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SUSTAINABILITY IN A RISKY WORLD

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ABSTRACT

We view sustainability as a requirement that welfare should not be expected to decline over time. We impose this requirement as a prior constraint on the consumption-savings-investment problem, and study its implications for saving, risky investment, and the social discount rate. The constraint does not distort portfolio choice, but it imposes an upper bound on the sustainable time preference rate and on the sustainable consumption-wealth ratio, which we show must lie between the riskless rate and the expected return on optimally invested wealth.

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1 Introduction

Do ethical considerations restrict the rate at which society consumes, or its preference for the present over the future? Economists have answered this question in different ways.

One view is that preferences, social or individual, must be taken as given. If society discounts the future at a high rate, strongly preferring present consumption over future consumption, that preference must be respected; and if it leads to high consumption today, declining over time, that outcome must be accepted.

An alternative view, famously expressed by Ramsey (1928), is that at least for long-term discounting over the lifetimes of multiple generations, society should not discount the future at all because to do so is unethically to privilege the generation alive today over those yet unborn. Recently, this view has found powerful expression in the Stern Review (Stern, 2006), which argues for aggressive action to combat climate change in large part on the basis of a social rate of time preference close to zero.

A third view is that social choices over consumption and saving should be subjected to an external "sustainability" constraint. Sustainability was defined by the World Commission on Environment and Development (1987) as a consumption plan that "meets the needs of the present without compromising the ability of future generations to meet their own needs." Economists including Pezzey (1992), Solow (1993), Howarth (1995), Arrow et al. (2004), Asheim (2007), and Llavador et al. (2015) have formalized this as a requirement that social value—the expected discounted value of utility from the present to the infinite future—should not decline over time. In the words of Solow (1993),

> "A sustainable national economy is one that allows every future generation the option of being as well off as its predecessors. The duty imposed by sustainability is to bequeath to posterity not any particular thing . . . but rather to endow them with whatever it takes to achieve a standard of living at least as good as our own and to look after their next generation similarly."

The concept of sustainability as a constraint, rather than an objective, is consistent with the

moral philosophy of Rawls (1999).[1] It can be understood as a prior principle that an ethical society should impose on itself because it would be agreed to by an individual who does not know into which of a sequence of generations they will be born. Since the time of birth is "morally arbitrary", it should not influence expected utility.

As Arrow et al. (2004) discuss, in a deterministic economy with a single form of capital that has a constant riskless rate of return the sustainability constraint requires that the social rate of time preference does not exceed the exogenous riskless interest rate. When the constraint binds, the constrained rate of time preference equals the riskless interest rate, implying that society consumes the riskless return generated by its wealth and leaves the capital stock unchanged. Wealth, consumption, the utility and marginal utility of current consumption, and social value are then all constant over time. Sustainable consumption is only feasible when the riskless interest rate is positive, and then the constrained social rate of time preference is also positive. The sustainability constraint responds to the availability of an investment opportunity with a positive rate of return by allowing a greater rate of time preference and higher current consumption than would be required by Ramsey.[2]

In this paper we extend the concept of sustainability to allow for risk. In a risky economy, with an uncertain return on capital, it is not possible to guarantee that social value remains constant over time. Instead, we impose a weaker sustainability constraint that social value—expected utility, which is itself a random variable because it is a function of current wealth—should not be *expected* to decline over time. This constraint, which has also been suggested though not formally analyzed by Howarth (1995), acknowledges the reality that social welfare is subject to random shocks, some of which cannot be controlled. In the deterministic case, our constraint reduces to the one considered

[1] Rawls (1999, §6) writes, "In justice as fairness, on the other hand, persons accept in advance a principle of equal liberty and they do this without knowledge of their more particular ends. They implicitly agree, therefore, to conform their conceptions of the good to what the principles of justice require, or at least not to press claims which directly violate them. ... The principles of right, and so of justice, put limits on which satisfactions have value; they impose restrictions on what are reasonable conceptions of one's good. In drawing up plans and deciding on aspirations men are to take these constraints into account."

[2] By adjusting the rate of time preference to the available rate of return, the sustainability constraint responds to a famous critique of Ramsey made by Koopmans (1960, 1967). Koopmans (1967) summarized his argument by writing: "The moral is, in my opinion, that one cannot adopt ethical principles without regard to ... the anticipated technological possibilities. Any proposed optimality criterion needs to be subjected to a mathematical screening, to determine whether it does indeed bear on the problem at hand, under the circumstances assumed. More specifically, too much weight given to generations far in the future turns out to be self-defeating. It does nobody any good. How much weight is too much has to be determined in each case."

by Arrow et al. (2004).

We study a continuous-time model with two forms of capital, one safe and one risky, so that society faces a portfolio choice problem as well as a consumption-savings decision. (We also consider a special case where the safe asset is in zero net supply, so the risk premium on the risky asset must adjust to ensure a risky portfolio share equal to one.) We assume that the assets have iid returns, and we allow the risky return to be driven both by a Brownian motion and by a compound Poisson jump process.[3] We assume that society has a standard time-separable power utility function defined over aggregate consumption, and we impose the sustainability constraint on this.

Our main results are as follows. First, our sustainability constraint does not distort portfolio choice, which is always the same whether or not the constraint binds. In the absence of jumps, the portfolio rule is the classic one derived by Merton (1969, 1971).

Second, when the sustainability constraint binds, the sustainable consumption-wealth ratio is independent of the unconstrained time preference rate.[4]

Third, the sustainable consumption-wealth ratio exceeds the riskless interest rate. In the absence of jumps, the sustainable consumption-wealth ratio is the riskless interest rate plus one-half the squared Sharpe ratio on the risky asset divided by risk aversion. In the presence of jumps, the solution is defined implicitly but is straightforward to calculate numerically.

Fourth, the constraint binds whenever the unconstrained social rate of time preference exceeds the sustainable consumption-wealth ratio. The constrained solution can be implemented by reducing the rate of time preference to equal the sustainable consumption-wealth ratio.

Fifth, the sustainable consumption-wealth ratio—equivalently, the sustainable rate of time preference—lies between the riskless interest rate and the expected return on optimally invested wealth. In the absence of jumps, it lies exactly at the midpoint between these two rates of return.

[3]Our assumption of iid returns is consistent with an endogenous growth or Ak model of capital accumulation with constant returns. It implies that there is a unique consumption-wealth ratio at which the sustainability constraint is binding. In a model with diminishing returns to capital, by contrast, any constant savings rate can be sustainable although different savings rates imply different levels of steady-state consumption.

[4]This is consistent with the view of Rawls (1999, §45), who writes: "Of course, a present or near future advantage may be counted more heavily on account of its greater certainty or probability ... But none of these things justifies our preferring a lesser present to a greater future good simply because of its nearer temporal position ... The just savings principle for society must not, then, be affected by pure time preference, since as before the different temporal position of persons and generations does not in itself justify treating them differently."

