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ABSTRACT

Governments face a trade-off between insuring bondholders and taxpayers. If the government fully insures bondholders by manufacturing risk-free debt, then it cannot also insure taxpayers against permanent macro-economic shocks over long horizons. Instead, taxpayers will pay more in taxes in bad times. Conversely, if the government fully insures taxpayers against adverse macro shocks, then the debt becomes risky; at least as risky as un-levered equity claim. Governments can only escape this trade-off if they enjoy a large and counter-cyclical convenience yield on their debt.

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Governments face a trade-off between insuring tax payers against adverse macro shocks and insuring bond holders. If a government provides more insurance to bond investors, who then require lower risk premia, then it can provide less insurance to taxpayers. Making government debt safer requires raising more tax revenue relative to GDP from tax payers in bad times. The larger the sovereign debt burden, the steeper this trade-off becomes.

Some countries, especially the U.S., pay a low risk premium on outstanding government debt. The portfolio is priced as if it is close to risk-free. Other countries, including most emerging market countries, pay a much larger risk premium to bond investors. The focus in the literature has been mostly on countries' willingness and ability to repay (see, e.g., Eaton and Gersovitz, 1981; Bulow and Rogoff, 1989; Aguiar and Gopinath, 2006; Arellano, 2008; DeMarzo, He, and Tourre, 2019, for examples). The trade-off between bondholder and taxpayer insurance applies regardless of whether a country contemplates default.

Despite its low government debt risk premium, the U.S. insures its taxpayers against output growth risk. That is, the U.S. appears to escape this trade-off. The sensitivity of federal government spending to GDP is much lower than the sensitivity of tax revenues to GDP over horizons between one and ten years in the U.S. data. Tax revenues rise and fall strongly with GDP growth. Spending moves inversely with GDP growth at short horizons and mildly positively at longer horizons. It appears that the U.S. government can insure taxpayers at all horizons by lowering their tax rates in recessions as well as by increasing its spending-to-output ratio in recessions. In the language of asset pricing, a claim to tax revenues has a higher beta than a claim to spending.

How can the Treasury manufacture debt that is risk-free? That requires a non-trivial feat of financial engineering. The Treasury's bond portfolio is backed by a long position in a claim to tax revenue and a short position in a claim to government spending. Both claims are exposed to output risk. The Treasury's long position in the tax claim exceeds the short position by the value of outstanding debt. To render the entire Treasury portfolio risk-free, the claim to tax revenues has to have a lower beta than the spending claim so as to ensure that the net beta of the Treasury portfolio is exactly zero. Recast in the language of Miller-Modigliani, the tax revenue claim is the unlevered version of the spending claim. The beta of the tax claim is the weighted average of the beta of the spending claim and the beta of the debt. If the debt beta is zero, the tax beta must be lower than the spending beta.

The tax claim has a low beta if the present discounted value (PDV) of future tax revenues increases in bad times, times in which the investor's marginal utility is high. The taxpayer is short the tax revenue claim since she pays the taxes. From the taxpayer's perspective, a low-beta tax claim is a risky tax liability. As a result, the government cannot insure taxpayers when it insures bondholders by keeping the debt risk-free. The larger the amount of outstanding government debt, the larger the gap between the betas of the spending and the tax claim needs to be to keep the debt risk-free. As the debt grows, the beta of the tax claim has to go to zero. The trade-off between insuring taxpayers and bondholders steepens.

Conversely, if the governments insists on insuring the tax payers by lowering tax rates in bad times, then government debt becomes risky for bond holders. It is the bondholders that now will be bearing the macro-economic risk. The empirical properties of tax revenue and spending are consistent with taxpayer insurance, and hence inconsistent with risk-free debt.

Our paper is the first one to characterize the trade-off between insuring taxpayers and bondholders at different horizons. To do so, we assume that the government commits to a counter-cyclical debt policy. We derive the restrictions that risk-free debt impose on the properties of the surplus/output ratio from the government budget constraint. When shocks to output are permanent, the government can only escape the trade-off over short horizons. It can issue more debt in response to a negative GDP growth shock rather than raise taxes. The surplus claim and tax revenue claim are risky over short horizons. Over longer horizons, the tax revenue claim has to become sufficiently safe for investors (risky for taxpayers) to offset the long-run risk in debt issuance. This long-run risk in debt arises from the long-run risk in output, as along as debt and output are co-integrated. How long the government can escape the trade-off depends on the persistence of the debt/output ratio.

A calibration suggests that an AR(2) model with a negative exposure to output shocks fits the U.S. government's debt/GDP dynamics over the post-war sample well. An AR(1) model for the government spending/GDP ratio with negative exposure to output shocks fits the dynamics of spending well. Insisting on risk-free debt then implies surplus/GDP and tax revenue/GDP ratios with the following properties. First, the surplus/ouput ratio can only remain negative for two periods in response to a negative output shock. There is a tight limit on how long the government can plug the fiscal hole by issuing more debt. After two years of deficits, the government must run a surplus for the next several years. The mean-reversion of surpluses the model implies is at odds with the persistence in surpluses observed in the data. The autocorrelation functions in model and data are far apart. Second, the government realizes these surpluses via higher taxation. The higher tax revenues in response to the adverse output shock make the tax claim safer at horizons past two years. If debt is to be risk-free, then the tax revenue beta has to drop below the spending beta at longer horizons. This is the opposite pattern as what we find in the data. We refer to this as the government debt risk premium puzzle: why is the portfolio of U.S. Treasurys priced as if it is close to risk-free, even though the government's cash flow fundamentals are risky? Third, the debt/output ratio has little or no predictive power for future surpluses in U.S. data at horizons from 1 to 10 years. In the model, the debt/output ratio is the single best predictor of future surpluses at longer horizons.

