilde{\tilde{\phi}}_2(c),c\right)$ monotonically decreases from a positive value to $-\infty$ as $z \to \infty$. This implies that function $V\left(z,\tilde{\tilde{\phi}}_2(c),c\right)$ crosses the horizontal axis $z = 0$ only once for some $\tilde{z} > z_2^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$. Thus, for $\phi_2 = \tilde{\tilde{\phi}}_2(c)$ there are two solutions to equation $V(z,\phi_2,c) = 0$: $z_1^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$ and $\tilde{z}$.

Next, as we have argued above, $V(z_1^*(\phi_2;c),\phi_2,c)$ is increasing in ϕ_2. Therefore, for $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ we have $V(z_1^*(\phi_2;c),\phi_2,c) > 0$, which implies that $V(z,\phi_2,c) > 0$ for all $z \in (0,z_2^*(\phi_2;c))$. And for $z > z_2^*(\phi_2;c)$, again, function $V(z,\phi_2,c)$ intersects the horizontal axis $z = 0$ once and only once. Thus, for $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ there is a unique solution to equation $V(z,\phi_2,c) = 0$.

Finally, for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ we have $V(z_1^*(\phi_2;c),\phi_2,c) < 0$. At the same time, we necessarily have $V(z_2^*(\phi_2;c),\phi_2,c) > 0$, because $V(z_2^*(\phi_2;c),\phi_2,c)$ is a decreasing function of ϕ_2 and $V\left(z_2^*\left(\tilde{\tilde{\phi}}_2(c);c\right),\tilde{\tilde{\phi}}_2(c),c\right) > 0$. Then, the facts that $V(0,\phi_2,c) > 0$ and that for any $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ we have $V(z_1^*(\phi_2;c),\phi_2,c) < 0$ and $V(z_2^*(\phi_2;c),\phi_2,c) > 0$ imply that for $z \in (0,z_2^*(\phi_2;c))$ function $V(z,\phi_2,c)$ intersects the horizontal axis $z = 0$ exactly two times. In addition to that, as in the cases with $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ and $\phi_2 = \tilde{\tilde{\phi}}_2(c)$, function $V(z,\phi_2,c)$ intersects the horizontal axis $z = 0$ one more time for some $\tilde{z} > z_2^*(\phi_2;c)$. Thus, for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ equation $V(z,\phi_2,c) = 0$ has three solutions.

Analysis of multiplicity of solutions of $V(z,\phi_2,c)$ for $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} < 1$ is similar to the above analysis with $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} > 1$. The difference is that for $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} < 1$

the horizontal axis $z = 0$ is tangent to the local maximum of $V\left(z, \tilde{\tilde{\phi}}_2(c), c\right)$ at point $\tilde{\tilde{z}}(c) = z_2^*\left(\tilde{\tilde{\phi}}_2(c); c\right)$.

The case with $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} = 1$ is special. In this case $\tilde{\tilde{\phi}}_2(c) = \tilde{\phi}_2(c)$. To see this, observe that $V(\bar{z}_0(c), \phi_2, c) = 0$ for any ϕ_2. We know that for any $\phi_2 \geq \tilde{\phi}_2(c)$ equation $V(z, \phi_2, c) = 0$ has a unique solution. Thus, for all $\phi_2 \geq \tilde{\phi}_2(c)$ the unique solution to $V(z, \phi_2, c) = 0$ is $\bar{z}_0(c)$. For $\phi_2 < \tilde{\phi}_2(c)$, we have that $z_1^*(\phi_2; c) < \bar{z}_0(c) < z_2^*(\phi_2; c)$ and that $V_1'(z, \phi_2, c) < 0$ for $z \in (0, z_1^*(\phi_2; c)) \cup (z_2^*(\phi_2; c), \infty)$ and $V_1'(z, \phi_2, c) > 0$ for $z \in (z_1^*(\phi_2; c), z_2^*(\phi_2; c))$. Therefore, $V(z_1^*(\phi_2; c), \phi_2, c) < 0 < V(z_2^*(\phi_2; c), \phi_2, c)$. Then, given that $V(0, \phi_2, c) > 0$ and $\lim_{z\to\infty} V(z, \phi_2, c) = -\infty$, we conclude that $V(z, \phi_2, c)$ intersects the horizontal axis $z = 0$ once for $z < z_1^*(\phi_2; c)$ and once for $z > z_2^*(\phi_2; c)$. Thus, overall, for $\phi_2 < \tilde{\phi}_2(c)$, equation $V(z, \phi_2, c) = 0$ has three solutions (one of which is $\bar{z}_0(c)$).

