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PRIVATE EQUITY AND GROWTH

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ABSTRACT

We study private equity in a dynamic general equilibrium model and ask two questions: (i) Why does the investment of venture funds respond more strongly to the business cycle than that of buyout funds? (ii) Why are venture funds returns higher than those of buyout? On (i), venture brings in new capital whereas buyout largely reorganizes existing capital; this can explain the stronger co-movement of venture with aggregate Tobin's Q. Regarding (ii), venture returns co-move more strongly with aggregate consumption and therefore pay a higher premium. Our model embodies this logic and fits the data on investment and returns well. At the estimated parameters, the two PE sectors together contribute between 14 and 21 percent of observed growth, relative to the extreme case where private equity is absent.

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1 Introduction

Private equity funds account for a growing share of real investment in the U.S. economy, averaging more than six percent of private domestic investment since 2001.[1] We build a model of private equity investment consistent with its cyclical properties, and fit the model to the observed levels and returns.

We focus on the empirical regularity that investments in both venture capital (VC) and buyout funds rise relative to other forms of investment as a function of aggregate Tobin's Q, but that venture responds more strongly. The left panel of Figure 1 shows the relative "intakes" of VC and buyout funds along with fluctuations in Q from 1987 to 2016, and the scatterplot in the right panel shows the positive relation between the two.[2] The left panel of Figure 2 shows the individual intakes and the closer link of venture with Q is apparent there as well.

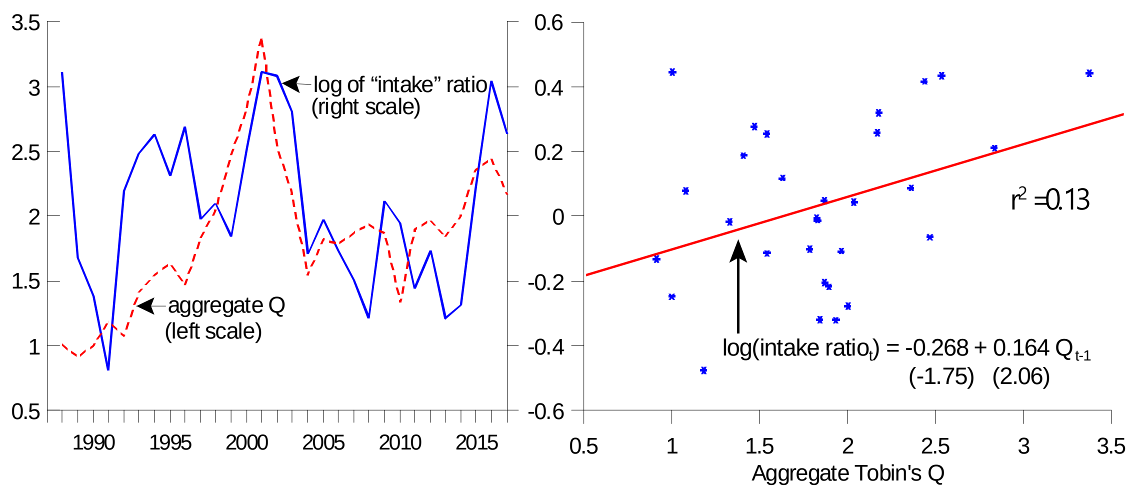

Figure 1: The relation between the log ratio of venture and buyout fund investment to aggregate Tobin's Q, 1987-2016

Why is intake more Q-elastic for VC than for buyouts? Our model implies it is because buyout funds reorganize existing capital, which is costlier to acquire when Q rises, whereas VC does not face this impediment (and indeed ordinary investment faces it to an even greater extent than buyout). The right panel of Figure 2 shows our second empirical regularity:

[1] According to authors' calculation. See the left panel of Figure 2 and Appendix A for details.
[2] See Appendix A for a description of all data and methods used in the empirical analysis.

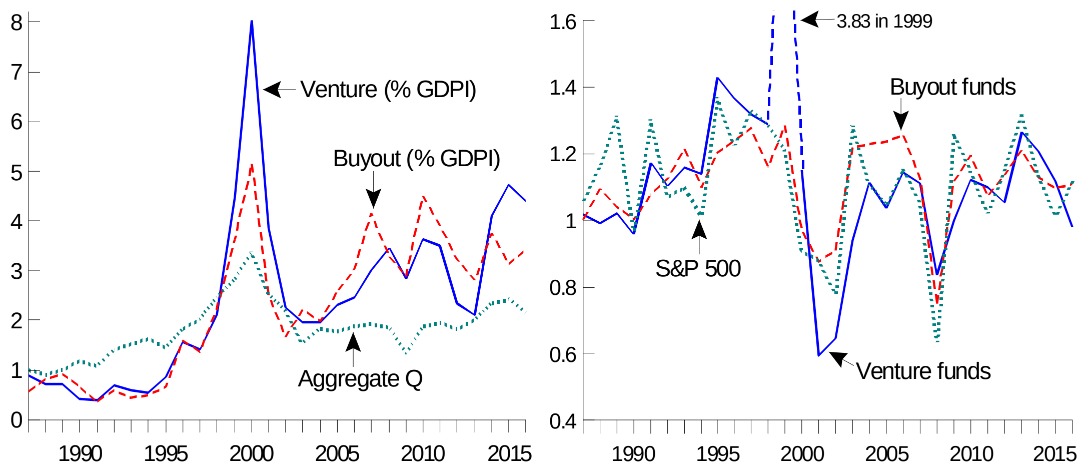

Figure 2: The intakes of venture and buyout funds with respect to gross private domestic investment and their respective returns, as related to Tobin's Q and the S&P 500, 1987-2016

payoffs to venture funds on average exceed those of buyout funds over the cycle. In our model this occurs because both ventured and bought-out capital collect the same amount, Q, per unit when sold to the public, and it is thus only the difference in costs that drives the stronger response of VC intakes. Moreover, both types of private equity outperform the S&P 500, and both co-move with it, yet venture returns respond more strongly to returns on the S&P 500.

The key feature of the model is that the returns to venture and buyout are drawn from different distributions with the returns known before investment decisions are made. This implies that the investment choice for each type of fund is described by a simple cutoff rule: invest as long as the return is sufficiently high. The cutoffs vary over the business cycle, and the relative returns, as well as the cyclical variation in relative intake of venture compared to buyout funds, is determined by the shapes of the distributions from which the returns are drawn. We show that, so long as the distribution of venture funds returns has a thinner tail than buyout returns, and the cutoffs are sufficiently close to each other, the model is consistent with the empirical regularities documented above. The model delivers premia over the S&P 500 return because private equity pays more when the cost of capital is high, but that's also the time when aggregate investment is low and consumption is high. Returns to private equity funds must therefore be higher at these times to compensate for the lower marginal utility of consumption. Earnings of the S&P 500 firms, on the other hand, depend primarily on TFP shocks, and there is no incentive to substitute toward consumption when TFP is high. Hence the premium for private equity.[3]

Why do these distinctions between the two types of private equity matter for macroeconomics? Although the two types together account for only 2-7% of total domestic investment, with the average exceeding 6%, this share is growing rapidly. With VC intakes and returns more procyclical than those for buyout, it is clear that VC can amplify business fluctuations, and importantly at times when activity is expanding and less promising ideas are getting implemented. At the same time, venture has higher external benefits than buyout that may offset the change in the average quality of ideas: Gompers *et al.* (2005), for example, show that founders of venture capital-backed start-ups disproportionately come from prior positions at previously venture-backed companies. This is learning-by-investing and therefore the Arrow (1962) type of effect that Searle (1945) and Lucas (1993, Figure 1) document, with such learning disproportionately associated with new ventures.

In particular, we embed VC and buyout funds in a traditional Ak model with two capital stocks and show, analytically and numerically, that their addition contributes to growth while retaining all of the standard implications of the Ak model. A stylized parametric example shows that PE can account for as much as half of observed growth. This effect falls to 14 percent of observed growth in numerical estimations with less extreme distributional assumptions about the quality of new projects and under Epstein-Zin preferences. Under CRRA preferences the effect is 21 percent of observed growth. In all cases the effect of VC on growth is larger than that of buyout, and the overall growth effects could be even larger if PE were embedded in a heterogeneous-idea model of the Lucas and Moll (2014) or Perla and Tonetti (2014) types.

Opp (2019) also models the implementation of heterogeneous ideas by VC funds, and in his model as in ours VC activity is procyclical, with funding standards declining during booms in the sense that the average quality of implemented ideas is countercyclical. He analyzes venture only, whereas we also analyze buyouts. As a result, we can study the movements in

[3]Not pictured is the relation between returns and investment, but Kaplan and Stromberg (2009, Table 3, panel B) find that private equity commitments rise as a function of returns realized over the previous year, and this occurs in our model as well.

the relative intakes (Figure 1) as well as the relative performance of venture and buyout funds (Figure 2). To the best of our knowledge, our paper is the first to endogenize both types of investments or "intakes" in a dynamic general equilibrium model. In addition, Opp (2019) quantifies VC funds' impact on aggregate growth and welfare while we study contributions to growth from both types of PE funds.

A complementary explanation for the premium in private equity returns is illiquidity, where there is an effective lock-up period of as much as ten to twelve years. Sorensen, Wang and Yang (2014) model the effects of these lock ups and find an annual premium of slightly more than one percent, which we subtract from the returns that we target when estimating the model because our premia are driven entirely by the correlation of PE returns with aggregate consumption. Ang, Papanikolaou and Westerfield (2014) also find the premium to be about one percent. Amihud and Mendelson (2006) argue that the low premia are the result of a selection effect whereby investors who can tolerate risk more easily are drawn into private equity – a preferred habitat view.

Our discussion of private equity is confined to PE *funds*. A broader measure of private equity includes occupation-specific investments in physical and human capital such as those made by self-employed people. Hamilton (2000) and Moskowitz and Vissing-Jorgensen (2002) find the risk-adjusted return to self employment insufficient to compensate for foregone wage earnings, and that perhaps non-pecuniary benefits play a major role. Such benefits are presumably not present when investing in PE funds. Vereshchagina and Hopenhayn (2009) point out, however, that the option of liquidating a private equity investment after realizing a low return can also lower its equilibrium return.

Earlier work finds that the mode of business investment varies across firms and over the cycle; in particular, young firms contribute a larger fraction of investment when stock prices are high. Gompers *et al.* (2008) find that VC investment is positively related to Q. Jovanovic and Rousseau (2014) show that a rise in aggregate Tobin's Q leads incumbents to reduce investment, thereby creating opportunities for investments by new firms.

The paper is organized as follows. Section 2 introduces our model. Section 3 characterizes the equilibrium and provides analytical examples. Section 4 discusses the empirical plausibility of the model by first providing some empirical evidence and further estimates of the model. Section 5 concludes. All data sources, estimation methods, and derivations are in the Appendix.

2 Model

We begin with a model of the real side of the economy in Sec. 2.1, and then add a private equity sector in Sec. 2.2.

2.1 An Ak Structure with two Capital Stocks

Output, y, is produced with physical capital, k, and human capital, h, as

$$y = Zk^{\alpha}h^{1-\alpha}, \tag{1}$$

where Z is a shock. Output is consumed, C, or invested in amounts X and X_{h} in the two types of capital at the unit cost of q and q_{h}. The income identity reads

$$y = C + qX + q_{\mathrm{h}}X_{\mathrm{h}}. \tag{2}$$

The laws of motion for k and h are

$$k' = (1-\delta)k + X \qquad \text{and} \tag{3}$$

$$h' = (1-\delta)h + X_{\mathrm{h}}, \tag{4}$$

where a prime, "$'$", denotes a variable's the next-period value, and δ is the common rate of depreciation.

Households.—Households are homogeneous with preferences are $E_0\left\{\sum_{t=0}^{\infty}\beta^t U(C_t)\right\}$, with $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$. They own h, which earns the competitive wage

$$w = (1-\alpha)Zk^{\alpha}h^{-\alpha},$$

and households receive the aggregate wage bill $wh = (1-\alpha)y$. Households own the firms, which pay out profits $\pi(k,h) = \alpha y$ as dividends back to households. Optimal investment in

h requires that the cost of a unit of human capital equals the present value of its expected wage payments. Written recursively, this condition reads

$$q_{\mathrm{h}} = \beta \int \left(\frac{C}{C'}\right)^{\gamma} \left((1-\alpha)\frac{y'}{h'} + (1-\delta)\, q'_{\mathrm{h}}\right) dF\left(s', s\right), \tag{5}$$

where y' is next period's output, and where the three aggregate shocks, $(Z, q_{\mathrm{h}}, q) \equiv s$, follow first-order Markov processes with the CDF of s' given by $F(s', s)$.

Firms.—They own k and optimal investment in k requires that the cost of a unit of physical capital, q, equals the present marginal value of expected dividends (discounted at the household's stochastic discount factor) of that unit. Expressed recursively, the condition reads

$$q = \beta \int \left(\frac{C'}{C}\right)^{-\gamma} \left(\alpha\frac{y'}{k'} + (1-\delta)\, q'\right) dF\left(s', s\right). \tag{6}$$

This too is the price of a unit of k, and the gross return on equity is $R_{\mathrm{E}} = \frac{1}{q}\left(\alpha\frac{y'}{k'} + (1-\delta)\, q'\right)$.

Let

$$\zeta \equiv \frac{q_{\mathrm{h}}}{q}. \tag{7}$$

If ζ is a constant, the model simplifies as k and h can be aggregated into a composite that is proportional to k. Let

$$\kappa = \frac{1-\alpha}{\alpha\zeta}, \tag{8}$$

and let z be the scaled TFP shock

$$z = Z\kappa^{1-\alpha}. \tag{9}$$

We summarize the results as follows:

Proposition 1 *If ζ is constant, then h is proportional to k,*

$$h = \kappa k, \tag{10}$$

output can be written as a function of k alone,

$$y = zk, \tag{11}$$

the goods cost of a unit of composite capital is

$$q\left(1 + \zeta\kappa\right) = \frac{q}{\alpha}, \tag{12}$$

and the income identity reduces to a function of C, k, and composite capital investment, X, reads $zk = C + \frac{q}{\alpha}X$, or in units of k,

$$z = c + \frac{q}{\alpha}x, \tag{13}$$

where $x = X/k$ is investment in physical capital.

Proof. Eqs. (8) and (10) imply that

$$(1-\alpha)\frac{y'}{h'} = \zeta\alpha\frac{y'}{k'}. \tag{14}$$

Substituting from (7) and (14) into (5) yields (6). Eq. (12) follows from (7) and (8). Finally, (2) reads

$$y = C + qX + q_{\mathrm{h}}X_{\mathrm{h}} = C + q\left(X + \zeta X_{\mathrm{h}}\right) = C + q\left(1 + \zeta\kappa\right)X$$

and division by k yields (13). ■

2.2 Private Equity and the Implementation of Ideas

The arrival process for new ideas.—Households get new ideas, and their number is proportional to the aggregate stock of human capital, h. The number of ideas implementable by VC funds is λh, and the number implementable by buyout funds is θh. Not all the ideas are implemented. The presence of h is external to the households; a household does not take it into account when choosing its human capital investment which still satisfies (5).

Formally, the production function (1) and the law of motion for h in (4) remain the same, but in (3), X is broadened as follows:

$$X = \sum_{j\in\{c,v,b\}} X_j - n_{\mathrm{b}}k, \tag{15}$$

where X_c represents investment in continuing projects, X_v is venture-backed investment in new projects, X_b is investment mediated by buyout funds, and n_b is the intake of buyout funds, which is formally defined in Sec. 2.2.2.