How then can we reconcile the observed insurance of taxpayers with the low risk premia on government debt in the U.S. data. Our paper offers two possibilities. The first is that the government's debt issuance policy imputes a unit root into the debt/output ratio. We show that this leads to a violation of the transversality condition when debt issuance is sufficiently counter-cyclical and the equity risk premium is sufficiently large. Second, the government also escapes the trade-off if government debt earns large and counter-cyclical convenience yields.

Related Literature Our paper applies a basic insight from the asset pricing literature to the fiscal policy literature. Modern asset pricing has consistently found that permanent cash-flow shocks receive a high price of risk in the market (e.g., Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Bansal and Yaron, 2004; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and Chernov, 2018). If GDP growth has a permanent component, which modern macro and econometrics recognizes to be the case, then the surplus process in levels S_t inherits that permanent component from Y_t. Surpluses have long-run risk. Because of the exposure of the surplus to long-run GDP risk, the claim to current and future surpluses will typically have a substantial risk premium. Since the value of the surplus claim equals the market value of outstanding debt, the portfolio of government debt is generally a risky asset. The properties of the stationary surplus/output ratios, which the literature focuses on, are irrelevant for the long-run discount rates of surpluses.[1] For long-run discount rates, only long-run risk matters (Backus, Boyarchenko, and Chernov, 2018). Therefore, even when debt is risk-free—in the sense that there is no news about current or future surpluses,— the risk-free rate is not the right discount rate for surpluses in the presence of permanent output risk.

There is an extensive literature which tests the government's inter-temporal budget constraint. Hansen, Roberds, and Sargent (1991); Hamilton and Flavin (1986); Trehan and Walsh (1988, 1991); Bohn (1998, 2007) derive time-series restrictions on the government revenue and spending processes that enforce the government's inter-temporal budget constraint.[2] This literature invariably uses the risk-free rate as the discount rate. This is not the right discount rate unless all shocks to output are temporary and the risk-free rate exceeds the growth rate of the economy. These authors really test the *joint hypothesis* that both the government budget constraint and the measurability condition—to render the debt risk-free—are satisfied. We derive restrictions on the surplus/output process that are compatible with the knife-edge case of risk-free debt. The answer depends crucially on whether GDP has a permanent component or not. In the realistic case where it does, the surplus/output ratio cannot be sufficiently autocorrelated compared to the data when we match the dynamics of the debt/output ratio to the data. Further, we show analytically that the substantial S-shaped impulse-responses of the surplus/output ratio discussed by Bohn (1998); Canzoneri, Cumby, and Diba (2001); Cochrane (2019, 2020) are not consistent with risk-free debt. Those require a debt/output ratio that has higher-order dynamics than those observed for

[1] For example, Cochrane (2020) completely abstracts from long-run output risk.

[2] Jiang (2019) derives the implications of the government's inter-temporal budget constraint for the nominal and real exchange rates, and finds support in the data.

debt/output in the data.

The U.S. government debt earns returns close to the risk-free rate, but the cash flow dynamics do not bear this out: the surpluses are too persistent, not predicted by the debt/GDP ratio and too risky. We call this the U.S. government risk premium puzzle. The U.S. government debt risk premium puzzle we document in this paper is distinct from, but related to the government debt valuation puzzle discussed by Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019), because the risk premium puzzle does not pertain to the first moments of future surpluses.

Our paper contributes to the normative literature on optimal government taxation and debt management, starting with Barro (1979)'s seminal work on tax smoothing. In the literature after Barro (1979), starting with Lucas and Stokey (1983), the risk-return tradeoff we highlight is present in the background, but is not explicitly analyzed. Importantly, most of these models do not have plausible asset pricing implications because they do not have permanent output risk. When markets are complete, the planner favors shifting the risk from taxpayers to bond investors (Lucas and Stokey, 1983). We do not derive the optimal tax rate, but show that, for any tax policy, the government can only truly insure taxpayers over short horizons, while keeping the debt risk-free. When the government accumulates assets rather than have debt, it can implement the complete markets Ramsey allocation, as shown by Aiyagari, Marcet, Sargent, and Seppälä (2002). Insuring taxpayers at all horizons against adverse macro shocks always comes at a large debt service cost to the Treasury in a model with plausible asset prices.

By changing the maturity composition of debt, the government may be able to get closer to the optimal tax policy when markets are incomplete, essentially by making the debt riskier (Angeletos, 2002; Buera and Nicolini, 2004; Lustig, Sleet, and Yeltekin, 2008; Arellano and Ramanarayanan, 2012; Bhandari, Evans, Golosov, and Sargent, 2017; Aguiar, Amador, Hopenhayn, and Werning, 2019), and shifting risk from taxpayers to bondholders. Our work is not focused on how the maturity choice of the government informs the riskiness of debt, but instead focuses directly on the fundamental determinants of the riskiness of the government's balance sheet.

In recent work, Mian, Straub, and Sufi (2020a,b) examine the distributional implications of government debt issuance, pointing out that the wealthy buy a large share of government (and private) debt. To the extent that the Gini coefficient of government debt holdings exceeds that of taxes, the government is trading off insuring the rich versus insuring the middle class.

