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OBSOLESCENCE OF CAPITAL AND INVESTMENT SPIKES

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ABSTRACT

The prospect of capital obsolescence inhibits investment. Investors thus become more optimistic when the obsolescence of their capital slows down. We propose a model with no fixed costs of investment, and random technological progress that induces obsolescence of capital in place. Spikes occur precisely when technological progress slows down. Moreover, the more variable the progress, the larger are the spikes. Cross-industry data show that where price of capital declines are more variable, investment spikes are larger.

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1 Introduction

The cost of capital has been declining in most industries as documented by Cummins and Violante (2002). The rate of decline and, in particular, the variability over time of that decline varies across industries. This paper shows that this behavior of capital goods prices can lead to investment spikes as measured e.g., by Doms and Dunne (1993), Ilyina and Samaniego (2011) and Kehrig and Vincent (2018).

We model an industry that may be competitive or a monopoly in which, as a result of technological progress the cost of capital declines exogenously and irregularly over time. Innovations in the capital-goods producing industries reduce the cost of investment for a while, but then the phase of decline ends and costs of investment remain constant until the next innovation phase starts. The model's three main implications are investment spikes, a rising inter-spike hazard, and a coexistence of falling costs of capital with investment inactivity; let us outline them in turn.

Investment spikes occur when the rate of progress slows down. The intuition is that declining investment costs imply declining product prices and represent bad news for owners of capital and makes investment relatively unattractive. First, as costs are declining, prices of output are expected to decline as well, which reduces the present value of investment. Second, as costs will be lower tomorrow, it is better to postpone investment till then. Once the rate of progress slows down, investment becomes more attractive, hence the spike.

The inter-spike hazard rises because after the spike, once costs begin to decline again, as time passes, the cost is more and more likely to reach the point where it will again be profitable to invest.

When investment is irreversible, periods of zero investment coexisting with falling costs of capital arise in our model because, following a spike, the value of capital declines below its cost. Thus investment only becomes profitable again once the cost of capital drops sufficiently to catch up.

These phenomena have been shown to also arise for other reasons. The leading explanation is that there are fixed costs of investment. Cooper and Haltiwanger (2006), Caballero and Engel (2000), Khan and Thomas (2008) and Jovanovic and Stolyarov (2000) all associate spikes with fixed investment costs. A rising inter-spike hazard obtains because the passage of time generally raises the distance between the actual and desired capital stocks, and in the simplest version of the models, between spikes there is no investment.

Second, vintage-capital models can generate spikes in the form of capital replacement echoes, as Boucekkine, Germain and Licandro (1997) and Jovanovic and Tse (2011) show. In those models, the inter-spike hazard falls as less productive or more labor intensive older vintages become increasingly unprofitable.

Third, demand shocks and productivity shocks can cause spikes. Guo, Miao, and Morellec (2005) generate spikes from positive surprises in demand growth, and Abel and Eberly (2012) generate them when the productivity of capital jumps up discretely. By contrast, in our model it is a drop in the rate of technological progress that causes the spike. Moreover, we show that this occurs even when the drop in technological progress is fully foreseen. Related, Jorgenson (1963) shows that when capital investment is reversible, changes in the rate of capital obsolescence can lead to sudden changes in the capital stock, but not to a coexistence between declining capital goods prices and an inactivity of investment.

Fourth, cyclical variation in product demand has been shown to affect the occurrence of spikes. In a fixed-cost model Caballero and Hammour (1996) find that reallocation should occur during recessions, when the opportunity cost of doing so is lowest. And in a vintage capital models in which the productivity of a new vintage depends on learning by doing, Klenow (1998) finds that technology updates are more likely in a boom than in a recession since a high rate of production enables faster learning. Cooper, Haltiwanger and Power (1999) incorporate both vintage effects and fixed costs and find that spikes are procyclical.

In our model, the presence of spikes requires that there be variability and persistence in the rate of decline of capital-goods prices. In particular, without fixed costs, no investment spikes occur when capital prices decline steadily. By contrast, in the fixed-cost model, a faster decline in the cost of capital relative to revenues raises the desired capital stock faster and produces more frequent spikes. And in vintage capital models a steady rise in the quality of capital causes more frequent replacement echoes. Our evidence shows that at the industry level, higher variability positively affects the frequency of spikes and thier size.

The plan of the paper is as follows: Section 2 describes the model and derives some of its properties. Section 3 shows that the implications are robust with respect to having reversible investment and physical depreciation, to adding monopoly power, and that if modified to include idiosyncratic productivity shocks it can generate differences between firm-level and industry-level spikes. Section 4 reports tests of some of the implications, and a more detailed comparison to what other models imply, and Section 5 concludes the paper. The Appendix contains most of the proofs.

2 A competitive industry

We formulate the model in continuous time so that spikes are jumps in the capital stock which otherwise changes continuously over time.

There is a competitive industry with free entry in which each unit of capital invested produces one unit of a homogenous product in perpetuity at marginal cost zero. Investment is irreversible; we discuss the implications of reversible investment in Section 3.1.

To highlight the effect of value obsolescence, we assume that capital does not depreciate. The accumulated capital stock at date t is k_t, and investment at date t is denoted $x_t = dk_t/dt$. The market price of output at any date is determined as

$$p_t = D(k_t). \tag{1}$$

where D is a decreasing demand function. By k we mean capital held collectively by all the measure-zero firms in the industry, i.e., $k = \int k(i)\, di$.

Value of capital in place.—Conditional on the output-price sequence, at date t this value is

$$v_t = E_t \int_t^\infty e^{-r(s-t)} p_s ds. \tag{2}$$

where E_t is the expectation operator as of date t, and r is the rate of interest.

Exogenous technological progress.—The cost of capital declines at the rate g_t so that

$$c_t = c_0 \exp\left(-\int_0^t g_s ds\right). \tag{3}$$

We assume that there are two levels for g and we normalize the lower one to be zero: $g_t \in \{0, g\}$. When $g_t = 0$, the cost of capital is "stagnant," whereas when $g_t = g > 0$, the cost is declining and we have "progress."

Since there are no adjustment costs and the industry is competitive,

$$x_t > 0 \Rightarrow v_t = c_t, \tag{4}$$

and

$$v_t < c_t \Rightarrow x_t = 0.$$

Definition of a spike.—An investment spike is a discontinuity in k_t and p_t; an upward jump in k_t and a downward jump in p_t.

Let $T_1, T_2, ...$ be exogenous *transition dates* at which costs stop declining and become stagnant, i.e., g_t falls from g to 0 at T_n. Let u_n be the duration of the n'th epoch of stagnation, i.e., $g_t = 0$ from T_n until $T_n + u_n$. Let $w_n > 0$ be the duration of the nth epoch of cost decline, i.e., $g_t = g$ from $T_n + u_n$ until $T_n + u_n + w_n = T_{n+1}$.

Let the transition dates be unknown and assume that w_n and u_n are random so that T_n are random as well. Specifically, we assume that w_n are i.i.d. with CDF $1 - e^{-\lambda_\mathrm{L} t}$, and u_n are i.i.d. with CDF $1 - e^{-\lambda_\mathrm{H} t}$. In other words the hazard of escaping from stagnation is λ_L and the hazard of termination of progress is λ_H. We sometimes refer to epochs of declining costs as state H and epochs of stagnant costs as state L.

Let v_t^{L} and v_t^{H} denote the values of capital in the two technological states. Then for $i, j \in \{L, H\}$, Eq. (2) reads

$$v_t^{\mathrm{i}} = \int_t^{\infty} \left(\int_t^T e^{-r(s-t)} p_s ds + e^{-r(T-t)} v_T^{\mathrm{j}} \right) \lambda_{\mathrm{i}} e^{-\lambda_{\mathrm{i}}(T-t)} dT. \tag{5}$$

Given $(p_t)_{t=0}^{\infty}$, the function v_t^{i} is differentiable in t, and satisfies the Bellman equations

$$r v_t^{\mathrm{i}} = p_t + \lambda_{\mathrm{i}} \left(v_t^{\mathrm{j}} - v_t^{\mathrm{i}} \right) + \frac{d v_t^{\mathrm{i}}}{dt}. \tag{6}$$

Lemma 1 *For $t \in (T_n, T_n + u_n]$*

$$x_t = 0. \tag{7}$$

Proof. The escape hazard λ_{L} is constant. Since for $t \in (T_n, T_n + u_n]$ $c_t = c_{\mathrm{T}_n}$, there can be no investment at $t \in (T_n, T_n + u_n]$ that was not profitable at date T_n, and which therefore would have happened at T_n. ■

We now establish that except at transition dates there are no spikes. Appendix A proves

Lemma 2 *A spike cannot occur at $t \neq T_n$.*

The following proposition states that if by a certain time during a period of progress investment has taken place, the very next transition into stagnation will generate an investment spike. Appendix B proves the following:

Proposition 1 *If $x_t > 0$ for some $t \in (T_{n-1} + u_{n-1}, T_n)$, an investment spike will then occur at $t = T_n$.*

Proposition 1 establishes that there are spikes. Spike size is defined as

$$\frac{k_T - k_{T-}}{k_{T-}} = \frac{D^{-1}(p_T)}{D^{-1}(p_{T-})} - 1 > 0, \tag{8}$$

where $k_{T-} = \lim_{t \nearrow T} k_t$ and $p_{T-} = \lim_{t \nearrow T} p_t$. Before proceeding with the analysis of the main model, we can obtain some insight by considering a simpler example in which the transition dates are deterministic.

2.1 The case where $u_{\mathbf{n}} = u$ and $w_{\mathbf{n}} = w$ are known constants

Suppose that the durations $u_n = u > 0$, and $w_n = w > 0$ are repeating constant durations that are known with certainty in advance as then are the transition dates T_n. We will show that investment spikes occur when costs actually stop declining, each time that this event occurs, even though it is fully anticipated beforehand.

v

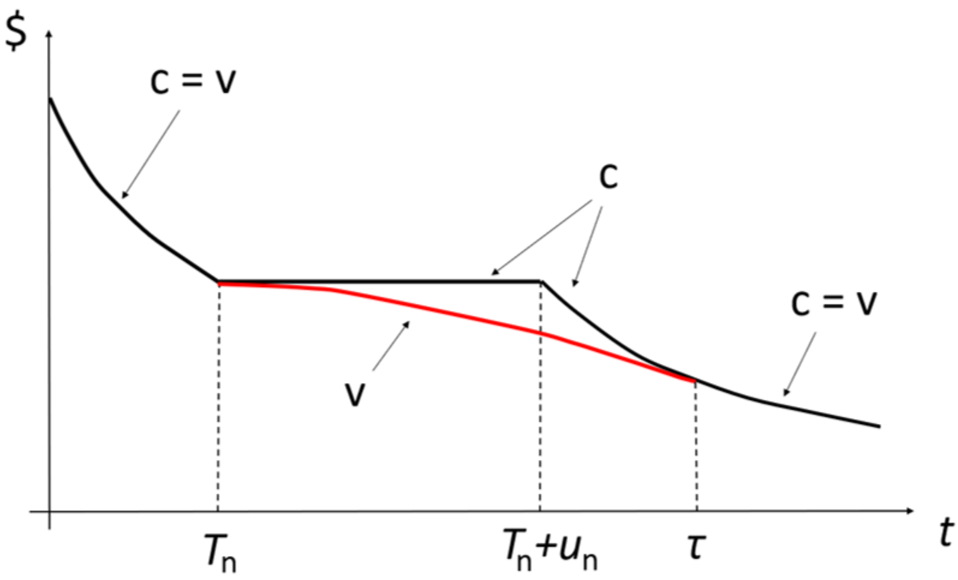

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Figure 1: $c_t > v_t$ ON THE INTERVAL $(0, \tau)$

The behavior of c and v is depicted in Fig. 1. First, $v_t < c_t$ after the spike because the spike causes the price to drop while the cost initially remains high. Then v_t is decreasing even before progress resumes because as we draw nearer to the resumption of progress - and declining costs - investment becomes even less valuable. Therefore, as established by lemma 1, there is no investment for $t \in (T_n, T_n + u_n]$ and moreover, as $v < c$ even after progress resumes at $T_n + u_n$, investment only begins again at $\tau > T_n + u_n$. This is stated more formally in proposition 2 and proved in Appendix C.

Proposition 2 *If w is sufficiently large, (i) there is an investment spike at each transition date T_n, and (ii) continuous investment resumes τ periods later, where $\tau > u$ uniquely solves:*

$$\frac{r}{g} = \frac{1 - e^{-r\tau}}{e^{g(\tau - u)} - 1}. \tag{9}$$

(iii) τ is decreasing in g and in r, (iv) $\partial\tau/\partial u > 1$, and (v) the spike size in (8) is increasing in u.