In other words, all three methods create the same commodity – physical capital, but their production functions differ. We now describe each in turn:

(*i*) *Continuing projects.*—X_{c} is created via existing projects and its total cost is qX_{c} in units of consumption.

(*ii*) *VC-backed projects.*—X_{v} denotes efficiency units of k created by implementing new projects. A project uses as inputs a unit of the consumption good and an idea. As output it delivers ε units of capital ready for use in the next period. The quality of the project, ε, is known at the start. New projects are born each period, and their quality is distributed with a CDF $G^{\mathrm{v}}(\varepsilon)$. Households get ideas at the rate λh so that the unnormalized distribution of new ideas is $\lambda h G^{\mathrm{v}}(\varepsilon)$.

(*iii*) *Upgraded projects.*—X_{b} denotes efficiency units of k created by buyout funds; θh units of existing physical capital k can be upgraded at a cost τ per unit. When upgraded, its efficiency changes from unity to ε. Idea qualities for upgrading are described by ε's drawn from the CDF G^{b}. The unnormalized distribution of new ideas is $\theta h G^{\mathrm{b}}(\varepsilon)$. As with the venture funds, ε is known before the fact.

Contracting between agents and PE funds.—Funds raise new subscriptions each year and close the following period. Each vintage of investors thus receives a return on their investments alone – there is no mixing of dividends among investments of different vintages. Costs predate returns by a period.[4] We assume that the PE fund owners get all the rents from the projects in which they invest.[5] It takes one period for the projects to mature, and a fund's revenue from selling the project is $q'\alpha\frac{y'}{k'}\varepsilon$. Capital created though new projects is sold by venture funds to incumbent firms or floated by IPO at a price equal to the cost, q, of making capital via the incumbents' technology. Capital created through buyout activity results in an effectively larger capital stock in the hands of existing firms.

2.2.1 Venture Funds

Let ε_{v} be the minimum project quality accepted by a VC fund. We shall measure intake and payout relative to k, and using (10), relative to k the number of ideas is

$$\frac{1}{k}\lambda h = \lambda\kappa.$$

[4] The typical fund lasts 10-12 years but our data will allow us to infer the year-to-year returns.

[5] For VC funds Jovanovic and Szentes (2013) obtain this outcome if VCs are scarce relative to founders of new firms.

The intake of VC funds.—Each implemented idea costs one unit of consumption, and the total VC fund investment then is the same as the number of projects

$$n_{\mathrm{v}} \equiv \#\text{ VC-backed projects} = \lambda\kappa \left[1 - G^{\mathrm{v}}\left(\varepsilon_{\mathrm{v}}\right)\right]. \tag{16}$$

VC fund payout.—Since capital is delivered in time for next-period production, i.e., next period dividends are the closing revenue

$$D_{\mathrm{v}}\left(s'\right) = \left(\alpha z' + \left(1-\delta\right) q'\right) x_{\mathrm{v}},$$

where

$$x_{\mathrm{v}} = \lambda\kappa \int_{\varepsilon_{\mathrm{v}}}^{\infty} \varepsilon dG^{\mathrm{v}}. \tag{17}$$

The VC fund's project-portfolio decision.—A VC fund chooses ε_{v} to maximize the expected utility of its investors.

$$\max_{\varepsilon_{\mathrm{v}}} \left\{\beta \int \left(\frac{C}{C'}\right)^{\gamma} D_{\mathrm{v}}\left(s'\right) dF\left(s', s\right) - n_{\mathrm{v}}\right\}. \tag{18}$$

Since $\int \left(\frac{C}{C'}\right)^{\gamma} D_{\mathrm{v}} dF = x_{\mathrm{v}} \int \left(\frac{C}{C'}\right)^{\gamma} \left(\alpha \frac{y'}{k'} + \left(1-\delta\right) q'\right) dF = \frac{q}{\beta}\lambda\kappa \int_{\varepsilon_{\mathrm{v}}}^{\infty} \varepsilon dG^{\mathrm{v}}$, the problem in (18) reduces to

$$\pi_{\mathrm{v}}\left(q\right) \equiv \max_{\varepsilon_{\mathrm{v}}} \left(q\lambda\kappa \int_{\varepsilon_{\mathrm{v}}}^{\infty} \varepsilon dG^{\mathrm{v}} - n_{\mathrm{v}}\right). \tag{19}$$

As a result, the minimal accepted quality of VC-backed projects satisfies,

$$\varepsilon_{\mathrm{v}} = \frac{1}{q}. \tag{20}$$

2.2.2 Buyout Funds

Let ε_{b} be the minimum project quality accepted by a VC fund.

The intake of Buyout funds.—Each implemented idea costs τ units of consumption and one unit of physical capital the price of which is q. The total buyout fund investment then is

$$n_{\mathrm{b}} \equiv \left(\tau + q\right) \theta\kappa \left[1 - G^{\mathrm{b}}\left(\varepsilon_{\mathrm{b}}\right)\right]. \tag{21}$$

Buyout fund payout.—Next period dividends are the closing revenue

$$D_{\mathrm{b}}\left(s'\right) = \left(\alpha \frac{y'}{k'} + \left(1-\delta\right) q'\right) x_{\mathrm{b}},$$

where

$$x_{\mathrm{b}} = \theta\kappa \int_{\varepsilon_{\mathrm{b}}}^{\infty} \varepsilon dG^{\mathrm{b}}. \tag{22}$$

Buyout fund's project-portfolio decision.—The fund chooses ε_{b} to maximize the expected utility of its investors,

$$\beta \int \left(\frac{C}{C'}\right)^{\gamma} D_{\mathrm{b}}\left(s'\right) dF\left(s', s\right) - n_{\mathrm{b}},$$

and using the same logic as that behind the proof of (20), we get the buyout fund's decision problem

$$\pi_{\mathrm{b}}\left(q\right) \equiv \max_{\varepsilon_{\mathrm{b}}} \left(q\theta\kappa \int_{\varepsilon_{\mathrm{b}}}^{\infty} \varepsilon dG^{\mathrm{b}} - n_{\mathrm{b}}\right), \quad \text{subject to (21)}, \tag{23}$$

with its optimal cutoff rule

$$\varepsilon_{\mathrm{b}} = \frac{\tau + q}{q}. \tag{24}$$

To summarize: Firms live forever and are publicly owned with share prices q in (6). Funds live for one period and are not publicly tradable. The timing within the period is as follows:

(*i*) $s \equiv (z, q)$ and the profile of ε's are realized at the start of the period;

(*ii*) Production and ideas implementation occur, determining $\varepsilon_{\mathrm{v}}, \varepsilon_{\mathrm{b}}$, x_{c} and $\Gamma \equiv k'/k$;

(*iii*) The equity market opens; firms invest, converting goods into k and buying up the PE-mediated capital at the price q. PE funds distribute their profits to households and consumption takes place.

2.2.3 Differences between the two funds

In the model, venture funds create new physical capital whereas buyout funds transform and upgrade existing physical capital units into new ones subject to an implementation cost τ. One can interpret a VC fund in our model as an investment vehicle that provides equity financing to *startups* and growth firms in their early stage. In our sample for empirical analysis, the majority of VC funds (1,070 out of 1,680 funds) are in the early stage and earn large and positive net-of-fee returns; this is consistent with Korteweg and Nagel (2016), who document that VC start-up investments earn large positive abnormal net returns whereas those in the later stage earn net returns close to zero.[6]

On the other hand, buyout funds can be interpreted as investment vehicles that acquire *existing* firms. As a result, the implementation cost τ could include any transaction fees charged when a buyout fund buys or sells a company, similar to the M&A fees charged by banks. Metrick and Yasuda (2010) argue that transaction fees are common features for buyout funds but are rare for VC funds.[7] A buyout fund therefore does not pay τ to its capital providers; rather, τ is as a real cost that may in fact compensate fund managers (i.e., the general partners) for the due diligence they perform with takeover deals. That is why the cutoff quality ε_{b} depends positively on τ.

3 Equilibrium and its Properties

Proposition 1 continues to hold with the two forms of private equity except the accounting equations change slightly. First, the law of motion for k in (3) changes to

$$\frac{k'}{k} = 1 - \delta + i, \tag{25}$$

where corresponding to (15),

$$i = x_{\mathrm{c}} + x_{\mathrm{v}} + x_{\mathrm{b}} - n_{\mathrm{b}}. \tag{26}$$

Second, in our new notation, the RHS of the income identity (13) reads $z = c + \frac{q}{\alpha} i$, the presence of α reflecting the investment of goods into h, which itself is proportional to k. The two PE funds, however, generate k at a cost lower than q and the difference is reflected in their profits $\pi_{\mathrm{v}} + \pi_{\mathrm{b}}$ that are added to the LHS of (13) along with $\frac{q}{\tau+q} n_{\mathrm{b}}$ of the buyout funds' costs, which are transfer payments. Thus the income identity becomes

$$z + \pi_{\mathrm{v}} + \pi_{\mathrm{b}} + \frac{q}{\tau + q} n_{\mathrm{b}} = c + \frac{q}{\alpha} i. \tag{27}$$

With both k and h dropping out of the equations, the state of the economy is $s = (z, q)$.

[6] Similarly, Cochrane (2005), Hall and Woodward (2007), and Korteweg and Sorensen (2010) find positive abnormal round-to-round returns (gross of fees) from start-up VC investment. We therefore adjust PE returns for early vs. later-stage funds in the empirics in Sec. 4.2.

[7] This difference is reflected in the RHS of Eqs. (16) and (21). Depending on the funding terms, transaction fees for BO funds range from 1.68 to 3.37 cents per dollar of committed capital (Metrick and Yasuda 2010).

Using (11), we re-write (6) as

$$q = \beta \int \left(\frac{c(s')}{c(s)} \Gamma(s) \right)^{-\gamma} (\alpha z' + (1-\delta) q') \, dF(s', s). \tag{28}$$

The functions $(\varepsilon_i, n_i, \pi_i)$ are defined in terms of primitives, and the remaining unknowns are $c(s)$, $x_{\text{c}}(s)$, and $\Gamma(s)$. These functions solve Eqs. (27), (25) and (28) for all s.

Existence and characterization.—Let the constant L solve

$$L = \left[\beta \int \frac{\left(1 + q^{1-1/\gamma} L\right)^{\gamma} (z + (1-\delta) q)}{\left(z + \sum \pi_i(q) + \theta\kappa q \left[1 - G^{\text{b}}\left(1 + \frac{\tau}{q}\right)\right] + (1-\delta) q\right)^{\gamma}} dF(z, q) \right]^{1/\gamma}. \tag{29}$$

Appendix B1 shows that if

$$\beta^{1/\gamma} \int \left(1 - \delta + \frac{z}{q}\right)^{1-\gamma} dF(s) \leq 1, \tag{30}$$

a unique solution for L to Eq. (29) exists. Then Appendix B1 also proves

Proposition 2 *If Eq. (30) holds, the equilibrium c and i are*

$$c = \frac{z + \sum \pi_i(q) + \theta\kappa q \left[1 - G^{b}\left(1 + \frac{\tau}{q}\right)\right] + (1-\delta) q}{1 + q^{1-1/\gamma} L}, \tag{31}$$

and

$$i = \alpha \frac{\left(z + \sum \pi_i(q) + \theta\kappa q \left[1 - G^{b}\left(1 + \frac{\tau}{q}\right)\right]\right) q^{-1/\gamma} L - (1-\delta)}{1 + q^{1-1/\gamma} L}. \tag{32}$$

We use both c and i from Eqs. (31) and (32) to estimate parameters of the model, and when we generalize the model in Sec. 4.4.1 to include Epstein Zin preferences, Eqs. (31) and (32) will continue to hold with only the constant L differing.

3.1 Private Equity and Growth

According to (11), y grows at the same rate as k in the long run. From (25), the growth rate of k is

$$g \equiv i - \delta.$$

We first consider an analytical example when the distribution of project qualities follow the Pareto distribution.

Example: Pareto G^{v} *and* G^{b}.—For $i \in \{v, b\}$, let $\rho_i > 1$. And for $\varepsilon \geq \varepsilon_{\mathrm{i},\,0}$, let

$$G^{\mathrm{i}}(\varepsilon) = 1 - \left(\frac{\varepsilon}{\varepsilon_{\mathrm{i},\,0}}\right)^{-\rho_i}. \tag{33}$$

Appendix B3 then proves the following result in the deterministic case where z and q are fixed:

Proposition 3 *If households have log preferences so that* $\gamma = 1$ *and if* G^v *and* G^b *follow the Pareto distribution in Eq. (33), with lower bounds satisfying* $\varepsilon_{v,\,0} < \frac{1}{q}$ *and* $\varepsilon_{b,\,0} < \frac{\tau+q}{q}$*, then the growth rate is*

$$g = \alpha \frac{\left(z + \nu + \theta\kappa q \left(\frac{\tau+q}{q\varepsilon_{b,\,0}}\right)^{-\rho_b}\right) q^{-1}\beta - q(1-\delta)\left(1 - \beta + [z + (1-\delta)q]^{-1}\nu\right)}{1 + [z + (1-\delta)q]^{-1}\nu} - \delta, \tag{34}$$

where

$$\nu = q\kappa \left(\frac{\theta}{\rho_b - 1}\left(\frac{\tau+q}{q\varepsilon_{b,\,0}}\right)^{1-\rho_b} + \frac{\lambda}{\rho_v - 1}\left(\frac{1}{q\varepsilon_{v,\,0}}\right)^{1-\rho_v}\right). \tag{35}$$

Growth is increasing in z *and in the thickness of each tail,*

$$\frac{\partial g}{\partial z} > 0,\ \frac{\partial g}{\partial \rho_b} < 0,\ \frac{\partial g}{\partial \rho_v} < 0,$$

and decreasing in q *and in the implementation cost of buyout funds* τ*,*

$$\frac{\partial g}{\partial q} < 0,\ \frac{\partial g}{\partial \tau} < 0.$$

The solution in (34) is of the same general character as that of other Ak models of growth, being increasing in z and decreasing in q. What we add is a role for PE. When q is high, so also are the costs of reorganizing capital through buyouts relative to VC, and while both forms of PE rise to partially offset the decline in investment generally, VC rises more. These points become clearer in the linear approximation; Appendix B4 further shows the following:

Corollary 1 *A first-order Taylor approximation around* $(\lambda, \theta) = (0, 0)$ *yields the following expression for the growth rate,*

$$g = \alpha\left(\tilde{A} - a\right) - \delta, \tag{36}$$

where

$$\tilde{A} = q^{-1}\beta\left(\underbrace{z}_{TFP\ term} + \underbrace{\tilde{\lambda}\left[\frac{q\kappa}{\rho_v - 1}\left(\frac{1}{q\varepsilon_{v,\,0}}\right)^{1-\rho_v}\right]}_{Value\ added\ by\ Venture} + \underbrace{\tilde{\theta}\left[\frac{q\kappa\rho_b}{\rho_b - 1}\left(\frac{\tau + q}{q\varepsilon_{b,\,0}}\right)^{1-\rho_b}\right]}_{Value\ added\ by\ Buyout}\right) \quad and$$

$$a = q\alpha\beta\left(1-\delta\right)\left(\underbrace{\frac{1-\beta}{\beta}}_{Discount\ Rate\ without\ PE} - \left[z + \left(1-\delta\right)q\right]^{-1} v\right)$$

are the PE-adjusted TFP term and the PE-adjusted discount rates, and where

$$\tilde{\lambda} \equiv \lambda\frac{\left(1-\delta\right)q}{z + \left(1-\delta\right)q} \quad and \quad \tilde{\theta} \equiv \theta\frac{\left(1-\delta\right)q}{z + \left(1-\delta\right)q}$$

are proportional to the arrival rate λ *and* θ*. Moreover,* $\tilde{A}$ *increases with* z *and decreases with* q*, and the opposite is true of* a*.*

The growth rate depends on the TFP term z and the cost q of physical capital as in all Ak models, but also depends on value added from the two PE sectors in our model. Value added by buyout is linear in $\rho_{\mathrm{b}}/\left(\rho_{\mathrm{b}} - 1\right)$, which is the mean of ε_{b}. The value added from two PE funds becomes zero in the limit case where λ and θ are zero so that there are no implementable ideas in the PE sectors, or when the distribution of ε converges to a degenerate distribution, as discussed next.