Finally, convenience yields may help explain why emerging economies with more sovereign risk typically have more pro-cyclical fiscal policies (Bianchi, Ottonello, and Presno, 2019). These countries do not benefit from the convenience yields, and hence cannot escape the trade-off. It may also help to explain the government debt risk premium puzzle. In international economics, there is a growing literature that emphasizes the U.S. role as the world's safe asset supplier (see Gourinchas and Rey, 2007; Caballero, Farhi, and Gourinchas, 2008; Caballero and Krishnamurthy, 2009; Maggiori, 2017; He, Krishnamurthy, and Milbradt, 2018; Gopinath and Stein, 2018; Krishnamurthy and Lustig, 2019; Jiang, Krishnamurthy, and Lustig, 2018a, 2019; Liu, Schmid, and Yaron, 2019; Koijen and Yogo, 2019).

The paper is organized as follows. Section 1 derives the general trade-off between the insurance of bondholders and taxpayers. When the government commits to plausible spending and tax revenue policies, the debt will generally be risky. We characterize these risk premia in closed form. Section 2 develops a simple model with permanent shocks to output and to the investor's marginal utility. The governments commits to a spending policy and a debt policy. We solve for the tax policy that keeps the debt risk-free. We start with the case of constant debt/output ratios. Section 3 introduces time-varying debt/output ratios. Section 4 characterizes the trade-off faced by the government at different horizons. Finally, Section 5 introduces convenience yields. Appendix A generalizes the results in a continuous time version of the model that allows for risky debt and convenience yields. Section B provides some additional risk premium results. Section C develops a version of the model without permanent shocks. With only transitory shocks to output and marginal utility, the government is able to insure taxpayers over longer horizons. However, these models have counterfactual asset pricing implications. The only model in which the government can insure taxpayers at all horizons is one in which the output shocks are transitory, but they are priced as if they are permanent.

1 The General Trade-off between Insuring Bondholders and Taxpayers

We use T_t to denote government revenue, and G_t to denote government spending. M_t denotes the stochastic discount factor. We assume that debt is fairly priced and does not earn any convenience yields.

Let B_t denote the market value of outstanding government debt at the beginning of period t, before expiring debt is paid off and new debt is issued. The debt can be long-term or short-term, and it can be nominal or real. In fact, it can be any contingent claim. In Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019), we show that the value of the government debt equals the sum of the expected present values of future tax revenues minus future government spending:

$$B_t = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (T_{t+j} - G_{t+j}) \right], \tag{1}$$

provided that there is no arbitrage opportunity and a transversality condition holds. This result does not rely on complete markets, and it still applies even when the government can default on its debt. See Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) for a proof. This result relies on the absence of arbitrage in bond markets and the transversality condition $\lim_{k\to\infty} \mathbb{E}_t M_{t,t+k} B_{t+k} = 0$.[3]

[3] While there are equilibrium models that generate violations of the TVC (see Samuelson, 1958; Diamond, 1965; Blanchard and Watson, 1982; Brunnermeier, Merkel, and Sannikov, 2020), these violations typically show up in all

Let $P_t^T = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j} T_{t+j}\right]$ and $P_t^G = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j} G_{t+j}\right]$ denote the present values of the "cum-dividend" tax claim and spending claim. Value additivity then implies that $B_t = P_t^T - P_t^G$.

1.1 The Government Debt Risk Premium

For notational convenience, let $D_t = B_t - S_t$ denote the difference between the market value of outstanding government debt and the government surplus. By the government budget condition, D_t is the market value of outstanding government debt at the end of period t, after expiring debt is paid off and new debt is issued.

Let R_{t+1}^D, R_{t+1}^T and R_{t+1}^G denote the holding period returns on the bond portfolio, the tax claim, and the spending claim, respectively:

$$R_{t+1}^D = \frac{B_{t+1}}{B_t - S_t}, \qquad R_{t+1}^T = \frac{P_{t+1}^T}{P_t^T - T_t}, \qquad R_{t+1}^G = \frac{P_{t+1}^G}{P_t^G - G_t}.$$

In Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019), we also show that the government debt portfolio return is the return on a portfolio that goes long in the tax claim and short in the spending claim:

$$\mathbb{E}_t\left[R_{t+1}^D - R_t^f\right] \;=\; \frac{P_t^T - T_t}{D_t}\mathbb{E}_t\left[R_{t+1}^T - R_t^f\right] - \frac{P_t^G - G_t}{D_t}\mathbb{E}_t\left[R_{t+1}^G - R_t^f\right]. \tag{2}$$

This result only relies on Eq. (1) and additivity. The value of a claim to surpluses equals the value of a claim to taxes minus the value of a claim to spending.

The government bond risk premium varies dramatically across countries. In some countries, such as the U.S., this risk premium $\mathbb{E}_t\left[R_{t+1}^D - R_t^f\right]$ is small. Hall and Sargent (2011) compute a real return of 168 basis points on all U.S. Treasurys, while Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) compute a risk premium of 111 basis points for the U.S. government portfolio. The returns on debt issued by peripheral or developing countries are estimated to be much higher; Using EMBI indices on a short sample, Borri and Verdelhan (2011) estimate annual excess returns between 4% and 15%. On a much longer sample going back to the 19th century, Meyer, Reinhart, and Trebesch (2019) estimate excess returns of around 4% above U.S. and U.K bond returns, taking into account defaults.

long-lived assets, including stocks, not just government debt, and these models typically do not feature long-lived investors.

