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ABSTRACT

In a deterministic overlapping-generations economy with production and physical capital, the price of debt can be positive without any budget surpluses being in the offing, because debt incorporates a rational bubble. Yet the dynamics of debt remain a function of the dynamics of the primary budget deficit. As a way to study their joint behavior, we endogenize a structural deficit in the form of an underfunded social-security scheme. We define debt capacity as the level of debt that can be just sustained without a change of policy all the way to an unstable steady state. When it starts below the capacity, the debt converges to a stable steady state, in which the bubble is sustained. Above capacity the bubble unravels and the deficit cannot be financed. In several realistic scenarios occurring in economies, we calculate the needed policy response, which is the true "fiscal cost" of exceeding debt capacity.

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Blanchard (2019) suggests that an increase of government debt may be feasible and beneficial at the current time when the rate of interest stands below the rate of growth of the economy.[1] He writes "If the interest rate paid by the government is less than the growth rate, then the intertemporal budget constraint facing the government no longer binds. What the government can and should do in this case is definitely worth exploring." This statement is best interpreted by means of an overlapping-generations (OLG) model of debt à la Diamond (1965), in which a Tirole (1985) bubble can exist. Because the debt incorporates a rational bubble, the valuation of the debt can be finite even though the discount rate is below the growth rate, and can differ from the present value of future primary surpluses. The bubble makes it even possible for the total market value of government debt to be strictly positive even when there is no prospect of a government surplus in the future.[2]

In a recent talk, Sims (2020),[3] adds one important qualification: he refers to debt issuance when the rate of interest is below the rate of growth as "zero fiscal cost debt." He states: "When the real interest rate on debt is below the growth rate of the economy, the government can issue and roll over debt forever without backing it by new taxes and still see the debt-to-GDP ratio shrink over time. But this only applies when the government makes a one-time increase in debt unaccompanied by increased taxation. This is not the situation we're in, even when it concerns pandemic relief spending. Given past, steady spending increases without tax increases, we unfortunately cannot view pandemic relief spending as a single, wartime increase in debt. Ultimately, these steady increases will affect the interest rate on debt and require dynamic solutions." The lesson is that, to reflect the true fiscal cost of government debt issuance, one needs a dynamic model with interest rate being affected by the government primary deficit/surplus directly or indirectly, while the government deficit/surplus itself is endogenously driven. Even when the market value of the debt is not equal to the present discounted value of future surpluses, the dynamics of the debt remain a function of the dynamics of the budget deficit.

Our paper aims at filling the void in the literature that Sims points out. We study debt sustainability with an endogenous primary government deficit. Extant models, building on Diamond's deterministic OLG model, assume an exogenous constant deficit. We assume that the deficit is driven by the fundamentals of the economy, namely, preferences, production technology, capital accumulation, population growth rate and the terms of a social-security scheme.[4] Our model brings together two important but isolated topics: sustainability of government debt and imbalance of social-security systems. It is in this sense and in full consistency with Sims' view that our model can help interpret the current debt situation of the governments of "high-income" countries.

[1] See also Furman and Summers (2020).

[2] While there is some uncertainty about whether interest rates are below growth in the data, Blanchard (2019) contains strong and robust evidence in favor of the view that interest rates are below growth. For instance, in the United States since 1950 even the nominal 10-year rate with an average of 5.6 percent is below the nominal GDP growth with an average of 6.3 percent.

[3] See Christopher Sims: "How to worry about government debt" at https://bcf.princeton.edu/events/christopher-sims-how-to-worry-about-government-debt/.

[4] There are many applied studies of social security. See for example Auerbach and Kotlikoff (1987a) and Auerbach and Kotlikoff (1987b). Although these models allow for government deficits in any given period, alternative social security responses are assumed to be such that the government budget is balanced over time.

We investigate the law of motion of the debt for a given taxation and spending policy and conclude that there exists an initial level such that the amount of debt would necessarily explode in the future if its starting level is above that initial level. We call "debt capacity" the threshold level. Too much debt can put the economy on a potentially explosive path, which, however, would unravel from the start.

Under conditions in which the rate of interest is below the growth rate, steady states, when they exist, come in pairs: one unstable determining the debt capacity and one stable with a lower deficit and, therefore, lower debt issuance.[5] For a given initial capital stock, there is one unique path emanating from a unique level of government debt that leads to the unstable steady state whereas many paths, from a range of values for an initial value of government debt, lead to the stable steady state. In our variant of the Diamond model, it is not sufficient to look at the (real or nominal) interest rate to determine whether debt is sustainable or desirable. The amount of outstanding debt (captured by the Debt-to-GDP ratio) also matters. It is for this reason that the notion of debt capacity and the knowledge of its value are essential.

We extend the model and its concept of debt capacity to two policy-relevant settings. First, one might be concerned that a high, potentially explosive level of nominal debt produces high inflation. To investigate this issue, we allow the government, in addition to collecting taxes and paying benefits, to intervene in the money market following a Taylor rule, as a way to fix the nominal rate of interest. Explosive real debt also leads to explosive inflation. Furthermore, although the rate of inflation is well determined, the price level is undetermined, within bounds.

Second, as debt capacity depends on growth, one might hope that endogenous productivity increases would raise the limit. We, therefore, ask whether a government can increase its debt capacity by subsidizing innovation which ultimately raises productivity and growth. To answer this question, we adjust our framework along the lines of a revised Romerian approach but let the government finance R&D in addition to paying for social-security. Overall, our numerical illustrations under these varied scenarios show that it is challenging to maintain debt-to-GDP ratios much above one, even with innovation.

In policy experiments, we use the notion of debt capacity to explore the responses that are needed in case debt exceeds it. These represent the true "fiscal cost" of exceeding debt capacity. We also explain how debt could come to exceed debt capacity. For that, we develop a scenario of population-growth declines. We again calculate the needed policy response. The more delayed the response, the larger it has to be. Our model can arguably be interpreted as saying that it is better to implement a policy that reduces the debt automatically during normal times, as a way to aim towards the stable steady state.

Over the last seventy-five years, the debt of high-income nations has mostly increased, while prospects for government surpluses are dimming.[6] The 2019 Long-Term Budget Out-

[5] The two steady states can coincide, in which case stability prevails when approaching the steady state from one side only.

[6] Historically, sovereigns have borrowed to finance wars. Only in the 19th century governments started to systemically borrow to build ports, railways, roads, schools and universities. Despite this history, there are only three successful debt reduction episodes: Great Britain after the Napoleonic Wars, the United States in the last third of the 19th century, and France in the decades leading up to 1913; see Eichengreen, El-Ganainy, Esteves, and Mitchener (2019).

look by the Congressional Budget Office of the U.S. predicts steady deficits, if not rising ones, for thirty years. In the same vein, the slowdown or even decline in population growth (Jones (2020)) as well the reduction in technological progress mirrored in reduced GDP growth rates would seem to make it unlikely that successful debt reduction is possible in the foreseeable future. Only a few countries seem to defy this trend. With its "debt brake" constitutional amendment, Germany is actively doing that:

> "Die Haushalte von Bund und Ländern sind grundsätzlich ohne Einnahmen aus Krediten auszugleichen. Bund und Länder können Regelungen zur im Auf- und Abschwung symmetrischen Berücksichtigung der Auswirkungen einer von der Normallage abweichenden konjunkturellen Entwicklung sowie eine Ausnahmeregelung für Naturkatastrophen oder außergewöhnliche Notsituationen, die sich der Kontrolle des Staates entziehen und die staatliche Finanzlage erheblich beeinträchtigen, vorsehen. Für die Ausnahmeregelung ist eine entsprechende Tilgungsregelung vorzusehen.[7]

Our paper relates to Diamond (1965), Tirole (1985), Chalk (2000), Blanchard (2019), and Farmer and Zabczyk (2020) all of which are OLG models building on Diamond.[8] Diamond (1965) sets the debt per capita to be constant and introduces the taxes needed to pay for the cost of financing. In Tirole (1985), there is no deficit and, therefore, also no government debt; however, as we show, the bubble in his model can be interpreted as a one time issuance of government debt. Chalk (2000) is perhaps the first to point out that government debt is determined by the real side of the economy and that there is an upper bound on its size. Yet, he also assumes that the exogenously determined deficit per capita (occasioned by a wasteful expenditure) is constant. Further, in his variant of Diamond's model agents work all periods and, therefore, there is no role for social security.[9]

More recently, Blanchard (2019) extends Diamond (1965) by introducing uncertainty to study the cost of an exogenously set public debt in an environment with low interest rates. While the joint understanding of public debt and endogenous deficits under uncertainty is of interest in its own right, we must first understand it under certainty, a question we have not seen addressed in the literature.

Farmer and Zabczyk (2020) show that, in the OLG model, the Fiscal Theory of the Price Level (FTPL) cannot generally be used to determine the price level uniquely. Similarly to our work, they find several steady states and multiple values for government debt, and question established views about what constitutes good government policies. For instance, they state "As long as the primary deficit or the primary surplus is not too large, the fiscal

[7] "The federal and state budgets shall in general be balanced without proceeds from borrowing. The federal and state governments can provide for rules to take into account the effects of deviations from normal cyclical developments, as well as a derogation for natural disasters or exceptional emergency situations that are beyond the control of the state and significantly affect the state's financial situation. A corresponding repayment plan must be provided for any derogation." Constitution of the Fedearl Republic of Germany, Article 109, Section 3.

[8] For a treatise on dynamics and policies in overlapping-generations models see De la Croix and Michel (2002).

[9] In Appendix A, we compare our model to previously published ones.

authority can conduct policies that are unresponsive to endogenous changes in the level of its outstanding debt."

A recent study by Jiang, Lustig, Nieuwerburgh, and Xiaolan (2021) aims to explain, as we do, the valuation gap of government debt, which is the gap between the value of the debt observed in the financial market and the present value of future primary surpluses. It features a thorough empirical investigation of the stochastic process of government surplus, postulates an exponential affine stochastic discount factor for infinitely lived investors and derives the risk premium of government debt. The authors dismiss a rational bubble as a possible explanation of the gap on the grounds that the value of the debt would become infinite if the transversality condition on debt were violated. We show below that, in the presence of a perpetual deficit and a rate of interest lower than the rate of growth, the debt per capita in an OLG model can be positive and finite at all times while containing a bubble that tends to positive infinity. The reason is that the present value of future surpluses tends to minus infinity, the sum of the two remaining finite. They also state that "In rational bubble models, the debt/GDP ratio declines over time," which is inconsistent with a valuation gap that grows empirically faster than GDP. In our model, the ratio can rise either temporarily on the way to the stable steady state or permanently on the way to the unstable one.

Jiang, Lustig, Nieuwerburgh, and Xiaolan (2021) propose a convenience yield on government debt as an explanation of the valuation gap but acknowledge that it falls short of the mark quantitatively.[10] They also acknowledge that the convenience yield does not explain a positive value for the debt when the primary surplus is permanently negative (see their Figure 10, Panel (a)). In our paper, we propose an explanation that is complementary to theirs and show that the more traditional OLG rational bubble is capable of explaining that last fact. To develop these arguments, a deterministic setting is sufficient and more transparent. The choice we make of a deterministic setting is not meant to deny the relevance of a stochastic extension, which we intend to pursue.[11]

The balance of the paper is organized as follows. Section 1 presents a deterministic model with perpetual refinancing of a social-security driven government deficit. Section 2 contains our definition of debt capacity and a study of the comparative statics of it. Section 3 contains policy experiments and describes the consequences of a decline in the population growth rate. In Section 4, we turn to two extensions of the model: first, we examine the implications for inflation of a high, potentially explosive, level of nominal government debt; second, we envision the possibility that the government may sustain its debt by subsidizing R&D, in addition to supporting a social-security scheme. In Section 5, by studying intergenerational lifetime utility, we justify the social-security scheme that we have postulated. The final section contains the conclusion.

[10]In an international context, Brunnermeier, Merkel, and Sannikov (2020) consider an incomplete market, which makes it possible for a bubble to exist. Their model features infinitely-lived agents instead of overlapping generations. It can be interpreted as providing micro-foundation for convenience yields, or more broadly "seigniorage," in bond prices.

[11]Corhay, Kind and Kung (2021) using a stochastic model, with an exogenous, affine pricing kernel, show how expected inflation absorbs changes in the maturity structure of government bonds.

1 A deterministic model of perpetual refinancing

1.1 The components of the system

We build an overlapping-generations model with population growth and physical capital accumulation. The economy comprises a production sector, a household sector and a government sector.

