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ABSTRACT

In this note, we consider a broad class of network models where a large number of heterogeneous agents simultaneously interact in many ways. We provide an iterative algorithm for calculating an equilibrium and offer sufficient and "globally necessary" conditions under which the equilibrium is unique. The results arise from a multi-dimensional extension of the contraction mapping theorem which allows for the separate treatment of the different types of interactions. We illustrate that a wide variety of heterogeneous agent economies – characterized by spatial, production, or social networks – yield equilibrium representations amenable to our theorem's characterization.

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1 Introduction

The twenty first century has witnessed the rise of big data and big models in the social sciences. Exponential growth in computational capacity combined with access to new micro-level datasets have allowed the empirical implementation of models where large numbers of heterogeneous agents interact simultaneously with each other in myriad ways. While the rise of big data and big models has introduced empirical content to traditionally theoretical fields, important questions about the positive properties of these big models remain unresolved. Two concerns – critical for applied work – are particularly pressing: How can we compute the solution of an equilibrium system with hundreds or thousands of heterogeneous agents efficiently? And even if we do calculate a solution, how do we know that the equilibrium we find is the only possible one?

In this note, we answer these questions for a large class of models where many heterogeneous agents simultaneously interact in many ways. In particular, we consider systems where N heterogeneous agents engage in H types of interactions whose equilibrium can be reduced to a set of $N \times H$ equations of the following form:

$$x_{ih} = \sum_{j=1}^{N} f_{ijh}\left(x_{j1}, ..., x_{jH}\right), \tag{1}$$

where $\{x_{ih}\} \in \mathbb{R}_{++}^{N\times H}$ reflect the (strictly positive) equilibrium outcomes for each agent of each interaction and $f_{ijh} : \mathbb{R}_{++}^{H} \to \mathbb{R}_{++}$ are the known (differentiable) functions that govern the interactions between different agents. In particular, f_{ijh} is the function that governs the impact that an interaction with agent j has on agent i's equilibrium outcome of type h.[1] As we illustrate, this formulation is sufficiently general to capture models of many different economic networks – from firm linkages to social networks to the spatial structure of cities.

The contribution of the paper is to provide conditions under which an equilibrium satisfying equation (1) is unique and can be calculated using an iterative algorithm. The key insight, loosely speaking, is to simplify the analysis by abstracting from agent heterogeneity and focusing on the strength of economic interactions. Formally, rather than focusing on the $N^2 \times H$ functions $\{f_{ijh}\}$, we instead focus on the $H \times H$ matrix of the uniform bounds of the elasticities $\epsilon_{hh'} \equiv \sup_{i,j,\{x_{jh}\}}\left(\left|\frac{\partial \ln f_{ijh}\left(\{x_{jh}\}_h\right)}{\partial \ln x_{jh'}}\right|\right)$. The conditions provided depend only on a single statistic of this matrix: its spectral radius being less than one (or, with additional restrictions on $\{f_{ijh}\}$, equal to one).[2] Moreover, the conditions provided are shown to be

[1]These interactions could be market interactions or non-market interactions (as discussed by Glaeser, Sacerdote, and Scheinkman (2003); Glaeser and Scheinkman (2002)).

[2]The spectral radius plays a number of important roles in economics, e.g. in the characterization of

"globally necessary", i.e. they are the best possible conditions that are agnostic about the heterogeneity across agents: formally, we show that if the conditions are not satisfied, there exist $\{f_{ijh}\}$ where multiplicity is assured.

Our main result relies on a multi-dimensional extension of the contraction mapping theorem, which – to our knowledge – is new and of independent interest in its own right. The insight of this extension is that it is possible to partition the space of endogenous variables into subsets, each of which operates in a different metric subspace. This partition is particularly helpful in economic models where heterogeneous agents interact in many ways (i.e. H is large), as it allows us to separate the study of each type of interaction.

To illustrate the versatility of our approach, consider two alternative strategies often employed to analyze the equilibrium properties of a system. The first alternative strategy is to recursively apply a process of substitution to re-define the equilibrium system as a function of fewer economic interactions. For example, in a simple exchange economy with multiple agents and multiple goods, there are two interactions – buying and selling, which in equilibrium can be summarized by the value of each agent's endowment (wages) and consumption bundle (price index). Alvarez and Lucas (2007) characterize the equilibrium of such a system by first substituting wages into the price index and then analyzing the structure of the model only in terms of wages.[3] While feasible for small H, the complexity of this strategy increases exponentially with the number of interactions in the model, creating a curse of dimensionality for large H.

The second alternative strategy is to "stack" all economic outcomes into a single $NH \times 1$ vector and apply standard contraction mapping arguments. The disadvantage of such an approach is that it treats different types of economic outcomes identically – despite the fact that they may play very different roles in the equilibrium system. The results in a loss of information and introduces the possibility that the sufficient conditions may fail despite the system being unique.[4] In contrast, our approach both avoids the curse of dimensionality of

macro-economic stability (see e.g. Hawkins and Simon (1949)) and the solution of linearized DSGE models (Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016)). More recently, Elliott and Golub (2019) shows that the spectral radius characterizes the efficiency of public goods provision in networks with non uniform externalities. To our knowledge, this note is the first to show that the spectral radius of a matrix of elasticities of economic interactions characterizes the uniqueness of (and the speed of convergence of an iterative algorithm to) the equilibrium of a network model with many heterogenous agents.

[3] Indeed, Allen, Arkolakis, and Takahashi (2020) show that the sufficient conditions presented in Alvarez and Lucas (2007) – which rely on showing the gross substitutes property of the system, c.f. Mas-Colell, Whinston, and Green (1995) – can be relaxed when treating wages and the price index separately. The results here extend those of Allen, Arkolakis, and Takahashi (2020) both by allowing for general (non-constant elasticity) functional forms and by allowing for more than two types of economic interactions.

[4] A simple example is the following system where $N = 1$ and $H = 2$: $x_{11} = x_{11}^{\frac{1}{2}} x_{12}^{2} + 1$, $x_{12} = x_{12}^{\frac{1}{2}} + 1$. It is straightforward to show that by treating x_{11} and x_{12} as a single vector variable, the standard contraction conditions that the matrix norm (induced by the vector norm) of the system's Jacobian matrix is strictly

the first strategy and the loss of information inherent to the second, permitting an analysis of economic systems with large numbers of interactions.

We provide additional results for a special case of equation (1) where the elasticities $\frac{\partial \ln f_{ijh}\left(\left\{x_{jh}\right\}_h\right)}{\partial \ln x_{jh'}}$ are constant and identical across agents. This case has emerged as the de-facto benchmark in the "quantitative" spatial literature, spanning the fields of international trade, economic geography, and urban economics (see e.g. the excellent review articles by Costinot and Rodriguez-Clare (2013) and Redding and Rossi-Hansberg (2017)). We also offer results that facilitate the analytical characterization of the spectral radius condition and, as a result, the parametric region where uniqueness and computation is feasible.

We finally apply our theorem to offer new results and extensions of seminal models from disparate fields in economics, illustrating its broad applicability. In particular, in the field of spatial economics, we provide uniqueness conditions for quantitative urban models in the spirit of Ahlfeldt, Redding, Sturm, and Wolf (2015) in the presence of spatial productivity and amenity spillovers. In the field of macroeconomics, we provide uniqueness results for the sectoral production network in the spirit of Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012) but generalized to allow for non unit elasticities of substitution as in Carvalho, Nirei, Saito, and Tahbaz-Salehi (2019). In the field of social networks, we provide uniqueness conditions for a model of discrete choice with social interactions in the spirit of Brock and Durlauf (2001) but generalized to allow for many choices and arbitrary weights on others' actions.

A voluminous literature in economic theory has used fixed point theorems to analyze existence and uniqueness of solutions of economic models. The literature has offered three main approaches in order to characterize the positive properties of economic models: (1) use of the contraction mapping theorem; (2) conditions on the Jacobian matrix such as it satisfying gross substitution or it being an M-matrix (see e.g. Mas-Colell, Whinston, and Green (1995) chapter seventeen, Arrow, Hahn, et al. (1971) chapter nine, and Gale and Nikaido (1965)); or (3) the Index Theorem.[5] This paper follows the first approach. While

less than one (see e.g. Olver (2008) Chapter 9) are not satisfied, whereas the multi-dimensional contraction mapping conditions we provide are satisfied. See Online Appendix B.3.1 for details.

[5]Notice that substitutability conditions are effectively conditions on the cross-derivatives of the Jacobian. Berry, Gandhi, and Haile (2013) show that a relaxed form of substitutability, weak gross-substitutes, together with strict connectedness are sufficient for invertibility (in our context, uniqueness). In a setup that maintains the assumptions of a typical Walrasian economy, Iritani (1981) shows that Weak Indecomposability is necessary and sufficient for uniqueness. He also shows that a stronger form of Weak Indecomposibility implies Weak Gross substitutability so these analysis are intimately related. Kennan (2001) shows that concave monotonically increasing functions have a unique positive fixed point; here, we make no restrictions that the functions be monotonic, increasing, or concave (although the condition that the spectral radius of the matrix of bounds of the elasticities be no greater than one does simplify to a requirement of quasi-concavity in the special case where $N = H = 1$ and the function being considered is monotonically increasing).

the latter two approaches are powerful, they are often impractical to apply to situations where many agents interact in many ways. For example, the Jacobian of equation (1) is of size $HN^2 \times HN^2$, making it difficult to characterize; in contrast, the conditions below depend on a single statistic of an $H \times H$ matrix.[6] Similarly, the the Index Theorem has typically proven impractical to apply to production economies.[7] Our contribution to this literature is to show that for a general class of models with heterogeneous agents and multiple interactions a multi-dimensional extension of the contraction mapping theorem can be a powerful tool in characterizing their properties. The resulting theorem provides easy-to-verify conditions for uniqueness of an equilibrium and an algorithm for its computation.

The structure of the remainder of the note is as follows: Section 2 presents the multi-dimensional contraction mapping extension (Lemma 1), offers the main result (Theorem 1), and makes five remarks. Section 3 presents three applications of the result to the fields of spatial networks, sectoral production networks, and social networks, respectively. For brevity, the proof of Lemma 1 and Theorem 1 are presented in the Appendix, and details of the remarks and applications are presented in the Online Appendix.

2 Main Results

We start our presentation by offering a multi-dimensional extension of the standard contraction mapping theorem. While of interest in itself, it also facilitates the proof of Theorem 1 below.

Lemma 1. *Let $\{(X_h, d_h)\}_{h=1,2,...,H}$ be H metric spaces where X_h is a set and d_h is its corresponding metric. Define $X \equiv X_1 \times X_2 \times ... \times X_H$, and $d : X \times X \to \mathbb{R}_+^H$ such that for $x = (x_{1,...,}x_H), x' = \left(x'_{1,...,}x'_H\right) \in X$, $d(x, x') = \begin{pmatrix} d_1(x_1, x'_1) \\ ... \\ d_H(x_H, x'_H) \end{pmatrix}$. Given operator $T : X \to X$, suppose for any $x, x' \in X$*

$$d(T(x), T(x')) \leq \mathbf{A} d(x, x'), \tag{2}$$

[6] Even when the Jacobian can be characterized, the conditions required to establish uniqueness may be too stringent. For example, consider the system $x_i = \sum_{j=1}^{N} K_{ij} x_j^{\alpha}$ for $K_{ij} > 0$ and $\alpha \in (0, 1)$. The i^{th} diagonal term of its Jacobian is $1 - \alpha K_{ii} x_i^{\alpha-1}$ which can be negative or positive, violating e.g. the classical condition of Gale and Nikaido (1965) that all principal submatrices of the Jacobian have positive determinants. In contrast, the spectral radius of the elasticity is $\alpha < 1$, so uniqueness is established immediately by the Theorem presented here.

[7] See an extensive discussion on the applications of the index theorem to exchange and production economies in Kehoe (1985); Kehoe, Whalley, et al. (1985). While mathematically powerful, the index theorem conditions typically lose their sufficiency when attempted to translate them in economically interpretable conditions.

where $\mathbf{A}$ *is a non-negative matrix and the inequality is entry-wise. Denote* $\rho(\mathbf{A})$ *as the spectral radius (largest eigenvalue in absolute value) of* $\mathbf{A}$.

