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Dennis Epple
Francisco Martinez-Mora
Richard Romano

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ABSTRACT

We model charter school entry and choice of educational practices. Student achievement depends on cognitive ability, motivation, effort, and match of school curriculum to ability. Exercising charter school autonomy over curriculum, to maximize achievement gains, the charter sets curriculum to attract the highest ability students. Achievement gains are modest, consistent with empirical evidence. We next investigate a no-excuses charter that not only chooses curriculum but also enforces an effort minimum. Consistent with the evidence, highly motivated students select into the charter, achievement gains are large, and the largest gains accrue to those who would be lower performers in public school.

Dennis Epple
Carnegie Mellon University
Tepper School of Business
Room 4126
Pittsburgh, PA 15213
and NBER
epple@cmu.edu

Francisco Martinez-Mora
University of Leicester
Brookfield Campus
266 London Road
LE2 1RQ
Leicester
United Kingdom
fmm14@le.ac.uk

Richard Romano
University of Florida
Department of Economics
PO Box 117140
Gainesville, FL 32611
romanor@ufl.edu

An appendix is available at: http://www.nber.org/data-appendix/w29529

1. Introduction

Charter schools are playing an increasingly important role in US education. The first charter school opened in Minnesota in 1992, and they currently operate in 44 states.[1] From fall 2009 to fall 2018, the percentage of public school students attending charter schools increased from 3 to 7 percent.[2] This share is much higher in many, especially urban, areas, for example exceeding 15 percent in 20 cities in the 2010-11 school year.[3] Moreover, more than 10% of public school students attend charter schools in 6 states. Charters are tuition-free public schools and are not permitted to impose selective criteria for admission. If oversubscribed, a charter must admit by lottery from its applicant pool. Charter schools are, however, permitted to adopt curriculum and educational practices different from those of traditional public schools. Much research has been devoted to comparing the performance of charter and public schools and to documenting educational approaches of charter schools. This research has established that there are large differences across charter schools in the extent to which they improve educational outcomes relative to the public schools their students would otherwise attend.

In contrast to this impressive body of empirical research, there has been little formal modeling of the decisions about curriculum and educational practices of a prospective charter school entrant, how those decisions vary with the educational environment the charter would enter, or the impacts on students of differing capabilities and socioeconomic background. There is, to the best of our knowledge, no existing framework that might be employed to address these fundamental issues.

In this paper, we develop such a framework, and apply it to analyze decisions about curriculum and educational practices made by a charter entrant seeking to maximize the educational gains of the students it attracts. We investigate how such choices vary with the characteristics of the student population from which the charter school draws its students. This in turn requires modeling the school choice decisions of students of differing capabilities and socioeconomic backgrounds. It also requires comparison of student outcomes in the equilibrium with charter entry relative to student outcomes in the equilibrium without charter entry. While charter schools cannot impose selective criteria for admission, and so must randomly admit students among their applicants, their curriculum choice maximizes the achievement gains of their students, both by targeting their capabilities and by affecting student self-selection into the school.

Apart from considering traditional charter schools, a key part of our analysis models "no-excuses charter schools," which have been quite successful in increasing student achievement.[4] In

[1] With some variation as to their educational and funding models, charter schools are playing an increasingly important role in other countries too. Among them are Australia, Canada, Chile, New Zealand, Spain, Sweden and the United Kingdom.
[2] https://nces.ed.gov/programs/coe/indicator/cgb
[3] See Figure 4, panel C of Epple, Romano, and Zimmer (2016).
[4] The evidence is discussed in Section 2.

addition to their curriculum choice, these charter schools enforce student work standards with parental support.

Our model features some elements that, while being central to the educational process, have not received much attention in the theoretical literature. Students differ along three dimensions: academic ability and motivation, which can be interpreted as the child's cognitive and non-cognitive skills, respectively, as well as household income. A student's achievement at school depends on the quality of the education received, on her academic ability and the (costly) effort she exerts. Costs of effort in turn depend on her motivation. There are three key educational variables that shape a school's unique educational process: the academic standards, the school curriculum and, in the case of a no-excuses charter school, the required effort minimum. Standards determine how demanding it is for a student to graduate from a school and are set by the educational authority. Setting higher standards may push some students to exert greater effort than they would normally do, but it may also discourage others from exerting any effort at all. A school's curriculum establishes the academic ability the school targets with its teaching, and thereby affects the quality of education received by every student. Thus, capturing an important aspect of the educational process, the quality of the educational services provided by a school in our model is specific to each student and depends on the alignment between the student's own academic ability and the school's curriculum.

We follow with a computational counterpart to our theoretical model. For calibration, we utilize recent econometric evidence regarding the roles of different types of skills in determining educational outcomes. We couple this with information about the distributions of student ability and student achievement. Using this computational model, we compute the gains in student achievement from different charter school designs and for different student populations. We compare the results of our computational analysis to the magnitudes estimated in empirical evaluations of charter school impacts. Our calibration does not use the latter estimates. Hence, these comparisons provide informative evidence about the validity of the model. There is a close congruence between the magnitudes we obtain from our computational model and those in the empirical literature.

Our paper proceeds as follows. Section 2 discusses the literature. Sections 3 and 4 present, respectively, our theoretical framework and results. Our computational model and its findings follow in Section 5. Section 6 concludes. Some technical analysis is in the appendix.

2. Literature Discussion.

Given their relative newness and on-going growth as an education provider, research on charter schools has exploded. The focus has been on their performance in increasing student achievement relative to traditional public schools, with the primary identification challenge being student selection into charter schools on unobservables.[5] We make no attempt to provide an in-depth survey of the findings and techniques, but refer the reader to Epple, Romano, and Zimmer (2016) for a comprehensive review of this research up through about 2015, and to Chabrier, Cohodes, and Oreopoulos (2016) which focuses on lottery studies of charter schools.[6] The thrust of the literature is that: (i) Overall, charter schools have mixed performance in increasing achievement, but with an upward trend and average positive gains in urban areas (see especially National Charter School Study, CREDO 2015[7]). (ii) Charter schools that follow the "No-Excuses" approach dramatically increase student achievement, especially for students with lower prior scores and for black students.[8] For example, Abdulkadiroglu, et.al (2011), exploiting the lottery setting, find local-average-treatment-effect estimates of attending a no-excuses school equal to .198σ and .359σ in middle school ELA and math and .265σ and .364σ in high school ELA and math.[9]

We next discuss papers more relevant to our theoretical modelling of charter schools and student choices. In Section 5.3, we also discuss the predictions of our theoretical model in light of the empirical research. We present a model of the micro-foundations of charter school differentiation based on curriculum focus, with an explicit charter objective, selection by students that differ in ability, motivation and socioeconomic background and student effort choice. Glomm, Harris, and Lo (2005) appeal to horizontal charter school differentiation, generally defined, to explain why more charter schools entered districts in Michigan and California that are characterized by more widely educated parental and more racially diverse populations. Exogenous differentiation of charter schools plays a role in several empirical papers on charter schools. Singleton (2019) estimates a structural model of charter school entry allowing costs to vary among profit, not-for-profit, and no-excuses charter schools. Gilraine, Petronijevic, and Singleton (2019) investigate how achievement gains vary among charter schools whose applications specify that "learning is experiential or project-based as opposed to focused on core skills through traditional instruction." Bau (2019), though studying private schools in Pakistan, has schools specialize to serve rich or poor students, this related to our curriculum specialization to student abilities Geographic locational choice is central to several empirical models of charter school entry, these taking residences of racially and socio-economically diverse populations as given, with students facing travel costs to attend schools. This includes

[5] Empirical research on charter schools also investigates the impact on educational attainment and non-educational outcomes, where charter schools locate and what students they serve, and competitive effects on neighboring traditional public schools. See Epple, Romano, and Zimmer (2016) for citations and discussion.
[6] Lottery studies exploit over-subscribed charter schools that are required to admit students randomly, essentially comparing the educational outcomes of lottery winners and losers, thus controlling for selection (though then able to use only over-subscribed charter schools).
[7] https://credo.stanford.edu/publications/national-charter-school-study. Another very recent national study finding average gains and a positive trend is Shakeel and Peterson (2021).
[8] See, for example, Abdulkadiroglu et al. (2011), Angrist et al. (2012), Dobbie and Fryer (2013), Angrist, Pathak, and Walters (2013), and Walters (2018).
[9] These LATE estimates are per year spent in the charter school and control for demographics and baseline scores. While dramatic, these are typical among studies of no-excuses schools.

Singleton (2019), Ferreyra and Kosenok (2018), and Mehta (2017). Ferreyra and Kosenok (2018) also have schools differentiated by whether core, language-oriented, arts, vocational, or "other." Mehta (2017) has schools dictate student effort, which is similar to what we do in our model of no-excuses schools.[10]

No-excuses charter schools differ from other schools by their educational practices. Those practices, which vary somewhat among the schools so labeled, include: (i) a longer school day and year; (ii) frequent testing; (iii) an ethos of comportment and a strong student work ethic;[11] (iv) extensive tutoring; (v) emphasis on core math and reading; and (vi) selective teacher hiring and professional development. Dobbie and Fryer (2013) construct an index of teacher feedback, use of data to guide instruction, tutoring, instructional time, and high expectations of students that explains 45 percent of the variation in charter school effectiveness. Angrist, Pathak, and Walters (2013) identify emphasis on discipline, school uniforms, cold-calling, adherence to school-wide standards, and employment of Teach-for-America teachers as the five practices that best explain effectiveness of no-excuses schools. As noted, we discuss further empirical findings below as related to our model's predictions.

3. Model and Preliminary Results

3.1 The Model

Households, Education Production, and Achievement. We study a continuum of households with mass normalized to 1. Each household consists of one school-aged child and a parent or parents. Our model reflects and synthesizes the latest available evidence on skill formation and the determinants of school achievement (see, for instance, Cunha et al. 2010, Heckman and Kautz, 2012, Heckman et al., 2006, and Knudsen et al., 2006). Households differ along three dimensions: parental income (y), the child's intellectual or cognitive ability (b), and motivation (m). The characteristics (b,m) of the child can be more broadly conceptualized as cognitive and non-cognitive skills, respectively. Students of greater cognitive ability have higher educational achievement for given school characteristics and effort (e) in school. Students who are more motivated are more efficient at or more inclined to study, which we capture by assuming they face lower costs of exerting schooling effort. For expositional ease, we refer to b as just "ability." We assume parents know their own child's ability and motivation.

[10] An interesting element of Mehta's model is that, when charter schools require more student effort, the school's cost rises, because teachers will then have more work. We do not consider this. On the other hand, we allow students to choose more than the required effort in our no excuses schools, while this is not considered by Mehta.

[11] In Knowledge is Power Program (KIPP) charter schools, parents and students sign contracts including a pledge of timely attendance, hard work, and completion of assignments. KIPP is the largest charter management organization in the nation, with 255 schools at the time of this writing.

The probability density function of student types in the population is denoted with f(b,m,y), and is positive on its support $[b_m, b_x]\times[m_m, m_x]\times[y_m, y_x]$. Let $\bar{b}$, $\bar{m}$ and $\bar{y}$ denote the population mean ability, motivation, and income, and b_{med}, m_{med}, y_{med} denote their respective median values. In our computational analysis, we study cases where the household population is representative of the US population and, alternatively, a poverty population. This is to investigate charter school practices if entering a school district with a representative population (e.g., a typical town) or if entering a poor district that arises from Tiebout sorting (e.g., an inner-city district).

Education production combines the student's study effort with ability and school quality to produce achievement (a). School quality varies among students depending on the match of the student's ability to the level of ability targeted in the school's instruction or curriculum (B). Specifically, the school's quality to a student of ability b equals $Q\Gamma(b,B)$, consisting of two elements: (i) a common quality component Q, a function of spending per pupil, and (ii) a student-specific component, which depends on the distance between the school's curriculum target and the student's ability. The function $\Gamma(b,B)$, whose value falls with the distance between individual ability b and the curriculum ability target B, captures the latter:

(1) $$\Gamma(b,B) \equiv \Gamma_m Z(b,B) \equiv \Gamma_m\left[\alpha + (1-\alpha)\exp\left\{-\frac{(b-B)^2}{\tau^2}\right\}\right], \ \alpha \in [0,1).$$

Note that Γ reaches its global maximum at b = B with value $\alpha\Gamma^m$, and declines as the distance from b to B increases.[12] A lower α implies a stronger curriculum effect and a lower τ^2 that a student's achievement declines more steeply as the curriculum ability target moves away from her ability.

Achievement is given by: $a(b,e,B,Q) = b^{\beta}Q\Gamma(b,B)e.$ To ensure that achievement increases with ability even if the distance to the school's curriculum target increases with ability (i.e., for b > B), we assume:

Assumption 1 (A1): $b^{\beta}\Gamma(b,B)$ increases with b.

The main analysis will have one (traditional) public school and one charter school, with limited capacity in the charter school denoted by κ.[13] We assume the charter school needs to fill to capacity to cover its fixed costs. Parents choose the school to maximize their child's achievement, or the school the student prefers if either school would lead to the same achievement. As discussed below, parental and child school preferences can diverge. Thus, parents apply to the charter school if attending would strictly increase achievement or lead to the same achievement and is preferred by the

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[12] This is straightforward to verify: $\frac{\partial\Gamma(b,B)}{\partial(b-B)^2} = \frac{-\Gamma_m(1-\alpha)}{\tau^2}\exp\left\{-\frac{(b-B)^2}{\tau^2}\right\} < 0.$

[13] While charter schools are publicly funded and legally public schools, we refer to the traditional public school as the public school.

child. If applications exceed capacity at the charter, the school conducts a lottery to decide which applicants receive an offer. We assume further that only a fraction $\mu \in (0,1]$ will accept an offer to attend and that the matriculation probability (μ) is constant among students. This allows parents to change their minds about sending their child to the charter and is empirically motivated.[14]

While parents choose the school, the child chooses her effort in the school attended. Children care about their achievement, but also about their effort costs (c). We assume effort costs are given by $c(e, m) = e^2/2m$, satisfying $\partial c/\partial e > 0$, $\partial c^2/\partial e^2 > 0$ and $\partial c/\partial m < 0$. Children choose effort to maximize their utility:

(2) $U = a - c.$

The child then may not prefer the school that maximizes achievement and, if both lead to the same achievement, prefers the school where she needs to exert less effort. A constraint on the child's effort choice arises in the case of a "no-excuses charter school," which is able to enforce an effort minimum with parental help.

Schools. In addition to adopting a curriculum, a school has a "standard" S, the minimum achievement necessary for the student to pass or to "graduate." We normalize the achievement of non-graduates to zero. The utility a student of type (b,m) derives from attending school j is therefore given by:

(3) $$U\left(b,m,e,B^j,Q^j,S^j\right)=\begin{cases} a\left(b,e,B^j,Q^j\right)-c\left(e,m\right), & \text{if } a\left(b,e,B^j,Q^j\right)\geq S^j \\ -c\left(e,m\right), & \text{if } a\left(b,e,B^j,Q^j\right)<S^j \end{cases}$$

where the superscripts denote values for school j. Conditional on not graduating, students optimally choose to exert zero effort and "drop out."[15] The utility a drop-out receives is therefore equal to zero.

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[14] The lottery randomly orders applicants by assigning each of them a number $\ell \in [0,1]$. If the school knew the matriculation rate, it would make an offer to every applicant with a lottery number $\ell \leq \overline{\ell} \equiv \frac{\kappa}{|A|\mu}$, where $|A|$ denotes the cardinality of the applicant set, and so it would fill its seats in a single round of offers. If, on the contrary, the charter did not know μ, it could initially make enough offers to fill its capacity under the (possibly incorrect) assumption that $\mu = 1$. That is, it may offer a seat to every applicant with lottery number $\ell \leq \ell_1 \equiv \frac{\kappa}{|A|}$. The school's working assumption that $\mu = 1$ guarantees that no one with an offer is left without a seat. If $\mu < 1$, though, only $\ell_1 |A| \mu < \kappa$ students would accept the offer. Were the school to know that the matriculation rate is the same for any random subset of applicants, it could deduce the true value of μ at that point and complete the process in one more round. Otherwise, in each successive round, the charter could make offers to just enough applicants to fill the remaining seats under the assumption that $\mu = 1$ until no seats remained unassigned. In all of these cases, it would assign its seats to those applicants with lottery number $\ell \leq \overline{\ell}$ who accepted the school's offer.