The production function is

$$Y_t = F(K_t, \Lambda_t)$$

where K_t and Λ_t are the inputs of capital and labor. The function F is continuous, twice differentiable, homogeneous of degree 1 in its two arguments, strictly increasing and strictly concave in each. To exploit the homogeneity, we write

$$k_t \triangleq \frac{K_t}{L_t}; f(k_t) \triangleq F(k_t, 1)$$

In applications, the production function will be of the constant-elasticity of substitution (CES) type

$$Y_t = A \times \left[\alpha K_t^\rho + (1-\alpha)\Lambda_t^\rho\right]^{\frac{1}{\rho}}; \rho < 1 \tag{1}$$

where $\rho = (\eta - 1)/\eta$ and η is the elasticity of technical substitution (ETS) between capital and labor, and $A > 0$, $0 < \alpha < 1$, $K > 0$, $\Lambda > 0$. At time 0, the economy starts with a capital stock equal to K_0. At $t \neq 0$, the capital stock K_t is set aside at time $t-1$ and chosen by the generation born at time $t-1$. It depreciates at the rate $\delta < 1$ per period.

The households/investors: Like in Diamond (1965), we introduce the following notation and assumptions: c_t^t is the consumption at date t of one household of the generation born at date t while c_{t+1}^t is the consumption at date $t+1$ of the generation born at date t. L_t is the exogenous number of individuals in the generation born at time t. Their lives are summarized with two periods and they work at the first point in time only; their supply of labor is inelastic. L^t grows at the constant rate n per period. The number n stands for population growth but possibly also for labor-augmenting technical progress. It is a catch-all for all forms of exogenous perpetual growth. We examine technical progress explicitly in Section 4.2.

Generations are born with an endowment of only one kind: their labor force. They collect a wage bill $w_t L_t$. At time 0, there are only "old" people born notionally at time -1 with arbitrary consumption c_0^{-1}.

The two-points in time utility functions of all generations are the same:

$$U\left(c_t^t, c_{t+1}^t\right) = u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right); t \geq 0$$

where u is a continuous, twice differentiable, strictly increasing and strictly concave function. In applications, the function u will be a power function $u(c) = c^{1-\zeta}/(1-\zeta); \zeta > 0$ with, therefore, an elasticity of intertemporal substitution (EIS) equal to $1/\zeta$.

The financial market: Two assets are traded in the financial market, the maturity of which is immaterial. One is a bond, which is a claim on the government and the other is the direct ownership of the capital that serves as input into the production system, and which can be rented to production facilities. In this deterministic world, the young households are indifferent between physical capital and government debt, so that we let them choose not each one separately but their sum which is called "savings" s_t. In total, they save an amount $s_t L_t$ at time t.

In other words,

$$s_t \times L_t \triangleq K_{t+1} + G_{t+1}$$

in which G_{t+1} is the time-t exiting amount of government held, and which can be restated on a per capita basis as:

$$s_t \triangleq (1+n)(k_{t+1} + g_{t+1}) \tag{2}$$

where $g_t \triangleq G_t / L_t$.

The two assets being perfect substitutes in demand, they bring the same rate of return. The one-period rate of return or rate of interest quoted at time t is called r_{t+1}.

Taxation and spending: Taxation is in the form of a contribution to the social-security system. The time-t young make a total social-security contribution of $L_t \tau w_t$, where τ is the social-security tax rate.

Government spending is in the form of social-security defined benefits paid to the old households on the basis of the wages they were earning when young. Specifically, at time t the old receive a total social-security benefit of $L_{t-1} \theta w_{t-1}$, where θ is the social-security benefit ratio. In Section 5 below, we show that social-security is a welfare-improving form of spending.[12] Throughout we consider the case in which the primary budget deficit is structural: $\tau < \theta \times (1+n)$. In Section 5 below, we provide the rationale for choosing that form of government spending: it imporves welfare.

With this notation, the simultaneous flow budget constraints of the households and the government at time t are as follows:
young household

$$c_t^t + s_t = (1-\tau) w_t$$

old household

$$c_t^{t-1} = s_{t-1} \times (1 + r_t) + \theta \times w_{t-1}$$

government

$$-G_{t+1} + \theta \times w_{t-1} \times L_{t-1} = \tau \times w_t \times L_t - (1 + r_t) G_t$$

where G_t is the total debt with which the government enters time t and G_{t+1} is the debt with which it exits time t.

Market clearing: The labor market clears

$$\Lambda_t = L_t$$

and the market for goods clears

$$L_t \times c_t^t + L_{t-1} \times c_t^{t-1} + K_{t+1} = F(K_t, L_t) + (1-\delta) K_t$$

[12]See the policy recommendation of L. Summers, *Washington Post*, Jan 7, 2020.

1.2 Difference equations and steady states

By writing the first-order conditions of a household, who owns the production facility and rents it out, and imposing market clearing, the difference-equations system governing the evolution of the economy, stated on a per capita basis, is[13]

$$
\begin{aligned}
\frac{\frac{\partial}{\partial c_{t+1}^t} U\left(c_t^t, c_{t+1}^t\right)}{\frac{\partial}{\partial c_t^t} U\left(c_t^t, c_{t+1}^t\right)} &= \frac{1}{1+r_{t+1}} && (3)\\
f'\left(k_t\right) - \delta &= r_t && (4)\\
f\left(k_t\right) - k_t f'\left(k_t\right) &= w_t && (5)\\
c_t^t + s_t &= (1-\tau)\, w_t && (6)\\
c_t^{t-1} &= s_{t-1} \times \left(1+r_t\right) + \theta \times w_{t-1} && (7)\\
-(1+n)\, g_{t+1} + \theta \times w_{t-1} \frac{1}{1+n} &= \tau w_t - \left(1+r_t\right) g_t && (8)\\
c_t^t + \frac{1}{1+n} c_t^{t-1} + (1+n)\, k_{t+1} &= f\left(k_t\right) + (1-\delta)\, k_t && (9)
\end{aligned}
$$

For each generation, it would be suboptimal to finish its life with total savings of capital and government-debt holdings greater than zero. And default is not allowed, so that their total savings at the end of their life are set exactly to zero. These are the only terminal conditions of optimality that are present in the model; they have already been coded into the system of equations above. In this overlapping-generations setting, no terminal conditions should be imposed at "the end of time," assumed to be infinite. No so-called transversality conditions apply at infinity.[14,15] For that reason, the system is a "forward" system of equations, with initial conditions to be specified only. It is entirely backward looking; this is a case of rational myopia, in which future events beyond one period need not be considered by economic agents. We introduce an exception to the myopia principle in Remark 1 below.

Equations (4) and (5) allow us to define r_t and w_t as functions $r\left(k_t\right)$ and $w\left(k_t\right)$:

$$r\left(k_t\right) = f'\left(k_t\right) - \delta;\ \ w\left(k_t\right) = f\left(k_t\right) - k_t f'\left(k_t\right)$$

The function $r\left(k_t\right)$ is strictly decreasing and bounded below by $-\delta$. Substituting Equation (7) shifted forward and Equation (6) into Equation (3) gives

$$\beta \frac{u'\left(s_t \times \left(1+r_{t+1}\right) + \theta \times w_t\right)}{u'\left((1-\tau)\, w_t - s_t\right)} = \frac{1}{1+r_{t+1}}$$

and, given the monotonicity of the function u' and customary Inada assumptions allows us

[13]In accordance with Walras law, the system made of equations ((2),(3)-(9)) contains a redundant equation: (3)-(9) implies (2) and ((2),(3)-(8)) implies (9).

[14]They apply as necessary conditions of optimality when an agent *with an infinite lifetime* maximizes his lifetime utility. There is no such agent in this economy.

[15]As a technical point, one could note that, if the "end of time" were finite, some terminal conditions could be imposed, and the solution could be calculated backward, resulting in a unique initial situation. Increasing the end of time forever in the backward solution with a range of terminal values gives the same result as does the forward solution.

to find a solution for s_t and define a supply of savings – demand for assets function

$$s_t = s\left(w_t, r_{t+1}\right)$$

At the end of its two-period life each household consumes its entire wealth, including the value of the capital stock, which is part of its savings, which it sells to the young, leaving nothing behind.

Equations (6), (7) and (9) form a linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \frac{1}{1+n} & 0 \end{bmatrix} \times \begin{bmatrix} c_t^t \\ c_t^{t-1} \\ s_t \end{bmatrix} = \begin{bmatrix} (1-\tau)\, w_t \\ s_{t-1} \times (1+r_t) + \theta \times w_{t-1} \\ f\left(k_t\right) + (1-\delta)\, k_t - (1+n)k_{t+1} \end{bmatrix}$$

which can be solved easily to provide the demand for savings (supply of assets)

$$s_t = \begin{bmatrix} 1 & \frac{1}{1+n} & -1 \end{bmatrix} \times \begin{bmatrix} (1-\tau)\, w_t \\ s_{t-1} \times (1+r_t) + \theta \times w_{t-1} \\ f\left(k_t\right) + (1-\delta)\, k_t - (1+n)k_{t+1} \end{bmatrix}$$

We are ready to reduce the eight-equation system to two difference equations or even one of second-order. That can be done in two, equivalent ways. We can, first, equate demand and supply and get an equation relating k_{t+1} to k_t and k_{t-1}, which is, therefore, a stand-alone second-degree difference equation:[16]

$$\begin{aligned} s\left[w\left(k_t\right), r\left(k_{t+1}\right)\right] &= \begin{bmatrix} 1 & \frac{1}{1+n} & -1 \end{bmatrix} \\ &\times \begin{bmatrix} (1-\tau)\, w\left(k_t\right) \\ s\left[w\left(k_{t-1}\right), r\left(k_t\right)\right] \times \left(1+r\left(k_t\right)\right) + \theta \times w\left(k_{t-1}\right) \\ f\left(k_t\right) + (1-\delta)\, k_t - (1+n)k_{t+1} \end{bmatrix} \end{aligned} \tag{10}$$

Initial conditions are set by the initial capital stock $k_0 = K_0/L_0$. Suppose for a moment that $k_1 = K_1/L_1$ were also given. Equation (10) would then provide the path of the capital stock autonomously and the evolution of government debt would just follow from difference Equation (8).

Since, however, k_0 and k_1 are not given jointly, debt must be part of initial conditions and the government budget equation (8) must be brought in. For easier interpretation, define the social-security deficit as[17]

$$d_t \triangleq d\left(k_{t-1}, k_t\right) = \frac{\theta}{1+n} w\left(k_{t-1}\right) - \tau \times w\left(k_t\right) \tag{11}$$

[16]As in the linear multiplier-accelerator model of Samuelson (1939).

[17]There are two ways to specify social-security deficit d_t in our model: benefits indexed on wage earned while working (Equation (11)) or benefits indexed on current wage $d_t = \theta w\left(k_t\right) / (1+n) - \tau w\left(k_t\right)$. The first way is more consistent with reality but causes the model to be second order in capital (involving k_{t-1}, k_t, k_{t+1}).

The two-equation systems is

$$\begin{aligned} s\left(w\left(k_t\right), r\left(k_{t+1}\right)\right) &= (1+n)\left(k_{t+1}+g_{t+1}\right) \\ (1+n) g_{t+1} &= \left(1+r\left(k_t\right)\right) g_t+d\left(k_{t-1}, k_t\right) \end{aligned} \tag{12}$$

Formulation (10) is sufficient for the study of steady states while formulation (12) highlights the interaction along a path between government debt and the capital stock, and is more traditional in the literature.[18] In Appendix A, we show the novelty of the latter formulation as compared to extant models.

We define a steady state as a situation in which K_t/L_t is a constant k over time. A steady state must solve Equation (10) with $k_{t+1} = k_t = k_{t-1}$. Figure 1 illustrates the way in which steady states are determined.

Example 1. *Particularizing the problem, there exists an analytical solution for the steady states in the special case of logarithmic utility (isoelastic with* $\zeta = 1$*) and Cobb-Douglas production function: Equation (1) with* $\rho \to 0$ *gives*

$$F(k, 1) = Ak^\alpha;\ w = A(1-\alpha) k^\alpha;\ r = A\alpha k^{\alpha-1} - \delta \tag{13}$$

It is a bit simpler to rewrite the equation system with $x \triangleq r + \delta$ *as an unknown. The equation is cubic in* x*. There exist explicit but fairly cumbersome formulae for the roots of a cubic equation. Here, we just give a numerical example. Suppose that a period of the model is equal to twenty-five years and that:* $n = (1+0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$.
The cubic equation of x *has two positive real roots:*
Root 1: $x = 1.27512$ *(*3.343%*/year),* $k = 0.0987074$*,* $r = 1.198$%*/year, locally stable.*
Root 2: $x = 1.55597$ *(*3.825%*/year),* $k = 0.0769635$*,* $r = 1.968$%*/year, locally unstable.*
Roots 1 and 2 are displayed in Figure 1.
In the absence of social security and debt ($\theta = 0, \tau = 0$*), there would have been only one root at* $k = 0.144993$.