Limiting case.— As $\lambda \to 0$ and $\theta \to 0$ the supply of PE projects disappears. And as $\rho_b \to \infty$ and $\rho_v \to \infty$, the two Pareto distributions collapse to a degenerate distribution at $\varepsilon_{0,\mathrm{v}}$ and $\varepsilon_{0,\mathrm{b}}$. In these limiting cases PE plays no role in the model and

$$g \to \alpha\left[zq^{-1}\beta - \left(1-\beta\right)\left(1-\delta\right)q\right] - \delta, \tag{37}$$

which is the same as Eq. (2) of Rebelo (1991) when his CRRA parameter $\sigma = 1$.[8]

[8]When $q = 1$, equation (37) becomes

$$\alpha\beta\left[z - \delta\frac{1-\beta}{\beta} - \frac{1-\beta}{\beta}\right],$$

which is the discrete-time version of Eq. (2) of Rebelo (1991) with $A = z$, $\delta_z = \delta\frac{1-\beta}{\beta}$, and $\rho = \frac{1-\beta}{\beta}$.

Numerical example – We now provide a simple quantitative illustration of the impact from the PE sectors to growth in the model using (34) and the Pareto assumptions above. This is meant as a benchmark – we later refine the model to consider alternative and more realistic distributions for G^{v} and G^{b}. For now, we target the moments listed in Table 3 reported in Sec. 4 using the Pareto with $\rho_b = \rho_v = 1.6$, which Jovanovic and Szentes (2013) used in their simulations.[9] The estimated growth rate from Eq. (36) is 2.66% per annum, and is reduced to 2.36% and 1.68% when we shut down the buyout funds ($\theta = 0$) and venture funds ($\lambda = 0$), respectively. When there are no PE funds at all ($\lambda = \theta = 0$), the annual growth rate is halved to 1.38%. From this parametric example, the PE sectors contribute to almost 50% of the growth in total and the effect of venture on growth is larger than that of buyout; the latter is consistent with the model's implication that venture co-moves more strongly with the business cycle. Figure A1 in Appendix D further shows that the marginal impact of PE funds on growth flattens out as ρ_v and ρ_b increase, which indicates that the PE sector has a larger impact on the real side of the economy when the distributions of its projects' qualities are more dispersed.

While these relatively large growth effects and the contrast between VC and buyout rely on log utility and Pareto distributions, they represent closed-form analytical solutions for how much PE may affect the economy. We shall relax these assumptions in Sec. 4, and further connect the model to the macro data and evaluate its empirical plausibility. We derive the model's implications for asset prices next.

3.2 Asset returns

We first derive the general expressions for returns, and then provide three examples.

Return on equity.—By (11) we have $y'/k' = z'$ and therefore the gross return on equity is

$$R_{\text{E}} = \frac{1}{q}\left(\alpha z' + (1-\delta)\, q'\right). \tag{38}$$

[9] Different from parameters in Table 3, we set $\gamma = 1$ as used in the proposition 3 and further set the depreciation rate $\delta = 8\%$ to achieve a reasonable range of growth rates. Similar to Table 3, we preset $\alpha = 0.25$ and $\beta = 0.95$. The estimated parameters are $\lambda = 1.58$, $\tau = 0.91$, $\theta = 0.12$, and $\kappa = 0.05$.

Gross return of VC and Buyout funds.—Realized returns are

$$R_{\mathrm{v}} \equiv \frac{D_{\mathrm{v}}(s')}{n_{\mathrm{v}}} \quad \text{and} \quad R_{\mathrm{b}} \equiv \frac{D_{\mathrm{b}}(s')}{n_{\mathrm{b}}}. \tag{39}$$

With the return on equity, R_{E}, defined in (38) and the project-selection rules ε_{v} and ε_{b} defined in (20) and (24), returns relative to R_{E} are

$$\begin{aligned} \frac{R_{\mathrm{v}}}{R_{\mathrm{E}}} &= \frac{1}{1-G^{\mathrm{v}}(1/q)} \int_{1/q}^{\infty} \varepsilon dG^{\mathrm{v}}(\varepsilon) \qquad \text{and} \\ \frac{R_{\mathrm{b}}}{R_{\mathrm{E}}} &\equiv \frac{1}{1-G^{\mathrm{b}}([\tau+q]/q)} \int_{[\tau+q]/q}^{\infty} \frac{\varepsilon}{\tau+q} dG^{\mathrm{b}}(\varepsilon) \ . \end{aligned} \tag{40}$$

Cyclical implications.—The model has the following cyclical implications for the ratios in (40). The means of the ε's and their truncation points are scaled by their costs which are unity for venture and $\tau + q$ for buyout.

Three examples now follow – the derivations are in Appendix B2.

Example 1: *Exponential G^{v} and G^{b}.*—For $\varepsilon \geq 0$, let $G^{\mathrm{i}}(\varepsilon) = 1 - \exp(-\lambda_{\mathrm{i}}\varepsilon)$ for $i \in \{v, b\}$.

$$\frac{R_{\mathrm{v}}}{R_{\mathrm{E}}} = \frac{1}{q} + \frac{1}{\lambda_{\mathrm{v}}} \quad \text{and} \quad \frac{R_{\mathrm{b}}}{R_{\mathrm{E}}} = \frac{1}{q} + \frac{1}{\lambda_{\mathrm{b}}(\tau+q)}. \tag{41}$$

so that

$$R_{\mathrm{v}} \gtrless R_{\mathrm{b}} \qquad \Leftrightarrow \qquad (\tau+q)\lambda_{\mathrm{b}} \gtrless \lambda_{\mathrm{v}}. \tag{42}$$

Example 2: *The Pareto* example again.—Consider again the Pareto example in Eq. (33). Then

$$\frac{R_{\mathrm{v}}}{R_{\mathrm{E}}} = \frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{v}}-1}\frac{1}{q} \quad \text{and} \quad \frac{R_{\mathrm{b}}}{R_{\mathrm{E}}} = \frac{\rho_{\mathrm{b}}}{\rho_{\mathrm{b}}-1}\frac{1}{q}. \tag{43}$$

As a result, the distribution with the thicker tail yields the higher returns:

$$R_{\mathrm{v}} \gtrless R_{\mathrm{b}} \qquad \Leftrightarrow \qquad \rho_{\mathrm{b}} \gtrless \rho_{\mathrm{v}}. \tag{44}$$

Example 3: *Normal G^{v} and G^{b}.*—Let $\varepsilon_{\mathrm{v}} \sim N(\mu_{\mathrm{v}}, \sigma_{\mathrm{v}}^2)$ and $\varepsilon_{\mathrm{b}} \sim N(\mu_{\mathrm{b}}, \sigma_{\mathrm{b}}^2)$. Then for the generic Normal distribution $N(\mu, \sigma^2)$, the inverse Mills ratio is expressed as

$$\mathrm{MR}_{\mathrm{i}}(\varepsilon_{\mathrm{i,min}}) = E(\varepsilon \mid \varepsilon > \varepsilon_{i,\min}) = \mu_i + \sigma_i \frac{\phi\left(\frac{\varepsilon_{i,\min}-\mu_i}{\sigma_i}\right)}{1-\Phi\left(\frac{\varepsilon_{i,\min}-\mu_i}{\sigma_i}\right)}, \ i = \mathrm{v,b}$$

where ϕ and Φ are PDF and CDF of standardized normal distribution $N(0,1)$, respectively. In our model the truncation points are $\varepsilon_{\text{v, min}} = 1/q$ and $\varepsilon_{\text{b, min}} = (\tau + q)/q$. The ratio of return is proportional to the ratio of inverse Mills ratio,

$$\frac{R_{\text{v}}}{R_{\text{b}}} = \frac{\text{MR}_v(\varepsilon_{\text{v, min}})}{\text{MR}_b(\varepsilon_{\text{b, min}})}(\tau + q). \tag{45}$$

Choice of G^{v} *and* G^{b} While both the Exponential and Pareto cases provide closed-form expressions of returns, (42) and (44) imply that higher returns are associated with thicker tails. Nevertheless, we will show in Sec. 4 that the cross-section data of buyout and VC returns reveal the *opposite*: a lower average buyout return coexists with a thicker cross-sectional tail. On the other hand, when G^{v} and G^{b} follow Normal distributions, Equation (45) shows that even if VC has a thinner tail (smaller σ) than buyout, VC could still have a higher mean return $E[R_{\text{v}}] > E[R_{\text{b}}]$ if $E[\text{MR}_v(\varepsilon_{\text{v, min}})(\tau + q)] > E[\text{MR}_b(\varepsilon_{\text{b, min}})]$ for sufficiently large implementation cost τ. We find that this inequality indeed holds at the estimated parameter values reported in Sec. 4.

In the Pareto case, Equation (43) implies that the relative returns R_v/R_b are driven *entirely* by the parameter choices of distributions and thus do not depend on q or aggregate risk – this is less supported empirically as we will show in the next section. This, however, is true for the Pareto case *only*. Under the assumption of normal distributions, Equation (45) shows not only that the relative returns depend on the distribution parameters, but that they also covary with q and hence with consumption according to equation (31). We summarize this finding in the following Lemma, which is proven in Appendix B2.

Lemma 1 *When the project quality ε_v and ε_b follows normal distributions, venture returns pay a higher premium than buyout if VC returns co-move more strongly with aggregate consumption.*

For the reasons above, we shall assume G^{v} and G^{b} follow normal distributions in the estimation, as discussed next.

4 Empirical Evidence and Model Estimation

Figures 1 and 2 in the introduction provide some preliminary and distinct features of buyout and venture funds. We begin this section by describing the empirical evidence in more detail and follow with estimation of the model to verify its ability to fit the aggregate time series of private equity investment and returns, as well as some key macro moments. Appendix A includes details of the data sources and estimation methods.

4.1 Empirical Evidence

We begin with regression tests of our model's predictions. We first verify the implications for observed intakes and returns to private equity. Table 1 reports the results of time-series regressions of the logs of venture and buyout intakes, as well as their log ratio, with respect to aggregate Tobin's Q_{t-1}. The table shows that both log intakes are positively related at the 1% level to aggregate Q_{t-1} (i.e., measured at the start of the period), but that venture intakes are more responsive to Q_{t-1}. The regressions in the right-most panel show that the log ratio of the intakes is also positively related to Q_{t-1} at the 1% level when we include a linear time trend and at the 5% level without a trend. These results offer empirical evidence that venture activities co-move more strongly with the business cycle and therefore should pay higher premia than buyout.

Table 2 reports time-series regressions for the returns to the two funds and their ratio on aggregate Q and the productivity shock z_t. Returns to both venture and buyout funds are related negatively to start-of-period Q_{t-1} and positively to end-of-period Q_t as the model predicts, and typically at the 5% level or less, and returns to venture are more Q-elastic than buyout returns. Consistent with the model in showing no significant relation between the ratio $R_{\mathrm{v}}/R_{\mathrm{b}}$ and Q_{t-1}, the regression in the right-most panel indicates that the relation with end-of-period Q_t is positive but is imprecisely estimated.

Comparing to the findings in the literature, Kaplan and Stromberg (2009) measure returns over the previous year and find in their Table 3 that they relate positively to current period commitments. Our model also predicts this: commitments rise in q and so do returns over the previous period – see Equation (43). Gompers *et al.* (2005) also find that an increase in initial

Table 1. Private Equity Intake Regressions, 1987-2016

Dependent variable: log intakes

	$\ln(n_t^V)$		$\ln(n_t^B)$		$\ln(n_t^V/n_t^B)$	
Q_{t-1}	1.411***	0.941***	1.248***	0.691***	0.164**	0.250***
	(7.23)	(6.57)	(5.47)	(4.23)	(2.07)	(2.87)
trend		0.060***		0.071***		-0.011**
		(6.52)		(6.77)		(-1.98)
constant	7.763***	7.640***	8.030***	7.884***	-0.268*	-0.244
	(20.66)	(31.94)	(18.28)	(28.85)	(1.77)	(1.68)
R^2	0.651	0.865	0.517	0.821	0.133	0.242

Note: Estimation is by OLS, with robust T-statistics in parentheses. Q_{t-1} is aggregate Tobin's Q measured at the start of year t. The variables n_t^V and n_t^B are in millions of constant 2000 dollars.

Table 2. Private Equity Fund Return Regressions, 1987-2016

Dependent variable: fee and inflation-adjusted returns

	$\mathrm{R}_{VC,t}$		$\mathrm{R}_{B,t}$		$\mathrm{R}_{VC}/\mathrm{R}_B$	
Q_{t-1}	-0.514**	-0.531**	-0.254***	-0.256***	-0.211	-0.222
	(-2.15)	(-2.25)	(-5.26)	(-5.31)	(-1.16)	(-1.16)
Q_t	1.014***	0.966***	0.312***	0.304***	0.557	0.527
	(4.10)	(3.92)	(6.24)	(6.03)	(1.46)	(1.45)
z_t		4.258		0.671		2.708
		(1.34)		(1.03)		(1.24)
constant	0.212	-1.310	0.996***	0.756***	0.381	-0.586
	(0.76)	(-1.12)	(17.61)	(3.17)	(0.95)	(-0.62)
R^2	0.443	0.479	0.591	0.607	0.314	0.342

Note: Estimation is by OLS, with robust T-statistics in parentheses. Q_{t-1} is aggregate Tobin's Q measured at the start of year t.

public offering (IPO) valuations leads venture capital firms to raise more funds, an effect that is particularly strong among younger venture firms (Kaplan and Schoar, 2005).

4.2 Model Estimation

In this section we estimate the model and evaluate its performance with respect to the available time series for private equity returns, intakes and other variables of interest. To do this, we assign values to α, β, γ, and δ and then choose G_{b},G_{v} (distribution parameters), λ, θ, τ, and κ jointly to target the means of c, n_{v}/i, n_{b}/i, R_{VC}, R_B, and $R_{\mathrm{S\&P}}$.[10]

For q, we use fourth quarter observations underlying Hall (2011) for 1987-1999, and then join them with estimates underlying Abel and Eberly (2011) for post-1999 periods. More details on the construction of the series are in Appendix A. The National Income and Product Accounts (NIPA) provide us with z_{k}, the ratio of output to physical capital. From it, we compute z using Eqs. (7)-(9) as follows:

$$z = z_{\mathrm{k}} + \frac{1-\alpha}{\alpha} qx.$$

Gross domestic product is then defined as

$$y = zk = \left(z_{\mathrm{k}} + \frac{1-\alpha}{\alpha} qx \right) k,$$

such that the "true" output and output-capital ratio are higher after bringing human capital into the measurement.[11]

Table 3 reports the values we assign for parameters for α, β, γ, and δ, along with those we estimate. We assume the distributions of ε's are normal: $\varepsilon_{\mathrm{v}} \sim N\left(\mu_{\mathrm{v}}, \sigma_{\mathrm{v}}^2\right)$ and $\varepsilon_{\mathrm{b}} \sim N\left(\mu_{\mathrm{b}}, \sigma_{\mathrm{b}}^2\right)$. We set the model to an annual frequency with $\beta = 0.95$ and a 1% rate of capital depreciation.[12] Table 3 shows that all estimated parameters are statistically significant, with the exception of human capital ratio κ, at the 5% percent level or less based upon 95 percent GMM confidence

[10] We take the S&P 500 return, denoted by $R_{\mathrm{S\&P}}$, as our observed measure of the return on equity in (38)

[11] This is also how Prescott and McGrattan (2010) treat unmeasured investment – it is "lost output" as in the Ben-Porath model where h is built by withholding time from production and not using goods that are produced.