More generally, we characterize the position of the sustainable consumption-wealth ratio within these bounds using methods derived from Martin (2013).

The risky model we consider is different in two fundamental ways from the riskless model considered by the previous literature. First, in the presence of risk a zero drift for social value does not imply zero drifts for consumption, wealth, or the marginal utility of consumption. We show that consumption, wealth, and marginal utility all have positive drifts in the constrained equilibrium. Intuitively, this is because risky investment causes the distribution of consumption and wealth to spread out over time, imposing more risk on later generations. To prevent risk from reducing the welfare of later generations relative to earlier ones, later generations must be compensated by higher average levels of consumption and wealth. This implies that our sustainability constraint is different from the zero drifts in consumption and wealth, or log consumption and log wealth, imposed by Campbell and Sigalov (2021). The Campbell and Sigalov constraints generally distort portfolio choice, whereas our sustainability constraint does not.

Second, the sustainable rate of time preference is not the same as the sustainable discount rate that society should apply to riskless investment projects. That discount rate is given by the riskless interest rate in the sustainable equilibrium, which is lower than the sustainable rate of time preference when the economy is exposed to risk. As a salient example, investments to mitigate climate change can rationally be discounted at low rates if the investments are riskless and the sustainable equilibrium has a low riskless interest rate. They should be discounted at even lower rates if climate investments pay off in bad states of the world—that is, if they are analogous to insurance policies—an important point emphasized by Weitzman (2009) and Gollier (2021).

Our main analysis defines utility over aggregate consumption, in effect treating each generation equally regardless of population. This is only equivalent to treating each individual equally if population is constant over time. Population growth creates notoriously difficult issues for intertemporal ethics (Parfit 1984, Dasgupta 2001), particularly when population is itself a choice variable. However we show that if population growth is exogenous and constant, then we can modify the sustainability constraint to prevent the expected utility of an individual from declining over time. This is equivalent to subtracting the rate of population growth from all rates of return, and therefore from the sustainable consumption-wealth ratio and the sustainable rate of time preference.

The literature on discounting and sustainability is enormous, and we do not attempt a complete review here. Dasgupta (2008, 2021) and Zeckhauser and Viscusi (2008) provide recent surveys.

Within the literature on climate change, there has been debate between those such as Cline (1992) and Stern (2006, 2016) who argue for a very low social rate of time preference, and Nordhaus (1994) who uses a higher rate of time preference. Our analysis implies that a substantial rate of time preference can be consistent with the ethical criterion of sustainability in a risky world.

The organization of the paper is as follows. Section 2 sets up our unconstrained continuous-time model with portfolio choice over a safe and a risky asset. Section 3 introduces the sustainability constraint and solves the constrained model. Section 4 imposes the equilibrium condition that the risky asset share equals one. Section 5 extends our concept of sustainability to allow for a constant rate of population growth. Section 6 concludes. An appendix presents details of key derivations.

2 Unconstrained Consumption and Portfolio Choice

We consider a representative investor faced with two assets, one riskless and one risky. The investor chooses society's aggregate consumption, C_t, and risky portfolio share, α, to maximize the expected discounted value of a power utility function,

$$U_0 = \mathrm{E} \int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} \, dt. \tag{1}$$

We take as given this representation of utility derived from aggregate consumption. It can be derived from individual utility of individual consumption under assumptions that permit aggregation across consumers. For example, we could assume that individuals have a constant probability of death following Blanchard (1985), that they have power utility defined over their own consumption, that they are unable to annuitize their wealth, and that the wealth of those who die is allocated to an equal number of newly born individuals. In this case utility at each point of time is both the welfare of the generation born at that moment and the welfare of all agents alive at that time. This microfoundation for equation (1) assumes a constant population; we consider a case with population growth in section 5.

We assume that the rate of time preference $\rho > 0$. If individuals have a constant probability of death and do not care about their descendants, then as Blanchard (1985) shows, ρ is the sum of the pure individual rate of time preference and the probability of death. In a more general model

with intergenerational altruism, ρ will also be affected by (and declining in) the degree of altruism.

We assume that the coefficient of relative risk aversion $\gamma > 1$. In the appendix we show that all the results extend in the expected way to the log utility case, $\gamma = 1$. It would also be easy to handle the case $0 < \gamma < 1$, but as this case requires occasional sign flips in our logic below, we rule it out to streamline the exposition.

The riskless asset has gross return R_f. It will generally be convenient to think in terms of the log riskless rate, $r_f = \log R_f$. The risky asset has expected excess return $\mu = \log(\mathrm{E}R/R_f) > 0$, Brownian volatility σ, and is exposed to jumps arriving according to a Poisson counting process N_t with constant arrival rate ω; we assume that μ, σ, and ω are each constant. We write W_t for wealth at time t and $\theta = C_t/W_t$ for the consumption-wealth ratio. Under our assumptions, θ is also constant. Thus

$$\frac{dC}{C} = \frac{dW}{W} = \left[r_f + \alpha \underbrace{(\mu + \omega \mathrm{E}L)}_{\widehat{\mu}} - \theta\right] dt + \alpha\sigma\, dZ - \alpha L\, dN, \tag{2}$$

where we suppress time subscripts on random variables for simplicity.

Jumps are captured by the third term on the right-hand side of equation (2). When a jump occurs, an agent who is fully invested in the risky asset loses a fraction L of her capital. We assume that L is a random variable that is drawn in iid fashion each time a jump occurs. We also assume (with one eye on an equilibrium in which $\alpha = 1$) that L is strictly less than 1, so that someone who invests fully in the risky asset is not bankrupted. We can allow L to take negative values; these represent good news for the risky asset. We write $\widehat{\mu} = \mu + \omega \mathrm{E}L$ to denote the expected excess return in the absence of jumps.

It follows from (2) that

$$d\log C = \left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 - \theta\right) dt + \alpha\sigma\, dZ + \log\left(1 - \alpha L\right) dN. \tag{3}$$

This equation is derived in the Appendix.