If $\rho(\mathbf{A}) < 1$ *and for all* $h = 1, 2, ..., H$, (X_h, d_h) *is complete, there exists a unique fixed point of* T, *and for any* $x \in X$, *the sequence of* x, $T(x)$, $T(T(x))$, ... *converges to the fixed point of* T.

Proof. See Appendix A.1. □

Lemma 1 extends the standard contraction mapping result to multiple dimensions by replacing the contraction constant with the matrix $\mathbf{A}$. It then states that a simple sufficient statistic of that matrix – its spectral radius $\rho(\mathbf{A})$ – replaces the role of the contraction constant in determining the contraction of the system. This sufficient statistic succinctly summarizes the role of the asymmetry of the impact of the different variables in determining the positive properties of the system: as long as the spectral radius is less than one there exists a unique fixed point, and it can be computed by applying the mapping $T(x)$ iteratively, which converges to the fixed point at a rate $\rho(\mathbf{A})$. Intuitively, a spectral radius of less than one holds if and only if the sequence $\lim_{k\to\infty} \mathbf{A}^k$ converges to zero so that repeated applications of the operator eventually bound the set of points of the sequence arbitrarily close to the fixed point. Note that Lemma 1 reduces to the standard contraction mapping theorem if $H = 1$ (see e.g. Theorem 3.2 of Lucas and Stokey (1989)).

2.1 Main Theorem

As mentioned in the introduction, the main result of the paper concerns systems whose equilibrium can be written as in equation (1). Before presenting our main result, some additional notation is in order. Let $\mathcal{N} \equiv \{1, ..., N\}$ and $\mathcal{H} \equiv \{1, ..., H\}$ correspond to the set of economic agents and the set of economic interactions, respectively. Let x be an N-by-H matrix of endogenous economic outcomes, where for $i \in \mathcal{N}$ and $h \in \mathcal{H}$, we (slightly abuse notation) and let x_i denote x's ith row and $x_{.h}$ to denote x's hth column. We restrict our attention to strictly positive $\{x_{ih}\}_{i\in\mathcal{N},h\in\mathcal{H}} \in \mathbb{R}^{N\times H}_{++}$ and strictly positive and differentiable $\{f_{ijh}\}$. Finally, define the elasticity $\epsilon_{ijh,jh'}(x_j) \equiv \frac{\partial \ln f_{ijh}(x_j)}{\partial \ln x_{jh'}}$, i.e. $\epsilon_{ijh,jh'}(x_j)$ is the impact of agent $j's$ outcome of type h' on agent $i's$ outcome of type h.

Theorem 1. *Suppose there exists an* H*-by-*H *matrix* $\mathbf{A}$ *such that for all* $i, j \in \mathcal{N}$, $h, h' \in \mathcal{H}$, *and* $x_j \in \mathbb{R}^H_{++}$ $|\epsilon_{ijh,jh'}(x_j)| \leq (\mathbf{A})_{hh'}$. *Then:*

(i). If $\rho(\mathbf{A}) < 1$, *then there exists a unique solution to equation (1) and the unique solution can be computed by iteratively applying equation (1) with a rate of convergence* $\rho(\mathbf{A})$;

(ii). If $\rho(\mathbf{A}) = 1$ and:

a. If $|\epsilon_{ijh,jh'}(x_j)| < (\mathbf{A})_{hh'}$ for all $i, j \in \mathcal{N}$ and $h, h' \in \mathcal{H}$ when $(\mathbf{A})_{hh'} \neq 0$, then equation (1) has at most one solution x;

b. If $\epsilon_{ijh,jh'}(x_j) = \alpha_{hh'} \in \mathbb{R}$ where $|\alpha_{hh'}| = (\mathbf{A})_{hh'}$ for all $i, j \in \mathcal{N}$ and $h, h' \in \mathcal{H}$ i.e. $f_{ijh}(x_j) = K_{ijh} \prod_{h' \in \mathcal{H}} x_{jh'}^{\alpha_{hh'}}$ for some $K_{ijh} > 0$–then equation (1)'s solution is column-wise up-to-scale unique, i.e. for any $h \in \mathcal{H}$ and solutions x and x' it must be $x'_{.h} = c_h x_{.h}$ for some scalar $c_h > 0$;

(iii). If $\rho(\mathbf{A}) > 1$, $N \geq 2H + 1$, and $f_{ijh}(x_j) = K_{ijh} \prod_{h' \in \mathcal{H}} x_{jh'}^{\alpha_{hh'}}$, then there exists some $\{K_{ijh} > 0\}_{i,j \in \mathcal{N}, h \in \mathcal{H}}$ such that equation (1) has multiple solutions that are column-wise up-to-scale different.

Proof. See Appendix A.2. □

It is important to emphasize that the conditions provided in the Theorem 1 abstract from the particular heterogeneity of agents – i.e. the particular functions $\{f_{ijh}\}$ – and instead focus on the magnitude of the economic interactions across all agents, i.e. the uniform bounds on elasticities $|\epsilon_{ijh,jh'}(x_j)| \leq (\mathbf{A})_{hh'}$. Loosely speaking, the matrix $(\mathbf{A})_{hh'}$ captures the degree to which the economic outcome of any agent of type h' can impact any other agents' economic outcome of type h. Such conditions that focus on the strength of the economic interactions rather than the heterogeneity of the agents themselves are advantageous in settings where the same economic model may be applied to different empirical contexts. For example, in spatial models, the heterogeneity of agents captures such things like the specific underlying geography (e.g. trade costs) which are highly context dependent; in contrast, the elasticities govern the strength of economic interactions (e.g. the elasticity of demand) that may be similar across locations.

Part (i) of the Theorem applies Lemma 1 to show that there exists a unique solution and that solution can be computed with an iterative algorithm that converges at a rate $\rho(\mathbf{A})$. In particular, denote equation (1) as $x = T(x)$; then for any initial "guess" of a positive solution $x^0 \in \mathbb{R}_{++}^{N \times H}$, one simply iterates $x^1 = T(x^0)$, $x^2 = T(x^1)$, $x^3 = T(x^2)$, ... until convergence. The restriction that $f_{ijh} : \mathbb{R}_{++}^N \to \mathbb{R}_{++}$ further guarantees that the solution is strictly positive (something not guaranteed by the original Lemma).

Part (ii) of the Theorem deals with the case of $\rho(\mathbf{A}) = 1$, which turns out to be a common phenomenon in economic modeling (see Remark 4 below). It establishes uniqueness by imposing extra conditions on the elasticities $\epsilon_{ih,jh'}(x_j)$: if either the elasticities are strictly smaller than their bounds (part ii.a) or the elasticities are constant (part ii.b) then uniqueness can be assured.

Finally, since whether or not a system of the form of equation (1) has a unique solution in general depends on the particular specification of heterogeneity of agents, our choice to abstract from agent heterogeneity comes at the cost of preventing us from providing necessary conditions for uniqueness. Nonetheless, part (iii) of Theorem 1 shows that the conditions provided are "globally necessary". That is, for any matrix of elasticity bounds $\mathbf{A}$ such that $\rho(\mathbf{A}) > 1$, one can construct a set functions that govern the interactions $\{f_{ijh}\}$ with a corresponding $\mathbf{A}$ where multiple equilibria are assured.[8] Such functions can be constructed even restricting attention only to functions with constant elasticities. Put another way, the sufficient conditions for uniqueness provided in the Theorem 1 are the best that can be provided when abstracting from agent heterogeneity.

2.2 Remarks

We provide below five remarks that both facilitate the implementation and extend Theorem 1. Details are presented in Online Appendix B.1.

The first two remarks provide extensions to Theorem 1.

Remark 1. ***(Generalized Domain)*** Although above we define $f_{ijh}(\cdot)$ as a function solely of x_j, Theorem 1 can be extended to allow $f_{ijh}(\cdot)$ to be a function of the full set of equilibrium outcomes x for all j i.e. $f_{ijh} : \mathbb{R}_{++}^{N\times H} \to \mathbb{R}_{++}$. Doing so requires replacing the condition on elasticity $|\epsilon_{ijh,jh'}(x_j)| \leq (\mathbf{A})_{hh'}$ with $\sum_{m\in\mathcal{N}} \left|\frac{\partial \ln f_{ijh}(x)}{\partial \ln x_{mh'}}\right| \leq (\mathbf{A})_{hh'}$. The remainder of Theorem 1 and its proof is unchanged. This generalization allows that the impact that agent j has on agent i through an interaction of type h can depend on the equilibrium outcomes of any other agents (including i's own outcomes).

Remark 2. ***(Presence of Endogenous Scalars)*** In addition to equilibrium outcomes for each agent and interaction, certain economic systems also contain an endogenous scalar that reflects e.g. the aggregate welfare of the system, as in:

$$\lambda_h x_{ih} = \sum_{j=1}^{N} f_{ijh}(x_{j1}, ..., x_{jH}), \tag{3}$$

where $\lambda_h > 0$ is endogenous. We offer two results for such systems.

The first result concern the equilibrium system (3) with constant elasticities (as in Theorem 1 part(ii)b). For this form, if $\rho(\mathbf{A}) = 1$, we have the same conclusion as in part (ii)b: the $\{x_{ih}\}$ of any solution is column-wise up-to-scale unique. If $\rho(\mathbf{A}) < 1$, it is possible to subsume the endogenous scalars into the equilibrium outcomes through a change in variables, expressing equation (3) as in equation (1), which in turn implies that the $\{x_{ih}\}$ are

[8] Part (iii) of Theorem 1 extends the result of Allen and Donaldson (2018) to equilibrium systems with more than two equilibrium interactions (i.e. $H > 2$).

column-wise up-to-scale unique. (Separating the $\{x_{ih}\}$ and $\{\lambda_h\}$ to determine the scale of $\{x_{ih}\}$ requires the imposition of further equilibrium conditions, e.g. aggregate labor market clearing conditions).

The second result concerns the the equilibrium system (3) with H additional aggregate constraints of the form $\sum_{i=1}^{N} x_{ih} = c_h$ for known constants $c_h > 0$. This system has a unique solution as long as $\rho(\mathbf{A}) < \frac{1}{2}$, where $\mathbf{A}$ is defined as in Theorem 1. Intuitively, $\rho(\mathbf{A}) < \frac{1}{2}$ ensures that the feedback effect from changes in the endogenous scalar are small enough to continue to ensure a contraction.

The next remark facilitates implementation of Theorem 1.

Remark 3. ***(Change of variables)*** It is often useful to consider a change of variables of one's original equilibrium system when writing it in the form of equation (1). A particularly important example that has found widespread use in spatial economics[9] is the following economic system in which the elasticities are constant:

$$\prod_{h'\in\mathcal{H}} x_{ih'}^{\gamma_{hh'}} = \sum_{j\in\mathcal{N}} K_{ijh} \prod_{h'\in\mathcal{H}} x_{ih}^{\kappa_{hh'}} x_{jh'}^{\beta_{hh'}} . \tag{4}$$

for all $i \in \mathcal{N}$ and $h' \in \mathcal{H}$ where $\gamma_{hh'}$, $\kappa_{hh'}$, and $\beta_{hh'}$ are (h, h')th cells of matrix $\mathbf{\Gamma}$, $\boldsymbol{K}$, and $\mathbf{B}$, respectively. To transform equation (4) to the form of equation (1), if $\mathbf{\Gamma} - \boldsymbol{K}$ is invertible, we can redefine $\tilde{x}_{ih} \equiv \prod_{h'\in\mathcal{H}} x_{ih'}^{\gamma_{hh'}-\kappa_{hh'}}$. Substituting this definition into the right-hand-side we obtain $\tilde{x}_{ih} = \sum_{j\in\mathcal{N}} K_{ijh} \prod_{h'\in\mathcal{H}} \tilde{x}_{jh'}^{\alpha_{hh'}}$, where $\alpha_{hh'}$ is the corresponding element of matrix $\mathbf{B}(\mathbf{\Gamma}-\mathbf{K})^{-1}$, which is in the form of (1) with $(\mathbf{A})_{hh'} = |\alpha_{hh'}|$. Note that a change of variables is not just analytically convenient: the presence of the absolute value operator in Theorem 1 means that a change of variables may reduce the spectral radius, making it more likely that the sufficient conditions for uniqueness are satisfied.