Example 2. *With logarithmic utility, CES production (1) with* $\rho = -1$ *(ETS equal to* 1/2*), and other parameters as in the previous example there exists only one real root of the steady-state equation, which is:*
$x = 1.56976$ *(*3.847%*/year),* $k = 0.196178$*,* $r = 2.002$%*/year.*

In general cases, non linear Equation (10) with $k_{t+1} = k_t = k_{t-1} = k$ is not in polynomial form. One can still solve it numerically with the limitation that one does not know how many roots it has in total. For that reason, we assume that the relationship between k_{t+1} and the two arguments (k_t, k_{t-1}), which is implicit in (10), is concave against k_t and that this concavity dominates, in case they are contrary, over the curvature against k_{t-1}. Then

[18] Formulation (12) can be shifted forward to produce a first-order system in (k_{t+1}, g_{t+2}):

$$\begin{aligned} s\left(w\left(k_t\right), r\left(k_{t+1}\right)\right) &= (1+n)\left(k_{t+1}+g_{t+1}\right) \\ (1+n) g_{t+2} &= \left(1+r\left(k_{t+1}\right)\right) g_{t+1}+d\left(k_{t+1}, k_{t+1}\right) \end{aligned}$$

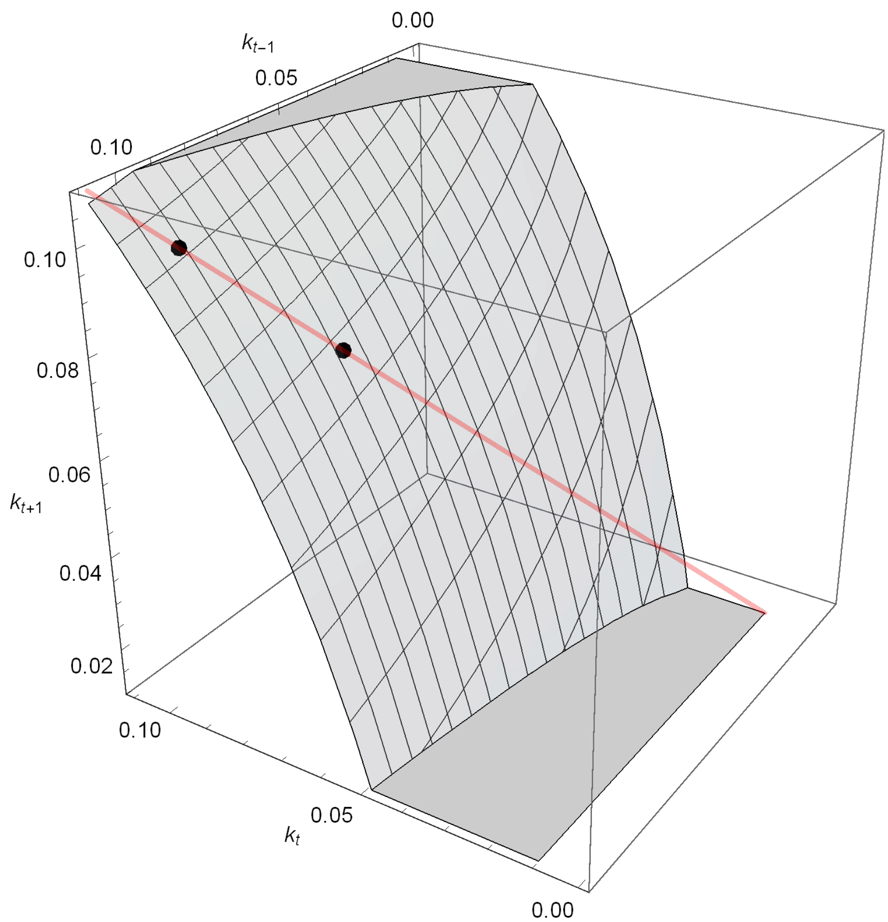

Figure 1: **The dynamics of the capital stock. Illustration with log utility and Cobb-Douglas production function**. The surface represents the relationship $k_{t+1}(k_t, k_{t-1})$. Parameter values are: $n = (1 + 0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1 - 0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$.

there exist at most two steady states, the steady state with the lower value of k being locally unstable.

So far, we have derived the law of motion of k starting from given k_0 and k_1. This solution procedure has allowed us to determine the steady states without any consideration of the behavior of government debt. It remains to endogenize k_1 if k_0 is given or k_0 if k_1 is given. Assume that the initial amount of debt, set by history, is contractually denominated as a real amount.[19] Then k_1 actually follows from k_0 and the definition of savings (2) written at time 0:

$$s\left(f\left(k_0\right)-k_0 f'\left(k_0\right), f'\left(k_1\right)-\delta\right)=(1+n)\left(k_1+g_1\right)$$

where it is most convenient to take g_1 (as opposed to g_0) as initial condition for the government debt.[20] In total, initial conditions are set by the initial capital stock $k_0=K_0/L_0$ and the amount of debt $g_1=G_1/L_1$ with which the government leaves time 0 or, equivalently, by k_1 and g_1. The law of motion of government debt then follows from the government budget constraint, Equation (8).

Rolling over Equation (8) one gets

Proposition 1. *The value of debt is*

$$\begin{aligned} g_1 &= \frac{1}{1+n}\sum_{t=1}^{s-1}\frac{-d_t}{\prod_{u=1}^{t}\frac{1+r_u}{1+n}}+\frac{g_s}{\prod_{u=1}^{s-1}\frac{1+r_u}{1+n}}\forall s>1 \\ &= \lim_{s\to+\infty}\underbrace{\frac{1}{1+n}\sum_{t=1}^{s-1}\frac{-d_t}{\prod_{u=1}^{t}\frac{1+r_u}{1+n}}}_{PV\ of\ current\ and\ future\ surpluses}+\underbrace{\frac{g_s}{\prod_{u=1}^{s-1}\frac{1+r_u}{1+n}}}_{Bubble} \end{aligned} \tag{14}$$

The proof is in Appendix B. Proposition 1 gives a decomposition of the time-1 value of debt: with $r(k)<n$, the first component becomes negative infinite, and the second positive infinite.

There sum can very well be finite. Indeed, call g a steady-state per-capita value of government debt; as per Equation (8):

$$g=\frac{-d(k)}{r(k)-n};d(k)=\frac{\theta}{1+n}w(k)-\tau\times w(k) \tag{15}$$

where the formula is valid if $n\neq r(k)$. In the Cobb-Douglas special case 1,

$$g=\frac{y}{r(k)-n}\left(\tau-\frac{\theta}{1+n}\right)(1-\alpha) \tag{16}$$

When there exists a steady state k of the capital stock per capita, then there exists a steady state value g of government debt per capita.

In a steady state, it may very well happen that the government's social-security budget per capita is permanently in deficit ($d>0$), while, if $r(k)<n$, government debt still has

[19] The alternative of nominal denomination is explored in Section 4.1 below.

[20] Otherwise, one would have to specify in an ad hoc fashion the amount of benefits paid to the old generation at time 0.

positive market value. That, in a nutshell, is the point made by Blanchard (2019). Obviously, although the formula looks like the Gordon formula or the summation of a geometric series, the value of government debt, in the case $r(k) < n$, cannot be equal to the present value of future primary surpluses. The government debt contains a positive rational bubble, along the lines of Tirole (1985).

With a zero deficit, Tirole (1985) shows that there exists a starting value of the bubble for which a saddle path leads to a steady-state, non-zero, per-capita finite bubble, which lasts forever and brings about the efficient (Golden rule) outcome $r(k) = n$. Below that starting value, the bubble eventually reaches zero (at the stable steady state). In contrast, in our model, in every period with an endogenous social-security deficit new debt is issued and old debt is rolled over. In a steady state, the per-capita value of debt is given by the Gordon growth formula (15). Whenever there is a deficit, $d > 0$ that is, and the interest rate is smaller than growth, the government finances each and every deficit with more debt. Deficits occur forever, total debt grows forever, and, hence, it contains a rational bubble. Deficits in the steady states imply that $r(k) < n$ and vice versa, a configuration that best reproduces current economic conditions. To the opposite, when the steady-state social-security scheme produces surpluses, we have $r(k) > n$.[21] Whether in the stable or in the unstable steady state, if there is a permanent deficit, the bubble component of debt is positive and larger, in absolute value, than the present value of surpluses.

2 The definition of debt capacity

In this section, we continue to assume that the initial amount of debt, set by history, is contractually denominated as a real amount and we turn to the important matter of global convergence or divergence. In the case of a first-degree difference equation, the matter is obvious: if the initial value is between a locally stable steady state and a locally unstable one, the convergence occurs towards the stable steady state. In the case of second-degree difference equations, the matter is a bit more complicated; it is not sufficient to compare the initial conditions to the level of the steady state.

Consider a parametrization in which a stable steady state produces a deficit period by period financed by a never ending issuance of debt while the real rate of interest is lower than the real rate of growth of the economy. The economy might be (already) at the stable steady state or on its way to it. Figure 2 illustrates the fact that many paths with many levels of deficit-to-GDP (or debt to GDP) ratios all lead to the same stable steady state.

Example 3. *Example 1 continued: the joint dynamics of the capital stock and the debt are illustrated in the diagram of paths Figure 2. The diagram also shows the two loci $g_t = g_{t+1}$ and $k_{t-1} = k_t = k_{t+1}$.[22] Their two intersections are the two steady states, where the stable steady state is marked "S" and the unstable one "U".*

Definition 1. *Debt capacity for a given level of k_0 is the highest level of g_1 such that convergence occurs* without any change of policy parameters (θ, τ).

[21] Farmer and Zabczyk (2020) state that, if the economy is dynamic efficient ($r > n$), the bubble term vanishes as long as per-capita debt g_s does not explode as $s \to +\infty$.

[22] These are the loci as per the second formulation. As per the first formulation, the locus $k_{t-1} = k_t = k_{t+1}$ is made of two vertical straight lines and the locus $g_t = g_{t+1}$ of two horizontal lines.

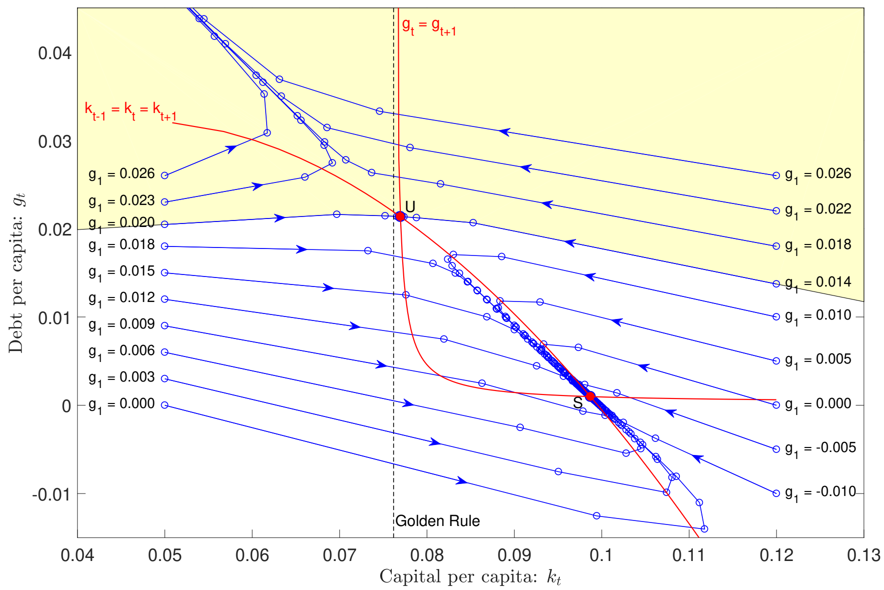

Figure 2: **The paths of the debt per capita and the capital stock per capita for two initial values k_1 and several initial values g_1.** Illustration with log utility and Cobb-Douglas production function. Parameter values are: $n = (1 + 0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1 - 0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$. The stable steady state is marked "S" and the unstable one "U".