[15] The student may just fail, rather than actually drop out. However, we refer to these students as dropouts. No students will apply to a charter school where they would drop out.

We consider three types of schools: a public school (P), a traditional charter school (TCS) and a no-excuses charter school (NEC). All schools must adopt the same standard, which we assume the district or a higher level of government sets.[16] We also assume all schools receive the same funding per student, which is becoming more common. Thus, Q is identical across schools. The public school targets its curriculum at the population median ability. It is not capacity constrained and accepts all students that do not apply to the charter school, or that apply but do not attend either because they lose the lottery or have a change of mind. The TCS chooses its curriculum target to maximize the *achievement gains* of its students, relative to what they would achieve in the public school. Aggregate and average achievement gains from attending the charter school are denoted with Δ and $\bar{\Delta}$, respectively. Any charter school is subject to a size and viability constraint: it must attract enough students to fill its capacity κ but cannot exceed it.[17] We assume the charter capacity is not large. Charter schools must accept all applicants if they have enough seats, or run a fair lottery if over-subscribed with the matriculation proviso discussed above. The NEC school is identical to the TCS except that, besides choosing the ability its curriculum will target (to maximize Δ), it also establishes an enforceable minimum effort requirement that its students must satisfy to attend the school.[18] We denote this minimum effort with e^m. Schools know the distribution of (b,m) in their district but do not observe individual values. We use a j superscript to identify non-common values over schools, with j $\in$ {P, T, N} indicating, respectively, the public school, a traditional charter school, and a no-excuses charter school.

Timing. The public policies (B^P,S) are given. The charter school chooses and announces its curriculum ability target B^j, $j \in \{T, N\}$, and, in the case of the NEC school, its minimum effort requirement e^m. Parents then decide whether to apply to the charter school, knowing the school policies and their child's characteristics (b,m). If oversubscribed, the charter school runs a fair lottery, and each winner decides whether to matriculate or not. Finally, once assigned to schools, children decide how much effort to exert and payoffs result. Note that parents anticipate their child's effort choices when deciding to apply to the charter school or not.

3.2 Preliminary Results: Single Public School

As a baseline and to better understand the model, we begin by examining the case in which only the public school exists. The public school's curriculum B^P targets the median student ability, while we will later calibrate the graduation standard S and the common component of quality Q to match key

[16] An extension would consider charter schools that adopt a different standard, perhaps at least as high as is the public school's.
[17] Having a simple capacity constraint for school viability avoids introducing a cost function and strikes us as a reasonable approximation.
[18] See the discussion of the literature above.

variables of US demographics and its school marketplace. We drop the superscript identifying the school in the remainder of this section.

Student Effort Choice. We start by defining and identifying three effort levels of a student: (i) the optimal (unconstrained) effort choice (e^*); (ii) the effort necessary to reach the school's standards (e^S); and (iii) the effort cutoff above which the student would rather drop out (e^0).

The unconstrained utility-maximization problem is:

(4) $$\max_{e} \; U(b,m,e,B,Q) = b^{\beta} Q \Gamma(b,B) e - \frac{1}{2m} e^2,$$

From the FOC, we obtain:

(5) $$e^*(b,m,B,Q) = b^{\beta} Q \Gamma(b,B) m$$

One can see by inspection that the unconstrained optimal effort level increases with motivation (because more motivated students face lower effort costs), with the common component of school quality (because higher quality implies greater productivity of effort) and, for the same reason, with the student-specific element of quality. Thus, effort increases as the curriculum target gets closer to the ability of the student. Moreover, under A1, optimal effort also goes up with ability, even when greater ability implies a greater distance to the school's curriculum target (i.e. for students whose ability is larger than the curriculum target).[19] Let a^* denote the utility-maximizing achievement assuming it exceeds the standard:

(6) $$a^*(b,m,B,Q) \equiv a\left(b, e^*(b,m,B,Q), B, Q\right)$$

The graduation standard constrains the behavior of those students whose most-preferred level of effort is not enough to reach it. The minimum effort a student (b,m) requires to attain the school's standard S satisfies $b^{\beta} Q \Gamma(b,B) e_S \equiv S$. Solving for e_S we obtain:

(7) $$e_S(b,B,Q,S) \equiv \frac{S}{b^{\beta} Q \Gamma(b,B)}.$$

e^S falls with the quality of education received by the student and, under A1, also with her academic ability. It increases with the school's graduation standard and is independent of student motivation.

The maximum effort the student can bear without dropping out satisfies

$b^{\beta} Q \Gamma(b,B) e_0 - \frac{1}{2m}(e_0)^2 \equiv 0$. Solving for e_0:

(8) $$e_0(b,m,B,Q) \equiv 2 b^{\beta} Q \Gamma(b,B) m.$$

19 $\frac{\partial e^*}{\partial b} = Qm \frac{\partial [b^{\beta} \Gamma(b,B)]}{\partial b} > 0$, by A1.

Notice that $e_0(b,m,B,Q) = 2e^*(b,m,B,Q)$. These three effort values (e^*, e_S, e_0) allow us to classify the students of a school into three groups: high achievers (those with $e^* > e_S$), effort-constrained students (those with $e^* \le e_S \le e_0$), and dropouts (those with $e^* < e_0 < e_S$). The following two loci partition the (b,m) plane into three regions: one with the school's high achievers, another with its effort-constrained students, and a third one with the dropouts. The first locus, denoted with $\hat{m}_S$, separates high-achievers from effort-constrained students and is defined by $e_S(b,B,Q,S) \equiv e^*(b,\hat{m}^S,B,Q)$. Solving for m, we obtain:

$$\text{(9)} \qquad \hat{m}_S(b,B,Q,S) \equiv \frac{S}{\left[b^\beta Q\Gamma(b,B)\right]^2}$$

It is straightforward to show that $\hat{m}_S$ is negatively sloped in the (b,m) plane using A1. The second locus, denoted with $\hat{m}_0$, separates effort-constrained students from dropouts and is defined by $e_S(b,B,Q,S) \equiv e_0(b,\hat{m}^0,B,Q)$. Solving for m again yields:

$$\text{(10)} \qquad \hat{m}_0(b,B,Q,S) \equiv \frac{\hat{m}_S(b,B,Q,S)}{2}.$$

Lemma 1 examines the characteristics and effort choices of each of these groups.

Lemma 1. Given the school curriculum target B, graduation standard S and common quality component Q:

(i) *High achievers* are students who achieve above the graduation standards: $a^*(b,m,B,Q) > S$. They satisfy: $e^*(b,m,B,Q) > e^S(b,B,Q,S)$ and have types (b,m) such that $m > \hat{m}_S(b,B,Q,S)$.

(ii) *Effort-constrained students* are students who exert the minimum effort they require to graduate and so obtain achievement equal to S. They satisfy: $e^*(b,m,B,Q) \le e_S(b,B,Q,S) \le e_0(b,m,B,Q)$ and have types (b,m) such that $m \in \left[\hat{m}_0(b,B,Q,S), \hat{m}_S(b,B,Q,S)\right]$.

(iii) *Dropouts* are students who drop out of school and exert no effort. They satisfy: $e_S(b,B,Q,S) > e_0(b,m,B,Q) > e^*(b,m,B,Q)$ and have types (b,m) such that $m < \hat{m}_0(b,B,Q,S)$.

Proof. The proof is simple and in the online appendix.

We close this section by defining the *induced achievement function*. This is denoted by $\tilde{a}$ and equals the achievement of students of type (b, m) conditional on the school inputs (B, S, Q) and on utility-maximizing effort choices:

$$(11) \qquad \tilde{a}(b,m,B,S,Q)=\begin{cases} \left(b^{\beta}Q\Gamma(b,B)\right)^2 m; & \text{if } m>\hat{m}_S(b,B,S,Q) \\ S; & \text{if } m \in \left[\hat{m}_0(b,B,S,Q),\hat{m}_S(b,B,S,Q)\right] \\ 0; & \text{if } m<\hat{m}_0(b,B,S,Q) \end{cases}$$

Figure 1 illustrates the student partition and implied achievement levels.[20]

4. Charter Schools

4.1. Traditional Charter School. We begin by studying competition between the public school and a traditional charter school. The analysis proceeds by backward induction. First, we establish which households apply to the charter school as a function of its curriculum target. Second, we determine the school's optimal curriculum choice. The TCS selects its ability target to maximize the achievement gains of its students, subject to the viability constraint and given public school policies (Q^P, S^P, B^P).

School Choice. The educational quality a student enjoys at a particular school is higher the closer is the student's ability to the school's curriculum target. Given that the student component of school quality is a function of $(b-B)^2$, the level of cognitive ability equidistant to the schools' targets, B^T and B^P, separates students who receive higher quality from one school and the other. Let $\tilde{b}(B^T,B^P)\equiv(B^T+B^P)/2$ define this cutoff ability.

The next lemma orders the loci separating high-achievers, effort-constrained students and dropouts in the two schools to study household preferences for schools. Refer to Figure 2. Recall that the public and charter schools receive the same level of funding per student, so $Q^T = Q^P = Q$. Also, note that policy requires the same graduation standards, so $S^T = S^P = S$. Let $\hat{m}_S^j(b,B^j,S,Q)$, j = P, T, denote, respectively for the public and the TCS, the locus separating high-achievers from effort-constrained students; and $\hat{m}_0^j(b,B^j,S,Q)$, j = P, T, denote the locus separating the latter group from the dropouts for the relevant school:

Lemma 2.[21] Suppose that $B^T < B^P$. Then:

(i) $e_S\left(b,B^T,S,Q\right)\begin{Bmatrix}<\\=\\>\end{Bmatrix}e_S\left(b,B^P,S,Q\right)\ \forall b\begin{Bmatrix}<\\=\\>\end{Bmatrix}\tilde{b},$

(ii) $\hat{m}_0^T\left(b,B^T,S,Q\right)\begin{Bmatrix}<\\=\\>\end{Bmatrix}\hat{m}_0^P\left(b,B^P,S,Q\right)\ \forall b\begin{Bmatrix}<\\=\\>\end{Bmatrix}\tilde{b},$ and

[20] The numerical values of m and b in Figures 1 – 7 are from a simple example developed to illustrate student partitions. Later we present a seriously calibrated model.
[21] An analogous result holds with $B^T > B^P$: simply flip the latter set of inequalities in every part of the lemma.

(iii) $\hat{m}_S^T\left(b, B^T, S, Q\right)\begin{Bmatrix} < \\ = \\ > \end{Bmatrix}\hat{m}_S^P\left(b, B^P, S, Q\right) \ \ \forall b\begin{Bmatrix} < \\ = \\ > \end{Bmatrix}\tilde{b}.$

Proof. The proof follows easily from the definition of $\Gamma(b, B)$, which implies:

$|b - B^T| < (>)(=) |b - B^P|$ and $\Gamma(b, B^T) > (<)(=) \ \Gamma(b, B^P)$ for $b < (>)(=)\tilde{b}$. ■

Taking the example of students with ability below $\tilde{b}$, when $B^T < B^P$, they require less effort to graduate from the charter school than from the public one (part i), and so they need less motivation to graduate (part ii) or to achieve above the graduation standards (part iii). (See Figure 2).

The following lemma characterizes the set of students whose parents apply to the TCS. We denote such a set with $A^T (B^T, B^P)$. Figure 2 shows the applicant set.

Lemma 3. (Applicant set.) Suppose the TCS enters the market with curriculum target $B^T \in [b_m, b_x]$. Then, every student with ability closer to B^T than to B^P and sufficiently motivated to not drop out of the TCS applies. Formally:

(12) $$A^T(B^T, B^P) = \{(b, m) \,\big|\, |b - B^T| < |b - B^P|, \ m > \hat{m}_0^T(b, B^T, \cdot)\}.$$

The proof is in the appendix, while we provide some intuition here. Observe in Figure 2 that the ability of TCS applicants is better suited to its curriculum target, B^T, than to the public school's one, B^P, all these students having ability closer to B^T. In addition, their motivation is strong enough to not drop out of the charter school. Several groups of students make up this applicant set. Those with $m > m_s^T$ are not effort constrained in the charter school and have higher achievement there. Those with $m \in [\hat{m}_o^T, \hat{m}_s^T]$ just pass at the charter school and either do the same at the public school or drop out. The former gain utility by attending the charter school since they exert less effort to do so, while the latter gain achievement and utility.

Remark 1. By lemma 3, $B^T = B^P \Leftrightarrow |b - B^T| = |b - B^P| \ \forall b \Leftrightarrow A^T(B^P, B^P) = \varnothing$. That is, the applicant set, $A^T (B^T, B^P)$, is empty whenever $B^T = B^P$. *Therefore, to be viable, the TCS must differentiate its curriculum from the public school's one.*

Remark 2. *Parental and student preferences between the public school and the TCS coincide, though student preferences include their effort costs.* On the one hand, for student types whose achievement differs among schools, the school that maximizes achievement also maximizes student utility since it offers them greater quality. On the other hand, for those whose achievement is identical in the two schools, recall that parents prefer the school the child prefers.

<u>Curriculum Choice</u>. The charter school's objective is to maximize the aggregate achievement *gains* relative to attending the public school. Given random admissions if over-subscribed, this is

tantamount to maximizing the aggregate achievement gains of the anticipated applicant set. Moreover, since the charter school has fixed capacity, maximizing average achievement gains is also equivalent. We then write the curriculum choice problem as follows:[22]

(13)

$$\underset{B^T}{\text{Max}}\ \overline{\Delta} = \left[\begin{array}{l} \int_{y_m}^{y_x}\int_{b_m}^{\tilde{b}(B^T)}\int_{\hat{m}_0^T(b,B^T,\cdot)}^{\hat{m}_0^P(b,\cdot)} S f(b,m,y)\,db\,dm\,dy + \int_{y_m}^{y_x}\int_{b_m}^{\tilde{b}(B^T)}\int_{\hat{m}_S^T(b,B^T,\cdot)}^{\hat{m}_S^P(b,\cdot)} \left[a^*(b,m,B^T,\cdot) - S\right] f(b,m,y)\,dm\,db\,dy + \\ \int_{y_m}^{y_x}\int_{b_m}^{\tilde{b}(B^T)}\int_{\hat{m}_S^P(b,\cdot)}^{m_x} \left[a^*(b,m,B^T,\cdot) - a^*(b,m,B^p,\cdot)\right] f(b,m,y)\,dm\,db\,dy \end{array}\right] / N^T,$$

$$\text{s.t.} \qquad \mu N^T \equiv \mu \int_{y_m}^{y_x}\int_{b_m}^{\tilde{b}(B^T)}\int_{\hat{m}_0^T(b,B^T,\cdot)}^{m_x} f(b,m,y)\,db\,dm\,dy \geq \kappa,$$

where N^T defines the mass of students that apply to the charter school (lemma 3).[23] The constraint is the viability or size constraint. To streamline the presentation, we focus on cases where $B^T \leq B^P$ while noting that analog results arise if $B^T \geq B^P$ as further explained below.

To have a well-behaved objective function, we assume:

Assumption 2 (A2): N^T is increasing with B^T on $[b_x, B^P)$.[24]

The assumption is that increasing the curriculum ability target toward the public school's one will always increase the size of the applicant pool. Such a curriculum change implies that more students have ability levels closer to the charter school's target, increasing the demand to attend. Moreover, it entails a better match to the ability of some students, further increasing demand. However, the reverse holds for lower ability students. The assumption is then that the demand enhancing effects dominate the latter effect.

To gain insight into the TCS objective function, refer again to Figure 2. Applicants to the school have ability closer to its curriculum target than to the public school's, $b < \tilde{b}$, and sufficient motivation to not drop out, $m > \hat{m}_0^C(b, B^T, \cdot)$. Each of the three colored areas of the graph corresponds to one addend in the numerator of (13). The first addend aggregates the achievement gains of students in the lower marked area (i.e. those with motivation $m \in [\hat{m}_0^T(b, B^T, \cdot), \hat{m}_0^P(b, B^P, \cdot)]$). These students would graduate from the TCS but would drop out of the public alternative. Thus, if they matriculate at the TCS, they experience gains given by the graduation standard, S. The second addend corresponds to students in the upper intermediate marked area (i.e. those with motivation

[22] Note that problem (13) drops the functional arguments that remain fixed throughout the analysis (B^P, Q, S).