Integrating forwards, exponentiating, using $C_0 = \theta W_0$, and raising to the power $1 - \gamma$, we have

$$C_t^{1-\gamma} = W_0^{1-\gamma}\theta^{1-\gamma} \exp\left\{(1-\gamma)\left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 - \theta\right) t + \alpha(1-\gamma)\sigma Z_t\right\} \prod_{i=1}^{N_t} (1 - \alpha L_i)^{1-\gamma}. \tag{4}$$

Writing L for a representative of the iid L_i, the Appendix shows that

$$\mathrm{E}C_t^{1-\gamma} = W_0^{1-\gamma}\theta^{1-\gamma}\exp\left\{(1-\gamma)\left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2 - \theta\right)t + \omega\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right]t\right\}. \tag{5}$$

Hence the objective function can be evaluated explicitly, as

$$U_0 = \frac{W_0^{1-\gamma}}{1-\gamma}\frac{\theta^{1-\gamma}}{\rho - (1-\gamma)\left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2 - \theta\right) - \omega\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right]}. \tag{6}$$

The optimal investment and consumption choices are identified by maximizing (6) with respect to α and θ. The optimal consumption-wealth ratio is

$$\theta_{\text{unc}} = \frac{\rho + (\gamma-1)\left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2\right) - \omega\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right]}{\gamma}. \tag{7}$$

We assume that θ_{unc} is positive. This implies that the denominator of (6) is positive when α and θ are chosen optimally, and hence that the integral in the definition of expected utility converges.

The optimal risky portfolio share is defined implicitly by

$$\widehat{\mu} - \alpha\gamma\sigma^2 = \omega\mathrm{E}\left[L\left(1-\alpha L\right)^{-\gamma}\right]. \tag{8}$$

In the absence of jumps, where the risky asset follows a pure Brownian motion, this simplifies to the classic Merton formula,

$$\alpha_{\text{BM}} = \frac{\mu}{\gamma\sigma^2}. \tag{9}$$

Using (8) to eliminate $\widehat{\mu}$ in (7), we can also write

$$\theta_{\text{unc}} = \frac{\rho + (\gamma-1)\left(r_f + \frac{1}{2}\gamma\alpha^2\sigma^2\right) - \omega\mathrm{E}\left[(1-\alpha\gamma L)\left(1-\alpha L\right)^{-\gamma} - 1\right]}{\gamma}. \tag{10}$$

3 A Sustainability Constraint

We now introduce a *sustainability constraint* that the representative agent may choose to impose on herself. The constraint requires that expected utility should not be allowed to decline, in expectation, over time. (Expected utility is itself a random variable, because it is a function of current wealth.) If the representative agent is thought of as the currently living generation in an infinite dynasty, then the constraint is appropriate if she does not want her descendants to expect a lower quality of life than she does.

One might imagine imposing other types of constraint. We could, for example, allow for a type of risk aversion over future expected utility by requiring that some concave function of future expected utility should have non-decreasing expectation. This is analytically intractable in the constant relative risk aversion (power) case, however; and indeed it is infeasible in the limit as risk aversion over future expected utility approaches infinity, as it would require expected utility—and hence wealth itself—to be non-decreasing, which is not possible unless society can entirely eliminate risk.

Equation (6) shows that expected utility at time t, U_t, is proportional to $W_t^{1-\gamma}/(1-\gamma)$. We multiply by $(1-\gamma)$, which is negative under our maintained assumption that $\gamma > 1$, and work with a rescaled variable $X_t = W_t^{1-\gamma}$. This follows the process

$$\frac{dX}{X} = (1-\gamma)\left(r_f + \alpha\widehat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2\right)dt + (1-\gamma)\alpha\sigma dZ + \left[(1-\alpha L)^{1-\gamma} - 1\right]dN\,. \tag{11}$$

The derivation of equation (11) is given in the Appendix.

The sustainability constraint dictates that the drift of X should be nonpositive, and hence that

$$\theta \leq r_f + \alpha\widehat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2 + \omega\frac{\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right]}{1-\gamma}, \tag{12}$$

recalling that $\mathrm{E}dN = \omega\, dt$.

Let us write θ_{con} for the case in which the constraint binds,

$$\theta_{\mathrm{con}} = r_f + \alpha\widehat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2 + \omega\frac{\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right]}{1-\gamma}, \tag{13}$$

and θ_{unc} for the case in which the constraint does not bind, so that the optimal choice is given by equation (10). The optimal consumption-wealth ratio is independent of ρ if the constraint binds; this is not true of the unconstrained case.

If the constraint binds, then we can use it to eliminate θ from the objective function (6), giving

$$U_{\text{con},0} = \frac{W_0^{1-\gamma}}{1-\gamma} \frac{\left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2 + \omega \frac{\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right]}{1-\gamma} \right)^{1-\gamma}}{\rho}. \tag{14}$$

Conveniently, the optimal investment choice is unaffected by the presence of the sustainability constraint. Maximizing equation (14) with respect to α, we find the same first-order condition as before, equation (8). (As a corollary, the sustainable strategy is Pareto-efficient because it is identical to the unconstrained-optimal strategy for some choice of ρ.)

Intuitively, it is not optimal to distort portfolio choice in the presence of a sustainability constraint because doing so affects expected utility in the same way in all periods. Distorting the portfolio choice decision away from the unconstrained optimum therefore does not relax the constraint; nor (by definition) does it directly benefit the objective function. By contrast, in papers such as Dybvig (1995) or Campbell and Sigalov (2021) that feature constraints on consumption as opposed to welfare, it may be optimal to distort portfolio choice relative to the unconstrained optimum in order to relax the constraint.

We can use the condition (8) to eliminate $\widehat{\mu}$ from equation (13), giving

$$\theta_{\text{con}} = r_f + \frac{1}{2}\gamma\alpha^2\sigma^2 + \omega \frac{\mathrm{E}\left[(1-\alpha\gamma L)\,(1-\alpha L)^{-\gamma} - 1\right]}{1-\gamma}. \tag{15}$$

When risk and the risky portfolio share are positive, the second two terms on the right hand side of equation (15) are positive. Hence (15) shows that the constrained consumption-wealth ratio θ_{con} exceeds the riskfree interest rate r_f in a risky economy. Conversely, it follows from conditions (8) and (15) that θ_{con} is less than the expected return on optimally invested wealth, $r_f + \alpha\mu$.

3.1 Comparing the constrained and unconstrained solutions

The optimal sustainable consumption-wealth ratio, θ_{sus}, is given by whichever of θ_{con} and θ_{unc} is smaller. If the unconstrained case features a lower consumption-wealth ratio, then it certainly satisfies the constraint and delivers higher utility. If not, the unconstrained case does not satisfy the constraint, so that θ_{con} is the best we can do. Thus

$$\theta_{\text{sus}} = \min\left\{\theta_{\text{unc}}, \theta_{\text{con}}\right\}. \tag{16}$$

Equivalently, θ_{con} is the highest possible sustainable consumption-wealth ratio.