The condition in Proposition 2 that w be "sufficiently large" is parallel to the condition in Proposition 1 that " $x_t > 0$ for some $t \in (T_{n-1} + u_{n-1}, T_n)$". In both cases, a spike will occur at the end of a period of progress only if progress has gone on long enough for investment to become profitable again. In the deterministic case, this is guaranteed if w is sufficiently long. Otherwise, several epochs of progress may be needed before a spike will occur, which means that there will be investment spikes at some but not all T_n. Such will also be the case in the stochastic version, where w is random.

The intuition for (iv) in the proposition is that the longer is the stagnation delay, u, the more costs have to decline before investment begins again. Therefore we have to wait longer until c is low enough for investment to be profitable again. And the longer the duration during which there will be no post-spike investment, the more attractive investment is at the spike date, and therefore the greater the spike. The comparative statics in the Proposition 2 is consistent with our simulation results for the stochastic case (where we cannot derive the comparative statics analytically).

Having explained why there is no investment following a spike, we now explain why there are spikes in the first place; when progress stops the value of investment jumps up, therefore investment also increases discontinuously, thus the capital stock spikes. We have $v_t = c_t$ whenever investment occurs. Differentiating (2) at the resumption date leads to the Jorgenson user cost formula

$$p_t = (r + g_t)\, c_t. \tag{10}$$

If there was no investment spike a drop in g would cause a downward jump in p, but that would be impossible unless k were to jump up, i.e., unless there was an investment spike.

2.2 Waiting time distributions

In this section we seek to characterize the distribution of the no-investment epoch and the inter-spike distribution. We begin, however, with the random variable $w + u$, the "inter-kink" waiting time which has a known closed form solution.

2.2.1 The inter kink distribution

Let H denote the inter-spike CDF. We do not have a closed form expression for H but we do have it for the inter-kink CDF that we shall denote by G. This is the distribution of $w + u$, i.e., the time between the successive dates T.[1]

Proposition 3 *The inter-kink distribution is that of a sum of two exponentially distributed random variables. It has CDF G,density g, and hazard η given by*

$$\begin{aligned} G(t) &= 1 - \left(\frac{\lambda_w e^{-\lambda_L t} - \lambda_u e^{-\lambda_H t}}{\lambda_H - \lambda_L} \right), & (11) \\ g(t) &= \lambda_L \lambda_H \left(\frac{e^{-\lambda_L t} - e^{-\lambda_H t}}{\lambda_H - \lambda_L} \right), \quad \textit{and} & (12) \\ \eta(t) &= \lambda_L \lambda_H \left(\frac{e^{-\lambda_L t} - e^{-\lambda_H t}}{\lambda_H e^{-\lambda_L t} - \lambda_L e^{-\lambda_H t}} \right). & (13) \end{aligned}$$

Two properties of η.—The following are derived in Appendix E:

[1] The derivations of Eqs. (11)-(13) are in Oguntunde *et al.* (2014).

1. $\eta(0) = 0$, $\eta' > 0$, $\eta'' < 0$ and

$$\lim_{t\to\infty} \eta(t) = \begin{cases} \lambda_{\mathrm{L}} & \text{if } \lambda_{\mathrm{H}} > \lambda_{\mathrm{L}} \\ \lambda_{\mathrm{H}} & \text{if } \lambda_{\mathrm{H}} < \lambda_{\mathrm{L}} \end{cases}.$$

   Thus, although we have assumed constant hazards λ_{L} and λ_{H}, we get an increasing inter-kink hazard.

2. When $\lambda_{\mathrm{L}} = \lambda_{\mathrm{H}}$,

$$\eta(t) = \lambda\left(1 - \frac{1}{1+\lambda t}\right) \to \lambda \text{ as } t \to \infty.$$

We shall plot η in Fig. 2 along with the inter-spike hazard to which we turn next.

2.2.2 The inter-spike distribution

To simplify the notation, we shall normalize the spike date $T_n = 0$, in which case τ becomes the duration of the no-investment epoch, the CDF of which will be denoted by $F(\tau)$. Let s be the cumulative time in state H from the spike date until τ when investment resumes and let

$$\theta = e^{-gs} = \frac{c_\tau}{c_0}, \tag{14}$$

and let $F(\tau \mid \theta)$ be the CDF of τ conditional on θ, i.e., the probability that c_t reaches θc_0 before date τ. Appendix D proves the result in Proposition 4 and shows that θ satisfies

$$\theta^{-1} = 1 + \frac{g}{r}\int_0^\infty \left(1 - e^{-r\tau}\right) dF(\tau \mid \theta). \tag{15}$$

Proposition 4 *Investment does not resume immediately when c_t starts to decline again, i.e.*

$$\tau > u. \tag{16}$$

This result implies that we cannot guarantee that x_t will be positive in each period of progress, because the latter could be quite short and could end before investment becomes profitable again.

Deriving $F(\tau)$.—There are two building blocks to deriving F. Because τ can span more than one epoch H and because there is an L epoch between any two H epochs we shall use a renewal equation approach. Let $\omega_{\mathrm{I,\ t}}$ be the cumulative time spent in state H up to date t starting in state I $\in$ {L,H}. Let

$$\begin{aligned} \Psi(\omega \mid t) &= \Pr(\omega_{L,t} \leq \omega) \quad \text{and} \\ A(\omega \mid t) &= \Pr(\omega_{H,t} \leq \omega). \end{aligned}$$

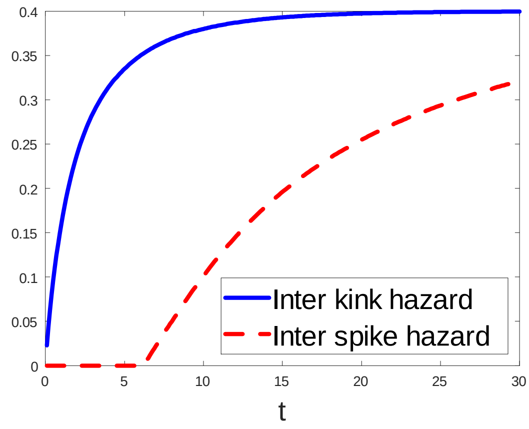

Figure 2: $\eta(t)$ (solid) and $H'/(1-H)$ (dashed) at $\lambda_{\mathrm{L}} = 0.6, \lambda_{\mathrm{H}} = 0.4, r = 0.05$, and $g = 0.1$

The relation between the two probabilities is in the renewal equation

$$\Psi(\omega \mid t) = e^{-\lambda_{\mathrm{L}}(t-\omega)} + \int_0^{t-\omega} A(\omega \mid t-u)\, \lambda_{\mathrm{L}} e^{-\lambda_{\mathrm{L}} u} du, \tag{17}$$

because, if we start in state L, we spend less than ω periods in H if we have spent at least $t-\omega$ periods in L in our current visit to L (the probability of which is $e^{-\lambda_{\mathrm{L}}(t-\omega)}$) or we spent less than ω periods in H after our first visit to H.

Starting in L, we spent at least $-\frac{1}{g}\ln\theta$ periods in state H before date τ was reached. Therefore

$$F(\tau \mid \theta) = 1 - \Psi\left(-\frac{1}{g}\ln\theta \mid \tau\right), \tag{18}$$

We shall use F to derive the inter-spike distribution that underlies the plots in Figs. 2, 3, 4 and 5. We note that neither θ nor s are random variables but τ *is*. Appendix F solves for the value θ^* that satisfies Eq. (15) and Eq. (18), and then we write $F(\tau \mid \theta^*) \equiv F(\tau)$.

The inter-spike CDF H.—No investment takes place for $t < \tau$. The probability that the next spike will occur before date t is

$$H(t) = \int_0^t \int_0^{t-\tau} \lambda_{\mathrm{H}} e^{-\lambda_{\mathrm{H}} w} dw dF(\tau). \tag{19}$$

After date τ, the very next transition to state L will generate a new spike, which explains the inner integral. Note that $F(\tau) = 0$ for $\tau < s$ and therefore $H(t) = 0$ for $t < s$. Remarkably, the preceding solution for the vector (θ, s, F, H) does not depend on the form of the demand function $D()$.We plot $\eta(t)$ in (13) as the solid curve in Fig. 2; the dashed curve is the inter-spike hazard, $H'(t)/(1-H(t))$ with H defined in Eq. (19).

The inter-spike hazard rises because as time passes following a spike, c_t likely gets closer to c_τ and to the resumption of investment. By contrast, in the lumpy cost models, the hazard rises because as time passes the distance between the actual and desired capital

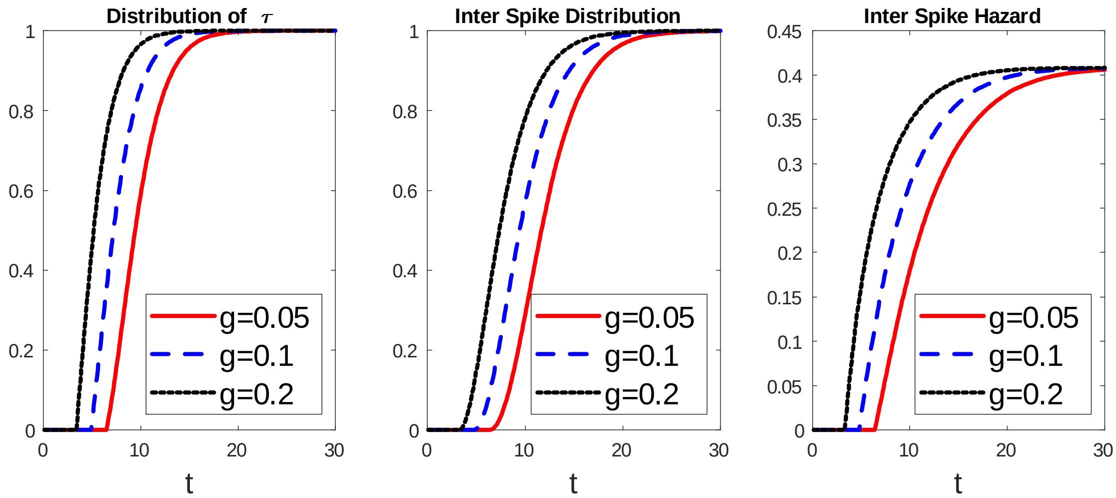

Figure 3: The effect of g at $\lambda_L = 1, \lambda_H = 0.4, r = 0.05$

stock rises, justifying new investment. At the parameter values in Fig. 2, long-run average progress is $E(g) = g\lambda_{\mathrm{L}}/(\lambda_{\mathrm{L}} + \lambda_{\mathrm{H}}) = 0.06$.

2.3 Comparative statics results

This subsection describes how equilibrium responds to changes in g, in λ_{L} and in r.

2.3.1 Varying g

Fig. 3 illustrates the effect of changes in g on distribution of τ, the inter spike distribution and the inter spike hazard. As g increases, the distribution of τ shifts to the left, implying a shorter waiting time until investment resumes. This is the analog of part (iii) of Proposition 2 in the deterministic section. There will be a spike after investment resumes, therefore, when the distribution of τ shifts to the left, the inter spike distribution and inter spike hazard will both shift to the left, as shown in Fig. 3.

Spike size and g.—At the post-spike capital stock k', the equilibrium price (at T and at τ, both) is $D(k') = p_0 = (r+g)\theta c_0$. On the other hand, at the pre-spike capital stock k, the equilibrium price is $D(k) = (r+g)c_0$. If demand is isoelastic, i.e. $D(k) = k^{-\gamma}$, with $\gamma < 1$, then we have

$$\left(\frac{k'}{k}\right)^{-\gamma} = \frac{(r+g)\theta c_0}{(r+g)c_0} \Rightarrow \frac{k'}{k} = \theta^{-1/\gamma}. \tag{20}$$

Thus, the size of the spike is decreasing in θ. Table 1 shows that spike size rises with g, and that s,the cumulative time in state H from the spike date until investment resumes, decreases with g:

Table 1: The effect of g

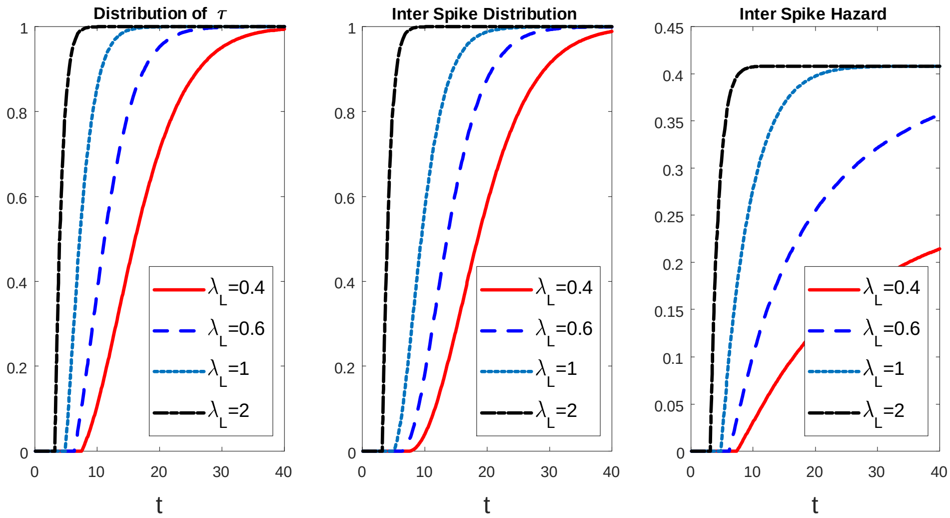

Figure 4: The effect of λ_{L} at $\lambda_H = 0.4, g = 0.1, r = 0.05$

	$\lambda_L = 1, \lambda_H = 0.4, r = 0.05$		
	$g = 0.05$	$g = 0.1$	$g = 0.2$
θ	0.723	0.613	0.507
s	6.5	4.9	3.4

This is the analog of Proposition 2 in the deterministic section. That θ declines means that the decline in c must be larger, but this is more than offset by the faster decline in costs due to the larger g.

