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ABSTRACT

We propose a tractable model of dynamic investment, division sales (spinoffs), financing, and risk management for a multi-division firm facing costly external finance. Our main results are: (1) within-firm resource allocation is based not only on the divisions' productivity—as in standard “winner picking” models—but also their risk; (2) firms may choose to voluntarily spin off productive divisions to increase liquidity; (3) diversification can reduce firm value in low liquidity states, as it hampers liquidity management;(4) corporate socialism makes liquidity less valuable; (5) division investment is determined by the ratio between marginal *q* and marginal value of cash.

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1 Introduction

Multi-division firms—i.e., firms that operate two or more divisions and allocate resources to their divisions through an internal capital market—play an important role in the economy. For example, Maksimovic and Phillips (2002) estimate that multi-division firms account for about three-fourths of total output in the U.S. manufacturing sector.

The empirical literature shows that multi-division firms behave very differently compared to stand-alone firms. These differences are found across firm policies, including the “holy grail” of corporate finance—that is, cash management, financing, and investment decisions. For example, multi-division firms tend to hold less cash (Duchin, 2010), are more resilient when external capital markets are disrupted (Matvos and Seru, 2014), and actively reallocate resources across divisions (Giroud and Mueller, 2015). The objective of this paper is to propose a tractable dynamic framework that sheds light on the mechanics of multi-division firms, taking into account the complex and intertwined nature of their risk management, financing, and investment decisions.

Broadly speaking, the theory literature on multi-division firms can be classified into two camps: the “bright side” and “dark side” theories of internal capital markets. The bright side theories highlight the winner picking role of headquarters (Alchian, 1969; Williamson, 1975; Stein, 1997). In these models, headquarters can create value by reallocating resources from the less productive divisions toward the more productive ones (the “winners”). In contrast, dark side theories argue that internal capital markets are plagued with agency conflicts, as they give rise to internal politics in the allocation of resources. This notion was first proposed by Coase (1937), who argued that power within a hierarchy impacts internal policies, and later formalized in the models of influence activities (Milgrom, 1988; Milgrom and Roberts, 1988; Meyer, Milgrom, and Roberts, 1992). In these models, managers of weaker divisions have an incentive to lobby headquarters for more resources, in an attempt to distort the resource allocation in their favor. To mitigate such inefficient lobbying, headquarters may find it optimal to tilt the resource allocation towards “corporate socialism” such that stronger divisions end up cross-subsidizing the weaker ones (Rajan, Servaes, and Zingales, 2000; Scharfstein and Stein, 2000).

While these models have been influential, they are subject to two main limitations. First, they typically take other policies (e.g., cash management) as given, and hence do

not account for the interdependence across these policies. As we show, allowing for a joint determination of these policies often reverses the predictions from simpler models featuring fewer policies. Second, these models are static, and hence do not account for the changing conditions companies face in a dynamic environment. These limitations are non-trivial. In a dynamic world, firms can run low on cash, which generates a need for state-contingent and time-varying risk management policy. In turn, this can affect the way internal capital markets operate. For example, the notion of winner picking mentioned above—albeit well-established in the literature—may need to be qualified. When companies run low on cash, the shareholder-value maximizing policy may no longer be to allocate resources to high-productivity divisions, but instead to low-risk divisions. Or companies may decide to spin off entire divisions, preferring higher corporate cash holdings over diversification benefits (by retaining more divisions). More broadly, as these examples illustrate, it is important to consider the dynamic and intertwined nature of multi-division firms’ policies when formulating a theory of internal capital markets.

This paper aims to fill this gap, by providing a tractable dynamic framework in which cash management, external financing, dividend payout, division sale (spinoff), and investment (including cross-divisional transfers) are characterized simultaneously for a multi-division firm that faces costly external finance. Our framework builds on the model of Bolton, Chen, and Wang (2011), henceforth BCW (2011), for stand-alone firms. Compared to BCW (2011), our framework has two main innovations. First, we consider a firm with two divisions. As such, our model has two key state variables: (1) the ratio of capital stock between the two divisions, denoted by z , which is new in our model, and (2) the ratio between the liquid asset (cash) and the illiquid productive capital stock (the sum of capital stock in the two divisions), denoted by w . Second, we allow for lobbying frictions at the division level, in the spirit of the dark side models of internal capital markets. As we will show, our parsimonious framework captures many situations that multi-division firms face in practice, and yields a rich set of prescriptions.

Our main results are as follow. First, starting with the case without corporate socialism, we find that multi-division firms hold less cash, require lower amounts of external financing, and can more easily pay dividends compared to stand-alone firms. These prediction are intuitive—diversification decreases the volatility of the firm’s cash flows, and hence reduces

the need for liquidity. This lower need for liquidity is consistent with Duchin's (2010) finding that multi-division firms tend to hold less cash than stand-alone firms.

Second, when firms run out of cash, they may optimally choose to spin off one of their divisions. Given the lumpy nature of division sales, the spinoff can generate more cash than what the firm needs to efficiently operate the remaining division, in which case the excess amount is paid out to shareholders as a special dividend. This is consistent with Dittmar's (2004) finding that firms often pay a special dividend subsequent to a spinoff.¹ Another implication of the model is that diversification can make future division sales (when the firm runs low on cash and has to increase cash holdings via division sale) more costly. Taking both dimensions into account, our model implies a dark and bright side of diversification from the perspective of liquidity management, even for a shareholder-value maximizing conglomerate. In good times, diversification reduces the need for liquidity. In bad times, diversification can hamper the firm's liquidity management by making division sales more costly.

Third, we find that, when companies are flush with cash, they allocate more of their resources to the high-productivity division, as predicted by static models of winner picking. However, when cash is scarce, the risk management motive dominates and companies allocate more of their resources to the low-risk division. Taking both aspects into account motivates a broader formulation of the “winner picking” role of internal capital markets: when headquarters allocates resources to divisions, it does so not only based on productivity, but also based on risk. In this regard, the within-firm allocation of resources is analogous to a dynamic portfolio choice problem, in which funding is allocated based on the risk-return profile of the individual securities. Unlike the standard portfolio choice problem (e.g., Merton, 1971), the risk-neutral firm in our setting is endogenously risk averse. As we will show, this endogenous risk aversion depends not only on the firm's scaled cash balance, w , but also the ratio of capital stock between the two divisions, z , as well as the cost of external financing and the cost of liquidating a division.

This insight has important implications for capital budgeting. Indeed, contrary to

¹This result also speaks to the literature on leveraged buyouts (LBOs) that finds that, following LBO deals, LBO investors often sell entire divisions and subsequently pay out large dividends (Eckbo and Thorburn, 2008). While this is often seen as a form of asset stripping in the interest of the LBO investors, our framework offers a potential shareholder-value maximizing interpretation.

the textbook view, ignoring idiosyncratic risk and the balance sheet of the conglomerate when doing capital budgeting is incorrect; depending on the firm's liquidity, it may be optimal to invest in a lower-NPV project if the project's idiosyncratic risk is sufficiently low. Introducing a project (division) changes the firm's entire balance sheet composition and risk profile. As such, the firm should value the new project by computing the net value difference caused by introducing the new project into the firm, as opposed to evaluating the project as if it were a stand-alone project.

Fourth, we find that corporate socialism reduces the value of the firm, exacerbates underinvestment, and hampers the winner picking. While these results are intuitive, one subtle implication of socialism is that division sales become less costly with socialism than without, as spinning off a division (and becoming a stand-alone firm) eliminates socialism frictions and hence is more valuable for a conglomerate with socialism. This has implications for liquidity management. Indeed, with socialism, liquidity is less valuable since it is less costly to replenish the firm's liquidity through a spinoff.

Fifth, our model offers insights on the q theory of investment for a conglomerate. In neoclassical settings where the Modigliani-Miller (MM) theorem holds, the value of a conglomerate is simply the sum of its divisions' values. In contrast, in our model, the conglomerate's division-level investment decisions depend on not only liquidity (as in BCW, 2011), but also the relative (capital stock) size of the two divisions. With convex adjustment costs, a financially constrained conglomerate equates the ratio between the marginal q and the marginal cost of investing in each division to the marginal value of cash.² This is a generalized version of the q theory of investment for a financially constrained single-division firm analyzed in BCW (2011).

Finally, our paper also makes two methodological contributions. First, we characterize the solution of a diversified firm's two-dimensional optimization problem by using a variational-inequality method and provide a verification theorem. Second, we develop a penalty-function-based iterative procedure to solve the variational inequality.

Related Literature. Our paper is related to several strands of the literature. First, it is related to the few but notable studies that use dynamic modeling to study the behavior

²The assumption of convex adjustment costs allows us to simplify the analysis by leaving out the possibility that inaction is optimal.

of multi-division firms. In particular, Gomes and Livdan (2004) use a dynamic model to examine the valuation implications of diversification.³ Matvos and Seru (2014) estimate a structural dynamic model that quantifies the extent to which internal capital markets helped offset the financial market disruptions that occurred during the financial crisis of 2007-2010. Bakke and Gu (2015) examine the rationales as to why multi-division firms hold less cash than stand-alone firms, estimating a structural dynamic model that quantifies the respective importance of selection (when a stand-alone firm endogenously becomes a multi-division firm) and diversification. Compared to these articles, our paper focuses on the interconnections between the various policies of multi-division firms. This allows us to provide a rich set of prescriptions that speak to various aspects of internal capital markets, ranging from the validity of winner picking predictions to the economics of spinoffs.

Second, our paper is related to the large literature that uses dynamic models of financially constrained firms to characterize their investment, financing, and risk management decisions (e.g., Cooley and Quadrini, 2001; Gomes, 2001; Hennessy and Whited, 2007; Ridick and Whited, 2009; Bolton, Chen, and Wang, 2011; Décamps, Mariotti, Rochet, and Villeneuve, 2011; Hugonnier, Malamud and Morellec, 2015; Nikolov, Schmid, and Steri, 2019; Abel and Panageas, 2020). Also related is the work of Malenko (2019), who uses a dynamic model to study the optimal capital budgeting mechanism in a single-division firm.

While our framework shares various features with the models proposed in this literature, the key difference is our focus on a two-division firm as opposed to a representative single-division firm modeled in these papers. As a result, our model, while parsimonious, is inevitably a two-dimensional problem involving partial differential equations (PDEs). This is a key difference from almost all existing models in the literature, whose formulations can be simplified to one-dimensional problems whose solutions are characterized by ordinary differential equations (ODEs). Despite the richness of our model, we offer a theoretical framework that remains analytically tractable and economically intuitive, provide proofs of the key results, and numerically solve the model with high accuracy.

Third, our paper is related to the large empirical literature that studies the mechanics

³In their model, stand-alone firms diversify only when they become relatively unproductive in their current activities. This relates to the earlier models by Matsusaka (2001) and Maksimovic and Phillips (2002), who predict that firms diversify into industries that match their organizational capabilities and managerial resources, respectively.

of internal capital markets. This literature finds support for both the bright and dark side views. In particular, the findings of Maksimovic and Phillips (2002), Guedj and Scharfstein (2004), and Giroud and Mueller (2015) indicate that companies allocate resources in a value-enhancing manner. Naturally, this need not imply that internal capital markets achieve the first-best allocations. Indeed, the shareholder-value maximizing formulation of our model does not deliver first-best allocations. In this regard, Shin and Stulz (1998), Rajan, Servaes, and Zingales (2000), and Ozbas and Scharfstein (2010) find evidence for distortions that are consistent with the models of corporate socialism.⁴ Overall, the empirical evidence suggests that our model, which combines both the bright and dark sides—i.e., firms striving to allocate resources in a value-maximizing fashion, while facing rent-seeking behavior of their division managers—might provide a realistic characterization of internal capital markets.⁵

Lastly, our paper is related to the literature on corporate spinoffs (e.g., Maksimovic and Phillips, 2001; Dittmar, 2004; Eckbo and Thorburn, 2008). In particular, and as mentioned above, our predictions that firms tend to pay out a special dividend following a spinoff is consistent with the empirical findings of Dittmar (2004).

2 Model

In the following, we introduce the diversified firm’s production and investment technology, describe the firm’s financing opportunities, and state the firm’s optimization problem.

2.1 Firm and Division Technologies

A diversified firm has two divisions, a and b . Each division employs capital as its factor of production. The price of capital is normalized to unity. We denote by K_t^s and I_t^s the level of capital stock and gross investment in division s at time t , respectively, where $s = a, b$.

⁴Direct evidence of influence activities at the division level is provided by Duchin and Sosyura (2013) and Glaser, Lopez-de-Silanes, and Sautner (2013). Relatedly, Graham, Harvey, and Puri (2015) provide survey evidence suggesting that the capital allocation is often based on the personal reputation of the division managers.

⁵For a detailed review of the empirical literature on internal capital markets, see Maksimovic and Phillips (2013).

The capital stock K_t^s of division s evolves according to

$$dK_t^s = (I_t^s - \delta_s K_t^s)dt, \quad (1)$$

where δ_s is the constant depreciation rate of the capital stock of division s .

The operating revenue generated by division s is proportional to its capital stock K_t^s and is given by $K_t^s dA_t^s$, where dA_t^s is the productivity shock for division s over time interval $(t, t + dt)$. We assume that, under the risk-neutral measure \mathbb{Q} (i.e., on a risk-adjusted basis), the cumulative (undiscounted) productivity of division s , A_t^s , follows an arithmetic Brownian motion process:

$$dA_t^s = \mu_s dt + \sigma_s d\mathcal{Z}_t^s, \quad s = a, b, \quad (2)$$

where \mathcal{Z}_t^s is a standard Brownian motion under \mathbb{Q} , and μ_s and σ_s denote the mean and volatility of the division's productivity for a unit of time under the risk-adjusted measure.⁶ We denote by ρ the constant correlation coefficient between the productivity shocks of the two divisions. That is, the quadratic co-variation between \mathcal{Z}_t^a and \mathcal{Z}_t^b , $d[\mathcal{Z}^a, \mathcal{Z}^b]_t$, is equal to ρdt . Note that the firm's productivity process in our model is a two-division generalization of the one used in BCW (2011).⁷

Let dY_t^s denote the operating profit generated by division $s = a, b$ over increment dt :

$$dY_t^s = K_t^s dA_t^s - I_t^s dt - G_t^s dt. \quad (3)$$

There are three terms contributing to the change in the division's operating profit dY_t^s . The first term in (3) is the division's operating revenue, the second term is the investment (capital acquisition) cost, and the last term describes the capital adjustment cost.

As in the q theory of investment (Lucas and Prescott, 1971; Hayashi, 1982; Abel and Eberly, 1994), we assume that the capital adjustment cost depends on investment and capital stock. That is, the capital adjustment cost in division s takes the form $G_t^s =$

⁶By directly specifying the joint productivity process for the firm's divisions under the risk-neutral measure, we incorporate the effect of the risk premium on the firm's decisions and valuation.

⁷The same i.i.d. productivity assumption is also made in dynamic contracting models, e.g., DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), DeMarzo and Fishman (2007), and DeMarzo, Fishman, He, and Wang (2012).

$$G^s(I_t^s, K_t^s).$$

For analytical tractability, we assume that the adjustment costs for both divisions are homogeneous of degree one in their divisional I and K , so that

$$G_t^s = G^s(I_t^s, K_t^s) = g_s(i_t^s) \cdot K_t^s, \quad (4)$$

where $i_t^s = I_t^s/K_t^s$ denotes the investment-capital ratio of division s at time t . (The firm engages in asset sales at the division level when investment i_t is negative.) We apply this homogeneity property, which was first proposed by Lucas and Prescott (1971) and Hayashi (1982) for corporate investment, to investment at the division level.⁸ We make the standard intuitive assumptions that $g_s(i)$ is increasing, smooth, and convex in i , as in the literature on the q theory of investment. Additionally, $g_s(0) = 0$.

The firm may choose to liquidate one or more divisions over time. show, it is suboptimal to liquidate both divisions at the same time. After liquidating division s , the firm receives liquidation value L_t^s and continues to operate as a going concern with the remaining division. Note that which division to liquidate at what time is endogenous. To preserve our model's homogeneity property, we assume that

$$L_t^s = \ell_s K_t^s, \quad (5)$$

where $\ell_s > 0$. The lower the value of ℓ_s , the more inefficient the liquidation technology for division s . Of course, the firm may eventually die as it may be optimal to also liquidate the remaining division in the future.

To focus on the economically interesting case, we impose the following conditions:

$$\mu_a > \ell_a \cdot (r + \delta_a) \quad \text{and} \quad \mu_b > \ell_b \cdot (r + \delta_b). \quad (6)$$

Otherwise, the firm prefers immediate liquidation without using its production technology.

The firm's operating cash flow, dY_t , over time increment dt is given by

$$dY_t = dY_t^a + dY_t^b = (K_t^a dA_t^a + K_t^b dA_t^b) - (I_t^a + I_t^b + G_t^a + G_t^b) dt. \quad (7)$$

⁸Eberly, Rebelo, and Vincent (2009) provide empirical evidence in support of the Hayashi homogeneity assumption for the upper-size quartile of Compustat firms.

Let τ_L denote the firm's (stochastic) liquidation/death time. If $\tau_L = \infty$, the firm never dies but may operate with only one division. For a single-division firm, $\tau_L = \tau_s$, as there is only one division s . However, for a two-division firm, division sale and firm liquidation are very different events. We thus need to differentiate between three stopping times: τ_a , τ_b , and τ_L .

2.2 External Financing Costs and Cash Management

Neoclassical investment models (Hayashi, 1982) assume that the firm faces frictionless capital markets and that the Modigliani and Miller (1958) theorem holds. In reality, however, firms often face external financing costs due to various financial frictions, e.g., transaction costs, asymmetric information, and managerial incentive problems.⁹

External financing costs. We do not explicitly model the micro foundations of financing costs. Instead, we directly specify equity issuance costs as in the literature. Specifically, as in BCW (2011), we assume that a firm incurs both a fixed cost Φ and a proportional (marginal) cost γ whenever it chooses to issue external equity. Together, these costs imply that the firm will optimally tap equity markets only intermittently, and when doing so it raises funds in lumps, consistent with observed firm behavior.

To preserve our model's homogeneity property, we assume that the firm's fixed cost of issuing equity at t is proportional to its total capital stock K_t . That is, the fixed cost of equity issuance, Φ_t , is given by

$$\Phi_t = \phi K_t = \phi \cdot (K_t^a + K_t^b), \quad (8)$$

where $\phi > 0$ is a constant measuring the fixed equity issuance cost.

In practice, external costs of financing scaled by firm size may decrease with firm size. With this caveat in mind, we point out that there are conceptual, mathematical, and economic reasons for modeling these costs as proportional to firm size. First, by modeling the fixed financing costs proportional to firm size, we ensure that the firm does not grow out of the fixed costs.¹⁰ Second, the information and incentive costs of external financing may

⁹The classic writings include Jensen and Meckling (1976), Leland and Pyle (1977), and Myers and Majluf (1984).

¹⁰Indeed, this is a common assumption in the investment literature. See Cooper and Haltiwanger (2006)

to some extent be proportional to firm size. Indeed, the negative announcement effect of a new equity issue affects the firm's entire capitalization. Similarly, the negative incentive effect of a more diluted ownership may have costs that are proportional to firm size. Finally, this assumption keeps the model tractable and generates stationary dynamics for the firm's cash-capital ratio.¹¹

We denote by H_t the firm's cumulative external financing up to time t with $H_0 = 0$ and by dH_t the firm's incremental external financing over time interval $(t, t + dt)$. Similarly, let X_t denote the cumulative costs of external financing up to time t with $X_0 = 0$, and dX_t the incremental costs of raising incremental external funds dH_t . The cumulative external equity issuance H and the associated cumulative costs X are stochastic controls chosen by the firm.