[23] Not every one of those students will attend the TCS in general, either because not every student who would gain matriculates or because there is excess demand and the charter school conducts a lottery to assign its seats. Still, $\overline{\Delta}$ in (13) yields average achievement gains because we assume that the probability of not matriculating is constant across types and that lotteries are fair.

[24] We make the analogous assumption when considering ability targets above B^P.

$m \in [\hat{m}_S^T(b, B^T, \cdot), \hat{m}_S^P(b, B^P, \cdot)]$). These students would be high-achievers in the TCS but just graduate from the public school. Therefore, if they matriculate at the charter school, they realize achievement gains equal to $a^*(b, m, B^T, \cdot) - S$. The third addend captures the achievement gains of students in the upper marked area of the graph, those sufficiently motivated to be high-achievers in both schools (i.e. with $m > \hat{m}_S^P(b, B^P, \cdot)$). They apply and, if they matriculate, derive gains equal to $a^*(b, m, B^T, \cdot) - a^*(b, m, B^P, \cdot)$. The remaining students among those with ability closer to B^T than to B^P would either dropout of the charter school and so do not apply (those with $m < \hat{m}_0^c(b, B^T, \cdot)$) or would just graduate from both the charter and public school (i.e., those with $m \in [\hat{m}_0^P(b, B^P, \cdot), \hat{m}_S^T(b, B^T, \cdot)]$). Students in the latter group still apply to the charter school because they can exert less effort there. They experience no achievement gains and vanish from the numerator of the objective but count as part of the denominator.

The TCS may target high or low abilities. Indeed, its optimization problem has two local solutions, one with entry at the high-end of the ability distribution –that is, where the curriculum targets high ability students, $B^T > B^P$– and another with entry at the low-end, $B^T < B^P$. Targeting high ability students is globally optimal in our computational examples. However, we also study entry at the low-end motivated by a potential additional objective to help weaker students. In that case, the TCS still maximizes achievement gains but assuming the target population has ability below the median, that is, with the additional constraint that $B^T \leq B^P$.

A student's income plays no direct role in achievement. However, we examine different income populations that have different conditional distributions on (b,m). We then specify the objective using the joint distribution on (b,m,y).

Let B_m^T (B_x^T) denote the minimum (maximum) value of $B^T < (>) B^P$ that would generate just enough demand to attend the TCS for it to be viable:

(14) $$\mu \int_{y_m}^{y_x} \int_{b_m}^{\tilde{b}(B_m^T)} \int_{\hat{m}_0^T(b, B_m^T, \cdot)}^{m_x} f(b, m, y) dm db dy \equiv \kappa.$$

(15) $$\mu \int_{y_m}^{y_x} \int_{\tilde{b}(B_x^T)}^{b_x} \int_{\hat{m}_0^T(b, B_x^T, \cdot)}^{m_x} f(b, m, y) dm db dy \equiv \kappa.$$

Assumption (A2) or its analogue for $B^T > B^P$ imply unique solutions to (14) and (15). A value of $B^T \in (B_m^T, B^P)$ or $B^T \in (B^P, B_x^T)$ would attract more than κ students and so would imply an excess demand and a need for a lottery. On the other hand, a value of $B^T < B_m^T$ or $B^T > B_x^T$ cannot be optimal since the capacity constraint would not be met and the school would not be viable. Proposition 2 characterizes the solution to the curriculum choice problem.

Proposition 2. (TCS Curriculum Choice.) The traditional charter school problem has at least two (local) maxima, at least one with entry at the low end, $B^{T*} < B^P$, and at least one with entry at the high end, $B^{T**} > B^P$. To avoid tedium, we assume a unique local maximum on both sides of B^P.[25] We have:

1. Low-end entry: Assuming $B^T \le B^P$, $B^{T*} = B^T_m$ if and only if

$$(16)\quad \left[\begin{array}{l} \int_{y_m}^{y_x} \int_{b_m}^{\tilde{b}(B^T_m)} \left\{ (\bar{\Delta} - S) \dfrac{\partial \hat{m}^T_s(b, B^T_m)}{\partial B^T} f(b, \hat{m}^T_s(b, B^C_m), y) db \right\} dy \\ + \int_{y_m}^{y_x} \int_{b_m}^{\tilde{b}(B^T_m)} \int_{\hat{m}^T_S(b, B^T_m)}^{m_x} \dfrac{\partial a^*(b, m, B^T_m)}{\partial B^T} f(b, m, y) dm db dy - \bar{\Delta} \cdot \dfrac{\partial \tilde{b}}{\partial B^T} \int_{y_m}^{y_x} \int_{\hat{m}^T_S(\tilde{b}, B^T_m)}^{m_x} f(\tilde{b}, m, y) dm dy \end{array} \right] \le 0.$$

Otherwise, the (interior) local maximum is $B^{T*} = B^T_I \in (B^T_m, B^p)$ such that

$$(16')\quad \left[\begin{array}{l} \int_{y_m}^{y_x} \int_{b_m}^{\tilde{b}(B^T_I)} \left\{ (\bar{\Delta} - S) \dfrac{\partial \hat{m}^T_s(b, B^T_I)}{\partial B^c} f(b, \hat{m}^T_s(b, B^T_I), y) db \right\} dy \\ + \int_{y_m}^{y_x} \int_{b_m}^{\tilde{b}(B^T_I)} \int_{\hat{m}^T_S(b, B^T_I)}^{m_x} \dfrac{\partial a^*(b, m, B^T_I)}{\partial B^T} f(b, m, y) db dm dy - \bar{\Delta} \cdot \dfrac{\partial \tilde{b}}{\partial B^T} \int_{y_m}^{y_x} \int_{\hat{m}^T_S(\tilde{b}, B^T_I)}^{m_x} f(\tilde{b}, m, y) dm dy \end{array} \right] = 0.$$

2. High-end entry: Assuming $B^T \ge B^P$, $B^{T**} = B^T_x$ if and only if

$$(17)\quad \left[\begin{array}{l} \int_{y_m}^{y_x} \int_{\tilde{b}(B^T_x)}^{b_x} \left\{ (\bar{\Delta} - S) \dfrac{\partial \hat{m}^T_s(b, B^T_x)}{\partial B^c} f(b, \hat{m}^T_s(b, B^T_x), y) db \right\} dy \\ + \int_{y_m}^{y_x} \int_{\tilde{b}(B^T_x)}^{b_x} \int_{\hat{m}^T_s(b, B^T_x)}^{m_x} \dfrac{\partial a^*(b, m, B^T_x)}{\partial B^c} f(b, m, y) dm db dy - \bar{\Delta} \cdot \dfrac{\partial \tilde{b}}{\partial B^T} \int_{y_m}^{y_x} \int_{\hat{m}^T_s(\tilde{b}, B^T_x)}^{m_x} f(\tilde{b}, m, y) dm dy \end{array} \right] \le 0.$$

Otherwise, the (interior) high-end local optimum is at $B^{T**} = B^c_{II} \in (B^p, B^T_x)$ such that

$$(17')\quad \left[\begin{array}{l} \int_{y_m}^{y_x} \int_{\tilde{b}(B^T_{II})}^{b_x} \left\{ (\bar{\Delta} - S) \dfrac{\partial \hat{m}^T_s(b, B^T_{II})}{\partial B^T} f(b, \hat{m}^c_0(b, B^T_{II}), y) db \right\} dy \\ + \int_{y_m}^{y_x} \int_{\tilde{b}(B^T_{II})}^{b_x} \int_{\hat{m}T(b, B^T_{II})}^{m_x} \dfrac{\partial a^*(b, m, B^T_{II})}{\partial B^T} f(b, m, y) dm db dy - \bar{\Delta} \cdot \dfrac{\partial \tilde{b}}{\partial B^T} \int_{y_m}^{y_x} \int_{\hat{m}^T_S(\tilde{b}, B^T_{II})}^{m_x} f(\tilde{b}, m, y) dm dy \end{array} \right] = 0.$$

[25] We find this in our computational model below. Notice that we are also assuming that κ is small relative to the student population so that viable entry at the high and low ends is feasible.

The complete proof if in the appendix. Taking the case of $B^T < B^P$, the proof shows in part that the sign of the LHS of (16) (or (16)') coincides with $\text{Sign}\left[\frac{\partial \bar{\Delta}}{\partial B^T}\right]$. The expression in (16) contains the effects of changing B^T on the charter objective. To interpret it, consider an increase in B^T and refer to Figure 3. There are three effects on average achievement gains. The increase in B^T shifts marginally to the right the vertical lines at B^T and $\tilde{b}$, i.e., the latter value also increases. The third term in the expression is the *loss* of bringing in students in the vicinity of $\tilde{b}$. These students add essentially 0 to the average achievement gain, while taking away seats, then reducing the average achievement gain $(\bar{\Delta})$ by an amount proportional to the average achievement gain (multiplied by the magnitude of their increased numbers). The second term corresponds to the effect on all students who are unconstrained in the charter, i.e., those who have achievement equal to $a^*(b,m,B^T)$. These are students in the charter with $b < \tilde{b}$ and $m > \hat{m}_s^T(b, B^T)$. Among them, those with $b > (<) B^c$ have reduced (increased) achievement and so an achievement gain (loss) results from the increase in B^T. Thus, the second term in (16) is of ambiguous sign. The first term captures the last effect on achievement gains. This stems from a steepening of $\hat{m}_s^T$ at (approximately) point B in Figure 3. (The locus also shifts marginally, which is not shown, this having second-order effects.) Some of those with $b < B^T$ drop out, resulting in reduced achievement gain of S. The reverse is true for some of those with $b > B^T$. The change in $\bar{\Delta}$ is the gain or loss of S relative to $\bar{\Delta}$. It is likely that $S > \bar{\Delta}$. Therefore, the students within this group attracted to the school increase the average gain, while those that drop out decrease it. Thus, the first term is of ambiguous sign too. We note that $\hat{m}_s^T$ will also steepen as shown at (approximately) point A. But this has no first-order effect on $\bar{\Delta}$, since these students have $a^*(b, m, B^T) \approx S$.

To summarize, the first two terms on the LHS of (16) are of ambiguous sign, while the third is negative. In our computational analysis, we find that the net effect at $B^T = B_m^T$ is negative, implying the corner solution where a minimum number of applicants is optimal.

4.2. No-Excuses Charter School.

Now we examine competition between an NEC school and the public school. In addition to selecting its curriculum, denoted B^N, the NEC can enforce a minimum effort of its students, denoted e^m. NEC schools require significant parental involvement, and so we assume they can successfully enlist parental help in enforcing the minimum effort. Otherwise, the NEC school works as the TCS. It maximizes achievement gains of its students, its students must meet the same graduation standard, it faces the same size-viability constraint, and it must admit students randomly if over-subscribed. In short, the NEC is the same as the TCS but can enforce a minimum effort requirement.

Effort Choice. An NEC school student chooses the largest of three effort levels: the unconstrained optimal effort, e^*, the effort needed to graduate, e_S, or the minimum effort demanded by the school, e^m. This is provided such effort level is no greater than the maximum the student is willing to exert, e_0. If it is, the student would rather drop out and exert zero effort. Formally, let $e^N(b,m,B^N,Q,S)$ denote the effort exerted by students of type (b,m) at the NEC school. Using that $e_0 > e^*$, we have:

$$(18) \qquad e^N(b,m,B^N,\cdot) = \begin{cases} \text{Max}\left[e^*(b,m,B^N,\cdot), e_S(b,B^N,\cdot), e^m\right] & \text{if } e_0(b,m,B^N,\cdot) \geq \text{Max}[e_S(b,B^N,\cdot), e^m] \\ 0 & \text{if } e_0(b,m,B^N,\cdot) < \text{Max}[e_S(b,B^N,\cdot), e^m]. \end{cases}$$

The minimum-effort policy of the NEC school constrains the choices of student types (b,m) for whom $e^m > \text{Max}[e^*(b,m,B^N,\cdot), e_S(b,B^N,\cdot)]$. Among them, those that satisfy $e^m \leq e_0(b,m,B^N,\cdot)$ would choose to exert the minimum effort, while the rest would experience negative utility if they did, then prefer to drop out and would not apply to the charter school.

Applicant Set Given Choice of the Curriculum. An interesting difference from the TCS is that the NEC will adopt the same curriculum as the public school at its optimum. To show this, we must, of course, examine the applicant sets for both $B^N = B^P$ and for $B^N \neq B^P$. An analogous argument proves that the school prefers $B^N = B^P$ over any $B^N > B^P$ or any $B^N < B^P$. We then focus on the alternative with $B^N > B^P$ in developing the argument. We will prove that $B^N = B^P$ with optimal minimal effort choice is a local optimum. We assume:

Assumption 3 (A3): The NEC's local maximum at $B^N = B^P$ is the global optimum.

We do not have general conditions for this assumption, but we do verify it holds in our computational analysis. We will also show that the NEC does adopt an effort minimum that strictly binds students, in fact, all its students!

Turning to the analysis, while we provide the math, it is notable that one can understand well the several findings with the help of a few graphs. Refer to Figure 4. Let $\hat{m}_0^N(b,B^N,\cdot)$ and $\hat{m}_S^N(b,B^N,\cdot)$ denote, respectively, the loci in the (b,m)-plane separating students that would or would not drop out from the NEC and those that would just meet the standard or exceed it *assuming effort unconstrained by the NEC effort minimum*. The definition of these loci is analogous to that given for the public and TCS schools above, with $B = B^N$. Figure 4 depicts two iso-effort loci that identify students who would choose a given effort if they were attending the NEC school unconstrained by the effort minimum. The values of b_i, i = 1,2, satisfy $e_i = S/[b_i^\beta Q\Gamma(b_i, B^N)]$, where e_i is the constant effort defining the locus. The vertical portion of an iso-effort locus above b_i has students choose e_i to just meet the standard. The downward-sloping section of each locus contains students for whom $e^* = e_i$, who optimally choose the same effort level but with achievement that exceeds the standard.

Letting $\hat{m}_e^N(b, B^N, e_i, Q)$ denote the value of m along the latter part of an iso-effort locus, it is straightforward to confirm (using (5)) that it is given by:

(19) $$\hat{m}_e^N(b, B^N, e_i, Q) \equiv \frac{e_i}{b^\beta Q\Gamma(b, B^N)}.$$

It is also straightforward to confirm that the latter locus slopes down and lies above $\hat{m}_S^N(b, B^N, \cdot)$ for b > b_i. Note that effort *decreases* as the iso-effort loci shift rightward.

Consider a case with $B^N > B^P$ and with the NEC setting an effort minimum that would *not* constrain the $\tilde{b}$ type, recall the ability level that separates those who receive higher and lower quality at the charter school relative to the public school.[26] Figure 5 depicts the applicant set. The assumed effort minimum is that along the iso-effort locus with the vertical portion above ability denoted b_N. Another locus $\hat{m}_0^e = \frac{\hat{m}_e^N}{2}$ separates students that choose not to attend the NEC because the effort minimum exceeds the effort that would leave them with 0 utility (e_0).[27] As identified in Figure 5, three groups apply to the NEC school. One of them consists of students not constrained by the effort minimum who achieve the standard. Another has students neither constrained by the effort minimum nor by the need to meet the standard. These two groups are exactly as for a TCS with $B = B^N$. The third group of applicants consists of those effort constrained by e^m. For them, the effort minimum increases achievement, but the effort minimum does not induce them to drop out. Those "driven out" of the NEC due to the effort constraint may or may not attend the public school (depending on whether they are above or below the $\hat{m}_0^P$ locus, which Figure 5 does not depict).

If $B^N > B^P$ and the NEC school selects an effort minimum that would constrain the $\tilde{b}$ type, the applicant set changes. Figure 6 illustrates this case, with b_N, now less than $\tilde{b}$, again the ability of a student who would choose effort equal to the NEC minimum and just graduate (over a range of values of m). Some students with ability b below $\tilde{b}$ - hence offered a less efficient curriculum by the charter school - will still attend the NEC because the effort minimum increases their achievement. Here, the $\hat{m}_s^P$ locus and yet another one, denoted $\hat{m}_a^N$, come into play. Proposition 3 presents the details, but it is possible to provide the reader with an intuitive explanation using Figure 6. Three groups of applicants arise. For those with $b > \tilde{b}$, one group consists of types with relatively lower m-values constrained by the effort minimum. The second group optimally chooses effort that exceeds the minimum. The latter two groups have their analogues in the previous case with $b_N > \tilde{b}$ (see Figure 5). The third group of applicants have $b < \tilde{b}$, but higher achievement in the NEC school because they

[26] Figure 5 does not show the values of (B^P, B^N).