2.3.2 Varying λ_L

Fig. 4 shows how changes in λ_L affect the distribution of τ, the inter spike distribution and the inter spike hazard. As λ_L increases, the distribution of τ shifts to the left, implying a shorter waiting time until investment resumes. Analogously to part (v) of Proposition 2 which showed that τ is increasing in u, here a larger λ_L means the time spent in state L is shorter, i.e., u is smaller.

Spike size and λ_{L}.—Table 2 shows that spike size decreases with λ_{L}:

Table 2: The effect of λ_{L}

	$\lambda_H = 0.4, g = 0.1, r = 0.05$			
	$\lambda_L = 0.4$	$\lambda_L = 0.6$	$\lambda_L = 1$	$\lambda_L = 2$
θ	0.472	0.534	0.613	0.726
s	7.5	6.3	4.9	3.2

As λ_{L} rises, θ rises and s falls. This is because a larger λ_{L} makes investment less attractive because obsolescence will set in earlier. The spike size will, as a result, be smaller: At the

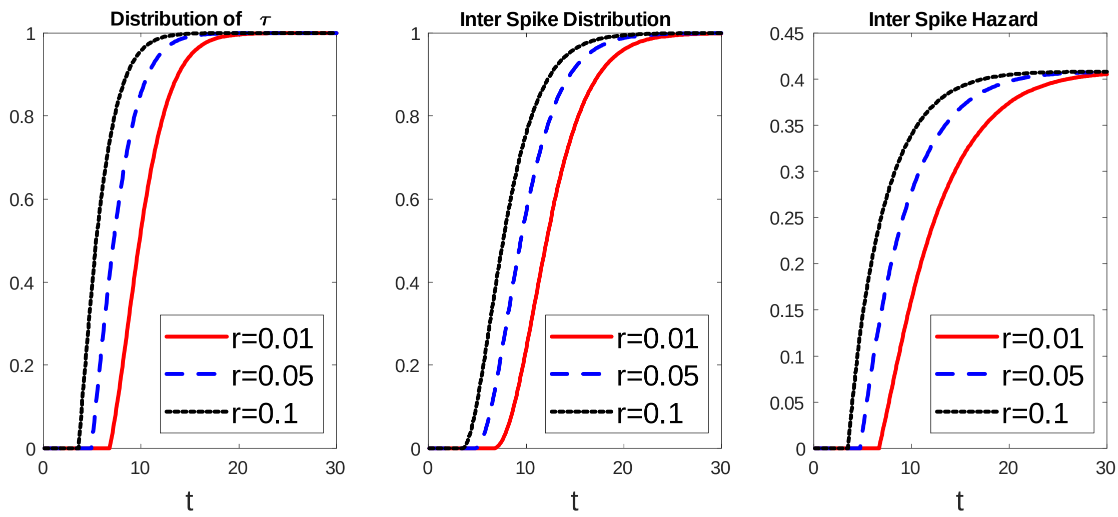

Figure 5: The effect of r at $\lambda_L = 1, \lambda_H = 0.4, g = 0.1$

post-spike capital stock k', the equilibrium price (at T and at τ, both) is $D(k') = p_0 = (r+g)\theta c_0$. On the other hand, at the pre-spike capital stock k, the equilibrium price is $D(k) = (r+g)c_0$.Thus, the size of the spike is

$$\frac{k'}{k} = \frac{D^{-1}((r+g)\theta c_0)}{D^{-1}((r+g)c_0)},$$

which is decreasing in θ. Thus, the size of the spike is decreasing with λ_{L}.

Since the spike size is decreasing in λ_{L}, this confirms Dixit and Pindyck (1994) who argued that irreversible investment is discouraged by the ability to resolve uncertainty by waiting.

2.3.3 Varying r

Fig. 5 shows the effect of changes in r on the distribution of τ, the inter spike distribution and the inter spike hazard. As r increases, the distribution of τ shifts to the left, implying a shorter waiting time until investment resumes. This is the analog of part *(iii)* of Proposition 2.

Spike size declines with r.—This is seen in Table 3:

Table 3: The effect of r

	$\lambda_L = 1, \lambda_H = 0.4, g = 0.1$		
	$r = 0.01$	$r = 0.05$	$r = 0.1$
θ	0.507	0.613	0.698
s	6.8	4.9	3.6

Raising r reduces investors' patience. The table shows that $\frac{\partial\theta}{\partial r} > 0$. The less patient one is, the less valuable investment is, thus the less one wants to invest at the spike date and therefore the target c at which investment resumes rises, waiting time falls. If demand is $D(k) = k^{-\gamma}$ with $\gamma < 1$, then once again $\frac{k'}{k} = \theta^{-1/\gamma}$, and the size of the spike is decreasing in θ and, since $\frac{\partial\theta}{\partial r} > 0$, decreasing in r.

3 Robustness

We show robustness to four possible changes to the model: adding reversibility of investment, having a monopoly instead of competition, adding physical depreciation of capital, and to adding firm specific TFP shocks and resulting investment rates and spikes.

3.1 Reversible investment

We now show that the reversible-k case will also be characterized by investment spikes (positive and negative ones), but not by periods of falling costs accompanied by zero investment as Proposition 4 showed. Reversibility implies that the value of capital must be the same inside the industry and outside it. I.e., regardless of whether $x_t > 0$, we must have

$$v_t^i = c_t, \qquad \text{for} \quad i \in \{\text{L,H}\}$$

for all t which together with Eq. (6) means that

$$rc_t = p_t + \frac{dc_t}{dt} = p_t - g_t c_t.$$

because $v_t^{\mathrm{j}} - v_t^{\mathrm{i}} = 0$. And, since $p = D(k) \Rightarrow k = D^{-1}(p)$,

$$k_t = D^{-1}([r + g_t] c_t)$$

E.g., let $D(Q) = Q^{-\alpha}$. Then $k_t = ([r + g_t] c_t)^{-1/\alpha}$.

Since c_t is continuous (and differentiable) in t whereas g_t jumps between zero and g, whenever g_t jumps (either up or down), k_t jumps in the opposite direction. Therefore, when we enter stagnation and g_t drops from g to zero, k_t jumps up (i.e., there is a spike) and when g_t rises from zero to g, k_t drops, as firms take their capital and put it to other use (because it is fully reversible). This is the dashed red line in Fig. 6.

3.2 Monopoly

The preceding has analyzed a competitive industry with free entry, in which individual firms and the industry are the same in the sense that they produce the same product and that they face the same cost of capital. To address firm level heterogeneity more in line with the data, we would need a model of an imperfectly competitive industry. This brief

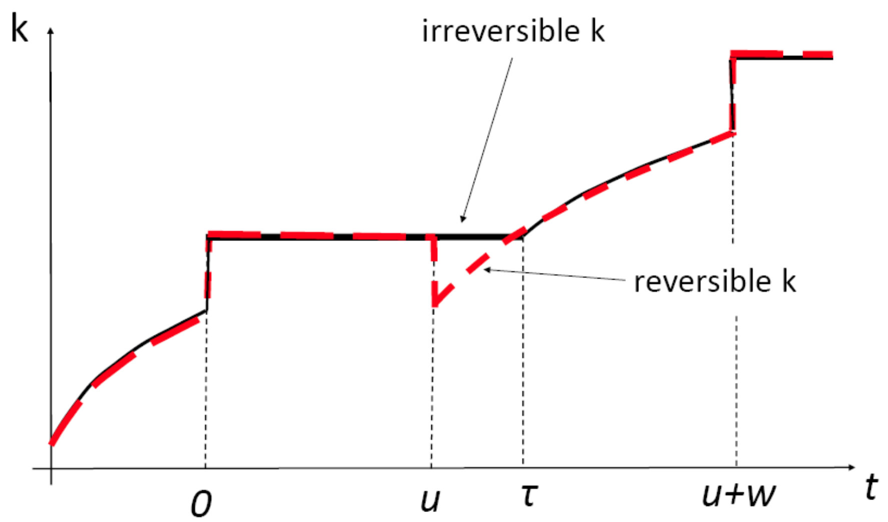

Figure 6: REVERSIBLE k (DASHED RED LINE) AND IRREVERSIBLE k (BLACK LINE)

section shows that the model could be applied to such a setup by showing that the basic results in Sec. 2 apply to monopoly firms as well.

We now extend the model to the monopoly case. Let k denote a monopolist's capital stock in a market with demand function $D(k)$. We now define the monopolist's marginal revenue as

$$m(k) = \frac{d}{dk} kD(k) = D(k) + kD'(k). \tag{21}$$

and the marginal present value of capital as

$$M_t = E_t \int_t^\infty e^{-r(s-t)} m_s ds. \tag{22}$$

Then the same equations hold for the monopoly:

Proposition 5 *A monopolist has the same Bellman equations (6) in state H and the same spike conditions as a competitive industry, but with $p = D(k)$ replaced by $m(k)$ defined in (21) and v_t replaced by M_t defined in (22).*

Proof. We start by taking the random sequence (k_t) as given. Then we will derive the conditions that k_t must satisfy in the case of monopoly. Let

$$m_t = m(k_t)$$

Conditional on the marginal revenue sequence, at date t, Then let M_t^{L} and M_t^{H} denote the monopoly's discounted marginal revenue products of capital in the two states. Then for $i, j \in \{L, H\}$, (22) reads

$$M_t^{\mathrm{i}} = \int_t^\infty \left(\int_t^T e^{-r(s-t)} m_s ds + e^{-r(T-t)} M_T^{\mathrm{j}} \right) \lambda_{\mathrm{i}} e^{-\lambda_{\mathrm{i}}(T-t)} dT. \tag{23}$$

In differential form the Bellman equations are

$$rM_t^{\mathrm{i}} = m_t + \lambda_{\mathrm{i}} \left(M_t^{\mathrm{j}} - M_t^{\mathrm{i}} \right) + \frac{dM_t^{\mathrm{i}}}{dt}. \tag{24}$$

Eqs. (23) and (24) are same functional form as (5) and (6) with $(p_t, v_t)_0^\infty$ replaced by $(m_t, M_t)_0^\infty$. ■

In other words, given the same $(c_t)_0^\infty$, the model yields the same solution for $(m_t)_0^\infty$ as for $(p_t)_0^\infty$. And therefore the same spike dates apply. The θ analysis of Eqs. (52)-(54) applies with p_0 replaced by $m(k_0)$

The next result refers to two different market structures (i.e., monopoly and competition) facing the same demand function.

Corollary 1 *If $D^M(k) = D^C(k) = k^{-\gamma}$, and if the capital cost $(c_t)_0^\infty$ sequence is the same, then*

$$k_t^M = (1-\gamma)^{1/\gamma} k_t^C. \tag{25}$$

Proof. By Proposition 5 we know that $\left(k_t^{\mathrm{C}}, k_t^{\mathrm{M}}\right)$ must be such that

$$m\left(k_t^{\mathrm{M}}\right) = p_t = \left(k_t^{\mathrm{C}}\right)^{-\gamma}.$$

We also have

$$m\left(k_t^{\mathrm{M}}\right) = D^{\mathrm{M}}\left(k_t^{\mathrm{M}}\right) + k_t^{\mathrm{M}} D^{\mathrm{M}\prime}\left(k_t^{\mathrm{M}}\right) = (1-\gamma)\left(k_t^{\mathrm{M}}\right)^{-\gamma}.$$

Combining the two equations gives:

$$(1-\gamma)\left(k_t^{\mathrm{M}}\right)^{-\gamma} = \left(k_t^{\mathrm{C}}\right)^{-\gamma},$$

i.e., (25). ■

Thus, given the same demand function, the monopolist will at each date have a smaller k_t, which implies that the spikes (measured as proportion of the stock) will be the same for a monopoly as for a competitive market. Given a particular (c_t) series, spikes dates are the same for monopoly and competition.