Technically, due to the fixed equity issuance costs, the firm's external financing policy can be described as a tuple $\nu = \{\tau^{(1)}, \tau^{(2)}, \dots; M^{(1)}, M^{(2)}, \dots\}$, where $\tau^{(i)}$ represents the i -th external financing (stopping) time, and $M^{(i)} > 0$ represents the corresponding net financing amount at the i -th financing time. When the firm issues no equity, i.e., $t \neq \tau^{(i)}$, we have $dH_t = dX_t = 0$. When the firm issues equity, i.e., $t = \tau^{(i)}$, we have

$$H_{\tau^{(i)}} = H_{\tau^{(i)}-} + M^{(i)}, \quad (9)$$

$$X_{\tau^{(i)}} = X_{\tau^{(i)}-} + \Phi_{\tau^{(i)}} + \gamma M^{(i)}. \quad (10)$$

Equations (9)-(10) imply that the net equity raised is $dH_t = M^{(i)}$ and the associated cost of financing is $dX_t = \Phi_{\tau^{(i)}} + \gamma M^{(i)}$ at $t = \tau^{(i)}$. Here, $\tau^{(i)}-$ refers to the time immediately before $\tau^{(i)}$.

Cash carry costs and cash management. Let W_t denote the firm's cash balance at t . If the firm's cash is positive, it survives with probability one. However, if the firm runs out of cash ($W_t = 0$), it has to either raise external funds to continue operating, or liquidate one of its divisions to replenish cash.

and Riddick and Whited (2009), among others. If the fixed cost is independent of firm size, it will not matter when firms become sufficiently large in the long run.

¹¹A potential limitation of our model is that it will be misspecified as a structural model of firms' outside equity issue decisions. As such, the model is likely to work best when applied to mature firms as opposed to start-ups and small entrepreneurial firms. Nevertheless, this limitation is mitigated in our setting, since conglomerates (or, more generally, multi-division firms) tend to fit the former category.

If the firm chooses to raise external funds, it incurs both the fixed and marginal financing costs specified above. In some situations the firm may prefer selling one of its divisions even before exhausting its cash balance. As we discuss in detail later, this result is a novel insight from our multi-division firm model. In contrast, it is not optimal for a single-division firm to liquidate itself as long as it still has a positive cash balance (BCW, 2011).

As in most cash management models, the rate of return that the firm earns on its cash balance is the risk-free rate r minus a carry cost $\lambda > 0$ that captures in a simple way the agency costs that may be associated with free cash inside the firm.¹² However, paying out cash also reduces the firm's cash balance, which potentially exposes the firm to current and future underinvestment, and future external financing costs. This tradeoff, which has been widely analyzed in the literature, determines the optimal payout policy. We denote by U_t the firm's cumulative (nondecreasing) payout to shareholders up to time t , and by dU_t the incremental payout over time interval dt . Distributing cash to shareholders may take the form of a special dividend or a share repurchase.

Combining cash flows from operations dY_t given in (7) with the firm's financing policy given by the cumulative payout process U and the cumulative external financing process H , in the region where the firm neither sells a division nor liquidates, its cash balance W evolves as follows:

$$dW_t = dY_t + (r - \lambda) W_t dt + dH_t - dU_t, \quad (11)$$

where the second term is the interest income (net of the carry cost λ), the third term dH_t is the cash inflow from external financing, and the last term dU_t is the cash outflow to investors, so that $(dH_t - dU_t)$ is the net cash flow from financing. As equity issuance is costly, it is not optimal to simultaneously issue equity and pay out a dividend. That is, at all t , either $dH_t = 0$ or $dU_t = 0$. As raising external financing is costly, the firm is often financially constrained; it neither issues equity nor pays out a dividend ($dH_t = dU_t = 0$), even though saving inside the firm is also costly ($\lambda > 0$).

¹²Alternatively, the cost of carrying cash may arise from tax distortions (e.g., Graham, 2000).

2.3 Firm Optimization

2.3.1 Single-Division Firm Optimization

Next, we state the optimization problem for a single-division firm, proposed by BCW (2011), which serves as an important benchmark for at least two reasons. First, it allows us to characterize how having more than one division changes a firm's decisions and valuation. Second, as a multi-division firm may sell one or more of its divisions, the solution for a single-division firm naturally enters into our analysis of the optimization problem for a multi-division firm.

Let $P^s(K^s, W)$ denote the value of a single-division firm with division s , and let $\{K_t^s; t \geq 0\}$ be the firm's capital stock process and $\{W_t; t \geq 0\}$ its cash balance process. The firm chooses its investment I^s , payout policy U^s , external financing policy H^s , and liquidation time $\tau_L = \tau_s$ to maximize shareholder value by solving

$$P^s(K^s, W) = \max \mathbb{E} \left[\int_0^{\tau_s} e^{-rt} (dU_t^s - dH_t^s - dX_t^s) + e^{-r\tau_s} (L_{\tau_s}^s + W_{\tau_s}) \right]. \quad (12)$$

The expectation takes risk into account (i.e., under the risk-neutral measure \mathbb{Q}). The first term is the discounted value of the net payouts to shareholders, and the second term is the discounted value from liquidation. The firm may never liquidate (i.e., $\tau_s = \infty$).

2.3.2 Multi-Division Firm Optimization

Unlike a single-division firm, which ceases to exist upon liquidating its only division, a multi-division firm can sell one or more divisions to replenish its cash balance, and continue operating as a going concern with the remaining divisions.

When selling division s at time τ_s , the conglomerate's cash balance increases from W_{τ_s-} by a discrete amount $L_{\tau_s-}^s$ to

$$W_{\tau_s} = W_{\tau_s-} + L_{\tau_s-}^s = W_{\tau_s-} + \ell_s K_{\tau_s-}^s. \quad (13)$$

For a two-division firm, after the division sale, the conglomerate becomes a single-division firm that behaves as in BCW (2011).

Let $F(K^a, K^b, W)$ denote the conglomerate's shareholder value. In Section 5.3 we show that it is never optimal for the firm to simultaneously sell both divisions (see Proposition

5.1). This is because the option value of keeping at least one division alive is strictly positive. We can therefore divide the conglomerate's optimization problem into two subproblems: one after it sells one of its divisions at stochastic time τ_s , and the other before the sale of the division. We solve the problem via backward induction.

Shareholders choose investment levels (I^a, I^b) , division sale timing (τ_a, τ_b) , payout policy U , and external financing H to maximize the conglomerate's value by solving

$$F(K^a, K^b, W) = \max_{I^a, I^b, U, H, \tau} \mathbb{E} \left[\int_0^\tau e^{-rt} (dU_t - dH_t - dX_t) + e^{-r\tau} \{ P^a(K_\tau^a, W_\tau) 1_{\{\tau=\tau_b\}} + P^b(K_\tau^b, W_\tau) 1_{\{\tau=\tau_a\}} \} \right], \quad (14)$$

where 1_A is the indicator function¹³ and $P^s(K_\tau^s, W_\tau^s)$ is the value of the single-division firm defined in equation (12). The conglomerate spins off a division at stopping time τ given by $\tau = \min\{\tau_a, \tau_b\}$ and liquidates itself at $\tau_L = \max\{\tau_a, \tau_b\}$. In sum, the firm's optimization problem is a combined convex control (investment), singular control (payout), impulse control (equity issuance), and optimal stopping (division sale) problem (see Appendix A).

Finally, we define the average q for a conglomerate as follows:

$$q_t = \frac{F(K_t^a, K_t^b, W_t) - W_t}{K_t^a + K_t^b}. \quad (15)$$

3 Corporate Socialism

While putting two divisions together as a firm provides diversification benefits, doing so may also give rise to agency costs. In particular, in the spirit of the models of influence activities (e.g., Milgrom, 1988; Milgrom and Roberts, 1988; Meyer, Milgrom, and Roberts, 1992), division managers may lobby headquarters to channel more of the firm's resources toward their division. In these models, division managers prefer larger resource allocations due to rent-seeking motives (e.g., if financial compensation, perquisite consumption, or outside job opportunities are linked to the size of the division they manage) or "empire building" preferences (e.g., if managers enjoy the power and status of managing a larger division), and lobby headquarters accordingly. This lobbying incentive is especially pronounced for

¹³The indicator function 1_A is equal to one if and only if the event A occurs and zero otherwise.

managers of weak divisions that face a higher risk of being downsized; their managers have incentives to overstate the division's true prospects in an attempt to gain access to corporate resources that can be used to prevent or delay the downsizing.

Such lobbying activities are costly to the firm, as division managers devote time and effort lobbying headquarters at the expense of more productive activities. In the models of corporate socialism (e.g., Rajan, Servaes, and Zingales, 2000; Scharfstein and Stein, 2000; Matvos and Seru, 2014), headquarters can mitigate this lobbying behavior by tilting the resource allocation toward weaker divisions at the expense of the stronger ones—analogous to a “socialist” outcome in which stronger divisions cross-subsidize the weaker ones.¹⁴

We model inefficient resource allocation within a firm by assuming that there is an additional cost that the firm pays by having two divisions inside the firm. Let G_t^c denote this cost, where c refers to the conglomerate. This cost can be interpreted as influence cost, which lowers divisions' productivity and causes output losses.

To be precise, the firm's operating profit, dY_t , over time increment dt is then given by

$$dY_t = dY_t^a + dY_t^b - G_t^c dt = (K_t^a dA_t^a + K_t^b dA_t^b) - (I_t^a + I_t^b + G_t^a + G_t^b) dt - G_t^c dt. \quad (16)$$

We focus on socialism for the case where the productivities of the two divisions are different. Without loss of generality, we refer to division a as the stronger division throughout the paper (i.e., $\mu_a > \mu_b$), whenever we study corporate socialism. Rather than using the (risk-adjusted) true expected productivity μ_a for the stronger division a , headquarters uses the (risk-adjusted) compromised productivity $\hat{\mu}_a \leq \mu_a$.

For division b , we set $\hat{\mu}_b = \mu_b$. This setup is in the spirit of the models of influence activities, in which internal politics is more costly to the high-productivity division. Importantly, due to internal politics, the compromised productivities are closer to one another compared to the uncompromised productivities.

As in Matvos and Seru (2014), we assume that the (flow) cost of being a conglomerate takes the form of:

$$G_t^c = K_t^a (\mu_a - \hat{\mu}_a) \quad (17)$$

¹⁴Several empirical studies find that multi-division firms tend to overinvest in divisions with low investment opportunities and underinvest in those with high investment opportunities (e.g., Shin and Stulz, 1998; Rajan, Servaes, and Zingales, 2000; Ozbas and Scharfstein, 2010), consistent with the models of corporate socialism.

where $\hat{\mu}_a$ satisfies

$$\hat{\mu}_a = \mu_a + \theta_c (\bar{\mu} - \mu_a) \quad (18)$$

with $\theta_c \in [0, 1]$ measuring the degree of corporate socialism and

$$\bar{\mu} = \frac{\mu_a + \mu_b}{2}. \quad (19)$$

The case with $\theta_c = 0$ corresponds to our baseline model of Section 2 with no corporate socialism. The higher the value of θ_c , the stronger corporate socialism. The case with $\theta_c = 1$ corresponds to the one with perfect socialism, in that headquarters allocates capital as if the two divisions were equally productive.

In summary, when optimizing, headquarters uses the following “compromised” risk-adjusted productivity process \hat{A}_t^s for division $s = a, b$:

$$d\hat{A}_t^s = \hat{\mu}_s dt + \sigma_s d\mathcal{Z}_t^s. \quad (20)$$

Substituting equation (17) into equation (16), we obtain the following dynamics for dY_t :

$$dY_t = dY_t^a + dY_t^b - G_t^c dt = (K_t^a d\hat{A}_t^a + K_t^b d\hat{A}_t^b) - (I_t^a + I_t^b + G_t^a + G_t^b) dt.$$

Importantly, corporate socialism disappears if headquarters sells one of the divisions, as doing so avoids the $G_t^c dt$ costs. As such, division sale is an important way for the firm to mitigate corporate socialism.

Headquarters chooses investment levels (\hat{I}^a, \hat{I}^b) , division sale timing $(\hat{\tau}_a, \hat{\tau}_b)$, payout \hat{U} , and external financing \hat{H} to solve

$$\begin{aligned} \hat{F}(K^a, K^b, W) &= \max_{\hat{I}^a, \hat{I}^b, \hat{\tau}_a, \hat{\tau}_b, \hat{U}, \hat{H}} \mathbb{E} \left[\int_0^{\hat{\tau}} e^{-rt} (d\hat{U}_t - d\hat{H}_t - d\hat{X}_t) \right. \\ &\quad \left. + e^{-r\hat{\tau}} \{ P^a(K_{\hat{\tau}}^a, W_{\hat{\tau}}) 1_{\{\hat{\tau}=\hat{\tau}_b\}} + P^b(K_{\hat{\tau}}^b, W_{\hat{\tau}}) 1_{\{\hat{\tau}=\hat{\tau}_a\}} \} \right], \end{aligned} \quad (21)$$

where $P^s(K_{\hat{\tau}}^s, W_{\hat{\tau}})$ is the value of a single-division firm defined in equation (12). As there is no corporate socialism for a single-division firm, the parameters for the value functions $P^a(K_{\hat{\tau}}^a, W_{\hat{\tau}})$ and $P^b(K_{\hat{\tau}}^b, W_{\hat{\tau}})$ are the original (true) parameter values μ_a and μ_b , rather than $\hat{\mu}_a$ and $\hat{\mu}_b$. The conglomerate spins off a division at time $\hat{\tau}$ given by $\hat{\tau} = \min\{\hat{\tau}_a, \hat{\tau}_b\}$.

The firm's liquidation time $\hat{\tau}_L$ is then given by $\hat{\tau}_L = \max\{\hat{\tau}_a, \hat{\tau}_b\}$.

Naturally, firm value is lower with socialism than without:

$$\hat{F}(K^a, K^b, W) \leq F(K^a, K^b, W; \mu_a, \mu_b). \quad (22)$$

Additionally, we can bound from below the firm's value with socialism, $\hat{F}(K^a, K^b, W)$, as follows:

$$\hat{F}(K^a, K^b, W) \geq F(K^a, K^b, W; \hat{\mu}_a, \hat{\mu}_b), \quad (23)$$

where $F(K^a, K^b, W; \hat{\mu}_a, \hat{\mu}_b)$ is the value of a hypothetical conglomerate with (risk-adjusted) productivities $\hat{\mu}_a$ and $\hat{\mu}_b$ for the two divisions under no socialism. While socialism lowers the conglomerate's productivity for the more productive division from μ_a to $\hat{\mu}_a$, it has a valuable division spinoff option, which allows the firm to eliminate socialism and increase productivity after becoming a single-division firm.

4 An MM First-Best Benchmark

Before solving our model for a financially constrained conglomerate, it is helpful to consider the special case where equity issuance is costless and there is no corporate socialism. In this case, both the Modigliani-Miller (MM) and Coase Theorems hold. Whether the two divisions are organized as units within a conglomerate or as two separate firms makes no economic difference. In either organizational structure, the first-best outcome is achievable as it is optimal for each division to choose its own first-best investment policy and financing is irrelevant.

Each division operates the same technology as in Hayashi (1982). Therefore, the average q for division s is equal to its marginal q and satisfies the following present-value relation:

$$q_s^{FB} = \max_{i^s} \frac{\mu_s - i_s - g_s(i^s)}{r + \delta_s - i^s}. \quad (24)$$

The first-order condition (FOC) for investment also implies that i_s^{FB} satisfies:

$$1 + g'_s(i_s^{FB}) = q_s^{FB}, \quad (25)$$

which states that the division's marginal cost of investing is equal to the marginal benefit

of investing, q_s^{FB} . Since adjustment costs are convex, (25) implies that the first-best investment is increasing in q . Note that q_s^{FB} is greater than unity, as the adjustment costs create a wedge between the value of installed capital and newly purchased capital.

The value of a conglomerate with cash W_t and divisional capital stocks K_t^a and K_t^b is given by:

$$F^{FB}(K_t^a, K_t^b, W_t) = q_a^{FB} K_t^a + q_b^{FB} K_t^b + W_t. \quad (26)$$

The enterprise value of the conglomerate is equal to the value of the conglomerate minus cash, $F^{FB}(K_t^a, K_t^b, W_t) - W_t$, which is independent of W_t since MM holds.

Let z_t denote the capital stock of division a divided by the firm's total capital stock:

$$z_t = \frac{K_t^a}{K_t^a + K_t^b}. \quad (27)$$

Moreover, let q_t^{FB} denote the conglomerate's average q in our first-best setting. Using the definition of average q given in equation (15), we obtain:

$$q_t^{FB} = z_t q_a^{FB} + (1 - z_t) q_b^{FB}. \quad (28)$$

That is, the average q of the conglomerate is simply a weighted average of the average q of its divisions, where the weights are the divisions' relative sizes, z_t and $(1 - z_t)$.

5 Solution: Bright Side of Internal Capital Markets

In this section, we solve the model proposed in Section 2. Firm value is a function of three state variables: the capital stock of each division (K^a and K^b) and the firm's cash balance (W). We solve the model by dividing the problem into three steps. First, we characterize the firm's decisions in the region where the marginal source of financing is its internal financing; second, we characterize the firm's optimal payout policies; finally, we analyze how a financially constrained conglomerate dynamically replenishes its cash by choosing between external financing, division sale, and firm liquidation.

5.1 Interior Region

In this region, firm value $F(K^a, K^b, W)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} rF(K^a, K^b, W) = \max_{I^a, I^b} & (I^a - \delta_a K^a) F_{K^a} + (I^b - \delta_b K^b) F_{K^b} \\ & + [(r - \lambda)W + \mu_a K^a + \mu_b K^b - (I^a + I^b + G^a + G^b)] F_W \\ & + \frac{1}{2} (\sigma_a^2 (K^a)^2 + \sigma_b^2 (K^b)^2 + 2\rho\sigma_a\sigma_b K^a K^b) F_{WW}. \end{aligned} \quad (29)$$

The first two terms (F_{K^a} and F_{K^b}) on the right-hand side of (29) capture the direct effects of investment on firm value; the third term (F_W) represents the effect of the firm's expected savings; and the last term (F_{WW}) captures the effect of the volatility of cash holdings W .

The firm finances its investment in both divisions out of the cash balance in this region. The divisional investment levels I^a and I^b satisfy the following interconnected FOCs:¹⁵

$$1 + G_{I^a}^a(I^a, K^a) = \frac{F_{K^a}(K^a, K^b, W)}{F_W(K^a, K^b, W)}, \quad (30)$$

$$1 + G_{I^b}^b(I^b, K^b) = \frac{F_{K^b}(K^a, K^b, W)}{F_W(K^a, K^b, W)}. \quad (31)$$

First, consider the special case with frictionless external and internal capital markets considered in Section 4 (i.e., the MM world). In this case, the marginal value of cash is $F_W = 1$, and the FOCs simplify to the neoclassical investment formula in (25)—that is, the firm's marginal q with respect to capital stock K^s in division s , $F_{K^s}(K^s, K^s, W)$, is equal to the firm's marginal cost of investing in division s , $1 + G_{I^s}^s$, and the two FOCs are independent of each other. In other words, one division's policy is independent of the other's (MM and Coase Theorems).

These properties no longer hold in our setup with financing frictions. The left-hand side of (30) is the firm's marginal cost of increasing a unit of capital in division a , $1 + G_{I^a}^a$. The right-hand side is the marginal benefit, which is equal to the marginal q for division a , $F_{K^a}(K^a, K^b, W)$, divided by the marginal cost of financing (or equivalently, the marginal value of cash), $F_W(K^a, K^b, W)$. Optimality requires that the two sides of (30) be equal.

¹⁵The convexity of the physical adjustment cost implies that the second-order condition is satisfied and the divisional investment decisions in our model admit interior solutions.