[27] One can find this locus by setting $e_i = e^m = e_0$ and using (8).

are effort constrained there. These students have ability $b \in [b_N, \tilde{b})$. A subset of them are relatively low m-types who would achieve the standard in the public school but above it in the NEC alternative due to its effort constraint (these have $m \in [\hat{m}_0^e, \hat{m}_s^P]$.) The other subset would exceed the standard in the public school, but the effort minimum imposed by the NEC school still induces higher achievement (these have $m \in [\hat{m}_s^P, \hat{m}_a^N]$.)

Two interesting observations about the latter applicant set are the following: (i) all applicants exceed the standard if matriculating to the NEC school; and (ii) unlike in the case with a TCS, it contains students who disagree with their parents about which school to attend. For example, all students with $b \in [b_N, \tilde{b})$ would prefer to attend the public school since it offers them a better curriculum match and does not push them to exert more effort than they would optimally choose.

Last, consider the case with $B^N = B^P$, which is simpler. *Only students that are effort constrained in the NEC would apply to it.* With the same curricula, the latter must hold for achievement to increase. *In turn, this implies choosing a binding effort constraint is optimal for the charter school if $B^N = B^P$.* Otherwise, the NEC has nothing to offer and would not attract any students.

It is easy to see that the applicant set is as shown in Figure 7, keeping in mind all students that would not be effort constrained in the NEC do not gain by attending the charter school and would not apply. *All applicants exceed the standard if they matriculate.* The nature of the applicant set is independent of how b_N compares to B_N.

Proposition 3 collects the main results, paying more attention to the $B^N = B^P$ case, which we show arises in equilibrium below.

Proposition 3. Let $A^N(B^N,e^m,B^P)$ denote the applicant set to the NEC.

1. Suppose the NEC school enters the market with curriculum target $B^N = B^P$. Then, $A^N(B^N, e^m, B^P) = \{(b,m) | b > b_N(B^N, e^m, Q, S),\ m \in [\hat{m}_0^N(b, B^N, e^m, Q), \hat{m}_e^N(b, B^N, e^m, Q)]\}$. Every student that matriculates chooses $e = e^m$ and exceeds the graduation standard.

2. Suppose the NEC school enters the market with curriculum target $B^N > B^P$ and minimum effort e^m such that $\tilde{b}(B^N, B^P) \le b_N(B^N, e^m, Q, S)$. Then:
$A^N(B^N, e^m, B^P) = \{(b,m) | b > \tilde{b}(B^N, B^P),\ m > \text{Max}[\hat{m}_0^N(b, B^N, e^m, Q), \hat{m}_0^e(b, B^N, Q, S)]\}$.

3. Suppose the NEC school enters the market with curriculum target $B^N > B^P$ and minimum effort e^m such that $b_N(B^N, e^m, Q, S) < \tilde{b}(B^N, B^P)$. For $b \in [b_N, \tilde{b}]$, define the locus

$\hat{m}_a^N(b, B^N, e^m, B^P, Q) \equiv \dfrac{\Gamma(b, B^N)e^m}{b^\beta Q \Gamma(b, B^P)^2}$, which satisfies $\hat{m}_a^N(\tilde{b}, B^N, e^m, B^P, Q) = \hat{m}_S^N(\tilde{b}, B^N, Q, S)$ and

$m_a^N(b_N, B^N, e^m, B^P, Q) = \hat{m}_S^P(b_N, B^N, Q, S)$. Then:

$$A^N(B^N,e^m,B^P)=\{(b,m)\big|\, b>\tilde{b}(B^N,B^P),\ m>\hat{m}_0^e(b,B^N,Q,S)]\}\cup$$
$$\{(b,m)\big|\, b\in(b_N,\tilde{b}(B^N,B^P)],\ m\in[\hat{m}_0^e(b,B^N,e^m,Q),\hat{m}_a^N(b,B^N,e^m,B^P,Q)]\}.$$

Proof of Proposition 3: See the appendix.

Choice of Curriculum and Effort Minimum. As we have already indicated, we show that setting the same curriculum as the public school and fixing e^m at its optimal value, denoted $e^{m}*$, yields a local maximum in (B^N,e^m). Suppose that $B^N > B^P$. Consider a case where e^m* would imply $b_N<\tilde{b}$, so that the application set would be as in Figure 6 and Part 3 of Proposition 3. Let AAG(B^N,e^m) denote the aggregate achievement gain that would result if all applicants to the NEC attended it. Also, let N(B^N,e^m) denote the measure of the applicant set (i.e., the "number" of applicants). The actual aggregate achievement gain (AG), taking account of random non-matriculation of applicants, random admissions, and the capacity constraint, is then:

(20) $$AG(B^N,e^m)=\kappa\frac{AAG(B^N,e^m)}{N(B^N,e^m)}.$$

The actual aggregate achievement gain equals the average among all applicants multiplied by the size of the charter school. Using Proposition 3 (or by inspection of Figure 6), for $B^N > B^P$ and $b_N>\tilde{b}$:

(21)
$$AAG(B^N,e^{m*})=\int_{y_m}^{y_x}\int_{b_N}^{\tilde{b}}\int_{\hat{m}_0^e}^{\hat{m}_a^N}[b^\beta Q\Gamma(b,B^N)e^{m*}-a^P(b,m)]f(y,b,m)dydbdm+$$
$$\int_{y_m}^{y_x}\int_{\tilde{b}}^{bx}\int_{\hat{m}_0^e}^{\hat{m}_e^N}[b^\beta Q\Gamma(b,B^N)e^{m*}-a^P(b,m)]f(y,b,m)dydbdm+$$
$$\int_{y_m}^{y_x}\int_{\tilde{b}}^{bx}\int_{\hat{m}_e^N}^{m_x}[b^\beta Q\Gamma(b,B^N)e^{*N}-a^P(b,m)]f(y,b,m)dydbdm$$

$$N(B^N,e^{m*})=\int_{y_m}^{y_x}\int_{b_N}^{\tilde{b}}\int_{\hat{m}_0^e}^{\hat{m}_a^N}f(y,b,m)dydbdm+$$
$$\int_{y_m}^{y_x}\int_{\tilde{b}}^{bx}\int_{\hat{m}_0^e}^{\hat{m}_e^N}f(y,b,m)dydbdm+$$
$$\int_{y_m}^{y_x}\int_{\tilde{b}}^{bx}\int_{\hat{m}_e^N}^{m_x}f(y,b,m)dydbdm$$

where a^P(b,m) denotes the achievement of student (b,m) in the public school (including 0 if a drop out) and e^{*j}, $j\in\{P,N\}$ is the unconstrained optimal effort in school j. Refer to Figure 6. The first integral term in AAG measures the achievement gain of the students in the left group of applicants. The second integral adds up the gains of the students that apply with $b>\tilde{b}$ who are constrained by the effort minimum, i.e. the applicants in the lower right group. The third integral collects the gains of the remaining applicants, these with $b>\tilde{b}$ but with motivation high enough for their unconstrained optimal effort choice in the NEC to exceed the effort minimum.

Now consider the impact on AAG, N, and AG as B^N declines toward B^P. The crucial effect is that the integrand of AAG in the third term converges toward 0 because students unconstrained in the

NEC will choose virtually the same effort and have virtually the same achievement in the public school. Also, the size of the first group will approach 0, because the measure of students with more efficient curriculum in the public school becomes vanishingly small. Thus, for δ near 0:

$$AAG(B^P+\delta, e^{m*}) \approx \int_{y_m}^{y_x}\int_{b_N}^{bx}\int_{\hat{m}_0^e}^{\hat{m}_e^N}[b^\beta Q\Gamma(b,B^P)e^{m*} - a^P(b,m)]f(y,b,m)dydbdm$$

(22)

$$N(B^P+\delta, e^{m*}) \approx \int_{y_m}^{y_x}\int_{b_N}^{bx}\int_{\hat{m}_0^e}^{\hat{m}_e^N} f(y,b,m)dydbdm + \int_{y_m}^{y_x}\int_{b_N}^{bx}\int_{\hat{m}_e^N}^{m_x} f(y,b,m)dydbdm$$

Note that the group that adds almost no achievement gains while vanishing from the approximate AAG in (22) (corresponding to the third term in the expression for AAG in (21)) is of strictly positive measure and part of approximate N: While these students add almost nothing to the NEC's objective, they take up space in the charter school.

If $B^N = B^P$, the latter group does not apply, and the expressions for AAG and N become:

$$AAG(B^P, e^{m*}) = \int_{y_m}^{y_x}\int_{b_N}^{bx}\int_{\hat{m}_0^e}^{\hat{m}_e^N}[b^\beta Q\Gamma(b,B^P)e^{m*} - a^P(b,m)]f(y,b,m)dydbdm$$

(23)

$$N(B^P, e^{m*}) = \int_{y_m}^{y_x}\int_{b_N}^{bx}\int_{\hat{m}_0^N}^{\hat{m}_e^N} f(y,b,m)dydbdm$$

The integrand in AAG in (22) and (23) is strictly positive. It equals the achievement gain to a student facing the same curriculum in the public school but who is constrained by the effort minimum in the NEC. Using (20), (22), and (23), one sees that $AG(B^P, e^{M*}) > AG(B^P+\delta, e^{M*})$ because $N(B^N, e^{m*}) < N(B^N+\delta, e^{m*})$. An analogous argument shows the same if e^{m*} is such that $b_N > \tilde{b}$ when B^N strictly exceeds B^P or if we consider B^N strictly less than (but arbitrarily close to) B^P.[28] Therefore, we have:

Proposition 4:

1. Choice by the NEC of $B^N = B^P$ and $e^m = e^{m*}$ is a local maximum.
2. Under (A3), it is the global maximum.

Proof. We have shown that $AG(B^N, e^{m*})$ strictly exceeds $AG(B^N+\delta, e^{m*})$ for δ near 0. The fact that $AG(B^N, e^m)$ is continuous in (B^N, e^m) for $B^N = B^P + \delta,\ \delta > 0$ completes the proof of Part 1. Part 2 follows trivially.

[28] When one writes out the terms of AAG and N for $b_N > \tilde{b}$ and $B^N > B^P$, one uses Part 2 of Proposition 3 or Figure 5. On the other hand, in cases where $B^N < B^P$, lower ability students attracted to the NEC school will be the ones that do not contribute to the objective.

The result is intuitive: *By choosing the same curriculum as the public school, the NEC avoids applicants who would otherwise only gain slightly in achievement. Such applicants would be of high ability and motivation and not be effort constrained. As we have already observed, with $B^N = B^P$, the effort minimum constrains the effort choices of all students in the NEC school.*

The NEC's choice of effort minimum maximizes (20) but subject to generating sufficient demand to meet the capacity requirement, i.e., $\mu N(B^N, e^m) \geq \kappa$. Refer to Figure 7 and consider a change in e^m. Ignoring the viability constraint for the moment, differentiating (20) and using that the achievement gain is 0 on the left and upper boundary of the applicant set, one finds:

(24)

$$\text{sign}\left(\frac{\partial AG}{\partial e^m}\right) = \text{sign}\left(\frac{\partial AAG}{\partial e^m} - \frac{AAG}{N}\frac{\partial N}{\partial e^m}\right)$$

$$\frac{\partial AAG}{\partial e^m} = \int_{y_m}^{y_x}\int_{b_N}^{b_x}\int_{\hat{m}_0^e}^{\hat{m}_e^N} b^\beta Q\Gamma f dm db dy - \int_{y_m}^{y_x}\int_{b_N}^{b_x}[2b^\beta Q\Gamma]^{-1}(b^\beta Q\Gamma e^m - a^P(b,\hat{m}_0^e)) f(y,b,\hat{m}_0^e) db dy$$

$$\frac{\partial N}{\partial e^m} = \int_{y_m}^{y_x}\int_{b_N}^{b_x}[b^\beta Q\Gamma]^{-1} f(y,b,\hat{m}_e^N) db dy - \int_{y_m}^{y_x}\int_{b_N}^{b_x}[2b^\beta Q\Gamma]^{-1} f(y,b,\hat{m}_0^e) db dy - \int_{y_m}^{y_x}\int_{\hat{m}_0^e}^{\hat{m}_e^N}\frac{\partial b_N}{\partial e^m} f(y,b^N,m) dm dy$$

Three effects arise. First an increase in e^m increases the achievement of all students in the interior of the applicant set, an effect captured by the first integral term in $\partial AAG / \partial e^m$. Second, an increase in e^m also shifts up the lower bound of the applicant set (i.e., $\hat{m}_0^e$ shifts up). The higher minimum effort required drives out some students whose achievement gains are then sacrificed. The second integral in $\partial AAG / \partial e^m$ captures this effect. The third effect is that the shifts in the lower, left, and upper boundary of the applicant set imply a net change in the number of applicants.[29] Given random matriculation and admission, the effect of changing N is proportional to the average applicant achievement gain. This is captured by the second term in the upper line of (24), along with $\partial N / \partial e^m$.

An interior solution would have the expression in the upper line of (24) vanish. We find a corner solution in our computational model. Here e^m is driven up to the point where the viability constraint just binds. At that pointN is decreasing in e^m and the sign of the expression in the upper line of (24) is positive.

5 Computational Model and Analysis.

5.1 Calibration of the Computational Model. The building blocks of our model are the joint distribution of income, cognitive and non-cognitive skills, the achievement production function, and the cost function for student effort. Education policy establishes curriculum and sets standards for graduation. These policies influence students' effort decisions which, coupled with the distribution of

[29] The shifts in the left and upper boundaries do not affect AAG because the achievement gains along these bounds are 0.

skills and the achievement function, determine the realized distribution of achievement and the realized dropout rate. Our approach to the calibration is to construct a model of a small nationally representative area with one public high school and no charter school. The area has household distribution of income corresponding to that in the US, and with a student population for which the joint distribution of cognitive and non-cognitive skills is representative of the student population in the US and for whom attending the public high school yields an achievement distribution representative of the 12th grade achievement distribution in the US.

The parameters of our model are the following:[30]

μ_y, σ_y : mean and standard deviation of the distribution of the ln(y)
μ_b, σ_b : mean and standard deviation of the distribution of the ln(b)
μ_m, σ_m : mean and standard deviation of the distribution of ln(m)
ρ_{by} : correlation of ln(b) and ln(y)
ρ_{bm} : correlation of ln(b) and ln(m)
β: exponent of b in the achievement function, Equation (25) below
Ω: product of Γ_m and Q^p in the achievement function, Equation (25) below
α, τ : parameters of the curriculum function, Z(b,B), Equation (1)
B^p, S^p: ability target in curriculum of, and standard in, the public school

Additional targeted values employed to determine the parameters are:

μ_a, σ_a : mean and standard deviation of the distribution of ln(a)
ω: effect in standard deviations of the logarithm of achievement of being at distance ($b_{.975} - Bp$) from the curriculum target.

We next summarize the empirical evidence employed in selecting parameters for our model. We estimate the mean and standard deviation of the distribution of the logarithm of income as follows. The US income distribution is well approximated by a lognormal distribution. Let y denote income. Then $\ln(y) \sim N(\mu_y, \sigma_y)$. Note that we let μ_y denote the mean of ln(y) and σ_y denote the standard deviation of ln(y). We use this same subscripting convention throughout. Mean and median household income in 2014 were, respectively $72,641 and $53,657.[31] The mean of a lognormal distribution is the log of the median of the unlogged distribution. Hence, $\mu_y = \log(53{,}657) = 10.89$. The mean of the unlogged distribution is $\exp(\mu_y + \sigma_y^2/2)$. Solving this for σ_y, substituting values above, and rounding to two digits after the decimal point, we obtain $\sigma_y = (2*(\ln(72641) - \mu_y))^{.5} = .78$. In 2014, 14.8% of households were below the poverty line.[32] Hence, when we analyze equilibrium with a poverty population, as when charter schools commonly enter in a central city, we take that population to be households below the 14.8th percentile of the income distribution.