Corresponding to a $D(\cdot)$ for the competitive case, one can find an alternative demand function that would generate identical Bellman equations for the monopoly case and identical solutions for $(k_t)_0^\infty$.

Example.—Suppose the demand of the competitive industry is $D(k) = k^{-\gamma}$ with $\gamma < 1$ so that demand is elastic. Then $p_t = D(k_t)$. Let $D^{\mathrm{M}}(k)$ be the demand for the monopolist's product. Then if $D^{\mathrm{M}}(k) = \frac{1}{1-\gamma} D(k)$, then

$$m(k_t) = D(k_t) = p_t. \tag{26}$$

3.3 Physical depreciation

We now show that the main implications survive adding physical depreciation of capital $\delta > 0$ so that instead of Eq. (2), the value of capital is

$$v_t = E_t \int_t^\infty e^{-(r+\delta)(s-t)} p_s ds. \tag{27}$$

We then consider the following three implications

1. *Spikes happen only at the end of an episode of progress.*—This is still true; p is constant and if a spike did not occur at T_n, p would remain constant for $t \in (T_n, T_n + u_n]$ (see point 2 below) and v_{T_n} would rise discretely and exceed c_{T_n}

2. *No flow investment occurs while c is constant.*—This changes to "*only replacement investment while c is constant.*" Since now $x_t = \delta k_t > 0$ for $t \in (T_n, T_n + u_n]$, differentiating (27) and setting $v_t = c_t$ leads to $(r + \delta) c_t = p_t + \frac{dc_t}{dt}$. And since for $t \in (T_n, T_n + u_n]$ $dc_t/dt = 0$, price is also constant at $p_t = (r + \delta) c_t$ which implies that k_t is also constant so that only replacement investment occurs.

3. *No investment for a while even after c starts to fall again.*—Still true in the general version because v falls discretely to $v^{\mathrm{H}}_{T_n+u_n} < v^{\mathrm{L}}_{T_n+u_n}$, a period of zero investment exists. It will be shorter the larger is δ, but it is positive no matter how large δ is.

3.4 Firm vs. industry spikes

The main model of section 2 is a representative firm model with no well-defined firm size and, whether under perfect competition or monopoly, any time a firm experiences a spike or zero investment, so does the industry. With heterogeneity among firms (or even among plants of a single firm as Kehrig and Vincent 2018 document) we would expect a fraction of the spikes to wash out with aggregation up to the industry level.

Suppose the production function shows diminishing returns in k and a serially uncorrelated firm-specific shock z, and that

$$\text{output} = y = zk^\alpha, \tag{28}$$

where $\alpha < 1$ and so that profits are $y - cx$. Let ρ be the Poisson event at which a new z' is drawn independently from the CDF Ψ' and so for I, J $\in$ {L, H},

$$rv^{\mathrm{I}}_t(z,k) = \max_{x \geq 0} \left\{ zk^\alpha - c_t x + \lambda_{\mathrm{I}} \left(v^{\mathrm{J}}_t(z,k) - v^{\mathrm{I}}_t(z,k) \right) + \rho \left(\int v^{\mathrm{I}}_t(z',k)\, d\Psi - v^{\mathrm{I}}_t(z,k) \right) + \frac{\partial v^{\mathrm{I}}}{\partial t} + \frac{\partial v^{\mathrm{I}}}{\partial k} x \right\}$$

Firms all face the same c, but now irreversibility of k implies that some low-z firms will be at a corner where $\frac{\partial v^{\mathrm{I}}}{\partial k} < c$, i.e., where the value of an additional unit of k is less than the cost. By contrast, in the model of Sec. 2, all firms have the same incentives to invest.

For the same (z, k), investment is more valuable at a spike than otherwise, for the same reason as before. Therefore, although an individual firms might invest more during stagnation than at a kink if it is then hit with a high z, on average aggregate investment should be greater when progress stops. Thus it will still be true that investment spikes at the kinks at the industry level, but some low-z firms will not take part in that spike. Another difference will be that during epoch L some firms will draw high zs and their capital will rise, thereby raising industry output. Thus, instead of being a constant during epoch L as was the case in the homogeneous firm model of Sec. 2, p_t would now decline although more slowly than it did in state H. Nevertheless, such a slower decline in p invites the spike.

Thus in spite of the idiosyncrasies of the productivity shocks, common shocks to the price of capital still create spikes at the industry level. A completely stated discrete time version of this extended model is in Appendix H

4 Tests

The results in Proposition 2 for the deterministic case and in Table 1 for the stochastic case state that spike size increases in g, the rate of cost decline in state H. To test this, we need a measure of spikes, and a measure of the process g for an industry (as in Sec. 2) or a firm (as in Sec. 3).

4.1 Three measures of spikes

We use three industry-specific definitions of spikes: One is from Ilyina and Samaniego (2011, henceforth IS) and two are from Kehrig and Vincent (2018, henceforth KV-A and KV-B)

1. IS measure the lumpiness of investment as the fraction of years that an industry experiences investment in excess of 30% of the capital stock. The Ilyina-Samaniego data cover 1970-1999.

2. KV-A: $\mathrm{E}[x_t/k_t \mid x_t/k_t > 0.2]$; i.e., a spike means simply that investment rates are large.

3. KV-B: $\mathrm{E}[x_t/k_t \mid x_t/k_t > 0.2,\ x_t/k_t > 2x_{t-1}/k_{t-1},\ x_{t+1}/k_{t+1} < 0.5x_t/k_t]$; i.e. the investment in a spike year must be at least twice as large as the investment rate in the adjacent years, i.e., *and* jump up and down over time (which may be more consistent with a fixed investment adjustment cost).[2]

[2] Since the Census moments have not been officially disclosed, these were computed using Compustat data. The investment spikes according to definition B are quite similar in the Compustat and Census data; investment spikes according to Definition A are positively correlated but not as close.

Note that the data are not direct measures of industry-level spikes but aggregates of spikes in more micro data. Kehrig & Vincent's data are annual Census data on manufacturing establishments, and they define spikes as establishment investment rates exceeding 15 percent. Ilyina & Samaniego's spikes are large investments undertaken by Compustat firms, in particular, annual capital expenditures exceeding 30 percent of a firm's stock of fixed assets; they arrive at industry measures by reporting the average or median firm statistics for each industry.

Table 4 reports the cross-industry correlations between the variables. Definition A is not highly correlated with the other two definitions.

Table 4: Correlations of the three Spike Measures

	IS	KV-A	.KV-B
IS	1		
KV-A	0.217	1	
KV-B	0.776	0.221	1

The high correlation between IS and KV-B suggests that years in which the investment rate exceeds 0.3 also are years when it is at least twice as large as investment in the adjacent years.

4.2 Two measures of g

To correspond to the model we shall classify each industry's observations into state H, and state L, and to estimate the parameter g for each industry. Our tests will assume that λ_{L} and λ_{H} are identical across industries, and that only g differs among them. Quality-adjusted capital-goods prices at the industry level are provided by Cummins and Violante (2002 – henceforth CV). We now reports test results based on these industry-level data which provide annual time series for 2-digit SIC industries for the years 1947- 1999 except for motor vehicles which is three digit (SIC 371).

We use the capital-goods prices as a measure of technological change at the industry level, i.e., as a measure of c_t in different industries. These data will serve as a measure of the of $\tilde{g}_{i,t}$, the random growth rate for industry i. That is, we use the CV quality-adjusted price $p_{i,t}$, and then compute $g_{i,t} = -\ln\left(p_{i,t}/p_{i,t-1}\right)$.

We construct two alternative industry measures that conform to the two-states $\tilde{g}_{i,t} \in \{0, \hat{g}_i\}$ as follows: [3]

$$\hat{g}_{i,1} = E\left[g_{i,t} \mid g_{i,t} > 0\right].$$

I.e., our first measure, $\hat{g}_{i,1}$, simply calculates an average, over all dates, among the positive realizations of $g_{i,t}$ in industry i. According to this measure, $g_{i,t} > 0$ corresponds to state

[3] We now use a hat to denote the industry's g_i in state H. I.e., $\hat{g}_i$ corresponds to $g > 0$ in the model.

Figure 7: IS SPIKES AND $\hat{g}_1$

H, and $g_{i,t} \leq 0$ corresponds to state L.

Our second measure is similar to the first measure, except that $\hat{g}_{i,2}$ is the average, again over all dates, but among the realizations of $g_{i,t}$ that are above the median:

$$\hat{g}_{i,2} = E\left[g_{i,t} \mid g_{i,t} > g_{i,\text{ median}}\right]$$

As for $\hat{g}_2$, we note that the stationary probability of being in state H is $\lambda_{\mathrm{L}}/\left(\lambda_{\mathrm{L}} + \lambda_{\mathrm{H}}\right)$; then $\hat{g}_2$ would be the correct measure if $\lambda_{\mathrm{L}} = \lambda_{\mathrm{H}}$ so that exactly half of the observations are predicted to be in the high state.

4.3 Results

Both $\hat{g}_1$ and $\hat{g}_2$ show a positive relationship with each spike measure, consistent with our comparative statics results. Figs 7, 8, and 9 plot the relationship between $\hat{g}_1$ and spike size. The regressions are reported in Table 5.

Figs 10, 11, and 12 plot the relationship between $\hat{g}_2$ and spike size. The regressions are also reported in Table 5.

Weighted regressions.—In our regressions industries are weighted by their size. We use an industry's total investment expenditure coming from the NAICS-CES Manufacturing

Figure 8: KV-A SPIKES AND $\hat{g}_1$

Figure 9: KV-B SPIKES AND $\hat{g}_1$

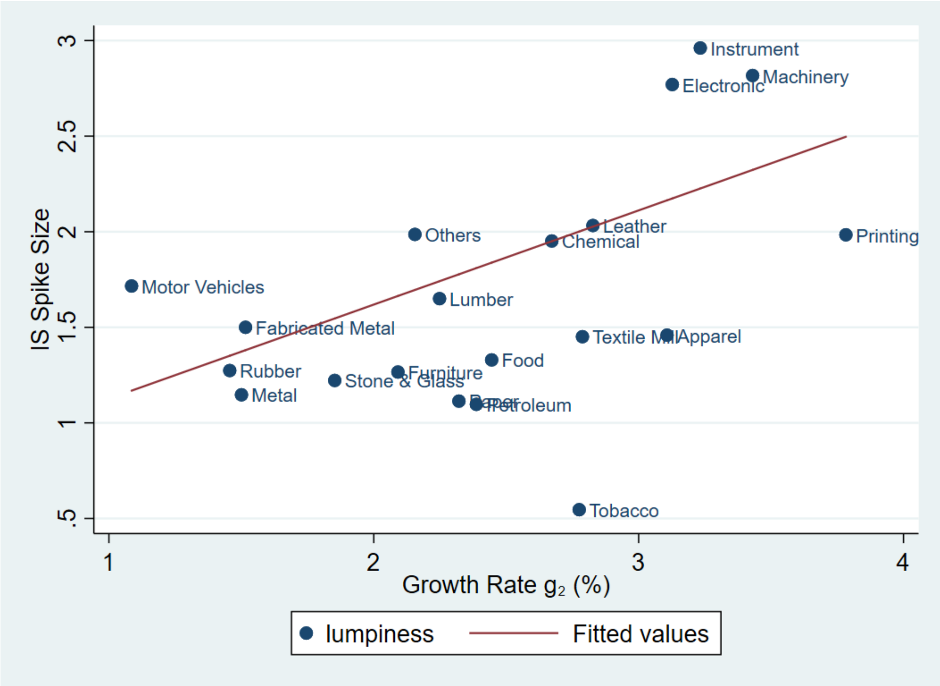

Figure 10: IS spikes and $\hat{g}_2$

Figure 11: KV-A spikes and $\hat{g}_2$

Figure 12: KV-B SPIKES AND $\hat{g}_2$

database as a proxy for its size.[4]

Table 5: Regression results using $\hat{g}_1$, $\hat{g}_2$ and $\bar{g}$

	IS	KV-A	KV-B	IS	KV-A	KV-B	IS	KV-A	KV-B
$\hat{g}_1$	1.376***	0.059***	0.105***						
	(0.266)	(0.016)	(0.027)						
$\hat{g}_2$				0.492**	0.010	0.044***			
				(0.204)	(0.012)	(0.015)			
$\bar{g}$							0.332*	0.004	0.033**
							(0.176)	(0.011)	(0.013)
Const.	-1.49**	0.16***	0.24***	0.63	0.28***	0.39***	1.91***	0.31***	0.50***
	(0.611)	(0.041)	(0.064)	(0.493)	(0.028)	(0.035)	(0.177)	(0.011)	(0.015)
# Obs.	20	20	20	20	20	20	20	20	20
R^2	0.52	0.39	0.38	0.34	0.05	0.34	0.18	0.01	0.22

The model performs well: The estimated coefficients remain significant except for the effect of $\hat{g}_2$ on KV-A. The results are similar if we do not use industry weights – these results are in Appendix G, Table A1. The estimated coefficients remain significant except for the effect of $\hat{g}_2$ on KV-A.