1.2 Characterizing the Trade-Off with Return Betas

Next, we rearrange equation (2) and derive the following expression for the risk premium on the tax claim:

$$\mathbb{E}_t\left[R^T_{t+1}-R^f_t\right]=\frac{P^G_t-G_t}{D_t+(P^G_t-G_t)}\mathbb{E}_t\left[R^G_{t+1}-R^f_t\right]+\frac{D_t}{D_t+(P^G_t-G_t)}E_t\left[R^D_{t+1}-R^f_t\right]. \quad (3)$$

Governments typically want a counter-cyclical spending claim, i.e. they want to spend more in recessions. On the other hand, they also want a risky tax claim, because they want to reduce the tax burden in recessions. As a result, the tax claim's risk premium $\mathbb{E}_t\left[R^T_{t+1}-R^f_t\right]$ is high and the spending claim's risk premium $\mathbb{E}_t\left[R^G_{t+1}-R^f_t\right]$ is low. When the debt value D_t is positive, the fraction $\frac{P^G_t-G_t}{D_t+(P^G_t-G_t)}$ is between 0 and 1. Then, for Eq. (3) to hold, it requires a high risk premium $\mathbb{E}_t\left[R^D_{t+1}-R^f_t\right]$ on the government debt portfolio. As the debt risk premium is a measure of the risk premium or insurance premium charged by bondholders, the government's debt portfolio has to be very risky.

According to equation (3), the tax revenue claim is the unlevered version of the spending claim, or, equivalently, the spending claim is the levered version of the tax claim. This result is analogous to the Miller-Modigliani relation between the unlevered return on equity (the return on the tax claim) and the levered return on equity (the return on the spending claim).

We define the beta of an asset i as:

$$\beta^i_t = \frac{-cov_t\left(M_{t+1},R^i_{t+1}\right)}{var_t(M_{t+1})}.$$

By the investor's Euler equation, β^i_t determines the conditional risk premium of this asset

$$\mathbb{E}_t\left[R^i_{t+1}-R^f_t\right] = \beta^i_t\lambda_t,$$

where the price of risk is $\lambda_t = R^f_t var_t(M_{t+1})$.

Let β^D_t, β^T_t and β^G_t denote the beta of the bond portfolio, the tax claim, and the spending claim, respectively. We assume $\beta^Y_t > 0$, so that the output claim has a positive risk premium. The following proposition characterizes the relationship of their risk exposures.

Proposition 1. *The beta on the tax claim is a weighted average of the beta of the spending claim and the beta of the debt:*

$$\beta^T_t = \frac{P^G_t-G_t}{D_t+(P^G_t-G_t)}\beta^G_t+\frac{D_t}{D_t+(P^G_t-G_t)}\beta^D_t.$$

Governments want to provide insurance to transfer recipients by choosing $\beta_t^G < \beta_t^Y$, but they also want to provide insurance to taxpayers by choosing $\beta_t^T > \beta_t^Y$. However, the following corollary states that this is impossible if the government debt is risk-free.

Corollary 1. *In order for debt to be risk-free ($\beta_t^D = 0$), the beta of the tax claim needs to equal the unlevered beta of the spending claim:*

$$\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)} \beta_t^G.$$

If the government has a positive amount of risk-free debt $D_t > 0$, there is no scope to insure taxpayers. Instead, the taxpayers provide insurance to the rest of the economy. We start our analysis with the case in which the spending claim has a positive beta ($\beta_t^G > 0$). Then, the government engineers risk-free debt by lowering the beta of the tax claim relative to that of the spending claim: $\beta_t^T < \beta_t^G$. A low beta for the tax claim means that tax revenue must fall by less than GDP in a recession. Tax rates must rise in recessions. The more debt outstanding, the lower the beta of the tax claim needs to be relative to that of the spending claim. With more debt, the trade-off between insuring bondholders and taxpayers becomes steeper.

The restriction on the betas holds true regardless of the specific dynamics of the tax and spending process. In the next section, we will derive restrictions on the underlying cash flows by committing to particular processes for debt/output and spending/output.

The only way the government can provide insurance to debt holders, while keeping the debt risk-free, is by saving—choosing $D_t < 0$. In other words, the government can only insure taxpayers at the expense of bondholders.[4] On the other hand, if the spending claim has a negative beta ($\beta_t^G < 0$), then the tax claim also has a negative beta: $\beta_t^T < 0$. The taxpayers have large tax payments during recessions.

This discussion implicitly assumes that taxpayers are long-lived households who value a dollar in each aggregate state in the same way as the marginal investor in Treasury markets. When markets are incomplete, agents may have different IMRS. Even when markets are incomplete, the aggregate component of households' IMRS will be common and risk premia are identical to those in the equivalent representative agent economy, , as long as the conditional distribution of idiosyncratic risk does not depend on the aggregate state of the economy (Krueger and Lustig, 2010; Werning, 2015).[5]

[4] Aiyagari, Marcet, Sargent, and Seppälä (2002) show that it is optimal for a government issuing only risk-free one period debt to accumulate savings $D_t \ll 0$ in the limit. This makes perfect sense, because that allows the government to choose $\beta_t^T \gg \beta_t^G$ and insure tax payers against macro shocks. In the limit, by accumulating sufficient assets, the government can implement the Lucas and Stokey (1983) complete markets allocation.

[5] The only effect is that the risk-free rate is lower in the model with incomplete markets.

1.3 Characterizing the Trade-Off with Cash Flow Betas

Thus far, we have characterized the return betas of the tax and spending claims. We can get further insight on what restrictions risk-free debt imposes on surplus dynamics by studying cash-flow betas for the surplus claim.

Let ε_t denote the shocks to the economy. We use $\varepsilon_t^l = (\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-l+1})$ to describe its history in the past l periods. Define the conditional beta of a generic stream of discounted cash flows Z as:

$$\beta_t^{Z,CF}(h) \equiv -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{j=1}^{h+1} M_{t+1,t+j} Z_{t+j}\right) = 0.$$

We refer to this object as the cash-flow beta for short.