The last two remarks offer details about the spectral radius.

Remark 4. ***(Spectral Radius of 1)*** In practice, $\rho(\mathbf{A}) = 1$ is a general phenomenon in economic systems which include nominal variables (e.g. prices). Indeed, any economic system of the form (4) that is homogeneous of degree 0 in at least one of its arguments will have spectral radius $\rho(\mathbf{A})$ equal to 1 or larger. This implies that part (i) of Theorem 1 is applicable to economic systems where all economic interactions are real, whereas part (ii) of Theorem 1 is applicable to economic systems where some economic interactions are nominal.

[9]See e.g. Eaton and Kortum (2002); Alvarez and Lucas (2007); Chaney (2008); Arkolakis, Costinot, and Rodríguez-Clare (2012); Allen and Arkolakis (2014); Redding (2016); Monte, Redding, and Rossi-Hansberg (2018).

Remark 5. ***(Characterization of the Spectral Radius)*** While it is straightforward to numerically calculate $\rho(\mathbf{A})$ to apply the results of Theorem 1, analytical characterizations are also possible. We offer two results to facilitate such characterization. The first is well known: the Collatz–Wielandt Formula (e.g. see Page 670 in Meyer (2000)), implies that if the summation of each row (or column) of $\mathbf{A}$ is less than 1, then $\rho(\mathbf{A}) \leq 1$.

The second is, to our knowledge, new. Define $g(s)$ as the determinant of matrix $sI - \mathbf{A}$ i.e. $g(s) = |sI - \mathbf{A}|$ and denote its k-th derivative as $g^{(k)}(s)$. For any constant $s > 0$, $\rho(\mathbf{A}) \leq s$ if and only if $g^{(k)}(s) \geq 0$ for all $k = 0, 1, 2, ..., n-1$.

3 Applications

In this Section, we apply Theorem 1 to provide new results to three seminal papers examining spatial networks, production networks, and social networks, respectively. For brevity, we present only a brief summary of the results here, relegating a more detailed discussion of each application to Online Appendix B.2.

3.1 Spatial Networks

The first example we consider is one of a urban spatial network. We follow the seminal work of Ahlfeldt, Redding, Sturm, and Wolf (2015), where agents choose where to reside and work in a city subject to commuting costs in the presence of spatial agglomeration spillovers which decay over space. In that paper, uniqueness is proven only in the absence of these spillovers. Here, we use Theorem 1 to provide conditions for uniqueness in the presence of agglomeration spillovers. Unlike Ahlfeldt, Redding, Sturm, and Wolf (2015), however, we assume residential and commercial floor spaces are exogenously given. Interpreting the spatial network model through the lens of our framework, an economic agent is a city block and there are three types interactions between agents: interactions through the goods market, interactions through the labor (commuting) market, and interactions through the spatial productivity spillovers. These interactions in turn determine the three types of equilibrium (strictly positive) outcomes for each agent: the residential floor price, the number of workers employed, and the productivity. As in the original paper, let α denote the labor share in the production function, $\varepsilon > 0$ denote the commuting elasticity, and λ denote the strength of the agglomeration spillover. Applying Theorem 1, a sufficient condition for uniqueness is $\lambda \leq \min\left(1 - \alpha, \frac{\alpha}{1+\epsilon}\right)$, i.e. uniqueness is guaranteed as long as the agglomeration spillovers are not too large and are bounded above by a combination of the land share and the commuting elasticity.

We note that this commuting model is one example of how to apply theorem Theorem 1 to spatial networks. In Online Appendix B.2 we also apply Theorem 1 to (1) trade models with tariffs and and input-output interactions (extending the parameter range provided by Alvarez and Lucas (2007) where uniqueness is assured); and (2) economic geography models with agglomeration productivity spillovers that decay across space (extending the frameworks of Allen and Arkolakis (2014) and Redding (2016), where spillovers are assumed to only be local).

3.2 Production Networks

The second example we consider is one of a sectoral production network. We follow the seminal work of Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012), who consider a production economy where each sector uses intermediate inputs from every other sector. In that paper, the production function is assumed to be Cobb-Douglas between labor and intermediates and Cobb-Douglas across intermediates. Here, we use Theorem 1 to provide conditions for uniqueness when we allow for a more general production function with non-unit elasticities of substitution both between labor and intermediates and across intermediates.[10]

Interpreting this production network through our framework, an economic agent is a sector, and the interactions are through intermediate input usage. Using Theorem 1, we can show that the equilibrium is always unique, regardless of the unit elasticity of substitution.

3.3 Social Networks

The third example we consider is one of a social network. We follow the seminal work of Brock and Durlauf (2001), where agents make a discrete choice over a set of actions and their payoffs of each actions depends on the choices on others in their social network. In that paper, conditions for uniqueness are provided when agents have a choice set of two actions and the effect of others' actions on an agent's payoffs is summarized by their mean actions. Here, we apply Theorem 1 to an extension with an arbitrary number of actions in the choice set and where the effect of others' actions on an agent's payoffs is summarized by a generalized weighted mean, where weights can be individual specific, i.e. we allow for an arbitrary social network. Unlike Brock and Durlauf (2001), however, we assume private and social component of utility are proportional rather than additive.

Through the lens of our framework, each individual in the social network is an economic agent and each of the actions in the choice set comprises a different economic interaction.

[10] Carvalho, Nirei, Saito, and Tahbaz-Salehi (2019) consider this general formulation and Carvalho and Tahbaz-Salehi (2019) the case with unit elasticities between labor and intermediates.

Each of these interactions in turn result in an equilibrium outcomes for each agent, which is the expected payoff of choosing each action. As in the original paper, let β denote the shape value of the extreme value distribution (which governs the relative importance of the random utility coefficient in agent's payoff) and let J denote the strength of social spillovers. Applying Theorem 1, a sufficient condition for uniqueness is $\beta J < \frac{1}{H}$ where H is the number of actions in the choice set, i.e. the greater the number of economic interactions, the weaker the social spillovers must be to ensure uniqueness.

4 Conclusion

In this note, we provide sufficient conditions for the uniqueness and computation of the equilibrium for a broad class of models with large numbers of heterogeneous agents simultaneously interacting in a large number of ways. The conditions are written in terms of the elasticities of the economic interactions across agents. These results are based on a multi-dimensional extension of the contraction mapping theorem which allows for the separate treatment of the different types of these interactions. We illustrate that a wide variety of heterogeneous agent economies – characterized by spatial, production, or social networks – yield equilibrium representations amenable to our theorem's characterization.

By construction, the conditions provided here depend only on the uniform bound of the elasticities of agent's interactions on each other's outcomes rather than the particular form of the network model; that is, the conditions provided abstract from agent heterogeneity. We show that should the conditions provided not hold, there exist network models for which multiplicity is guaranteed, i.e. our conditions are "globally" necessary. However, an outstanding and important question remains about how agent heterogeneity itself shapes the positive properties of model equilibria.

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A Appendix

A.1 Proof of Lemma 1

Proof. We prove that the sequence generated by the operator converges to a unique point.

To prove convergence we first prove that the sequence is a Cauchy sequence on a complete metric space. Define $d_{max}(x, x') = \max(d(x, x'))$ as the metric in space X. Clearly (X, d_{max}) is complete. Now consider any $x \in X$. Denote $x^0 = x$ and for integer $n \geq 1$ $x^n = T(x^{n-1})$. For integers n and m, suppose $n < m$. We have

$$\begin{aligned} d(x^n, x^m) &\leq d\left(x^n, x^{n+1}\right) + d\left(x^{n+1}, x^{n+2}\right) + ... + d\left(x^{m-1}, x^m\right) \\ &< \left(\mathbf{A}^n + \mathbf{A}^{n+1} + ... + \mathbf{A}^{m-1}\right) d\left(x^0, x^1\right) \\ &\leq \left(\mathbf{A}^n + \mathbf{A}^{n+1} + ... + \mathbf{A}^{m-1} + \mathbf{A}^m + ...\right) d\left(x^0, x^1\right) \\ &\leq \mathbf{A}^n (\mathbf{I} - \mathbf{A})^{-1} d\left(x^0, x^1\right). \end{aligned} \tag{5}$$

Notice if $\rho(\mathbf{A}) < 1$ then $\mathbf{A}^n$ converges to zero matrix and $(\mathbf{I} - \mathbf{A})^{-1}$ is finite. Furthermore, for $n < m$, $d_{max}(x^n, x^m) \to 0$ as $n \to \infty$. Therefore $\{x^n\}_{n=1,2,...}$ is a Cauchy sequence on a complete metric space and it has a limit.

To prove existence denote the limit of the sequence $y = \lim_{n\to\infty} x^n$ in X. We claim $T(y) = y$. This is because $T(\cdot)$ is continuous, which is implied by the following formula

$$\begin{aligned} d_{max}(T(x), T(x')) &\leq \max(\mathbf{A} d(x, x')) \\ &\leq H\bar{a} \max(d(x, x')) \\ &= H\bar{a} d_{max}(x, x') \end{aligned}$$

where $\bar{a}$ is the largest element of matrix $\mathbf{A}$. Finally, by a standard contradiction argument the point has to be unique. We thus have established convergence, existence, and uniqueness.

□

A.2 Proof of Theorem 1.

Proof. Define $y = \ln x$ i.e. for any $h \in \mathcal{H}$ $i \in \mathcal{N}$ $y_{ik} = \ln x_{ik}$. Thus, equation (1) can be equivalently rewritten as $y_{ih} = \ln \sum_{j\in\mathcal{N}} f_{ijh}(\exp y_j)$. Denote its right side as function $g_{ih}(y)$, thus

$$\frac{\partial g_{ih}}{\partial y_{jh'}} = \frac{\epsilon_{ijh,jh'}(\exp y_j) f_{ijh}(\exp y_j)}{\sum_{j\in\mathcal{N}} f_{ijh}(\exp y_j)} \tag{6}$$

For any y and y', according to mean value theorem, there exists some $t_{ih} \in [0,1]$ such that $\hat{y} = (1 - t_{ih})\, y + t_{ih} y'$ satisfies for each i and h

$$\begin{aligned} g_{ih}(y) - g_{ih}(y') &= \nabla g_{ih}(\hat{y})(y - y') \\ &= \sum_{j\in\mathcal{N}, h'\in\mathcal{H}} \frac{\partial g_{ih}(\hat{y})}{\partial y_{jh'}} \left(y_{jh'} - y'_{jh'}\right) \end{aligned} \tag{7}$$

Part (i): Combine the above two equations (6) and (7) with condition $|\epsilon_{ih,jh'}(x_j)| \le (\mathbf{A})_{hh'}$, we have

$$\begin{aligned} |g_{ih}(y) - g_{ih}(y')| &\le \sum_{j\in\mathcal{N}, h'\in\mathcal{H}} \frac{(\mathbf{A})_{hh'} f_{ijh}(\exp y_j)}{\sum_{j\in\mathcal{N}} f_{ijh}(\exp y_j)} \left|y_{jh'} - y'_{jh'}\right| \\ &\le \sum_{h'\in\mathcal{H}} (\mathbf{A})_{hh'} \max_{j\in\mathcal{N}} \left|y_{jh'} - y'_{jh'}\right|. \end{aligned} \tag{8}$$

For any $h \in H$, define $d_h(y_h, y'_h) = \max\limits_{j\in\mathcal{N}} \left|y_{jh} - y'_{jh}\right|$ and $Y_h = \mathbb{R}^N$. $d_h(\cdot,\cdot)$ is a metric on Y_h. Furthermore, define $Y = Y_1 \times Y_2 \times ... \times Y_H$ and $d(y, y') = \begin{pmatrix} d_1(y_1, y'_1) \\ ... \\ d_H(y_H, y'_H) \end{pmatrix}$ for $y, y' \in Y$. Notice that inequality (8) then becomes $d(g(y), g(y')) \leqq \mathbf{A} d(y, y')$. Thus we can apply Lemma 1 to obtain the desired results (existence, uniqueness and computation).