[4]The use of other weights such as capital stock, value added, and total sales leads to similar results.

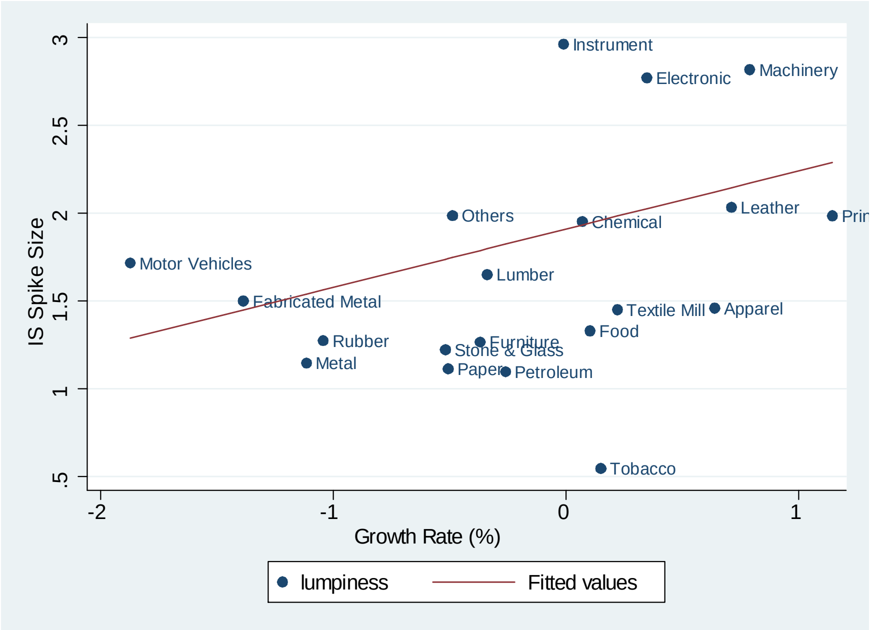

Figure 13: IS SPIKES AND $\bar{g}$

4.3.1 Spikes and average progress, $\bar{g}_i$

We now plot the relationship of spike size and average growth rate $\bar{g}_i$ over the period 1947 to 1999. The reason for doing so is that both the echoes and the fixed costs models imply there should be more spikes when $\bar{g}$ rises whereas in our model the effect of $\bar{g}$ on spikes is ambiguous.

The effect of $\bar{g}$ in our model.—The stationary probability of being in state H is $\lambda_{\mathrm{L}}/\left(\lambda_{\mathrm{L}}+\lambda_{\mathrm{H}}\right)$; the model predicts the mean of g to be

$$E\left(\bar{g}\right)=\frac{\lambda_{\mathrm{L}}}{\lambda_{\mathrm{L}}+\lambda_{\mathrm{H}}}\hat{g}. \tag{29}$$

Thus, if $\bar{g}$ rises, that does not mean that $\hat{g}$ rises. The two become identical if $\lambda_{\mathrm{H}}=0$ so that progress is uninterrupted. Indeed, when $\lambda_{\mathrm{H}}=0$ our model implies *no* spikes because there never is a switch to the no-progress state. More generally $\bar{g}$ can be higher due to higher λ_{L} or higher $\hat{g}$, but the two channels have opposite effect on spike size, as a result, the effect of $\bar{g}$ on spike size is ambiguous.

Echoes and $\bar{g}$.—A higher $\bar{g}$ should speed up the growth of capital. If there are vintage-capital related echoes, the fraction of capital that is of older vintages is smaller, and this should reduce the fraction of capital that exhibits spikes due to replacement. Both the scatter plots and the last three columns of Table 5 show that spikes seem to be larger in industries where the price of capital declines faster.

Fixed costs of investment and $\bar{g}$.—The fixed cost model yields spikes if growth in the desired capital stock, call it k_t^*, is positive. To take a simple example, let C be the fixed cost of adjustment of capital. If competition yields a product price declining at the same

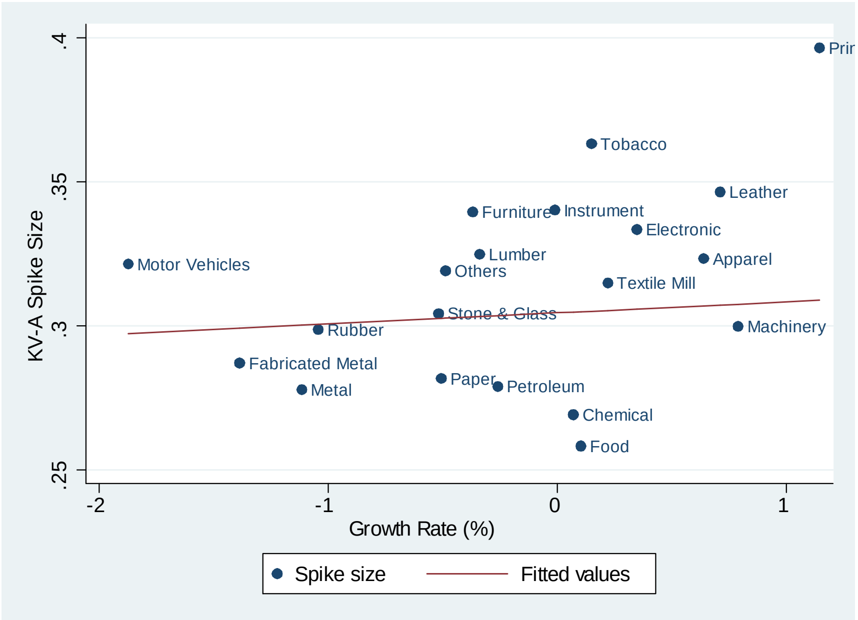

Figure 14: KV-A SPIKES AND $\bar{g}$

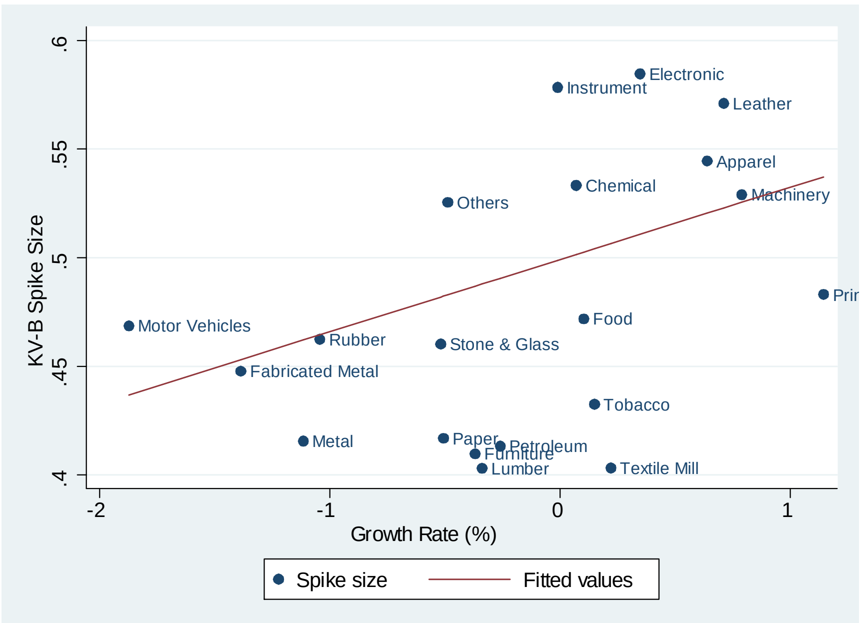

Figure 15: KV-B SPIKES AND $\bar{g}$

rate as c_t so that $p_t = e^{-\bar{g}t}$, and if $D(p) = 1 - \ln p$, then with each firm's k_t^* growing in proportion to quantity supplied we would have $\frac{dk^*}{dt} = \bar{g}$, also constant. The Baumol-Tobin formula states that each firm would raise its capital at discrete intervals,[5] as follows:

$$\text{jump up by } \Delta = \sqrt{\frac{2C\bar{g}}{r}} \text{ every } \frac{\Delta}{\bar{g}} = \sqrt{\frac{2C}{r\bar{g}}} \text{ periods.} \tag{30}$$

In other words, while our model generates spikes only if technological progress is variable, the fixed-cost model generates them even if progress is constant.

Comparing the regression results in Tables 5, $\hat{g}_1$ does a better job explaining spikes than $\bar{g}$ – the effect of $\bar{g}$ on spike size is insignificant except for the KV-B measure.

5 Conclusion

In a model with no fixed investment costs we have shown that spikes occur when a sudden reduction in technological progress occurs, both at the level of the firm and at industry level. In our model investment spikes occur when progress stops. Following a spike, the decline in price causes investment to stop until costs decline sufficiently to justify further investment. As time passes after a spike, the hazard rate for the next spike rises, just as it does in the lumpy costs models.

The model predicts that higher variability leads to greater investment spikes and the data confirm it; in industries where the variability of progress is higher, spikes are larger. Future empirical work may consider combining other motives for spikes such as fixed costs, vintage capital, and demand shifts, and assess the relative contribution of each.

A Appendix

A. Proof of Lemma 2

Proof. Suppose that, on the contrary, a spike did occur at $t \neq T_{\mathrm{n}}$. By Proposition 1, $x_t = 0$ on $(T_n, T_n + u_n]$. Therefore the only place where such a hypothesized spike can occur is on $(T_n + u_n, T_{n+1})$. That is, the spike must occur in the interior of the epoch of progress, i.e., in state H. Let $\hat{t}$ be the spike date, and by contradiction let $\hat{t} \in (T_{\mathrm{n}} + u_{\mathrm{n}}, T_{\mathrm{n}+1})$ so that $\frac{dc}{dt} = -gc_{\hat{t}} < 0$. Now, letting E_t denote the expectation operator conditional on information available at t,

$$v_{\hat{t}}^{\mathrm{H}} = \int_{\hat{t}}^{\infty} e^{-r(s-\hat{t})} E_{\hat{t}}(p_s)\, ds$$

[5] Jumps Δ would be spread out over firms so as not to create any jumps in the industry's total capital stock.

so that v is everywhere differentiable and its derivative at $\hat{t}$ is $dv^{\mathrm{H}}/dt = rv_{\hat{t}} - p_{\hat{t}}$. If there is a spike at $\hat{t}$, then $v_{\hat{t}} = c_{\hat{t}}$, and $\lim_{t\nearrow\hat{t}} p_t > p_{\hat{t}}$. Then

$$\lim_{t\nearrow\hat{t}} \frac{dv}{dt} = rv_t - \lim_{t\nearrow\hat{t}} p_t < \lim_{t\searrow\hat{t}} \frac{dv}{dt} = rv_t - \lim_{t\searrow\hat{t}} p_t$$

But $\lim_{t\searrow\hat{t}} c_t = c_{\hat{t}}$ since c is continuous and since there is a spike at $\hat{t}$. This means that $\lim_{t\searrow\hat{t}}(dv/dt) > \lim_{t\searrow\hat{t}} c_t$, which implies $v_{\hat{t}+\varepsilon} > c_{\hat{t}+\varepsilon}$ for ε small enough, and this contradicts investment incentives. In other words, the derivative of v has an upward jump at the hypothesized spike date, and if that date is in the interior of the progress epoch, this leads to the contradiction. ■

B. Proof of Proposition 1

Normalize time so that $g_t = g$ at date $t = 0$, and let T be the date at which g_t switches to $g = 0$. To prove the proposition 1, we first establish the following two Lemma.

Lemma 3 *If for some $t < T$, $x_t > 0$, then*

$$x_s > 0, \tag{31}$$

for all $s \in (t, T)$ and

$$v_s^H = c_s. \tag{32}$$

for all $s \in (t, T]$.