The same reasoning applies to equation (31) with respect to division b . With costly external financing, the firm deploys cash optimally to both divisions so that the FOCs (30) and (31) for the two divisions hold and become interconnected.

Re-writing (30) and (31), we have the following FOCs:

$$\frac{F_{K^a}(K^a, K^b, W)}{1 + G_{I^a}^a(I^a, K^a)} = \frac{F_{K^b}(K^a, K^b, W)}{1 + G_{I^b}^b(I^b, K^b)} = F_W(K^a, K^b, W). \quad (32)$$

The first equality states that the ratio between marginal q and the marginal cost of investing is equal for the two divisions—the implication of the intratemporal optimal allocation. The second equality describes the intertemporal optimal savings: the ratio between marginal q and the marginal cost of investing in all divisions is equal to the marginal value of savings (cash), $F_W(K^a, K^b, W)$. Note that, while marginal q is well defined for each division, it is unclear how to define a meaningful marginal q at the conglomerate level, as $K^a + K^b$ is not a state variable; instead, both K^a and K^b are state variables.

By using the homogeneity property of our model and applying Euler's theorem, we obtain the following expression:

$$F(K^a, K^b, W) = F_{K^a}(K^a, K^b, W)K^a + F_{K^b}(K^a, K^b, W)K^b + F_W(K^a, K^b, W)W, \quad (33)$$

which links the book values of key balance sheet items (W , K^a , and K^b) to the firm's market value. Multiplying cash (W) and divisional capital stocks (K^a and K^b) by their respective marginal (shadow) values (i.e., the marginal value of cash F_W , the marginal q of division a 's capital stock F_{K^a} , and the marginal q of division b 's capital stock F_{K^b}) and then summing up the three terms, we obtain the conglomerate's market value $F(K^a, K^b, W)$.

The homogeneity property also allows us to equivalently express the firm's three-state-variable value function as a two-state-variable value function:

$$F(K_t^a, K_t^b, W_t) = (K_t^a + K_t^b) \cdot f(z_t, w_t), \quad (34)$$

where z_t is given in (27) and w_t is the firm's cash-capital ratio defined as

$$w_t = \frac{W_t}{K_t^a + K_t^b}. \quad (35)$$

The ratio w between cash balance W_t and the firm's total physical capital stock ($K_t^a + K_t^b$) is the key state variable measuring the firm's degree of financing constraints. Using Ito's Lemma, we obtain the following dynamics for w_t :

$$\begin{aligned} dw_t = & (r - \lambda)w_t dt + [z_t (\mu_a dt + \sigma_a d\mathcal{Z}_t^a) + (1 - z_t) (\mu_b dt + \sigma_b d\mathcal{Z}_t^b)] \\ & - [(i_t^a + g_a(i_t^a)) z_t + (i_t^b + g_b(i_t^b)) (1 - z_t)] dt \\ & - w_t [z_t(i_t^a - \delta_a) + (1 - z_t)(i_t^b - \delta_b)] dt. \end{aligned} \quad (36)$$

The first term reflects the firm's net interest income; the second term captures the operating revenues from the two divisions; the third term captures the total (flow) costs of investing; and the last term captures the impact of changes in the divisions' capital stock.

In addition to w_t , the capital stock of division a as a fraction of the firm's total capital stock, $z_t = K_t^a/(K_t^a + K_t^b)$, measures the distribution of illiquid productive capital stocks between the two divisions, which is the other key state variable. Using the dynamics of K^a and K^b , we obtain the following process for $z_t \in [0, 1]$:

$$dz_t = z_t(1 - z_t) [(i_t^a - \delta_a) - (i_t^b - \delta_b)] dt. \quad (37)$$

If and only if the growth rate of division a exceeds that of division b , i.e., $(i_t^a - \delta_a) > (i_t^b - \delta_b)$, the relative size of division a grows, that is, z_t increases.

By using our model's homogeneity property—e.g., equation (34)—we can simplify the HJB equation (29) and obtain the following partial differential equation (PDE) for $f(z, w)$:

$$\mathcal{L}f(z, w) = 0, \quad (38)$$

where

$$\begin{aligned} \mathcal{L}f(z, w) = & \max_{i^a, i^b} (i^a - \delta_a)z [f(z, w) + (1 - z)f_z(z, w) - wf_w(z, w)] \\ & + (i^b - \delta_b)(1 - z) [f(z, w) - zf_z(z, w) - wf_w(z, w)] \\ & + [(r - \lambda)w + (\mu_a - i^a - g_a(i^a))z + (\mu_b - i^b - g_b(i^b))(1 - z)] f_w(z, w) \\ & + \frac{1}{2} [\sigma_a^2 z^2 + \sigma_b^2 (1 - z)^2 + 2z(1 - z)\rho\sigma_a\sigma_b] f_{ww}(z, w) - rf(z, w). \end{aligned} \quad (39)$$

The first and second terms on the right-hand side of equation (38) capture the effects of investment in divisions a and b on firm value; the third term captures the effect of cash management; and the fourth term captures the volatility effects (from both divisions and their correlation). The sum of these four terms represents the expected change in firm value. Subtracting $rf(z, w)$, the annuity value of $f(z, w)$, provides the net change in $f(z, w)$.

Note that it is optimal for the firm to solely rely on internal funds to finance both divisions' investments in this region. The conglomerate optimally chooses its divisional investments i^a and i^b so that the net change in its (scaled) value $f(z, w)$, $\mathcal{L}f(z, w)$, is zero.

The FOCs for divisional investment decisions can be simplified as follows:

$$1 + g'_a(i^a) = \frac{f(z, w) + (1 - z)f_z(z, w)}{f_w(z, w)} - w, \quad (40)$$

$$1 + g'_b(i^b) = \frac{f(z, w) - zf_z(z, w)}{f_w(z, w)} - w. \quad (41)$$

The left-hand side of equation (40) is the marginal cost of investing in division a . The right-hand side is the marginal q of K^a divided by the marginal value of cash $f_w(z, w)$. The firm optimally chooses i^a by equating the two sides of (40). The same applies to division b in equation (41). As discussed above, the divisional investment decisions are interconnected because of financial constraints.

In order to fully characterize the value function $f(z, w)$, we must also analyze the firm's payout, external financing, and division sale decisions. We show that there are two other regions in the state space of (z, w) : (1) a payout region where the firm also actively pays out a dividend to shareholders; and (2) an external financing/division sale region in which the firm replenishes its cash by choosing external financing, division sale, or liquidation of the whole firm.

5.2 Payout Region

To determine the firm's payout region, we first consider the case in which the firm's cash holdings are very large relative to its size. In this case, the firm is better off paying out its excess cash to shareholders to avoid the cash-carrying costs. Let \bar{w}_t denote the (stochastic) level of the cash-capital ratio w_t above which the firm pays out cash. As cum-dividend firm

value $f(z_t, w_t)$ must be continuous at all t , the following value-matching condition holds:

$$f(z_t, w_t) = f(z_t, \bar{w}_t) + (w_t - \bar{w}_t), \quad \text{for } w_t > \bar{w}_t. \quad (42)$$

By taking the limit $w_t \rightarrow \bar{w}_t$ and calculating the derivative with respect to w_t , we obtain the following equation in the region where $w_t \geq \bar{w}_t$:

$$f_w(z_t, w_t) = 1. \quad (43)$$

That is, at the time of payout, the marginal value of cash is one. As the payout is optimally chosen, the following “super contact” condition must also hold (Dumas, 1991):

$$f_{ww}(z_t, \bar{w}_t) = 0. \quad (44)$$

5.3 Division Sale, External Financing, and Liquidation Regions

When the firm is short on cash, it may raise costly external equity or liquidate a division to replenish its cash stock. An important difference from BCW (2011) is that a conglomerate may issue equity or liquidate a productive division to replenish its cash holdings before exhausting its cash. That is, the firm may move preemptively, as doing so alleviates even larger distortions in the future. Another key difference from BCW (2011) is that a conglomerate may choose equity issuance and division sale under different circumstances, while in BCW (2011) the firm either issues equity or liquidates itself when running out of cash. We show that the liquidation of the whole firm is the firm’s last resort, as doing so wipes out its going-concern value (see Appendix B).

Proposition 5.1. *Under the conditions given in equation (6), a.) in the first-best world, it is never optimal to liquidate either division; b.) in a world with costly external financing, it is optimal for the firm to sequentially sell its divisions rather than liquidate the firm in its entirety.*

When the firm replenishes its cash via either equity issuance or division sale, it chooses the less costly option that yields a higher firm value. The choice depends not only on the liquidity ratio w_t , but also the relative size of the two divisions, captured by z_t , in addition to the structural parameters of the model.

Costly external equity issuance. First, we calculate firm value conditional on issuing external equity at t . Let $J(K_t^a, K_t^b, W_t)$ denote this conditional firm value, which is given by

$$J(K_t^a, K_t^b, W_t) = \max_{M_t > 0} F(K_t^a, K_t^b, W_t + M_t) - [\phi \cdot (K_t^a + K_t^b) + (1 + \gamma)M_t]. \quad (45)$$

The first term on the right-hand side of equation (45) is the post-equity-issuance firm value, and the second term is the sum of net equity issuance M_t and the total cost of equity issuance, which includes the fixed equity issuance cost, $\phi \cdot (K_t^a + K_t^b)$, and the proportional issuance cost γM_t . Again, note that the value $J(K_t^a, K_t^b, W_t)$ is conditional on equity issuance, but equity issuance may not be optimal.

Let $\tilde{F}(K_t^a, K_t^b, W_t)$ denote firm value conditional on external financing or division sale being optimal. That is, we have the following condition:

$$\tilde{F}(K_t^a, K_t^b, W_t) = \max \{P^a(K_t^a, L_t^b + W_t), P^b(K_t^b, L_t^a + W_t), J(K_t^a, K_t^b, W_t)\}. \quad (46)$$

Equation (46) states that the firm selects one of the three mutually exclusive discrete choices to maximize its value. If sale of division a or b is optimal, firm value is given by the first and second term, respectively, on the right side of equation (46). If equity issuance is optimal, firm value is equal to $J(K_t^a, K_t^b, W_t)$ given in equation (45). The value function for the single-division firm is the same as in BCW (2011) and reported in Section 2.

Let w_t^a denote the cash-capital ratio immediately after the conglomerate sells division b and becomes a stand-alone firm with only division a : $w_t^a = W_t^a/K_t^a$, where $W_t^a = L_t^b + W_t$. We define w_t^b analogously. Using the homogeneity property, we obtain:

$$w_t^a = \frac{\ell_b(1 - z_t) + w_t}{z_t} \quad \text{and} \quad w_t^b = \frac{\ell_a z_t + w_t}{1 - z_t}. \quad (47)$$

Let m_t denote the scaled net equity issuance, $m_t = M_t/(K_t^a + K_t^b)$, and $j(z_t, w_t)$ denote firm value scaled by $(K_t^a + K_t^b)$, that is, $j(z_t, w_t) = J(K_t^a, K_t^b, W_t)/(K_t^a + K_t^b)$. Equation (45) can be simplified as:

$$j(z_t, w_t) = \max_{m_t > 0} f(z_t, w_t + m_t) - \phi - (1 + \gamma)m_t. \quad (48)$$

Let $\tilde{f}(z_t, w_t) = \tilde{F}(K_t^a, K_t^b, W_t)/(K_t^a + K_t^b)$. Using the homogeneity property to simplify equation (46), we obtain

$$\tilde{f}(z_t, w_t) = \max \left\{ z_t p^a(w_t^a), (1 - z_t) p^b(w_t^b), j(z_t, w_t) \right\}, \quad (49)$$

where w_t^a and w_t^b are given in equation (47).

The equation that defines the external financing/division sale regions is then given by

$$f(z_t, w_t) = \tilde{f}(z_t, w_t). \quad (50)$$

5.4 Summary

In Appendix A, we prove that the firm's (scaled) value $f(z, w)$ associated with the firm's optimal policies satisfies the following variational inequality:

$$\max \left\{ \mathcal{L}f(z, w), 1 - f_w(z, w), \tilde{f}(z, w) - f(z, w) \right\} = 0 \quad (51)$$

in the two-dimensional region defined by $z \in (0, 1)$ and $w \geq 0$.

Intuitively, the firm finds itself in one of the three regions. When the first term in equation (51) is equal to zero (and the other two terms are strictly negative), the firm is in the interior region and optimally chooses its investment-capital ratios for divisions a and b as prescribed by (40) and (41). When the second term is equal to zero, the conglomerate optimally makes a payout to shareholders as described by (42), (43), and (44). Finally, when the last term is equal to zero, the firm optimally chooses either division sale or costly external financing, as captured by (48) and (49). We numerically solve the variational inequality in equation (51) by using a penalty-function-based iterative procedure described in Appendix C.

6 Solution with Corporate Socialism

In this section, we analyze our generalized model with corporate socialism introduced in Section 3.

The key change from our baseline model is that corporate socialism effectively lowers the productivity of the productive division a from μ_a to $\hat{\mu}_a$. Once one of the divisions is

sold, the conglomerate becomes a single-division firm and no longer incurs the cost of being a socialistic conglomerate. Therefore, eliminating the dark side of internal capital markets provides an incentive for the conglomerate to engage in division sale.

Specifically, the cash-capital ratio under socialism, $w_t = W_t/(K^a + K_t^b)$, is given by

$$dw_t = (r - \lambda)w_t dt + [z_t (\hat{\mu}_a dt + \sigma_a d\mathcal{Z}_t^a) + (1 - z_t) (\mu_b dt + \sigma_b d\mathcal{Z}_t^b)] - [(i_t^a + g_a(i_t^a)) z_t + (i_t^b + g_b(i_t^b)) (1 - z_t)] dt - w_t [z_t(i_t^a - \delta_a) + (1 - z_t)(i_t^b - \delta_b)] dt. \quad (52)$$

As corporate socialism lowers μ_a to $\hat{\mu}_a$, the conglomerate's cash balance w_t has a lower drift than without socialism.

The solution of the headquarters' problem satisfies the following variational inequalities:

$$\max \left\{ \hat{\mathcal{L}}\hat{f}(z, w), 1 - \hat{f}_w(z, w), \tilde{f}(z, w) - \hat{f}(z, w) \right\} = 0, \quad z \in (0, 1), \quad w \geq 0, \quad (53)$$

where

$$\begin{aligned} \hat{\mathcal{L}}\hat{f}(z, w) &= \max_{i^a, i^b} (i^a - \delta_a)z \left[\hat{f}(z, w) + (1 - z)\hat{f}_z(z, w) - w\hat{f}_w(z, w) \right] \\ &\quad + (i^b - \delta_b)(1 - z) \left[\hat{f}(z, w) - z\hat{f}_z(z, w) - w\hat{f}_w(z, w) \right] \\ &\quad + [(r - \lambda)w + (\hat{\mu}_a - i^a - g_a(i^a))z + (\mu_b - i^b - g_b(i^b))(1 - z)]\hat{f}_w(z, w) \\ &\quad + \frac{1}{2} [\sigma_a^2 z^2 + \sigma_b^2 (1 - z)^2 + 2z(1 - z)\rho\sigma_a\sigma_b]\hat{f}_{ww}(z, w) - r\hat{f}(z, w) \end{aligned} \quad (54)$$

and $\tilde{f}(z, w)$ is the scaled value when the conglomerate chooses to either sell a division or issue equity:

$$\tilde{f}(z, w) = \max \left\{ z p^a(w^a), (1 - z) p^b(w^b), \hat{j}(z, w) \right\}. \quad (55)$$

As in our baseline model without socialism, $p^a(w^a)$ and $p^b(w^b)$ are the value of the firm with a single division, a and b , respectively. Note that, as division sale eliminates socialism, the productivity parameter for $p^a(\cdot)$ is μ_a rather than $\hat{\mu}_a$. That is, to analyze the headquarters' problem, the firm needs to use both the true risk-adjusted productivity μ_a and the compromised productivity $\hat{\mu}_a$.¹⁶ In equation (55), $\hat{j}(z, w)$ denotes the conglomerate value

¹⁶While in Matvos and Seru (2014), it is sufficient for the conglomerate to use compromised productivities, as their model features no division sale, this is no longer the case here since selling a division eliminates

conditional on external financing, which is given by

$$\hat{j}(z, w) = \max_{\hat{m} > 0} \quad \hat{f}(z, w + \hat{m}) - [\phi + (1 + \gamma)\hat{m}] . \quad (56)$$

As we show later, the solution features three regions: (1) the interior region where $\hat{\mathcal{L}}\hat{f} = 0, \hat{f}_w > 1, \hat{f} > \tilde{f}\}$; (2) the payout region: $\{\hat{f}_w = 1\}$; and (3) the external financing/liquidation region: $\{\hat{f} = \tilde{f}\}$.

7 Quantitative Analysis: Without Socialism

We now turn to the quantitative analysis of the model. In this section, we consider the case without socialism ($\theta_c = 0$) for a firm with two symmetric divisions, that is, two divisions whose productivity shocks have the same mean and volatility ($\mu_a = \mu_b$ and $\sigma_a = \sigma_b$), but are not perfectly correlated ($\rho = 10\%$).¹⁷

The parameters used in the benchmark case are provided in Table 1. We set the annual mean and volatility of the productivity shocks to $\mu_a = \mu_b = 20\%$ and $\sigma_a = \sigma_b = 9\%$, respectively, which are in line with the estimates of Eberly, Rebelo, and Vincent (2009) for large U.S. firms.

While our analyses do not depend on the specific functional form of $g_s(i^s)$ for division $s = a, b$, for simplicity, we adopt the following widely used quadratic form:

$$g_s(i^s) = \frac{\theta_s(i^s)^2}{2} , \quad (57)$$

where the parameter θ_s measures the degree of the adjustment cost for division s . For our baseline calculations, we assume that both divisions have the same adjustment cost parameter, which we set to $\theta_a = \theta_b = 8$ as in Shapiro (1986) and Hall (2001). We further assume that both divisions have the same annual depreciation rate, which we set $\delta_a = \delta_b = 9\%$. Moreover, we assume that the liquidation value for a division is proportional to the market value (on a first-best basis) of the division's capital stock. This assumption captures the idea that, when acquiring a division, the buyer uses as benchmark the value

socialism in our model.

¹⁷In Section 8, we consider the case with socialism ($\theta_c > 0$) for a firm with asymmetric divisions.

Table 1: SUMMARY OF PARAMETERS. This table summarizes the symbols for the key parameters and the values used in our quantitative analysis. The values in the “symmetric” column are for the case where the two divisions have the same parameter values. The values in the “asymmetric” column are for the case where the two divisions have different parameter values. Whenever applicable, parameter values are annualized.

Parameters	Symbol	Symmetric	Asymmetric
Risk-free rate	r	6%	6%
Proportional cash-carrying cost	λ	1%	1%
Proportional financing cost	γ	6%	6%
Fixed financing cost	ϕ	1%	1%
Correlation of two divisions	ρ	10%	10%
Risk-neutral mean productivity shock for division a	μ_a	20%	24%
Risk-neutral mean productivity shock for division b	μ_b	20%	16%
Volatility of productivity shock for division a	σ_a	9%	9%
Volatility of productivity shock for division b	σ_b	9%	9%
Rate of depreciation for division a	δ_a	9%	9%
Rate of depreciation for division b	δ_b	9%	9%
Adjustment cost parameter for division a	θ_a	8	8
Adjustment cost parameter for division b	θ_b	8	8
Socialism parameter	θ_c	0	0.686
Capital liquidation value for division a	ℓ_a/q_a^{FB}	0.6	0.6
Capital liquidation value for division b	ℓ_b/q_b^{FB}	0.6	0.6

that the division can potentially create.¹⁸ As the two divisions have the same production parameters, we set $\ell_a/q_a^{FB} = \ell_b/q_b^{FB} = 0.6$.¹⁹

As in BCW (2011), we set the annual risk-free rate to $r = 6\%$, the proportional cash-carrying cost to $\lambda = 1\%$, the proportional financing cost to $\gamma = 6\%$, and the fixed financing cost to $\phi = 1\%$ of the firm’s total capital stock. With these parameter values, the first-best average q in the neoclassical model is $q^{FB} = 1.4$ and the corresponding first-best investment-capital ratio is $i^{FB} = 0.05$ in both divisions.²⁰

¹⁸In this regard, we differ from BCW (2011) who assume that the liquidation value is proportional to the book value of the division’s capital stock.