It follows from equations (10) and (15) that

$$\theta_{\text{unc}} = \frac{1}{\gamma}\rho + \left(1 - \frac{1}{\gamma}\right)\theta_{\text{con}}. \tag{17}$$

Equations (16) and (17) have several interesting implications. First, the sustainability constraint binds if and only if $\rho > \theta_{\text{con}}$, or equivalently if and only if $\rho > \theta_{\text{unc}}$. Related to this, we can show that in the absence of a sustainability constraint,

$$\mathrm{E}_0 X_t = X_0 e^{(\rho - \theta_{\text{unc}})t}. \tag{18}$$

If impatience is sufficiently high that $\rho > \theta_{\text{unc}}$, then $X_t = W_t^{1-\gamma}$ is expected to grow without limit in an unconstrained equilibrium and expected utility is expected to decline without limit.[5] The sustainability constraint, which rules out declining expected utility, binds in this circumstance.

Second, equation (17) shows that the moderating influence of ρ makes θ_{unc} less sensitive than θ_{con} to changes in other parameters of the model, holding ρ fixed.

Third, equation (17) implies that the behavior of an extremely risk-averse individual is little affected by the presence or absence of a sustainability constraint, as $\theta_{\text{unc}} \approx \theta_{\text{con}}$ if γ is large. This reflects the fact that highly concave utility leads an agent to choose a flat consumption path that is close to sustainable, regardless of the level of ρ.

[5] Expected wealth will decline toward zero if ρ is sufficiently large; but if ρ is sufficiently close to θ_{con} then wealth has positive drift despite the negative drift in expected utility.

Fourth, equations (16) and (17) show that θ_{sus} and θ_{unc} can easily be calculated from knowledge of θ_{con}, so we can focus our analysis on the determinants of θ_{con}.

Finally, we note that a planner could implement the sustainable optimum by using a modified "social discount rate" $\widehat{\rho}$, chosen to satisfy

$$\theta_{\text{unc}}\left(\widehat{\rho}\right) = \theta_{\text{sus}} = \min\left\{\theta_{\text{unc}}\left(\rho\right), \theta_{\text{con}}\right\}. \tag{19}$$

(The first equality defines the social discount rate. The second is equation (16). The notation emphasizes the dependence of θ_{unc} on the discount rate.) If the sustainability constraint is not binding, $\theta_{\text{unc}}\left(\rho\right) \leq \theta_{\text{con}}$, then we can set $\widehat{\rho} = \rho$. If the constraint does bind, then we must ensure that $\theta_{\text{unc}}\left(\widehat{\rho}\right) = \theta_{\text{con}}$. By equation (17), this requires that $\widehat{\rho} = \theta_{\text{con}}$. As the constraint binds precisely when $\rho > \theta_{\text{con}}$, we can summarize all this by saying that we should set

$$\widehat{\rho} = \min\left\{\rho, \theta_{\text{con}}\right\}. \tag{20}$$

3.2 Sustainable drifts in wealth and marginal utility

A binding sustainability constraint implies zero drift in expected utility, but this does not imply a zero drift in wealth. Instead wealth drifts upwards over time under our assumption that $\gamma > 1$, because

$$\frac{dW}{W} = \left\{\frac{1}{2}\gamma\alpha^2\sigma^2 - \frac{\omega}{1-\gamma}\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right]\right\} dt + \alpha\sigma\, dZ - \alpha L\, dN, \tag{21}$$

so the drift is

$$\mathrm{E}\frac{dW}{W} = \left(\frac{1}{2}\gamma\alpha^2\sigma^2 + \frac{\omega}{\gamma-1}\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1 + \alpha L(1-\gamma)\right]\right) dt, \tag{22}$$

and both terms in the brackets are positive.

We can also show the stronger result that log wealth drifts upwards over time under a binding sustainability constraint. We have

$$\mathrm{E}\, d\log W = \left(\frac{1}{2}(\gamma-1)\alpha^2\sigma^2 + \frac{\omega}{\gamma-1}\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1 - (1-\gamma)\log(1-\alpha L)\right]\right) dt\,, \tag{23}$$

and again both terms in the brackets are positive.

These facts illustrate the distinction between our sustainability constraint and the arithmetic and geometric constraints considered by Campbell and Sigalov (2021), which impose zero drift in wealth or in log wealth, respectively. Their constraints distort portfolio choice, unlike the sustainability constraint we consider.

A binding sustainability constraint also does not imply zero drift in marginal utility. The process for marginal utility, $M = W^{-\gamma}$, is

$$\frac{dM}{M} = \left\{ \frac{1}{2}\gamma\alpha^2\sigma^2 + \frac{\omega\gamma}{1-\gamma}\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right] \right\} dt - \gamma\alpha\sigma\, dZ + \left[(1-\alpha L)^{-\gamma} - 1\right]\, dN\,, \tag{24}$$

so the drift is

$$\mathrm{E}\frac{dM}{M} = \left(\frac{1}{2}\gamma\alpha^2\sigma^2 + \frac{\omega}{1-\gamma}\left\{ \gamma\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right] + (1-\gamma)\mathrm{E}\left[(1-\alpha L)^{-\gamma} - 1\right] \right\} \right) dt, \tag{25}$$

which is positive when $\gamma > 1$. The drift in marginal utility equals the drift in wealth in the Brownian case where $L = 0$, but differs from it in the general case with jumps.

The positive drift in marginal utility in the constrained economy is another way to understand the result that the constrained consumption-wealth ratio exceeds the riskfree interest rate. The first-order condition for optimal investment in a riskless asset implies that discounted marginal utility drifts downward at the riskfree interest rate, and hence that the drift in undiscounted marginal utility is the constrained social rate of time preference (equivalently, the constrained consumption-wealth ratio) less the riskfree interest rate. That is, we have

$$\mathrm{E}\frac{dM}{M} = \theta_{\mathrm{con}} - r_f. \tag{26}$$

Our solutions have this property, as can be verified by comparing the right hand sides of equations (15) and (25).

With power utility, the driftless variable $X = W^{1-\gamma}$ is the product of marginal utility and wealth: $X = MW$. At first sight it might seem surprising that X has no drift while both M and W have positive drift. But we must also take into account comovement in M and W, whose effect

is visible in the product rule:

$$\frac{dX}{X} = \frac{dM}{M} + \frac{dW}{W} + \frac{dM}{M}\frac{dW}{W}. \tag{27}$$

The product is

$$\frac{dM}{M}\frac{dW}{W} = -\gamma\alpha^2\sigma^2\,dt - \alpha L\left[(1-\alpha L)^{-\gamma} - 1\right]\,dN\,, \tag{28}$$

and the fact that this quantity is negative makes it possible for $X = MW$ to be driftless even though M and W each have positive drift:

$$\mathrm{E}\frac{dX}{X} = \mathrm{E}\frac{dM}{M} + \mathrm{E}\frac{dW}{W} + \mathrm{E}\left(\frac{dM}{M}\frac{dW}{W}\right) = 0\,. \tag{29}$$

We conclude by noting one other implication of the upwards drift in log wealth shown in equation (23). Although the sustainability constraint requires that the average future value of expected utility U_t is the same as expected utility today, as one looks into the far-distant future expected utility is overwhelmingly likely to be higher than its current value. This counterintuitive fact, which echoes the result of Martin (2012), follows from the fact that $\log W_t$ has positive drift, so that expected utility at time t approaches its upper bound of zero almost surely as t approaches infinity. The invariance of expected utility to the horizon is achieved by a vanishingly small number of paths in which expected utility in the future is arbitrarily low.