C Exhaustive Analysis of Regular Equilibria in the Krugman Case

Taking logarithms from both sides of equation (10) in Lemma 1 for $\alpha = 1$ and collecting all terms on one side, we get equation $V(x) = 0$, where

$$V(x) \equiv -\ln G + \ln\left(\frac{\phi_1}{\phi_2}\right) - \mu \ln \frac{1 + \phi_1 x}{\phi_2 + x} - \ln \frac{1 + d_1 x}{d_2 + x}.$$

Taking the first derivative, we get

$$V'(x) = \frac{\mu(1 - \phi_1\phi_2)}{(1 + \phi_1 x)(\phi_2 + x)} + \frac{1 - d_1 d_2}{(1 + d_1 x)(d_2 + x)}$$

We have $V'(x) = 0$ if and only if $W(x) = 0$, where $W(x) \equiv Ax^2 + Bx + C$ with

$$\begin{aligned}
A &\equiv \mu(1 - \phi_1\phi_2) d_1 + \phi_1(1 - d_1 d_2), \\
B &\equiv \mu(1 - \phi_1\phi_2)(1 + d_1 d_2) + (1 - d_1 d_2)(1 + \phi_1\phi_2), \\
C &\equiv \mu(1 - \phi_1\phi_2) d_2 + (1 - d_1 d_2)\phi_2.
\end{aligned}$$

Case 1. If $d_1 d_2 \leq 1$, then $W(x) > 0$ for all $x > 0$ and thus $V(x)$ is an increasing function. Hence, in this case there exists at most one regular equilibrium. This equilibrium exists if and only if $V(0) < 0 < V(\infty)$.

Case 2. Suppose that $d_1 d_2 > 1$.

Case 2.1. Suppose that $A > 0$ and $C > 0$. In this case $W(x)$ is a convex function that achieves its minimum at $x^* = -\frac{B}{2A}$.

Case 2.1.1. If $B^2 \leq 4AC$ then $W(x) > 0$ for all x except for, maybe, $x = -B/(2A)$. In this case $V(x)$ is increasing. As in Case 1, there exists a unique regular equilibrium if $V(0) < 0 < V(\infty)$, otherwise there are no regular equilibria.

Case 2.1.2. If $B > 2\sqrt{AC}$, then $W(x) < 0$ if and only if $x \in (x_1^*, x_2^*)$, where

$$x_1^* = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \quad \text{and} \quad x_2^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

Having $B > 2\sqrt{AC}$ implies that $x_1^* < x_2^* < 0$ and that $W(x) > 0$ for $x > 0$. Hence, again, there exists a unique regular equilibrium if $V(0) < 0 < V(\infty)$, otherwise there are no regular equilibria.

Case 2.1.3. Suppose that $B < -2\sqrt{AC}$. Then $0 < x_1^* < x_2^*$ and, therefore, $V(x)$ is increasing for $x < x_1^*$ and $x > x_2^*$, and $V(x)$ is decreasing for $x \in (x_1^*, x_2^*)$.

Case 2.1.3.1. Suppose that $V(0) < 0 < V(\infty)$. If $V(x_1^*) < 0$ then there is a unique equilibrium. If $V(x_1^*) = 0$, then there are two equilibria. And if $V(x_1^*) > 0$, then there are three equilibria.

Case 2.1.3.2. Suppose that $V(\infty) \leq 0 \leq V(0)$. Then there is a unique equilibrium.

Case 2.1.3.3. Suppose that $V(0) < 0$ and $V(\infty) \leq 0$. If $V(x_1^*) < 0$ then there are no regular equilibria. If $V(x_1^*) = 0$, then there is a unique regular equilibrium. If $V(x_1^*) > 0$ then there are two regular equilibria.

Case 2.1.3.4. Suppose that $V(0) \geq 0$ and $V(\infty) > 0$. If $V(x_2^*) > 0$ then there are no regular equilibria. If $V(x_2^*) = 0$, then there is a unique regular equilibrium. If $V(x_2^*) < 0$ then there are two regular equilibria.

Case 2.2. Suppose that $A > 0$ and $C \leq 0$. In this case $x_1^* \leq 0 \leq x_2^*$. Then $W(x) < 0$ for $0 < x < x_2^*$ and $W(x) > 0$ for $x > x_2^*$. Thus, x_2^* is a global minimum of $V(x)$.