¹⁹In Sections 7.3 and 7.4, we consider alternative values of the liquidation parameters.

²⁰The first-best investment-capital ratio is given by $i^{FB} = r + \delta_s - \sqrt{(r + \delta_s)^2 - 2(\mu_s - (r + \delta_s))/\theta_s}$, and the first-best q by $q^{FB} = 1 + \theta_s i^{FB}$. See Appendix B for details.

7.1 Diversified vs. Stand-Alone Firm

In Figure 1, we compare the diversified firm (that is, the firm with the two symmetric divisions described above) with a stand-alone firm. Both firms have the same total capital stock and the same parameter values. The only difference is that the diversified firm has an internal capital market (allowing headquarters to reallocate resources across the two symmetric divisions with imperfectly correlated productivity shocks) and the option to spin off a division at any time it chooses. For now, we assume that the diversified firm has two divisions of equal size ($z = 0.5$).

In the figure, the left-hand panels (A1-A3) pertain to the liquidation case, where we assume that, if the firm runs out of cash, the only option is to liquidate a division or the entire firm (that is, the firm cannot issue external financing). Alternatively, this can be seen as an extreme case of costly external financing—the cost is so high that firms do not resort to it. We relax this assumption in the right-hand panels (B1-B3), pertaining to the refinancing case, where firms have the option to issue equity to replenish their liquidity. In what follows, we discuss both cases, starting with the liquidation case.

Liquidation case.

In panel A1, we plot the average q for the diversified firm (solid line) and stand-alone firm (dashed line), along with the first-best benchmark implied by the neoclassical model, which is $q^{FB} = 1.4$ for our parameter values (dotted line). The horizontal axis plots the cash-to-capital ratio w . The vertical lines mark the endogenous payout boundary at which the firm pays out cash to shareholders ($\bar{w} = 0.13$ for the diversified firm, and $\bar{w} = 0.21$ for the stand-alone firm, respectively).

When the firm runs out of cash (reaching $w = 0$), the diversified firm spins off one of the divisions.²¹ Since the divisions are symmetric and the firm starts with $z_0 = 0.5$, the investment levels of both divisions are identical at all times, and hence $z_t = 0.5$ for all t before the firm liquidates a division. (For this reason, the choice of which division to liquidate is immaterial.) Assuming that division a is spun off, the firm receives the liquidation value $\ell_a K^a$ and becomes a single-division firm with a cash-to-capital ratio of

²¹For our parameter values, it is never optimal to liquidate a division when $w > 0$. In Section 7.4, we consider alternative parameterizations, under which early liquidation can be optimal. Note that, as shown in Section 5.3, it is never optimal for the diversified firm to liquidate both divisions at once.

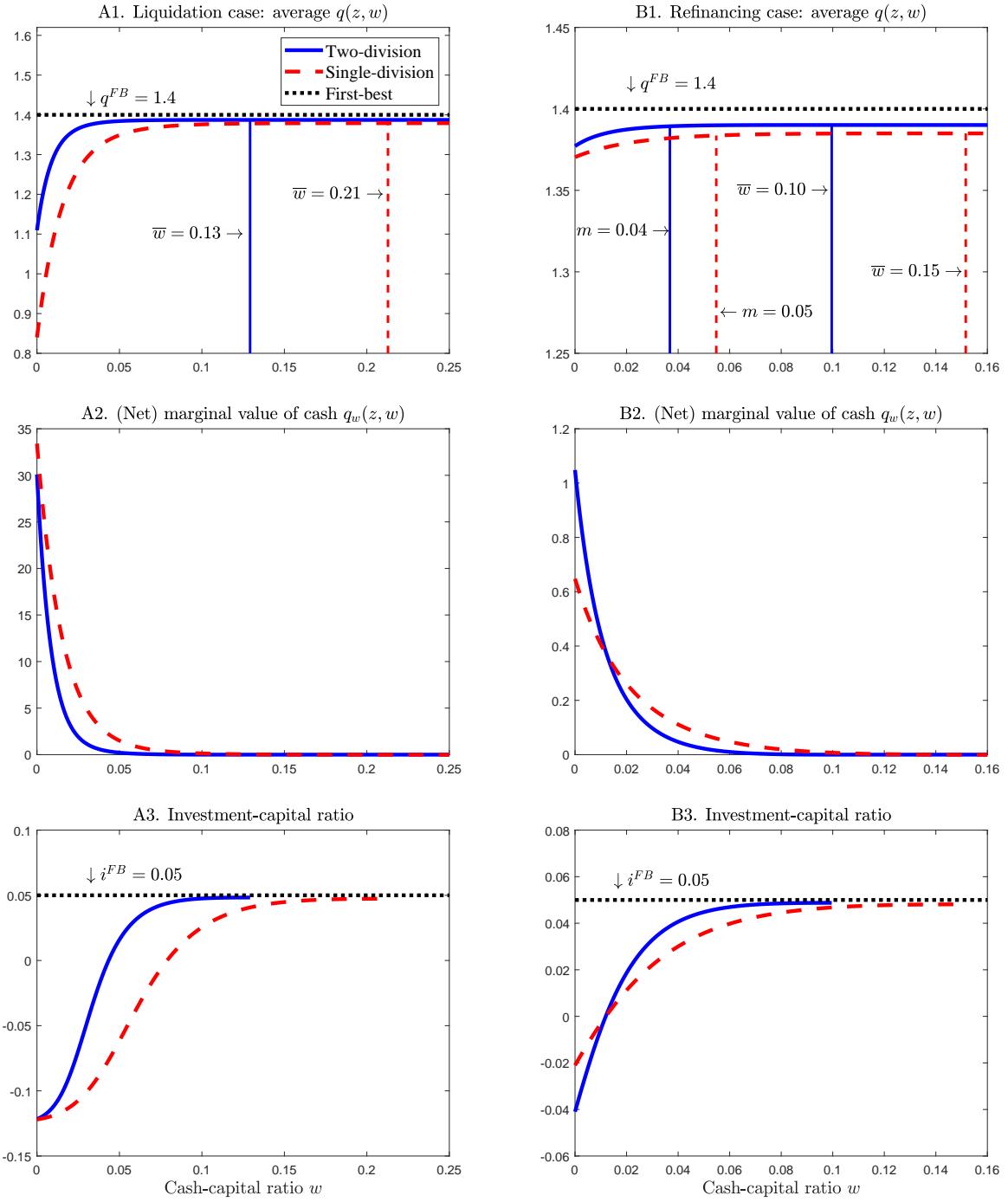


Figure 1: Comparison of a diversified firm (with $z = 0.5$) and a single-division firm. The left-hand panels (A1-A3) pertain to the liquidation case, the right-hand panels (B1-B3) to the refinancing case. In panel A1, the vertical lines mark the payout boundary \bar{w} . In panel B1, the vertical lines mark the payout boundary \bar{w} and the equity issuance amount m , respectively.

$\frac{\ell_a K^a}{K^b} = \ell_b = 0.84$. Since this ratio exceeds the dividend payout boundary of the single-division firm ($\bar{w} = 0.21$), it will optimally pay out the difference of 0.63 to shareholders, and then operate as a stand-alone firm with the remaining liquidity.

Note that this sequence is consistent with Dittmar's (2004) finding that firms often pay a special dividend subsequent to a spinoff. Given this optimal response, the diversified firm's value at $w = 0$ is 1.11.²² In contrast, when $w = 0$, the stand-alone firm has no choice but to liquidate the entire firm. Given our assumption that the liquidation value is 0.6 times the (first-best) market value, this implies that the stand-alone firm's value at $w = 0$ is $0.6 \times q^{FB} = 0.84$. As these calculations illustrate, liquidation is more costly for the stand-alone firm (as it permanently forgoes all future growth opportunities) compared to the diversified firm (as it liquidates a division in lieu of the entire firm).

As can be seen, we find that the diversified firm achieves a higher valuation compared to the stand-alone firm, especially in bad times when the firm is low on cash.²³ The rationale is twofold. First, the diversified firm has the option to spin off a division to replenish its liquidity. This option is more valuable for lower values of w .²⁴ Second, diversification reduces the volatility of the firm's productivity shocks and hence the likelihood of costly liquidation. This benefit from diversification is especially valuable in bad times when w is low.²⁵ We further observe that the payout boundary \bar{w} is lower for the diversified firm. The same two rationales explain this finding. That is, the conglomerate's option to spin off a division, and the lower volatility of the productivity shocks reduce the value of holding cash. As such, the diversified firm can more easily afford to pay a dividend. This finding is consistent with the evidence of Duchin (2010) who documents that diversified firms hold less cash compared to stand-alone firms.

Panel A2 plots the (net) marginal value of cash q_w . As is shown, the marginal value

²²Formally, the diversified firm's value at $w = 0$ satisfies $F(K^a, K^b, W) = P^b(K^b, \ell_a K^a)$, and hence $f(z, w) = (1 - z)p^b(\ell_a)$. For $z = 0.5$ and our parameter values, $f(0.5, 0) = 1.1$.

²³This pattern is consistent with the empirical evidence of Matvos and Seru (2014) and Kuppuswamy and Villalonga (2016), who find that the value of diversification was higher during the recent financial crisis. It also echoes Matvos, Seru, and Silva's (2018) finding that firms aim to diversify their operations in times of capital market disruptions.

²⁴Indeed, this spinoff option makes $q(z, w)$ at $w = 0$ exceed the liquidation value for the stand-alone firm, $\ell = 0.6 \times 1.4 = 0.84$.

²⁵Note that our setup is conservative in that it likely underestimates the value gains from diversification. This is because of our assumption of i.i.d. productivity shocks. In reality, shocks are likely to exhibit some degree of persistence, which increases the benefits from diversification.

of cash increases as the firm becomes more constrained and liquidation more likely. Since liquidation is more costly for the stand-alone firm, the marginal value of cash is higher for the stand-alone firm compared to the diversified firm. Note that costly liquidation induces both firms to be de facto “risk averse,” as the average q of both firms is concave in w . Thus, holding cash today (below the payout boundary \bar{w}) maximizes firm value and reduces the likelihood of liquidation in the future. Intuitively, cash is valuable as it helps keep the firm away from costly liquidation.

Panel A3 plots the investment-capital ratio for both firms, along with the first-best level ($i^{FB} = 0.05$ for our parameter values). Due to financing constraints, investment is below the first-best for both firms. Importantly, underinvestment is more severe for the stand-alone firm. This mirrors the pattern in panel A2. As liquidity is more valuable for the stand-alone firm (compared to the diversified firm), it has a greater demand for cash, and hence reduces investment more aggressively.²⁶

Refinancing case.

In the right-hand panels of Figure 1, we consider the refinancing case. Specifically, when the firms run out of cash ($w = 0$), they now replenish their liquidity by raising external equity. Doing so is costly, as the firms incur fixed ($\phi = 1\%$) and variable ($\gamma = 6\%$) financing costs. In panel B1, we plot the average q for both the diversified and single-division firms. Because the firms can now issue equity (at a cost), they avoid inefficient liquidation even under financial distress. As a result, for both firms, the average q is higher in the refinancing case compared to the liquidation case in panel A1. Moreover, the respective payout boundaries (\bar{w}) are lower than in the liquidation case. This is because firms are more willing to pay out cash when they can raise new funds in the future.

We continue to find that the diversified firm is more valuable than the stand-alone firm. As diversification reduces the volatility of the firm’s cash flows, it reduces the likelihood of running out of cash and resorting to (costly) equity issuance. As a result, the diversified firm has higher value, can more easily afford to pay out cash ($\bar{w} = 0.10$ for the diversified firm, and $\bar{w} = 0.15$ for the stand-alone firm), and issues less equity in the event of refinancing ($m = 0.04$ for the diversified firm, and $m = 0.05$ for the stand-alone firm).

²⁶Note that, close to $w = 0$, investment is negative for both firms (and even more so for the stand-alone firm). Intuitively, the firms disinvest in order to raise cash and stay away from the liquidation boundary.

In panels B2 and B3, we find that, when liquidity is abundant, the diversified firm is less prone to underinvestment, and has a lower marginal value of cash, compared to the stand-alone firm. This is consistent with our previous analysis for the liquidation case. As diversification reduces the volatility of the firm’s cash flows, the diversified firm has less of a need for liquidity (i.e., liquidity is less valuable), and is more inclined to invest instead of hoarding cash.

Interestingly, the opposite results hold when w is low (the two curves cross in both panels B2 and B3). The intuition is as follows. As the volatility of the firm’s cash flows is reduced through diversification, an extra dollar of cash becomes more effective in helping the diversified firm avoid costly equity issuance. When w is sufficiently low, the (marginal) value of cash is larger for the diversified firm than the stand-alone firm, as the value-add of preserving both divisions is very high. Therefore, the diversified firm reduces investment more than the single-division firm. As a result, diversification can lead to a paradoxically higher demand for precautionary savings, and hence more underinvestment, when the cash situation is sufficiently dire.

7.2 Relative Size of Divisions: z

In Figure 1, we considered a diversified firm with equal-sized divisions ($z = 0.5$). Recall that in this case, $z = 0.5$ is an absorbing state. Therefore, the solution for the diversified firm boils down to a single state variable (w) problem as in BCW (2011) but with a lower volatility (due to the imperfect correlation between the two divisions’ productivity shocks).

Albeit insightful, the symmetric-division case with $z_0 = 0.5$ is a rather special case. In what follows, we examine the case in which $z_0 \neq 0.5$. We show that, even with two symmetric divisions, the model generates very rich dynamics along both the w_t and z_t margins. In Figure 2 (liquidation) and Figure 3 (refinancing), we consider $z = 0.1$, which means that division a accounts for 10% of the firm’s capital stock, while division b accounts for the remaining 90%.²⁷

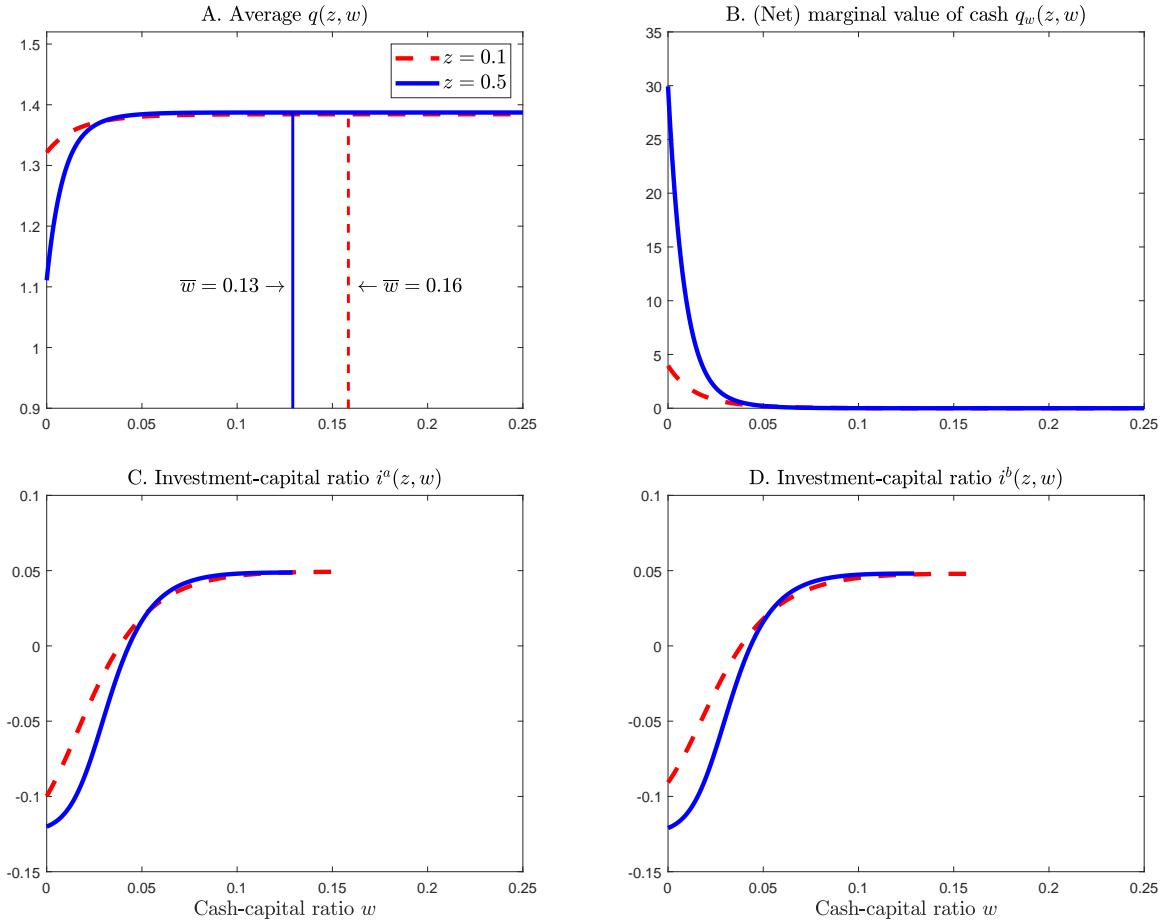


Figure 2: Liquidation case—comparison of diversified firms with $z = 0.1$ and $z = 0.5$. In panel A, the vertical lines mark the payout boundary \bar{w} .

Liquidation case.

Panel A of Figure 2 plots the value of the firm for the $z = 0.1$ case, and compares it with the $z = 0.5$ case analyzed earlier. When cash is abundant (high w), the balanced firm ($z = 0.5$) is more valuable. This finding is intuitive—the diversification gains are highest at $z = 0.5$, which translates in higher firm value. In contrast, in bad times (low w), the balanced firm is less valuable. This is because, closer to $w = 0$, liquidation is more likely, and liquidation is more costly when $z = 0.5$, as the firm would forgo half of its productive assets. In contrast, in the event of liquidation, the $z = 0.1$ firm would optimally spin off the smaller division (division a), and only forgo 10% of its productive assets.

²⁷Since the divisions are symmetric (i.e., they have the same mean and volatility of the productivity shocks), we obtain the same results for $z = 0.9$, switching divisions a and b in the figures.

Panel B plots the marginal value of cash, and panels C and D the investment-capital ratio for divisions a and b , respectively. The observed patterns are consistent with the above interpretation. In high- w states, liquidity is less valuable for the more balanced firm, as it faces lower volatility due to diversification. Given the lower need for cash, it is able to invest more compared to the less balanced firm ($z = 0.1$). In contrast, in low- w states, firms worry about liquidation. Since liquidation is more costly to the more diversified firm ($z = 0.5$), it is more eager to prevent this scenario from happening. As a result, compared to the $z = 0.1$ firm, the more diversified firm reduces investment more aggressively to preserve cash, and cash has a higher marginal value.

Overall, the patterns from Figure 2 imply a dark and bright side of diversification from the perspective of liquidity management, even for a value-maximizing conglomerate. In good times, diversification reduces the need for liquidity and creates value. In bad times, diversification can hamper the conglomerate's liquidity management by making spinoffs more costly and destroys value.

Refinancing case.

In Figure 3, we analyze the refinancing case. When equity financing is less costly than liquidation, both firms choose to issue equity when they run out of cash ($w = 0$). Thus, the more balanced firm ($z = 0.5$) no longer bears the higher liquidation costs arising from the liquidation of a relatively large division. As a result, the benefits from diversification—i.e., the lower volatility of the balanced firm's cash flows—dominate, and the more balanced firm is always more valuable than the $z = 0.1$ firm, as shown in panel A.