[12] We set $\delta = 1\%$ to match the level of equity returns; we later show that the results are robust to depreciation rates of 4% as in Karabarbounis and Neiman (2014), 5% as in Jovanovic and Rousseau (2014), and 6% as in Nadiri and Prucha (1996).

intervals. The estimated distribution of ε for buyout has a larger mean ($\mu_b > \mu_v$) and a thicker tail than VC ($\sigma_b > \sigma_v$).

Table 3. Parameters for the Estimation

Pre-set Parameters			
Capital Share	Discount Rate	Risk Aversion	Depreciation Rate
α	β	γ	δ
0.25	0.95	3	0.01
Estimated Parameters			
Average VC Quality	SD of VC Quality	Average BO Quality	SD of VC Quality
μ_{v}	σ_{v}	μ_{b}	σ_{b}
-0.48	1.03	-0.18	2.26
$[-0.96, -0.01]$	$[0.68, 2.73]$	$[-0.40, -0.04]$	$[1.56, 2.96]$
Arrival Rate of VC Idea	BO Implementation Cost	Arrival Rate of BO Upgrade	Human Capital Ratio
λ	τ	θ	κ
0.72	0.72	0.17	0.01
$[0.67, 0.77]$	$[0.08, 1.35]$	$[0.02, 0.35]$	$[-3.42, 3.45]$

Note: The table reports parameters for the estimation, with GMM 95 percent confidence intervals in brackets.

Table 4 reports the means of the various series of interest as estimated and in the data for 1987-2016. The table incorporates two adjustments to the data:

Adjustment for early vs. late-stage.—For PE returns, Korteweg and Nagel (2016) document that VC *start-up* investments earn large positive abnormal returns whereas those in later stages earn excess returns near zero, net of fees. To address this, we adjust VC returns as follows

$$R_{VC} = \omega_{VC}\tilde{R}_{VC} + (1 - \omega_{VC})\, R_{S\&P},$$

where $\omega_{VC} = \frac{\#\text{ VC funds in early stage}}{\#\text{ Total VC funds}}$ and $\tilde{R}_{VC}$ is the combined VC return (i.e., without adjusting for early versus later stage funds).[13] In our sample, 1,070 out of 1,680 VC funds are early stage, indicating that $\omega_{\text{VC}} = 65\%$. We also note that the average combined VC return, $\tilde{R}_{VC}$, in our sample is 18.1%, which is in line with the estimates in Table IA.II of Harris, Jenkinson, and Kaplan (2015).

Adjustment for the liquidity premium.—We further subtract 1.05 percent from buyout

[13] The average $\tilde{R}_{VC}$ in our sample is 18.1% ; this is in line with the estimates in Table IA.II of Harris, Jenkinson, and Kaplan (2015). Table 7 in the robustness section later shows that our model fits the data well using combined VC returns as well.

and venture returns, which is the liquidity premium reported by Sorensen, Wang and Wang (2014) for these types of funds. This value was also subtracted from the series shown in the top two panels in Figure 3. The 1.05 premium is similar to the value of 0.9 percent reported by Ang, Papanikolaou and Westerfield (2014).

Table 4. Fit of model to observed means

	Targeted			**Non-targeted**	
	Data	Model		Data	Model
mean R_{VC}	1.139	1.139	mean g	1.020	1.054
mean R_{BO}	1.105	1.105	mean i	0.050	0.064
mean $R_{\text{S\&P}}$	1.086	1.089	Volatility R_{VC}	0.550	0.394
mean n_{v}/i	0.024	0.024	Volatility R_{BO}	0.128	0.314
mean n_{b}/i	0.023	0.023	Volatility $R_{\text{S\&P}}$	0.172	0.178
mean c	0.300	0.300	mean $qR_{VC}/R_{\text{S\&P}}$	2.670	2.416

Note: The table reports targeted and non-targeted moments of the variables of interest for the baseline estimation. The empirical moments are computed over the sample from 1987 to 2016.

Table 4 shows that the model matches most of the means in the data well, and matches the second moments of asset returns and average investment i reasonably well. The model predicts larger average returns for VC than for buyout because the estimated average q-adjusted Mills ratio for VC, $E\left[\text{MR}_v\left(\tau+q\right)\right]$, is larger than for buyout $E[\text{MR}_b]$, as was discussed in example 3 in Sec. 2.3. The model, however, overestimates the average growth rate in order to match average stock returns based on (38). Using the estimated parameters, the PE sectors together contribute to 21% of growth. Table 8 reported in the robustness section later will show that the model can match the average growth rate g with capital depreciation δ set to 4% per annum but at a cost of slightly lower fitted average stock returns. In addition, the model overpredicts the volatility for buyout whereas the estimated venture return is less volatile than the data.

The last non-targeted moment in Table 4 relates to Kortum and Lerner's (2000) findings that, from 1983 to 1992, the ratio of venture intake to R&D spending averaged less than 3% and venture generated 8% of industrial innovations. Assuming that the return to R&D would be the same as to ordinary investment, our model indicates that the ratio of the average to marginal product of VC, ε_v, was $8/3 \approx 2.67$. Our model would interpret this as follows: Eqs.

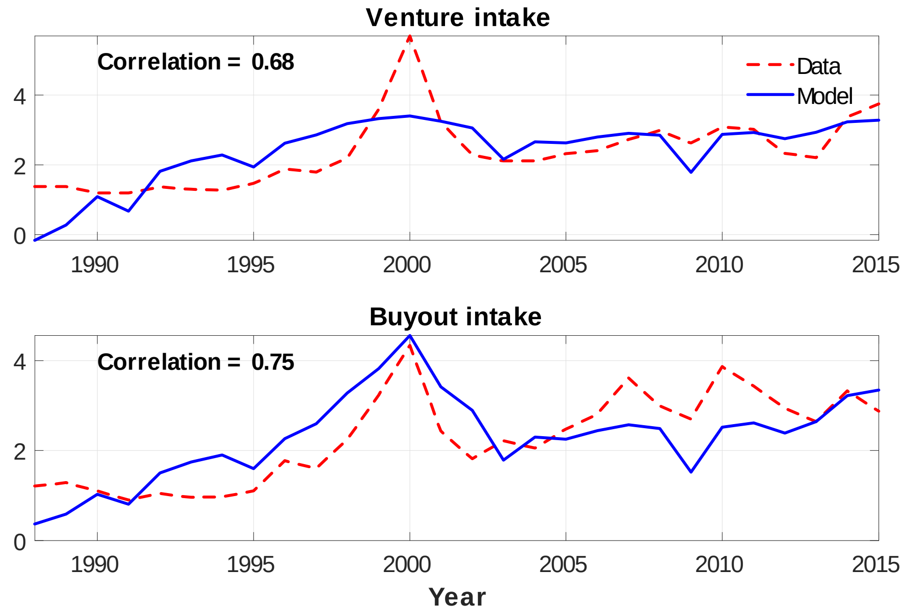

Figure 3: Intakes of venture and buyout in the model and the data, 1987-2015

(20) and (40) would imply

$$\frac{E(\varepsilon \mid \varepsilon \geq \varepsilon_{\mathrm{v}})}{\varepsilon_{\mathrm{v}}} = \frac{q}{1 - G(\varepsilon_{\mathrm{v}})} \int_{\varepsilon_{\mathrm{v}}} \varepsilon dG^{\mathrm{v}}(\varepsilon) = q \frac{R_{\mathrm{v}}}{R_{\mathrm{E}}} = 2.67.$$

Our data on average show that $q = 1.83$ and $\frac{R_{\mathrm{VC}}}{R_{\mathrm{S\&P}}} = 1.32$ and the product is 2.416, which as shown in the last row of Table 4, is a slight underestimate.

Figure 3 shows the intakes of venture and buyout funds as percentages of gross private domestic investment in the data and in the model.[14] The upper panel of Figure 3 shows that the model fits the cyclical properties of the intake for venture investment well, with a correlation of 0.68 between model and data. The series in the bottom panel have a correlation of 0.75, and indicate the model can reproduce the spike in buyout funds that occurred in the year 2000, albeit overly so.

Role of Spikes in 2000 – It is worth noting that the high correlation between the model estimate and the data is not driven by the spike: when we exclude 2000 in the sample, the top and bottom panels continue to exhibit high correlations of 0.73 and 0.69 respectively.

Figure 4 shows the fit between the model and data with respect to returns. The model once again fits venture well with a correlation of 0.65 (upper panel) when we include the spike

[14] To produce the model-based time series, we insert historical (q, z) into the corresponding policy functions in the model.

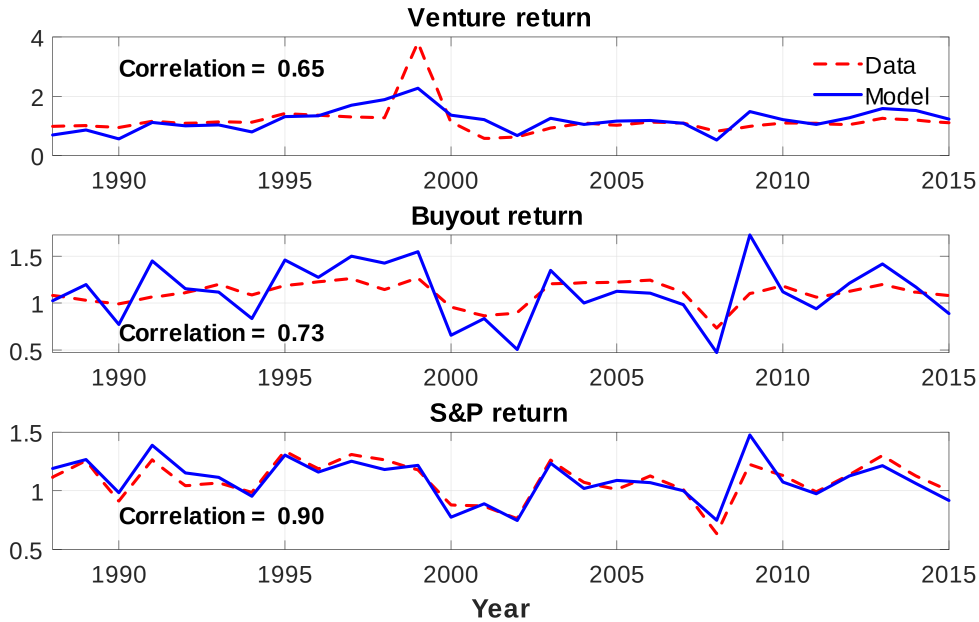

Figure 4: Returns to buyout, venture, and the S&P 500 in the model and the data, 1987-2015

in 2000 and 0.64 without the spike, and fits the returns to buyout funds even more closely with the correlation of model and data of 0.73 (middle panel). The bottom panel shows the fit to the S&P 500 return, where the correlation of the model with the data is 0.90.

In sum, our model is able to match the dynamics of returns to buyout, venture and stock well using data on the macro series (z, q).

4.3 Empirical Validation: Estimation of Distributions

One of the key distinctions between the venture and buyout funds in the model is the difference in the distributions $G^{\mathrm{v}}(\varepsilon)$ and $G^{\mathrm{b}}(\varepsilon)$. In fact, under the Pareto case (example 2 in Sec. 2.3), we show that the relative returns are only driven by the parameters of the distributions, ρ_v and ρ_b. In general, the model requires buyout ε_{b} to have a thicker tail than venture ε_{v} to match the mean returns in the data. Does this hold empirically? To answer this question, given that systematic data on the individual projects of venture and buyout funds are generally unavailable, we construct an empirical counterpart of $G^{\mathrm{v}}(\varepsilon)$ and $G^{\mathrm{b}}(\varepsilon)$ using actual IPOs and acquisitions from the Securities Data Company (SDC) Platinum and CRSP/Compustat Merged (CCM) databases.

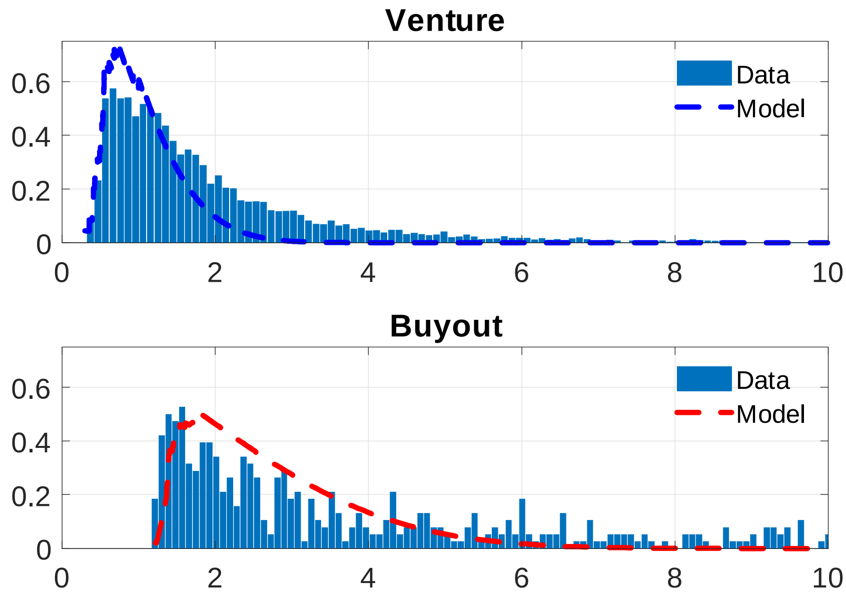

Figure 5: The pooled data and their estimated distributions

We define a firm's value by that of its common stock and trim the top and bottom 2.5% of firm-year observations to avoid extreme values that may reflect data errors.[15] We examined various other definitions of firm value such as total assets (common stock plus cash, debt, and preferred stock), but chose to work with the value of common stock to obtain the largest possible number of venture and buyout observations, although results using total assets are very similar. The final sample contains 8,209 venture and 665 buyout observations spanning the period from 1986 to 2017. Appendix A provides detailed descriptions of data and sources.

When a firm has an IPO or is taken over, its ε_{v} or ε_{b} is computed as follows:

$$\varepsilon_{\mathrm{v}} = \frac{\text{IPO Value}}{q \times \text{total Assets}} \tag{46}$$

$$\varepsilon_{\mathrm{b}} = \frac{\text{Combined Value} - \text{Acquirer Value}}{q \times \text{Target's Assets}} \tag{47}$$

where both the IPO value and "Combined Value" are defined as the number of common shares outstanding at year end multiplied by the annual closing price. Following the model, we further truncate the distribution of ε_{v} and ε_{b} by cutoffs $1/q$ and $1 + \tau/q$.