For the purpose of the computation, instead of applying the iterative procedure in the space $Y = \mathbb{R}^{N\times H}$ according to Lemma 1, it is equivalent to do so in the space where x lies on, i.e. $\mathbb{R}^{N\times H}_{++}$.

Part (ii.a):Suppose there are two distinct solutions y and y' i.e. $y_{ih} = g_{ih}(y)$ and $y'_{ih} = g_{ih}(y')$. We will arrive at a contradiction. Substitute these two solutions into equation (7). Also $|\epsilon_{ih,jh'}(x_j)| < (\mathbf{A})_{hh'}$ when $(\mathbf{A})_{hh'}$, as long as the right side of equation (8) is not zero, we have

$$|y_{ih} - y'_{ih}| < \sum_{h'\in\mathcal{H}} (\mathbf{A})_{hh'} \max_{j\in\mathcal{N}} |y_{ih'} - y'_{ih'}|. \tag{9}$$

Thus we have $d(y, y') \le \mathbf{A} d(y, y')$ and the inequality strictly holds as long as the right side is not zero. Since y and y' are distinct. We must have $d(y, y')$ as a nonzero nonnegative vector. Thus according to the Collatz–Wielandt Formula ($\rho(\mathbf{A}) = \max_{d\in\mathbb{R}^H_+, y\neq 0} \min_{\substack{1\le h\le H \\ d_h\neq 0}} \frac{(\mathbf{A}d)_h}{z_h}$ Page 670 in Meyer (2000)), we have $\rho(\mathbf{A}) > 1$. A contradiction.

Part (ii.b): We will again argue by contradiction. Suppose a pair of solutions x and x' to equation (1) exists that are column-wise up-to-scale different. That is $d = \begin{pmatrix} d_1 \\ ... \\ d_H \end{pmatrix}$ is a nonzero vector where $d_h = \min\limits_{s\in\mathbb{R}} \max\limits_{j\in\mathcal{N}} \left|y_{jh} - y'_{jh} + s\right|$. For any $h \in \mathcal{H}$, we can suppose we have s_h and j_h such that $d_h = \left|y_{j_h h} - y'_{j_h h} + s_h\right|$.

Combine the above two equations (6) and (7) with condition $\epsilon_{ih,jh'}(x_j) = \alpha_{hh'}$ where

$|\alpha_{hh'}| = (\mathbf{A})_{hh'}$, we have

$$|g_{ih}(y) - g_{ih}(y') + \hat{s}_h| = \left| \sum_{h' \in \mathcal{H}} \alpha_{hh'} \sum_{j \in \mathcal{N}} \frac{f_{ijh}(\exp \hat{y}_j)}{\sum_{j \in \mathcal{N}} f_{ijh}(\exp \hat{y}_j)} \left(y_{jh'} - y'_{jh'} + s_{h'} \right) \right|$$
$$\leq \sum_{h' \in \mathcal{H}} |\alpha_{hh'}| \, d_{h'} \tag{10}$$

where $\hat{s}_h = \sum_{h' \in \mathcal{H}} \alpha_{hh'} s_{h'}$. Notice that $d_h \leq \max_{i \in \mathcal{N}} |y_{ih} - y'_{ih} + \hat{s}_h|$. Therefore we have $d_h \leq \sum_{h' \in \mathcal{H}} |\alpha_{hh'}| \, d_{h'} \leq \sum_{h' \in \mathcal{H}} (\mathbf{A})_{hh'} \, d_{h'}$ i.e.

$$d \leqq \mathbf{A} d. \tag{11}$$

If $d_h > 0$, there there must exists h' such that $d_{h'} > 0$ and $\alpha_{hh'} \neq 0$. For any h' $d_{h'} > 0$, according to the definition of $d_{h'}$ there must exist some $j \in \mathcal{N}$ such that $\left| y_{jh'} - y'_{jh'} + s_{h'} \right| < d_{h'}$. Thus inequality (10) must strictly hold for all $i \in \mathcal{N}$ whenever $d_h > 0$. Therefore $d_h < \sum_{h' \in \mathcal{H}} |\alpha_{hh'}| \, d_{h'} \leq \sum_{h' \in \mathcal{H}} (\mathbf{A})_{hh'} \, d_{h'}$. Thus, again, according to the Collatz–Wielandt Formula, we have $\rho(\mathbf{A}) > 1$, which is a contradiction.

Part (iii): Consider $\{K_{ijh} > 0\}_{i,j \in \mathcal{N}, h \in \mathcal{H}}$ which satisfies $\sum_{j \in \mathcal{N}} K_{ijh} = 1$ for any i. Obviously, $x = 1$ is one solution of equation (4). In the following we are going to construct $\{K_{ijh} > 0\}_{i,j \in \mathcal{N}, h \in \mathcal{H}}$ such that there exists another different solution.

As we have $\rho(\mathbf{A}) > 1$, suppose z is $\mathbf{A}$'s non-negative eigenvector such that $\rho(\mathbf{A}) z = \mathbf{A} z$. For a given h, divide $\mathcal{H} = \{1, 2, ..., H\}$ into two sets $\mathcal{H}_h^- = \{h' | \alpha_{hh'} \leq 0\}$ and $\mathcal{H}_h^+ = \{h' | \alpha_{hh'} > 0\}$; also arbitrarily divide $\mathcal{N} = \{1, 2, ..., N\}$ into $2H + 1$ non-empty disjoint sets $\{\mathcal{N}_h^+, \mathcal{N}_h^-\}_{h \in H}$ and $\mathcal{N}^0$.

Now we construct $\bar{x} \in \mathbb{R}_{++}^{N \times H}$. If $j \in \mathcal{N}^0$, for any h', $\bar{x}_{jh'} = 1$; if $j \in \mathcal{N}_h^+$, $\bar{x}_{jh'} = \begin{cases} \exp(z_h) & h' \in \mathcal{H}_h^+ \\ \exp(-z_h) & h' \in \mathcal{H}_h^- \end{cases}$; if $j \in \mathcal{N}_h^-$, $\bar{x}_{jh'} = \begin{cases} \exp(-z_h) & h' \in \mathcal{H}_h^+ \\ \exp(+z_h) & h' \in \mathcal{H}_h^- \end{cases}$. Obviously, x' is column-wise up-to-scale different from x. In below, we show there exists $\{K_{ijh} > 0\}_{i,j \in \mathcal{N}, h \in \mathcal{H}}$ such that $\bar{x}$ is also a solution of equation (1).Notice that

$$\begin{aligned}
& \sum_{j \in \mathcal{N}} K_{ijh} \prod_{h' \in \mathcal{H}} \bar{x}_{jh'}^{\alpha_{hh'}} = \\
= \; & \sum_{j \in \mathcal{N}_h^+} K_{ijh} \prod_{h' \in \mathcal{H}} \bar{x}_{jh'}^{\alpha_{hh'}} + \sum_{j \in \mathcal{N}_h^-} K_{ijh} \prod_{h' = \mathcal{H}} \bar{x}_{jh'}^{\alpha_{hh'}} + \sum_{j \notin \mathcal{N}_h^+, \mathcal{N}_h^-} K_{ijh} \prod_{h' \in \mathcal{H}} \bar{x}_{jh'}^{\alpha_{hh'}} \\
= \; & \exp\left(\sum_{h' = \mathcal{H}} |\alpha_{hh'}| \, z_{h'} \right) \sum_{j \in \mathcal{N}_h^+} K_{ijh} + \exp\left(- \sum_{h' = \mathcal{H}} |\alpha_{hh'}| \, z_{h'} \right) \sum_{j \in \mathcal{N}_h^-} K_{ijh} + \\
& + \sum_{j \notin \mathcal{N}_h^+ \cup \mathcal{N}_h^-} K_{ijh} \prod_{h' \in \mathcal{H}} \bar{x}_{jh'}^{\alpha_{hh'}}
\end{aligned} \tag{12}$$

In the last term of above equation, for any $j \notin \mathcal{N}_h^+ \cup \mathcal{N}_h^-$, we have $\exp\left(\sum_{h' \in \mathcal{H}} |\alpha_{hh'}| \, z_h \right) \geq$

$\sum_{h'\in\mathcal{H}} \bar{x}_{jh'}^{\alpha_{hh'}} \geq \exp\left(-\sum_{h'\in\mathcal{H}} |\alpha_{hh'}|\, z_h\right)$. Notice that $\exp\left(\sum_{h'\in\mathcal{H}} |\alpha_{hh'}|\, z_h\right) = \exp\left(\rho\left(\mathbf{A}\right) z_h\right)$ where $\rho\left(\mathbf{A}\right) > 1$. Therefore, we can adjust the value of $\{K_{ijh}\}_{j\in I}$ while keeping $\sum_{j\in\mathcal{N}} K_{ijh} = 1$ such that equation (12) is equal to $\exp\left(z_h\right)$ or $\exp\left(-z_h\right)$. That is we have $\sum_{j\in\mathcal{N}} K_{ijh} \prod_{h'\in\mathcal{H}} \bar{x}_{jh'}^{\alpha_{hh'}} = \bar{x}_{ih}$ as desired.

$\square$

B Online Appendix (not for publication)

B.1 Further Details of Remarks

In this section, we provide further details for the remarks discussed in the paper.

B.1.1 Remark 1

Extending the domain of f_{ijh} to all x requires only a small change to the proof of Theorem 1, where equality (6) and inequality (8) respectively become $\frac{\partial g_{ih}}{\partial y_{jh'}} = \frac{\sum_m \frac{\partial \ln f_{imh}(x)}{\partial \ln x_{jh'}} f_{imh}(\exp y)}{\sum_{j\in\mathcal{N}} f_{ijh}(\exp y)}$ and

$$\begin{aligned}
|g_{ih}(y) - g_{ih}(y')| &\leq \sum_{h'\in\mathcal{H}} \max_{j\in\mathcal{N}} \left|y_{jh'} - y'_{jh'}\right| \frac{\sum_{j\in\mathcal{N}}\sum_m \left[\left|\frac{\partial \ln f_{imh}(x)}{\partial \ln x_{jh'}}\right|\right] f_{imh}(\exp y)}{\sum_{j\in\mathcal{N}} f_{ijh}(\exp y)} \\
&= \sum_{h'\in\mathcal{H}} \max_{j\in\mathcal{N}} \left|y_{jh'} - y'_{jh'}\right| \frac{\sum_m \left|\sum_{j\in\mathcal{N}} \frac{\partial \ln f_{imh}(x)}{\partial \ln x_{jh'}}\right| f_{imh}(\exp y)}{\sum_{j\in\mathcal{N}} f_{ijh}(\exp y)} \\
&\leq \sum_{h'\in\mathcal{H}} (\mathbf{A})_{hh'} \max_{j\in\mathcal{N}} \left|y_{jh'} - y'_{jh'}\right|.
\end{aligned}$$

The rest of the proof of Theorem 1 remains unchanged.

B.1.2 Remark 2

Consider first the equilibrium system (3) with constant elasticities, which can be written as follows:

$$\lambda_h x_{ih} = \sum_{j\in\mathcal{N}} K_{ijh} \prod_{h'\in\mathcal{H}} x_{jh'}^{\alpha_{hh'}}, \tag{13}$$

where $\lambda_h > 0$ is endogenous. In the case that $\rho(\mathbf{A}) = 1$, we have the same conclusion as in part (ii)b: the $\{x_{ih}\}$ of any solution is column-wise up-to-scale unique. The proof of this result is exactly the same as part (ii)b of Theorem 1.