Case 2.2.1. If $V(0) \leq 0 < V(\infty)$, then there is a unique equilibrium.

Case 2.2.2. If $V(\infty) \leq 0 < V(0)$, then there is a unique equilibrium.

Case 2.2.3. If $V(0) \leq 0$ and $V(\infty) \leq 0$, then there are no regular equilibria.

Case 2.2.4. Suppose that $V(0) > 0$ and $V(\infty) > 0$. If $V(x_2^*) > 0$ then there are no regular equilibria. If $V(x_2^*) = 0$, then there is a unique regular equilibrium. If $V(x_2^*) < 0$ then there are two regular equilibria.

Case 2.3: Suppose that $A < 0$ and $C < 0$. In this case $W(x)$ is a concave function that achieves its maximum at $x^* = -\frac{B}{2A}$.

Case 2.3.1. If $B^2 \leq 4AC$ then $W(x) < 0$ for all x except for, maybe, $x = -B/(2A)$. In this case $V(x)$ is decreasing. There exists a unique regular equilibrium if $V(\infty) < 0 < V(0)$, otherwise there are no regular equilibria.

Case 2.3.2. If $B > 2\sqrt{AC}$, then $x_1^* < x_2^* < 0$ and $W(x) < 0$ for $x > 0$. Hence, again, there exists a unique regular equilibrium if $V(\infty) < 0 < V(0)$, otherwise there are no regular equilibria.

Case 2.3.3. Suppose that $B < -2\sqrt{AC}$. Then $0 < x_1^* < x_2^*$, and $V(x)$ is decreasing for $x < x_1^*$ and $x > x_2^*$, and $V(x)$ is increasing for $x \in (x_1^*, x_2^*)$.

Case 2.3.3.1. Suppose that $V(\infty) < 0 < V(0)$. Then if $V(x_1^*) > 0$ then there is a unique equilibrium. If $V(x_1^*) = 0$, then there are two equilibria. And if $V(x_1^*) < 0$, then there are three equilibria.

Case 2.3.3.2. If $V(0) \leq 0 \leq V(\infty)$ then there is a unique equilibrium.

Case 2.3.3.3. Suppose that $V(0) > 0$ and $V(\infty) \geq 0$. If $V(x_1^*) > 0$ then there are no regular equilibria. If $V(x_1^*) = 0$, then there is a unique regular equilibrium. If $V(x_1^*) < 0$ then there are two regular equilibria.

Case 2.3.3.4. Suppose that $V(0) \leq 0$ and $V(\infty) < 0$. If $V(x_2^*) < 0$ then there are no regular equilibria. If $V(x_2^*) = 0$, then there is a unique regular equilibrium. If $V(x_2^*) > 0$ then there are two regular equilibria.

Case 2.4. Suppose that $A < 0$ and $C \geq 0$. In this case $x_1^* \leq 0 \leq x_2^*$. Then $W(x) > 0$ for $0 < x < x_2^*$ and $W(x) < 0$ for $x > x_2^*$. Thus, x_2^* is a global maximum of $V(x)$.

Case 2.4.1. If $V(\infty) < 0 \leq V(0)$ then there is a unique equilibrium.

Case 2.4.2. If $V(0) < 0 \leq V(\infty)$ then there is a unique equilibrium.

Case 2.4.3. If $V(0) \geq 0$ and $V(\infty) \geq 0$ then there are no regular equilibria.

Case 2.4.4. Suppose that $V(0) < 0$ and $V(\infty) < 0$. If $V(x_2^*) < 0$ then there are no regular equilibria. If $V(x_2^*) = 0$, then there is a unique regular equilibrium. If $V(x_2^*) > 0$ then there are two regular equilibria.

Case 2.5. Suppose that $A = 0$. In this case, $W(x) = Bx + C$.

Case 2.5.1. If $B < 0$ and $C \geq 0$ then $V(x)$ is increasing for $x < x^*$ and decreasing for $x > x^* \equiv -C/B$. This is the same as case 2.4.

Case 2.5.2. If $B > 0$ and $C \leq 0$ then $V(x)$ is decreasing for $x < x^*$ and increasing for $x > x^* \equiv -C/B$. This is the same as case 2.2.

Case 2.5.3. Suppose that $B < 0$ and $C \leq 0$. Then $V(x)$ is decreasing for all $x > 0$. If $V(\infty) < 0 < V(0)$, then there exists a unique equilibrium, otherwise there are no regular equilibria.