Figure 5 shows the estimated conditional probability densities of ε for venture (blue dashed

[15] Results are highly robust to how we trim the sample.

line in the top panel) and buyout (red dashed line in the bottom panel), using the parameters reported in Table 3, against their corresponding empirical distributions from the SDC database. The data are pooled over years and more observations tend to come from years when q was high. The model implies that in year t, ε_v would be included only if $\varepsilon_{\text{v}} > 1/q_t$, and ε_{b} would be included only if $\varepsilon_{\text{b}} > (\tau + q_t)/q_t$. The predicted number of IPOs in year t_0 is therefore $\left(1 - G^{\text{v}}\left(\frac{1}{q_0}\right)\right) / \sum_{t=1}^{T}\left(1 - G^{\text{v}}\left(\frac{1}{q_t}\right)\right)$, and similarly for the number of buyouts. Thus the predicted distributions are calculated as

$$\hat{g}^{v}(\varepsilon_{\text{v}}) = \frac{\sum_{t=1}^{T} g^{v}(\varepsilon_{\text{v}}) \cdot I\left(\varepsilon_{\text{v}} > \frac{1}{q_t}\right)}{\sum_{t=1}^{T}\left(1 - G^{\text{v}}\left(\frac{1}{q_t}\right)\right)} \quad \text{and}$$

$$\hat{g}^{b}(\varepsilon_{\text{b}}) = \frac{\sum_{t=1}^{T} g^{b}(\varepsilon_{\text{b}}) \cdot I\left(\varepsilon_{\text{b}} > \frac{\tau+q_t}{q_t}\right)}{\sum_{t=1}^{T}\left(1 - G^{\text{b}}\left(\frac{\tau+q_t}{q_t}\right)\right)},$$

where g^{i} is the PDF of $N(\mu_i, \sigma_i)$ with the values μ_i and σ_i reported in Table 3 for $i =$ v, b, and G^{v} and G^{b} are the associated CDFs. $I(\cdot)$ is an indicator function.[16]

Figure 5 shows that, consistent with model's implication, the distribution of project qualities for buyout indeed has a thicker tail than that for venture. This is consistent with Figure F.3 of Gupta and Van Nieuwerburgh (2019), which shows the profit distribution of buyout has a thicker tail than venture. Other evidence on fat tails in PE returns include Scherer (2000) and Silverberg and Verspagen (2007).

Before we conclude this section, it's worth noting that we generate the estimated distributions by fitting the same moments used in Table 4 rather than by fitting the targeted empirical distributions *directly*. Therefore, Figure 5 should be considered as further empirical validation of the model.[17]

[16] By construction then, $\int \hat{g}^{i}(\varepsilon)\, d\varepsilon = 1$ for $i =$ v, b.

[17] An additional validation of distributional assumption is based on Jovanovic and Szentes (2013), who use the Pareto distribution for their analog of ε_{v}. When fitting distributions of waiting times to successful exit and to termination of venture projects, in their Table 1 for the parameter ρ_{v} (for which their analog was titled λ) they use $1.55, 1.6$, and 1.73. At the middle value of $\rho_{\text{v}} = 1.6$ and the average q of 1.83, Eq. (43) from our model implies $\frac{R_{\text{v}}}{R_{\text{E}}} = \frac{\rho_{\text{v}}}{\rho_{\text{v}}-1}\frac{1}{q} = 1.46$, which is close to the ratio $\frac{R_{\text{VC}}}{R_{\text{S\&P}}} = 1.32$ in the data.

4.4 Additional Robustness Checks

In this section, we consider alternative estimations of model and different targets of the calibrations to check the robustness of results.

4.4.1 Model Extension: Recursive Preferences

In the model, households are assumed to have power utility over consumption C. Even though this assumption substantially simplifies the analysis and allows for *closed-form* solutions of PE returns under certain distributional assumptions, it also leads to counterfactually high volatility of the implied risk-free rate and the quantitatively implausibly low equity risk premia analyzed by Mehra and Prescott (1985). In this section, we address the issue by relaxing the assumption of power utility using Epstein and Zin (EZ, 1989) preferences that separate parameters of risk aversion and elasticity of intertemporal substitution.

More specifically, we extend the model to allow for EZ recursive preferences,

$$U_t = \left[(1-\beta)C_t^{1-\psi} + \beta\left(E_t[U_{t+1}^{1-\gamma}]\right)^{\frac{1-\psi}{1-\gamma}}\right]^{\frac{1}{1-\psi}}. \tag{48}$$

where ψ is the inverse of the elasticity of intertemporal substitution (EIS) and γ is the curvature parameter. Our baseline power utility is a special case where $\psi = \gamma$. The stochastic discount factor m_{t+1} is

$$m_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\psi}\left(\frac{U_{t+1}}{(E_t(U_{t+1})^{1-\gamma})^{\frac{1}{1-\gamma}}}\right)^{\psi-\gamma}. \tag{49}$$

As in the baseline model, households own the firms and optimal investment in h requires that the cost of a unit of human capital equals the present value of its expected wage payments. Using Eq. (49) and written recursively, this condition now reads

$$q_{\mathrm{h}} = \beta\int\left(\frac{C'}{C}\right)^{-\psi}\left(\frac{V(k',s')}{(E_t(V(k',s'^{1-\gamma})^{\frac{1}{1-\gamma}}}\right)^{\psi-\gamma}\left((1-\alpha)\frac{y'}{h'} + (1-\delta)\,q'_{\mathrm{h}}\right)dF\left(s',s\right),$$

where $V(k,s) \equiv U_t$ denotes the value function as a function of $(k,s)'$. Similarly, for the firms, optimal investment in k requires that the cost of a unit of physical capital, q, equals the present marginal value of expected dividends (discounted at the household's stochastic discount factor) of that unit. The condition now reads

$$q = \beta \int \left(\frac{C'}{C} \right)^{-\psi} \left(\frac{V(k', s')}{(E_t(V(k', s'^{1-\gamma})^{\frac{1}{1-\gamma}}} \right)^{\psi - \gamma} \left(\alpha \frac{y'}{k'} + (1 - \delta) q' \right) dF(s', s) .$$

We follow the baseline model and define $\zeta \equiv \frac{q_h}{q}$ to be the ratio of two prices. If ζ is a constant, the model simplifies as k and h can be aggregated into a composite that is proportional to k.

The following proposition, proven in Appendix B5, shows that the Ak property is preserved under EZ preferences:

Proposition 4 *Proposition 1 holds under EZ preferences and the value function takes the form $V(k, z, q) = v(z, q)k$*

Next, let the constant L_{EZ} satisfy the following equation:

$$L_{\text{EZ}} = \left(\frac{\alpha\beta}{1-\beta} \right)^{1/\psi} \left[\int v(z,q)^{1-\gamma} dF \right]^{\frac{1-\psi}{\psi(1-\gamma)}}, \qquad \text{where} \tag{50}$$

$$v(z,q) = \left[(1-\beta) \left(z + \sum \pi_i(q) + \theta\kappa q \left[1 - G^{\text{b}} \left(1 + \frac{\tau}{q} \right) \right] + (1-\delta) q \right)^{1-\psi} (1 + q^{1-1/\psi} L_{\text{EZ}})^{\psi} \right]^{\frac{1}{1-\psi}} .$$

Then Appendix B6 proves that the optimal policies take on the same functional form under EZ preferences as they do under CRRA preferences except that the constant L is different. In particular, analogously to Proposition 2, we have:

Proposition 5 *With recursive preferences in Eq. (48), the solutions for c and i in Eqs. (31) and (32) remain valid but with ψ substituted for γ, and with L_{EZ} substituted for L.*

Comparison to Proposition 2.— When $\gamma \neq \psi$, the only effect of the additional parameter on the policies c and i is through L_{EZ} being different from L. Eqs. (31) and (32) remain the same. In Appendix B6, we show that $L_{\text{EZ}} = L$ when $\gamma = \psi$, and so are c and i.

We next estimate the extended model and again evaluate its performance with respect to the available time series for private equity returns, intakes and other variables of interest. Similar to the baseline estimation with power utility, we preset $\alpha = 0.25$ and $\beta = 0.95$, but choose a higher depreciation rate $\delta = 5\%$ as in Jovanovic and Rousseau (2014). We further set $1/\psi$, the elasticity of intertemporal substitution (EIS), to be 0.95 which is in line with studies such as Hall (1988), and let the data determine the curvature parameter γ. Table A2 in Appendix C shows an alternative estimation result when we freely estimate both γ and ψ; the estimated EIS $1/\psi$ remains less than 1.

As in the baseline estimation, we choose G_{b},G_{v}, λ, θ, τ, and κ to target the means of c, n_{v}/i, n_{b}/i, R_{VC}, R_B, and $R_{\text{S\&P}}$. We again assume the distributions of ε's are normal: $\varepsilon_{\text{v}} \sim N(\mu_{\text{v}}, \sigma_{\text{v}}^2)$ and $\varepsilon_{\text{b}} \sim N(\mu_{\text{b}}, \sigma_{\text{b}}^2)$. The additional parameter γ is jointly estimated with other parameters by targeting the mean of the risk free rate defined as

$$R_{\text{f}} = \left[\beta \int \left(\frac{C'}{C}\right)^{-\psi} \left(\frac{V(k', s')}{(E_t(V(k', s'^{1-\gamma})^{\frac{1}{1-\gamma}}}\right)^{\psi-\gamma} dF(s', s)\right]^{-1}.$$

Table 5 reports that all estimated parameters are statistically significant, with the exception of κ, at the five percent level or less based upon 95 percent confidence intervals. Similar to the results reported under power utility, the estimated distribution of ε for buyout has a larger mean and a thicker tail than VC.

Table 5. Parameters for the Estimation under EZ Preferences

Pre-set Parameters			
Capital Share	Discount Rate	Depreciation Rate	EIS
α	β	δ	$1/\psi$
0.25	0.95	0.05	0.95
Estimated Parameters			
Average VC Quality	SD of VC Quality	Average BO Quality	
μ_{v}	σ_{v}	μ_{b}	
-0.54	1.09	-0.07	
$[-0.67, -0.40]$	$[0.61, 1.58]$	$[-0.14, -0.00]$	
SD of BO Quality	Arrival Rate of VC Idea	BO Implementation Cost	
σ_{b}	λ	τ	
2.31	0.69	0.66	
$[2.10, 2.51]$	$[0.68, 0.71]$	$[0.47, 0.86]$	
Arrival Rate of BO Upgrade	Human Capital Ratio	Curvature Parameter	
θ	κ	γ	
0.15	0.01	19.27	
$[0.09, 0.21]$	$[-0.99, 1.02]$	$[7.81, 31, 91]$	

Note: The table reports parameters for the estimation under EZ preferences, with GMM 95 percent confidence intervals in brackets.

Table 6 shows that the extended model matches most of the means in the data well and improves baseline model's fit. The model is able to match the non-targeted average growth, mean investment, and volatility of both PE returns and equity return $R_{\text{S\&P}}$. More importantly, as an improvement over power utility, the use of EZ preferences provides a much closer to fit to both mean risk-free rate (targeted) as well as its non-targeted volatility, as highlighted in the last row of the table. Therefore, the estimated model under EZ can jointly fit both volatility of the risk free rate and the equity risk premium. In addition, the model is able to match the non-target average growth well (1.9% in the model versus 2.0% in the data), and under the estimated parameters, the two PE sectors together contribute 14 percent of observed growth, relative to the extreme case when both λ and θ are set to zero.

Table 6. Model vs. Data, Estimation under EZ Preferences

	Target			**Non-target**	
	Data	Model		Data	Model
mean R_{VC}	1.139	1.139	mean g	1.020	1.019
mean R_{BO}	1.105	1.105	mean i	0.050	0.064
mean $R_{\text{S\&P}}$	1.086	1.052	Volatility R_{VC}	0.550	0.499
mean n_{v}/i	0.024	0.024	Volatility R_{BO}	0.128	0.214
mean n_{b}/i	0.023	0.023	Volatility $R_{\text{S\&P}}$	0.172	0.172
mean c	0.300	0.300	mean $qR_{VC}/R_{\text{S\&P}}$	2.670	2.891
Addition to the Baseline Estimation					
mean R_{f}	1.034	1.034	Volatility R_f	0.025	0.031

Note: The table reports targeted and non-targeted moments of the variables of interest for the alternative estimation under EZ preferences. The empirical moments are computed over the sample from 1987 to 2016.

4.4.2 Combined VC Returns

Table 7 reports the results using combined VC returns without adjusting for early versus later stage funds. In this case, average venture returns rise to 17% from 13.9%. We now use the same pre-set parameters and targets in the baseline calibration and re-estimate parameters, and Table 7 shows our model continues to fit the data well.

4.4.3 Higher Capital Depreciation Rate Under Power Utility

Table 8 reports the results when we increase the depreciation rate from 1% to 4% as in Karabarbounis and Neiman (2014) and 6% as estimated by Nadiri and Prucha (1996), respec-

Table 7. Alternative Estimation under Power Utility, Combined VC Returns

	Target			**Non-target**	
	Data	Model		Data	Model
mean R_{VC}	1.170	1.170	mean g	1.020	1.055
mean R_{BO}	1.105	1.105	mean i	0.050	0.065
mean $R_{\mathrm{S\&P}}$	1.086	1.089	Volatility R_{VC}	0.550	0.403
mean n_{v}/i	0.024	0.024	Volatility R_{BO}	0.128	0.314
mean n_{b}/i	0.023	0.023	Volatility $R_{\mathrm{S\&P}}$	0.172	0.178
mean c	0.300	0.300	mean $qR_{VC}/R_{\mathrm{S\&P}}$	2.670	2.828

Note: The table reports targeted and non-targeted moments of the variables of interest for the alternative estimation under power utility with combined VC returns and without adjusting for early versus later stage funds.

tively. The third and seventh columns show that, when δ is set to be 4%, the model can now generate 2.5% annual growth, which is closer to the data counterpart (2%) than the baseline estimate, albeit at a cost of slightly lower returns on the S&P 500. The fourth and last columns of the table report results when the depreciation rate is set at 6%. While the model can still match mean growth, investment, and the mean and volatility of various returns well, we find that the average equity return is about half of that observed in the data. As shown earlier in Table 6, however, this issue can be partially resolved under general EZ preferences.

Table 8. Alternative Estimation under Power Utility, Higher Depreciation rate

	Target				**Non-target**		
	Data	Model			Data	Model	
		$\delta = 4\%$	$\delta = 6\%$			$\delta = 4\%$	$\delta = 6\%$
mean R_{VC}	1.139	1.139	1.139	mean g	1.020	**1.025**	**1.014**
mean R_{BO}	1.105	1.105	1.105	mean i	0.050	0.065	0.065
mean $R_{\mathrm{S\&P}}$	1.086	**1.058**	**1.037**	Volatility R_{VC}	0.550	0.389	0.389
mean n_{v}/i	0.024	0.024	0.024	Volatility R_{BO}	0.128	0.313	0.313
mean n_{b}/i	0.023	0.023	0.023	Volatility $R_{\mathrm{S\&P}}$	0.172	0.173	0.170
mean c	0.300	0.300	0.300	mean $qR_{VC}/R_{\mathrm{S\&P}}$	2.670	2.858	2.916

Note: The table reports targeted and non-targeted moments of the variables of interest for the alternative estimation under power utility with depreciation rates set at 4 and 6 percent, respectively.

5 Conclusion

We document that the returns to venture funds are higher than those of buyout funds, and that venture funds' intake responds more strongly to the business cycle than buyout funds' intake. The model assumes that venture brings in new capital whereas buyout largely reorganizes existing capital, leading venture intake to co-move more strongly with aggregate Tobin's Q.

A thicker right tail for venture ideas also relates to venture's higher returns. More venture activity is drawn into the economy when q rises and in the model agents consume more when q is high. Venture returns are thus more strongly correlated with aggregate consumption than buyout and must pay a higher return.