Proposition 2. *When the debt is risk-free,*
(a) the cash flow beta of all future surpluses is zero: $\beta_t^{S,CF}(\infty) = 0$.
(b) If the government chooses a policy for the debt/output ratio that is a function of a finite history of past shocks, $d_t(\varepsilon_t^h)$*, then the cash flow beta of surpluses over the next* $h+1$ *periods is zero:* $\beta_t^{S,CF}(h) = 0$.

Part (a) states that for debt to be risk-free, the beta of the entire discounted surplus stream must be zero. This follows directly from the fact that the debt D_t equals the present discounted value of all future surpluses, and from the fact that risk-free debt imposes that the return beta of debt be zero: $\beta_t^D = 0$. For debt to be risk-free today, the discounted sum of surpluses must not respond to any shock at some future date.

Part (b) derives tighter restrictions on the cash-flow betas of the surplus when the government commits to a debt issuance policy that depends on a finite history of shocks. Note that such debt/output policy implies that the surplus/output ratio depends on the same history of shocks. When the government surplus is only allowed to depend on the shocks in the past h periods, the cash-flow beta of the surplus must be zero over shorter periods for debt to remain risk-free.

For example, if the debt/output process depends only on the current output shock ($h = 1$), then the government can raise more debt in response to a negative output shock. This allows it to run a deficit for one period. However, to keep the debt risk-free, the surplus must turn positive in the next period and fully offset the prior negative surplus (after discounting).

The government can smooth out the adverse shock over more periods only if it adopts a debt issuance policy that depends on a longer history of shocks. The issuance decision is the only source of state-contingency with risk-free debt. There is no independent surplus policy. Surpluses over the next h periods are fully pinned down by the risk-free nature of the debt. In the limit, as the debt policy depends on the entire history of shocks, we end up with the standard restriction for risk-free debt in part (a) of the proposition. Below, we will analyze realistic processes for the debt policy and analyze the quantitative implications for the surplus/output ratio.

2 The Insurance Trade-off in a Benchmark Economy

We characterize the trade-off between insuring debtholders and taxpayers in a canonical macro-finance model in the tradition of Breeden (2005); Lucas (1978); Rubinstein (1974). We reverse-engineer the revenue process T that keeps the debt risk-free. We do so under simple spending and debt policies in this section and more complex policies in the next section.

2.1 Characterizing the Trade-off

In the main text, we consider an economy with permanent output shocks and a homoscedastic stochastic discount factor (SDF):

Assumption 1. *(a) Let Y_t and $y_t = \log Y_t$ denote output and its log. All output shocks are i.i.d. and permanent:*

$$y_{t+1} \quad = \quad \mu + y_t + \sigma\varepsilon_{t+1},$$

where ε_{t+1} denotes the innovation to output growth that is i.i.d. normally distributed with mean zero and standard deviation one.
(b) The log pricing kernel is given by:

$$m_{t,t+1} \quad = \quad -\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}.$$

(c) The government only issues one-period real risk-free debt.

The appendix studies models with transitory output risk and with transitory output risk that is perceived to be permanent. Note that the real one-period risk-free rate in this model is constant and equal to ρ.

To build intuition for the general trade-off between insurance of bondholders and taxpayers, we start by considering the simplest case of constant spending/output and debt/output ratio policies.

Assumption 2. *(a) The government commits to a constant spending/output ratio $x = G_t/Y_t$.*
(b) The government commits to a constant debt/output ratio $d = D_t/Y_t$.

Under Assumption 2, the government budget constraint implies a counter-cyclical process for tax revenue-to-GDP (the tax rate):

$$\frac{T_t}{Y_t} \quad = \quad \frac{G_t}{Y_t} - \frac{D_t}{Y_t} + R^f_{t-1}\frac{D_{t-1}}{Y_{t-1}} = x - d\left(1 - \exp\left\{-(\mu - \rho + \sigma\varepsilon_t)\right\}\right).$$

To perfectly insure the bondholders by keeping the debt risk-free, the government must make the tax revenue claim counter-cyclical. When the growth rate of output is low, tax revenue needs to increase as a fraction of GDP. Tax rates must rise in recessions. The magnitude of the counter-cyclical exposure is increasing in the debt-to-GDP ratio d.

Similarly, the primary surplus/output ratio is counter-cyclical:

$$s_t = \frac{S_t}{Y_t} = \frac{T_t - G_t}{Y_t} = -d\left(1 - \exp\left\{-(\mu - \rho + \sigma\varepsilon_t)\right\}\right). \tag{4}$$

When the growth rate of output exceeds the risk-free rate ($\mu > \rho$), the government runs a primary deficit on average. But whenever there are negative shocks such that $\mu - \rho < -\sigma\varepsilon$, the government must run a primary surplus.

In this benchmark model, the government cannot run persistent deficits. The conditional auto-covariance of the surplus/output ratio is zero: $cov_t(s_t, s_{t-1}) = 0$. When $\sigma \to 0$, the government always runs deficits. But $\mu > \rho$ now implies a violation of the TVC, as we show below. This result is more general. With risk-free debt, the autocorrelation of the surpluses tends to zero as the persistence of the debt/output ratio tends to one.

The restrictions on the surplus and tax processes described above were independent on the SDF model. Next, we turn to valuing the debt as the expected present-discounted value of future surpluses.