Case 2.5.4. Suppose that $B > 0$ and $C \geq 0$. Then $V(x)$ is increasing for all $x > 0$. If $V(0) < 0 < V(\infty)$ then there exists a unique equilibrium, otherwise there are no regular equilibria.

Now we can collect outcomes of all cases above and write conditions for different number of regular equilibria. Let us denote:

$$F(x) \equiv G^{-1}\left(\frac{\phi_1}{\phi_2}\right)\left(\frac{1+\phi_1 x}{\phi_2+x}\right)^{\mu}\left(\frac{1+d_1 x}{d_2+x}\right)^{-1};$$

$$F_0 \equiv \lim_{x\to 0} F(x) = G^{-1}\left(\frac{\phi_1}{\phi_2}\right)\phi_2^{\mu} d_2,$$
$$F_\infty \equiv \lim_{x\to\infty} F(x) = G^{-1}\left(\frac{\phi_1}{\phi_2}\right)\phi_1^{-\mu} d_1^{-1};$$

$$\bar{x}_1 \equiv \frac{-B-\sqrt{B^2-4AC}}{2A} \text{ for } A \neq 0;$$
$$\bar{x}_2 \equiv \begin{cases} \dfrac{-B+\sqrt{B^2-4AC}}{2A} & \text{if } A>0, \\ -\dfrac{C}{B} & \text{if } A=0. \end{cases}$$

1. **No regular equilibria:**
   (a) $d_1 d_2 \leq 1$ and either $F_0 \geq 1$ or $F_\infty \leq 1$;
   (b) $d_1 d_2 > 1$, $A > 0$, $C > 0$, and one of the following conditions:
      i. $B \geq -2\sqrt{AC}$, and either $F_0 \geq 1$ or $F_\infty \leq 1$; or
      ii. $B < -2\sqrt{AC}$, $F_0 < 1$ and $F_\infty \leq 1$, and $F(\bar{x}_1) < 1$; or
      iii. $B < -2\sqrt{AC}$, $F_0 \geq 1$ and $F_\infty > 1$, and $F(\bar{x}_2) > 1$;
   (c) $d_1 d_2 > 1$, $A < 0$, $C < 0$, and one of the following conditions:
      i. $B \geq -2\sqrt{AC}$, and either $F_0 \leq 1$ or $F_\infty \geq 1$; or
      ii. $B < -2\sqrt{AC}$, $F_0 > 1$ and $F_\infty \geq 1$, and $F(\bar{x}_1) > 1$; or
      iii. $B < -2\sqrt{AC}$, $F_0 \leq 1$ and $F_\infty < 1$, and $F(\bar{x}_2) < 1$;
   (d) $d_1 d_2 > 1$, $A > 0$, $C \leq 0$, and one of the following conditions:
      i. $F_0 \leq 1$ and $F_\infty \leq 1$; or
      ii. $F_0 > 1$ and $F_\infty > 1$, and $F(\bar{x}_2) > 1$;
   (e) $d_1 d_2 > 1$, $A = 0$, $B > 0$, $C \leq 0$, and one of the conditions (1.d.i)-(1.d.ii).
   (f) $d_1 d_2 > 1$, $A < 0$, $C \geq 0$, and one of the following conditions:
      i. $F_0 \geq 1$ and $F_\infty \geq 1$; or
      ii. $F_0 < 1$ and $F_\infty < 1$, and $F(\bar{x}_2) < 1$;

(g) $d_1 d_2 > 1$, $A = 0$, $B < 0$, $C \geq 0$, and one of the conditions (1.f.i)-(1.f.ii).

2. **Unique regular equilibrium:**

   (a) $d_1 d_2 \leq 1$ and $F_0 < 1 < F_\infty$;

   (b) $d_1 d_2 > 1$, $A > 0$, $C > 0$, and one the following conditions:

      i. $B \geq -2\sqrt{AC}$ and $F_0 < 1 < F_\infty$; or
      ii. $B < -2\sqrt{AC}$, $F_0 < 1 < F_\infty$, and $F(\bar{x}_1) < 1$; or
      iii. $B < -2\sqrt{AC}$ and $F_\infty \leq 1 \leq F_0$; or
      iv. $B < -2\sqrt{AC}$, $F_0 < 1$ and $F_\infty \leq 1$, and $F(\bar{x}_1) = 1$; or
      v. $B < -2\sqrt{AC}$, $F_0 \geq 1$ and $F_\infty > 1$, and $F(\bar{x}_2) = 1$;