Interestingly, panel B shows that the marginal value of cash is lower for the more balanced firm in good times (high w), but higher in bad times (low w). Moreover, as shown in panels C and D, the more balanced firm cuts investment in low- w states.

While this pattern mirrors the one in Figure 2, the rationale is different for low w . In bad times—due to the higher degree of diversification—an extra dollar of cash is more effective in helping the more balanced firm avoid issuing costly equity. As a result, the more balanced firm has a stronger preference for liquidity closer to $w = 0$.

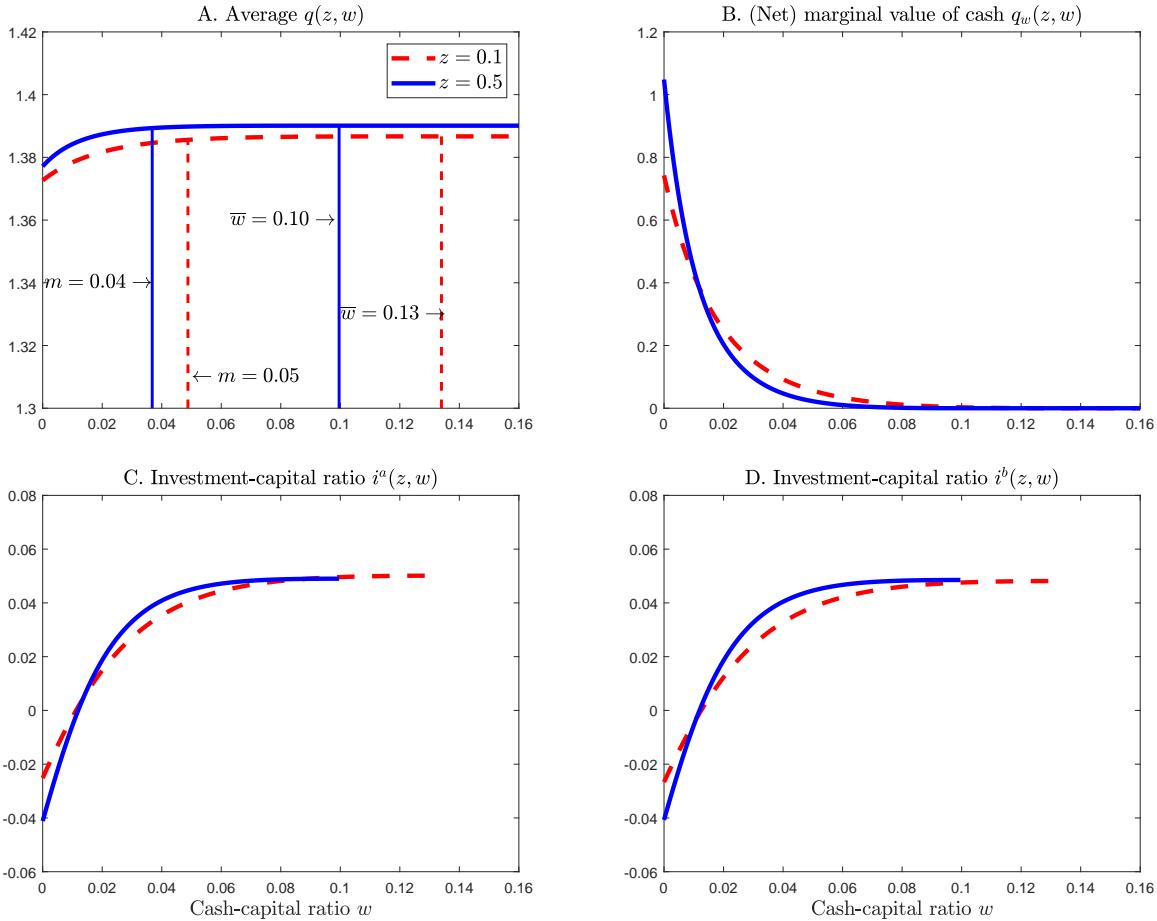


Figure 3: Refinancing case—comparison of diversified firms with $z = 0.1$ and $z = 0.5$. In panel A, the vertical lines mark the optimal payout boundary \bar{w} and the equity issuance amount m .

7.3 Characterization of Solution Regions

In Figure 4, we characterize our model’s solution by regions for the liquidation case (panel A) and two variants of the refinancing case (panels B and C, along with the respective equity issuance amount m reported in panel D). As we have two state variables, the cash-to-capital ratio w and the relative size z of division a , all regions are defined by (w, z) . The horizontal and vertical axes correspond to w and z , respectively. (When $z = 0$ or $z = 1$, the firm is a stand-alone firm.) The solid line corresponds to the payout boundary \bar{w} as a function of z , $\bar{w}(z)$. The function $\bar{w}(z)$ is part of the “payout region” and separates this region from the “interior region.” If $w_t \geq \bar{w}(z_t)$, the firm is in the payout region and pays out its excess cash $w_t - \bar{w}(z_t)$ to the shareholders.

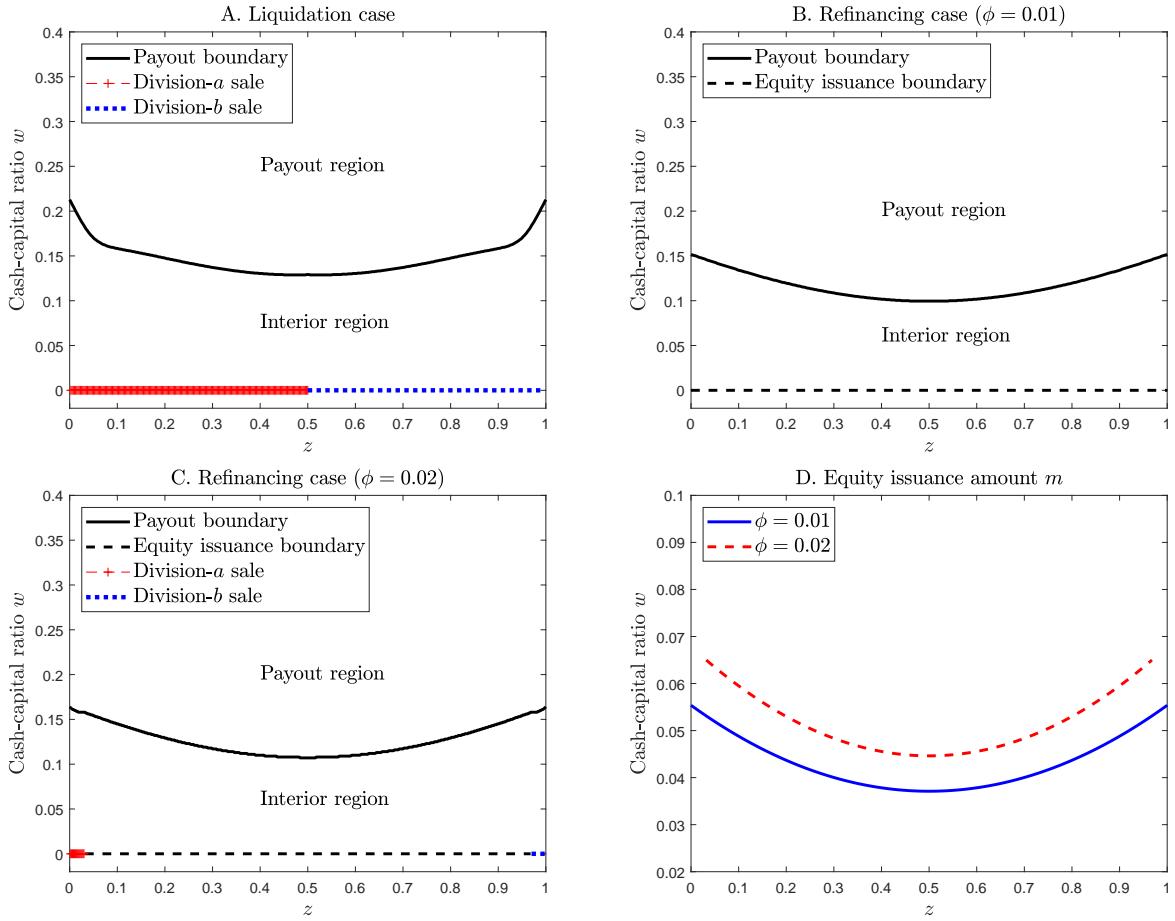


Figure 4: Solution regions for a firm with two symmetric divisions in the liquidation (panel A) and refinancing (panels B and C) cases. Panel D plots the equity issuance amount m .

Liquidation case.

In panel A, we consider the liquidation case. As can be seen, the payout boundary is the lowest when the firm's z reaches the absorbing state $z = 0.5$, and increases as the firm becomes less balanced. Intuitively, since the volatility of the firm's cash flows is lowest at $z = 0.5$, the balanced firm has the lowest demand for precautionary savings; the more unbalanced the firm is, the higher the volatility, and the higher the demand for precautionary savings.

When the firm runs out of cash ($w = 0$), it hits the liquidation boundary. Since liquidation is more costly for larger divisions (as the firm forgoes more of its productive assets), the firm optimally liquidates the smaller of the two divisions—that is, the firm liquidates division a when $z < 0.5$ (represented by the line with the + markers), and

division b when $z > 0.5$ (dotted line).²⁸ This prediction is consistent with the empirical evidence of Maksimovic and Phillips (2001), who find that multi-division firms are more likely to spin off their smaller divisions.

Refinancing case.

In panel B, we turn to the refinancing case. For our baseline parameters, refinancing is less costly than liquidation. Accordingly, when the firm runs out of cash ($w = 0$), it now hits the refinancing boundary (represented by the dashed line), and responds by issuing equity to replenish its cash. The firm has no incentive to issue equity sooner, as doing so would forgo the option of avoiding costly equity issuance.

We plot the corresponding equity issuance amount m in panel D (solid line). As is shown, both the payout boundary \bar{w} and the equity issuance amount m are the lowest at $z = 0.5$, and are higher the higher the imbalance between the two divisions. The intuition is the same as in the liquidation case—since the balanced firm is better diversified (and hence faces lower volatility), it can more easily afford to pay a dividend, and needs less financing conditional on issuing equity.

In panel C, we depart from our baseline parameters by assuming that equity financing has a higher fixed cost ($\phi = 2\%$). In this case, we find that refinancing is not always preferred to liquidation. When one of the divisions is sufficiently small (specifically, when $z < 0.03$ or $z > 0.97$), liquidation is less costly (as the firm only forgoes a relatively small fraction of its productive assets), and the firm prefers to spin off the smaller division as opposed to issuing costly equity. When $z \in [0.03, 0.97]$, the firm issues equity when it exhausts its cash holdings.

Finally, we observe in panel D that the equity issuance amount is always higher when $\phi = 2\%$ (compared to the baseline case with $\phi = 1\%$). Due to the higher fixed costs of issuing equity, firms resort to higher amounts in order to reduce the odds of bearing the fixed costs again in the future.

Note that our finding that the firm may find both external financing and division sale (liquidation) to be optimal on the equilibrium path is not possible in BCW (2011). This is because our model features two divisions, and whether equity financing or liquidation

²⁸When $z = 0.5$, the firm is indifferent between spinning off either division a or b .

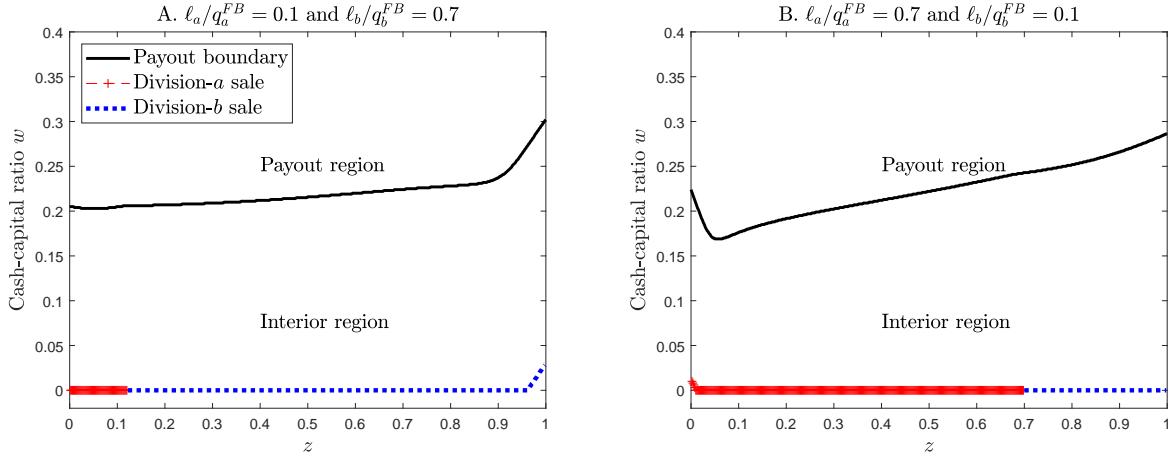


Figure 5: Early liquidation. This figure plots the solution regions for a diversified firm with asymmetric divisions with $\mu_a = 24\%$, $\mu_b = 16\%$, and $\rho = 0.9$. In panel A, the more productive division (division a) has a lower liquidation value ($\ell_a/q_a^{FB} = 0.1$ and $\ell_b/q_b^{FB} = 0.7$). In panel B, it has a higher liquidation value ($\ell_a/q_a^{FB} = 0.7$ and $\ell_b/q_b^{FB} = 0.1$).

is optimal for a financially distressed firm depends on z (in addition to w). This result illustrates how, at the conceptual level, the analysis of a financially constrained multi-division firm can be fundamentally different compared to that of a single-division firm.

7.4 Early Liquidation

With two symmetric divisions (i.e., the structural parameters μ , σ , δ , θ , and ℓ are the same for both divisions), liquidating a division when $w > 0$ (“early” liquidation) is never optimal. However, with asymmetric divisions, early liquidation can be optimal.

We consider such a parameterization in Figure 5. Specifically, we assume that the two divisions have different productivity ($\mu_a = 24\%$ and $\mu_b = 16\%$) and different liquidation values. Moreover, we set the correlation coefficient between the divisional productivity shocks to $\rho = 0.9$ so that the diversification benefits are lower than in our baseline analysis. The other parameters are the same as in Table 1 pertaining to the firm with symmetric divisions.

In panel A, we consider the case where the more productive division (division a) has a lower liquidation value than the other division ($\ell_a/q_a^{FB} = 0.1$ and $\ell_b/q_b^{FB} = 0.7$). In this case, it is optimal for the firm to voluntarily liquidate division b before running out of cash. This is because the diversification benefit of keeping division b is limited (due to the high

ρ and small size $1 - z$ of division b), and the cost of liquidating division b is relatively low (due to the relatively high liquidation value ℓ_b/q_b^{FB} and small size $1 - z$ of division b). Both forces make the firm willing to liquidate early in order to enhance its liquidity and mitigate underinvestment going forward, especially in low- w states.

In panel B, we turn to the more realistic scenario in which the more productive division (division a) also has a higher liquidation value ($\ell_a/q_a^{FB} = 0.7$ and $\ell_b/q_b^{FB} = 0.1$). The firm now liquidates division a before running out of cash when division a is sufficiently small (z close to 0). Note that, in this case, the firm liquidates the division with the *higher* productivity in low- w states.²⁹ This is because liquidating the division with a higher liquidation value generates more cash, which is highly valuable in low- w states, even though doing so eliminates the division with the higher productivity.

Importantly, this finding illustrates that, when taking risk management considerations into account, it can be misleading to rank divisions solely based on their productivity (μ). We discuss this point in more detail in the next section.

7.5 Resource Allocation with Different Volatilities

In Figure 6, we examine the resource (re-)allocation across two asymmetric divisions. A real-world relevant case is when one division has high μ and high σ , while the other has low μ and low σ , which provides an economically meaningful tradeoff. We set $\mu_a = 20.5\%$ and $\sigma_a = 40\%$ for division a , and $\mu_b = 19.5\%$ and $\sigma_b = 9\%$ for division b . All other parameters are the same as in our baseline, and we consider policies at $z = 0.5$.

Panel A plots the investment-capital ratio for both divisions in the liquidation case. Perhaps surprisingly, headquarters channels more of the firm's resources toward the low- μ and low- σ division, as opposed to the high- μ and high- σ division. This pattern is especially pronounced when the firm is low on cash (low w). Intuitively, the firm prefers to reduce its risk exposure and preserves its going concern value by investing in the low- μ and low- σ division, especially in low- w states where the likelihood of liquidation is higher.

²⁹This prediction is consistent with the empirical evidence of Schlingemann, Stulz, and Walkling (2002), who find that firms in need of cash tend to liquidate their more liquid divisions (that is, their divisions with higher liquidation values), even if those are their more productive units. Relatedly, this is consistent with the empirical evidence of Ma, Tong, and Wang (2020), who find that, in financial distress, innovative firms are more likely to liquidate their most productive assets (specifically, their core patents) if doing so allows them to raise more cash.

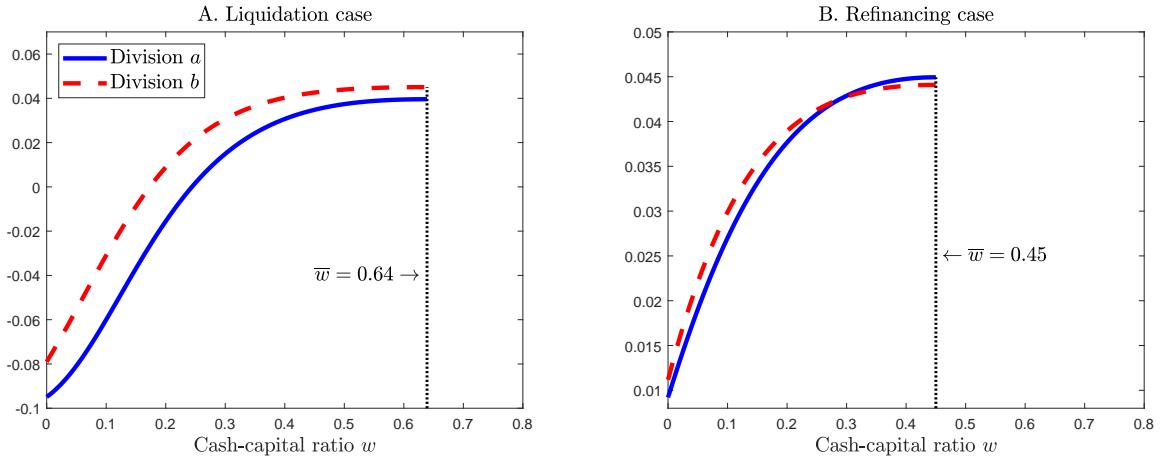


Figure 6: Resource allocation with different volatilities. This figure plots division-specific investment-capital ratios as a function of w (fixing $z = 0.5$) for a firm with asymmetric divisions with $\mu_a = 20.5\%$ and $\sigma_a = 40\%$ for division a , and $\mu_b = 19.5\%$ and $\sigma_b = 9\%$ for division b , respectively. The dotted line marks the payout boundary \bar{w} .

Panel B considers the refinancing case. As can be seen, the optimal capital allocation differs depending on w . In bad times (low w), more resources are allocated toward the low- σ division. The intuition is analogous to the liquidation case. In low- w states, the firm's primary concern is to manage risk and reduce the probability of tapping costly external equity financing, rather than generate higher expected cash flows. In contrast, in good times (high w), headquarters allocates more of the firm's resources toward the high- μ division—as in static models of “winner picking” (e.g., Stein, 1997). The reason we obtain the static model intuition in high- w states is because, when cash is abundant, profit-generating considerations dominate risk management considerations. Conversely, the reason the two lines do not cross in panel A is because liquidation is too costly and hence risk management considerations dominate for all levels of w .