We take the joint distribution of the logarithms of ability and income to be normally distributed. In modeling the relationship of income and ability, we take the correlation of ln(y) and

[30] For simplicity, we have assumed the correlation of m and y equals 0.
[31] https://en.wikipedia.org/wiki/Household_income_in_the_United_States#Mean_household_income
[32] https://www.census.gov/library/publications/2015/demo/p60-252.html

ln(b), ρ_{by}, to be .4. This in turn implies that a household with ability one-tenth of a standard deviation above the median will have earnings 3.5% higher than a household at the median. This corresponds closely to the 3.3% estimate of the gain in annual income at age 50 from a one-tenth standard deviation increase in cognitive ability in Table 3 of Lin, Lutter, and Ruhm (2018).

Findings of Cunha, Heckman, and Schennach (CHS) (2010) and Borghans, Golsteyn, Heckman, and Humphries (BGHH) (2016) play a central role in our calibration of non-cognitive skills, the relationship of non-cognitive to cognitive skills, and the roles of the two types of skills in determining achievement. From CHS, we obtain $\rho_{bm} = .2165$ and $\sigma_m / \sigma_b = \sqrt{.75}$. The details are presented in the accompanying appendix.

BGHH Appendix Table 7.8 columns (1) and (2) provide regressions of achievement on IQ and achievement on non-cognitive skills respectively. The ratio of the R^2 statistic in column (2) to that in column (1) is .173/.489 = .35. Our calibration requires the counterpart regressions in our simulated sample of 20,000 public-school students to produce the same R^2 ratio. In particular, let R^2_{am} be the R^2 obtained in our model from regressing ln(a) on ln(m) and let R^2_{ab} be the R^2 obtained from regressing ln(a) on ln(b). Our calibration requires that, in equilibrium, $(R^2_{am}/ R^2_{ab})=.35$.

In their regression including both cognitive and non-cognitive skills, (Table 7.8, column 3), BGHH obtain an $R^2 = .53$. As they note, measurement errors and/or unmeasured factors account for the unexplained variance. In our calibration, we assume the R^2 would be .75 if all relevant elements on the right-hand-side were measured without error. This implies that the variance of achievement, σ_a^2, can be obtained from the variance of measured achievement, σ_x^2, as follows: $\sigma_a^2 = .75\sigma_x^2$. The latter is also required in our calibration.

Our distribution of the logarithm of ability is as follows. As noted above, we take ability to be lognormally distributed, and we take the logarithm of ability to be IQ. The standard measure of IQ has mean 100 and standard deviation 15. The ratio of the latter to the former is .15. Hence, in our model, the ratio of the standard deviation of log ability to the mean of log ability is $\sigma_b / \mu_b = .15$.[33]

To obtain empirical-counterpart values for the mean (μ_a) and standard deviation of the logarithm of achievement (σ_a), we utilize data for the distribution of 12th grade scores from the National Assessment of Educational Progress, NAEP[34]. Median NAEP scores and standard deviations are reported for mathematics, reading, and writing. Mathematics, reading, and writing have median scores of 152, 289, and 148 respectively with standard deviations of 34, 35, and 41 respectively.

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[33] We exploit the fact that the quantiles of a normal distribution are invariant to the choice of mean as long as the ratio of the standard deviation to the mean is preserved. For example, IQ can be measured with a mean of 1 and standard deviation of .15, a mean of 2 and standard deviation .3, etc. We exploit this feature at several points in our calibration.

[34] We obtained this NAEP information from: https://www.nationsreportcard.gov/ndecore/xplore/NDE

NAEP scores are approximately normally distributed.[35] Hence, we use the reported medians as means. We rescale all three variables to have a common standard deviation, 34, yielding mean scores of 152, 240, and 148 respectively for mathematics, reading, and writing. We then create a weighted average with .5 weight for mathematics and .25 each for reading and writing. The weighted mean is 180. Hence, the composite score is distributed normally with mean 180 and standard deviation 34. To provide convenient magnitudes for reporting simulation results, and without loss of generality, we divide NAEP achievement by 10, giving mean achievement equal to 18. Hence, we set μ_a, such that $\exp(\mu_a)=18,$ implying $\mu_a = \log(18) \approx 2.89.$ Letting σ_x denote the standard deviation of the composite NAEP score, allowing for measurement error, we obtain $\sigma_x / \mu_a = 34/180 = .189.$ This and $\sigma_a^2 = .75\sigma_x^2$ imply $\sigma_a / \mu_a = (.189\sqrt{.75}) = .164 \approx 1/6$. This and $\mu_a = \log(18) \approx 2.89.$ imply $\sigma_a = \log(18)/6 \approx .48.$ Thus, the logarithm of achievement in our model has mean 2.89 and standard deviation .48. Note that this derivation of μ_a and σ_a does not require that the logarithm of achievement be normally distributed. We discuss this further below.

Another targeted empirical value is the high school dropout rate. From the National Center for Education Statistics[36], we obtain the high school dropout rate of 5.4%.

The following equations will be used in detailing the calibration of the curriculum function in our computational model. Our achievement function can be expressed as:

(25) $$a(b,e,B,Q) = b^{\beta}Q\Gamma(b,B)e = b^{\beta}Q\Gamma_m Z(b,B)e = b^{\beta}\Omega Z(b,B)e.$$

Recall that $Z(b,B) = \left[\alpha + (1-\alpha)\exp\left\{-\frac{(b-B)^2}{\tau^2}\right\}\right]$. All else constant, the impact on the logarithm of achievement for a student with ability at the curriculum target, ln(B), relative to a student with logarithm of ability $\ln(B)+\delta$ is the following:

(26) $$\begin{aligned}&\ln\left[Z(e^{\ln(B)+\delta}, e^{\ln(B)})\right] - \ln\left[Z(e^{\ln(B)}, e^{\ln(B)})\right] = \ln\left[Z(e^{\ln(B)+\delta}, e^{\ln(B)})\right] - \ln\left[Z(1,1)\right]\\ &= \ln\left[Z(e^{\ln(B)+\delta}, e^{\ln(B)})\right]\end{aligned}$$

We assume the public school targets students of median ability. Hence, $B^P = e^{\mu_b}$. Consider a student at the 97.5th percentile of the logarithm of ability, $b_{.975}$. We choose parameters α and τ of the Z(b,B) function so that the effect of being at the 97.5th percentile of ability relative to median ability is .04 standard deviations of the logarithm of achievement. That is, we choose α and τ with the objective of having ω in the equation below be .04. We have not found direct empirical evidence to calibrate this measure. Hence, this choice of ω is largely based on our judgment.

[35] See https://nces.ed.gov/nationsreportcard/tdw/analysis/trans_compare.aspx. For mathematics and reading, see respectively https://nces.ed.gov/nationsreportcard/tdw/analysis/2013/trans_compare_math2013g12.aspx and https://nces.ed.gov/nationsreportcard/tdw/analysis/2013/trans_compare_read2013g12.aspx.
[36] This was the dropout rate in 2017 reported in Table 2.1 of https://nces.ed.gov/pubs2020/2020117.pdf

(27) $\omega\sigma_a = \ln[Z(b_{.975}, B^P)]$

The latter pins down one of the two parameters that characterize the Z(b,B) function. Recall that B denotes the ability level to which the curriculum is targeted. Students drop out if their achievement falls below the standard, S^P. We set α=.85. Hence, other things constant, a student at the extreme distance from the curriculum target, $(B-b)^2 \to \infty$, suffers a decrement of .15 standard deviations of achievement relative to the achievement the student would have if the curriculum were targeted to that student's ability.

For calibrating β in the achievement function (25), it is useful to consider a student (b,m) not constrained by the standard. Equation (5) shows that optimal effort implies:

$$e^*(b,m,B,Q) = b^\beta \Omega Z(b,B) m$$

Hence, achievement for an unconstrained student is:

$$a^* = b^\beta \Omega Z(b,B) e^* = [b^\beta \Omega Z(b,B)]^2 m$$

Taking logs, we obtain the following[37].

(28) $\ln(a^*) = 2\beta \ln(b) + 2\ln(\Omega) + 2[\ln(Z(b,B))] + \ln(m)$

Taking expectations then yields:

$$\begin{aligned} E\ln(a^*) &= 2\beta E\ln(b) + 2\ln(\Omega) + 2E[\ln(Z(b,B))] + E\ln(m) \\ &= 2\beta\mu_b + 2\ln(\Omega) + 2E[\ln(Z(b,B))] + \mu_m \end{aligned}$$

As the preceding relationship shows, $\beta\mu_b$ scales the relationship between ln(b) and ln(a*). Thus, $\beta\mu_b$ can be treated as a single parameter. Hence, we set β=1 while preserving in our calibration the requirement derived above that specifies σ_b / μ_b =.15.

To complete our calibration, we employ an inner-loop, outer-loop approach with the inner loop solving for equilibrium in a simulated sample of 20,000 students and the outer-loop varying the parameters to be calibrated. Assembling the items above, in addition to fixing the already calibrated values of the joint income-ability distribution, our calibration objective is to have the following ten conditions be satisfied in equilibrium:

ρ_{bm} =.2165, $\sigma_m = \sigma_b\sqrt{.75}$, σ_b / μ_b =.15, μ_a = log(18), $\ln(B^P) = \mu_b$, α = .85, σ_a / μ_a =.167, (R^2_{am}/R^2_{ab})=.35, dropout rate=5.4%, and $\omega_{.975}$=.04. We proceed as follows. We impose the first six of the

[37] Earlier, we chose the mean and standard deviation of the logarithm of achievement, μ_a and σ_a, by benchmarking those parameters to the NAEP. In doing so, we noted that we did not impose the requirement that the logarithm of achievement in our model be normally distributed. As the preceding equation demonstrates, the logarithm of achievement is not normally distributed. While both ln(b) and ln(m) are normally distributed, ln(Z(b,B)) is not. The above equation is for students not constrained by the curriculum standard. There are also students who are constrained by the standard, including some who drop out. These factors also gives rise to a departure of the achievement distribution from lognormality. We emphasize that these departures from lognormality are not of concern. It is inevitable that a realistic model with curriculum standards and dropouts will have an achievement distribution that is not lognormal. Indeed, the NAEP itself is administered to students who are in school, and hence does not measure achievement of those who dropout.

latter conditions. We then search for values of the remaining parameters targeting the last four conditions in equilibrium, i.e., σ_a / μ_a =.167, (R^2_{am}/ R^2_{ab})=.35, dropout rate=5.4%, and $\omega_{.975}$=.04. In particular, we search for values of Ω, μ_b, μ_m, τ, and S^P. The chosen parameters must be consistent with the equilibrium requirement that all students choose their optimal effort given their type (b,m), the curriculum B^P, and the graduation standard S^P. Satisfying the optimality conditions for the entire set of students limits the set of parameters that meet the targets, but we succeed in closely approximating these conditions.

A fine-grained grid search yields the following parameter values:

$$\Omega = .625;\ \mu_b = 1.8; \mu_m = .023; \tau = 11.25; S^P = 10.6; \sigma_b = .27; \sigma_m = .0234; \rho^{bm} = .2165; \alpha = .85;$$
$$\text{and } B^p = e^{\mu_b} = 6.05.$$

Measures of fit are in Table 1. The comparison of targets to outcomes illustrates that the calibration satisfies these conditions quite well.[38]

Table 1: Model Fit

Measure	Target	Model
R^2_{am}/ R^2_{ab}	.35	.326
Dropout Rate	.054	.055
σ_a / μ_a	.167	.163
$\omega_{.975}$	.4	.413

5.2 Computational Findings.

In our computational analysis, we investigate five alternative cases of charter schools. Results are presented in Table 2. Columns (1) and (2) report results for traditional charter schools serving a population representative of the US population. Results with optimal curriculum target below the public school curriculum are in Column (1) while results with optimal curriculum target above the public school's are in Column (2). Recall that two local optima exist, one with curriculum target above and one with curriculum target below that at the public school. Columns (3) and (4) report results for traditional charter schools serving a population representative of a poverty population, with curriculum target below, Column (3), and above, Column (4), that of the public school. In both cases of the student populations, the global optimum is that with curriculum target above the public school's target. However, if a charter school has a primary preference to target lower ability students and secondary preference to maximize achievement gains, then the local maximum with $B^c \leq B^p$ is of interest. Column (5) presents results for a no-excuses charter school serving a poverty population.

[38] Recall that we targeted $\omega_{.975}$=.04. With the above calibration, a student at the 2.5th percentile would then experience a .015σ_a decrement relative to a curriculum targeted to the student's ability, $\omega_{.025}$=.015. The asymmetry in the magnitude of the two ω values arises because achievement is lognormally distributed. Hence, a student at the 2.5th percentile is closer to median ability than a student at the 97.5th percentile of ability.

	Table 2: Computational Results				
	(1)	**(2)**	**(3)**	**(4)**	**(5)**
	TCS Income Nat Rep $B^C \leq B^P$	**TCS Income Nat Rep** $B^C \geq B^P$	**TCS Income Pov** $B^C \leq B^P$	**TCS Income Pov** $B^C \geq B^P$	**NEC Income Pov**
$S^P = S^C$	10.6	10.6	10.6	10.6	10.6
B^C, ln(B^C)	4.08, 1.41	9.10, 2.21	3.49, 1.25	6.68, 1.9	6.05, 1.8
B^C Percentile	.072	.935	.020	.644	.5
B^P Percentile[39]	.5	.5	.5	.5	.5
Prop. Attend Ch	.1	.1	.1	.1	.1
Prop. Apply Ch	.2	.2	.2	.2	.2
Med Ch Inc./ Med Pop Inc.	.72	1.60	.97	1.08	1.06
Achieve Gain*	.000σ_N	.022σ_N	.000σ_P	.003σ_P	.52 σ_P
Prop Effort Constrained Charter	NA	NA	NA	NA	100%
Prop Charter Exceed Standard	14.4%	100%	3.8%	99.5%	100%
Prop Dropouts if charter	5.41%	5.50%	13.76%	13.87%	13.87%
Prop Dropouts if No charter	5.50%	5.50%	13.87%	13.87%	13.87%

* Standard deviations of public school achievement: National population, σ_N=.785; Poverty population, σ_P=.976.

5.2.1 Traditional Charter Schools. Column (1) of Table 2 is for a TCS in a neighborhood with population representative of the national population and with curriculum optimally chosen to serve low-ability students in that population. Looking down the column, we see that the TCS curriculum target is to a student at the 7.2th percentile of ability. Recall that the public school targets the student of 50th percentile ability. The charter school has capacity to serve 10% of the population in the school zone and 20% percent of students apply. The median income of charter school students is roughly three-fourths the median income of the neighborhood population. The mean achievement gain of students attending the charter school is exceedingly small relative to those who applied but lost the charter school lottery. Most charter school students just meet the standard for graduation and most of them would do the same if attending the public school. However, 14.4% exceed the standard. Moreover, by targeting lower-ability students, the charter school helps reduce the dropout rate somewhat. The dropout rate with only public schooling is 5.50%. With the charter school, the population dropout rate decreases to 5.41%. This reduction in dropout rate is achieved by a charter school serving 10% of the population. The attendance spaces for this equilibrium are shown in Figure 8a. The ability level targeted by the curriculum is shown by the left vertical line. This vertical line

[39] Note that this is for nationally representative population across row.

intersects the horizontal axis the value of $\ln(B^C)=1.41$. This target, and the curriculum targets for the other cases, are shown in Table 2. Charter school applicants occupy the wedge in the upper-left region of Figure 8a, with right border midway between the public and charter curriculum targets.[40] This is comparable to Figure 2 from the theoretical development, though Figure 2 uses hypothetical parameters.[41] The widening of the wedge as one moves vertically arises because highly motivated low-ability students are willing to make the effort to meet the achievement standard required to be retained (graduate) in school. Students of lower motivation below and to the left of the wedge prefer to drop out and do not attend the charter school. Figure 8b shows detail for the charter school applicants, with additional detail for the charter attendees (lottery winners). Charter attendees in the upper left triangle in the figure are those who exceed the standard, comprising, as noted above, 14.4% of charter school attendees. All remaining charter attendees perform at the minimum effort required to just meet the standard.