An extension we do not pursue is that of returns to self employment. Perhaps the modeling distinction we make between how VC and buyout funds add value to capital also applies in the domain of self employment choices. One can open a store in an entirely new location, or one can buy someone else out or simply take over a location that someone else has vacated. Another extension would be to an open economy; U.S. private equity firms are active abroad and our estimate that private equity contributes between 14 and 21 percent of observed growth does not include the effects that private equity activity in or from the U.S. has on growth elsewhere in the world.

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Appendix A. Data and Methods

In this appendix we document the data sources and methods used to construct the series depicted in our figures and included in the empirical analysis.

Figure 1. —The "intakes" are the sum of investments made annually in U.S. venture capital and buyout funds, divided by annual estimates of gross private domestic investment from the BEA (2017, Table 5.2.5, line 4). Venture and buyout investments are from the April 2017 version of Thomson One's *VentureXpert* database, and are the sum of all investments made in a given calendar year at any stage or round across funds of each type. The "intake ratio," $n_{\mathrm{v}}/n_{\mathrm{b}}$, is the ratio of the respective investment sums in each year.

For aggregate q_t, we use fourth quarter observations underlying Hall (2001) for 1987-1999, and then join them with estimates underlying Abel and Eberly (2011) for 1999 to 2005. Abel and Eberly derive aggregate Tobin's Q from the Federal Reserve Board's *Flow of Funds Accounts* as the ratio of total market value of equity and bonds to private fixed assets in the non-financial corporate sector. We bring these estimates forward through 2016 using the same sources. Hall's measure of Q in 1999 is higher than that of Abel and Eberly (3.376 vs. 1.819), so we use a ratio splicing factor of 1.856 to adjust the series from 2000 forward.

Figure 2.—Aggregate returns to U.S. venture capital funds for 1987-2016 are from Cambridge Associates (2016a), and are annualized returns constructed by compounding one quarter horizon pooled returns. Aggregate returns to U.S. buyout funds for 1987-2016 are from Cambridge Associates (2016b), and are also annualized returns constructed from single quarter horizon pooled returns. Both series are net of fees, expenses, and carried interest.[18] It's worth noting that the majority of the VC funds (1,070 out of 1,680) in our sample are in the early stage and earn large and positive returns. This is consistent with Korteweg and Nagel (2016), who document that VC start-up investments earn large positive abnormal returns whereas those in the later stage earn net returns close to zero. In addition, the positive VC returns could also reflect the possibility that general partners have considerable equity in the

[18] The fees that go to general partners absorb most of the rents and are thus not compensation going to capital providers. While an investor obtains the return on the S&P 500 almost fully (an ETF costs a few basis points annually), an investors' PE investment comes with a hefty fee, likely in excess of 10% (Metrick and Yasuda 2010).

projects in addition to collecting fees. The average combined VC return $\tilde{R}_{VC}$ in our sample is 18.1%; this is in line with the estimates in Table IA.II of Harris, Jenkinson, and Kaplan (2015).

We convert each series into ex-post real returns using the annual growth of the consumer price index from the National Income and Product accounts. Annual returns to the S&P 500 are from Damodaran (2018),[19] and deflated by the consumer price index. We then subtract 1.05 percent from both venture and buyout returns, which is liquidity premium reported in Sorensen, Wang and Wang (2014).

Figures 3 and 4.—For z_t, we use private output, defined as GDP less government expenditures on consumption and investment from the BEA (2017) for 1987-2016. We then divide the result by K_{t-1} after adjusting it for inflation during year $t-1$ by averaging the annual inflation factors across the two years that overlap $t-1$ and then using its square root as a deflator. The K_t are end-year stocks of private fixed assets from the BEA (2017, table 6.1, line 1) for 1987 through 2016. The aggregate investment rate i_t is constructed as annual gross private domestic investment from the Bureau of Economic Analysis (BEA 2017, table 5.2.5, line 4) for 1987-2016 divided by K_{t-1}.

Figure 5.— We obtained the data on Venture and Buyout from the Securities Data Company (SDC) Platinum and CRSP/Compustat Merged (CCM) Database.

For buyouts, the SDC lists the acquirer's CUSIP and the market value before the merger. It also lists the year the merger occurred and the target's assets. In the CCM, there are multiple CUSIPs per year in both the SDC and the CRSP/Compustat. The multiple CUSIPs per year in the SDC are due to an acquirer completing multiple mergers in a year. The multiple CUSIPs per year in the CRSP/Compustat are due to firms with different permanent numbers having the same CUSIPs. Unfortunately, the SDC does not contain permanent numbers (lpermno) so we can only match via CUSIPs. As a result, all observations in both datasets in which the same CUSIP appeared multiple times in a year were dropped from both datasets. We then merged the CCM and SDC based on the CUSIP and year of the merger. The combined value was then the shares outstanding at end of the year of the merger (CSHO

[19] URL: http://www.stern.nyu.edu/~adamodar/pc/datasets/histretSP.xls

from the CCM data) times the calendar year closing price (PRCC_C from CCM).

For venture, we use the firms from CCM, only keeping data for the first year a firm appears in the CCM data, and defining IPO value as the shares outstanding at the end of that year (CSHO) times the closing price (PRCC_C).[20]

For both buyouts and venture, we further trim the sample based on firm's value of common stock at 95% level (i.e. observations at the bottom 2.5% and top 2.5% of common stock values are dropped). Our final sample contains 8,209 venture observations and 665 buyout observations. The annual data spans the periods from 1986 to 2017. For each venture and buyout funds in our sample, we compute the ε_v and ε_b as defined in Eqs. (46) and (47), where IPO and combined value are defined as CSHO times the PRCC_C. We truncate the distribution of ε_v and ε_b by $\frac{1}{q_t}$ and $\frac{q_t+\tau}{q_t}$, respectively, for each time t.

Appendix B. Proofs

B1: Proof of Proposition 2

Let L satisfy the equation

$$L = \left(\beta \int \left(\frac{1+q^{1-1/\gamma}L}{z+\sum \pi_j + \theta\kappa q\left[1-G^\text{b}\left(1+\frac{\tau}{q}\right)\right] + (1-\delta)\,q}\right)^\gamma [z+(1-\delta)\,q]\,dF\,(s)\right)^{1/\gamma}. \tag{A1}$$

First, suppose that a solution for L exists (its existence will be shown at the end of this proof). Since $\frac{C'}{C} = \frac{c'}{c}\frac{k'}{k}$, (28) implies

$$qc^{-\gamma}\left(\frac{k}{k'}\right)^{-\gamma} = \beta \int (c')^{-\gamma}\,[z'+(1-\delta)\,q']\,dF\,(s') \tag{A2}$$

Therefore

$$c\frac{k}{k'} = \left(\frac{\beta}{q}\int (c')^{-\gamma}\,[z'+(1-\delta)\,q']\,dF\,(s')\right)^{-1/\gamma}$$

where k'/k is defined in (25).

To simplify notation, we now omit the input of the function and denote $G \equiv G^\text{b}\left(1+\frac{\tau}{q}\right)$. From income identity (27)

$$i = \alpha\frac{z+\sum \pi_j + \theta\kappa q\,(1-G) - c}{q}$$

[20] We also examined the CRSP Delistings Data, results do not change.

thus we have

$$\begin{aligned} c\frac{k}{k'} &= \frac{c}{1-\delta+\frac{\alpha}{q}\left[z+\sum\pi_j+\theta\kappa q\left(1-G\right)-c\right]} \\ &= \frac{1}{\frac{1}{c}\left(1-\delta+\frac{\alpha}{q}\left(z+\sum\pi_j+\theta\kappa q\left(1-G\right)\right)\right)-\frac{1}{q}}, \end{aligned}$$

where the second line uses the identity (13) which implies $z-c=\frac{q}{\alpha}x$.

Therefore

$$\frac{1}{\frac{1}{c}\left(1-\delta+\frac{\alpha}{q}\left(z+\sum\pi_j+\theta\kappa q\left(1-G\right)\right)\right)-\frac{1}{q}} = \left(\frac{\beta}{q}\int\left(c'\right)^{-\gamma}\left[z'+\left(1-\delta\right)q'\right]dF\right)^{-1/\gamma}$$

i.e.,

$$\frac{1}{c}\left(1-\delta+\frac{\alpha}{q}\left(z+\sum\pi_j+\theta\kappa q\left(1-G\right)\right)\right)=\frac{1}{q}+\left(\frac{\beta}{q}\int\left(c'\right)^{-\gamma}\left[z'+\left(1-\delta\right)q'\right]dF\right)^{1/\gamma}$$

i.e.,

$$\begin{aligned} c &= \frac{1-\delta+\frac{\alpha}{q}\left(z+\sum\pi_j+\theta\kappa q\left(1-G\right)\right)}{\frac{1}{q}+\left(\frac{\beta}{q}\int\left(c'\right)^{-\gamma}\left[z'+\left(1-\delta\right)q'\right]dF\left(s'\right)\right)^{1/\gamma}} \\ &= \frac{z+\sum\pi_j+\theta\kappa q\left(1-G\right)+\left(1-\delta\right)q}{1+q^{1-1/\gamma}\left(\beta\int\left(c'\right)^{-\gamma}\left[z'+\left(1-\delta\right)q'\right]dF\left(s'\right)\right)^{1/\gamma}} \\ &= \frac{z+\sum\pi_i\left(q\right)+\theta\kappa q\left(1-G\right)+\left(1-\delta\right)q}{1+q^{1-1/\gamma}L} \end{aligned}$$

where L is defined in (A1), and

$$\begin{aligned} i &= \alpha\frac{z+\sum\pi_j+\theta\kappa q\left(1-G\right)-c}{q} \\ &= \alpha\frac{\left[z+\sum\pi_j+\theta\kappa q\left(1-G\right)\right]q^{-1/\gamma}L-\left(1-\delta\right)}{1+q^{1-1/\gamma}L}. \end{aligned}$$

Existence of solution for L in Eq. (29).—Next, we show that L exists when (30) holds. Divide both sides of (A1) by L to get

$$\begin{aligned} 1 &= \left(\beta\int L^{-\gamma}\left(\frac{1+q^{1-1/\gamma}L}{z+\sum\pi_j+\left(1-\delta\right)q}\right)^{\gamma}\left[z+\left(1-\delta\right)q\right]dF\right)^{1/\gamma} \\ &= \left(\beta\int\left(\frac{L^{-1}+q^{1-1/\gamma}}{z+\sum\pi_j+\left(1-\delta\right)q}\right)^{\gamma}\left[z+\left(1-\delta\right)q\right]dF\right)^{1/\gamma}. \end{aligned}$$

Since L^{-1} ranges from zero to infinity as L ranges over the positive line, and since $\gamma > 0$, a necessary and sufficient condition for a solution for L to exist is that

$$\left(\beta \int \left(\frac{q^{1-1/\gamma}}{z + \sum \pi_j + (1-\delta)\, q}\right)^{\gamma} [z + (1-\delta)\, q]\, dF\right)^{1/\gamma} \leq 1.$$

This is equivalent to

$$1 \geq \beta^{1/\gamma} \int q^{\gamma-1} \frac{z + (1-\delta)\, q}{\left(z + \sum \pi_j (q) + (1-\delta)\, q\right)^{\gamma}} dF. \tag{A3}$$

Since π_j and γ are positive, for (A3) to hold it suffices that

$$1 > \beta^{1/\gamma} \int q^{\gamma-1} \frac{z + (1-\delta)\, q}{\left(z + (1-\delta)\, q\right)^{\gamma}} dF = \beta^{1/\gamma} \int \left(1 - \delta + \frac{z}{q}\right)^{1-\gamma} dF,$$

i.e., (30).

B2: Derivations of Eqs. (41) and (43)

Derivation of (41).—We first show the derivation for venture. Since $1 - G^{\mathrm{v}}(\varepsilon_{\mathrm{v}}) = e^{-\lambda_{\mathrm{v}} \varepsilon_{\mathrm{v}}}$, the first equality in (40) reads

$$\frac{R_{\mathrm{v}}}{R_{\mathrm{E}}} = e^{\lambda_{\mathrm{v}} \varepsilon_{\mathrm{v}}} \lambda_{\mathrm{v}} \int_{\varepsilon_{\mathrm{v}}}^{\infty} \varepsilon e^{-\lambda_{\mathrm{v}} \varepsilon} d\varepsilon. \tag{A4}$$

By a change of variable from ε to $u = \varepsilon - \varepsilon_{\mathrm{v}}$, the RHS of (A4) reads

$$\lambda_{\mathrm{v}} e^{\lambda_{\mathrm{v}} \varepsilon_{\mathrm{v}}} \int_0^{\infty} (\varepsilon_{\mathrm{v}} + u)\, e^{-\lambda_{\mathrm{v}} (\varepsilon_{\mathrm{v}} + u)} du = \varepsilon_{\mathrm{v}} + \frac{1}{\lambda_{\mathrm{v}}}. \tag{A5}$$

Substitution for ε_{v} from (20) into (A5) yields the first equality in Eq. (41).

Next, the derivation for buyout. The second equality in (40) reads

$$\frac{1}{\tau + q} e^{\lambda_{\mathrm{b}} \varepsilon_{\mathrm{b}}} \lambda_{\mathrm{v}} \int_{\varepsilon_{\mathrm{b}}}^{\infty} \varepsilon e^{-\lambda_{\mathrm{b}} \varepsilon} d\varepsilon = \frac{1}{\tau + q} \left(\varepsilon_{\mathrm{b}} + \frac{1}{\lambda_{\mathrm{b}}}\right) = \frac{1}{\tau + q} \left(\frac{\tau + q}{q} + \frac{1}{\lambda_{\mathrm{b}}}\right),$$

i.e., the second equality in (41).

Derivation of (43). Since the parameter $\varepsilon_{\mathrm{i},\, 0}$ cancels from the ratio $(1 - G(\varepsilon_{\mathrm{i}}))\, dG^{\mathrm{i}} = \rho_{\mathrm{i}} \varepsilon^{-1-\rho_{\mathrm{i}}}$ for $i \in \{\mathrm{v}, \mathrm{b}\}$, the first equality in Eq. (40) reads

$$\frac{\mathrm{R}_{\mathrm{v}}}{R_{\mathrm{E}}} = \left(\frac{1}{q}\right)^{\rho_{\mathrm{v}}} \int_{1/q}^{\infty} \rho_{\mathrm{v}} \varepsilon^{-\rho_{\mathrm{v}}} d\varepsilon = \frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{v}} - 1} \left(\frac{1}{q}\right)^{\rho_{\mathrm{v}}} \left(\frac{1}{q}\right)^{1-\rho_v} = \frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{v}} - 1} \frac{1}{q}, \tag{A6}$$

The second equality in Eq. (40) reads

$$\frac{\mathrm{R_b}}{R_{\mathrm{E}}} = \left(\frac{\tau+q}{q}\right)^{\rho_{\mathrm{b}}} \frac{1}{\tau+q} \int_{(\tau+q)/q}^{\infty} \rho_{\mathrm{b}} \varepsilon^{-\rho_{\mathrm{b}}} d\varepsilon = \frac{\rho_{\mathrm{b}}}{\rho_{\mathrm{b}}-1} \left(\frac{\tau+q}{q}\right)^{\rho_{\mathrm{b}}} \frac{1}{\tau+q} \left(\frac{\tau+q}{q}\right)^{1-\rho_{\mathrm{b}}} = \frac{\rho_{\mathrm{b}}}{\rho_{\mathrm{b}}-1} \frac{1}{q}, \tag{A7}$$

i.e., Eq. (43).