3.3 Pure Brownian motion and pure jumps

There are some cases in which we can characterize the optimal investment choice explicitly. We can, for example, contrast the pure Brownian case, $\omega = 0$, with the pure jump case in which $\sigma = 0$ and $L \in (0, 1)$ is deterministic. In the pure Brownian case, the risky share satisfies the Merton formula (9). Substituting into equation (15) we have

$$\theta_{\mathrm{con,BM}} = r_f + \frac{1}{2}\frac{\mu^2}{\gamma\sigma^2}. \tag{30}$$

In this case the constrained consumption-wealth ratio is the riskless interest rate plus one half the squared Sharpe ratio of the risky asset, divided by risk aversion. This can be much larger than the riskless interest rate: for example, if the riskless rate is 2%, the Sharpe ratio of the risky asset is

0.4, and risk aversion is 2, then the constrained consumption-wealth ratio is 6%. The constrained consumption-wealth ratio only approaches the riskless rate, its value in a deterministic model, as risk aversion becomes extremely high.

In the pure jump case, we can solve equation (8) to give

$$\alpha_{\text{jump}} = \frac{1}{L}\left[1 - \left(\frac{\omega L}{\mu + \omega L}\right)^{1/\gamma}\right]. \tag{31}$$

This solution has αL invariant to L if μ/L is invariant to L. In other words, the portfolio exposure to jumps does not alter with jump size provided that the risk premium per unit of jump risk is constant. This is the jump equivalent of the well known property under Brownian motion that $\alpha\sigma$ is invariant to σ if the Sharpe ratio μ/σ is invariant to σ.

The pure jump model has two free parameters, ω and L, to compare with the single parameter σ in the Brownian case. To put them on the same footing, we choose ω and L so that variance is the same in each case, i.e., we set $\omega = \sigma^2/L^2$. We can imagine fixing σ and then choosing the parameter L freely. High values of L correspond to rare extreme disasters—ω moves in the opposite direction to L—whereas values of L close to zero correspond to frequent small jumps. Equation (31) then becomes

$$\alpha_{\text{jump}} = \frac{1}{L}\left[1 - \left(\frac{\sigma^2}{\mu L + \sigma^2}\right)^{1/\gamma}\right], \tag{32}$$

and the Appendix shows that

$$\alpha_{\text{jump}} < \alpha_{\text{BM}} \text{ if } \gamma \geq 1. \tag{33}$$

When jumps are larger and rarer—with mean and variance held constant—the Appendix shows that the investor trades less aggressively, that is,

$$\frac{\partial \alpha_{\text{jump}}}{\partial L} < 0. \tag{34}$$

Conversely, in the limit in which jumps are very small and very frequent, we recover the same allocation as in the Brownian motion case: by l'Hôpital's rule, $\lim_{L\to 0} \alpha_{\text{jump}} = \frac{\mu}{\gamma\sigma^2}$.

Substituting the expression for α given in equation (32) into the expression for θ_{con} given in

Table 1: Numerical examples in the Brownian case

Baseline calibration sets $r_f = 1\%, \mu = 8\%, \sigma = 20\%$.

γ	θ_{con}	α	$\mathrm{E}\,\frac{dW}{W}$	$\mathrm{E}\,d\log W$	$\mathrm{E}\,\frac{dW^{-\gamma}}{W^{-\gamma}}$
1	0.09	2	0.08	0	0.08
2	0.05	1	0.04	0.02	0.04
5	0.026	0.4	0.016	0.0128	0.016
10	0.018	0.2	0.008	0.0072	0.008

equation (15), we have

$$\theta_{\text{con}} = r_f + \frac{\sigma^2}{L^2}\frac{(1-\gamma)\,(K^{-\gamma}-1)+\gamma\,(K^{1-\gamma}-1)}{1-\gamma} \quad \text{where} \quad K = \left(\frac{\sigma^2}{\mu L+\sigma^2}\right)^{1/\gamma}. \tag{35}$$

This tends to the corresponding expression in the pure Brownian case (30) as $L \to 0$.

3.4 Numerical examples

In this section we present some numerical examples to illustrate the properties we have discussed. Table 1 reports numerical results for a Brownian model without jumps, in which the riskless interest rate r_f equals 1%, the risk premium μ equals 8%, and the standard deviation of the risky asset σ equals 20%, implying a Sharpe ratio of 0.4. The four rows of the table consider risk aversion coefficients γ of 1, 2, 5, and 10. The columns report the constrained consumption-wealth ratio θ_{con}, the risky portfolio share α , and the corresponding drifts in wealth, log wealth, and marginal utility.

In a Brownian model, the risky portfolio share is inversely proportional to risk aversion. Given our assumed parameters an investor with log utility ($\gamma = 1$) sets $\alpha = 2$, levering the risky asset two for one. The constrained consumption-wealth ratio is 9% and the corresponding expected growth rate of wealth is 8%. However, the expected growth rate of log wealth is zero since this is the growth rate of utility for a log investor so the sustainability constraint sets it to zero. The expected growth rate of marginal utility is the difference between the constrained consumption-wealth ratio and the riskfree rate, or 8%, and in the Brownian model it also equals the expected growth rate of wealth, as we noted earlier.

Table 2: Numerical examples with jumps

Baseline calibration sets $r_f = 2\%, \mu = 4\%, \sigma = 10\%, \omega = 4\%, L = 0.4, \gamma = 2$.

	γ	r_f	μ	σ	ω	L	θ_{con}	α	$\mathrm{E}\,\frac{dW}{W}$	$\mathrm{E}\,d\log W$	$\mathrm{E}\,\frac{dW^{-\gamma}}{W^{-\gamma}}$
Baseline	2	0.02	0.04	0.10	0.02	0.40	0.045	1.11	0.019	0.010	0.025
High γ	5						0.031	0.49	0.009	0.007	0.011
Low γ	1						0.065	1.83	0.028	0	0.045
High r_f		0.04					0.065	1.11	0.019	0.010	0.025
Low r_f		0					0.025	1.11	0.019	0.010	0.025
High μ			0.08				0.10	1.56	0.045	0.026	0.080
Low μ			0.02				0.027	0.66	0.006	0.003	0.007
High σ				0.15			0.035	0.72	0.014	0.007	0.015
Low σ				0.06			0.055	1.36	0.019	0.011	0.035
High ω					0.04		0.040	0.88	0.016	0.008	0.020
Low ω					0		0.060	2.00	0.040	0.020	0.040
High L						0.60	0.037	0.72	0.012	0.007	0.017
Low L						0.20	0.056	1.73	0.034	0.017	0.036
Negative L						-0.40	0.054	1.74	0.036	0.017	0.034

As risk aversion increases, the constrained consumption-wealth ratio and the risky portfolio share both decline. For example, when $\gamma = 2$ the constrained consumption-wealth ratio is 5% and the risky portfolio share $\alpha = 1$. However, the constrained consumption-wealth ratio declines slowly and is still 1.8%, almost twice the riskless interest rate, when $\gamma = 10$. The expected growth rates of wealth and marginal utility also decline with risk aversion, but the expected growth rate of log wealth is hump-shaped in risk aversion, first increasing and then ultimately declining towards zero.