These findings highlight the importance of considering a dynamic setting in order to characterize the mechanics of internal capital markets. In static models of winner picking, headquarters allocates resources to the more productive units (i.e., based on μ). In a dynamic setting, firms can run out of cash, and hence need to engage in risk management. As our model shows, when cash is scarce, headquarters may optimally channel resources toward the low-risk division (i.e., based on σ). These considerations motivate a broader formulation of the winner picking role of internal capital markets: when headquarters

allocates resources to divisions, it does so not only based on productivity, but also based on risk.

From this perspective, the within-firm allocation of resources can be viewed as a dynamic portfolio choice problem where different divisions offer very different risk-return tradeoffs. Unlike the standard portfolio choice problem (e.g., Merton, 1971), the risk-neutral firm in our model is endogenously risk averse, and the instruments used by the firm (e.g., equity issuance) are different from those used by households. As described above, this endogenous risk aversion depends on both the firm's liquidity (w) and the size distribution of its divisions (z).

Our findings have important implications for capital budgeting. Standard corporate finance textbooks prescribe that capital budgeting should be done based on the WACC of the stand-alone project, and the project's idiosyncratic risk should not matter. Our framework shows that this prescription is misguided. Indeed, ignoring idiosyncratic risk and the balance sheet of the firm when doing capital budgeting is incorrect; depending on the firm's liquidity, costs of external financing, and liquidation costs, it may be optimal to invest in a lower-NPV project (based on the project's WACC) if the project's idiosyncratic risk is sufficiently low. Adding a new project (or a new division) changes the firm's entire balance sheet composition and risk profile. As such, the firm should evaluate the new project by computing the net value difference caused by introducing the new project into the firm, as opposed to evaluating the project as if it were a stand-alone project.³⁰

8 Quantitative Analysis: Corporate Socialism

In this section, we incorporate the socialism cost induced by the division managers' rent-seeking behavior into the quantitative analysis presented in the preceding section. Corporate socialism is economically interesting when the divisions have different productivity. We assume that division a is more productive than division b . As described in Section 3,

³⁰Consider a setting where the unconditional CAPM holds under MM (i.e., for a financially unconstrained firm.) For a financially constrained firm, the unconditional CAPM does not hold but instead the conditional CAPM holds as the firm is endogenously risk averse (BCW, 2011). With financing constraints and multiple divisions, both w and z affect the firm's risk-return tradeoff, and hence influence corporate investment and capital budgeting decisions. As a result, the firm's cost of capital depends on both w and z for a multi-division firm. We can show that a conditional CAPM—where beta depends on both w and z —holds for the financially constrained multi-division firm.

corporate socialism makes the productive division less productive. Specifically, the productivity of division a is lowered from μ_a to $\hat{\mu}_a$, which translates into a revenue loss of $G^c = K^a(\mu_a - \hat{\mu}_a)$ in the model.

In what follows, we set the productivity of the two divisions to $\mu_a = 0.24$ and $\mu_b = 0.16$. Moreover, we set the socialism parameter to $\theta_c = 0.686$, using the estimate obtained by Matvos and Seru (2014). The other parameters are the same as in our baseline. The full set of parameters used in this section are summarized in the last column of Table 1.

Liquidation case.

In Figure 7, we consider the liquidation case with (solid line) and without (dashed line) socialism, for a firm with balanced divisions ($z = 0.5$). In panel A, we find that the value of the firm is always lower with socialism, consistent with the “dark side” models of internal capital markets. This finding is intuitive. Influence activities reduce the firm’s productivity, which in turn reduces the value of the firm. Moreover, we find that the payout boundary is lower with socialism. As socialism reduces the firm’s productivity, it makes payout more desirable from the shareholders’ perspective.

Interestingly, we find that the difference in valuation shrinks as we move closer to the liquidation boundary ($w = 0$). This reflects the lower cost of liquidating a division when the firm is subject to socialism—by spinning off a division, the firm becomes a stand-alone firm that is free of socialism frictions. Accordingly, the option to spin off a division is more valuable with socialism than without. Indeed, when $w = 0$, spinning off one division fully addresses the socialism problem, and hence the average q with socialism at $w = 0$ is the same as without socialism ($q = 1.31$).

This insight has implications for liquidity management. With socialism, it is less costly to replenish the firm’s liquidity through a spinoff. As a result, liquidity is less valuable with socialism than without, especially when the firm is low on cash. This is consistent with the pattern in panel B, showing that the marginal value of cash is lower with socialism, especially in low- w states. This also explains the pattern in panels C and D, showing that, with socialism, the firm is less prone to underinvestment in low- w states (as it has less of a need to preserve cash by reducing investment). In contrast, in high- w states, the lower productivity (due to socialism frictions) dominates, such that investment is lower for the

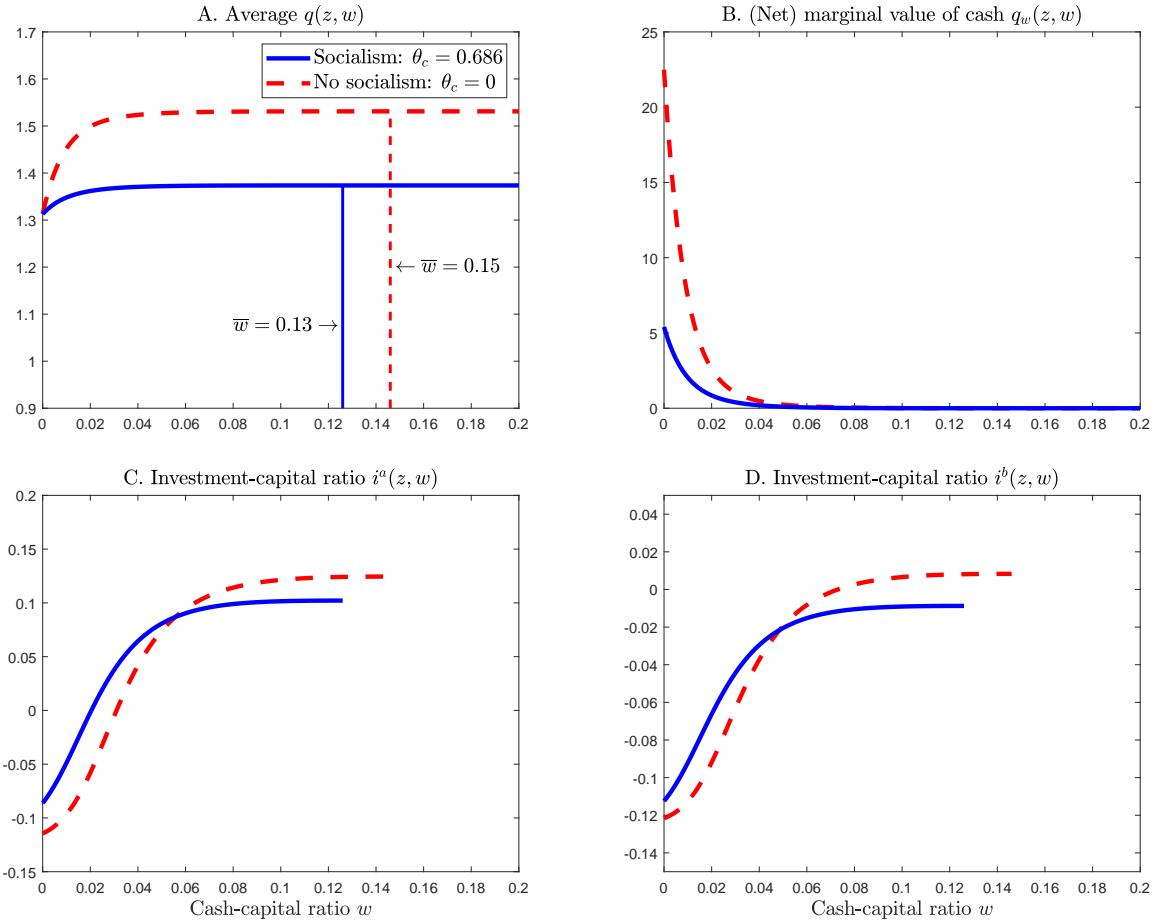


Figure 7: Liquidation case—comparison of diversified firms with and without socialism. The firms have asymmetric divisions with $\mu_a = 24\%$ and $\mu_b = 16\%$. The divisions are of equal size ($z = 0.5$). In panel A, the vertical lines mark the payout boundary \bar{w} .

firm with socialism.

In Figure 8, we plot the solution regions. Without socialism (panel A), the solution regions are similar to those in panel A of Figure 4, except that they are no longer symmetric around $z = 0.5$ due to the higher productivity of division a (compared to division b). In particular, when the firm runs out of cash ($w = 0$), it is now less likely to liquidate division a relative to division b . That is, the firm liquidates division a only when it exhausts its cash ($w = 0$) and it is sufficiently small compared to division b ($z \leq 0.38$).

Panel B reports the solution regions with socialism. When the firm is subject to socialism frictions, division b (the low-productivity division) is liquidated as soon as z exceeds a certain threshold (that is, when it is sufficiently small). This threshold is marked by the

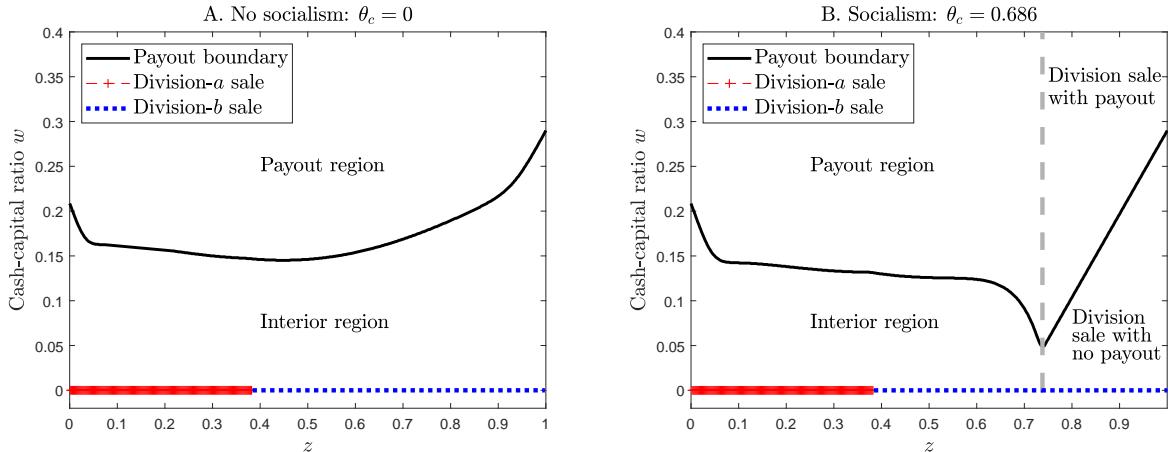


Figure 8: Liquidation case—comparison of policy regions for diversified firms with and without socialism. The firms have asymmetric divisions with $\mu_a = 24\%$ and $\mu_b = 16\%$.

dashed vertical line ($z = 0.738$). As long as $z \geq 0.738$, the firm instantly liquidates division b regardless of its w . Whether the firm makes a dividend payment or not when liquidating division b depends on the level of w . Formally, $\bar{w}(z)$ is given by

$$\bar{w}(z) = z\bar{w}^a - \ell_b(1-z), \quad (58)$$

where \bar{w}^a is the optimal payout boundary for a stand-alone firm with only division a , as in BCW (2011). Equation (58) defines the straight solid line in panel B.

If w is sufficiently large, in that $w > \bar{w}(z)$, the firm makes a one-time dividend payment $w - \bar{w}(z)$ when liquidating division b . Otherwise (i.e., if $w < \bar{w}(z)$), the firm pays no dividend when liquidating division b . These two regions are marked in Panel B as “division sale with payout” and “division sale with no payout,” respectively, above the $z = 0.738$ threshold.³¹

Below the threshold (i.e., when $z < 0.738$), liquidation only occurs at $w = 0$. The firm liquidates division a when $z \leq 0.38$ and liquidates division b when $0.38 \leq z < 0.738$. Moreover, we see that the firm is more willing to pay out as z increases, as it is more willing to spin off division b , which replenishes the firm’s liquidity and eliminates socialism.

³¹In our numerical example, the payout boundary for a stand-alone firm with only division a is $\bar{w}^a = 0.29$. A diversified firm with $z = 0.9$ and $w = 0.22$ immediately sells division b and becomes a stand-alone firm with division a and a cash-capital ratio of $w/z + \ell_b(1-z)/z = 0.22/0.9 + 0.07 = 0.31$, which is larger than $\bar{w}^a = 0.29$. Therefore, the firm pays out the excess cash $0.31 - 0.29 = 0.02$ per unit of division a ’s capital stock to shareholders.

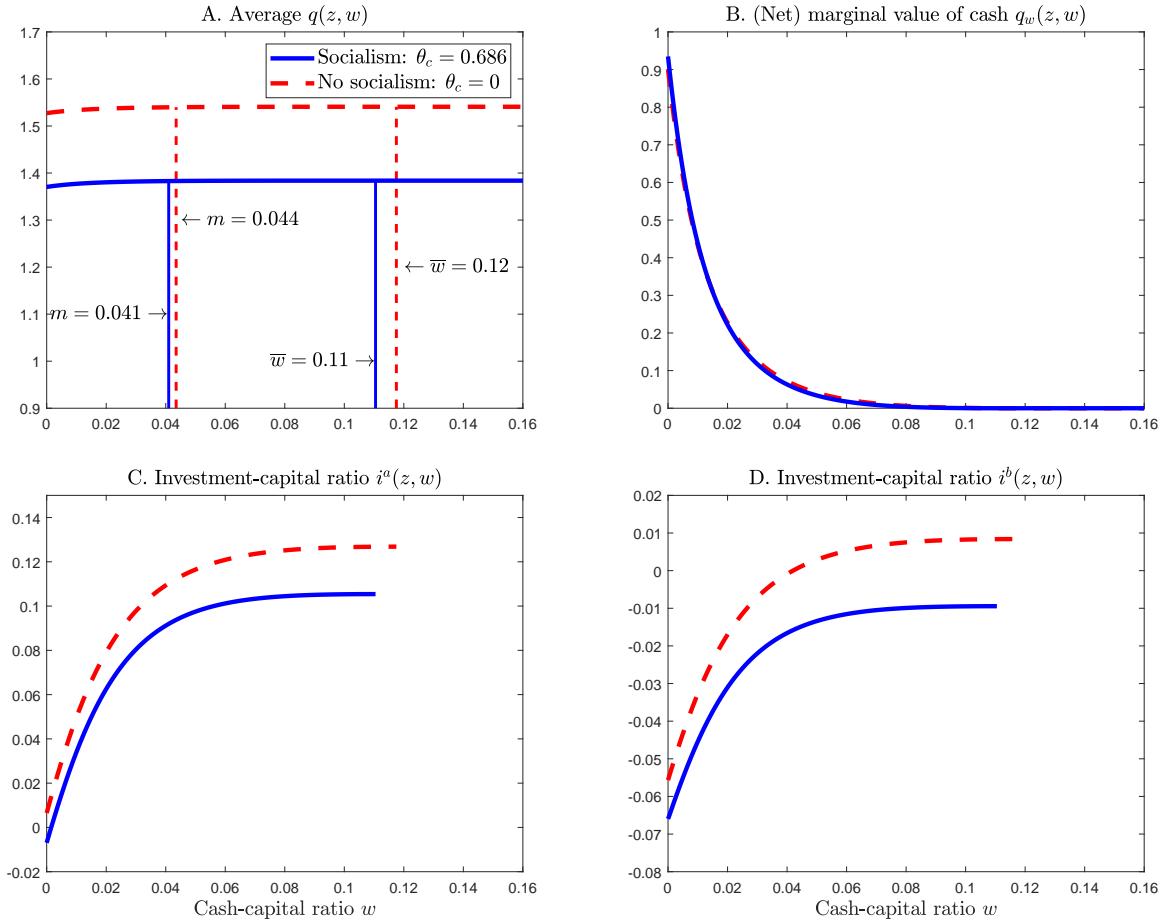


Figure 9: Refinancing case—comparison of diversified firms with and without socialism. The firms have asymmetric divisions with $\mu_a = 24\%$ and $\mu_b = 16\%$. The divisions are of equal size ($z = 0.5$). In panel A, the vertical lines mark the payout boundary \bar{w} and the equity issuance amount m , respectively.

Refinancing case.

In Figure 9, we turn to the refinancing case. In panel A, we observe again that the value of the firm is higher without socialism. In contrast to the liquidation case, we find that the wedge between the two curves barely changes, even in low- w states. This is because, for our parameter values, when firms run out of cash ($w = 0$), they prefer to issue equity as opposed to liquidating a division. For this reason, the firm with socialism no longer benefits from the lower cost of spinning off a division. This is further reflected in panel B, where we observe no noticeable difference in the marginal value of cash with and without socialism.

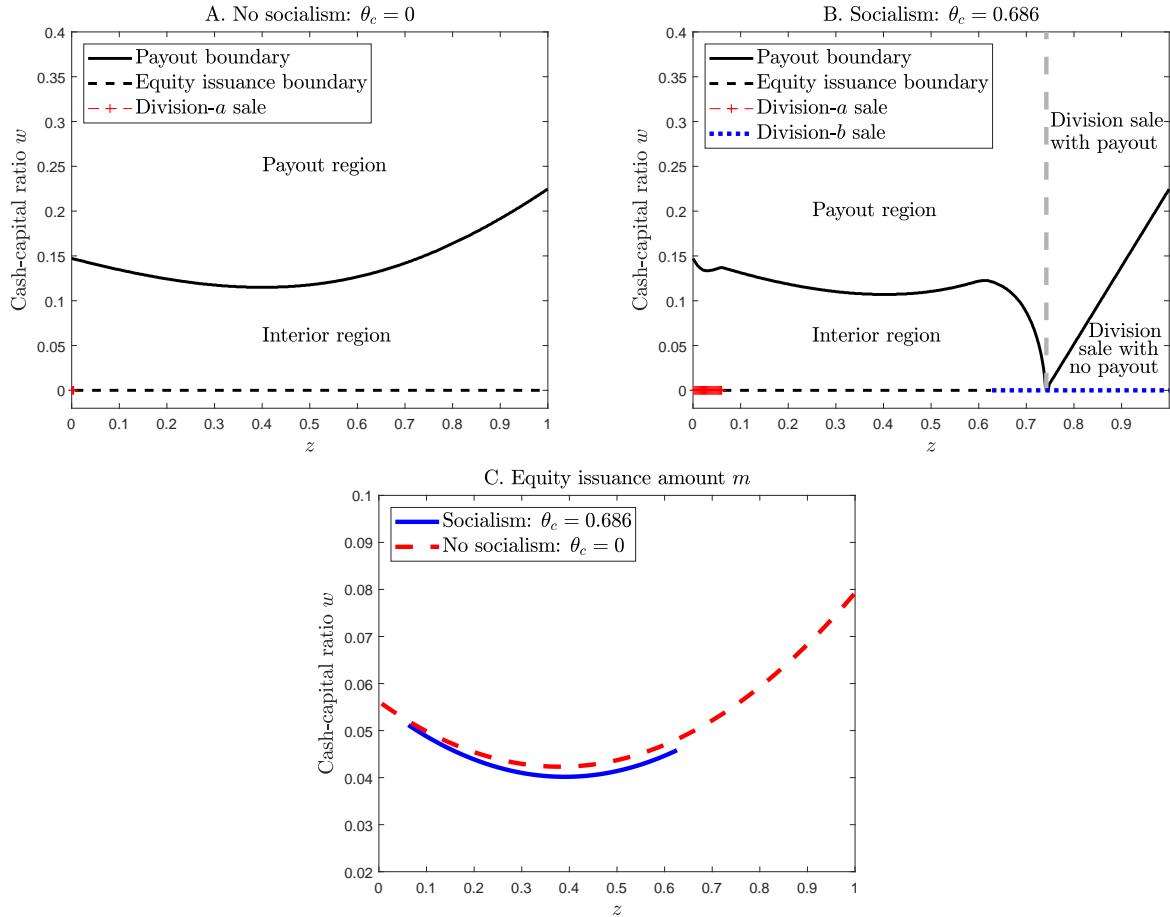


Figure 10: Refinancing case—comparison of solution regions for diversified firms with and without socialism. The firms have asymmetric divisions with $\mu_a = 24\%$ and $\mu_b = 16\%$.