Proof of Lemma 1. Because the ratio $\frac{R_v}{R_b}$ is increasing in q according to Equation (45), it's sufficient to prove this lemma by proving consumption c increases in q. Using equation (31) from proposition 2, we have,

$$\frac{\partial c}{\partial q} = \frac{\left\{ \begin{array}{c} \left(1+q^{1-1/\gamma}L\right) \left(\sum \pi_i'(q) + \theta\kappa \left[1 - G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right] + \theta\kappa g^{\mathrm{b}} \left(-\frac{\tau}{q^2}\right) + (1-\delta)\right) \\ -\left(1-\frac{1}{\gamma}\right) \left[\sum \pi_i(q) + \theta\kappa q \left[1 - G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right] + (1-\delta) q\right] q^{-1/\gamma} L \end{array} \right\}}{\left(1+q^{-1/\gamma}L\right)^2}.$$

After rearranging terms, the numerator can be expressed as

$$\left[\begin{array}{c} \left(\underbrace{ \begin{array}{c} \sum \pi_i'(q) + \theta\kappa \left[1 - G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right] \\ +\theta\kappa g^{\mathrm{b}}\left(-\frac{\tau}{q^2}\right) + (1-\delta) \end{array} }_{>0} \right) + q^{-1/\gamma} L\theta\kappa q \underbrace{g^{\mathrm{b}}\left(-\frac{\tau}{q^2}\right)}_{\geq 0} \\ +\frac{1}{\gamma} \left[\underbrace{\sum \pi_i(q) + \theta\kappa q \left[1 - G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right] + (1-\delta) q}_{>0} \right] q^{-1/\gamma} L \end{array} \right]$$
$$> 0$$

where the first term of the is positive because, by Leibnitz's rule, $\pi_b'(q) = \theta\kappa \int_{\varepsilon_{\mathrm{b}}}^{\infty} \varepsilon dG^{\mathrm{b}} + \theta\kappa q \int_{\varepsilon_{\mathrm{b}}}^{\infty} \varepsilon dG^{\mathrm{b}} \left(\frac{\tau}{q^2}\right) > 0$ and $\pi_v'(q) = \lambda\kappa \int_{\varepsilon_v}^{\infty} \varepsilon dG^{\mathrm{v}} + \lambda\kappa q \int_{1/q}^{\infty} \varepsilon dG^{\mathrm{b}} \left(\frac{1}{q^2}\right) > 0$. The second is non-negative because $g^{\mathrm{b}}(\cdot) \geq 0$. The last term is positive because $G^{\mathrm{b}} \in [0,1]$. Thus $\frac{\partial c}{\partial q} > 0$.

B3: Equilibrium Growth in Deterministic Case

When $\gamma = 1$, z and q are constant, Eq. (A1) becomes

$$L\left(z + \sum \pi_j + (1-\delta) q\right) = \beta(1+L)\left[z + (1-\delta) q\right].$$

This implies that

$$\begin{aligned} L &= \frac{\beta\left[z+(1-\delta)\,q\right]}{\left(z+\sum\pi_j+(1-\delta)\,q\right)-\beta\left[z+(1-\delta)\,q\right]} \\ &= \frac{\beta}{1-\beta+\left[z+(1-\delta)\,q\right]^{-1}\sum\pi_j}. \end{aligned}$$

We denote

$$\mu=\left[z+(1-\delta)\,q\right]^{-1},$$

then L can be expressed as,

$$L=\frac{\beta}{1-\beta+\mu\sum\pi_j}.$$

We then have

$$i=\alpha\frac{\left(z+q\theta\kappa\int_{\varepsilon_{\mathrm{b}}}^{\infty}\varepsilon dG^{\mathrm{b}}\right)q^{-1}\beta-q\,(1-\delta)\left(1-\beta+\mu\sum\pi_j\right)}{1+\mu\sum\pi_j}.$$

Using (16) and (21), we have

$$\begin{aligned} \sum\pi_j &= q\theta\kappa\int_{\varepsilon_{\mathrm{b}}}^{\infty}\varepsilon dG^{\mathrm{b}}-(\tau+q)\,\theta\kappa\left[1-G^{\mathrm{b}}\left(\varepsilon_{\mathrm{b}}\right)\right] \\ &\quad +q\lambda\kappa\int_{\varepsilon_{\mathrm{v}}}^{\infty}\varepsilon dG^{\mathrm{v}}-\lambda\kappa\left[1-G^{\mathrm{v}}\left(\varepsilon_{\mathrm{v}}\right)\right] \\ &= q\kappa\left(\frac{\theta}{\rho_{\mathrm{b}}-1}\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{1-\rho_{\mathrm{b}}}+\frac{\lambda}{\rho_{\mathrm{v}}-1}\left(\frac{1}{q\varepsilon_{\mathrm{v},\,0}}\right)^{1-\rho_v}\right). \end{aligned}$$

This implies

$$\begin{aligned} i &= \alpha\frac{\left(z+q\theta\kappa\int_{\varepsilon_{\mathrm{b}}}^{\infty}\varepsilon dG^{\mathrm{b}}\right)q^{-1}\beta-q\,(1-\delta)\left(1-\beta+\mu\sum\pi_j\right)}{1+\mu\sum\pi_j} \\ &= \alpha\frac{\left(z+\sum\pi_i+\theta\kappa q\left(1+\frac{\tau}{q\varepsilon_{\mathrm{b},\,0}}\right)^{-\rho_b}\right)q^{-1}\beta-q\,(1-\delta)\left(1-\beta+\mu\sum\pi_j\right)}{1+\mu\sum\pi_j}. \end{aligned}$$

Thus the growth $g=i-\delta$ becomes,

$$g=\alpha\frac{\left(z+\nu+\theta\kappa q\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{-\rho_b}\right)q^{-1}\beta-q\,(1-\delta)\,(1-\beta+\mu\nu)}{1+\mu\nu}-\delta,$$

where

$$\nu = q\kappa\left(\frac{\theta}{\rho_{\mathrm{b}}-1}\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{1-\rho_{\mathrm{b}}}+\frac{\lambda}{\rho_{\mathrm{v}}-1}\left(\frac{1}{q\varepsilon_{\mathrm{v},\,0}}\right)^{1-\rho_{v}}\right).$$

As for the comparative statics, first we have

$$\frac{\partial g}{\partial z} = \alpha q^{-1}\beta > 0$$

Next, we observe that,

$$\begin{aligned}\frac{\partial \nu}{\partial \rho_b} &= -q\kappa\theta\frac{\left[\ln\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)(\rho_b-1)+1\right]}{(\rho_{\mathrm{b}}-1)^2}\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{1-\rho_{\mathrm{b}}}\\ &< 0\,.\end{aligned}$$

The last line follows because $\varepsilon_{\mathrm{b},\,0} < \frac{\tau+q}{q}$ so that $\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}} > 1$ or $\ln\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right) > 0$. Therefore, we have,

$$\frac{\partial \nu}{\partial \rho_b} < 0.$$

Similarly,

$$\frac{\partial \nu}{\partial \rho_v} = -q\kappa\lambda\frac{\left[\ln\left(\frac{1}{q\varepsilon_{\mathrm{v},\,0}}\right)(\rho_v-1)+1\right]}{(\rho_{\mathrm{v}}-1)^2}\left(\frac{1}{q\varepsilon_{\mathrm{v},\,0}}\right)^{1-\rho_{\mathrm{b}}} < 0,$$

where the inequality follows because $\frac{1}{q\varepsilon_{\mathrm{v},\,0}} > 1$ or $\ln\left(\frac{1}{q\varepsilon_{\mathrm{v},\,0}}\right) > 0$. This implies that

$$\begin{aligned}\frac{\partial g}{\partial \nu} &= \left(q^{-1}\beta - q(1-\delta)\mu\right)(1+\mu\nu)\\ &\quad -\mu\left[\left(z+\nu+\theta\kappa q\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{-\rho_b}\right)q^{-1}\beta - q(1-\delta)(1-\beta+\mu\nu)\right]\\ &= q^{-1}\beta\left[1+\mu\nu-\mu\left(z+\nu+\theta\kappa q\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{-\rho_b}\right)\right]+q(1-\delta)\beta\mu\\ &= q^{-1}\beta\left[\underbrace{\frac{(1-\delta)q}{z+(1-\delta)q}+\theta\mu\kappa q\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{-\rho_b}}_{>0}\right]+q(1-\delta)\beta\left[z+(1-\delta)q\right]^{-1}\\ &> 0,\end{aligned}$$

which implies that

$$\begin{aligned}\frac{\partial g}{\partial \rho_v} &= \frac{\partial g}{\partial \nu}\frac{\partial \nu}{\partial \rho_v} < 0\\ \frac{\partial g}{\partial \rho_b} &= \frac{\partial g}{\partial \nu}\frac{\partial \nu}{\partial \rho_b} < 0.\end{aligned}$$

Similarly,

$$\frac{\partial \nu}{\partial \tau} = -\kappa\theta \left(\frac{\tau + q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{-\rho_{\mathrm{b}}} < 0,$$

as stated in the proposition.

B4: Taylor Approximation for the Growth

We consider a first-order Taylor expansion of the growth $g(\lambda, \theta)$ at $\lambda = \theta = 0$. First we notice that

$$v(0,0) = 0.$$

Therefore we have,

$$g(0,0) = \alpha z q^{-1}\beta - q\alpha(1-\delta)(1-\beta) - \delta.$$

The first-order Taylor approximation reads,

$$g = g(0,0) + g_{\lambda}(0,0)\,\lambda + g_{\theta}(0,0)\,\theta.$$

Note that,

$$v_{\lambda} = \frac{q\kappa}{\rho_{\mathrm{v}} - 1}\left(\frac{1}{q\varepsilon_{\mathrm{v},\,0}}\right)^{1-\rho_{v}} \tag{A8}$$

$$v_{\theta} = \frac{q\kappa}{\rho_{\mathrm{b}} - 1}\left(\frac{\tau + q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{1-\rho_{\mathrm{b}}}. \tag{A9}$$

Therefore, we have

$$\begin{aligned} g_{\lambda}(0,0) &= \frac{\alpha\left\{\begin{array}{c}\left[1 + [z + (1-\delta)q]^{-1}\nu\right](q^{-1}\beta\nu_{\lambda} - q(1-\delta)\mu v_{\lambda}) \\ -\left\{\begin{array}{c}\left(z + \nu + \theta\kappa q\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{-\rho_b}\right)q^{-1}\beta \\ -q(1-\delta)(1-\beta+\mu\nu)\end{array}\right\}\left[[z + (1-\delta)q]^{-1}\nu_{\lambda}\right]\end{array}\right\}}{\left[1 + [z + (1-\delta)q]^{-1}\nu\right]^{2}} \\ &= \alpha\left(q^{-1}\beta\nu_{\lambda} - q(1-\delta)\mu v_{\lambda}\right) - \alpha\left\{zq^{-1}\beta - q(1-\delta)(1-\beta)\right\}[\mu\nu_{\lambda}] \\ &= \frac{(1-\delta)q}{z + (1-\delta)q}v_{\lambda}\alpha q^{-1}\beta - \alpha q(1-\delta)\beta(\mu\nu_{\lambda}). \end{aligned}$$

Similarly,

$$\begin{aligned} g_{\theta}(0,0) &= \alpha\left\{\begin{array}{c}\left(\left(v_{\theta} + \kappa q\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{-\rho_b}\right)q^{-1}\beta - q(1-\delta)\left([z + (1-\delta)q]^{-1}\nu_{\theta}\right)\right) \\ -\left[zq^{-1}\beta - q(1-\delta)(1-\beta)\right][z + (1-\delta)q]^{-1}\nu_{\theta}\end{array}\right\} \\ &= \frac{(1-\delta)q}{z + (1-\delta)q}\left[\left(v_{\theta} + \kappa q\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{-\rho_b}\right)\right]\alpha q^{-1}\beta - \alpha q(1-\delta)\beta(\mu\nu_{\theta}), \end{aligned}$$

so we have

$$
\begin{aligned}
g &\approx \left(z+\frac{\lambda(1-\delta)q}{z+(1-\delta)q}v_\lambda+\frac{\theta(1-\delta)q}{z+(1-\delta)q}\left(v_\theta+\kappa q\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{-\rho_b}\right)\right)\alpha q^{-1}\beta \\
&\quad -q\alpha(1-\delta)(1-\beta-\beta(\mu\lambda\nu_\lambda)-\beta(\mu\theta\nu_\theta))-\delta \\
&= \left(z+\frac{\lambda(1-\delta)q}{z+(1-\delta)q}v_\lambda+\frac{\theta(1-\delta)\rho_b q}{z+(1-\delta)q}v_\theta\right)\alpha q^{-1}\beta-q\beta\alpha(1-\delta)\left[\frac{1-\beta}{\beta}-\mu v\right]-\delta,
\end{aligned}
$$

where the last line follows

$$\lambda\nu_\lambda+\theta\nu_\lambda=v.$$

Now we plug in (A8) and (A9), and get

$$
\begin{aligned}
g &\approx \alpha q^{-1}\beta\left(z+\frac{\lambda(1-\delta)q}{z+(1-\delta)q}\frac{q\kappa}{\rho_{\mathrm{v}}-1}\left(\frac{1}{q\varepsilon_{\mathrm{v},\,0}}\right)^{1-\rho_v}+\frac{\theta(1-\delta)q}{z+(1-\delta)q}\frac{q\kappa\rho_b}{\rho_{\mathrm{b}}-1}\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{1-\rho_{\mathrm{b}}}\right) \\
&\quad -q\beta\alpha(1-\delta)\left[\frac{1-\beta}{\beta}-\mu v\right]-\delta.
\end{aligned}
$$

We then get Eq. (36) using the following notations

$$
\begin{aligned}
\tilde{A} &= q^{-1}\beta\left(z+\frac{(1-\delta)q}{z+(1-\delta)q}\left[\frac{\lambda q\kappa}{\rho_{\mathrm{v}}-1}\left(\frac{1}{q\varepsilon_{\mathrm{v},\,0}}\right)^{1-\rho_v}\right]+\frac{(1-\delta)q}{z+(1-\delta)q}\left[\frac{\theta q\kappa\rho_b}{\rho_{\mathrm{b}}-1}\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{1-\rho_{\mathrm{b}}}\right]\right) \\
a &= q\beta(1-\delta)\left(\frac{1-\beta}{\beta}-[z+(1-\delta)q]^{-1}v\right).
\end{aligned}
$$

Last, we notice that

$$
\begin{aligned}
\frac{\partial\tilde{A}}{\partial q} &= \left[\begin{array}{c}
-\frac{z\beta}{q^2}-\frac{(1-\delta)q}{z+(1-\delta)q}\left[\underbrace{\lambda\kappa\left(\frac{1}{q\varepsilon_{\mathrm{v},\,0}}\right)^{-\rho_v}\frac{1}{q^2\varepsilon_{\mathrm{v},\,0}}+\theta\rho_b\kappa\left(\frac{\tau}{q\varepsilon_{\mathrm{b},\,0}}\right)^{-\rho_b}\frac{\tau+q}{q^2\varepsilon_{\mathrm{v},\,0}}}_{>0}\right] \\
-\frac{q^{-2}}{\left[z(1-\delta)^{-1}q^{-1}+1\right]^2}\left[\underbrace{\frac{\lambda q\kappa}{\rho_{\mathrm{v}}-1}\left(\frac{1}{q\varepsilon_{\mathrm{v},\,0}}\right)^{1-\rho_v}+\frac{\theta q\kappa\rho_b}{\rho_{\mathrm{b}}-1}\left(\frac{\tau+q}{q\varepsilon_{\mathrm{b},\,0}}\right)^{1-\rho_{\mathrm{b}}}}_{>0\text{ because }\rho_v>1\text{ and }\rho_b>1}\right]
\end{array}\right] \\
&< 0
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial a}{\partial q} &= (1-\beta)\beta(1-\delta)+\frac{\beta(1-\delta)q^{-2}}{(zq^{-1}+(1-\delta))^2}v \\
&> 0\text{ because }v>0.
\end{aligned}
$$

Therefore we have

$$\frac{\partial g}{\partial q} = \alpha \left(\underbrace{\frac{\partial \tilde{A}}{\partial q}}_{<0} - \underbrace{\frac{\partial a}{\partial q}}_{>0} \right) < 0$$

as stated in Proposition 3 and corollary 1.