Table 2 reports numerical results for a more general model allowing for jumps. Here, in the benchmark case the riskless interest rate r_f equals 2%, the risk premium μ equals 4%, and the Brownian standard deviation of the risky asset σ equals 10%. In addition a jump of size $L = 40\%$ occurs with a probability ω of 4% per period. The benchmark case sets $\gamma = 2$, in which case the optimal constrained consumption-wealth ratio is 4.5% and the risky portfolio share $\alpha = 1.11$. These values are fairly close to those in the Brownian case reported in Table 1, indicating that the calibration with jumps is broadly comparable in its overall level of risk.

The remaining rows of the table consider variations of the benchmark model with higher

and lower risk aversion, then higher and lower riskfree rates, risk premium, Brownian volatility, jump probability, and jump size. The results are intuitive and in all cases imply a sustainable consumption-wealth ratio well above the riskfree interest rate, or equivalently a substantial positive expected growth rate of marginal utility.

4 Sustainability Without a Riskless Asset

We might alternatively deal with α by requiring that it should equal one in equilibrium. In this case we are implicitly normalizing the risky asset return to equal the return on society's invested wealth.

It will be convenient to write $L = 1 - e^{-J}$ where J is a random variable that can take positive or negative values, and whose moment-generating function we write, following Martin (2013), as $\boldsymbol{m}(x) = \mathrm{E}e^{Jx}$. (Large positive J is very bad news, large negative J is very good news.) Setting $\alpha = 1$ in equation (8),

$$\mu = \gamma\sigma^2 + \omega\left\{\boldsymbol{m}(\gamma) - \boldsymbol{m}(\gamma-1) - [\boldsymbol{m}(0) - \boldsymbol{m}(-1)]\right\}. \tag{36}$$

Setting $\alpha = 1$ in (15),

$$\theta_{\mathrm{con}} = r_f + \frac{1}{2}\gamma\sigma^2 + \omega\left[\boldsymbol{m}(\gamma) - \boldsymbol{m}(\gamma-1) - \frac{\boldsymbol{m}(\gamma-1) - \boldsymbol{m}(0)}{\gamma-1}\right]. \tag{37}$$

The terms in square brackets on the right hand side of equation (37) are positive because $\boldsymbol{m}(x)$ is convex. This provides a lower bound on the maximum sustainable consumption-wealth ratio, θ_{con}, that is greater than the riskless rate.

To derive a bound in the opposite direction, we can use equation (36) to rewrite equation (37) as

$$\theta_{\mathrm{con}} = \mu + r_f - \frac{1}{2}\gamma\sigma^2 - \omega\left[\frac{\boldsymbol{m}(\gamma-1) - \boldsymbol{m}(0)}{\gamma-1} - \frac{\boldsymbol{m}(0) - \boldsymbol{m}(-1)}{1}\right]. \tag{38}$$

The terms in square brackets on the right hand side of equation (38) are also positive (by the convexity of $\boldsymbol{m}(x)$ once again), so equation (38) supplies an upper bound on the maximum sustainable

consumption-wealth ratio that is lower than the return on the risky asset. To sum up, we have

$$r_f + \frac{1}{2}\gamma\sigma^2 \leq \theta_{\text{con}} \leq \mu + r_f - \frac{1}{2}\gamma\sigma^2. \tag{39}$$

Moreover, it is possible to choose the distribution of J so that the terms in square brackets in (38) are close to zero while the terms in square brackets in (37) are not; then the upper bound is tight. This happens if the MGF of J is roughly flat between -1 and $\gamma - 1$ and steeply upward-sloping from $\gamma - 1$ to γ, as can happen in the frightening case with occasional *bad news* jumps.[6] Conversely, it is possible to arrange for the terms in square brackets in (37) to be close to zero, while those in (38) are not; then the lower bound binds. This happens if the MGF of J is steeply downward-sloping between -1 and 0 and roughly flat on the range $[0, \gamma]$, as can happen if there are occasional *good news* jumps. Meanwhile in the pure Brownian case, $\omega = 0$ or $J = 0$, the sustainable consumption-wealth ratio is the equally weighted average of r_f and $r_f + \mu$:

$$\theta_{\text{BM,con}} = r_f + \frac{1}{2}\gamma\sigma^2. \tag{40}$$

4.1 Numerical examples without a riskless asset

We illustrate these properties numerically in two different ways. Figure 1 shows the constrained consumption-wealth ratio and the upper and lower bounds given in equation (39) for a model with risk aversion $\gamma = 2$, an expected return on the risky asset $r_f + \mu = 6\%$, Brownian volatility $\sigma = 10\%$, a jump probability $\omega = 2\%$, and jumps of deterministic size L. The horizontal axis shows different values for L, where positive values correspond to negative jumps (losses) in wealth, and negative values correspond to positive jumps in wealth. In the left panel, the constrained consumption-wealth ratio θ_{con} is plotted along with the expected risky asset return $r_f + \mu$ (constant at 6%) and the riskfree interest rate r_f. The constrained consumption-wealth ratio is halfway between the two returns in the Brownian case; it is closer to the risky asset return in the bad-jump region where $L > 0$, and closer to the riskfree interest rate in the good-jump region where $L < 0$. In the right panel, the constrained consumption-wealth ratio is plotted along with the upper and lower bounds

[6] For any given $\gamma > 1$, we can can make θ_{con} as close as we like to either of the bounds by appropriately choosing some very small ω and a fixed jump size J. To make the lower bound tight, we will want to choose $J < 0$; to make the upper bound tight, we will want to choose $J > 0$.