In panels C and D, we find that investment in both divisions is higher for the firm without socialism. This finding is intuitive—without the socialism cost, the firm is more productive and hence generates higher cash flows that are used to sustain higher levels of current and future investment. Moreover, we observe that investment is relatively higher in division a , that is, the firm channels relatively more resources toward the more productive division.³²

Figure 10 plots the solution regions for the refinancing case. Panel A refers to the setting without socialism. Compared to the liquidation case (Panel A of Figure 8), the firm finds it optimal to spin off division a only when it is sufficiently small ($z \leq 0.004$), and chooses to issue equity for all other values of z . Because the firm can issue equity at a cost, the model

³²Note the different scale of the vertical axis in both panels.

generates a hump-shaped prediction for the equity issuance amount, m , as a function of z (represented by the dashed line in Panel C). The intuition is that a more diversified firm can afford to hold less cash, and hence has less of a need to issue large amounts. Because of the divisions' asymmetry, the firm with $z = 0.41$ has the lowest demand for cash. As the more productive division's relative size, z , increases beyond $z = 0.41$, the firm's increasing productivity calls for greater funding needs, which explains why the equity issuance amount m increases with z .

Panel B of Figure 10 characterizes the various regions with socialism. As in the liquidation case (panel B of Figure 8), we find that the low-productivity division b is liquidated when it becomes sufficiently small. If $z \geq 0.742$, where 0.742 is the threshold marked by the dashed vertical line, the firm instantly liquidates division b regardless of its w . Whether the firm immediately makes a dividend payment or not upon liquidating division b depends on its level of w . Similar to panel B of Figure 8, the straight line in panel B corresponds to the endogenous payout boundary $\bar{w}(z)$ defined by equation (58).³³

In the regions to the left of this threshold (i.e., $z < 0.742$), when the firm runs out of cash ($w = 0$), it liquidates division a for $z \leq 0.06$, liquidates division b for $z \geq 0.63$, and issues equity for values of $z \in (0.06, 0.63)$. Note that the range of z values for which divisions are liquidated is larger than in panel A. This again reflects the lower cost of spinning off a division when the firm is plagued with socialism frictions. Finally, the solid line in panel C represents the equity issuance amount m when $z \in (0.06, 0.63)$. The pattern is again hump-shaped. Note that the curve with socialism is lower than the one without socialism—due to the lower productivity of the firm plagued with socialism, the optimal refinancing amount is lower.³⁴

³³In our numerical example with socialism, the optimal payout boundary for a stand-alone firm with only division a is $\bar{w}^a = 0.22$.

³⁴Note the extensive non-monotonicity of the payout boundary in panel B, which reflects the richness of our model. On the left of the $z = 0.742$ threshold, \bar{w} follows a similar hump-shaped pattern as in panel A, except in the bordering regions. In particular, when z approaches the $z = 0.742$ threshold from the left, \bar{w} decreases sharply, as the firm anticipates the cash inflow from the likely spinoff of division b . Similarly, in the z region where liquidation of division a is potentially optimal ($z < 0.06$), the pattern follows another hump-shaped pattern that reflects the tradeoff between the expectation of higher liquidity through the likely spinoff of division a and the loss of diversification benefits (net of the gain from eliminating socialism frictions).

9 Conclusion

In this paper, we provide a tractable model in which investment, cross-divisional transfers, division sale (spinoff), cash management, external financing, and dividend payout are jointly characterized for a multi-division firm that faces costly external finance. Our model provides a rich set of novel predictions, ranging from a refined formulation of the “winner picking” role of internal capital markets to a characterization of the optimal spinoff decision. Moreover, we develop a q theory of investment for *financially constrained* multi-division firms, in which division-level investment is set such that the ratio between marginal q and the marginal cost of investing in each division equals the marginal value of cash.

Our model lends itself to several extensions. In particular, as in BCW (2011), one could extend the model by including debt financing (in the form of a credit line) and financial hedging (e.g., through the reliance on options and futures contracts). While doing so would enrich the model, adding these components would not reverse our main predictions.

Finally, our analysis leaves several avenues open for future research, of which we highlight two here. First, while our model allows for rent-seeking behavior of the division managers, it does not speak to the optimal contract design. In this regard, enriching our model with a dynamic contracting framework—e.g., of the type studied by Malenko (2019) for capital budgeting—could be a fruitful extension. Doing so would provide a characterization of the optimal contract that arises taking into account the complexity and intertwined nature of the multi-division firm’s policies.

Second, our model could be used to analyze the dynamics of M&As. In particular, a merger between two stand-alone firms that become a two-division firm could be studied by integrating key institutional features of M&As into a variant of the model developed in this paper. Such a model would provide insights into the role (and intertwined nature) of both financing and operating synergies. While the latter has received considerable attention in the literature, much less is known of the former, let alone the interconnections between the two synergies, and the dark side of internal capital markets for the newly formed conglomerate after a successful M&A.

References

- Abel, Andrew B., and Janice C. Eberly, 1994, A unified model of investment under uncertainty, *American Economic Review* 84(5), 1369-1384.
- Abel, Andrew B., and Stavros Panageas, 2020, Precautionary saving in a financially-constrained firm, NBER Working paper 26628.
- Alchian, Armen A., 1969, Corporate management and property rights, in Manne, Henry G., ed.: *Economic Policy and the Regulation of Corporate Securities* (American Enterprise Institute, Washington, DC).
- Bakke, Tor-Erik, and Tiantian Gu, 2017, Diversification and cash dynamics, *Journal of Financial Economics* 123(3), 580-601.
- Biais, Bruno, Thomas Mariotti, Guillaume Plantin, and Jean-Charles Rochet, 2007, Dynamic security design: Convergence to continuous time and asset pricing implications, *Review of Economic Studies* 74(2), 345-390.
- Bolton, Patrick, Hui Chen, and Neng Wang, 2011, A unified theory of Tobin's q , corporate investment, financing, and risk management, *Journal of Finance* 66(5), 1545-1578.
- Coase, Ronald H., 1937, The nature of the firm, *Economica* 4(16), 386-405.
- Cooley, Thomas F., and Vincenzo Quadrini, 2001, Financial markets and firm dynamics, *American Economic Review* 91(5), 1286-1310.
- Cooper, Russell W., and John C. Haltiwanger, 2006, On the nature of capital adjustment costs, *Review of Economic Studies* 73(3), 611-633.
- Dai, Min, and Yifei Zhong, 2010, Penalty methods for continuous-time portfolio selection with proportional transaction costs, *Journal of Computational Finance* 13(3), 1-31.
- Décamps, Jean-Paul, Thomas Mariotti, Jean-Charles Rochet, and Stphane Villeneuve, 2011, Free cash flow, issuance costs and stock prices, *Journal of Finance* 66(5), 1501-1544.
- DeMarzo, Peter, and Yuliy Sannikov, 2006, Optimal security design and dynamic capital structure in a continuous-time agency model, *Journal of Finance* 61(6), 2681-2724.
- DeMarzo, Peter, and Michael Fishman, 2007, Optimal long-term financial contracting, *Review of Financial Studies* 20(6), 2079-2128.
- DeMarzo, Peter, Michael Fishman, Zhiguo He, and Neng Wang, 2012, Dynamic agency and the q theory of investment, *Journal of Finance* 67(6), 2295-2340.
- Dittmar, Amy, 2004, Capital structure in corporate spinoffs, *Journal of Business* 77(1), 9-43.
- Duchin, Ran, 2010, Cash holdings and corporate diversification, *Journal of Finance* 65(3), 955-992.

- Duchin, Ran, and Denis Sosyura, 2013, Divisional managers and internal capital markets, *Journal of Finance* 68(2), 387-429.
- Dumas, Bernard, 1991, Super contact and related optimality conditions, *Journal of Economic Dynamics and Control* 15(4), 675-685.
- Eberly, Janice C., Sergio Rebelo, and Nicolas Vincent, 2009, Investment and value: A neoclassical benchmark, Working paper, Northwestern University.
- Eckbo, B. Espen, and Karin S. Thorburn, 2008, Corporate restructuring: Breakups and LBOs, in Eckbo, B. Espen, ed.: *Handbook of Empirical Corporate Finance* (North-Holland/Elsevier, Amsterdam).
- Evans, Lawrence C., 1979. A second order elliptic equation with gradient constraint, *Communications in Partial Differential Equations* 4(5), 555-572.
- Friedman, Avner, and Joel Spruck, 2012, Variational and free boundary problems (Springer Science & Business Media).
- Giroud, Xavier, and Holger M. Mueller, 2015, Capital and labor reallocation within firms, *Journal of Finance* 70(4), 1767-1804.
- Glaser, Markus, Florencio Lopez-De-Silanes, and Zacharias Sautner, 2013, Opening the black box: Internal capital markets and managerial power, *Journal of Finance* 68(4), 1577-1631.
- Gomes, Joao F., 2001, Financing investment, *American Economic Review* 91(5), 1263-1285.
- Gomes, Joao F., and Dmitry Livdan, 2004, Optimal diversification: Reconciling theory and evidence, *Journal of Finance* 59(2), 507-535.
- Graham, John R., 2000, How big are the tax benefits of debt? *Journal of Finance* 55(5), 1901-1941.
- Graham, John R., Harvey, Campbell R. and Manju Puri, 2015. Capital allocation and delegation of decision-making authority within firms, *Journal of Financial Economics* 115(3), 449-470.
- Guedj, Ilan, and David S. Scharfstein, 2004, Organizational scope and investment: Evidence from the drug development strategies and performance of biopharmaceutical firms, NBER Working paper 10933.
- Hall, Robert E., 2001, The stock market and capital accumulation, *American Economic Review* 91(5), 1185-1202.
- Hayashi, Fumio, 1982, Tobin's marginal q and average q : A neoclassical interpretation, *Econometrica* 50(1), 215-224.
- Hennessy, Christopher A., and Toni M. Whited, 2007, How costly is external financing? Evidence from a structural estimation, *Journal of Finance* 62(4), 1705-1745.

- Hugonnier, Julien, Semyon Malamud, and Erwan Morellec, 2015, Capital supply uncertainty, cash holdings, and investment, *Review of Financial Studies* 28(2), 391-445.
- Jensen, Michael C., and William H. Meckling, 1976, Theory of the firm: Managerial behavior, agency costs and ownership structure, *Journal of Financial Economics* 3(4), 305-360.
- Kuppuswamy, Venkat, and Belén Villalonga, 2016, Does diversification create value in the presence of external financing constraints? Evidence from the 2007-2009 financial crisis, *Management Science* 62(4), 905-923.
- Leland, Hayne E., and David H. Pyle, 1977, Informational asymmetries, financial structure, and financial intermediation, *Journal of Finance* 32(2), 371-387.
- Lucas, Robert E., Jr., and Edward C. Prescott, 1971, Investment under uncertainty, *Econometrica* 39(5), 659-681.
- Ma, Song, Joy T. Tong, and Wei Wang, 2020, Bankrupt innovative firms, Working paper, Yale University.
- Maksimovic, Vojislav, and Gordon M. Phillips, 2001, The market for corporate assets: Who engages in mergers and asset sales and are there efficiency gains? *Journal of Finance* 56(6), 2019-2065.
- Maksimovic, Vojislav, and Gordon M. Phillips, 2002, Do conglomerate firms allocate resources inefficiently across industries? Theory and evidence, *Journal of Finance* 57(2), 721-767.
- Maksimovic, Vojislav, and Gordon M. Phillips, 2013, Conglomerate firms, internal capital markets, and the theory of the firm, *Annual Review of Financial Economics* 5(1), 225-244.
- Malenko, Andrey, 2019, Optimal dynamic capital budgeting, *Review of Economic Studies* 86(4), 1747-1778.
- Matusaka, John G., 2001, Corporate diversification, value maximization, and organizational capabilities, *Journal of Business* 74(3), 409-431.
- Matvos, Gregor, and Amit Seru, 2014, Resource allocation within firms and financial market dislocation: Evidence from diversified conglomerates, *Review of Financial Studies* 27(4), 1143-1189.
- Matvos, Gregor, Amit Seru, and Rui C. Silva, 2018, Financial market frictions and diversification, *Journal of Financial Economics* 127(1), 21-50.
- Merton, Robert C., 1971, Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* 3(4), 373-413.
- Meyer, Margaret, Paul R. Milgrom, and John Roberts, 1992, Organizational prospects, influence costs, and ownership changes, *Journal of Economics and Management Strategy* 1(1), 9-35.

- Milgrom, Paul R., 1988, Employment contracts, influence activities, and efficient organization design, *Journal of Political Economy* 96(1), 42-60.
- Milgrom, Paul R., and John Roberts, 1988, An economic approach to influence activities in organizations, *American Journal of Sociology* 94(1), 154-179.
- Modigliani, Franco, and Merton Miller, 1958, The cost of capital, corporation finance, and the theory of investment, *American Economic Review* 48(3), 261-297.
- Myers, Stewart C., and Nicholas S. Majluf, 1984, Corporate financing and investment decisions when firms have information that investors do not have, *Journal of Financial Economics* 13(2), 187-221.
- Nikolov, Boris, Lukas Schmid, and Roberto Steri, 2019, Dynamic corporate liquidity, *Journal of Financial Economics* 132(1), 76-102.
- Ozbas, Oguzhan, and David S. Scharfstein, 2010, Evidence on the dark side of internal capital markets, *Review of Financial Studies* 23(2), 581-599.
- Rajan, Raghuram, Henri Servaes, and Luigi Zingales, 2000, The cost of diversity: The diversification discount and inefficient investment, *Journal of Finance* 55(1), 35-80.
- Riddick, Leigh A., and Toni M. Whited, 2009, The corporate propensity to save, *Journal of Finance* 64(4), 1729-1766.
- Scharfstein, David S., and Jeremy C. Stein, 2000, The dark side of internal capital markets: Divisional rent-seeking and inefficient investment, *Journal of Finance* 55(6), 2537-2564.
- Schlingemann, Frederik P., René M. Stulz, and Ralph A. Walkling, 2002, Divestitures and the liquidity of the market for corporate assets, *Journal of Financial Economics* 64(1), 117-144.
- Shapiro, Matthew D., 1986, The dynamic demand for capital and labor, *Quarterly Journal of Economics* 101(3), 513-542.
- Shin, Hyun-Han, and René M. Stulz, 1998, Are internal capital markets efficient? *Quarterly Journal of Economics* 113(2), 531-552.
- Stein, Jeremy C., 1997, Internal capital markets and the competition for corporate resources, *Journal of Finance* 52(1), 111-133.
- Williamson, Oliver E., 1975, Market and hierarchies: Analysis and antitrust implications (Free Press).

Appendices

A HJB Equation and Verification Theorem

In this appendix, we formulate our stochastic control model and then provide a verification theorem for the model solution characterizing both the firm's value function and its optimal decision rules.

Given an external financing policy $\nu = \{\tau^{(1)}, \tau^{(2)}, \dots; M^{(1)}, M^{(2)}, \dots\}$, the firm's cash balance process W satisfies

$$\begin{cases} dW_t = dY_t + (r - \lambda)W_t dt - dU_t & t \in (\tau^{(i)}, \tau^{(i+1)}); \\ W_{\tau^{(i)}} = W_{\tau^{(i)}-} + M^{(i)} & t = \tau^{(i)}. \end{cases} \quad (\text{A.1})$$

We may then express the optimization problem (14) as

$$\max_{I^a, I^b, U, \tau, \nu} \mathbb{E} \left[\int_0^\tau e^{-rt} dU_t - \sum_{\tau^{(i)} < \tau} e^{-r\tau^{(i)}} (\phi(K_{\tau^{(i)}}^a + K_{\tau^{(i)}}^b) + (1 + \gamma)M^{(i)}) \right. \\ \left. + e^{-r\tau} \{P^a(K_\tau^a, W_\tau)1_{\{\tau=\tau_b\}} + P^b(K_\tau^b, W_\tau)1_{\{\tau=\tau_a\}}\} \right], \quad (\text{A.2})$$

where the second term accounts for the financing amount $M^{(i)}$ and the financing cost $\phi \cdot (K_t^a + K_t^b) + \gamma M^{(i)}$ at $t = \tau^{(i)}$.

The associated HJB variational inequality for this optimization problem is

$$\max \left\{ \mathcal{L}_0 F, 1 - F_W, \tilde{F} - F \right\} = 0, \quad K^a \geq 0, \quad K^b \geq 0, \quad W \geq 0, \quad (\text{A.3})$$

where

$$\begin{aligned} \mathcal{L}_0 F &= \max_{I^a, I^b} \left\{ (I^a - \delta_a K^a) F_{K^a} + (I^b - \delta_b K^b) F_{K^b} \right. \\ &\quad + [(r - \lambda)W + \hat{\mu}_a K^a + \mu_b K^b - (I^a + I^b + G^a + G^b)] F_W \\ &\quad \left. + \frac{1}{2} [\sigma_a^2 (K^a)^2 + \sigma_b^2 (K^b)^2 + 2\rho\sigma_a\sigma_b K^a K^b] F_{WW} \right\} - rF. \end{aligned}$$

Here, $\tilde{F}(K^a, K^b, W)$ is the firm's value conditional on external financing or division sale given in equation (46).

By using $z_t = \frac{K_t^a}{K_t^a + K_t^b}$ and $w_t = \frac{W_t}{K_t^a + K_t^b}$, we obtain the following dynamics for (w_t, z_t) between the consecutive rounds of refinancing:

$$\begin{aligned} dw_t &= (r - \lambda)w_t dt + [z_t(\mu_a dt + \sigma_a d\mathcal{Z}_t^a) + (1 - z_t)(\mu_b dt + \sigma_b d\mathcal{Z}_t^b)] \\ &\quad - [(i_t^a + g_a(i_t^a))z_t + (i_t^b + g_b(i_t^b))(1 - z_t)] dt \\ &\quad - w_t[z_t(i_t^a - \delta_a) + (1 - z_t)(i_t^b - \delta_b)] dt - du_t, \end{aligned} \quad (\text{A.4})$$

$$dz_t = z_t(1 - z_t)[(i_t^a - \delta_a) - (i_t^b - \delta_b)] dt, \quad (\text{A.5})$$

where $i_t^s = I_t^s/K_t^s$ for $s = a, b$ and $du_t = \frac{dU_t}{K_t^a + K_t^b}$ is the incremental dividend payout dU_t scaled by the firm's total capital stock. At any refinancing time $t = \tau^{(i)}$, w_t satisfies:

$$w_{\tau^{(i)}} = w_{\tau^{(i)}-} + m^{(i)}, \quad (\text{A.6})$$

where $m^{(i)} = \frac{M^{(i)}}{K_{\tau^{(i)}}^a + K_{\tau^{(i)}}^b}$ denotes the scaled net financing at stopping time $\tau^{(i)}$. By using the model's homogeneity property, we can simplify the three-state variable HJB variational inequality HJB equation (A.3) to the two-state variable HJB equation (51) for $f(z, w)$.

We numerically solve the two-dimensional HJB equation (51) by using a penalty method that is efficient for singular/impulse control problems; see, e.g., Dai and Zhong (2010).