B5: Value Function Under EZ Preferences

Nothing in Proposition 1 changes with EZ preferences, so the proof of Proposition 1 holds.

As for the value function, we first notice that the resource constraint is

$$z + \pi_v + \pi_b + \frac{q}{\tau + q} n_b = c + \frac{q}{\alpha} i. \tag{A10}$$

The law of motion is

$$\frac{k'}{k} = 1 - \delta + i. \tag{A11}$$

Combining (A10) and (A11), we get

$$c = z - \frac{q}{\alpha}(\Gamma - 1 + \delta) + \pi_v + \pi_b + \frac{q}{\tau + q} n_b. \tag{A12}$$

We will verify that the value function takes the conjectured form:

$$\begin{aligned} v(z,q)k &= \max_{C,I} \left[(1-\beta)(C)^{1-\psi} + \beta \left(E[v(z',q')k']'^{1-\gamma} \right)^{\frac{1-\psi}{1-\gamma}} \right]^{\frac{1}{1-\psi}} \\ &= \max_{C,I} \left[(1-\beta)(ck)^{1-\psi} + \beta \left(E[v(z',q')\Gamma k]^{1-\gamma} \right)^{\frac{1-\psi}{1-\gamma}} \right]^{\frac{1}{1-\psi}}. \end{aligned}$$

Using Eq. (A12), then

$$\begin{aligned} v(z,q)k &\equiv \max_{\Gamma} \left[\begin{array}{c} (1-\beta) \left(\left(z - \frac{q}{\alpha}(\Gamma - 1 + \delta) + \pi_v + \pi_b + \frac{q}{\tau+q} n_b \right) k \right)^{1-\psi} \\ +\beta \left(E(v(z',q')\Gamma k)^{1-\gamma} \right)^{\frac{1-\psi}{1-\gamma}} \end{array} \right]^{\frac{1}{1-\psi}} \\ &= \max_{\Gamma} \left[\begin{array}{c} (1-\beta) \left(z - \frac{q}{\alpha}(\Gamma - 1 + \delta) + \pi_v + \pi_b + \frac{q}{\tau+q} n_b \right)^{1-\psi} k^{1-\psi} \\ +\beta \left(E(v(z',q)\Gamma)'^{1-\gamma} \right)^{\frac{1-\psi}{1-\gamma}} k^{1-\psi} \end{array} \right]^{\frac{1}{1-\psi}} \\ &= \max_{\Gamma} \left[\begin{array}{c} (1-\beta) \left(z - \frac{q}{\alpha}(\Gamma - 1 + \delta) + \pi_v + \pi_b + \frac{q}{\tau+q} n_b \right)^{1-\psi} \\ +\beta \left(E(v(z',q)\Gamma)'^{1-\gamma} \right)^{\frac{1-\psi}{1-\gamma}} \end{array} \right]^{\frac{1}{1-\psi}} k. \end{aligned}$$

Therefore, we define

$$v(z,q)=\max_{\Gamma}\left[\begin{array}{c}(1-\beta)\left(z-\frac{q}{\alpha}(\Gamma-1+\delta)+\pi_v+\pi_b+\frac{q}{\tau+q}n_b\right)^{1-\psi}\\+\beta\left(E(v(z',q')\Gamma)^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}}\end{array}\right]^{\frac{1}{1-\psi}} \tag{A13}$$

so that $v(z,q,k)=v(z,q)k$ as conjectured.

B6: Proof of Proposition 5

Taking first order conditions from A13:

$$\left[(1-\beta)(1-\psi)\left(z-\frac{q}{\alpha}(\Gamma-1+\delta)+\pi_v+\pi_b+\frac{q}{\tau+q}n_b\right)^{-\psi}\frac{q}{\alpha}\right]=\beta(1-\psi)\Gamma^{-\psi}\left(Ev(z',q')^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}},$$

this implies that

$$\Longrightarrow(1-\beta)\frac{q}{\alpha}c^{-\psi}=\beta\Gamma^{-\psi}\left(Ev(z',q')^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}}.$$

Substituting our conjectures:

$$\begin{aligned}&(1-\beta)\frac{q}{\alpha}\left(\frac{z+\sum\pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+(1-\delta)q}{1+q^{1-1/\psi}L}\right)^{-\psi}\\=\;&\beta\left(1-\delta+\frac{\left(z+\sum\pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]\right)q^{-1/\psi}L-(1-\delta)}{1+q^{1-1/\psi}L}\right)^{-\psi}\\&\times\left(Ev(z',q')^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}}\end{aligned}$$

and multiplying both sides by $(1+q^{1-\frac{1}{\psi}}L)^{-\psi}$

$$\begin{aligned}&(1-\beta)\frac{q}{\alpha}\left(z+\sum\pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+(1-\delta)q\right)^{-\psi}\\=\;&\beta\left(\left(z+\sum\pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]\right)q^{-1/\psi}L-(1-\delta)+(1-\delta)(1+q^{1-\frac{1}{\psi}}L)\right)^{-\psi}\\&\times\left(Ev(z',q')^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}},\end{aligned}$$

this implies that,

$$\begin{aligned}\Longrightarrow\;&(1-\beta)\frac{q}{\alpha}\left(z+\sum\pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+(1-\delta)q\right)^{-\psi}\\=\;&\beta\left(\left(z+\sum\pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]\right)q^{-1/\psi}L+(1-\delta)q^{1-\frac{1}{\psi}}L)\right)^{-\psi}\\&\times\left(Ev(z',q')^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}}.\end{aligned}$$

Multiplying both sides by L^{ψ}, we have

$$
\begin{aligned}
& (1-\beta)\frac{q}{\alpha}\left(z+\sum\pi_i\left(q\right)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+\left(1-\delta\right)q\right)^{-\psi}L^{\psi} \\
= \; & \beta\left(\left(z+\sum\pi_i\left(q\right)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]\right)q^{-1/\psi}+(1-\delta)q^{1-\frac{1}{\psi}})\right)^{-\psi} \\
& \times\left(Ev(z',q')^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}},
\end{aligned}
$$

or equivalently,

$$
L^{\psi}=\frac{\beta}{1-\beta}\alpha\left(Ev(z',q')^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}}.
$$

This delivers a recursion for L that satisfies the FOC.

Now we need to verify that (48) holds. Notice that

$$
\begin{aligned}
v(z,q) &= \left[(1-\beta)\left(z+\sum\pi_i\left(q\right)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+\left(1-\delta\right)q\right)^{1-\psi}(1+q^{1-1/\psi}L)^{\psi}\right]^{\frac{1}{1-\psi}} \\
&= \left[(1-\beta)\left(z+\sum\pi_i\left(q\right)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+\left(1-\delta\right)q\right)\right]^{\frac{1}{1-\psi}} \\
&\quad\times\left[\left(\frac{z+\sum\pi_i\left(q\right)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+\left(1-\delta\right)q}{(1+q^{1-1/\psi}L)}\right)^{-\psi}\right]^{\frac{1}{1-\psi}} \\
&= \left[(1-\beta)\left(z+\sum\pi_i\left(q\right)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+\left(1-\delta\right)q\right)c^{-\psi}\right]^{\frac{1}{1-\psi}}
\end{aligned}
$$

If our value function guess holds, then (31) implies

$$
\begin{aligned}
& \left[(1-\beta)\left(z+\sum\pi_i\left(q\right)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+\left(1-\delta\right)q\right)c^{-\psi}\right]^{\frac{1}{1-\psi}} \\
= \; & \left[(1-\beta)c^{1-\psi}+\beta\Gamma^{1-\psi}\left(Ev(z',q')^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}}\right]^{\frac{1}{1-\psi}}.
\end{aligned}
$$

This implies that,

$$
\begin{aligned}
\Longrightarrow \; & (1-\beta)\left(z+\sum\pi_i\left(q\right)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+\left(1-\delta\right)q\right)c^{-\psi} \\
= \; & (1-\beta)c^{1-\psi}+\beta\Gamma^{1-\psi}\left(Ev(z',q')^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}}.
\end{aligned}
$$

Or equivalently,

$$\Longrightarrow \quad \frac{z+\sum \pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+(1-\delta)q}{c} = 1+\left(\frac{\Gamma}{c}\right)^{1-\psi}\frac{\beta}{1-\beta}\left(Ev(z',q')^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}}$$

$$\Longrightarrow \quad \frac{z+\sum \pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+(1-\delta)q}{c} = 1+\left(\frac{\Gamma}{c}\right)^{1-\psi}L^{\psi}$$

Plugging the conjecture for consumption on the LHS:

$$1+q^{1-\frac{1}{\psi}}L = 1+\left(\frac{\Gamma}{c}\right)^{1-\psi}L^{\psi}$$

which implies

$$q^{-\frac{1}{\psi}}L = \frac{\Gamma}{c} \tag{A14}$$

Using the conjectures for c and Γ, we can write the ratio $\frac{\Gamma}{c}$ as

$$\begin{aligned}
\frac{g}{c} &= \left(\frac{z+\sum \pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+(1-\delta)q}{1+q^{1-\frac{1}{\psi}}L}\right)^{-1} \\
&\times\left(1-\delta+\frac{\left(z+\sum \pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]\right)q^{-\frac{1}{\psi}}L-(1-\delta)}{1+q^{1-\frac{1}{\psi}}L}\right) \\
&= \frac{(1-\delta)(1+q^{1-1/\psi}L)+\left(z+\sum \pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]\right)q^{-\frac{1}{\psi}}L-(1-\delta)}{z+\sum \pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+(1-\delta)q} \\
&= \frac{(1-\delta)q^{1-1/\psi}L+\left(z+\sum \pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]\right)q^{-\frac{1}{\psi}}L}{z+\sum \pi_i(q)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+(1-\delta)q} \\
&= q^{-\frac{1}{\psi}}L
\end{aligned}$$

From Eqs. (A14) and (30) it is clear that the conjectured decision rules satisfy the problem.

Last, from the budget constraint, we have

$$i = \alpha\frac{z+\sum \pi_i(q)+\theta\kappa q\left(1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right)-c}{q}$$

and we plug in $c=\frac{z+\sum \pi_i(q)+\theta\kappa q(1-G)+(1-\delta)q}{1+q^{1-1/\psi}L}$, we have

$$\begin{aligned}
i &= \alpha\frac{q^{1-1/\psi}L\left[z+\sum \pi_i(q)+\theta\kappa q\left(1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right)\right]-(1-\delta)q}{q\left(1+q^{1-1/\psi}L\right)} \\
&= \alpha\frac{\left[z+\sum \pi_i(q)+\theta\kappa q\left(1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right)\right]q^{-1/\psi}L-(1-\delta)}{1+q^{1-1/\psi}L}
\end{aligned}$$

as stated in the proposition.

Proof that $L_{\mathrm{EZ}} = L$ when $\psi = \gamma$.—To show that $L_{\mathrm{EZ}} = L$, we set $\psi = \gamma$ in Eq. (50), which then reads

$$L_{\mathrm{EZ}} = \left[\beta \int \frac{\alpha}{1-\beta} v(z,q)^{1-\gamma} dF\right]^{1/\gamma}, \tag{A15}$$

$$\text{where } v(z,q)^{1-\gamma} = \frac{(1-\beta)\left(z+(1-\delta)\,q\right)\left(1+q^{1-1/\gamma}L_{\mathrm{EZ}}\right)^{\gamma}}{\left[z+\sum \pi_i\left(q\right)+\theta\kappa q\left[1-G^{\mathrm{b}}\left(1+\frac{\tau}{q}\right)\right]+(1-\delta)\,q\right]^{\gamma}}. \tag{A16}$$

Substituting from (A16) into (A15) and simplifying makes the resulting equation the same as Eq. (29) in Proposition 2.

Appendix C: Additional Tables and Figures

Table A1. Parameters for the Estimation under EZ Preferences, Freely Estimate (γ, ψ)

Pre-set Parameters			
Capital Share	Discount Rate	Depreciation Rate	
α	β	δ	
0.25	0.95	0.05	
Estimated Parameters			
Average VC Quality	SD of VC Quality	Average BO Quality	SD of BO Quality
μ_{v}	σ_{v}	μ_{b}	σ_{b}
-1.40	1.31	-1.06	2.75
$[-1.51, -1.30]$	$[0.86, 1.76]$	$[-1.12, -1.01]$	$[2.57, 2.94]$
Arrival Rate of VC Idea	BO Implementation Cost	Arrival Rate of BO Upgrade	Human Capital Ratio
λ	τ	θ	κ
0.95	0.82	0.12	0.03
$[0.93, 0.97]$	$[0.62, 0.99]$	$[0.04, 0.20]$	$[-0.90, 0.95]$
Curvature Parameter	EIS		
γ	$1/\psi$		
27.91	0.68		
$[12.79, 40, 35]$	$[0.31, 1.21]$		

Note: This table reports the parameters for the estimation under EZ preferences when both γ and ψ are freely estimated. The GMM 95% confidence intervals are reported in brackets. The volatility of R_f is added as an additional target.

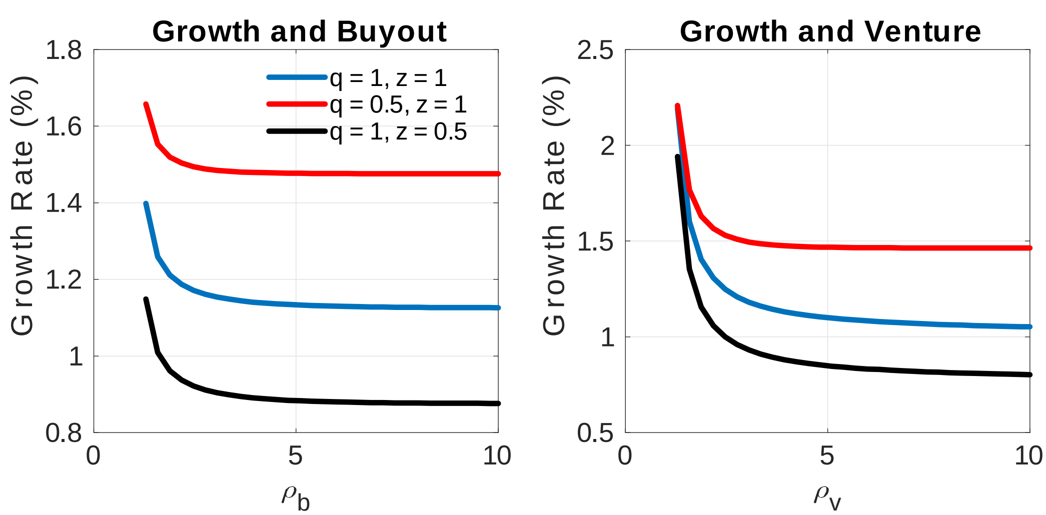

Figure A1. Growth and PE, A numerical example