If $\rho(\mathbf{A}) < 1$, it is possible to subsume the endogenous scalars into the equilibrium outcomes through a change in variables, expressing equation (13) as in equation (1). To do so, define $\tilde{x}_{ih} \equiv x_{ih} \prod_{h'\in\mathcal{H}} \lambda_{h'}^{d_{h'h}}$, where $d_{h'h}$is the $h'h^{th}$ element of the $H \times H$ matrix $(\mathbf{I} - \boldsymbol{\alpha})^{-1}$ and $\boldsymbol{\alpha} \equiv (\alpha_{hh'})$ (i.e. $\boldsymbol{\alpha}$ is the matrix of elasticities without the absolute value taken) so the system becomes:

$$\tilde{x}_{ih} = \sum_{j\in\mathcal{N}} K_{ijh} \prod_{h'\in\mathcal{H}} \tilde{x}_{jh'}^{\alpha_{hh'}}.$$

Note that because $\rho(\mathbf{A}) < 1$, then so too is $\rho(\boldsymbol{\alpha}) < 1$, so that $(\mathbf{I} - \boldsymbol{\alpha})^{-1}$ exists. From Theorem 1 part (i), the $\{\tilde{x}_{ih}\}$ are unique and can be calculated using an iterative algorithm, which in turn implies that the $\{x_{ih}\}$ are column-wise up-to-scale unique. (Separating the $\{x_{ih}\}$ and $\{\lambda_h\}$ to determine the scale of $\{x_{ih}\}$ requires the imposition of further equilibrium conditions, e.g. aggregate labor market clearing conditions).

Consider now equilibrium system (3) with H additional aggregate constraints $\sum_{i=1}^{N} x_{ih} =$

c_h for known constants $c_h > 0$.

The second result concerns the general case with an endogenous scalar:

$$\lambda_h x_{ih} = \sum_{j=1}^{N} f_{ijh}\left(x_{j1}, ..., x_{jH}\right)$$

with H additional aggregate constraints $\sum_{i=1}^{N} x_{ih} = c_h$ for known constants $c_h > 0$. Substituting in the aggregate constraints allows us to express the equilibrium system as:

$$x_{ih} = \sum_{j=1}^{N} \left(\frac{f_{ijh}\left(x_{j1}, ..., x_{jH}\right)}{\frac{1}{c_h} \sum_{i'=1}^{N} \sum_{j'=1}^{N} f_{i'j'h}\left(x_{j'1}, ..., x_{j'H}\right)} \right),$$

where the denominator is equal to the endogenous scalar, i.e. $\lambda_h = \frac{1}{c_h} \sum_{i'=1}^{N} \sum_{j'=1}^{N} f_{i'j'h}\left(x_{j'1}, ..., x_{j'H}\right)$. We can define the new function:

$$g_{ij,h}\left(x\right) \equiv \frac{f_{ijh}\left(x_{j1}, ..., x_{jH}\right)}{\frac{1}{c_h} \sum_{i'=1}^{N} \sum_{j'=1}^{N} f_{i'j'h}\left(x_{j'1}, ..., x_{j'H}\right)}$$

so that the equilibrium system becomes:

$$x_{ih} = \sum_{j=1}^{N} g_{ijh}\left(x\right).$$

We can then bound the elasticities, following Remark 1. Note:

$$\frac{\partial \ln g_{ij,h}}{\partial \ln x_{m,l}} = \begin{cases} \left(\frac{\partial \ln f_{ij,h}}{\partial \ln x_{j,l}}\right)\left(1 - \frac{f_{ij}\left(x_{p,l}\right)}{\sum_{o,p} f_{op}\left(\left\{x_{p,l}\right\}\right)}\right) & \text{if } m = j \\ -\sum_{o}\left(\frac{\partial \ln f_{om,h}}{\partial \ln x_{m,l}}\right) \frac{f_{om,k}\left(x_{p,l}\right)}{\sum_{o,p} f_{op,k}\left(\left\{x_{p,l}\right\}\right)} & \text{if } m \neq j \end{cases}$$

so that:

$$\left|\frac{\partial \ln g_{ij,h}}{\partial \ln x_{m,l}}\right| = \begin{cases} \left|\frac{\partial \ln f_{ij,h}}{\partial \ln x_{ml}}\right|\left(1 - \frac{f_{ij,k}\left(x_{p,l}\right)}{\sum_{o,p} f_{op}\left(\left\{x_{p,l}\right\}\right)}\right) & \text{if } m = j \\ \sum_{o}\left|\frac{\partial \ln f_{om,h}}{\partial \ln x_{m,l}}\right| \frac{f_{om,k}\left(x_{p,l}\right)}{\sum_{o,p} f_{op,k}\left(\left\{x_{p,l}\right\}\right)} & \text{if } m \neq j \end{cases}$$

so that:

$$\left|\frac{\partial \ln g_{ij,h}}{\partial \ln x_{m,l}}\right| \leq \begin{cases} \left|A_{kh}\right|\left(1 - \frac{f_{ij,k}\left(x_{p,l}\right)}{\sum_{o,p} f_{op}\left(\left\{x_{p,l}\right\}\right)}\right) & \text{if } m = j \\ \left|A_{kh}\right| \frac{\sum_{o} f_{om,k}\left(x_{p,l}\right)}{\sum_{o,p} f_{op,k}\left(\left\{x_{p,l}\right\}\right)} & \text{if } m \neq j \end{cases}$$

Finally, we can sum across all m locations to yield:

$$\sum_m \left|\frac{\partial \ln g_{ij,k}}{\partial \ln x_{m,l}}\right| \leq |A_{kl}| \left(1 - \frac{f_{ij,k}(x_{p,l})}{\sum_{o,p} f_{op}(\{x_{p,l}\})}\right) + \sum_{m\neq j}\left(|A_{kl}| \frac{\sum_o f_{om,k}(x_{p,l})}{\sum_{o,p} f_{op,k}(\{x_{p,l}\})}\right) \iff$$
$$\sum_m \left|\frac{\partial \ln g_{ij,k}}{\partial \ln x_{m,l}}\right| \leq |A_{kl}| \left(1 - \frac{f_{ij,k}(x_{p,l})}{\sum_{o,p} f_{op}(\{x_{p,l}\})} + \left(1 - \frac{f_{ij,k}(x_{p,l})}{\sum_{o,p} f_{op}(\{x_{p,l}\})}\right)\right) \iff$$
$$\sum_m \left|\frac{\partial \ln g_{ij,k}}{\partial \ln x_{m,l}}\right| \leq 2\,|A_{kl}|\,.$$

Hence, from Remark 1, we have uniqueness as long as $\rho(\mathbf{A}) < \frac{1}{2}$, as required.

B.1.3 Remark 3

Here we provide a simple example of the claim that "The presence of the absolute value operator in Theorem 1 means that a change of variables may reduce the spectral radius, making it more likely that the sufficient conditions for uniqueness are satisfied."

Consider the equilibrium system:

$$x_i = \sum_{j=1}^{N} K_{ij} x_i^{-\frac{1}{2}} x_j.$$

From Remark 1, a sufficient condition for uniqueness is that $\sum_{m\in N} \left|\frac{\partial \ln f_{ijh}(x)}{\partial \ln x_{mh'}}\right| \leq (\mathbf{A})_{hh'} = \left|-\frac{1}{2}\right| + |1| = \frac{3}{2}$. The transformed system $\tilde{x}_i = \sum_{j=1}^{N} K_{ij} \tilde{x}_j^{\frac{2}{3}}$, where $\tilde{x}_i = x_i^{\frac{3}{2}}$ has a spectral radius of $\frac{2}{3}$. Hence, the sufficient condition for uniqueness provided from Theorem 1 is satisfied for the transformed system but not the original system.

B.1.4 Remark 4

Consider equation (4). We will directly prove that $\rho(\mathbf{A}) = \rho\left(\mathbf{B}\boldsymbol{\Gamma}^{-1}\right) \geq 1$. Suppose for some $\bar{h} \geq 1$ that $\{x_{\cdot h}\}_{h=1,\ldots,\bar{h}}$ are nominal variables. Then if we construct $\{\bar{x}_{\cdot h}\}_{h\in\mathcal{H}}$ by scaling $\{x_{\cdot h}\}_{h=1,\ldots,\bar{h}}$ up to t times and keeping all other entries unchanged, the constructed $\{\bar{x}_{\cdot h}\}_{h\in\mathcal{H}}$ should still solve the equation. Therefore we can write

$$\boldsymbol{\Gamma} T = \mathbf{B} T,$$

where T is a H-by-1 vector and

$$T_h = \begin{cases} t & h \leq \bar{h} \\ 0 & \text{other case} \end{cases}.$$

Notice that this further implies $\boldsymbol{\Gamma}^{-1}\mathbf{B}$ has eigenvalue of 1. Furthermore, because $\mathbf{B}\boldsymbol{\Gamma}^{-1} = \boldsymbol{\Gamma}\left(\boldsymbol{\Gamma}^{-1}\mathbf{B}\right)\boldsymbol{\Gamma}^{-1}$, $\mathbf{B}\boldsymbol{\Gamma}^{-1}$ also has eigenvalue of 1. We define matrix $\mathbf{A}$ as the absolute value of $\mathbf{B}\boldsymbol{\Gamma}^{-1}$ (i.e. each entry of matrix $\mathbf{A}$ is the absolute value of the corresponding entry in matrix $\mathbf{B}\boldsymbol{\Gamma}^{-1}$). Therefore $\rho(\mathbf{A})$ must be weakly larger than 1 because $\rho(\mathbf{A}) = \lim_{n\to\infty} \|\mathbf{A}^n\|^{\frac{1}{n}} \geq$

$\lim_{n\to\infty}\left\|\left(\mathbf{B\Gamma^{-1}}\right)^{n}\right\|^{\frac{1}{n}}=\rho\left(\mathbf{B\Gamma^{-1}}\right)$.

B.1.5 Remark 5

We prove a necessary and sufficient condition such that $\rho\left(\mathbf{A}\right)\leq 1$.

Lemma 2. *Let* $\mathbf{A}$ *be a non-negative* $n\times n$ *matrix. The function* $f\left(\lambda\right)$ *is defined as the determinant of matrix* $\lambda I-\mathbf{A}$ *i.e.* $f\left(\lambda\right)=\left|\lambda I-\mathbf{A}\right|$ *, and its k-th derivative is denoted by* $f^{(k)}\left(\lambda\right)$. *Then* $\rho\left(\mathbf{A}\right)\leq s$ *if and only if* $f^{(k)}\left(s\right)\geq 0$ *for all* $k=0,1,2,...,n-1$.

Proof. **If part:** Notice that $f^{(n)}\left(s\right)=n!>0$. Then $f^{(n-1)}\left(\lambda\right)$ strictly increases with λ. So $f^{(n-1)}\left(\lambda\right)>0$ for $\lambda\in\left[s,\infty\right)$. Using deduction we obtain $f\left(\lambda\right)$ is strictly increasing and $f\left(\lambda\right)\geq 0$ for any $\lambda\in\left[s,\infty\right]$. According to Perron–Frobenius theorem, $\rho\left(\mathbf{A}\right)$ is $\mathbf{A}$'s largest eigenvalue, so that $f\left(\rho\left(\mathbf{A}\right)\right)=0$. Thus, by strict monotonicity it must be $\rho\left(\mathbf{A}\right)\leq s$.