The following verification theorem characterizes the firm's value function and optimal policies for the optimization problem (A.2).

Proposition A.1 (Verification theorem). *Let $f(z, w)$ be a solution to the HJB equation (51) satisfying certain regularity conditions. We define the interior region (IR), payout region (PR), and the external financing/division sale region (ED) as follows:*

$$\begin{aligned} IR &= \{(z, w) : \mathcal{L}f(z, w) = 0, f_w(z, w) > 1, f(z, w) > \tilde{f}(z, w)\}, \\ PR &= \{(z, w) : f_w(z, w) = 1, \mathcal{L}f(z, w) \leq 0, f(z, w) \geq \tilde{f}(z, w)\}, \\ ED &= \{(z, w) : f(z, w) = \tilde{f}(z, w), \mathcal{L}f(z, w) \leq 0, f_w(z, w) \geq 1\}, \end{aligned}$$

where $\tilde{f}(z, w)$ is as given in equation (49). The external financing/division sale region consists of two sub-regions: the division sale region (SR) and external financing region (ER):

$$\begin{aligned} SR &= \{(z, w) : f(z, w) = \tilde{f}(z, w), f(z, w) > j(z, w)\}, \\ ER &= \{(z, w) : f(z, w) = \tilde{f}(z, w), f(z, w) = j(z, w)\}, \end{aligned}$$

where $j(z, w)$ is as given in equation (48).

The payout boundary is the intersection of \overline{IR} , the complement of IR , and payout region (PR), i.e., $\partial P = \overline{IR} \cap PR$. Similarly, the division sale boundary is the intersection of \overline{IR} and SR : $\partial S = \overline{IR} \cap SR$, and the external financing boundary is the intersection of \overline{IR} and ER : $\partial E = \overline{IR} \cap ER$. The firm's value is given by $F(K^a, K^b, W) = (K^a + K^b) \cdot f(z, w)$, where $z = \frac{K^a}{K^a + K^b}$ and $w = \frac{W}{K^a + K^b}$.

In addition, the optimal strategy $(i_t^a, i_t^b, U_t, \nu, \tau)$ is given as follows:

(a) Optimal investment (i_t^a, i_t^b) in the interior region IR :

$$1 + g'_a(i_t^a) = \frac{f(z_t, w_t) + (1 - z_t)f_z(z_t, w_t)}{f_w(z_t, w_t)} - w_t, \quad (\text{A.7})$$

$$1 + g'_b(i_t^b) = \frac{f(z_t, w_t) - z_t f_z(z_t, w_t)}{f_w(z_t, w_t)} - w_t, \quad (\text{A.8})$$

where (z_t, w_t) is the solution of (A.4)-(A.6) associated with the optimal strategy;

(b) Payout strategy U_t :³⁵

$$U_t = \int_0^t 1_{\{(z_\xi, w_\xi) \in \partial P\}} dU_\xi; \quad (\text{A.9})$$

(c) External financing strategy $\nu = \{\tau^{(1)}, \tau^{(2)}, \dots; M^{(1)}, M^{(2)}, \dots\}$:

$$\tau^{(n+1)} = \inf \left\{ t \in (\tau^{(n)}, \tau] : (z_t, w_t) \in ER \right\}, \quad (\text{A.10})$$

$$M^{(n+1)} = (K^a + K^b) \arg \max_{m > 0} f(z_{\tau^{(n+1)}}, w_{\tau^{(n+1)}} + m) - \phi - (1 + \gamma)m, \quad (\text{A.11})$$

where $\tau^{(0)} = 0$;

(d) Division sale strategy $\tau = \min\{\tau_a, \tau_b\}$:

$$\tau_a = \inf \{t \geq 0 : (z_t, w_t) \in SR, f(z_t, w_t) > z_t p^a(w_t^a/z_t)\} \quad (\text{A.12})$$

$$\tau_b = \inf \{t \geq 0 : (z_t, w_t) \in SR, f(z_t, w_t) > (1 - z_t) p^b(w_t^b/(1 - z_t))\}. \quad (\text{A.13})$$

³⁵As our model is two-dimensional, the payout decision is described by a local time associated with a curve ∂P . In contrast, most models in the literature are one-dimensional, in which case the payout decision, while also described by a local time, is associated with a single point (the payout threshold) rather than a curve. For example, Hugonnier, Malamud, and Morellec (2015) formulate a one-dimensional model (with lumpy investment and uncertain equity issuance timing) for a financially constrained firm and provide a proof.

Proof. Define

$$N_t = \int_0^t e^{-rh} dU_h + e^{-rt} F(K_t^a, K_t^b, W_t). \quad (\text{A.14})$$

Let U_t^c be the continuous part of U_t and $\Delta U_h = U_h - U_{h-}$ be the discrete jump at time h . Using Ito's Lemma, we obtain:

$$\begin{aligned} N_t &= N_0 + \int_0^t e^{-rh} \mathcal{L}_0 F dh + \int_0^t e^{-rh} (1 - F_W) dU_h^c \\ &\quad + \sum_{0 \leq h \leq t} e^{-rh} \left(\Delta U_h + F(K_{h-}^a, K_{h-}^b, W_{h-} - \Delta U_h) - F(K_{h-}^a, K_{h-}^b, W_{h-}) \right). \end{aligned} \quad (\text{A.15})$$

First, we show that the last term in equation (A.15) is non-positive for any feasible strategy. By the mean-value theorem, there exists $u \in [0, \Delta U_h]$ such that

$$\begin{aligned} &\sum_{0 \leq h \leq t} e^{-rh} \left(\Delta U_h + F(K_{h-}^a, K_{h-}^b, W_{h-} - \Delta U_h) - F(K_{h-}^a, K_{h-}^b, W_{h-}) \right) \\ &= \sum_{0 \leq h \leq t} e^{-rh} \left(\Delta U_h - F_W(K_{h-}^a, K_{h-}^b, W_{h-} - u) \Delta U_h \right) \leq 0, \end{aligned}$$

where the inequality follows from the HJB equation.

As f is a solution to the HJB equation (51), we can verify that F satisfies the HJB variational inequality (A.3), which means that $\mathcal{L}_0 F \leq 0$, $F \geq \tilde{F}$, and $F_W \geq 1$ in the entire state space. Note that $dU_t^c \geq 0$. Then, the second and third terms in (A.15) are non-positive for any feasible strategy and equal to zero for the proposed strategy defined in equations (A.7)-(A.13). Therefore, we have shown that N_t is a martingale for the proposed strategy defined by equations (A.7)-(A.13) and is a supermartingale for any alternative (feasible) strategy.

Because N_t is a supermartingale, for a feasible strategy $(\check{I}_t^a, \check{I}_t^b, \check{U}_t, \check{\tau}^{(i)}, \check{M}^{(i)}, \check{\tau})$, we have

$$\begin{aligned} F(K_0^a, K_0^b, W_0) &\geq \mathbb{E}[N_{\check{\tau}^{(i)} \wedge \check{\tau}}] \\ &= \mathbb{E} \left[\int_0^{\check{\tau}^{(i)} \wedge \check{\tau}} e^{-rh} d\check{U}_h + e^{-r\check{\tau}^{(i)} \wedge \check{\tau}} F(K_{\check{\tau}^{(i)} \wedge \check{\tau}}^a, K_{\check{\tau}^{(i)} \wedge \check{\tau}}^b, W_{\check{\tau}^{(i)} \wedge \check{\tau}}) \right] \\ &\geq \mathbb{E} \left[\int_0^{\check{\tau}^{(i)} \wedge \check{\tau}} e^{-rh} d\check{U}_h + e^{-r\check{\tau}^{(i)} \wedge \check{\tau}} \tilde{F}(K_{\check{\tau}^{(i)} \wedge \check{\tau}}^a, K_{\check{\tau}^{(i)} \wedge \check{\tau}}^b, W_{\check{\tau}^{(i)} \wedge \check{\tau}}) \right], \end{aligned}$$

where $\tilde{F}(K_t^a, K_t^b, W_t) = \max \{P^a(K_t^a, L_t^b + W_t), P^b(K_t^b, L_t^a + W_t), J(K_t^a, K_t^b, W_t)\}$ and the last inequality follows from $F(K_t^a, K_t^b, W_t) \geq \tilde{F}(K_t^a, K_t^b, W_t)$ implied by equation (A.3). For

the proposed policy $(I_t^a, I_t^b, U_t, \tau^{(i)}, M^{(i)}, \tau)$ defined by equations (A.7)-(A.13), the above inequalities hold with equality, which implies the optimality of the proposed policy. \square

B Proof of Proposition 5.1

We first state a lemma that will be used in our proof of Proposition 5.1.

Lemma B.1. *Consider a financially constrained firm with a single division, s . Under the condition given in equation (6), i.e., $\mu_s > \ell_s(r + \delta_s)$, if the firm ever chooses to liquidate itself, it will only do so when exhausting its cash holding, i.e., when $W_\tau = 0$.*

We relegate the proof for this lemma to the Internet Appendix. Next, we prove Proposition 5.1.

Proof of Proposition 5.1. The firm can always set $i_t^a = i_t^b = 0$, although it is generally suboptimal. Therefore, in the first-best world, the average q for division s is at least larger than $\mu_s/(r + \delta_s)$ as we can see from equation (24). We thus conclude that liquidating a division in the first-best world is never optimal as long as the economically meaningful conditions given in equation (6), i.e., $\mu_s/(r + \delta_s) > \ell_s$ hold.

Consider three different liquidation strategies for a diversified firm. Recall $f(z, w)$ is the scaled firm value for the conglomerate and $p^s(w)$ is the scaled firm value for a single-division firm with division s .

First, liquidating both divisions simultaneously yields $f(z, w) = \ell_a z + \ell_b(1 - z) + w$. Second, liquidating division a yields $f(z, w) = (1 - z)p^b(w^b)$, where $w^b = (\ell_a z + w)/(1 - z)$. Third, liquidating division b yields $f(z, w) = zp^a(w^a)$, where $w^a = (\ell_b(1 - z) + w)/z$.

Lemma B.1 implies that $f(z, w) = zp^a(w^a) > z \cdot (\ell_a + w^a) = z\ell_a + \ell_b(1 - z) + w$ as liquidating division b only rather than liquidating both divisions simultaneously yields a higher payoff. Similarly, $f(z, w) = (1 - z)p^b(w^b) > (1 - z) \cdot (\ell_b + w^b) = (1 - z)\ell_b + \ell_a z + w$. As liquidating only one division yields a higher value of $f(z, w)$ than liquidating both divisions simultaneously, a multi-division firm always prefers selling one of its divisions than liquidating the whole firm. \square

C Numerical Procedure

We numerically solve the two-dimensional HJB equation (51) by using a penalty method that is efficient for singular/impulse control problems (e.g., Dai and Zhong, 2010). Specif-

ically, we use the following penalty method to solve the variational inequality (51):

$$\mathcal{L}_1(i^a, i^b)f + \mathcal{K}(1 - f_w)^+ + \mathcal{K}(\tilde{f} - f)^+ = 0, \quad (\text{C.1})$$

where the penalty parameter \mathcal{K} is a sufficiently large positive constant. The operator $\mathcal{L}_1(i^a, i^b)$, which corresponds to the operator \mathcal{L} in equation (51), satisfies

$$\begin{aligned} \mathcal{L}_1(i^a, i^b)f(z, w) &= (i^a - \delta_a)z[f(z, w) + (1 - z)f_z(z, w) - wf_w(z, w)] \\ &\quad + (i^b - \delta_b)(1 - z)[f(z, w) - zf_z(z, w) - wf_w(z, w)] \\ &\quad + [(r - \lambda)w + (\mu_a - i^a - g_a(i^a))z + (\mu_b - i^b - g_b(i^b))(1 - z)]f_w(z, w) \\ &\quad + \frac{1}{2}[\sigma_a^2 z^2 + \sigma_b^2(1 - z)^2 + 2z(1 - z)\rho\sigma_a\sigma_b]f_{ww}(z, w) - rf(z, w) \end{aligned}$$

where i^a and i^b satisfy the FOCs (40)-(41). Penalty methods are widely used to establish the existence of the variational-inequality solution by letting the penalty parameter \mathcal{K} approach infinity (e.g., Evans, 1979 and Friedman and Spruck, 1982).

To obtain a numerical solution, we restrict attention to a bounded domain $\{(z, w) \in [0, 1] \times [0, w^{\max}]\}$, where w^{\max} is a large finite number. We prescribe the following boundary conditions based on our economic analysis:

$$f_w = 1 \text{ at } w = w^{\max}, \quad f = \tilde{f} \text{ at } w = 0, \quad f = p^a \text{ at } z = 1, \quad \text{and} \quad f = p^b \text{ at } z = 0.$$

These conditions indicate that the firm pays dividends at $w = w^{\max}$, issues equity or spins off a division at $w = 0$, and becomes a single-division firm at $z = 0$ and $z = 1$, respectively. Then, we use a finite difference method similar to the one in Dai and Zhong (2010).

We use the following iteration algorithm in the given domain:

1. Choose an initial value of f, f^0 .
2. Given f^n from the n -th iteration, compute the division investment in the interior region, $(i^a)^n$ and $(i^b)^n$, by solving

$$\begin{aligned} 1 + g'_a((i^a)^n) &= \frac{f^n(z, w) + (1 - z)f_z^n(z, w)}{f_w^n(z, w)} - w, \\ 1 + g'_b((i^b)^n) &= \frac{f^n(z, w) - zf_z^n(z, w)}{f_w^n(z, w)} - w. \end{aligned}$$

Then calculate

$$\begin{aligned}\tilde{f}^n(z, w) &= \max \left\{ z p^a(w^a), (1-z) p^b(w^b), j^n(z, w) \right\}, \\ j^n(z, w) &= \max_{m>0} f^n(z, w+m) - \phi - (1+\gamma) m.\end{aligned}$$

3. Solve f^{n+1} by using $(i^a)^n, (i^b)^n, f_w^n, \tilde{f}^n$, and

$$\mathcal{L}_1((i^a)^n, (i^b)^n) f^{n+1} + \mathcal{K}(1 - f_w^{n+1}) 1_{\{1-f_w^n>0\}} + \mathcal{K}(\tilde{f}^n - f^{n+1}) 1_{\{\tilde{f}^n - f^n > 0\}} = 0,$$

with the following boundary conditions: $f_w^{n+1} = 1$ at $w = w^{\max}$, $f^{n+1} = \tilde{f}^n$ at $w = 0$, $f^{n+1} = p^a$ at $z = 1$, and $f^{n+1} = p^b$ at $z = 0$.

4. If $|f^{n+1} - f^n| < \epsilon$ where ϵ is a very small number (tolerance), then we have obtained the numerical solution. Otherwise, set $f^n = f^{n+1}$ and go to step 2.

Internet Appendix: Proof of Lemma B.1

In this internet appendix, we prove Lemma B.1 for a single-division firm's optimization problem, which is used in our proof of Proposition 5.1.

Lemma B.1. *Consider a financially constrained firm with a single division, s . Under the condition given in equation (6), i.e., $\mu_s > \ell_s(r + \delta_s)$, if the firm ever chooses to liquidate itself, it will only do so when exhausting its cash holding, i.e., when $W_\tau = 0$.*

Proof. Consider two feasible (suboptimal) strategies, D and \hat{D} for the single-division firm with (K_0, W_0) at time 0. Under strategy D , the firm liquidates its capital stock at time 0 and immediately obtains its liquidation value $\ell_s K_0 + W_0$. Under strategy \hat{D} , the firm does not liquidate itself at time 0, instead it immediately makes a payout with amount $(1 - \epsilon)W_0$ at time 0 for some $\epsilon \in (0, 1)$ satisfying³⁶

$$(\mu_s - \ell_s(r + \delta_s))K_0 > \lambda\epsilon W_0. \quad (\text{IA.1})$$

Additionally, under strategy \hat{D} , the firm pays no dividends to shareholders, does not invest over the time period $(0, \hat{t})$, and liquidates at \hat{t} , where \hat{t} will be defined later.

We may then write down the dynamics of K_t^s and W_t under strategy \hat{D} for $t \in (0, \hat{t})$ as:

$$dK_t^s = -\delta_s K_t^s dt, \quad (\text{IA.2})$$

$$\begin{aligned} dW_t &= (r - \lambda)W_t dt + K_t^s dA_t \\ &= (r - \lambda)W_t dt + K_t^s(\mu_s dt + \sigma_s dZ_t^s). \end{aligned} \quad (\text{IA.3})$$

Next, we define two stopping times $\tau_0 = \inf\{t > 0 : W_t = 0\}$ and $\hat{t} = \tau_0 \wedge \Delta$, where Δ is a sufficiently small positive constant satisfying

$$\Delta < \frac{(\mu_s - \ell_s(r + \delta_s))K_0 - \lambda\epsilon W_0}{\mu_s(r + \delta_s)K_0}. \quad (\text{IA.4})$$

Condition (IA.1) ensures that the right-hand side of equation (IA.4) is positive.

Next, we show that the firm's value at time 0 under strategy \hat{D} is higher than its value under strategy D . By integrating equations (IA.2)-(IA.3), we obtain the following at time

³⁶Importantly, under condition (6), i.e., $\mu_s > \ell_s(r + \delta_s)$, we know there exists a value of ϵ such that equation (IA.1) holds.

\hat{t} :

$$K_{\hat{t}}^s = e^{-\delta_s \hat{t}} K_0, \quad (\text{IA.5})$$

$$\begin{aligned} W_{\hat{t}} &= e^{(r-\lambda)\hat{t}} \epsilon W_0 + \frac{\mu_s}{r + \delta_s - \lambda} \left(e^{(r-\lambda)\hat{t}} - e^{-\delta_s \hat{t}} \right) K_0 \\ &\quad + e^{(r-\lambda)\hat{t}} K_0 \int_0^{\hat{t}} e^{-(r+\delta_s-\lambda)t} \sigma_s dZ_t^s. \end{aligned} \quad (\text{IA.6})$$

The firm's value under strategy \hat{D} at time 0 is

$$\begin{aligned} &(1 - \epsilon)W_0 + \mathbb{E}e^{-r\hat{t}} (\ell_s K_{\hat{t}}^s + W_{\hat{t}}) \\ &= W_0 + \ell_s K_0 + (e^{-(r+\delta_s)\hat{t}} - 1) \ell_s K_0 + (e^{-\lambda\hat{t}} - 1) \epsilon W_0 + \frac{\mu_s}{r + \delta_s - \lambda} \left(e^{-\lambda\hat{t}} - e^{-(r+\delta_s)\hat{t}} \right) K_0 \\ &\geq W_0 + \ell_s K_0 + (- (r + \delta_s) \hat{t}) \ell_s K_0 + (-\lambda \hat{t}) \epsilon W_0 \\ &\quad + \frac{\mu_s}{r + \delta_s - \lambda} (1 - (r + \delta_s) \hat{t}) ((r + \delta_s - \lambda) \hat{t}) K_0 \\ &= W_0 + \ell_s K_0 + \{ [(\mu_s - \ell_s(r + \delta_s)) K_0 - \lambda \epsilon W_0] - \mu_s(r + \delta_s) K_0 \hat{t} \} \hat{t} \\ &> W_0 + \ell_s K_0, \end{aligned} \quad (\text{IA.7})$$

where the first equality uses equations (IA.5)-(IA.6), the first inequality follows from the inequality $e^x > 1 + x$ for $x \in \mathbb{R}$, and the last inequality follows from condition (IA.4).

As the firm's value at time 0 under strategy D is $W_0 + \ell_s K_0$, strategy \hat{D} dominates strategy D . Even though strategy \hat{D} is not optimal, we have shown that postponing firm liquidation is necessarily part of the optimal strategy. In summary, a single-division firm should never liquidate itself before running out of cash. \square