Debt capacity is also the level of debt today that would lead to the unstable steady state along a saddle path. Provided debt starts from any level *strictly* within capacity, it converges to the stable steady state. The limitation on the level of debt exists despite the myopia of each individual generation. It is not due to concern about the welfare of future generations. Old generations and young generations trade with each other and hence one might think that through this trading current generations care, at least indirectly, for future generations. But they simply do not do that in an OLG setting.

To make more concrete the threshold between convergence and divergence, we display in Figure 3, for initial conditions (k_1, g_1), the contour that separates the area of convergence (upper, shaded area) from the area of divergence.

Example 4. *Example 1 continued: we include in the top plot in Figure 3 a diagram of paths in the plane of the ratio of debt to output g/y (y being output over* 25 *years) versus the real rate of interest r. The diagram shows that, for the higher of the two initial capital stocks displayed, a g_1 of* 0.014 *(which is about* 52% *of annual output), and for the lower one a g_1 of* 0.020 *(which is* 89% *of annual output), lead to the unstable steady state. At the unstable steady state itself, the debt is* 89.17% *of annual output. At the stable steady state, the debt is about* 4% *of annual output. We also display in the bottom plot in Figure 3 the paths in the plane of the ratio of budget deficit to output d/y versus the real rate of interest r. The ratio d/y can be viewed as a Maastricht criterion. We see that the deficit ratio is* 0.046% *in both steady states.*[23]

[23]For the Cobb-Douglas production function, $w(k)/f(k)$ is independent of k and the steady-state deficit-

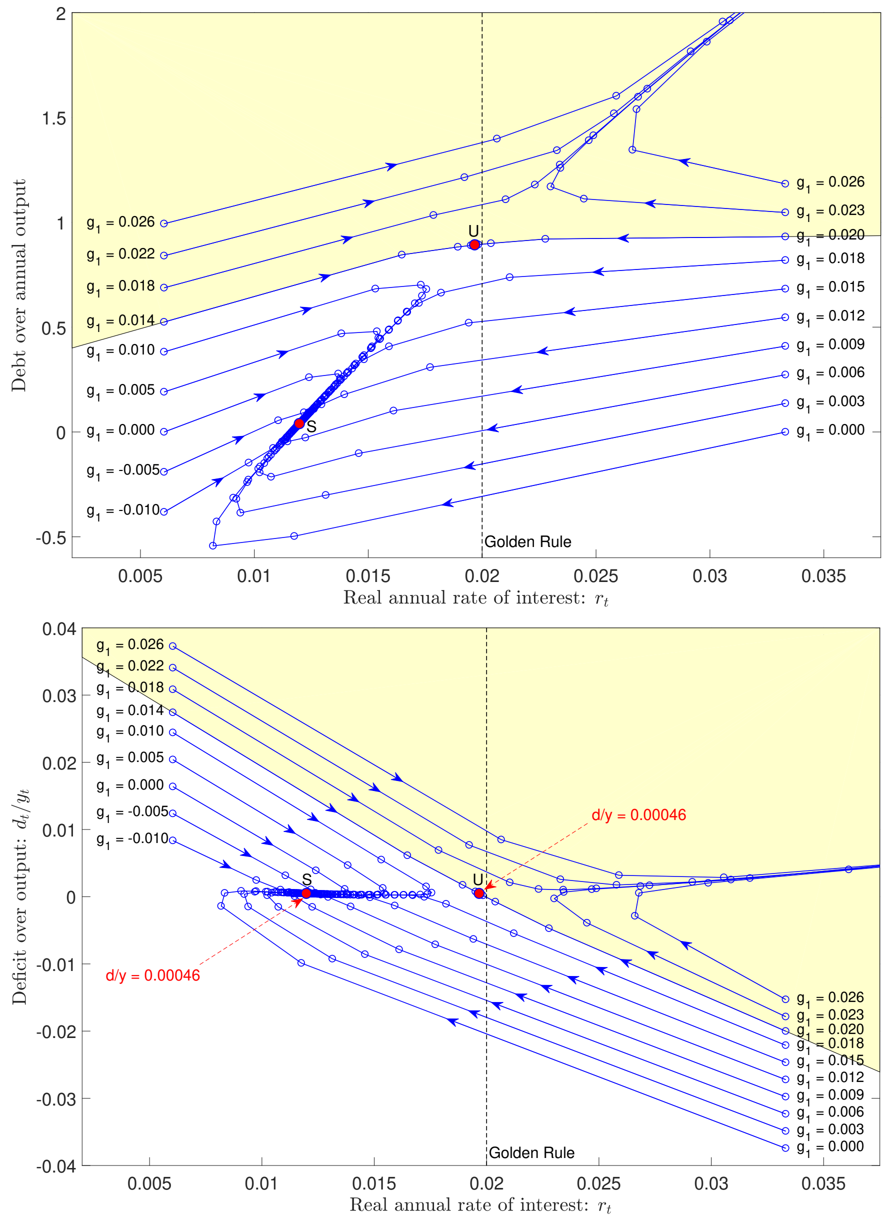

Figure 3: **The paths of the debt ratio and the real interest rate and of the deficit ratio and the real interest rate for two initial values k_1 and several initial values** g_1. Illustration with log utility and Cobb-Douglas production function. Parameter values are: $n = (1 + 0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1 - 0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$. The stable steady state is marked "S" and the unstable one "U". g is debt per capita; y is output per capita over 25 years and d is deficit over 25 years.

An economy can start above debt capacity, on a seemingly explosive path, if it can be anticipated that the government will, at some point, increase the tax rate, τ, decrease the social-security benefit ratio, θ, or both, in order to help return to a sustainable steady state. In Section 3 below, we examine such policy responses.

When the debt does not converge, one can call it "divergent" or "explosive". But it is more properly called "unsustainable" or, at most, "potentially explosive." As it exploded, the debt, inclusive of its bubble, would crowd out physical capital, since total saving cannot explode. Hence it could also be forecast that physical capital would become negative, which is impossible. Since this would be known to the last generation that lived just before this happened, the debt and the capital could not be sold to them; it would have zero market value. It then could not be sold to the previous generation and so on; hence, any equilibrium with explosive debt would unravel. This means that rational explosive paths cannot even begin: the debt cannot be sold, or has zero market value, even at the initial date.

Remark 1. *Although, when convergent, the debt is calculated forward (see the paragraph after Equations (3) to (9)), the bounds to be placed on it are forward looking, and, therefore, calculated backward. That is, indeed, true for the saddle path that leads to the unstable steady state. By the same token, when no steady state exists, there can be no debt at all, as all paths would be divergent.*

2.1 Existence

We now explore the existence of an unstable steady state and the comparative statics of it, under the assumption that the utility function of the private sector is isoelastic ($u(c) = c^{1-\zeta}/(1-\zeta); \zeta > 0$). The savings function is then explicit:

$$s[w_t, r_{t+1}] = w_t \frac{\beta^{\frac{1}{\zeta}}(1-\tau) - \theta(1+r_{t+1})^{-\frac{1}{\zeta}}}{\beta^{\frac{1}{\zeta}} + (1+r_{t+1})^{1-\frac{1}{\zeta}}} \tag{17}$$

More importantly, the wage w_t appears in it as a factor. That allows us to recast the steady-state equation (10) with $k_{t+1} = k_t = k_{t-1} = k$, in a way that places the preference parameters on one side of the equation and the production parameters on the other side, provided one puts r in the role of the unknown variable:

$$s[1, r] \times \frac{n-r}{1+n} = \kappa(r) \frac{n-r}{w(\kappa(r))} - \tau + \frac{\theta}{1+n} \tag{18}$$

where $\kappa(r)$ is defined as the function inverse of $f'(k) - \delta$ (a decreasing function). Equivalently, since $r = n$ is evidently not a solution as long as $-\tau + \theta/(1+n) \neq 0$, we can write the above as:

$$s[1, r] \times \frac{1}{1+n} = \kappa(r) \frac{1}{w(\kappa(r))} + \frac{-\tau + \frac{\theta}{1+n}}{n-r} \tag{19}$$

Savings divided by wage is equal to capital over wage plus government debt over wage. The function $s[1, r]$ depends on preference parameters β and ζ (and fiscal parameters τ, θ) while

output ratio is $(\theta/(1+n) - \tau)(1-\alpha)$.

$\kappa(r)$ and $w(\kappa(r))$ depend only on production parameters, and $-\tau+\theta/(1+n)$ is a composite budget deficit parameter.

We restrict our search to the class of steady states in which the rate of interest r satisfies $r < n$. The reason we invoke for this restriction is relevance to the current government-debt situation of developed countries. Referring to steady-state equation (15) above, these countries run a deficit ($d > 0$) and their debt has positive market value ($g > 0$). It must be, therefore, that $r < n$, with the result that the three terms of Equation (19) are all positive.[24]

When is there a steady state in that class? The answer to the question depends very much on the EIS $1/\zeta$ and on the ETS $\eta = 1/(1-\rho)$ between capital and labor.

The savings function (17) on the left-hand side of formulation (19) is well-known to be monotonic in the rate of interest. It is monotonically increasing when the EIS is greater than 1, so that the substitution effect dominates the income effect of the rate of interest, and monotonically decreasing otherwise. As the EIS changes, the graph of the function in the $(r, s[1,r])$ plane pivots around the point $(1/\beta - 1, (1-\tau-\theta)/(1+\beta^{-1}))$.

Assume further that the production function is CES, as in Equation (1) above. For that production function, the ratio w/k is equal to

$$(1-\alpha)\frac{1}{k}(1+\alpha\times(k^{\rho}-1))^{\frac{1-\rho}{\rho}}$$

which is a decreasing function of k. Hence, on the right-hand side of formulation (19), the first term is a decreasing function of r. The second term, however, rises sharply as r approaches n from below. As a result, the graph of the right-hand side goes through a minimum point. If and when the savings function of the left-hand side passes below that minimum, there does not exist a steady state. If and when the savings function of the left-hand side crosses the graph of the right-hand side above that minimum, there exist two steady states, in both of which $r < n$.

In Figure 4 we display the domain of existence of a steady state in the space of $1/\zeta$ (the EIS) on the x-axis and η (the ETS) on the y-axis. We have chosen a range for the EIS that is empirically relevant (see Hall (1988), Attanasio and Weber (1995) and Kaltenbrunner and Lochstoer (2010)). For the other parameter values indicated in the caption of the figure, the resulting minimal values of the ETS are fairly restrictive as they fall not far below 1, which is the traditional Cobb-Douglas reference case. Below those values there does not exist any debt capacity; government debt is explosive or unsustainable for any initial debt or asset level.

2.2 Comparative statics

Within the domain of existence, we plot on the x-axis of the top plot in Figure 5 the values of the rate of interest r at the locally stable (solid graph segments) and locally unstable (dotted graph segments) steady states for several values of the EIS and ETS. The value of the rate of interest at the stable steady state is a decreasing function of the ETS, the more

[24] Of course, we cannot assume that these countries are in a steady state. The assumptions we make is that the path of the rate of interest followed by an economy cannot get above n when initial conditions are $r_0 < n$ and g_0 or g_1 is below debt capacity. We verify these assumptions when we find a solution r below n.

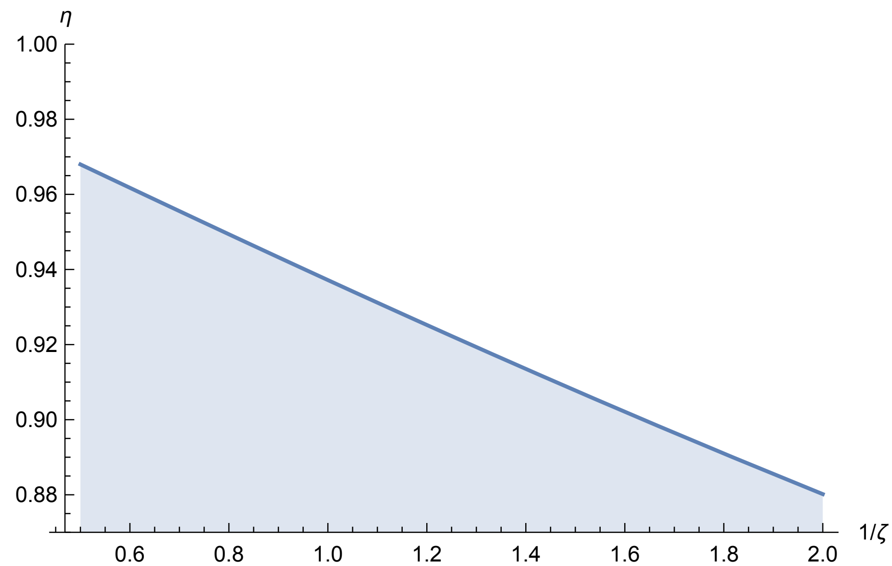

Figure 4: **Condition of existence of steady states with** $r < n$. For a given value of the elasticity of intertemporal substitution $1/\zeta$, the elasticity of technical substitution η must not fall in the shaded area below the frontier. The utility function is the isoelastic and the production function is CES. Parameter values other than EIS and ETS are: $n = (1 + 0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1 - 0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$.

so as the EIS is higher. As the ETS rises the value of the rate of interest at the unstable steady state rapidly approaches $r = n$, the more so as the EIS is higher.