Only If part: According to the Fundamental Theorem of Algebra (e.g. see Corollary 3.6.3 of Fine and Rosenberger (1997)), $f\left(\lambda\right)$ can be decomposed as $f\left(\lambda\right)=f_1\left(\lambda\right)f_2\left(\lambda\right)$ such that $f_1\left(\lambda\right)=\prod_{i\in C}\left(\lambda-\lambda_i\right)\left(\lambda-\overline{\lambda_i}\right)$ and $f_2\left(\lambda\right)=\prod_{i\in R}\left(\lambda-\lambda_i\right)$ where $\overline{\lambda_i}$ is conjugate of λ_i and C and R are set of indexes. For all $i\in C$, λ_i is a complex number and for all $i\in R$ λ_i is a real number. Clearly, λ_i and $\overline{\lambda_i}$ are eigenvalues of $\mathbf{A}$.Notice that $f^{(k)}\left(\lambda\right)=\sum_{(k_1,k_2)\in D_k,}f_1^{(k_1)}\left(\lambda\right)f_2^{(k_2)}\left(\lambda\right)$ where $D_k=\left\{k_1,k_2|k_1+k_2=k,k_1,k_2\geq 0\right\}$. When $i\in R$ $\lambda_i\leq\rho\left(\mathbf{A}\right)$ (from Perron–Frobenius theorem), we have $f_2^{(k_2)}\left(s\right)\geq 0$. Additionally, $f_1^{(k_1)}\left(\lambda\right)=\prod_{i\in C}\left[\lambda^2-\left(\lambda_i+\overline{\lambda_i}\right)\lambda+\lambda_i\overline{\lambda_i}\right]^{(k_{2,i})}$ where $k_{2,i}\geq 0$ and $\sum_{i\in C}k_{2,i}=k_2$. Notice that

$$\left[s^2-\left(\lambda_i+\overline{\lambda_i}\right)s+\lambda_i\overline{\lambda_i}\right]^{(k_{2,i})}=\begin{cases}s^2-\left(\lambda_i+\overline{\lambda_i}\right)s+\lambda_i\overline{\lambda_i}>0 & k_{2,i}=0\\ 2\left(s-\mathrm{Re}\left(\lambda_i\right)\right) & k_{2,i}=1\\ 2>0 & k_{2,i}=2\\ 0 & k_{2,i}>3\end{cases},$$

where $\mathrm{Re}\left(\lambda_i\right)$ is real part of λ_i. As $\mathrm{Re}\left(\lambda_i\right)<\left\|\lambda_i\right\|\leq\rho\left(\mathbf{A}\right)\leq s$ (the second inequality is also from Perron–Frobenius theorem), so $\left[s^2-\left(\lambda_i+\overline{\lambda_i}\right)s+\lambda_i\overline{\lambda_i}\right]^{(k_{2,i})}\geq 0$. In all, $f^{(k)}\left(s\right)\geq 0$ $k=0,1,2,...,n-1$.

□

B.2 Applications

In this section, we provide more detail for the three examples discussed in Section 3.

B.2.1 Spatial Networks

The first set of applications is examples where interactions across heterogeneous agents take place in space. We consider an urban model (extending the results of Ahlfeldt, Redding, Sturm, and Wolf (2015)), an economic geography model (extending the results of Allen and Arkolakis (2014)), and a trade model (extending the results of Alvarez and Lucas (2007)) in turn.

Here we prove the uniqueness of the quantitative urban framework of Ahlfeldt, Redding, Sturm, and Wolf (2015) with endogenous agglomeration spillovers but assume residential and commercial land are exogenously given. In terms of our framework, each city block is a different economic agent and there are three different economic interactions, each represented by an equilibrium condition. The first economic interaction is through the goods market, where we require the goods markets clear, i.e. the income in a city block is equal to its total sales:

$$I_i = \sum_{j=1}^{S} K_{ij} Q_i^{-\epsilon(1-\beta)} w_j^{1+\epsilon} \tag{14}$$

where $I_i = \frac{Q_i H_{Ri}}{\beta}$ is the total income of the residents living in location i, Q_i is the rental price in location i, w_j, is the wage in location j, and $K_{ij} = \Phi^{-1} T_i E_j d_{ij}^{-\epsilon} H > 0$ is a matrix incorporating the commuting costs between locations.

The second economic interaction is through the labor (commuting) market, where we require that the total number of agents working in a location, H_{Mi}, is equal to the number of workers choosing to commute there, i.e.:

$$H_{Mi} = \sum_{j=1}^{S} K_{ji} Q_j^{-\epsilon(1-\beta)} w_i^{\epsilon}, \tag{15}$$

Finally, the third economic interaction is through the spatial productivity spillover, where the productivity of a city block depends on the density of nearby workers, i.e:

$$A_i^{\frac{1}{\lambda}} = a_i^{\frac{1}{\lambda}} \sum_{j=1}^{S} \frac{e^{-\delta \tau_{ij}}}{K_j} H_{Mj}. \tag{16}$$

Given the assumed Cobb-Douglas production function and the assumed fixed amount of land in each location used for production, we substitute $w_i = \alpha A_i H_{Mi}^{\alpha-1} L_{Mi}^{1-\alpha}$ into above equations to create three equilibrium conditions that are a function of three outcomes: the price of residential land, the number of agents working in a location, and the productivity of a location. Observe that equations above are of the form of equation 4 with $\{Q_i, H_{Mi}, A_i\}_{i=1,\ldots,S}$ as endogenous outcome variables.[11] And the corresponding $\mathbf{\Gamma}$ and $\mathbf{B}$ are respectively $\begin{pmatrix} 1+\epsilon(1-\beta) & 0 & 0 \\ 0 & 1+\epsilon(1-\alpha) & -\epsilon \\ 0 & 0 & \frac{1}{\lambda} \end{pmatrix}$ and $\begin{pmatrix} 0 & (\alpha-1)(1+\epsilon) & 1+\epsilon \\ -\epsilon(1-\beta) & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Then we have

$$\mathbf{A} = \begin{pmatrix} 0 & \frac{(1-\alpha)(1+\epsilon)}{1+\epsilon(1-\alpha)} & \frac{\lambda(1+\epsilon)}{1+\epsilon(1-\alpha)} \\ \frac{\epsilon(1-\beta)}{1+\epsilon(1-\beta)} & 0 & 0 \\ 0 & \frac{1}{1+\epsilon(1-\alpha)} & \frac{\lambda\epsilon}{1+\epsilon(1-\alpha)} \end{pmatrix}.$$

Recall from Remark 5 that if the summation of each row of $\mathbf{A}$ is less than 1, then we have

[11] Although Φ is also an endogenous variable, it is not location specific. Treating it exogenously is equivalent with the equilibrium. (The equivalence can be shown by scaling $\{Q_i, H_{Mi}, A_i\}_{i=1,\ldots,S}$.)

$\rho(\mathbf{A}) \leq 1$. Specifically, from Theorem 1(i), $\mathbf{A}x \leq x$ holds as long as $\lambda \leq \min\left(1-\alpha, \frac{\alpha}{1+\epsilon}\right)$, as claimed.

We now consider the framework of Allen and Arkolakis (2014). The model yields the same mathematical equilibrium system as in Redding (2016) and Allen, Arkolakis, and Takahashi (2020) and thus the results apply in all these models. We extend that framework to allow for productivity spillovers that decay over space of the form:

$$A_i = \bar{A}_i \sum_{j=1}^{N} K_{ij}^{A} L_j^{\alpha},$$

where A_i represents the productivity of region i, $\bar{A}_i$ its exogenous component and L_i the labor in region i that is determined in equilibrium. K_{ij}^{A} represents spatial spillovers in productivity and α the spillover elasticity that is common across locations. Furthermore, appropriately replacing the equilibrium conditions (corresponding to equations 10 and 11 of Allen and Arkolakis (2014) that represent interactions through trade and the labor market) we obtain:

$$L_i A_i^{1-\sigma} w_i^{\sigma} = W^{1-\sigma} \sum_{j=1}^{N} T_{ij}^{1-\sigma} \bar{u}_j^{\sigma-1} L_j^{1+\beta(\sigma-1)} w_j^{\sigma}$$

$$L_i^{\beta(1-\sigma)} w_i^{1-\sigma} = W^{1-\sigma} \sum_{j=1}^{N} T_{ji}^{1-\sigma} \bar{u}_i^{\sigma-1} A_j^{\sigma-1} w_j^{1-\sigma},$$

where w_i is the wage in location i, $\bar{u}_i$ the exogenous amenity, β the local amenity spillover elasticity and σ the demand elasticity. T_{ij} represents the matrix of trade costs to ship goods across locations.[12]

We can write the parametric parametric matrices corresponding to Theorem 1 as

$$\mathbf{\Gamma} = \begin{pmatrix} 1 & 1-\sigma & \sigma \\ \beta(1-\sigma) & 0 & 1-\sigma \\ 0 & 1 & 0 \end{pmatrix}, \; \mathbf{B} = \begin{pmatrix} 1+\beta(\sigma-1) & 0 & \sigma \\ 0 & \sigma-1 & 1-\sigma \\ \alpha & 0 & 0 \end{pmatrix}.$$

Therefore,

$$\mathbf{\Gamma}^{-1} = \begin{pmatrix} 1 & \frac{\sigma}{\sigma-1} & \sigma-1 \\ \beta(\sigma-1) & \beta\sigma & 1+\beta\ (\sigma-1)^2 \\ 0 & -\frac{1}{\sigma-1} & 0 \end{pmatrix}$$

and

$$\mathbf{B\Gamma^{-1}} = \begin{pmatrix} 1+\beta(\sigma-1) & \beta\sigma & \sigma-1+\beta(\sigma-1)^2 \\ \beta(\sigma-1)^2 & \beta\sigma(\sigma-1)+1 & \sigma-1+\beta(\sigma-1)^3 \\ \alpha & \frac{\alpha\sigma}{\sigma-1} & \alpha(\sigma-1) \end{pmatrix}.$$

[12] Overall amenity of living in a location i is $u_i = \bar{u}_i L_i^{\beta}$,i.e. it depends on local population. The amenity is assumed to affect welfare of a location multiplicatively.

We consider the case that $\beta < 0 < \alpha$ which allows for the spectral radius to be less or equal than one. The case $\alpha, \beta \geq 0$ always implies a spectral radius bigger than one. When $\beta < 0$ the first two rows of $\mathbf{B\Gamma^{-1}}$ may be negative. Notice that $(\sigma-1)\left(\mathbf{B\Gamma^{-1}}\right)_{22} < (\sigma-1)\left(\mathbf{B\Gamma^{-1}}\right)_{11} = \left(\mathbf{B\Gamma^{-1}}\right)_{13}$ and $(\sigma-1)\left(\mathbf{B\Gamma^{-1}}\right)_{22} < \left(\mathbf{B\Gamma^{-1}}\right)_{23}$. There is a number of cases to discuss. Here we only consider the case $\left(\mathbf{B\Gamma^{-1}}\right)_{22} \geq 0$; other cases can be derived similarly.

If $\left(\mathbf{B\Gamma^{-1}}\right)_{22} > 0$ i.e. $\beta > -\frac{1}{\sigma(\sigma-1)}$, then we have

$$\left|\mathbf{B\Gamma^{-1}}\right| = \begin{pmatrix} 1+\beta(\sigma-1) & -\beta\sigma & \sigma-1+\beta(\sigma-1)^2 \\ -\beta(\sigma-1)^2 & \beta\sigma(\sigma-1)+1 & \sigma-1+\beta(\sigma-1)^3 \\ \alpha & \frac{\alpha\sigma}{\sigma-1} & \alpha(\sigma-1) \end{pmatrix}.$$

A sufficient condition for $\rho\left(\left|\mathbf{B\Gamma^{-1}}\right|\right) \leq 1$ is that the summation of each column is smaller than 1 (see Remark 5). Thus we have

$$\alpha + \beta(\sigma-1)(2-\sigma) \leq 0$$
$$\frac{\alpha\sigma}{\sigma-1} + \beta\sigma(2-\sigma) \leq 0$$
$$\alpha + \beta\sigma(\sigma-1) \leq \frac{1}{\sigma-1} - 2.$$

The three inequalities and $\beta > -\frac{1}{\sigma(\sigma-1)}$ can guarantee $\rho\left(\left|\mathbf{B\Gamma^{-1}}\right|\right) \leq 1$ and, therefore, uniqueness.

We now analyze the celebrated Ricardian model developed by Eaton and Kortum (2002) specified with tariffs and input-output network as in Alvarez and Lucas (2007).