Proof. Positive x_t implies $v_t = c_t$. As long as c_t is strictly decreasing and since, by Lemma 2 there are no investment spikes, $x_s > 0$ for all $s \in (t, T)$; if not, we would have $v_s^{\mathrm{H}} > c_s$. This implies that $v_s^{\mathrm{H}} = c_s$ for all $s \in (t, T)$ which proves (31) and (32) except for $s = T$. Finally, since v_t^{H} and c_t are both continuous, $v_T^{\mathrm{H}} \neq c_T$ would imply that $v_s^{\mathrm{H}} \neq c_s$ for some $s < T$, a contradiction. ■

Lemma 4 *If $x_t > 0$ just before T, then if there is no spike at T,*

$$v_T^L = c_T. \tag{33}$$

Proof. In equilibrium, $v_t^{\mathrm{L}} \leq c_t$. Suppose that $v_{\mathrm{T}}^{\mathrm{L}} < c_{\mathrm{T}}$, then $rv^{\mathrm{H}} = p + \lambda_{\mathrm{H}}\left(v^{\mathrm{L}} - v^{\mathrm{H}}\right) - gc < p - gc$, $\Rightarrow$

$$v_T^{\mathrm{H}} = c_T < \frac{p_T}{r+g}. \tag{34}$$

Then (34) implies

$$\frac{p_T}{r} > c_T. \tag{35}$$

Now Lemma 1 states that for $t \in [T, T+u)$, $x_t = 0$, which implies that $k_t = k_T$ and $p_t = p_T$. Then since Lemma 3 says that $v_{\mathrm{T}}^{\mathrm{H}} = c_{\mathrm{T}}$, and since there is no spike at T and since Lemma 1 says that $x_t = 0$ on the interval $(T, T+u)$, this means that $k_{T+u} = \lim_{t \nearrow T} k_t$ and then the state of the industry as described by the triple (c, k, I) is the same at $T+u$ as it was just prior to T and that therefore

$$v_{T+u}^{\mathrm{H}} = c_{\mathrm{T+u}} = c_{\mathrm{T}},$$

then

$$\begin{aligned} c_T &> v_T^{\mathrm{L}} = \int_0^\infty \left(p_T \int_T^{T+u} e^{-r(s-T)} ds + e^{-ru} v_{T+u}^{\mathrm{H}} \right) \lambda_{\mathrm{L}} e^{-\lambda_{\mathrm{L}} u} du \\ &= \int_0^\infty \left(p_T \int_T^{T+u} e^{-r(s-T)} ds + e^{-ru} c_{T+u} \right) \lambda_{\mathrm{L}} e^{-\lambda_{\mathrm{L}} u} du \end{aligned} \tag{36}$$

Then

$$\begin{aligned} v_T^{\mathrm{L}} &= \int \left(\frac{p_T}{r} \left(1 - e^{-ru}\right) + e^{-ru} c_T \right) \lambda_{\mathrm{L}} e^{-\lambda_{\mathrm{L}} u} du \\ &> \int \left(c_T \left(1 - e^{-ru}\right) + e^{-ru} c_T \right) \lambda_{\mathrm{L}} e^{-\lambda_{\mathrm{L}} u} du \quad \text{(using (35))} \\ &= c_T, \end{aligned}$$

which contradicts (36). ■

(PROOF OF THE PROPOSITION). For $s \in (t, T)$, Eq. (6) reads

$$r v_s^{\mathrm{H}} = p_s + \lambda_{\mathrm{H}} \left(v_s^{\mathrm{L}} - v_s^{\mathrm{H}} \right) - \frac{d v_s^{\mathrm{H}}}{dt}$$

Substituting for v_s^{H} from (32) implies that for $s \in (t, T)$,

$$r c_s = p_s + \lambda_{\mathrm{H}} \left(v_s^{\mathrm{L}} - c_s \right) - g c_s \tag{37}$$

Suppose no spike occurs at T. Then p_t is continuous at T. Taking limits in (37) as $s \nearrow T$,

$$r c_T = p_T + \lambda_{\mathrm{H}} \left(v_T^{\mathrm{L}} - c_T \right) - g c_T \tag{38}$$

Since v_t^{L} and v_t^{H} are differentiable, $\left(v_t^{\mathrm{J}} - v_t^{\mathrm{I}} \right)$ is continuous at T.

Evaluating Eq. (6) at $I = L$, for $s \in (T, T+u)$

$$\frac{d v_s^{\mathrm{L}}}{dt} = 0, \quad \text{as} \quad s \searrow T, \tag{39}$$

and $rv_{\mathrm{T}}^{\mathrm{L}} = p_{\mathrm{T}} + \lambda_{\mathrm{L}}\left(v_{\mathrm{T}}^{\mathrm{H}} - v_{\mathrm{T}}^{\mathrm{L}}\right)$. By (39), $dv_s^{\mathrm{L}}/dt = 0$, so that

$$rc_{\mathrm{T}} = p_{\mathrm{T}}. \tag{40}$$

But (38) implies that

$$rc_{\mathrm{T}} = p_{\mathrm{T}} - gc_T. \tag{41}$$

Thus (41) contradicts (40), i.e., it contradicts the supposition that no spike occurs at T.

C. Proof of Proposition 2

Proof. PROOF OF (*ii*).—Let $t = 0$ be a transition date into stagnation so that progress resumes at date u. By Lemma 1, $x = 0$ at $0 < t < u$. At any date $t \neq T$ at which $x > 0$, differentiating (2) and setting $v_t = c_t$ leads to[6]

$$rc_t = p_t + \frac{dc_t}{dt}. \tag{42}$$

And, since $dc_t/dt = -gc_t$, (42) implies

$$p_t = (r + g_t)c_t. \tag{43}$$

Since by assumption $x_t = 0$ for $t < \tau$, it must be that $p_t = p_0$ for $0 \leq t < \tau$. At date τ, when investment resumes, the PV of earnings $= c_\tau$. Thus the PV of earnings at date 0 satisfies:

$$c_0 = \int_0^\tau p_0 e^{-rt} dt + e^{-r\tau} c_\tau = p_0 \frac{1 - e^{-r\tau}}{r} + e^{-r\tau} c_0 e^{-g(\tau - u)},$$

Let $\zeta_0 \equiv \dfrac{p_0}{c_0}$. Then, dividing the preceding equation by c_0, gives:

$$1 = \frac{1 - e^{-r\tau}}{r} \zeta_0 + e^{-(r+g)\tau + gu}. \tag{44}$$

Also, since $p_\tau = p_0$, by (43), $p_\tau = p_0 = (r + g)\, c_\tau = (r + g)\, c_0 e^{-g(\tau - u)}$, i.e.,

$$\zeta_0 = (r + g)\, e^{-g(\tau - u)}. \tag{45}$$

Substituting ζ_0 from (45) into (44) gives:

$$1 = \frac{1 - e^{-r\tau}}{r} (r + g)\, e^{-g(\tau - u)} + e^{-(r+g)\tau + gu}.$$

[6] Eq. (42) leads to Jorgensen's (1963) user-cost formula.

Multiplying by $re^{g(\tau-u)}$, and rearranging gives

$$r\left(e^{g(\tau-u)}-1\right)=g\left(1-e^{-r\tau}\right),$$

i.e., (9).

Note that if $\tau \le u$, then the RHS of (9) ≤ 0, which, since $\frac{r}{g}>0$, is a contradiction. Thus $\tau > u$.

To complete the proof of (ii) we now show that if w is sufficiently large, a solution for $\tau \in (u, u+w)$ exists and is unique. To this end, note that the RHS of (9) is: i) a continuous function of τ, which $\to \infty$ as $\tau \to u$, and $\to 0$ as $\tau \to \infty$ and ii) strictly decreasing in τ. In fact:

$$\frac{dRHS}{d\tau}=\frac{re^{-r\tau}}{e^{g(\tau-u)}-1}\left(1-e^{g(\tau-u)+r\tau}\right)<0$$

because $g(\tau-u)+r\tau>0$. So, since τ doesn't depend on w, τ satisfying this constraint exists if w is large enough.

Proof of (i).—Let $w>\tau-u$ so that τ (the date at which investment resumes) occurs before the next transition date, i.e., before the next stagnation begins. Let $\zeta_T=\dfrac{p_T}{c_T}$. Then, by reasoning analogous to the derivation of ζ_0 in Eq. (45), $\zeta_T=\zeta_0$, i.e., ζ_T is just a function of τ and exogenous parameters, independent of T. Thus $p_T/c_T=p_0/c_0$. Since $c_T=c_0e^{-gw}$

$$\frac{p_T}{p_0}=\frac{c_0e^{-gw}}{c_0}=e^{-gw}\to p_T=e^{-gw}p_0. \tag{46}$$

Note that $p_t=p_0$ for all $t\in[0,\tau]$ and only starts to decline when investment starts at date τ. Thus $p_t=p_\tau e^{-g(t-\tau)}=p_0e^{-g(t-\tau)}$ for $t\in[\tau,T)$.

Let $p_{T-}\equiv \lim_{t\nearrow T} p_t$. Then

$$p_{T-}=p_0e^{-g(T-\tau)}=p_0e^{-g(u+w-\tau)}. \tag{47}$$

Thus, using (46) and (47),

$$\frac{p_T}{p_{T-}}=\frac{e^{-gw}p_0}{p_0e^{-g(u+w-\tau)}}=e^{-g(\tau-u)}<1. \tag{48}$$

That is, the price jumps down at the transition date T implying that capital jumps up, i.e., a spike.

Proof of (iii).—Cross multiplying in (9) we obtain $r(e^{g(\tau-u)}-1)=g\left(1-e^{-r\tau}\right)$. Differentiating it with respect to g,

$$\frac{\partial\tau}{\partial g}=\frac{e^{g(\tau-u)}\left(1-(\tau-u)\,g\right)-1}{g^2\left(e^{g(\tau-u)}-e^{-r\tau}\right)}. \tag{49}$$

the denominator is always positive. Then $\frac{\partial \tau}{\partial g} < 0$ iff the numerator is negative i.e. $e^{g(\tau-u)}\left(1-(\tau-u)\,g\right)-1<0$. Note that at $\tau=u$ the equality holds, whereas

$$\frac{d\text{numerator}}{d\tau} = -g^2 e^{g(\tau-u)}\,(\tau-u)\,g < 0$$

i.e. $e^{g(\tau-u)}\left(1-(\tau-u)\,g\right)-1<0$ for all $\tau>u$. So we conclude that $\frac{\partial \tau}{\partial g} < 0$.

Using the same steps as before but differentiating (9) with respect to r,

$$\frac{\partial \tau}{\partial r} = \frac{e^{-r\tau}\,(1+r\tau)-1}{r^2\left(e^{g(\tau-u)}-e^{-r\tau}\right)}$$

the denominator is again always positive. Then $\frac{\partial \tau}{\partial r} < 0$ iff the numerator is negative i.e. $e^{-r\tau}\,(1+r\tau)-1<0$. Note that at $\tau=0$ the equality holds, whereas

$$\frac{d\text{numerator}}{d\tau} = -r^2\tau e^{-r\tau} < 0,$$

i.e. $e^{-r\tau}\,(1+r\tau)-1<0$ for all $\tau>0$. So we conclude that $\frac{\partial \tau}{\partial r} < 0$.

Proof of (iv).—From (9) we have $r\left(e^{g(\tau-u)}-1\right) = g\left(1-e^{-r\tau}\right)$. Differentiating it with respect to u,

$$rge^{g(\tau-u)}\frac{\partial}{\partial u}(\tau-u) = rg\frac{\partial \tau}{\partial u} \Rightarrow \frac{\partial \tau}{\partial u} = \frac{e^{g(\tau-u)}}{e^{g(\tau-u)}-1} > 1.$$

or alternatively $\frac{\partial(\tau-u)}{\partial u} > 0$, i.e., (iv). Finally

$$\frac{\partial\,(\tau-u)}{\partial u} > 0 \Longrightarrow \frac{d}{du}\left(\frac{p_T}{p_{T-}}\right) < 0,$$

i.e., the fall in price at the spike rises with u and thus the size of the spike $(k_{\mathrm{T}}-k_{\mathrm{T}\text{-}})/k_{\mathrm{T}\text{-}}$ also rises with u, i.e., (v). ∎

D. Proof of Proposition 4.

Proof. Let s be the cumulative time in state H from the spike date T_n until τ when investment resumes, so that

$$c_\tau = c_0 e^{-gs}, \tag{50}$$

where c_0 is the cost at the spike date. I.e., s it is the waiting time (in state H) for c_t to reach $c_\tau = \theta c_0$. With θ defined in Eq. (14), we need to show that $\theta < 1$.

At the spike date cost equals discounted revenue. Then, as there is no further investment until τ, $p_t = p_0$ for $t \in [0,\tau]$. At that date we are in state H, and since continuous investment resumes at τ, $v_\tau^{\mathrm{H}} = c_\tau$. And since obsolescence is higher in state H, $v_t^{\mathrm{H}} \le v_t^{\mathrm{L}}$.