Proposition 3. *Under Assumptions 1 and 2, (a) if the transversality condition holds and the primary surplus satisfies (4), the government debt value is the sum of the values of the surplus strips:*

$$D_t = \mathbb{E}_t\left[\sum_{k=1}^{\infty} M_{t,t+k} S_{t+k}\right] = dY_t.$$

(b) Proposition 2 can be simplified to the following measurability constraint:

$$(\mathbb{E}_{t+1} - \mathbb{E}_t)\left[\sum_{k=1}^{\infty} M_{t,t+k} S_{t+k}\right] = 0.$$

This proposition confirms that the (ex-dividend) value of outstanding debt in period t is indeed a constant fraction of output. The proof solves for the price of a claim to a single future surplus realization (a surplus strip), and adding up the surplus strip prices at all horizons. The result in part (a) implies that there is no news about the present discounted value of future surpluses since output is already known at time t.[6]

To see why we cannot simply discount future surpluses at the risk-free rate, even when the

[6] Hansen, Roberds, and Sargent (1991) discuss a version of this condition that uses the risk-free rate when devising an econometric approach to testing the budget constraint: $(\mathbb{E}_{t+1} - \mathbb{E}_t)\left[\sum_{k=1}^{\infty} \exp(-r^f_{t,t+k}) S_{t+k}\right] = 0$. However, this condition is equivalent to the one in the Proposition, only if the risk-free rate exceeds the growth rate of the economy. If not, this equation may fail even when the condition in Proposition 3 holds.

debt is risk-free, consider the valuation equation for debt as a function of surplus/output ratios:

$$D_t = \mathbb{E}_t \left[\sum_{j=0}^{T} M_{t,t+j} Y_{t+j} s_{t+j} \right] + \mathbb{E}_t \left[M_{t,t+T} Y_{t+T} \frac{D_{t+T}}{Y_{t+T}} \right].$$

The debt/output ratio $\frac{D_{t+T}}{Y_{t+T}} = d$ is constant. This term $\mathbb{E}_t \left[M_{t,t+T} Y_{t+T} \right] \to 0$ as $T \to \infty$, i.e., the TVC will hold, even if $\rho < \mu$. A necessary and sufficient condition is that there is enough permanent, priced risk in output: $\gamma\sigma > \mu - \rho + \frac{1}{2}\sigma^2$. Note that $\rho < \mu$ implies a violation of TVC as $\sigma \to 0$. So, it is not the case that the government can always run deficits when $\rho < \mu$, at least not without violating the TVC.[7] The output risk premium matters even when debt is risk-free. The risk-free rate is not the correct discount rate for surpluses even when the debt is risk-free, in the presence of permanent output shocks. The correct TVC is given by:

$$\lim_{T \to \infty} \mathbb{E}_t \left[M_{t,t+T} D_{t+T} \right] = \lim_{T \to \infty} \exp \left\{ T(\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma) \right\} dY_t.$$

This TVC is satisfied if and only if $-\rho + \mu + \frac{1}{2}\sigma^2 - \gamma\sigma < 0$. The textbook condition $\rho < g$ is neither necessary nor sufficient for a TVC violation.

Next, we turn to the main result characterizing the expected return and beta of the tax claim.

Proposition 4. *(a) The ex-dividend values of the spending and revenue claims are given by:*

$$\begin{aligned} P_t^G - G_t &= x \frac{\xi_1}{1 - \xi_1} Y_t, \\ P_t^T - T_t &= \left(d + x \frac{\xi_1}{1 - \xi_1} \right) Y_t, \end{aligned}$$

with $\xi_1 = \exp\left\{ -\rho - \gamma\sigma + \mu + 0.5\sigma^2 \right\}$. *(b) The risk premia and betas on the tax claim and the spending claim satisfy:*

$$\mathbb{E}_t \left[R_{t+1}^T - R_t^f \right] = \frac{x \frac{\xi_1}{1-\xi_1}}{d + x \frac{\xi_1}{1-\xi_1}} \mathbb{E}_t \left[R_{t+1}^G - R_t^f \right], \quad (5)$$

$$\beta^T = \frac{x \frac{\xi_1}{1-\xi_1}}{d + x \frac{\xi_1}{1-\xi_1}} \beta^G < \beta^G. \quad (6)$$

The constant ξ_1 is the price/dividend ratio of a one-period output strip, a claim to GDP next year. The expected return on this output strip is given by $E_t \left[R_{t+1}^Y \right] = \frac{\exp(\mu + 0.5\sigma^2)}{\exp(-\rho - \gamma\sigma + \mu + 0.5\sigma^2)} = \exp(\rho + \gamma\sigma)$. Hence, the (log of the multiplicative) output risk premium is constant and equal to $\gamma\sigma$. Since spending is a constant fraction of output, the risk premium on the spending claim equals that of

[7] See Bohn (1995) for an early reference on why discounting at the risk-free may fail. However, Bohn (1995) refers to this case as one in which the government runs persistent deficits, while the deficits really are uncorrelated over time.

the output claim: $\mathbb{E}[R^G - R^f] = \mathbb{E}[R^Y - R^f]$. The beta of the spending claim equals the beta of the output claim: $\beta^G = \beta^Y > 0$.

The investor in government debt is long in a tax revenue claim and short in a spending claim. To make the debt risk-free, as long as the debt/output ratio d is positive, we need to render the government tax revenue process safer than the spending process. A positive d implies the fraction $\frac{x\frac{\xi_1}{1-\xi_1}}{d+x\frac{\xi_1}{1-\xi_1}}$ is between 0 and 1, which requires the return on the tax claim to be less risky than the return on the output claim: $0 < \beta^T < \beta^Y$. When output falls, tax revenues must fall by less. The tax rate increases. In other words, there is no scope to insure taxpayers. As the debt/output ratio d increases, the government needs to make the tax revenue increasingly safe. The tax claim is really a portfolio of a claim to government spending and risk-free debt. The larger the debt/output ratio d, the safer the tax claim needs to be. As the debt/output ratio approaches infinity, the beta of the tax claim tends to 0.