The equilibrium of their model can be characterized by the three equations below (corresponding to equations 3.8, 3.15, and 3.17 respectively in Alvarez and Lucas (2007)),

$$p_{mi} = \left[\sum_{j=1}^{n} \lambda_j \left(\frac{1}{\kappa_{ij}} \frac{AB}{\omega_{ij}}\right)^{-\theta} \left(w_j^{\beta} p_{mj}^{1-\beta}\right)^{-\theta}\right]^{-\frac{1}{\theta}}, \tag{17}$$

$$L_i w_i (1 - s_{fi}) = \sum_{j=1}^{n} L_j \frac{w_j \left(1 - s_j^f\right)}{F_j} D_{ji} \omega_{ji}, \tag{18}$$

$$F_i = \sum_{j=1}^{n} D_{ij} \omega_{ij}, \tag{19}$$

where $D_{ij} \equiv \frac{\left(w_j^{\beta} p_j^{1-\beta}\right)^{-\theta}}{p_{mi}^{-\theta}} \left(\frac{AB}{\kappa_{ij}\omega_{ij}}\right)^{-1/\theta} \lambda_j$ is country i's per capita spending on tradeables that is spent on goods from country j and $s_i^f = \frac{\alpha[1-(1-\beta)F_i]}{(1-\alpha)\beta F_i + \alpha[1-(1-\beta)F_i]}$ is labor's share in the production of final goods (equations 3.10 and 3.16 in Alvarez and Lucas (2007)) and the endogenous variables are: p_{mi}, the price index of tradeables in country i; F_i, the fraction of country i's spending on tradeables that reaches producers; and w_i, country i's wage. Finally, ω_{ij} is the bilateral tariff.

Now we show how to transform the equilibrium equations into the form of equation (4). First, raise both sides of equation (17) to the power of $-\theta$ and denote $\lambda_j\left(\frac{1}{\kappa_{ij}}\frac{AB}{\omega_{ij}}\right)^{-\theta}$ as K_{ij}^1, then we can rewrite equation (17) as

$$p_{mi}^{-\theta} = \sum_{j=1}^{n} K_{ij}^1 w_j^{-\beta\theta} p_{mj}^{-(1-\beta)\theta}; \tag{20}$$

Second, substitute the expression of D_{ij} into equation (19), multiply both sides by $p_{mi}^{-\theta}$, and denote $\omega_{ij}\lambda_j\left(\frac{1}{\kappa_{ij}}\frac{AB}{\omega_{ij}}\right)^{-\theta}$ as K_{ij}^2, then we can rewrite equation (19) as

$$p_{mi}^{-\theta} F_i = \sum_{j=1}^{n} K_{ij}^2 w_j^{-\beta\theta} p_{mj}^{-(1-\beta)\theta}; \tag{21}$$

Third, define $\tilde{F}_i \equiv \alpha + (\beta-\alpha)F_i$, substitute equation (19) into it, and notice that $\sum_{j=1}^{n} D_{ij} = 1$. Thus we have $\tilde{F}_i = \sum_{j=1}^{n} D_{ij}\left[\alpha + (\beta-\alpha)\omega_{ij}\right]$. Again, substitute the expression of D_{ij}, multiply both sides by $p_i^{-\theta}$, and denote $\left[\alpha + (\beta-\alpha)\omega_{ij}\right]\lambda_j\left(\frac{1}{\kappa_{ij}}\frac{AB}{\omega_{ij}}\right)^{-\theta}$ as K_{ij}^3, then we can have equation

$$p_{mi}^{-\theta}\tilde{F}_i = \sum_{j=1}^{n} K_{ij}^3 w_j^{-\beta\theta} p_{mj}^{-(1-\beta)\theta} \tag{22}$$

Last, substitute the expressions of s_{fi} and D_{ji} into equation (18) , subsequently replace $\alpha + (\beta-\alpha)F_i$ with $\tilde{F}_i$, multiply both sides by $p_{mi}^{(1-\beta)\theta} w_i^{\beta\theta}$ and define $\frac{L_j}{L_i}\omega_{ji}\lambda_j\left(\frac{1}{\kappa_{ij}}\frac{AB}{\omega_{ij}}\right)^{-\theta}$ as K_{ij}^4, then we can rewrite equation (18) as

$$p_{mi}^{(1-\beta)\theta} F_i \tilde{F}_i^{-1} w_i^{1+\beta\theta} = \sum_{j=1}^{n} K_{ij}^4 w_j \tilde{F}_j^{-1} p_{mj}^{\theta}. \tag{23}$$

Now we have transformed the equilibrium equations into the form (4) but with four set of endogenous variables $\left\{p_{mi}, F_i, \tilde{F}_i, w_i\right\}_{i=1,2,...,n}$. Notice that all the kernels, $K_{ij}^1, ..., K_{ij}^4$, defined above are positive when $\alpha, \beta, \theta > 0$ and $0 < \omega_{ij} \leq 1$. Then we have the corresponding parameter matrices

$$\Gamma = \begin{pmatrix} -\theta & 0 & 0 & 0 \\ -\theta & 1 & 0 & 0 \\ -\theta & 0 & 1 & 0 \\ (1-\beta)\theta & 1 & 1 & 1+\beta\theta \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} -(1-\beta)\theta & 0 & 0 & -\beta\theta \\ -(1-\beta)\theta & 0 & 0 & -\beta\theta \\ -(1-\beta)\theta & 0 & 0 & -\beta\theta \\ \theta & 0 & -1 & 1 \end{pmatrix}$$

The determinant of Γ is $-\frac{1}{\beta\theta^2+\theta} \neq 0$. This implies Γ is always invertible as long as $\theta > 0$. Therefore, we have

$$\left|\mathbf{B\Gamma^{-1}}\right| = \begin{pmatrix} 1-\beta & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1-(1-\beta)^2}{\beta+1/\theta} & 0 & \frac{1}{\beta\theta+1} & \frac{|1-(1-\beta)\beta\theta|}{\beta\theta+1} \end{pmatrix}$$

Here $1 \geq \theta(1-\beta)\beta$ or $\beta \geq \frac{1}{2}$ is sufficient for $\rho\left(\left|\mathbf{B\Gamma^{-1}}\right|\right) \leq 1$ i.e. we have (up-to-scale) uniqueness. In comparison, the essential conditions for uniqueness in Alvarez and Lucas (2007) are i) $(\min_{i,j=1,2,\ldots,n}\{\kappa_{ij}\}\min_{i,j=1,2,\ldots,n}\{\omega_{ij}\})^{\frac{2}{\theta}} \geq 1-\beta$; ii) $\alpha \geq \beta$; iii) $1-\min_{i,j=1,2,\ldots,n}\{\omega_{ij}\} \leq \frac{\theta}{\alpha-\beta}$ (see their Theorem 3).[13]

B.2.2 Production Networks

We next study economic interactions that arise from from input-output production linkages.

Constant Elasticity Among Intermediates

We first consider a direct extension of the framework by Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012) where the production function is Cobb-Douglas in labor and intermediates. Instead, we assume that intermediates across all sectors are aggregated through a constant elasticity of substitution aggregator different sectors with an elasticity σ. This extension is explicitly discussed in Carvalho and Tahbaz-Salehi (2019) as a special case of the nested CES case considered by Baqaee and Farhi (2018). Formally the production function is

$$y_i = z_i l_i^{\alpha_1}\left[\left(\sum_j x_{ji}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}\right]^{\alpha_2}$$

where z_i stands for the productivity and is exogenous, l_i is the labor, x_{ji} is the intermediate goods from sector j, and $\alpha_1+\alpha_2=1$.

Therefore, from cost minimization we have the price of the goods produced in sector i

$$p_i = \frac{\bar{\alpha}}{z_i} w^{\alpha_1}(P_i)^{\alpha_2} \tag{24}$$

where we define $\bar{\alpha} = \alpha_1^{\alpha_1}\alpha_2^{\alpha_2}$, w is the wage, and the price index of intermediate goods P_i is determined in the following equation

$$P_i^{1-\sigma} = \sum_j \tau_{ji}^{1-\sigma} p_j^{1-\sigma} \tag{25}$$

[13] If $1 \geq \theta(1-\beta)\beta$, we can solve explicitly the eigenvalues are $\left\{\begin{array}{cccc} 0, & 0, & 1, & \frac{(1-\beta)-\beta\theta}{1+\beta\theta}\end{array}\right\}$. Obviously, $\left|\frac{(1-\beta)-\beta\epsilon}{1+\beta\epsilon}\right| < 1$, thus the uniqueness holds. If $1 < \theta(1-\beta)\beta$, the characteristic polynomial is $f(x) = x^4 + \frac{2\beta^2-2\beta+\beta/\theta}{\beta+1/\theta}x^3 + \frac{2\beta^3-4\beta^2+\beta+\beta/\theta-1/\theta}{\beta+1/\theta}x^2$. According to Lemma 2, we can check the value of $f^{(k)}(1)$ for $k=0,1,2,3$, a sufficient condition to guarantee $\rho\left(\left|\mathbf{B\Gamma^{-1}}\right|\right) \leq 1$ is $\beta \geq \frac{1}{2}$. (In this case the sufficient and necessary condition is $4\beta^3-2\beta^2+2/\theta+5\beta/\theta \geq 0$ and $2\beta^3+2\beta^2+\beta+4\beta/\theta-1/\theta \geq 0$ when $1 < \theta(1-\beta)\beta$.)

where τ_{ji} stands for the standard iceberg trade cost but can be interpreted here as the cost of adaption of the good as an intermediate in another sector. Substitute the expression of $p_j = \frac{\bar{\alpha}}{z_j} w^{\alpha_1} P_j^{\alpha_2}$(equation (24)), into (25) we immediately obtain

$$P_i^{1-\sigma} = \sum_j \left(\frac{\bar{\alpha}}{z_j} w^{\alpha_1} \right)^{1-\sigma} \tau_{ji}^{1-\sigma} P_j^{\alpha_2(1-\sigma)}. \tag{26}$$

Normalize the wage w to be 1. Notice that since z_i is exogenous, this equation (for all i) determines the price indexes $\{P_i\}$. Therefore, as long as consumer utility function satisfies concavity condition, this equation alone can represent the equilibrium. Define $x_i \equiv P_i^{1-\sigma}$ and $f_{ij}(x_j) \equiv \left(\frac{\bar{\alpha}}{z_j} w^{\alpha_1} \right)^{1-\sigma} \tau_{ji}^{1-\sigma} x_j$, thus the above equation is the form of equation (1). We immediately have $\frac{\partial \ln f_{ij}}{\partial \ln x_j} = \alpha_2$, so uniqueness and convergence of an iterative operator require $|\alpha_2| < 1$, which is satisfied as long as labor is used in production.

We now consider the generalization of the production networks setup in Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012) as discussed in Carvalho, Nirei, Saito, and Tahbaz-Salehi (2019) to incorporate constant elasticity of substitution between intermediate goods.

Consider a static economy consisting of n competitive firms denoted by $\{1, 2, \cdots, n\}$, each of which producing a distinct product. Firms employ nested CES production technology

$$y_i = \left[\chi (1-\mu)^{\frac{1}{\sigma}} (z_i l_i)^{\frac{\sigma-1}{\sigma}} + \mu^{\frac{1}{\sigma}} M_i^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

where l_i is the amount of labor, z_i is the (exogenous) labor productivity, and the intermediate input bundle M_i is a CES aggregate of inputs purchased from other firms:

$$M_i = \left[\sum_{j=1}^{n} a_{ij}^{\frac{1}{\zeta}} x_{ij}^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}}.$$

We remark that Carvalho, Nirei, Saito, and Tahbaz-Salehi (2019) also include firm-specific capital in the production function; however, given that it is assumed to be supplied inelastically, it is isomorphic to the exogenous labor productivity term z_i.