Column (2) of Table 2 reports the results for a TCS targeting high ability students. This is the global optimum of achievement gains maximization. The optimally chosen target is the 95.5th of ability. The charter school attracts a clientele with household income substantially above that of the population; median income of the TCS students is 1.6 times that of the population. Charter school students gain .022 standard deviations in achievement relative to those who applied and lost the charter lottery. This is a modest but non-negligible gain. This TCS attracts students who would not drop out whether in public or charter school. Hence, entry of this TCS does not affect the overall dropout rate. As shown in Figure 9a, the charter school attracts high-ability students. Here, the left boundary of the space of charter school applicants lies halfway between the public school curriculum target and the charter school curriculum target. Figure 9b shows the charter school applicant space in more detail. All charter school applicants choose to exert effort well above that required to meet the retention standard. Hence, in contrast to the TCS serving low ability students, no students in the high-ability charter are constrained by the standard.

Column (3) of Table 2 presents results for a charter school targeting low-ability students in a neighborhood with a poverty population. Achievement of students attending the charter are slightly higher than for charter applicants who lost the lottery, and gains are slightly higher than for the comparable case with nationally representative population. As shown in Figure 10a, the curriculum target for the charter school is set at a quite low level, far below the public school curriculum target. By setting a low target, the charter differentiates itself enough from the public school to attract the lowest ability students and maximize achievement gains. The effort of almost all charter students, 96.2%, is constrained by the requirement to meet the achievement standard in order to be retained in school. Figure 10b shows the small number of students in the upper right corner of the charter school

[40] Here and in the remaining discussion, we use charter school applicants to refer to both those who win and those who lose the charter school lottery.
[41] Figure 2 also presents attendance choices in the (b,m) plane, rather than the (ln b,ln m) plane.

admission space who choose effort higher than that required to be retained in school. Entry of this TCS lowers the dropout somewhat because the targeting of low ability students enables some of them to learn material that they would not be able to learn with the public school curriculum target.

Column (4) of Table 2 reports results for a TCS serving a poverty population with curriculum target above that for the public school. The median income in this TCS is about 8 percent higher than the median income in the poverty neighborhood. The mean achievement gain of TCS students is modest, .003 standard deviations. Because the poverty population has a higher concentration of lower ability students than the nationally representative population, the charter curriculum target is not too far above the public school's target, even though optimally attracting higher ability students from the population. Almost all charter students, 99.5%, achieve above the standard required for retention in school. This selection is displayed in Figure 11a. Figure 11b shows the space of charter applicants in more detail. As the figure shows, all perform above the standard except the tiny fraction in the lower left corner who must exert effort above their preferred level in order to meet the standard for retention.

5.2.2 No Excuses Charter School. Column (5) of Table 2 displays results for a No-Excuses charter school serving a poverty neighborhood. We examine only the poverty population because NEC schools are usually urban. Recall that the optimum has $B^c = B^p$. Students attending this NEC obtain very large achievement gains, $.516\sigma_P$, relative to those who applied to the school but lost the lottery. Comparison of Figure 12a to Figure 11a shows that this NEC attracts a larger proportion of highly motivated students than the TCS despite the latter having a higher curriculum target, while the NEC does not attract students who have relatively high ability but low motivation. Students that attend the NEC must be sufficiently motivated to increase their effort to the required level.

5.2.3 Model Predictions and the Literature

Our theoretical model predicts that traditional charter schools will differentiate their curriculum to maximize the achievement gains of the students they attract. Whether the student population is poor or representative, the curriculum choice that maximizes achievement gains targets high ability students. Our computational model quantifies these predictions. As summarized above, we find, for a nationally representative neighborhood, that an entering TCS charter serving high-ability students population yields achievement gains of $.022\sigma_N$ whereas a TCS entrant serving low-ability students can induce only very small gains. For students representative of a poverty neighborhood, we find the same, but with the differential gain for a TCS serving high ability students in that neighborhood quite modest. Epple, Romano, and Zimmer (2016) provide an extensive review of charter school research. The evidence from this body of research is that traditional charter schools typically yield modest achievement gains, if any, relative to traditional public schools. The findings of our quantitative model accord with this evidence. In our calibrated model, curriculum targeting is effective in selecting students, but is not enough to generate large achievement gains.

By contrast, the evidence is that students in no-excuses charter schools have very large achievement gains. While the NEC in our model can vary the ability target of the curriculum relative to the public school curriculum target, our model predicts that it will not. Rather, the NEC focuses on getting students to work hard. Dobbie and Fryer (2013) investigate practices of no excuses charter schools in New York City. One finding is: "Surprisingly, lesson plans at high-achieving charter schools are *not* (italics added) more likely to be at or above grade level and do not have higher Bloom's Taxonomy scores (p. 38)."[42] Thus, in choosing curriculum, the NEC does not appear to target brighter or weaker students. This is not to suggest that they have the same learning expectations as does the public school. To the contrary, the high expectations ethos, longer contact times, frequent testing, and parental engagement of NEC schools are consistent with the focus on student effort our model predicts (see the discussion and references in Section 2).

Our model predicts much larger achievement gains from no excuses charter schools than from traditional charter schools. As shown in Column (5) of Table 2, our model predicts that a no-excuses charter school serving a neighborhood with a poverty population increases achievement by .52 standard deviations. Walters (2018) provides empirical estimates that no-excuses charter schools serving poverty population increase achievement for mathematics and reading by .71 and .52 standard deviations respectively.

In addition, the predicted achievement gain in the no excuses charter school is higher for the lower scoring students. This is also consistent with estimates (see Abdulkadiroglu, et.al., 2011 and Angrist, et.al., 2012, and Walters (2018)). In his analysis, Walters (2018) finds that the gains in achievement in no-excuses charter schools are greatest for students who were previously relatively low achievers. For a random sample of 400 charter lottery applicants from our model, we simulated their charter school scores if they won the charter lottery and their public school scores if they had lost the lottery. This sample size is comparable to that in Walters (2018). The regression in (29) below using this simulated data is the counterpart to the regressions presented by Walters in Table 7. As in Walters empirical estimates, we find that that the gains to charter school attendees are greatest for those who would otherwise have had the lowest achievement scores if they were in public school. In this simulated sample, these coefficients are both highly significant ($p<.001$). Our coefficient of -.46 is larger in absolute value than the estimates obtained by Walters. In practice, of course, an achievement test measures scores with error, which would tend to attenuate the magnitude of the estimated coefficient. In addition, Walters includes quite appropriately indicator variables for demographic groups. This may also reduce the magnitude of the coefficient estimate. Thus, while differences in magnitude are to be expected, our model delivers the kind of inverse relationship between achievement gains and public school achievement scores that is found by Walters in his

[42] Bloom's Taxonomy identifies six levels of thinking that are used to "organize objectives and create lesson plans with appropriate content and instruction …," https://tophat.com/blog/blooms-taxonomy/.

empirical analysis.[43] This is another important respect in which the implications of our no-excuses model are supported by empirical evidence.

(29) ln(Achievement Gain) = 1.94 -.46·ln(Achievement in Public School)

Finally, the students the no excuses charter school attracts in our model that draw from the poverty population have profile as found in urban no excuses charter schools. They are not as poor and have higher prior scores within the population of poverty students, with, again, the predicted score in the public school equated to prior scores. This is found e.g., by Abdulkadiroglu, et.al. (2011), Angrist, et.al. (2012), and Angrist, Pathak, and Walters (2013), and Walters (2018).[44] Thus, the key predictions of our no-excuses model conform nicely to the empirical evidence.

6. Concluding Remarks

We have developed a model to analyze charter school educational practices and entry. Charter schools aim to maximize achievement gains of their students, drawn from a student population differentiated by cognitive ability, motivation (or non-cognitive ability), and income. Students must put forth study effort to achieve, with achievement dependent on their cognitive ability, how well their school's curriculum matches their capability, and how hard they study. More motivated students are more inclined to study. We capture the autonomy of charter schools by allowing differentiation in the curriculum match to student ability. An entering charter school selects students by choosing the level of difficulty of its curriculum with the objective of maximizing student achievement gains subject to the requirement that it draw enough students to be viable. Two local optima exist, one with a charter school that enters with a curriculum targeted to higher ability students than the public school's curriculum, and another with curriculum targeted to lower ability students. In a calibrated computational counterpart model, the global optimum by a charter school targets higher ability students. Here, the charter school draws off the highest ability students along with highly motivated students who are of more modest ability. Achievement gains are modest, consistent with most of the empirical evidence.

Motivated by the practices of the no-excuses segment of charter schools, we then also investigate charter schools that enforce an effort minimum. Interestingly, a no excuses charter chooses not to differentiate its curriculum from the public school curriculum, and instead employs the effort minimum to induce selection into the charter by highly motivated students. No excuses charter

[43] Walters (2018) also makes the point that the finding that marginal students that select into no excuses charter schools gain the most implies potential for higher gains: "charter expansion is likely to be most effective when targeted to students who are currently unlikely to apply (p. 2182)." In our model, students just below the margin of application to the charter are unwilling to put in the required effort and would then fail at the charter school. To induce them to select in and succeed (i.e., to target them) would require a lower effort minimum, this adversely affecting the performance of more motivated students. An interesting question is whether several charter schools with different effort minimum requirements would improve average achievement gains in light of student selection.

[44] Another well established characteristic of no excuses charter students is that they have higher proportion black students, though race is not an element of our model.

school students must be willing to study a lot. Consistent with the evidence on such charter schools, we find in our computational model that the no excuses approach is highly effective in increasing student achievement. For some students, this occurs at the expense of student utility, which triggers a disagreement in the preferences over schools of those children and their parents. We also find that the largest achievement gains accrue to the lower achieving students among those that attend the charter school relative to their counterparts who applied for the charter school but lost the charter lottery. This is also consistent with the evidence.

Several extensions of the model for potential future research are of interest. Our model has a graduation or retention standard for achievement, which we assume is dictated by the educational authority and must be adhered to by both the public and charter school. Varying this standard, if allowed, is another potential dimension on which a charter school might pursue its objective. It would also be of interest to investigate whether public schools might seek to ward off competition from charter schools by adoption of tracking. Our model considers competition between just one charter school and one public school. Consideration of equilibrium with more schools is also of interest.

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Figure 1. Student Partition in Public School