2.2 Quantifying the Trade-off

Panel A of Table 1 proposes a calibration of the model that matches basic features of post-war U.S. data. We set γ to 1. This parameter measures the maximum Sharpe ratio in the economy. A long asset pricing literature suggests that this is a reasonable value given high average excess returns on a broad set of risky assets. The standard deviation of output is set to $\sigma = 0.05$. The growth rate of real GDP is set to its observed value: $\mu = 3.1\%$. The real risk-free rate ρ is set to 2%. Spending accounts for 10% of GDP in post-war data: $x = 0.10$. Note that in this calibration features a risk-free rate below the growth rate of output. However, per our discussion above, the TVC is satisfied because $-\rho + \mu + \frac{1}{2}\sigma^2 - \gamma\sigma = \log(\xi_1) = -0.0418 < 0$. The government cannot simply roll over the debt. The surpluses need to satisfy tight restrictions.

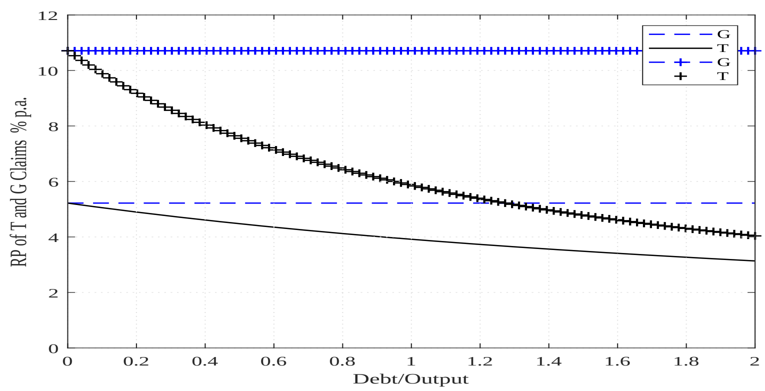
Figure 1 plots the risk premia on the tax and the spending claim as we vary the debt/output ratio d. The risk premium on the spending claim is 5% per annum. This is also the output risk premium, which we can think of as an unlevered equity premium. By Corollary 4, the risk premium on the tax claim is given by (5). The risk premium on the tax claim is 5% when $d = 0$. It falls to 4% when $d = 1$, and close to 3% when $d = 2$. As the government becomes more levered, the tax claims needs to become safer for debt to remain risk-free. The scope for taxpayer insurance disappears. This trade-off steepens when we increase the maximum Sharpe ratio γ from 1 to 2. When $\gamma = 2$, the risk premium on the spending claim is 10% per annum. The risk premium on the tax claim falls to 6% when $d = 1$ and close to 4% when $d = 2$.

Table 1: Benchmark Calibration

Panel A: Preferences and Output Dynamics		
γ	1	maximum annual Sharpe ratio
ρ	2.0%	real risk-free rate
μ	3.1%	mean of growth rate of output
σ	5.0%	std. of growth rate of output
Panel B: Debt/Output Ratio Dynamics		
λ	$1.94 \times \sigma$	sensitivity of debt/output to output innovations
$d = \exp\{\phi_0/(1-\phi_1-\phi_2)\}$	0.43	mean of debt/output
ϕ_1	1.40	AR(1) coeff of debt/output
ϕ_2	-0.48	AR(2) coeff of debt/output
Panel C: Government Spending/Output Ratio Dynamics		
β^g	$1.53 \times \sigma$	sensitivity of spending/output to output innovations
φ_1^g	0.88	AR(1) coeff of spending/output
$x = \exp\left\{\varphi_0^g/(1-\varphi_1^g)\right\}$	0.10	mean of govt. spending/output

Figure 1: Risk Premium of T and G Claims with $\gamma = 1, 2$

The figure plots the implied risk premium of the T and G claims when the debt/output ratio and spending/output ratio are constant. The figure plots two values for the maximum Sharpe ratio γ of 1 ($-$) and γ of 2 ($+$). The other parameters are given in Table 1.



3 Model with State-Contingent Debt/Output

The previous section showed that when the debt/output ratio is constant, there is no scope for insuring taxpayers at any horizon in the presence of permanent output shocks. Next, we allow the government to introduce state-contingent variation in the debt/output ratio. This will create limited opportunities for the government to temporarily insure taxpayers over short horizons.

3.1 Characterizing the Trade-Off with Counter-cyclical Debt/Output

We allow the government to vary the debt/output ratio counter-cyclically.

Assumption 3. *The government commits to a policy for the debt/output ratio $d_t = D_t/Y_t$ given by:*

$$\log d_t = \sum_{p=1}^{P} \phi_p \log d_{t-p} + \phi_0 - \lambda \varepsilon_t - \frac{1}{2}\lambda^2,$$

where $\lambda > 0$ so that the debt-output ratio increases in response to a negative output shock ε_t.

The results in Section 1 still apply and are a straightforward generalization of the results from the simple benchmark model of section 2. The value of the spending is unchanged and the value of the revenue claim now depends on the time-varying debt/output ratio d_t:

$$P_t^G - G_t = x\frac{\xi_1}{1-\xi_1}Y_t, \qquad P_t^T - T_t = \left(d_t + x\frac{\xi_1}{1-\xi_1}\right)Y_t.$$

The tax claim's conditional beta satisfies:

$$\beta_t^T = \frac{x\frac{\xi_1}{1-\xi_1}}{d_t + x\frac{\xi_1}{1-\xi_1}}\beta_t^G.$$

Can the government systematically issue more risk-free debt, instead of raising taxes, when the economy is hit by a permanent, adverse shock, in order to break the restriction on insurance of taxpayers? We consider two special cases for the debt/output dynamics.