And since v_t cannot exceed c_t this means that $v_\tau^{\mathrm{H}} = v_\tau^{\mathrm{L}} = c_\tau$. Eq. (6) then reads

$$rc_\tau = p_\tau - gc_\tau. \tag{51}$$

where $p_\tau = p_0$ is the price established at the last spike. When investment resumes at τ the value of the unit of capital created will equal c_τ, and thus with $F(\tau \mid \theta)$ the CDF of τ conditional on θ, we have

$$c_0 = \int_0^\infty \left(p_0 \frac{1 - e^{-r\tau}}{r} + e^{-r\tau} c_\tau \right) dF(\tau \mid \theta). \tag{52}$$

Using (50), Eq. (52) reads

$$1 = \int_0^\infty \left(\zeta \frac{1 - e^{-r\tau}}{r} + \theta e^{-r\tau} \right) dF(\tau \mid \theta), \tag{53}$$

where

$$\zeta \equiv \frac{p_0}{c_0}. \tag{54}$$

Since $p_0 = p_\tau = (r+g)\theta c_0$ where the second equality follows from (51), we have $\zeta = (r+g)\theta$ which, when substituted into (53) yields

$$1 = \theta \int_0^\infty \left(\left(1 + \frac{g}{r}\right)\left(1 - e^{-r\tau}\right) + e^{-r\tau} \right) dF(\tau \mid \theta).$$

i.e., we have an equation in θ alone. Expanding the value inside the integral, we have $\left(1 + \frac{g}{r}\right)\left(1 - e^{-r\tau}\right) + e^{-r\tau} = 1 + \frac{g}{r}\left(1 - e^{-r\tau}\right)$, and so the equation for θ is

$$\theta^{-1} = 1 + \frac{g}{r} \int_0^\infty \left(1 - e^{-r\tau}\right) dF(\tau \mid \theta)$$

i.e. Eq. (15) in the text. We shall complete the proof by showing that $\theta < 1$. Since $\tau \geq u$, and since the CDF of u is $1 - e^{-\lambda_{\mathrm{L}} u}$, $F(\tau \mid \theta) \leq 1 - e^{-\lambda_{\mathrm{L}} \tau}$. Since the term $1 - e^{-r\tau}$ is non-negative and strictly increasing in τ and since $F(\tau \mid \theta)$ first-order dominates $1 - e^{-\lambda_{\mathrm{L}} \tau}$,

$$\int_0^\infty \left(1 - e^{-r\tau}\right) dF(\tau \mid \theta) \geq \int_0^\infty \left(1 - e^{-r\tau}\right) \lambda_{\mathrm{L}} e^{-\lambda_{\mathrm{L}} \tau} d\tau = \frac{r}{\lambda_{\mathrm{L}} + r}$$

so

$$\theta^{-1} \geq 1 + \frac{g}{r} \frac{r}{\lambda_{\mathrm{L}} + r} \Rightarrow \theta \leq \frac{\lambda_{\mathrm{L}} + r}{\lambda_{\mathrm{L}} + r + g} < 1,$$

and that completes the proof of Proposition 4. ■

E. Properties of the inter-kink hazard (13)

Property 1: The hazard rate is increasing in t,

$$\eta'(t) = \lambda_{\mathrm{u}}\lambda_{\mathrm{w}}\left(\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}}\right)^2 \frac{e^{(\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}})t}}{\left(\lambda_{\mathrm{w}}e^{(\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}})t}-\lambda_{\mathrm{u}}\right)^2} \geq 0,$$

and it is concave in t:

$$\begin{aligned}\eta''(t) &= -\lambda_{\mathrm{u}}\lambda_{\mathrm{w}}\left(\frac{\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}}}{\lambda_{\mathrm{w}}e^{(\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}})t}-\lambda_{\mathrm{u}}}\right)^2 e^{(\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}})t}\left(\lambda_{\mathrm{w}}e^{(\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}})t}+\lambda_{\mathrm{u}}\right)\frac{\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}}}{\lambda_{\mathrm{w}}e^{(\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}})t}-\lambda_{\mathrm{u}}}\\ &\leq 0\end{aligned}$$

To derive the limit it helps to write $\eta(t)$ as follows:

$$\eta(t) = \lambda_{\mathrm{u}}\left(1-\frac{\left(\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}}\right)e^{-\lambda_{\mathrm{w}}t}}{\lambda_{\mathrm{w}}e^{-\lambda_{\mathrm{u}}t}-\lambda_{\mathrm{u}}e^{-\lambda_{\mathrm{w}}t}}\right) = \lambda_{\mathrm{u}}\left(1-\frac{\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}}}{\lambda_{\mathrm{w}}e^{(\lambda_{\mathrm{w}}-\lambda_{\mathrm{u}})t}-\lambda_{\mathrm{u}}}\right),$$

Property 2. The expression simplifies when $\lambda_{\mathrm{u}} = \lambda_{\mathrm{w}} = \lambda$; in that case

$$g(t) = \lambda^2 te^{-\lambda t}$$

because

$$\begin{aligned}G(t) &= \int_0^t s\lambda^2 e^{-\lambda s}ds = \lambda^2\left(s\frac{-e^{-\lambda s}}{\lambda}\Big|_0^t + \frac{1}{\lambda}\int_0^t e^{-\lambda s}ds\right) = \lambda^2\left(\frac{-te^{-\lambda t}}{\lambda} + \frac{-e^{-\lambda s}}{\lambda^2}\Big|_0^t\right)\\ &= 1-e^{-\lambda t}-\lambda te^{-\lambda t}\end{aligned}$$

and hazard rate simplifies to

$$\eta(t) = \frac{g}{1-G} = \frac{\lambda^2 te^{-\lambda t}}{e^{-\lambda t}+\lambda te^{-\lambda t}} = \frac{\lambda^2 t}{1+\lambda t} = \lambda\left(1-\frac{1}{1+\lambda t}\right)$$

which increases from $\eta(0) = 0$ to λ.

F. Steps taken to derive F

The definition of F in Eq. (18) requires that we have Ψ which, in turn, uses A. This appendix first present the solution for A following which it verifies the corner conditions and limiting properties of the resulting solutions for (A, Ψ).

Proposition 6

$$A(\omega \mid t) = \begin{cases} e^{-\lambda_H\omega-\lambda_L(t-\omega)}\sum_{n=1}^{\infty}\frac{(\lambda_H\omega)^n}{n!}\sum_{i=1}^{n}\frac{(\lambda_L(t-\omega))^{i-1}}{(i-1)!}. & \textit{for } 0<\omega<t\\ 1 & \textit{for } \omega \geq t\end{cases}. \tag{55}$$

Proof. Denote $H(t) = 1 - e^{-\lambda_{\mathrm{H}} t}$, and $L(t) = 1 - e^{-\lambda_{\mathrm{H}} t}$,using Theorem 2.1 of Zacks (2012), we have

$$A(\omega \mid t) = \begin{cases} 0 & \text{for } \omega < 0 \\ 1 - \sum_{n=0}^{\infty} [H^{(n)}(\omega) - H^{(n+1)}(\omega)] L^{(n)}(t-\omega) & \text{for } 0 < \omega < t \\ 1 & \text{for } \omega \geq t \end{cases} \tag{56}$$

where $H^{(n)}(\cdot)$ is the n-fold convolution of $H(\cdot)$. Consider any exponential distribution $F = 1 - e^{-\lambda t}$,

$$F^{(2)}(t) = \int_0^t \lambda e^{-\lambda w} \left(\int_0^{t-w} \lambda e^{-\lambda u} du \right) dw = 1 - e^{-\lambda t} - \int_0^t \lambda e^{-\lambda t} dw = F^{(1)}(t) - \lambda t e^{-\lambda t}$$

Similarly, substituting the value of $F^{(2)}(t)$ into $F^{(3)}(t)$ and simplifying gives

$$F^{(3)}(t) = \int_0^t \lambda e^{-\lambda w} F^{(2)}(t-w) dw = F^{(2)}(t) - \int_0^t \lambda^2 (t-w) e^{-\lambda t} dw = F^{(2)}(t) - \frac{1}{2} \lambda^2 t^2 e^{-\lambda t}$$

and

$$F^{(4)}(t) = \int_0^t \lambda e^{-\lambda w} F^{(3)}(t-w) dw = F^{(3)}(t) - \int_0^t \lambda \frac{1}{2} \lambda^2 (t-w)^2 e^{-\lambda t} dw = F^{(3)}(t) - \frac{1}{3!} \lambda^3 t^3 e^{-\lambda t}$$

We now guess that

$$F^{(n+1)}(t) = F^{(n)}(t) - \frac{(\lambda t)^n}{n!} e^{-\lambda t}. \tag{57}$$

If $F^{(n)}(t)$ satisfies (57), then we have:

$$\begin{aligned} F^{(n+1)}(t) &= \int_0^t \lambda e^{-\lambda w} F^{(n)}(t-w) dw \\ &= \int_0^t \lambda e^{-\lambda w} \left(F^{(n-1)}(t-w) - \frac{\lambda^{n-1} (t-w)^{n-1}}{(n-1)!} e^{-\lambda (t-w)} \right) dw \\ &= \int_0^t \lambda e^{-\lambda w} F^{(n-1)}(t-w) dw - \int_0^t \lambda e^{-\lambda w} \left(\frac{\lambda^{n-1} (t-w)^{n-1}}{(n-1)!} \right) e^{-\lambda (t-w)} dw \\ &= F^{(n)}(t) - \int_0^t \frac{\lambda^n (t-w)^{n-1}}{(n-1)!} e^{-\lambda t} dw = F^{(n)}(t) - \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \end{aligned}$$

which also satisfies (57). Thus, for $0 < \omega < t$, we have

$$\begin{aligned}
A\left(\omega \mid t\right) &= 1-\sum_{n=0}^{\infty}\frac{\left(\lambda_{\mathrm{H}}\omega\right)^{n}}{n!}e^{-\lambda_{\mathrm{H}}\omega}L^{(n)}(t-\omega) \\
&= 1-e^{-\lambda_{\mathrm{H}}\omega}L^{(0)}(t-\omega)-\sum_{n=1}^{\infty}\frac{\left(\lambda_{\mathrm{H}}\omega\right)^{n}}{n!}e^{-\lambda_{\mathrm{H}}\omega}L^{(n)}(t-\omega) \\
&= 1-e^{-\lambda_{\mathrm{H}}\omega}-\sum_{n=1}^{\infty}\frac{\left(\lambda_{\mathrm{H}}\omega\right)^{n}}{n!}e^{-\lambda_{\mathrm{H}}\omega}\left(1-\sum_{i=1}^{n}\frac{\left(\lambda_{\mathrm{L}}(t-\omega)\right)^{i-1}}{(i-1)!}e^{-\lambda_{\mathrm{L}}(t-\omega)}\right) \\
&= 1-e^{-\lambda_{\mathrm{H}}\omega}\sum_{n=0}^{\infty}\frac{\left(\lambda_{\mathrm{H}}\omega\right)^{n}}{n!}+\sum_{n=1}^{\infty}\frac{\left(\lambda_{\mathrm{H}}\omega\right)^{n}}{n!}e^{-\lambda_{\mathrm{H}}\omega}\sum_{i=1}^{n}\frac{\left(\lambda_{\mathrm{L}}(t-\omega)\right)^{i-1}}{(i-1)!}e^{-\lambda_{\mathrm{L}}(t-\omega)} \\
&= 1-e^{-\lambda_{\mathrm{H}}\omega}e^{\lambda_{\mathrm{H}}\omega}+\sum_{n=1}^{\infty}\frac{\left(\lambda_{\mathrm{H}}\omega\right)^{n}}{n!}e^{-\lambda_{\mathrm{H}}\omega}\sum_{i=1}^{n}\frac{\left(\lambda_{\mathrm{L}}(t-\omega)\right)^{i-1}}{(i-1)!}e^{-\lambda_{\mathrm{L}}(t-\omega)} \\
&= e^{-\lambda_{\mathrm{H}}\omega-\lambda_{\mathrm{L}}(t-\omega)}\sum_{n=1}^{\infty}\frac{\left(\lambda_{\mathrm{H}}\omega\right)^{n}}{n!}\sum_{i=1}^{n}\frac{\left(\lambda_{\mathrm{L}}(t-\omega)\right)^{i-1}}{(i-1)!}. \qquad (58)
\end{aligned}$$

■

REMARKS: We check that the solution in Eq. (55) satisfies several corner and limit conditions that $A(\omega \mid t)$ and the resulting $\Psi\left(\omega \mid t\right)$ in Eq. (17) should also satisfy

$$\begin{aligned}
\lim_{\lambda_{\mathrm{L}}\to\infty} A\left(\omega \mid t\right) &= \begin{cases} 0 & \text{for } \omega < t \\ 1 & \text{for } \omega = t \end{cases} \qquad (59) \\
\lim_{\lambda_{\mathrm{L}}\to\infty} \Psi\left(\omega \mid t\right) &= \begin{cases} 0 & \text{for } \omega < t \\ 1 & \text{for } \omega = t \end{cases}
\end{aligned}$$

$$\begin{aligned}
\lim_{\lambda_{\mathrm{L}}\to 0} A\left(\omega \mid t\right) &= 1-e^{-\lambda_{\mathrm{H}}\omega} \\
\lim_{\lambda_{\mathrm{L}}\to 0} \Psi\left(\omega \mid t\right) &= 1 \\
\lim_{\lambda_{\mathrm{H}}\to\infty} A\left(\omega \mid t\right) &= 1 \\
\lim_{\lambda_{\mathrm{H}}\to\infty} \Psi\left(\omega \mid t\right) &= 1 \\
\lim_{\lambda_{\mathrm{H}}\to 0} A\left(\omega \mid t\right) &= \begin{cases} 0 & \text{for } \omega < t \\ 1 & \text{for } \omega = t \end{cases} \\
\lim_{\lambda_{\mathrm{H}}\to 0} \Psi\left(\omega \mid t\right) &= e^{-\lambda_{\mathrm{L}}(t-\omega)} \qquad (60)
\end{aligned}$$

Conditions (59)-(60) were checked.