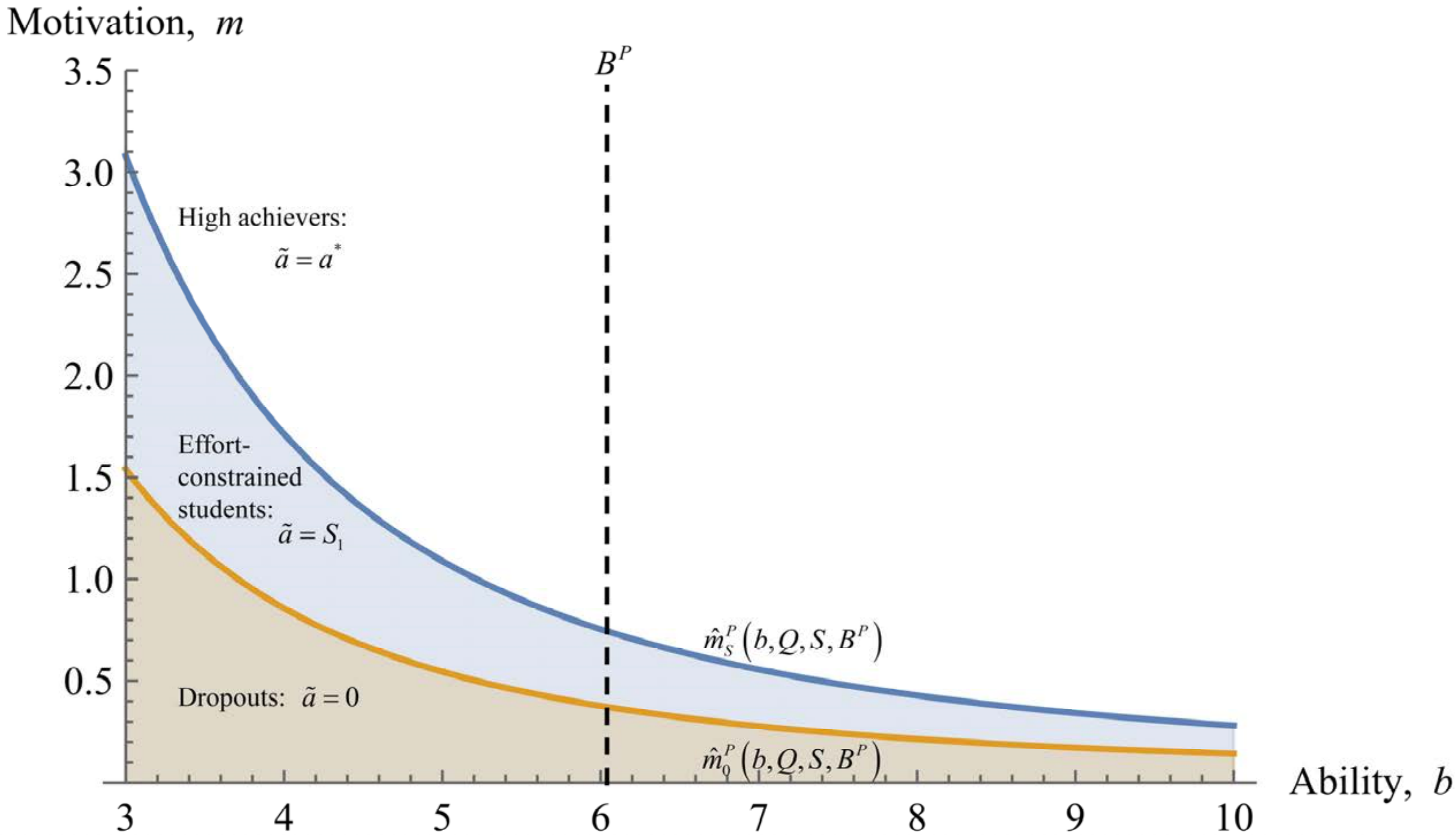

Figure 2. TCS Achievement Gains

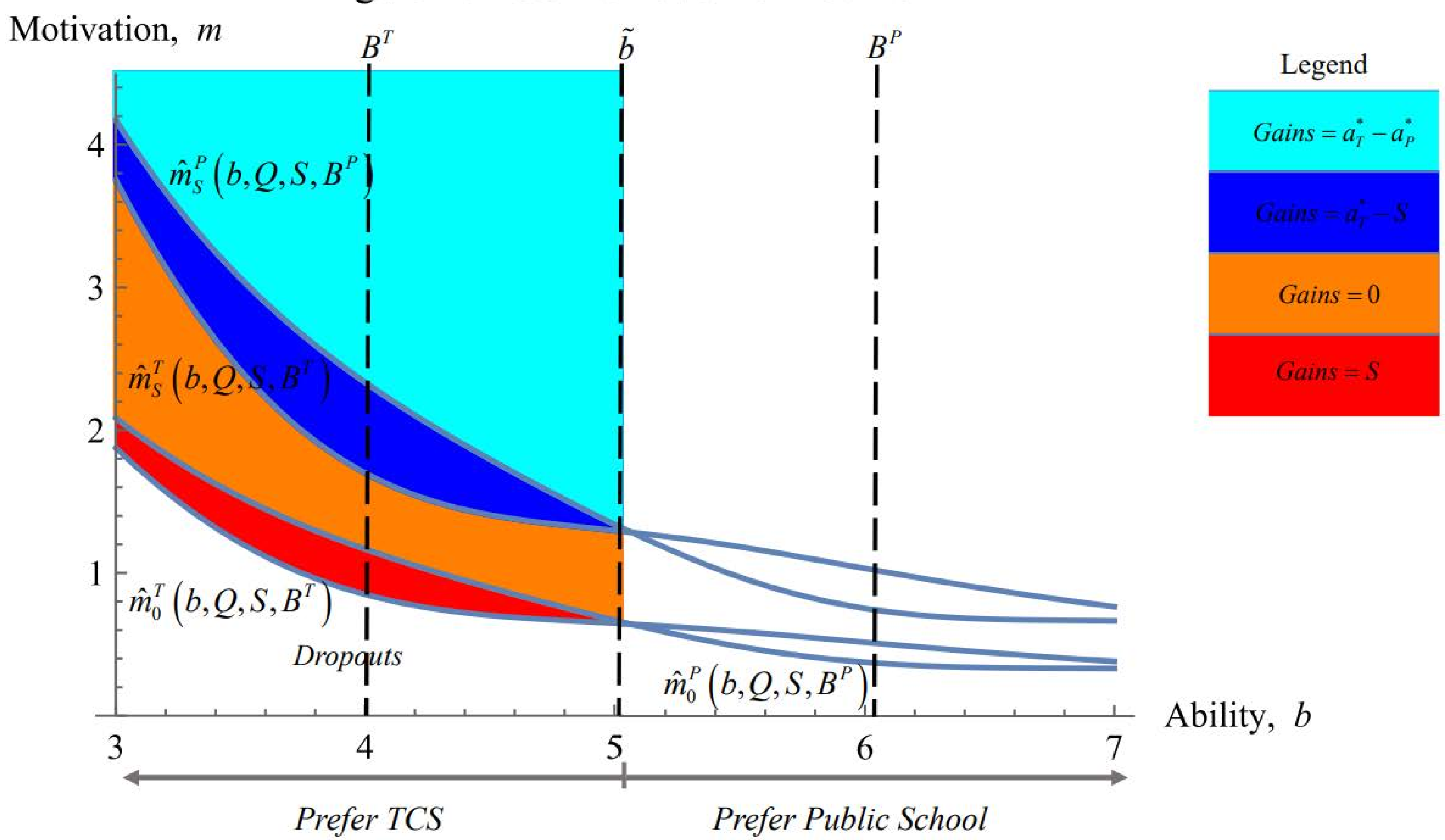

Figure 3. TCS Achievement Gains: Changes of B^T Increase

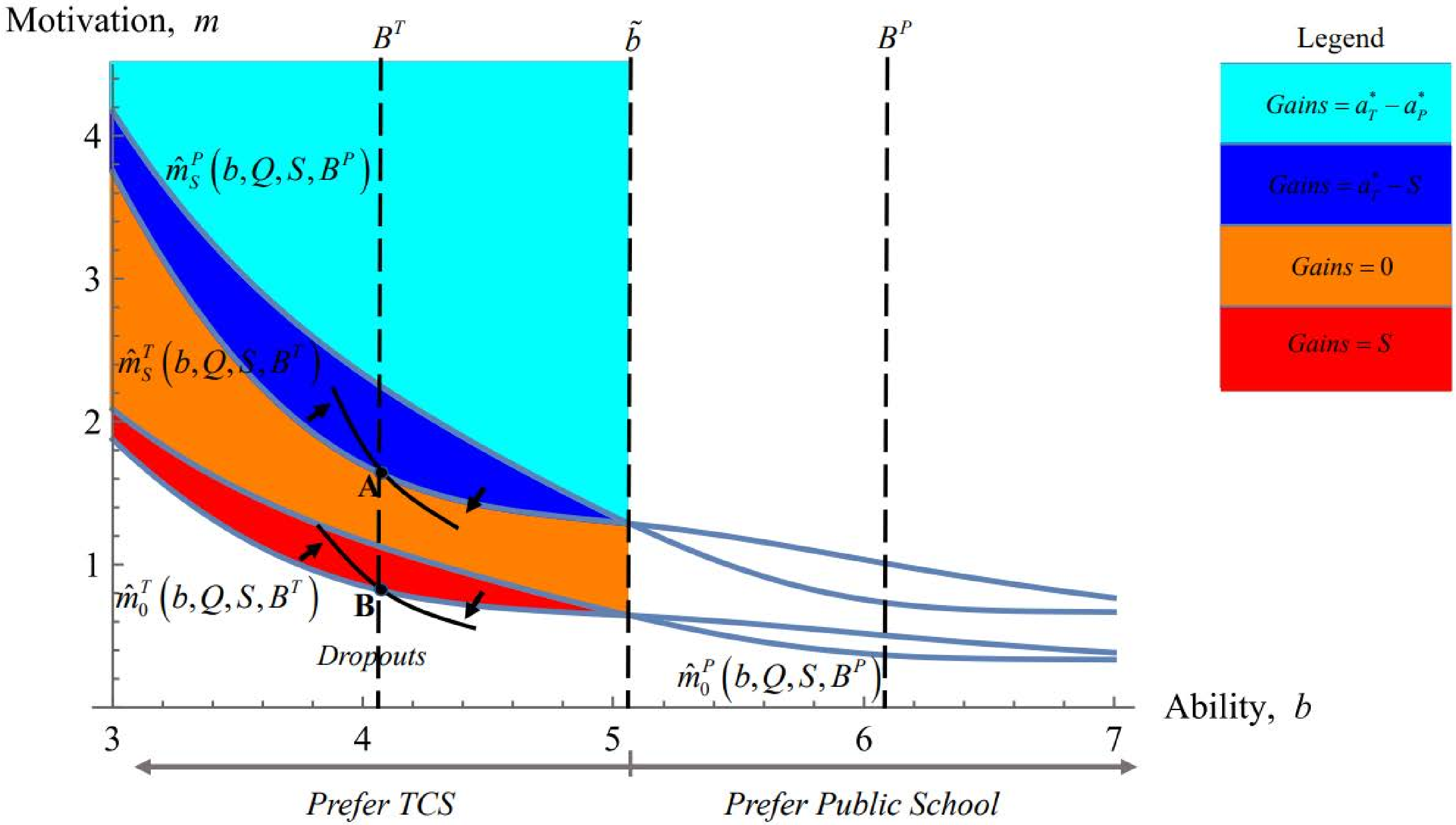

Figure 4. Iso – Effort Loci

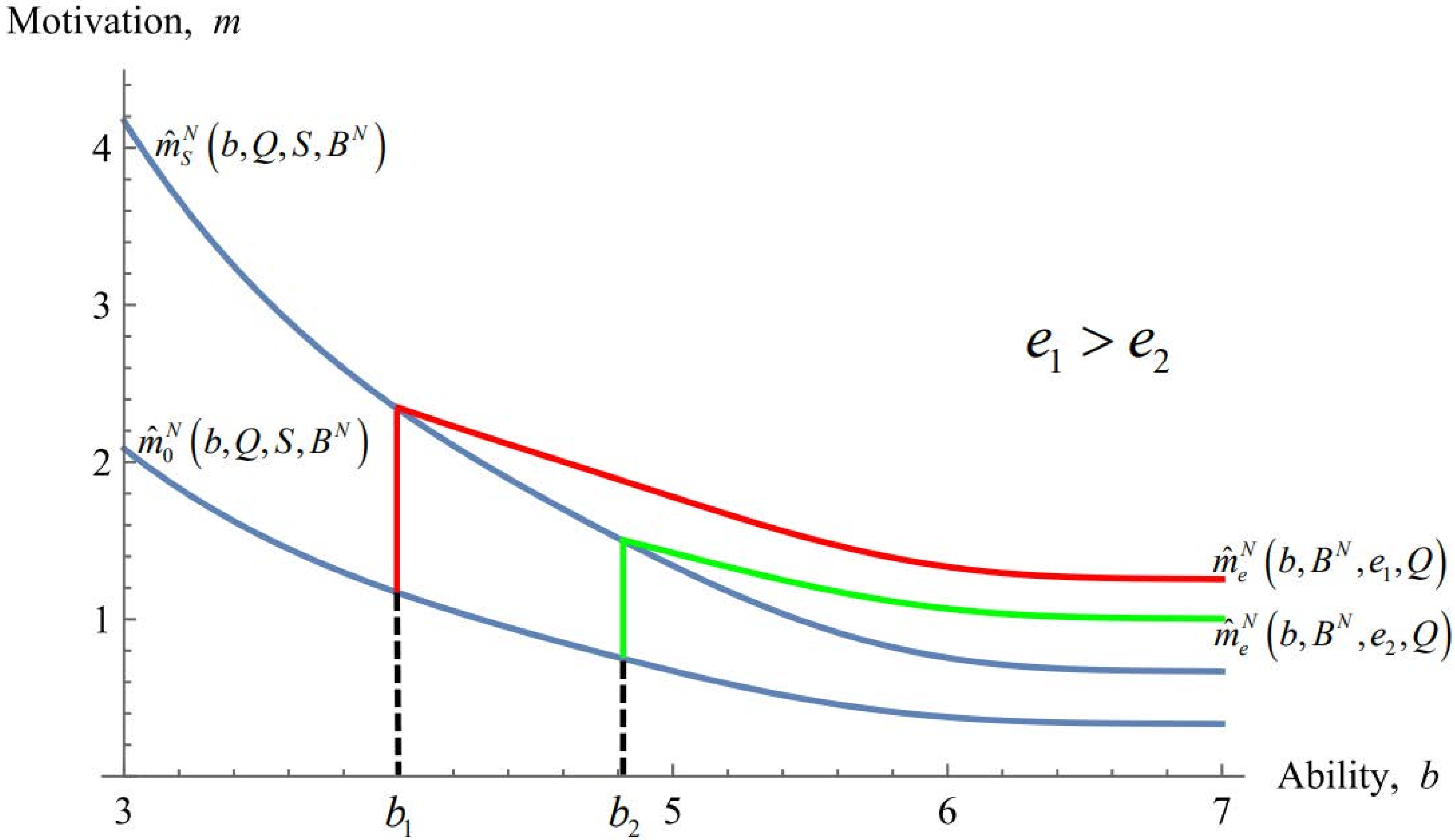

Figure 5. Applicants Set NEC $(B^N > B^P,\ b_N > \tilde{b})$

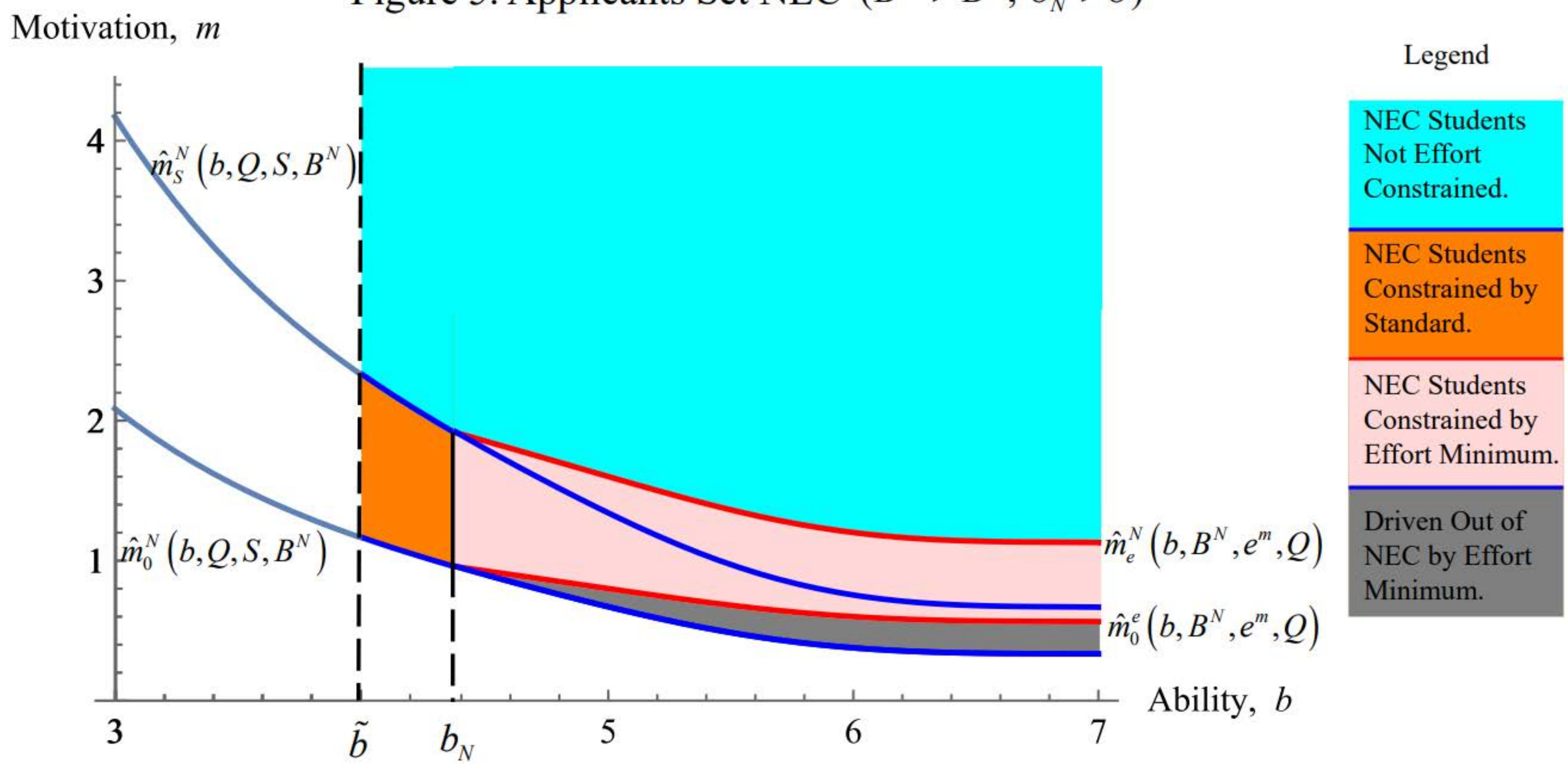

Figure 6. Applicants Set NEC $(B^N > B^P,\ b_N < \tilde{b})$

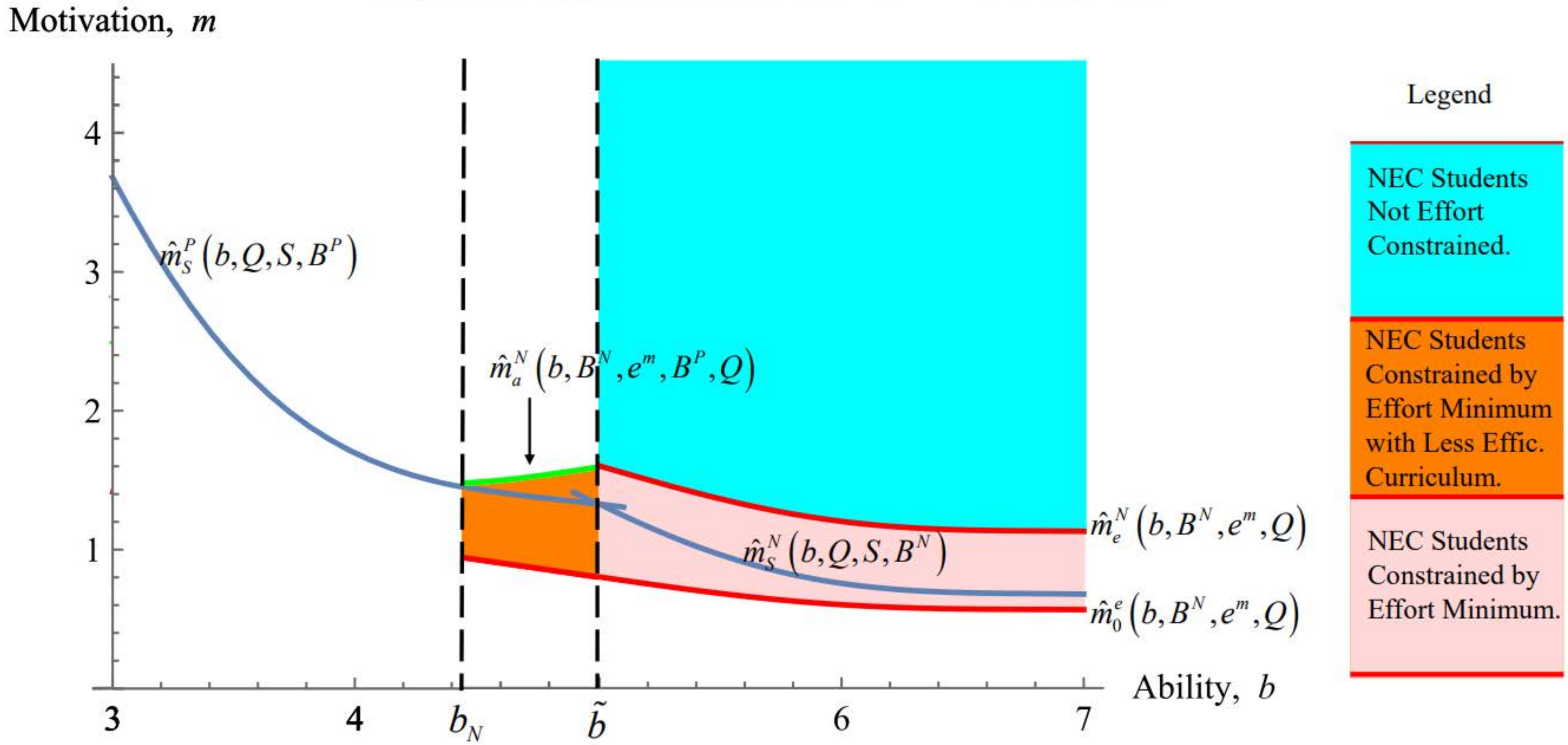

Figure 7. Applicants Set NEC $(B^N = B^P)$