   (c) $d_1 d_2 > 1$, $A < 0$, $C < 0$, and one the following conditions:

      i. $B \geq -2\sqrt{AC}$ and $F_\infty < 1 < F_0$; or
      ii. $B < -2\sqrt{AC}$, $F_\infty < 1 < F_0$, and $F(\bar{x}_1) > 1$; or
      iii. $B < -2\sqrt{AC}$, $F_0 \leq 1 \leq F_\infty$; or
      iv. $B < -2\sqrt{AC}$, $F_0 > 1$ and $F_\infty \geq 1$, and $F(\bar{x}_1) = 1$; or
      v. $B < -2\sqrt{AC}$, $F_0 \leq 1$ and $F_\infty < 1$, and $F(\bar{x}_2) = 1$;

   (d) $d_1 d_2 > 1$, $A > 0$, $C \leq 0$, and one the following conditions:

      i. $F_0 \leq 1 < F_\infty$; or
      ii. $F_\infty \leq 1 < F_0$; or
      iii. $F_0 > 1$ and $F_\infty > 1$, and $F(\bar{x}_2) = 1$;

   (e) $d_1 d_2 > 1$, $A = 0$, $B > 0$, $C \leq 0$, and one of the conditions (2.d.i)-(2.d.iii).

   (f) $d_1 d_2 > 1$, $A < 0$, $C \geq 0$, and one the following conditions:

      i. $F_\infty < 1 \leq F_0$; or
      ii. $F_0 < 1 \leq F_\infty$; or
      iii. $F_0 < 1$ and $F_\infty < 1$, and $F(\bar{x}_2) = 1$;

   (g) $d_1 d_2 > 1$, $A = 0$, $B < 0$, $C \geq 0$, and one of the conditions (2.f.i)-(2.f.iii).

3. **Two regular equilibria:**

   (a) $d_1 d_2 > 1$, $A > 0$, $C > 0$, and one of the following conditions:

      i. $B < -2\sqrt{AC}$, $F_0 < 1 < F_\infty$, and $F(\bar{x}_1) = 1$; or
      ii. $B < -2\sqrt{AC}$, $F_0 < 1$ and $F_\infty \leq 1$, and $F(\bar{x}_1) > 1$; or

iii. $B < -2\sqrt{AC}$, $F_0 \geq 1$ and $F_\infty > 1$, and $F\left(\bar{x}_2\right) < 1$;

(b) $d_1 d_2 > 1$, $A < 0$, $C < 0$, and one of the following conditions:

i. $B < -2\sqrt{AC}$, $F_\infty < 1 < F_0$ and $F_\infty < 1$, and $F\left(\bar{x}_1\right) = 1$; or

ii. $B < -2\sqrt{AC}$, $F_0 > 1$ and $F_\infty \geq 1$, and $F\left(\bar{x}_1\right) < 1$; or

iii. $B < -2\sqrt{AC}$, $F_0 \leq 1$ and $F_\infty < 1$, and $F\left(\bar{x}_2\right) > 1$;

(c) $d_1 d_2 > 1$, $A > 0$, $C \leq 0$, and $F_0 > 1$ and $F_\infty > 1$, and $F\left(\bar{x}_2\right) < 1$;

(d) $d_1 d_2 > 1$, $A = 0$, $B > 0$, $C \leq 0$, and $F_0 > 1$ and $F_\infty > 1$, and $F\left(\bar{x}_2\right) < 1$;

(e) $d_1 d_2 > 1$, $A < 0$, $C \geq 0$, and $F_0 < 1$ and $F_\infty < 1$, and $F\left(\bar{x}_2\right) > 1$;

(f) $d_1 d_2 > 1$, $A = 0$, $B < 0$, $C \geq 0$, and $F_0 < 1$ and $F_\infty < 1$, and $F\left(\bar{x}_2\right) > 1$;

4. **Three regular equilibria:**

(a) $d_1 d_2 > 1$, $A > 0$, $C > 0$, $B < -2\sqrt{AC}$, $F_0 < 1 < F_\infty$, and $F\left(\bar{x}_1\right) > 1$;

(b) $d_1 d_2 > 1$, $A < 0$, $C < 0$, $B < -2\sqrt{AC}$, $F_\infty < 1 < F_0$, and $F\left(\bar{x}_1\right) < 1$.