Simulation steps for Ψ, F, and H—We substitute into (17) from (55), we can solve for $\Psi\left(\omega \mid t\right)$. Using equation (18), we can get $F(\tau \mid \theta)$. Recall that θ satisfies

$$\theta^{-1} = 1 + \frac{g}{r}\int_0^\infty \left(1 - e^{-r\tau}\right) dF(\tau \mid \theta).$$

Denote the solution for the above by θ^*. Then the lower bound of τ is $\tau_{\min} = \frac{1}{g}\ln\theta^* = s$ and the distribution of τ is then $F(\tau \mid \theta^*)$ with $F(\tau \mid \theta)$ as defined in Eq. (18).

G. Unweighted Regressions

We now present the analog of Table 5, but without industry weights:

Table 6. Unweighted Regression results using $\hat{g}_1$ and $\hat{g}_2$

Dep. var.	IS	KV-A	KV-B	IS	KV-A	KV-B
$\hat{g}_1$	1.110***	0.063***	0.084**			
	(0.336)	(0.018)	(0.036)			
$\hat{g}_2$				0.438**	0.022*	0.043***
				(0.174)	(0.01)	(0.015)
Constant	-0.991	0.164***	0.277***	0.596	0.260***	0.374***
	(0.763)	(0.045)	(0.084)	(0.385)	(0.025)	(0.035)
Observations	20	20	20	20	20	20
R^2	0.35	0.38	0.20	0.25	0.22	0.24

As mentioned in the text, the estimates remain significant except for the effect of $\hat{g}_2$ on KV-A.

H. Firm vs. Industry spikes

We augment the discussion in Sec.3.4 with a discrete time model with an equilibrium with a Markov structure. There is a continuum of firms with no entry or exit. In state H, costs decline so that next period cost c' is

$$c' = \gamma^{\chi} c, \tag{61}$$

where $\gamma < 1$ and where

$$\chi = \begin{cases} 0 & \text{if } \mathrm{J} = \mathrm{L} \\ 1 & \text{if } \mathrm{J} = \mathrm{H} \end{cases}. \tag{62}$$

In other words, state L costs do not change and $c' = c$ so that $\chi = 0$ means there will be no progress and $\chi = 1$ means there will be progress. For I, J $\in$ {L, H}, let $\lambda_{\mathrm{I}} = \Pr(\mathrm{I} \rightarrow \mathrm{J})$ so that the Markov transition matrix for χ is

$$\chi'$$

$$\chi \qquad \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \left[\begin{array}{cc} 1-\lambda_{\mathrm{L}} & \lambda_{\mathrm{L}} \\ \lambda_{\mathrm{H}} & 1-\lambda_{\mathrm{H}} \end{array} \right] \end{array} \tag{63}$$

A firm's production function is as given in Eq. (28), i.e., zk^{α}, and its profit is $py - cx$, with x denoting the firm's investment. The firm's k evolves as

$$k' = k + x, \tag{64}$$

and its z is again drawn from the CDF Ψ each period, independently of other firms' z values and of its past z realizations. Next period's value z – call it z' – is perfectly foreseen one period ahead.

The industry state is $S \equiv (c, \chi, \Phi)$, and for the firm state we include $s \equiv (z, z', k)$.

Market clearing.—The market-clearing price solves for p in the equation

$$D(p) = \int zk^{\alpha} d\Phi(z, k). \tag{65}$$

It depends only on Φ and not on $\{c, \chi\}$ and is denoted by $p = p(\Phi)$.

The *state of the system* is the cost of capital, and its projected evolution and the cross-firm distribution $\Phi(z, k)$.

Evolution of the aggregate state.—The pair (c, χ) evolves exogenously as the sentence after Eq. (61) explains and as Eq. (63) specifies. Let us hypothesize the evolution for Φ to be

$$\Phi' = \xi(S) \tag{66}$$

and later make it consistent with firms' decisions. The Timing of decisions is as follows:—

1. When the period opens, everyone learns whether next period J will be H or L, and every firm learns its z'. Therefore the firm's state (s, S) is in place.

2. Each firm then chooses its x and collects its profit $pzk^{\alpha} - xc$

Then

$$v(s, S) = p(\Phi) zk^{\alpha} + \max_{k' \geq k} \left\{ -c(k' - k) + \frac{1}{1+r} E[v(s', S') \mid s, S] \right\}, \tag{67}$$

with the FOC

$$c \geq \frac{1}{1+r} \frac{\partial}{\partial k} E[v(s', S') \mid s, S], \tag{68}$$

with equality if $k' > k$. Applying the envelope theorem,

$$\frac{\partial v(s, S)}{\partial k} = p(\Phi) z\alpha k^{\alpha-1} + c. \tag{69}$$

Updating (s, S) to (s', S') on the RHS of Eq. (69) and rearranging, Eq. (68) becomes

$c \geq \frac{1}{1+r}\left(p\left(\xi\left(S\right)\right)z'\alpha\left(k'\right)^{\alpha-1} + \gamma^{\chi}c\right)$, i.e.,

$$p\left(\xi\left(S\right)\right)z'\alpha\left(k'\right)^{\alpha-1} \leq \left(r+1-\gamma^{\chi}\right)c. \tag{70}$$

The LHS of Eq. (74) is the expected marginal product of capital and the RHS is the Jorgenson user cost consisting of the interest cost rc plus the obsolescence term $\left(1-\gamma^{\chi}\right)c$. The inequality is strict if $k' < k$. Exact equality in Eq. (70) holds when $k' > k$; denote its solution for z' as a function of a hypothetical k' by

$$Z\left(k', S\right) = \frac{\left(r+1-\gamma^{\chi}\right)c}{p\left(\xi\left(S\right)\right)\alpha}\left(k'\right)^{1-\alpha}. \tag{71}$$

This solution exists for all k' if the support of Ψ is $\mathbb{R}R_{+}$.

The aggregate law of motion in Eq. (66).—For a given value k', any firm with $k > k'$ will have more capital than k' in the next period. And so will a firm with a value z' exceeding $Z\left(k'\right)$, and so

$$\xi\left(S\right) = \underbrace{\int_0^{z'} I_{\{z \leq Z(k',S)\}}\psi(z)dz}_{\text{firms with } \tilde{z}' \leq z' \text{ and } \tilde{z}' \leq Z(k')} \times \underbrace{\int_0^{k'}\int_0^{\infty}\phi_t\left(z,k\right)dzdk.}_{\text{firms with } k<k'} \tag{72}$$

Equilibrium consists of the 3 functions (p, v, ξ) describing aggregates and their law of motion, and the firms decision rules for k'.

Spikes.—To show that a spike results from the cessation of progress we start two hypothetical economies at the same (c, Φ), but one with $\chi = 1$ (tech. progress) and the other with $\chi = 0$. Solving (70) for k',

$$k' = \max\left(k, \frac{\alpha p\left(\xi\left(S\right)\right)z'}{\left(r+1-\gamma^{\chi}\right)c}\right). \tag{73}$$

When χ alone among the aggregate states changes, the user cost drops from $\left(r+1-\gamma\right)c$ to rc. Then if p did not fall, the RHS of Eq. (73) would rise for all firms, except some for which z' is significantly below the zero investment boundary. Such firms must exist if the lower bound of the support of Ψ is zero as would be the case, for example, if Ψ was the log normal distribution

The zero-investment boundary.—The value of z' at which the firm with current capital k is indifferent between investing and not investing at all is $Z\left(k, S\right)$, i.e., the expression in Eq. (71) evaluated at $k' = k$.

Size of spike.—Since $p\left(\xi\left(c, 0, \Phi\right)\right) < p\left(\xi\left(c, 0, \Phi\right)\right)$ the industry's output must rise; the rise in the industry's capital stock is larger the more elastic is the demand for the product.

One would expect that if χ remains at zero and the low-progress state persists for a long time, the RHS of (73) would remain unchanged at $\max\left(k, \frac{\alpha p z'}{(r+1-\gamma^{x})c}\right)$, firms with higher z' values will still invest driving the p further down (but by less than initially), but not by much so that firms with low z' realizations get further and further below their zero investment boundaries. Asymptotically investment should cease and p approach a constant.

References

Abel, Andrew B, and Janice C Eberly. 2012. "Investment, valuation, and growth options." *The Quarterly Journal of Finance*, 2(01): 1250001.

Boucekkine, Raouf, Marc Germain, and Omar Licandro. 1997. "Replacement echoes in the vintage capital growth model." *Journal of economic theory*, 74(2): 333–348.

Bresnahan, Timothy F, and Daniel MG Raff. 1991. "Intra-industry heterogeneity and the Great Depression: The American motor vehicles industry, 1929–1935." *The Journal of Economic History*, 51(2): 317–331.

Caballero, Ricardo J, and Eduardo MRA Engel. 1999. "Explaining investment dynamics in US manufacturing: a generalized (S, s) approach." *Econometrica*, 67(4): 783–826.

Caballero, Ricardo J, and Mohamad L Hammour. 1996. "On the timing and efficiency of creative destruction." *The Quarterly Journal of Economics*, 111(3): 805–852.

Cooper, Russell, John Haltiwanger, and Laura Power. 1999. "Machine replacement and the business cycle: lumps and bumps." *American Economic Review*, 89(4): 921–946.

Cooper, Russell W, and John C Haltiwanger. 2006. "On the nature of capital adjustment costs." *The Review of Economic Studies*, 73(3): 611–633.

Cummins, Jason G, and Giovanni L Violante. 2002. "Investment-specific technical change in the United States (1947–2000): Measurement and macroeconomic consequences." *Review of Economic dynamics*, 5(2): 243–284.

Dixit, Avinash K, Robert K Dixit, and Robert S Pindyck. 1994. *Investment under uncertainty.* Princeton university press.

Doms, Mark, and Timothy Dunne. 1993. "An investigation into capital and labor adjustment at the plant level." *Manuscript, Center for Economic Studies, US Bureau of the Census.*

Gourio, Francois, and Anil K Kashyap. 2007. "Investment spikes: New facts and a general equilibrium exploration." *Journal of Monetary Economics*, 54: 1–22.

Guo, Xin, Jianjun Miao, and Erwan Morellec. 2002. "Irreversible investment with regime shifts." *Simon School of Business Working Paper No. FR*, 02–20.

Ilyina, Anna, and Roberto Samaniego. 2011. "Technology and financial development." *Journal of Money, Credit and Banking*, 43(5): 899–921.

Jorgenson, Dale W. 1963. "Capital theory and investment behavior." *The American Economic Review*, 53(2): 247–259.

Jovanovic, Boyan, and Chung-Yi Tse. 2010. "Entry and exit echoes." *Review of Economic Dynamics*, 13(3): 514–536.

Jovanovic, Boyan, and Dmitriy Stolyarov. 2000. "Optimal adoption of complementary technologies." *American Economic Review*, 90(1): 15–29.

Kehrig, Matthias, and Nicolas Vincent. 2018. "Good dispersion, bad dispersion." National Bureau of Economic Research.

Khan, Aubhik, and Julia K Thomas. 2008. "Idiosyncratic shocks and the role of nonconvexities in plant and aggregate investment dynamics." *Econometrica*, 76(2): 395–436.

Klenow, Peter J. 1998. "Learning curves and the cyclical behavior of manufacturing industries." *Review of Economic dynamics*, 1(2): 531–550.

National Bureau of Economic Research. (1952-2009). "NBER-CES Manufacturing Industry Database." http://data.nber.org/nberces/ (accessed Oct 8, 2018).

Oguntunde, PE, OA Odetunmibi, and AO Adejumo. 2014. "On the Sum of exponentially distributed random variables: A convolution approach." *European Journal of Statistics and Probability*, 2(1): 1–8.

Sargent, Thomas J. 1980. "'Tobin's q' and the rate of investment in general equilibrium." Vol. 12, 107–154, Elsevier.

U.S. Census Bureau. (1987-2002). "North American Industry Classification System." https://www.census.gov/eos/www/naics/concordances/concordances.html (accessed Oct 8, 2018).