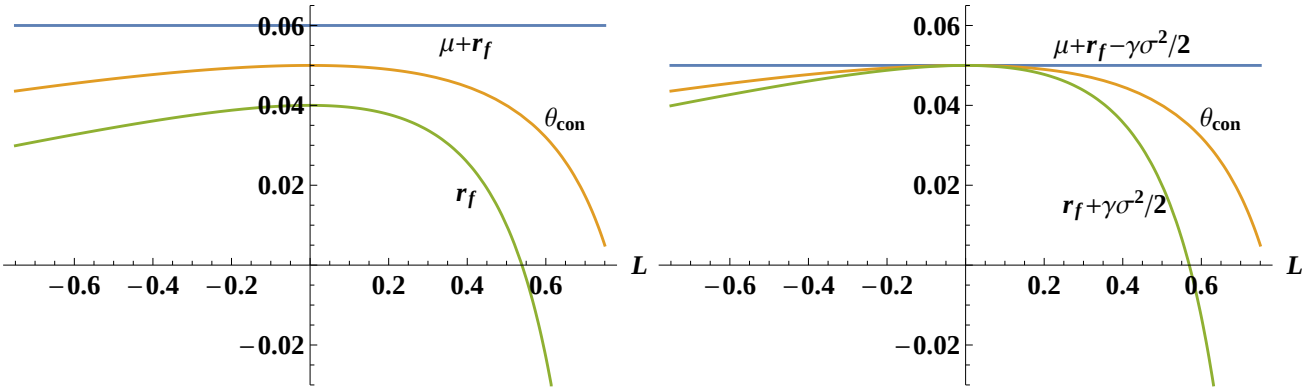

Figure 1: θ_{con} and the upper and lower bounds for various deterministic jump sizes L, with $\gamma = 2$, $\sigma = 0.1$, $\omega = 0.02$, $\mu + r_f = 0.06$. Jumps are bad news if L is positive and good news if L is negative.

from equation (39). The bounds are tight in the Brownian case, and widen out as the absolute jump size increases.

Table 3 reports numerical results for variations of this model. Here, in the benchmark case the jump size $L = 0.4$. The constrained consumption-wealth ratio equals 4.5% and the riskfree interest rate is only 2.6%. The remaining rows of the table consider variations of the benchmark model with higher and lower risk aversion, then higher and lower risky asset returns, Brownian volatility, jump probability, and jump size. The results are intuitive and in all cases imply a sustainable consumption-wealth ratio well above the riskfree interest rate, or equivalently a substantial positive expected growth rate of marginal utility.

5 Sustainability With Population Growth

The analysis so far has imposed sustainability on a social welfare function defined over aggregate consumption. This is equivalent to sustainability of individual utility only if the population is constant. In this section we modify our analysis to make individual utility sustainable given exogenous, constant population growth at rate g.

If there is population growth, then wealth at time t is shared between more people. Normalizing

Table 3: Numerical examples without a riskless asset

Baseline calibration sets $\mu + r_f = 6\%, \sigma = 10\%, \omega = 2\%, L = 0.4, \gamma = 2$ in equilibrium with $\alpha = 1$.

	γ	$\mu + r_f$	σ	ω	L	θ_{con}	r_f	μ	$\mathrm{E}\,\frac{dW}{W}$	$\mathrm{E}\,d\log W$	$\mathrm{E}\,\frac{dW^{-\gamma}}{W^{-\gamma}}$
Baseline	2	0.06	0.10	0.02	0.40	0.045	0.026	0.034	0.015	0.008	0.019
High γ	5					0.009	−0.085	0.145	0.051	0.043	0.094
Low γ	1					0.053	0.045	0.015	0.007	0	0.008
High $\mu + r_f$		0.10				0.085	0.066	0.034	0.015	0.008	0.019
Low $\mu + r_f$		0.02				0.005	−0.014	0.034	0.015	0.008	0.019
High σ			0.15			0.032	0.001	0.059	0.028	0.014	0.031
Low σ			0.06			0.051	0.039	0.021	0.009	0.005	0.012
High ω				0.04		0.039	0.012	0.048	0.021	0.011	0.028
Low ω				0		0.050	0.040	0.020	0.010	0.005	0.010
High L					0.50	0.040	0.010	0.050	0.020	0.011	0.030
Low L					0.30	0.047	0.034	0.026	0.013	0.006	0.014
Negative L					−0.40	0.048	0.036	0.024	0.012	0.006	0.012

the initial population size to 1, the wealth of an average individual at time t is $W_t e^{-gt}$, where $g > 0$ is the population growth rate. To ensure that such an average individual's expected utility is nondecreasing, we require that $\widetilde{X}_t$ has nonpositive drift, where $\widetilde{X}_t = e^{g(\gamma-1)t} X_t$.

This condition also ensures nondecreasing expected utility for any class of individuals who have a constant share of the wealth of society. For example, a Blanchard (1985) model with population growth implies that a newborn person has lower wealth than the average currently living person, because more people are born than die at each instant; however, with a constant population growth rate the wealth share of newborn individuals is constant over time. Thus, the constraint that $\widetilde{X}_t$ has nonpositive drift ensures that the expected utility of newborn individuals does not decline over time.

Noting that

$$\frac{d\widetilde{X}}{\widetilde{X}} = g(\gamma-1)\,dt + \frac{dX}{X} = (1-\gamma)\left(r_f + \alpha\widehat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2 - g\right)dt + (1-\gamma)\alpha\sigma dZ + \left[(1-\alpha L)^{1-\gamma} - 1\right]dN\,, \tag{41}$$

the condition that $\widetilde{X}_t$ has nonpositive drift modifies the sustainability constraint (12) to

$$\theta \leq r_f + \alpha\widehat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2 + \omega\frac{\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right]}{1-\gamma} - g. \tag{42}$$

Equation (42) subtracts the population growth rate g from the previous formula for the sustainable consumption-wealth ratio. Sustainability of individual utility is a more demanding requirement in the presence of population growth. However, for realistic population growth rates the central message of the paper remains unchanged: in the presence of risk, the sustainable consumption-wealth ratio exceeds the riskfree interest rate.

6 Conclusion

In this paper we have argued, in the spirit of Koopmans (1960, 1967), that the implication of an ethical criterion—sustainability—for social discounting and consumption decisions depends on the production technology available to society. Specifically, in a risky world with a binding sustainability constraint, the sustainable social rate of time preference and consumption-wealth ratio, which equal one another, are not equal to either the riskless interest rate or the risky return on invested wealth, but lie in between these two. In the special case where invested wealth has only Brownian risk and no jump risk, the sustainable social rate of time preference is the equal-weighted average of the riskless interest rate and the risky return.

We have made this point in the context of an extremely simple model with iid returns in which the parameters governing the distribution of returns are known. We have therefore ignored parameter uncertainty, a phenomenon emphasized by Weitzman (2001). We have also ignored the possibility that returns may not be iid, because expected returns or risks change over time. Models with non-iid returns in general imply time-varying consumption growth and a term structure of discount rates. When consumption growth is persistent, this term structure is generally downward-sloping for safe investments and upward-sloping for risky ones as in the long-run risk model of Bansal and Yaron (2004). Gollier (2002) emphasizes the potential importance of a downward-sloping term structure of discount rates for social discounting. Our iid model has discount rates that are invariant to the horizon of an investment.