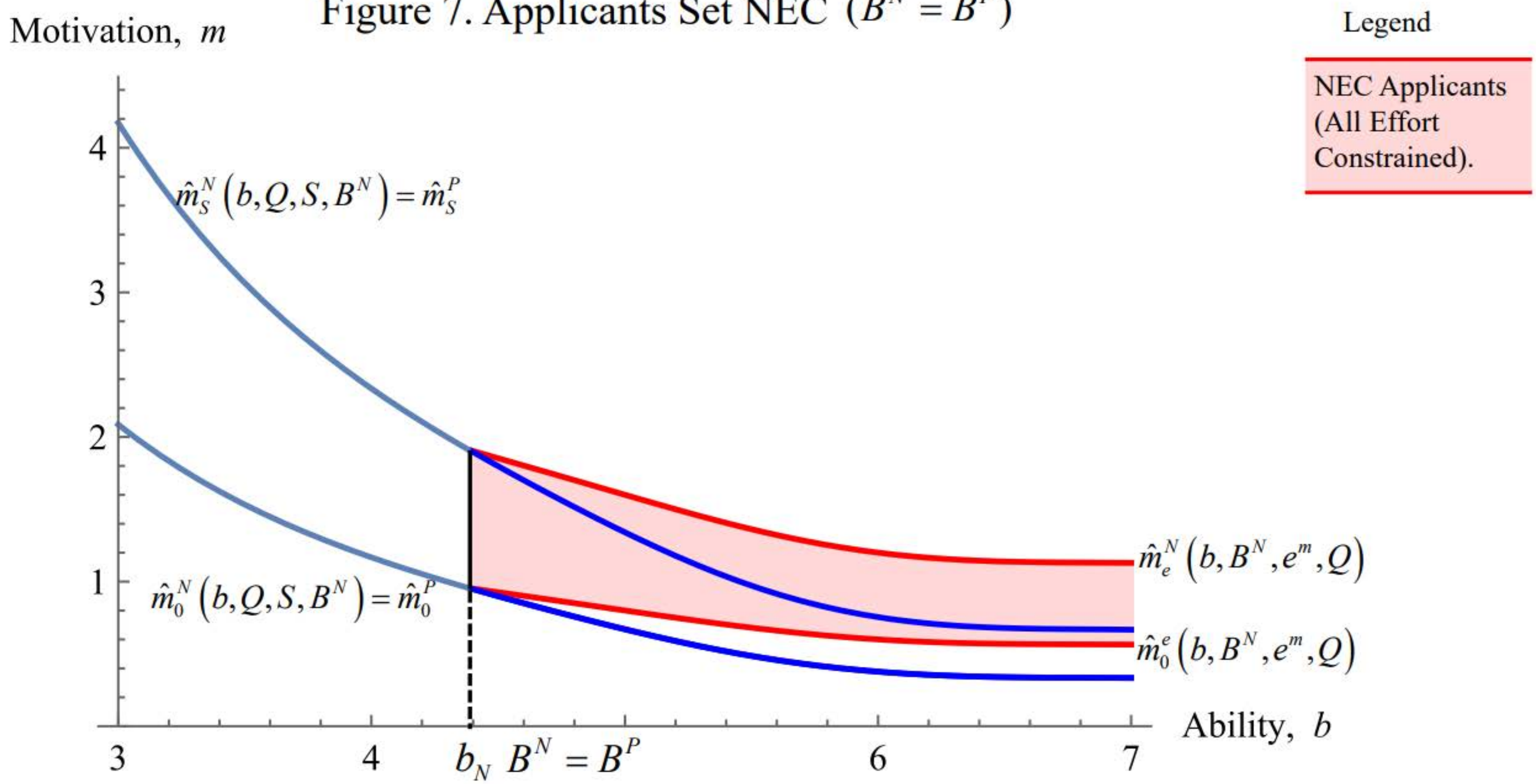

Figure 8a

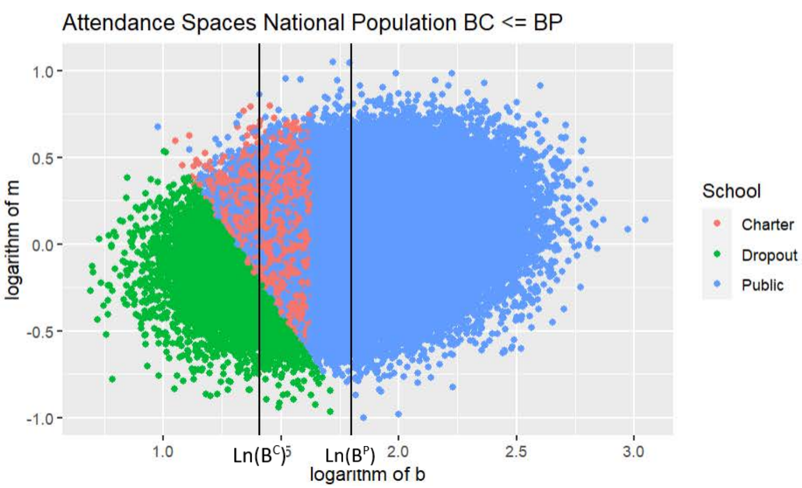

Figure 8b

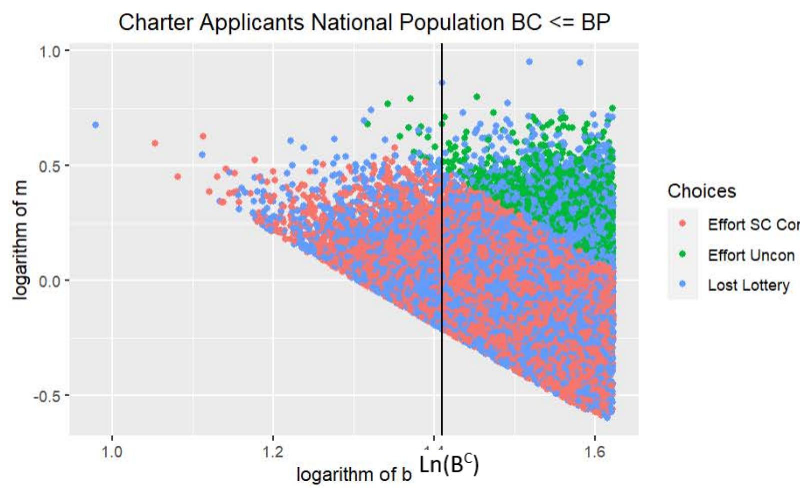

Figure 9a

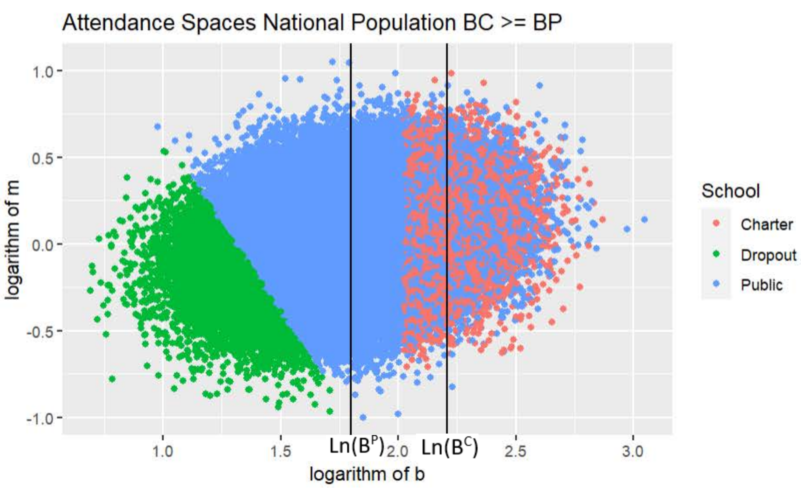

Figure 9b

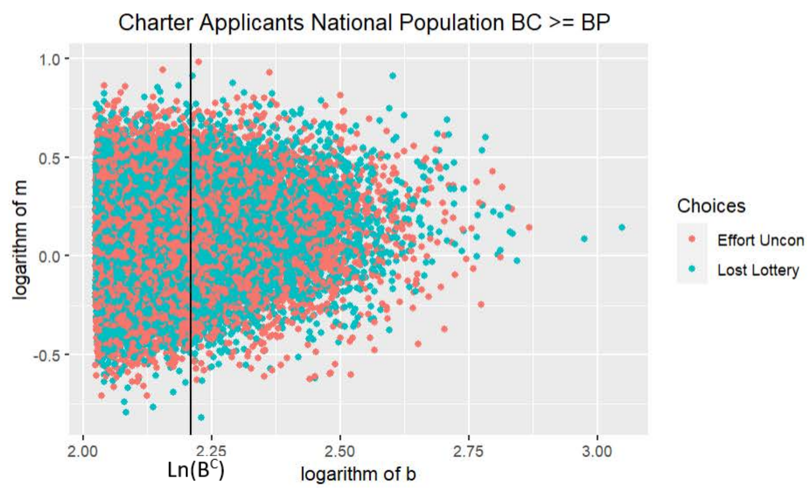

Figure 10a

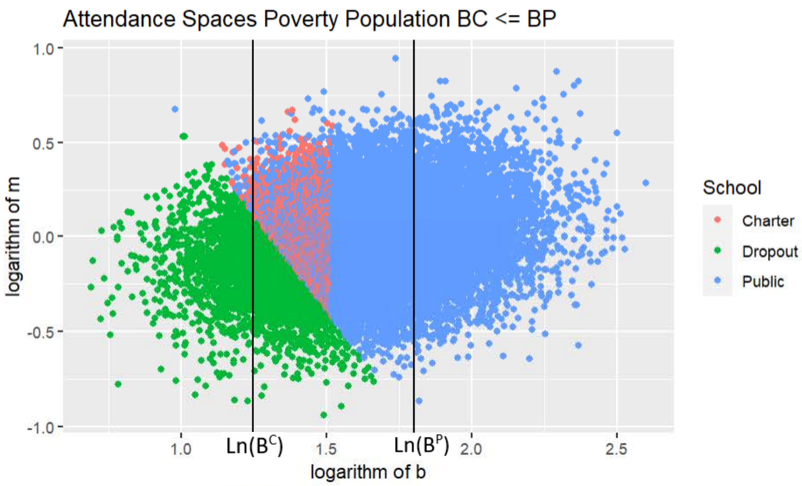

Figure 10b

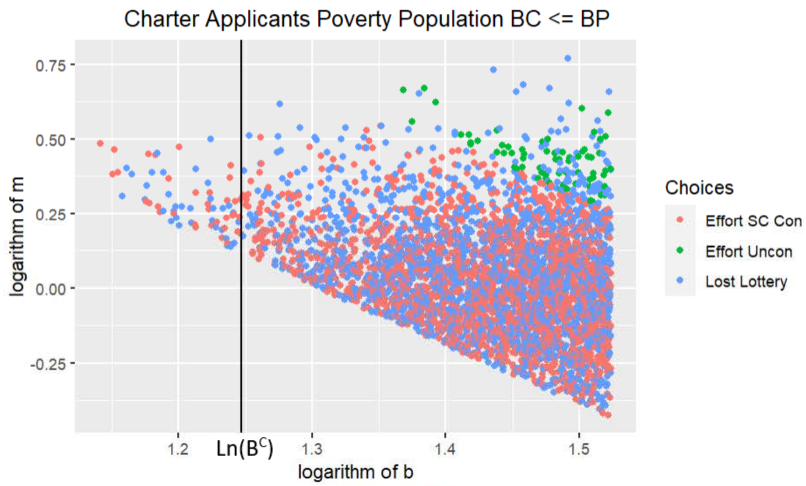

Figure 11a

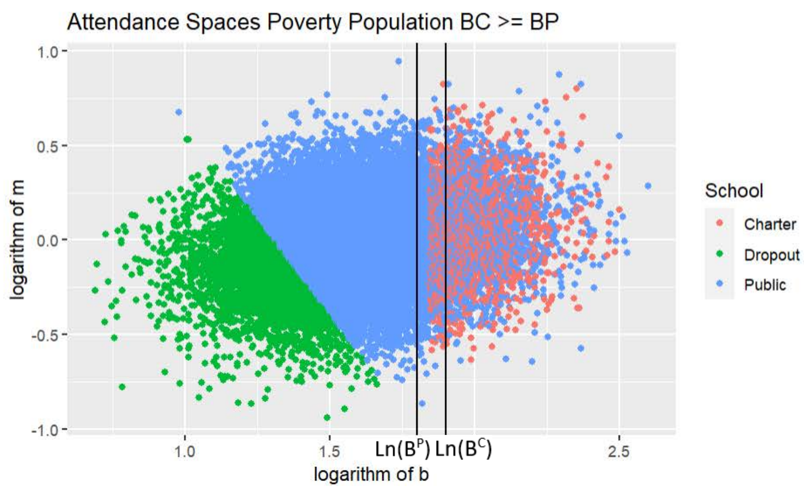

Figure 11b

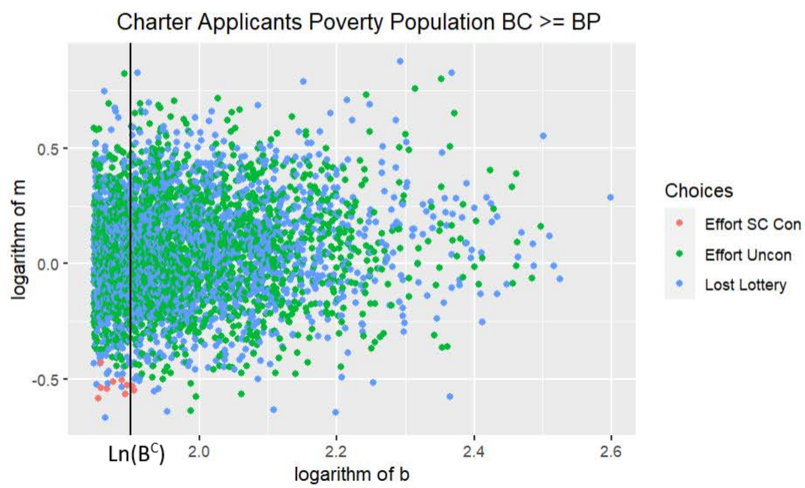

Figure 12a

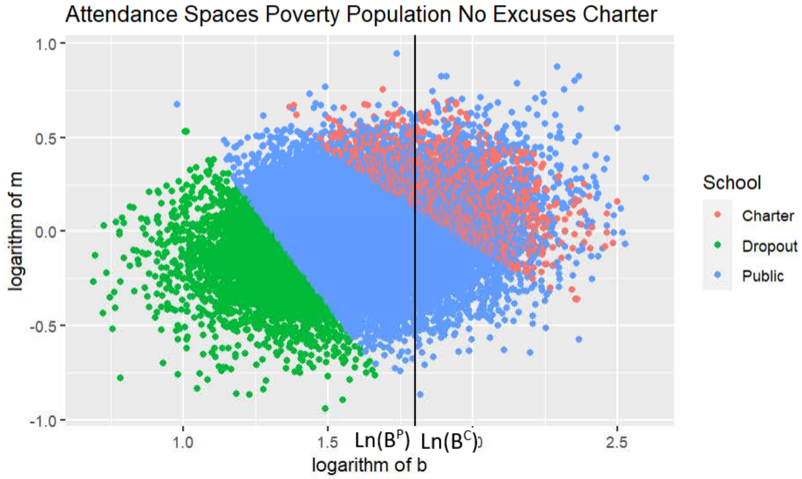

Figure 12b