In the bottom plot of Figure 5, we display (as the dotted graph segments) the corresponding values of the long-run debt capacity, which is the debt per capita at the unstable steady state. For low values of the ETS η, which are close to the frontier of non existence of a steady state, the debt capacity is not far from zero but it rises rapidly as the ETS rises toward its higher levels, the more so as the EIS is higher.

3 Policy experiments and demographic scenarios

We now exploit the concept of debt capacity to run policy experiments. If future policies that are stabilizing are anticipated, wrongly or rightly, debt may start above capacity on a seemingly explosive path. The stabilizing responses that are needed represent the true "fiscal cost" of exceeding debt capacity. We illustrate such scenarios in Figure 6. As before, the debt capacity per capita at the initial point in time is $g_1 = 0.0137$ (debt/output = 52.45%), which is on the saddle path leading to steady state marked U. If the debt starts above that level, it embarks on a seemingly explosive path that is rectified after one period of 25 years, by means of an increase in the wage tax rate τ, in order to put the economy on another saddle path. Notice a very important effect of this rectification. The steady state that follows rectification features a rate of interest that is above the Golden rule rate equal to the rate of population growth $n = 2\%$/year. Correspondingly, the new steady-state primary budget is

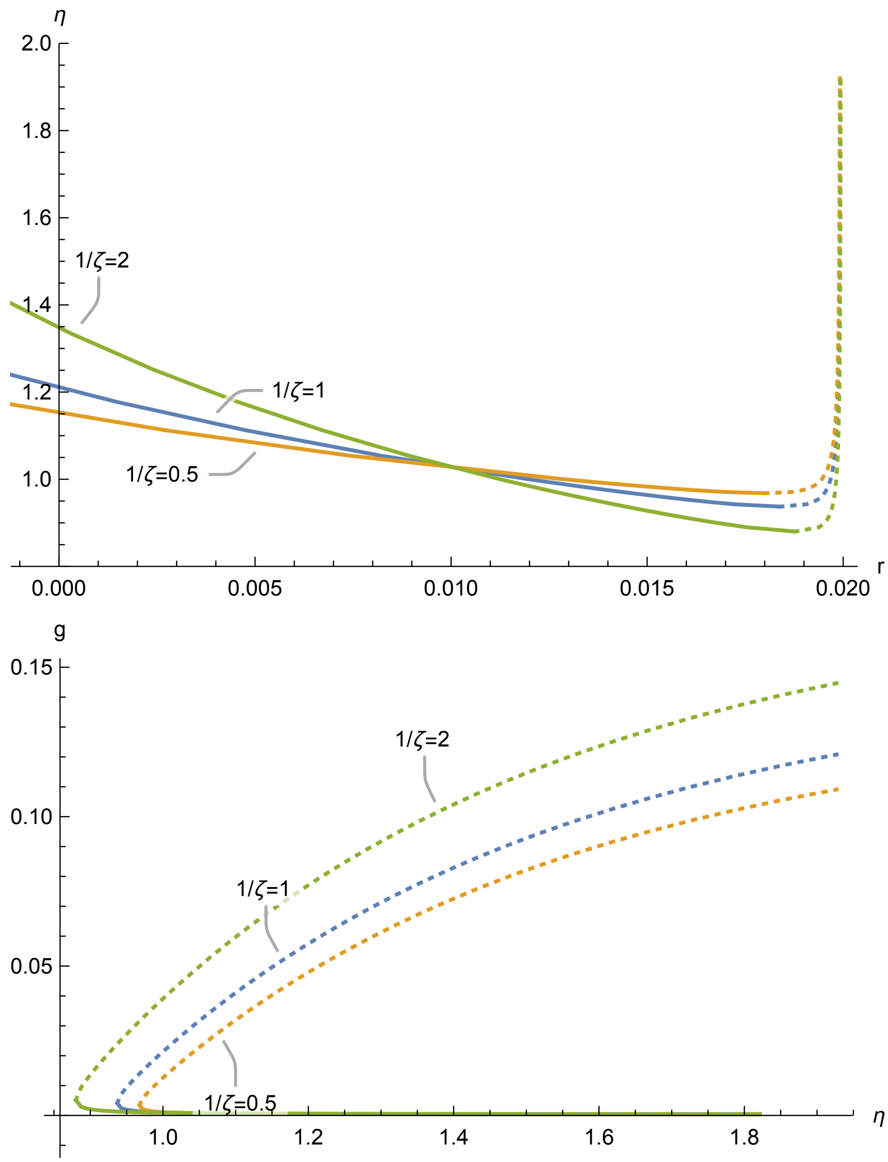

Figure 5: **Steady-state values of the yearly rate of interest r (on the x-axis) of the top plot and debt per capita (bottom plot) for different values of the elasticity of intertemporal substition $1/\zeta$ and of the elasticity of technical substitution** $\eta = 1/(1-\rho)$. The solid graph segments show the values at the stable steady state and the dotted segments the values at the unstable steady state. The utility function is the isoelastic and the production function is CES. Parameter values other than EIS and ETS are: $n = (1+0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$.

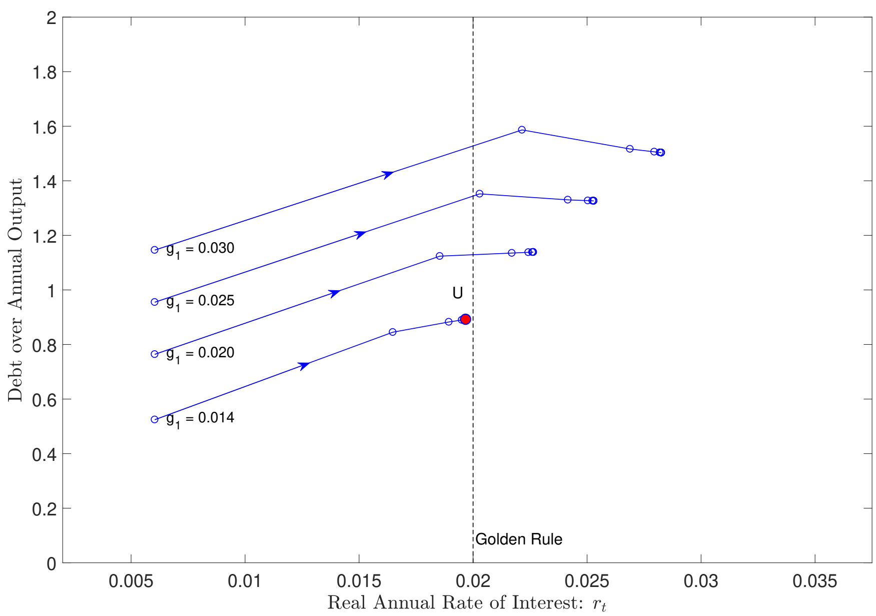

Figure 6: **The paths of debt/output and the rate of interest when debt starts above capacity and a policy response takes place at time** 2**.** Parameter values are: $n = (1+0.02)^{25}-1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1-(1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$. Initial capital per capita is $k = 0.12$ with $r = 0.006$, which is a high amount of capital and a low rate of interest in the context of our model illustration. Debt starts on or above the debt-capacity saddle path ($g_1 = 0.0137$, which is a debt-output ratio equal to 0.5245). The policy response is indicated in the column labelled "τ" in Table 1.

	Initial debt per capita	Initial debt /annual output	τ	Steady-state r/year	Steady-state debt /annual output	Steady-state deficit /output
$M = 0$	0.0300	1.1461	0.1183	0.0260	1.3714	−0.3552
	0.0250	0.9551	0.1111	0.0240	1.2394	−0.2098
	0.0200	0.7641	0.1052	0.0221	1.0940	−0.0934
	0.0137	0.5245	0.1000	0.0197	0.8917	0.0115
$M = 1$	0.0300	1.1461	0.1281	0.0283	1.5030	−0.5508
	0.0250	0.9551	0.1156	0.0253	1.3264	−0.3007
	0.0200	0.7641	0.1068	0.0226	1.1385	−0.1248
	0.0137	0.5245	0.1000	0.0197	0.8917	0.0115
$M = 2$	0.0300	1.1461	0.1569	0.0336	1.7442	−1.1263
	0.0250	0.9551	0.1264	0.0279	1.4821	−0.5157
	0.0200	0.7641	0.1099	0.0236	1.2130	−0.1856
	0.0137	0.5245	0.1000	0.0197	0.8917	0.0115

Table 1: **Tax rate, interest rate, and deficit responses to over-capacity initial debt.** Parameter values are: $n = (1+0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1-(1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$. Initial capital per labor is $k = 0.12$ with $r = 0.006$, which is a high amount of capital and a low rate of interest in the context of our model illustration. Debt starts on or above the debt-capapcity saddle path. The policy response can take place at the intitial point, $M = 0$, or with a delay, $M = \{1, 2\}$.

in surplus. The reason is that the initial steady state was close to n, leaving little room for extra debt. All the same, the delay in the response generally goes in the direction of forcing a higher tax increase.

In Table 1, we calculate the needed quantitative responses. When the response is immediate ($M = 0$), an initial debt equal to 114.61% of output requires a tax of 11.83%, which is above the 10% considered so far and leads to a steady-state rate of interest equal to 2.6%/year. When the response is delayed to the next generation ($M = 1$), the tax rate for the same initial debt must be raised to 12.81%, which leads to a much larger steady-state surplus and a larger steady-state interest equal to 2.83%/year. We see again that, as the government delays the policy response, the new tax rate may be sufficiently high to cause a switch to $r > n$ and to a huge primary surplus.

	τ	Steady-state r/year	Steady-state debt /annual output	Steady-state deficit/output
		Initial drop to 1%/year		
$M = 0$	0.1288	0.0101	0.6990	−0.0021
$M = 1$	0.1339	0.0130	1.0458	−0.104
$M = 2$	0.1500	0.0180	1.5215	−0.4275
		Initial drop to 1.5%/year then to 1%/year		
$M = 0$	0.1301	0.0111	0.8206	−0.0282
$M = 1$	0.1399	0.0152	1.2721	−0.2245
$M = 2$	0.1673	0.0218	1.7919	−0.7729

Table 2: **Tax rate, interest rate, debt and deficit responses to declining population growth.** Parameter values are: $n = (1+0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1 - (1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$. Initial capital per labor is $k = 0.12$ with $r = 0.006$, which is a high amount of capital and a low rate of interest in the context of the model. Debt starts on the saddle path, which means at capacity (debt per capita $g_1 = 0.0137$, which is debt/annual output = 0.5245). At the initial point in time, or at the initial point and then again at the second point in time, the annual population growth rate drops to the levels indicated. The policy response can take place immediately, $M = 0$, or with a delay, $M = \{1, 2, 3\}$.

We also explain how debt could come to exceed debt capacity. Debt may become unsustainable, for instance, because of a drop in the growth rate of the population. For that reason, we turn to a major concern that one might have regarding the debt capacity of a government. In real life, population growth has been in decline in every single industrialized economy. In our model, when the population growth rate falls the debt capacity shrinks and the economy may move to an exploding path, which, as we saw, would actually unravel. Here again, in case of explosion, the government could no longer sell its debt unless it increased taxes, or promised to do so, and thereafter embarked on a new saddle path.

To generalize the principle that the later the policy response, the larger it has to be, we develop scenarios of population-growth declines, starting debt exactly on the saddle path and varying the timing of the response. Specifically, we start with a high k (0.12) and with a debt at capacity (debt over annual output equal to 52.45%). In the first scenario, the rate of population growth is 1%/year instead of 2%/year. The policy response can take place immediately ($M = 0$) or with a delay of several quarter centuries ($M = \{1, 2\}$). The top panel of Table 2 indicates the value to which the government must raise the contribution or tax rate τ in order to stay on a saddle path and avoid an unsustainable situation. It is clear from this panel that, if the government increases the tax rate with delay ($M > 0$), it must raise it more: changing the tax rate in the same period as the decline in n requires an increase from 10% to 10.1% while two periods after the decline in the population growth rate an 18% tax is required.