Although we have emphasized the sustainable social rate of time preference in this paper, we conclude by noting that this is not the same as the appropriate social discount rate that should be applied to an investment project. That discount rate depends on the project's risk. For a riskless project, the appropriate discount rate is the riskless interest rate, which is lower than the sustainable social rate of time preference in a risky world; and for a project that has the same risk as society's invested wealth, the appropriate discount rate is the expected risky return, which is higher than the sustainable social rate of time preference. Some previous discussions of social discounting have obscured these distinctions by ignoring the risk that society faces. Our analysis is deliberately simple in order to achieve clarity about these issues.

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A Appendix

Derivation of equation (3). Equation (3) follows from (2) and Itô's formula for semimartingales. See, for example, Proposition 8.19 of Cont and Tankov (2004). Heuristically, we can derive it by writing

$$d\log C = \frac{1}{C}dC - \frac{1}{2!}\frac{1}{C^2}(dC)^2 + \frac{1}{3!}\frac{2}{C^3}(dC)^3 - \frac{1}{4!}\frac{6}{C^4}(dC)^4 + \cdots$$

and using the relationships $dt\,dN = dZ\,dN = 0$ and $dN^k = dN$ for all $k > 0$, in addition to the standard properties of dZ and the fact that $\log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$ if $|x| < 1$, which holds when $x = -\alpha L$ because the agent will never risk bankruptcy.

Derivation of equation (5). The calculation exploits the fact that N_t, Z_t, and L_i are independent. In particular, note that by the law of iterated expectations, the fact that N_t is a Poisson random variable with parameter ωt, the iid nature of the L_i, and the series definition of the exponential function,

$$\begin{aligned}
\mathrm{E}\prod_{i=1}^{N_t}(1-\alpha L_i)^{1-\gamma} &= \mathrm{E}\left[\mathrm{E}\left(\prod_{i=1}^{N_t}(1-\alpha L_i)^{1-\gamma} \;\middle|\; N_t\right)\right] \\
&= \sum_{n=0}^{\infty} e^{-\omega t}\frac{(\omega t)^n}{n!}\mathrm{E}\prod_{i=1}^{n}(1-\alpha L_i)^{1-\gamma} \\
&= \sum_{n=0}^{\infty} e^{-\omega t}\frac{(\omega t)^n}{n!}\left(\mathrm{E}\left[(1-\alpha L)^{1-\gamma}\right]\right)^n \\
&= \exp\left\{\omega \mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right]t\right\}.
\end{aligned}$$

Derivation of equation (11). We write

$$\begin{aligned}\frac{dX}{X} &= (1-\gamma)\frac{dW}{W} + \frac{\gamma(\gamma-1)}{2}\left(\frac{dW}{W}\right)^2 - \frac{\gamma(\gamma-1)(\gamma+1)}{6}\left(\frac{dW}{W}\right)^3 + \cdots \\ &= (1-\gamma)\left(r_f + \alpha\widehat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2\right)dt + (1-\gamma)\alpha\sigma dZ + \\ &\quad + \left[(\gamma-1)\alpha L + \frac{\gamma(\gamma-1)}{2}\alpha^2L^2 + \frac{\gamma(\gamma-1)(\gamma+1)}{6}\alpha^3L^3 + \cdots\right]dN \\ &= (1-\gamma)\left(r_f + \alpha\widehat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2\right)dt + (1-\gamma)\alpha\sigma dZ + \left[(1-\alpha L)^{1-\gamma} - 1\right]dN\,.\end{aligned}$$

Derivation of equation (33). This follows because

$$\left(\frac{\sigma^2}{\mu L+\sigma^2}\right)^{1/\gamma} = \left(1 - \frac{\mu L}{\mu L+\sigma^2}\right)^{1/\gamma} > \left(1 - \frac{\mu L}{\sigma^2}\right)^{1/\gamma} \geq 1 - \frac{\mu L}{\sigma^2} \quad \text{if } \gamma \geq 1.$$

Derivation of equation (34). We differentiate equation (32) with respect to L. The result is negative if and only if

$$1 + \frac{\mu L}{\gamma\left(\mu L+\sigma^2\right)} < \left(1 - \frac{\mu L}{\mu L+\sigma^2}\right)^{-1/\gamma}.$$

But this holds, because

$$1 + \frac{\mu L}{\gamma\left(\mu L+\sigma^2\right)} < \left(1 + \frac{\mu L}{\mu L+\sigma^2}\right)^{1/\gamma} < \left(\frac{1}{1-\frac{\mu L}{\mu L+\sigma^2}}\right)^{1/\gamma},$$

where the first inequality is Bernoulli's inequality.

The log utility case. With log utility, the investor's objective function is

$$U = \mathrm{E}\int_0^\infty e^{-\rho t}\log C_t\,dt\,, \quad \text{where } \rho > 0.$$

It follows from equation (3) that

$$\log C_t = \log C_0 + \left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 - \theta\right)dt + \alpha\sigma Z_t + \sum_{i=1}^{N_t}\log\left(1-\alpha L_i\right),$$

and hence

$$\mathrm{E}\log C_t = \log C_0 + \left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 - \theta\right)dt + \omega\mathrm{E}\left[\log\left(1-\alpha L\right)\right]t.$$

Thus the objective function can be evaluated explicitly as

$$U = \frac{\log W_0 + \log\theta}{\rho} + \frac{r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 - \theta + \omega\mathrm{E}\left[\log\left(1-\alpha L\right)\right]}{\rho^2}.$$

Maximizing with respect to θ and α we find the first-order conditions for an unconstrained optimum,

$$\theta = \rho \quad \text{and} \quad \widehat{\mu} - \alpha\sigma^2 = \omega\mathrm{E}\left[L\left(1-\alpha L\right)^{-1}\right].$$

The objective function at time t is affine in $\log W_t$, so the sustainability condition requires that $d\log W_t$, or equivalently $d\log C_t$, is driftless, i.e. that

$$\theta \leq r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 + \omega\mathrm{E}\left[\log\left(1-\alpha L\right)\right].$$

We define the constrained solution as before, giving

$$\theta_{\mathrm{con}} = r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 + \omega\mathrm{E}\left[\log\left(1-\alpha L\right)\right].$$

When the constraint binds, we have

$$U = \frac{\log W_0 + \log\theta}{\rho},$$

so α is chosen to maximize the constrained consumption-wealth ratio. We end up with the same first-order condition as in the unconstrained case. Thus the optimal investment choice is the same in the constrained and unconstrained cases, as before. Equations (16) and (17) also hold as before. Thus, all the results stated previously for risk aversion $\gamma > 1$ carry over to the log case where $\gamma = 1$.