Solving the cost minimization problem of the firm results in the following system of equations for equilibrium prices:

$$p_i^{1-\sigma} = (1-\mu)(z_i w)^{1-\sigma} + \mu \left(\sum_{m=1}^{n} a_{sm} p_m^{1-\zeta} \right)^{\frac{1-\sigma}{1-\zeta}}$$

which in turn can be written as:

$$\left(\frac{p_i^{1-\sigma}-(1-\mu)(z_i w)^{1-\sigma}}{\mu}\right)^{\frac{1-\zeta}{1-\sigma}}=\sum_{m=1}^{n} a_{sm} p_m^{1-\zeta}.$$

Normalizing the wage $w=1$ and defining $x_i \equiv \left(\frac{p_i^{1-\sigma}-(1-\mu)(z_i w)^{1-\sigma}}{\mu}\right)^{\frac{1-\zeta}{1-\sigma}}$, this becomes:

$$x_i=\sum_{j=1}^{N} a_{ij}\left(\mu x_j^{\frac{1-\sigma}{1-\chi}}+(1-\mu) z_j^{1-\sigma}\right)^{\frac{1-\zeta}{1-\sigma}},$$

which is a special case of equation (1) with $f_{ij} \equiv a_{ij}\left(\mu x_j^{\frac{1-\sigma}{1-\zeta}}+(1-\mu)(z_j)^{1-\sigma}\right)^{\frac{1-\zeta}{1-\sigma}}$.

Note that:

$$\frac{\partial \ln f_{ij}}{\partial \ln x_j}=\left(\frac{1-\zeta}{1-\sigma}\right)\left(\frac{1-\sigma}{1-\zeta}\right)\frac{\mu x_j^{\frac{1-\sigma}{1-\chi}}}{\mu x_j^{\frac{1-\sigma}{1-\chi}}+(1-\mu)(z_j)^{1-\sigma}} \Longrightarrow$$

$$\left|\frac{\partial \ln f_{ij}}{\partial \ln x_j}\right|=\frac{\mu x_j^{\frac{1-\sigma}{1-\chi}}}{\mu x_j^{\frac{1-\sigma}{1-\chi}}+(1-\mu)(z_j)^{1-\sigma}}<1,$$

so that by Theorem 1 (part ii.a), there exists at most one equilibrium.

B.2.3 Social Networks

Here we consider a discrete choice framework with social interactions as in Brock and Durlauf (2001), generalized to include a choice set of more than two actions. Suppose there are N individuals where each individual $i \in \{1, ..., N\}$ chooses from a set of H actions, where $h_i \in \{1, ..., H\}$ indicates her choice. Let the N-tuple $\boldsymbol{\omega} \equiv \{h_1, ..., h_N\}$ denote the actions by entire population and let $\boldsymbol{\omega}_{-i}$ denote the actions of all individuals except i.

Let agent $i's$ payoffs for choosing action h consists of three components:

$$V_{ih}=u_{ih}+S_{ih}(\boldsymbol{\omega}_{-i})+\varepsilon_{ih},$$

where u_{ih} is the private utility associated with choice h, $S_{ih}(\boldsymbol{\omega}_{-i})$ is the social utility associated with the choice, and ε_{ih} is a random utility term, independently and identically distributed across agents. In equilibrium, an agent will choose the action h_i that maximizes her payoffs given the actions of others, i.e:

$$h_i(\boldsymbol{\omega}_{-i}) \equiv \arg \max_{h \in \{1, ..., H\}} V_{ih}(\boldsymbol{\omega}_{-i}).$$

Define μ_{ijh} to be the conditional probability measure agent i places on the probability that agent j chooses action h. We assume that $S_{ih}(\boldsymbol{\omega}_{-i})$ takes the following form:

$$S_{ih}(\boldsymbol{\omega}_{-i}) = J \ln \left(\left(\sum_{j \neq i} \omega_{ij,h} (\mu_{ijh})^{\eta} \right)^{\frac{1}{\eta}} \right),$$

where J governs the strength of the social interaction, $\omega_{ij,h}$ (normalized so that $\sum_{j \neq i} \omega_{ij,h} = 1$) are weights that agent i places on agent j's choice of action h to capture heterogeneity in the social network connections, and the parameter $\eta \in (-\infty, \infty)$ determines what type of mean aggregation is used across other individuals (e.g. $\eta = -\infty$ is the minimum, $\eta = -1$ is the harmonic mean; $\eta = 0$ is the geometric mean; $\eta = 1$ is the arithmetic mean; and $\eta = \infty$ is the maximum). We note that the log transform on the social utility function – not present in the primary case considered by Brock and Durlauf (2001) – ensures that the uniqueness of the equilibrium can be characterized without reference to an (endogenous) threshold value (c.f. Brock and Durlauf (2001) Proposition 2).

The presence of weights $\omega_{ij,h}$ and the flexibility of the particular mean function (governed by parameter η) – both of which are absent in the particular functional forms characterized by Brock and Durlauf (2001) – allow for flexible social interactions between individuals in the network. However, the uniqueness conditions provided below turn out to only depend on the strength of the social interaction J. Note that without loss of generality we can define the private utility as follows $u_{ih} \equiv \ln v_{ih}$, which allows us to interpret J as the parameter which governs the extent to which social interactions determine the choice of agents. A value of $J = 0$ means that decisions are only made by private considerations of utility, whereas a value $J = 1$ means that social utility and private utility v_{ih} are given equal proportions in the utility function.

Retaining the assumption from Brock and Durlauf (2001) that the random utility term follows an extreme value distribution with shape parameter β and agent's conditional probabilities are rational (so that $\mu_{ijh} = \mu_{jh}$ for all $j \in \{1, ..., N\}$ and μ_{jh} is equal to the probability agent j actually chooses action h) results in the following equilibrium conditions for all $i \in \{1, ..., N\}$ and for all $h \in \{1, ..., N\}$:

$$\mu_{ih} = \frac{\exp(\beta u_{ih}) \times \left(\left(\sum_{j \neq i} \omega_{ij,h} (\mu_{jh})^{\eta} \right)^{\frac{1}{\eta}} \right)^{J\beta}}{\sum_{k=1}^{H} \exp(\beta u_{ik}) \times \left(\left(\sum_{j \neq i} \omega_{ij,k} (\mu_{jk})^{\eta} \right)^{\frac{1}{\eta}} \right)^{J\beta}} \tag{27}$$

Note this is a system of $N \times H$ equilibrium conditions in $N \times H$ unknown probabilities μ_{jh}. Equation (27) is a special case of (1). To see this, define $y_{ih} \equiv \mu_{ih}^{\frac{\eta}{J\beta}}$, so that equation (27) becomes:

$$y_{ih} = \frac{\exp\left(\frac{\eta}{J} u_{ih}\right) \times \sum_{j \neq i} \omega_{ij,h} y_{jh}^{J\beta}}{\left(\sum_{k=1}^{H} \left(\sum_{l \neq i} \exp\left(\frac{\eta}{J} u_{ik}\right) \omega_{il,k} y_{lk}^{J\beta} \right)^{\frac{J\beta}{\eta}} \right)^{\frac{\eta}{J\beta}}}$$

Furthermore, define $x_{ih} \equiv \sum_{l \neq i} \exp\left(\frac{\eta}{J} u_{ih}\right) \omega_{il,h} y_{lh}^{J\beta}$ so that equation (27) becomes:

$$y_{ih} = \frac{x_{ih}}{\left(\sum_{k=1}^{H} x_{ik}^{\frac{J\beta}{\eta}}\right)^{\frac{\eta}{J\beta}}}.$$

Then given the definition of x_{ih}, we have:

$$x_{ih} = \sum_{j \neq i} \exp\left(\frac{\eta}{J} u_{ih}\right) \omega_{ij,h} \left(\frac{x_{jh}}{\left(\sum_{k=1}^{H} x_{jk}^{\frac{J\beta}{\eta}}\right)^{\frac{\eta}{J\beta}}} \right)^{J\beta}. \tag{28}$$

Finally, defining $f_{ijh} \equiv \exp\left(\frac{\eta}{J} u_{ih}\right) \omega_{ij,h} \left(\frac{x_{jh}}{\left(\sum_{k=1}^{H} x_{jk}^{\frac{J\beta}{\eta}}\right)^{\frac{\eta}{J\beta}}} \right)^{J\beta}$ if $j \neq i$ and $f_{iih} = 0$ results in equation (28) be written as:

$$x_{ih} = \sum_{j=1}^{N} f_{ijh} \left(x_{j1}, ..., x_{jH}\right),$$

as in (1). It is straightforward to provide bounds on the elasticities of interactions as follows:

$$\frac{\partial \ln f_{ij,h}}{\partial \ln x_{j,h}} = J\beta \left(1 - \frac{x_{jh}^{\frac{J\beta}{\eta}}}{\sum_{k=1}^{H} x_{jk}^{\frac{J\beta}{\eta}}}\right) \in [0, \beta J]$$

and, for $h' \neq h$:

$$\frac{\partial \ln f_{ij,h}}{\partial \ln x_{j,h'}} = -J\beta \left(\frac{x_{jh'}^{\frac{J\beta}{\eta}}}{\sum_{k=1}^{H} x_{jk}^{\frac{J\beta}{\eta}}}\right) \in [-\beta J, 0]$$

So that if we define:

$$(\mathbf{A})_{hh'} \equiv \beta J$$

then we have for all h, h':

$$\left|\frac{\partial \ln f_{ij,h}}{\partial \ln x_{j,h'}}\right| \leq (\mathbf{A})_{hh'}$$

Since the largest eigenvalue of a constant positive square matrix is that constant divided by the number of rows, Theorem 1(i) implies that we have uniqueness as long as $\beta J < \frac{1}{H}$. Hence, as as the size of agent's choice set increases, guaranteeing uniqueness requires increasingly weak social spillovers.

B.3 Additional Remarks

B.3.1 Footnote 4

Here we illustrate the importance of treating the endogenous as H vectors with N elements instead of one giant variable with NH elements. To focus on the ideas, we set $N=1$. Consider the below example:

$$x_{11} = x_{11}^{\frac{1}{2}} x_{12}^{2} + 1$$
$$x_{12} = x_{12}^{\frac{1}{2}} + 1$$

(Here, in order to be consistent with the paper, we do not suppress the notation of N.) We show when the system is treated as a single 2×1 vector, it is not a contraction. We consider its log transformation by setting $y_1 = \ln x_{11}$ and $y_2 = \ln x_{12}$. Thus the above two equations become:

$$y_1 = \ln\left(e^{\frac{1}{2}y_1 + 2y_2} + 1\right) \tag{29}$$

$$y_2 = \ln\left(e^{\frac{1}{2}y_2} + 1\right). \tag{30}$$

Denote its right side as $T(\cdot): \mathbb{R}^2 \to \mathbb{R}^2$. Its Jacobian matrix is

$$J(y) = \begin{pmatrix} \frac{1}{2}\frac{e^{\frac{1}{2}y_1+2y_2}}{e^{\frac{1}{2}y_1+2y_2}+1} & \frac{2e^{\frac{1}{2}y_1+2y_2}}{e^{\frac{1}{2}y_1+2y_2}+1} \\ 0 & \frac{1}{2}\frac{e^{\frac{1}{2}y_2}}{e^{\frac{1}{2}y_2}+1} \end{pmatrix}.$$

Notice that the tight upper bound of the Jacobian matrix is

$$\boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & 2 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

For two y and y', applying the mean value theorem on the two single-valued functions of $T(\cdot)$, we have

$$|T(y) - T(y')| \leq \boldsymbol{A}\,|y - y'| \tag{31}$$

To apply the standard contraction mapping, we treat y_1 and y_2 as a single vector variable. We consider two natural choices of norms to serve for the metric used in the contraction mapping: 1. the max norm $\|y\|_{\max} = \max(y_1, y_2)$; 2. the Euclidean norm $\|y\| = \sqrt{y_1^2 + y_2^2}$.

For the first norm, according to inequality (31), we have

$$\|T(y) - T(y')\|_{\max} \leq 2\,\|y - y'\|_{\max}.$$

Clearly, the standard contraction mapping does not apply.

For the second norm, again according to inequality (31), we have

$$\|T(y) - T(y')\| \leq \|\boldsymbol{A}\| \, \|y - y'\|$$

where $\|\boldsymbol{A}\|$ is the $\boldsymbol{A}$'s matrix norm. Here $\|\boldsymbol{A}\| \approx 2.118$. Again, the standard contraction mapping does not apply.

In constrast, applying our multi-dimension contraction mapping, we treat y_1 and y_2 as two separate variables.We immediately have $\rho(\boldsymbol{A}) = \frac{1}{2}$, so that inequality (31) implies the uniqueness.