In the second scenario (bottom panel of Table 2), the rate of population growth is 1.5%/year initially and drops 1%/year at the next generation. If this demographic evolution is anticipated by the government so that it acts right at the initial point in time, the tax needed for sustainability is only 11.1%, whereas, if it waits till the second drop in population growth, the tax must be as high as 21.8%. In both scenarios, the need arises in case of delay to generate a huge steady-state budget surplus.

The true fiscal cost of excessive government debt issuance cannot be assessed from the current rate of interest or any current macroeconomic variable. Rather, it should be assessed in a dynamic context reflecting anticipated deficits and population growth going forward. A switch to surpluses may be needed in the future, either through increased tax rates, reduced social-security benefits, or both, all being politically painful. Our model can arguably be interpreted as saying that it is better to implement a policy that reduces the debt automatically during normal times, as a way to aim towards the stable steady state, so that a safety margin remains in case growth drops.

4 Extensions

In this section we consider two additional forms of intervention by the government. First, we aim to increase our model's degree of realism by introducing nominal considerations. For that, we allow the government (not distinguished from the central bank) to issue nominal debt and to buy and sell it as a way to implement a form of monetary policy. Second, we ask whether a government can increase its debt capacity by subsidizing innovation which ultimately raises productivity and growth.

4.1 Inflation and the price level

One might be concerned that a high, potentially explosive level of nominal debt would produce high inflation.

To investigate that issue, we now assume realistically that the initial amount of debt, set by history, is contractually denominated as a nominal amount. In the interest of simplicity, however, there are no actual money balances; money is just a unit of account that serves to specify the face value of the debt.[25] We allow the government to intervene in the money market, buying and selling bonds, as a way to fix the nominal rate of interest, in addition to collecting taxes and paying benefits. As before, government debt is a one-period debt. Let the nominal rate of interest be i_t.

[25]This is a so-called "cashless economy". See Woodford (2003).

The simultaneous budget constraints of the households and the government at time t are as follows:

- young household

$$c_t^t + s_t = (1-\tau) \times w_t$$

- old household

$$c_t^{t-1} = s_{t-1} \times (1+i_t)\frac{P_{t-1}}{P_t} + \theta \times w_{t-1}$$

- government

$$-G_{t+1} + \theta w_{t-1} L_{t-1} = \tau w_t L_t - (1+i_t)\frac{P_{t-1}}{P_t} G_t$$

  or

$$-(1+n) g_{t+1} + \theta w_{t-1} \frac{1}{1+n} = \tau w_t - (1+i_t)\frac{P_{t-1}}{P_t} g_t$$

  where G_t is the total debt in *real* units (total nominal debt deflated by P_{t-1}) with which the government enters time t and G_{t+1} is the debt in real units with which it exits time t, and $g_t \triangleq G_t/L_t$.

The behavior of the agents is dictated by:

- government (Taylor rule):

$$1 + i_{t+1} = (1+\bar{\imath}) \times \left(\frac{\frac{P_{t+1}}{P_t}}{1+\bar{\pi}}\right)^{\phi}; \phi \geq 0; \phi \neq 1 \tag{20}$$

- firms:

$$\begin{aligned} f'(k_t) - \delta &= r_t \\ f(k_t) - k_t f'(k_t) &= w_t \end{aligned}$$

- young households:

$$\frac{\frac{\partial}{\partial c_{t+1}^t} U\left(c_t^t, c_{t+1}^t\right)}{\frac{\partial}{\partial c_t^t} U\left(c_t^t, c_{t+1}^t\right)} = \frac{1}{1+i_{t+1}} \frac{P_{t+1}}{P_t}$$

Market clearing is unchanged. The initial conditions are set by the initial capital stock $k_0 = K_0/L_0$ and the amount of *nominal* debt $G_1 \times P_0$ (this product is a given number) with which the government leaves time 0.

It follows from this specification that, in economic equilibrium, all variables are identical in real units to what they were in Section 1 with:

$$1 + r_t = (1+i_t)\frac{P_{t-1}}{P_t}$$

It remains to solve for the rate of inflation, given the solution for the real variables (including in particular the real rate r_t). The Taylor rule provides a first-order difference equation that answers the question

$$\begin{aligned}
(1+r_{t+1})\,\frac{P_{t+1}}{P_t} &= (1+\bar{\imath}) \times \left(\frac{\frac{P_{t+1}}{P_t}}{1+\bar{\pi}}\right)^{\phi} \\
\frac{P_{t+1}}{P_t} &= \left[\frac{(1+r_{t+1}) \times (1+\bar{\pi})^{\phi}}{(1+\bar{\imath})}\right]^{\frac{1}{\phi-1}}
\end{aligned}$$

If ever debt exceeded debt capacity and the real rate were on a putative explosive path,[26] the rate of inflation would also turn to hyperinflation if $\phi > 1$. It would approach zero if, instead, $\phi < 1$. The two plots in 7 (Example 1 continued) illustrate the contrast in the plane $(G/Y, i)$. Inflation is debt inflation.

It would remain to determine the price level. In principle, since the initial conditions for public debt are set contractually in nominal terms, that could be achieved by means of the Fiscal Theory of the Price Level (FTPL), which compares the real discounted value of government debt G_1 with its contractual nominal value. But we have seen in Section 1, that, in this OLG model, the real discounted value of government debt is not uniquely determined.[27] For that reason, the price level is undetermined (within bounds).[28] We have no choice but to assume that it is inherited from history as a social convention.

Lastly, since the real value of the debt is fixed, an inconvenient truth is that the government cannot inflate away its debt.

4.2 Innovation

In the model developed so far, the only source of perpetual growth is population growth. However, in today's economy, productivity, thanks to innovation, keeps rising. As a second extension, we would like to know whether the government can increase its debt capacity by subsidizing innovation and fostering growth. To determine to what extent innovation can modify a government's debt capacity, we borrow a model from the literature on endogenous growth but let the government finance R&D, in addition to paying for social security. Strictly speaking, growth by innovation is endogenous when R&D is decided by the private sector. Our focus, however, is on government expenditure. We choose a model of innovation that allows policy to have a direct effect on growth.[29]

The modifications to the specification of the economy are as follows.

[26] Here again, explosive paths will unravel.

[27] Bassetto and Cui (2018) show for a broad class of economies, including for economies with dynamic inefficiency, that the FTPL is not a robust equilibrium selection criterion when the interest rate is persistently below the growth rate of the economy. Further, Farmer and Zabczyk (2020) show that it cannot be used to determine the price level uniquely.

[28] Plots in 7 are drawn with $P_0 = 1$.

[29] See Aghion and Howitt (1998), Dinopoulos and Thompson (1998), Peretto (1998), and Young (1998). We use a reduced form formulated by Jones (1999) that is also used in Laincz and Peretto (2006). That last paper, which is empirical, provides us with ranges for the values of parameters.

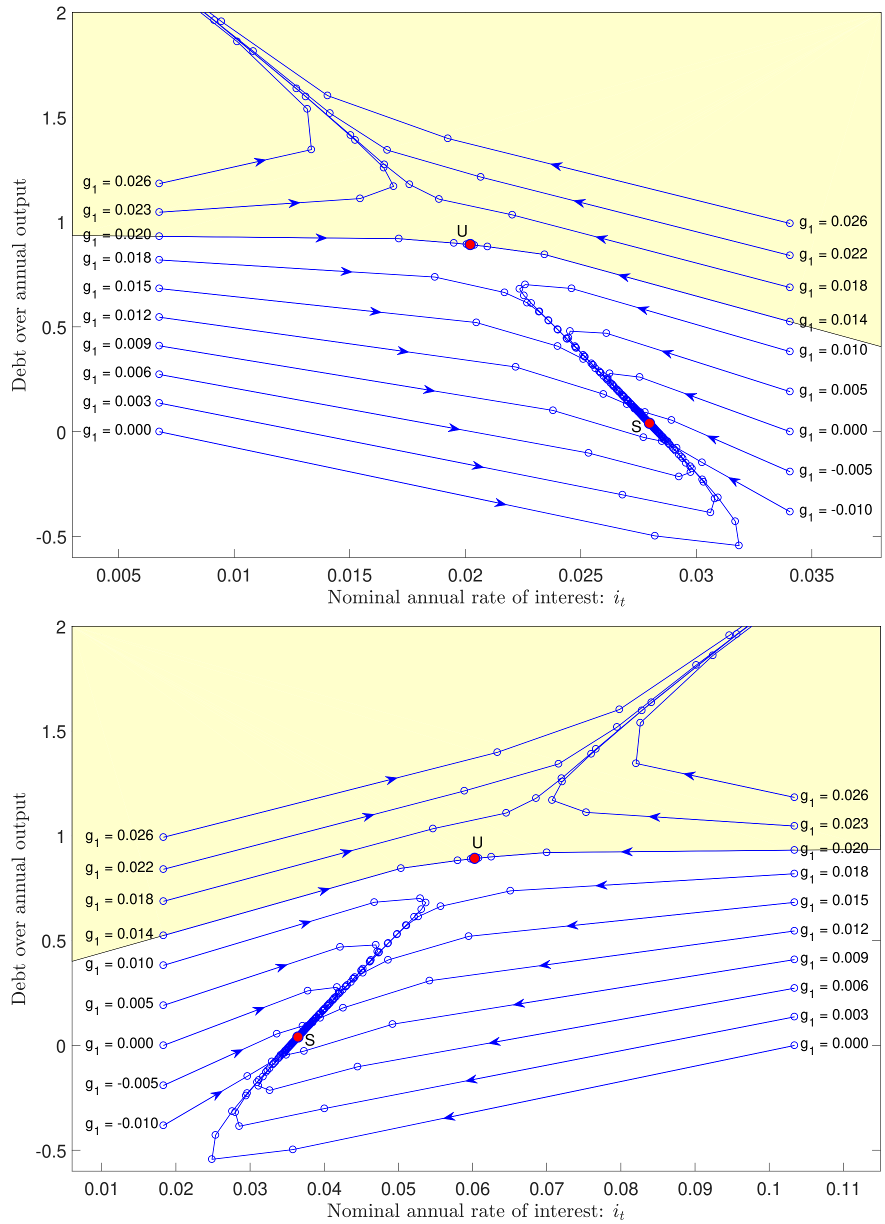

Figure 7: **The paths of the debt ratio and the nominal interest rate.** Illustration with log utility and Cobb-Douglas production function. Parameter values are: $n = (1+0.02)^{25}-1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\delta = 1-(1-0.1)^{25}$, $\theta = 0.165$, $\tau = 0.1$, $\phi = 0.5$ (top plot) and $\phi = 1.5$ (bottom plot), $\bar{\imath} = 0.03$, $\bar{\pi} = 0.02$, $P_0 = 1.0$. The stable steady-state is marked "S" and the unstable one "U".

The households/investors: intermediate goods are produced in varieties i, the cardinal number B_t of which grows like population ($B_t = L_t$). This assumption captures the fact that more people generate more varieties; see Jones (1999). For continuity with the previous specifications, in which the population growth rate n stood for all exogenous forms of productivity growth, we now reduce that number to make room for endogenous growth.[30]

Varieties of intermediate goods enter the utility function $U\left(c_t^t, c_{t+1}^t\right)$ in a final-good, composite CES form

$$C_t = L_t c_t^t + L_{t-1} c_t^{t-1} = \left(\int_0^{B_t} Y_{i,t}^{1/\theta_Y} di\right)^{\theta_Y}$$

where Y_{it} is the output of each variety and $\int_0^{B_t} di = B_t$. We look for an equilibrium that is symmetric across varieties: $Y_{i,t} = \bar{Y}_t$; $\int_0^{B_t} Y_{i,t} di = B_t \bar{Y}_t = Y_t$. In other words, total production of intermediate goods is $Y_t = B_t \bar{Y}_t$ while total consumption of final goods is:

$$C_t = \bar{Y}_t B_t^{\theta_Y} = Y_t B_t^{\theta_Y - 1}$$

The production function for each variety i is

$$Y_{i,t} = A_t^{\sigma} F\left(K_{i,t}, \Lambda_{i,Y,t}\right)$$

where $K_{i,t}$ and $\Lambda_{i,Y,t}$ are the inputs of physical capital and labor into the production process of variety i, $A_t > 0$ is knowledge capital applicable in a non rival way to the production of all varieties, and $0 < \sigma$. With symmetric use of labor for the production of each variety,

$$\bar{Y}_t = A_t^{\sigma} \frac{F\left(K_t, \Lambda_{Y,t}\right)}{B_t}$$

so that

$$\begin{aligned} Y_t &= A_t^{\sigma} F\left(K_t, \Lambda_{Y,t}\right) \\ C_t &= B_t^{\theta_Y - 1} A_t^{\sigma} F\left(K_t, \Lambda_{Y,t}\right) \end{aligned}$$

The production and accumulation of knowledge capital: is controlled by government expenditure. It evolves as

$$A_{t+1} - A_t = \frac{\theta_A L_{A,t} A_t}{B_t}$$

where $\theta_A > 0$ is the productivity of labor in knowledge production and $L_{A,t}$ is the amount of labor devoted by the government to knowledge production. As the number of varieties rises, more research labor is required to increase knowledge.

[30] That is, we solve for a new number n such that, when research activity is at a zero level, the overall growth rate, including population growth and varieties growth, remains what it was before. See Footnote 31.

Taxation and spending: the government budget constraint becomes

$$-G_{t+1} + \theta w_{t-1} L_{t-1} + w_t L_{A,t} = \tau w_t L_t - (1 + r_t) G_t$$

Market clearing: the labor market clears

$$\Lambda_{Y,t} + L_{A,t} = L_t$$

and the market for goods clears

$$L_t c_t^t + L_{t-1} c_t^{t-1} + K_{t+1} = B_t^{\theta_Y - 1} A_t^{\sigma} F(K_t, \Lambda_{Y,t}) + (1 - \delta) \times K_t$$

Difference equations and steady states: Suppose that the government pays for a constant proportion s_A of labor to be involved in R&D: $L_{A,t} = s_A \times L_t$. With that, the difference-equations system governing the evolution of the economy, stated on a per capita basis, still contains Equations (3,6,7) together with:

$$B_t^{\theta_Y - 1} A_t^{\sigma} f'\left(\frac{k_t}{1 - s_A}\right) - \delta = r_t \tag{21}$$

$$B_t^{\theta_Y - 1} A_t^{\sigma} \left[f\left(\frac{k_t}{1 - s_A}\right) - \frac{k_t}{1 - s_A} f'\left(\frac{k_t}{1 - s_A}\right)\right] = w_t \tag{22}$$

$$-(1 + n) g_{t+1} + \theta w_{t-1} \frac{1}{1 + n} + w_t s_A = \tau w_t - (1 + r_t) g_t \tag{23}$$

$$c_t^t + \frac{1}{1 + n} c_t^{t-1} + (1 + n) k_{t+1} = (1 - s_A) B_t^{\theta_Y - 1} A_t^{\sigma} f\left(\frac{k_t}{1 - s_A}\right) + (1 - \delta) \times k_t \tag{24}$$

$$A_{t+1} - A_t = \frac{\theta_A L_{A,t} A_t}{B_t} \tag{25}$$

Equations (21) and (22) allow us to define r_t and w_t as functions $r(k_t, A_t, B_t)$ and $w(k_t, A_t, B_t)$. The savings function $s_t = s(w_t, r_{t+1})$ is unchanged. Proceeding to equate demand and supply, as we did in Section 1, we get an equation relating k_{t+1} to k_t and k_{t-1}

$$s\left[w(k_t, A_t, B_t), r(k_{t+1}, A_t, B_t)\right] = \begin{bmatrix} 1 & \frac{1}{1+n} & -1 \end{bmatrix} \times \begin{bmatrix} (1 - \tau) w(k_t, A_t, B_t) \\ s\left[w(k_{t-1}, A_{t-1}, B_{t-1}), r(k_t, A_t, B_t)\right] \times (1 + r(k_t, A_t, B_t)) + \theta \times w(k_{t-1}, A_{t-1}, B_{t-1}) \\ (1 - s_A) B_t^{\theta_Y - 1} A_t^{\sigma} f\left(k_t \frac{1}{1 - s_A}\right) + (1 - \delta) k_t - (1 + n) k_{t+1} \end{bmatrix} \tag{26}$$

The evolution of the debt follows from Equation (23).

Given the similarity of equation system (26) with the previous one (10), one can safely state that, under similar conditions, there will be again two steady states, in each of which, however, growth per capita is no longer zero. That is, there are two "expansion paths" with the same growth rate (see below the calculation of the growth rates) but differing rates of interest rate. One of them is stable as all paths that start in the debt-capacity region (to be determined) approach it; the other is unstable as all paths that do not start within the debt-capacity region diverge from it.

We turn to the calculation of growth rates on a steady-state path. The stock of knowledge capital A_t evolves autonomously as does the population. From Equation (25), for a constant policy s_A, its growth rate, denoted ω_A, is equal to $\theta_A s_A$ (independently of a steady-state assumption). For a Cobb-Douglas production function, Equation (21) says that $B_t^{\theta_Y - 1} A_t^{\sigma} \left[k_t / \left(1 - s_A\right)\right]^{\alpha - 1}$ is constant:

$$\left(1 + \omega_A\right)^{\sigma} \left(1 + n\right)^{\theta_Y - 1} \left(1 + \omega\right)^{\alpha - 1} = 1$$

which gives the steady-state rate of growth ω of capital per capita k_t (a rate which, in previous sections, was equal to 0)[31]

$$1 + \omega = \left[\left(1 + \omega_A\right)^{\sigma} \left(1 + n\right)^{\theta_Y - 1}\right]^{\frac{1}{1-\alpha}} = \left[\left(1 + \theta_A s_A\right)^{\sigma} \left(1 + n\right)^{\theta_Y - 1}\right]^{\frac{1}{1-\alpha}}$$

Output per capita y, debt per capita g, deficit per capita d and the wage rate w all grow at that same rate at any steady state.

As we did before in the case of a Cobb-Douglas production function, where:

$$\begin{aligned} r\left(k_t, A_t, B_t\right) &= B_t^{\theta_Y - 1} A_t^{\sigma} \alpha \left(\frac{k_t}{1 - s_A}\right)^{\alpha - 1} \frac{1}{1 - s_A} - \delta \\ w\left(k_t, A_t, B_t\right) &= B_t^{\theta_Y - 1} A_t^{\sigma} \times \left(\frac{k_t}{1 - s_A}\right)^{\alpha} - \frac{k_t}{1 - s_A} \left[r\left(k_t, A_t, B_t\right) + \delta\right] \end{aligned}$$

we can write the equation for steady-state interest rates r (analogous to Equation (19))

$$\begin{aligned} \frac{1}{\left(1+n\right)\left(1+\omega\right)} s\left(1, r\right) &= \frac{1 - s_A}{r + \delta} \frac{\alpha}{1 - \alpha} \\ &+ \frac{1}{\left(1+n\right)\left(1+\omega\right) - \left(1+r\right)} \left(\frac{\theta}{\left(1+n\right)\left(1+\omega\right)} - \left(\tau - s_A\right)\right) \end{aligned} \tag{27}$$

and calculate the steady-state debt-capacity ratio, which is debt per output at the unstable steady-state, as we did before in Equation (16):

$$\frac{g}{y} = \frac{1}{\left(1+n\right)\left(1+\omega\right) - \left(1+r\right)} \left(\frac{\theta}{\left(1+n\right)\left(1+\omega\right)} - \left(\tau - s_A\right)\right) \frac{1 - \alpha}{1 - s_A}$$

In Figure 8, we display the way it changes as one varies the policy parameter s_A and other parameters. One would expect two opposing effects: R&D enhances growth but deepens the

[31] In these equations, as explained above, n is reduced to a number n_{new} such that

$$\left(1 + \omega|_{s_A = 0}\right)\left(1 + n_{new}\right) = 1 + n_{old}$$

$$\left(1 + n_{new}\right)^{\frac{\theta_Y - 1}{1 - \alpha} + 1} = 1 + n_{old}$$

For $1 + n_{old} = \left(1.02\right)^{25}$, $n_{new} = 0.4183$ (over 25 years), which is $0.01408/year$.

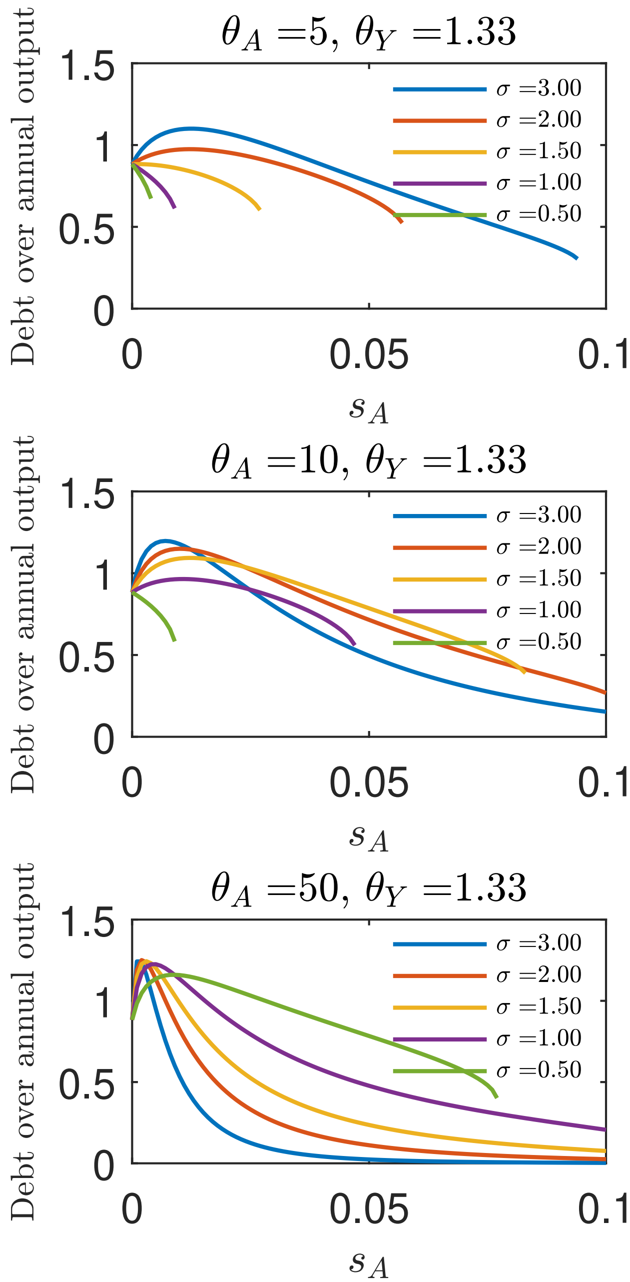

Figure 8: **Debt capacity as a function of s_A.** Illustration with log utility and Cobb-Douglas production function. Across the plots, we vary θ_A from $\theta_A = 5$ (top) to $\theta_A = 50$ (bottom). Then, in each plot of g/y we vary σ and s_A. The line stops at combinations of parameters for which the steady-state does not exist. In these examples we set the population growth rate such that the compounded growth rate is at 0.02 for $s_A = 0$; see footnote 31. We use $\alpha = 0.2$ and $\theta_Y = 4/3$ implying $n_{new} = 0.01408/year$ (0.4183 over 25 years). The other parameter values are identical to what they are in the other figures.

deficit of the government. Specifically, one would hope for a hump shaped relation where initially increasing s_A the growth effect of an increase in R&D dominates and thus the debt capacity increases. For some large enough value of s_A the deficit increases faster than growth implying that the debt capacity declines.

The plots in Figure 8 confirm this intuition. These are drawn for various values of θ_A (the degree to which the growth of knowledge responds to R&D labor input) and σ (the elasticity of the output of intermediate goods with respect to the capital of knowledge) and for $\theta_Y = 1.33$, which corresponds to an elasticity of substitution between varieties equal to 4, a number accepted often by macroeconomists (see, for instance, Galí (2015)). We see that for some specification the debt capacity increases all the way to 1.25. We do see a hump shaped relation but the hump occurs for small values of s_A of about 2%. We suspect that most high-income countries already spend more than 2% on R&D. Therefore, for most parameter configurations, steady-state debt capacity is not increased, or is even reduced, by an increase in public R&D spending beyond what it is already. Overall, this exercise does not show that public R&D spending miraculously lifts debt capacity.

5 Social security

In this section, we provide the rationale for having chosen to incorporate social security in our model as the form of government spending. We verify that, when the rate of interest is below the rate of growth, a social-security scheme, balanced or unbalanced, can be welfare improving, which is the reason for which we chose that form of government spending as an illustration. We focus on the steady-state lifetime welfare, which we define as in Diamond (1965). In either the stable or unstable steady state, the lifetime utility of one person is constant, generation after generation.

Five configurations are considered here, the first two being viewed as benchmarks: the Diamond (1965) equilibrium with no social security and no debt, the bubbly equilibrium of Tirole (1985) with zero deficit and no debt, the equilibrium with balanced security and zero government debt as in Blanchard and Fischer (1989), the equilibrium with balanced social security with pure roll-over government debt, and finally equilibrium with social security in deficit, financed by government debt.

The Diamond equilibrium is inefficient for the well-known reason that each generation, in order to finance their retirement, saves in excess of what they would if the welfare of all generations were optimized. As there is too much capital,[32] the steady-state utility is strictly smaller than in the Golden-rule equilibrium, which can be reached in the Tirole bubbly, zero-deficit equilibrium. These two facts are reflected in our Figure 9 (Example 1 continued) by the solid green line, which is below the solid blue line.

Because the government is infinitely lived, it alone can issue debt that can be perpetually refinanced and, for that reason, can contain a bubble component. When the stock of capital is too high,[33] our Figure 9, – plotted against the level of benefits and for two levels of

[32]This comes with the caveat that in case of endogenous productivity growth, there are two kinds of capital: physical capital and knowledge capital. Knowledge capital may be too low, while physical capital is too high. See Section 4.2.

[33]If, to the opposite, the stock of capital were below the welfare optimum (the rate of interest is above the

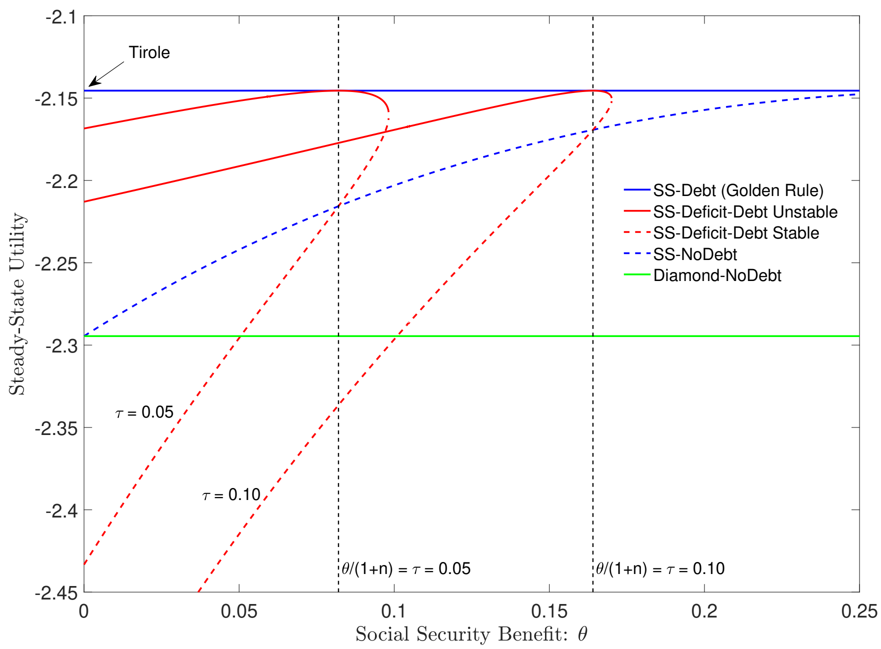

Figure 9: **Steady-state utilities.** Illustration with log utility and Cobb-Douglas production function. The parameter values are identical to what they are in the other figures. In the plot, we vary the social-security benefit, θ, and show the resulting steady-state utilities in the Diamond and Tirole models and in our stable and unstable steady-states, and for a special case without debt. Five configurations are considered: the Diamond (1965) equilibrium with no social security and no debt, the bubbly equilibrium of Tirole (1985) with zero deficit and no debt, the equilibrium with fully funded social security and zero government debt, the equilibrium with fully funded social security with pure roll-over government debt, and finally equilibrium with social security in deficit, financed by government debt.

contributions (taxes) of 5% and 10% –, illustrates the fact that a budget deficit generated by social security and financed by debt can be a welfare-improving form of spending, relative to the competitive Diamond equilibrium.

The figure shows the special case of the equilibrium with balanced social security and no debt. That configuration can only approach the welfare optimum.

More importantly, the figure shows that, with deficit social security, the unstable steady state, when it exists and assuming one can stay there, produces a larger utility per labor than the competitive equilibrium of Diamond (1965). For the special cases with zero deficit, such as $\theta/(1+n) = \tau = 0.05$ and $\theta/(1+n) = \tau = 0.10$, the unstable steady state with bubbly debt can actually reach the Golden Rule equilibrium, as does Tirole's bubble.

The stable steady states of equilibria with deficit social security also produce a welfare improvement, but only for sufficiently high values of the benefits. For lower values of the benefits, it is also possible for the stable steady state to exhibit smaller utility per capita than the competitive equilibrium of Diamond (1965).

6 Conclusion

In an overlapping-generations economy with capital accumulation and a realistic social-security scheme, where debt covers deficits from the scheme, debt is welfare improving and can have positive market value even if the government budget is forever in deficit. This is because government debt contains a rational bubble. Of two steady states we found, the unstable one has higher debt to GDP ratio and is closer to the Golden-rule economy. In that sense, it appears to be a good idea to let the debt rise if it is not already at its capacity level.

We have shown that, even when the cost of financing is very low, one cannot push the level of debt beyond some amount. We have defined debt capacity as the level of debt that leads to an unstable steady state. Whenever the market value of debt is below debt capacity, the debt converges to a stable steady state. If it is above, it is unsustainable. We have followed the economy along an explosive path and shown that government debt crowds out physical capital to extinction, so that by anticipation such paths actually unravel, which means that debt is unsustainable.

Steady states, however, may not exist. When none exists, there is no capacity for debt. We have explored the issue of existence among equilibria for which the real rate of interest is below the real rate of growth, in accordance with the situation in today's world. And, when steady states exist, we have shown how debt capacity varies with the parameters of the model, the key ones being the elasticity of intertemporal substitution in the lifetime utility and the elasticity of substitution between capital and labor.

We have used this basic idea to run policy experiments. If future policies that are stabilizing are anticipated, wrongly or rightly, debt may start above capacity on a seemingly explosive path, which may be slow and last several generations. The stabilizing responses that are needed sooner or later represent the true "fiscal cost" of exceeding debt capacity. By way of illustration, we have examined demographic scenarios, which can lead to debt

one in the welfare optimum), the proceeds of government debt issue could, of course, be used for investment. But that is not the case considered here.

becoming unsustainable.[34]

We have extended the model and its concept of debt capacity to two policy-relevant settings. First, we have shown in a nominal, cashless version of the economy that debt explosion means inflation explosion as well, while at the initial point in time the price level is indeterminate. Second, adding growth by innovation to our model, we have shown that, in all cases considered, a government R&D subsidy raises debt capacity for small subsidy amounts but, for larger subsidy rates, worsens it.

The first policy implication of our model is that it is not enough to compare interest rates to growth rates to draw any conclusion about the sustainability of debt; amounts of debt outstanding also matter. They must remain within debt capacity.

Secondly, the debt capacity is related to parameters of the economy. When near a cliff edge, it is useful to find out where the edge is located. Our model is a first attempt, in a realistic policy setting, at locating the edge.

Are the high-income country government debt levels close to debt capacity right now? Careful econometric estimation of our model will have to be carried out before precise, quantitative answers can be given. We have provided some illustrative numerical examples; these suggest that the current debt levels of high-income countries are close to debt capacity right now. We see steady increases in deficits and government debt levels relative to GDP. We do not, so far, see that social-security benefits, or other governmental services, or governmental spending more generally, are being reduced. To make things worse, population growth rates are declining and even turning negative. At least, they are predicted to turn negative. All of these developments but, especially, the reduced population and economic growth rates, leave little debt capacity to spare.

[34]There is evidence suggesting that productivity has declined over the last few decades. The effect of such a decline is similar.

Appendixes

A Comparison of the second formulation (12) with previous models

In an attempt to enhance policy realism, our model generalizes a number of preexisting models described below:

- **Diamond (1965):** $d(k_{t-1}, k_t) = -\varphi$. The deficit is negative; a constant tax on the young yields a government surplus.

$$\begin{aligned} s(w(k_t), r(k_{t+1})) &= (1+n)(k_{t+1} + g_{t+1}) \\ (1+n)g_{t+1} &= (1 + r(k_t))\, g_t - \varphi \end{aligned}$$

- **Tirole (1985):** $d(k_{t-1}, k_t) \equiv 0$**.** A bubble g is present while the government pays and collect nothing.

- **Chalk (2000)**: $d(k_{t-1}, k_t) = d > 0$. The deficit arises from a constant, wasteful expenditure.

$$\begin{aligned} s(w(k_t), r(k_{t+1})) &= (1+n)(k_{t+1} + g_{t+1}) \\ (1+n)g_{t+1} &= (1 + r(k_t))\, g_t + d \end{aligned}$$

- Tirole (1985) focuses on a special case of Diamond with $d \equiv 0$. Chalk (2000) focuses on a special case opposite to Diamond where $d > 0$ and is constant (i.e., negative tax $\varphi < 0$).

- **De la Croix and Michel (2002)** contains a synthesis of the models that existed at the time of their writing.

- **Our Model**: $d(k_{t-1}, k_t) = \frac{\theta}{1+n} w(k_{t-1}) - \tau w(k_t)$. A tax $\tau w(k_t)$ is levied on the young and a benefit $\frac{\theta}{1+n} w(k_{t-1})$ is paid to the old.

$$\begin{aligned} s(w(k_t), r(k_{t+1})) &= (1+n)(k_{t+1} + g_{t+1}) \\ (1+n)g_{t+1} &= (1 + r(k_t))\, g_t + d(k_{t-1}, k_t) \end{aligned}$$

B Proof of Proposition 1

The government budget or debt evolution is

$$-G_{t+1} + \theta_t w_{t-1} L_{t-1} = \tau_t w_t L_t - (1 + r_t)\, G_t$$

which we first rewrite as

$$-\left(1+n\right)g_{t+1}+\theta_t w_{t-1}\frac{1}{1+n}=\tau_t w_t-\left(1+r_t\right)g_t$$

then as

$$\left(1+n\right)g_{t+1}=\left(1+r_t\right)g_t+d_t \quad \text{where} \quad d_t=\theta_t w_{t-1}\frac{1}{1+n}-\tau_t w_t$$

Rearranging leads to

$$g_t=\frac{1+n}{1+r_t}\left(g_{t+1}-\frac{d_t}{1+n}\right)$$

Rolling over from $t=1$ to $s>1$ leads to

$$\begin{aligned}
g_1&=\frac{1+n}{1+r_1}\left(g_2-\frac{d_1}{1+n}\right)\\
&=\frac{1+n}{1+r_1}\frac{1+n}{1+r_2}g_3-\frac{1+n}{1+r_1}\frac{1+n}{1+r_2}\frac{d_2}{1+n}-\frac{1+n}{1+r_1}\frac{d_1}{1+n}\\
&=\frac{1+n}{1+r_1}\frac{1+n}{1+r_2}\frac{1+n}{1+r_3}g_4-\frac{1+n}{1+r_1}\frac{1+n}{1+r_2}\frac{1+n}{1+r_3}\frac{d_3}{1+n}-\frac{1+n}{1+r_1}\frac{1+n}{1+r_2}\frac{d_2}{1+n}-\frac{1+n}{1+r_1}\frac{d_1}{1+n}\\
&\vdots\\
g_1&=\frac{1}{1+n}\sum_{t=1}^{s-1}\frac{-d_t}{\prod_{u=1}^{t}\frac{1+r_u}{1+n}}+\frac{g_s}{\prod_{u=1}^{s-1}\frac{1+r_u}{1+n}} \qquad \forall s>1\\
g_1\prod_{u=1}^{s-1}\frac{1+r_u}{1+n}&-\frac{1}{1+n}\sum_{t=1}^{s-1}\prod_{u=1}^{s-1}\frac{1+r_u}{1+n}\frac{-d_t}{\prod_{u=1}^{t}\frac{1+r_u}{1+n}}=g_s\\
g_s&=g_1\prod_{u=1}^{s-1}\frac{1+r_u}{1+n}+\frac{1}{1+n}\sum_{t=1}^{s-1}\left(\prod_{u=t+1}^{s-1}\frac{1+r_u}{1+n}\right)d_t
\end{aligned}$$

Finally, letting $s\to+\infty$ completes the proof.

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