Case 1: AR(1) Assume that the debt/output ratio evolves according to an $AR(1)$-process:

$$\log d_t = \phi_0 + \phi_1 \log d_{t-1} - \lambda \varepsilon_t - \frac{1}{2}\lambda^2.$$

There are two sub-cases. First, when $0 < \phi_1 < 1$, the debt/output process is mean-reverting. Second, when $\phi_1 = 1$ and $\phi_0 = 0$, the debt/output process is a martingale. In both cases, a positive λ means that the debt/output ratio increases when the shock ε_t is negative, implying a counter-cyclical debt policy.

First, we need to make sure the transversality (TVC) is satisfied. How persistent can debt be without violating TVC?

Proposition 5. *Apply Assumptions 1 and 3 with $P = 1$.*
(a) when $0 < \phi_1 < 1$, the TVC condition is satisfied if and only if:

$$\log(\xi_1) = -\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) < 0.$$

(b) When $\phi = 1$ and $\phi_0 = 0$, then the TVC condition is satisfied if and only if:

$$\log(\xi_1) + \lambda(\gamma - \sigma) = -\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) + \lambda(\gamma - \sigma) < 0.$$

For the case of $0 < \phi_1 < 1$, the TVC is satisfied whenever the price-dividend ratio of a claim to next period's output is less than one. That is, when investors are willing to pay less than Y_t today for a claim to Y_{t+1}. This requires the discount rate to exceed the growth rate of GDP (modulo a Jensen adjustment). This condition can be satisfied even when $\rho < \mu$, as long as the risk premium $\gamma\sigma$ is large enough.

For the case where $\phi_1 = 1$, the same condition ensures that the TVC is satisfied when the government does not pursue counter-cyclical stabilization ($\lambda = 0$). If the government does pursue counter-cyclical stabilization ($\lambda > 0$), then the TVC is only satisfied if

$$\gamma\sigma - \lambda(\gamma - \sigma) > -\rho + \mu + \frac{1}{2}\sigma^2 \Leftrightarrow \lambda < \frac{\rho + \gamma\sigma - \mu - \frac{1}{2}\sigma^2}{\gamma - \sigma}.$$

The left-hand side of the first inequality is now lower than before when the Sharpe ratio of the economy exceeds the volatility of output ($\gamma > \sigma$). When debt issuance is sufficiently counter-cyclical, $\lambda > \sigma$, the expression on the left-hand side is decreasing in the economy's maximum Sharpe ratio γ. For high enough γ, the TVC is violated. Intuitively, when investors are risk averse enough, the insurance provided by the counter-cyclical debt issuance policy is so valuable that the price of a claim to the debt outstanding in the distant future $d_{t+T}Y_{t+T}$ fails to converge to zero. This claim is a terrific hedge. This is the first important insight contributed by asset pricing theory. If output is subject to permanent, priced risk and we want to rule out arbitrage opportunities then there have to be limits to the government's ability to pursue counter-cyclical debt issuance. This bound on λ is shown in the second inequality. When the government exceeds this bound, it has granted itself an arbitrage opportunity.

Case 2: AR(2) As we show below, a better description of the debt/output ratio is the data is an AR(2) process:

$$\log d_t = \phi_0 + \phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} - \lambda\varepsilon_t - \frac{1}{2}\lambda^2. \tag{7}$$

When the roots of the characteristic equation $1 - \phi_1 z - \phi_2 z^2 = 0$ lie outside the unit circle, the debt/output process is mean-reverting. The result of part (a) of Proposition 5 applies. If one or both roots are smaller than one, the result in part (b) of Proposition 5 applies.

Response of the Surplus to Adverse Shock We can compute the impulse-response functions (IRF) of the surpluses with respect to an output shock in closed form when the government issues risk-free debt. These moments are particularly informative because these do not depend on the properties of the SDF.

We start from the following expressions for the surplus/output ratios in period $t+j$ for $j \geq 1$:

$$s_{t+j} = \frac{S_{t+j}}{Y_{t+j}} \quad = \quad d_{t+j-1} \exp(\rho - \mu - \sigma \varepsilon_{t+j}) - d_{t+j}.$$

If we assume that the risk-free rate equals the growth rate of the economy ($\mu = \rho$), we obtain closed-form expression for the IRF of the surplus with respect to an output shock.

Proposition 6. *Under If Assumptions 1 and 3 hold, the debt is risk-free, the TVC is satisfied, and $\rho = \mu$, (a) when the debt/output ratio follows an $AR(1)$ process, the IRF of the surplus/output ratio is given by:*

$$\begin{aligned}
\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma)d, \text{ for } j = 1, \\
&= \lambda \phi_1^{j-1}(\phi_1 - 1)d, \text{ for } j > 1.
\end{aligned}$$

(b) when the debt/output ratio follows an $AR(2)$ process, the IRF of the surplus output ratio is given by:

$$\begin{aligned}
\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma)d, \text{ for } j = 1, \\
&= \lambda(\phi_1 - 1)d, \text{ for } j = 2, \\
&= \lambda(\psi_{j-1} - \psi_{j-2})d, \text{ for } j > 2.
\end{aligned}$$

where $\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2}, j > 2; \psi_2 = \phi_2 + \phi_1 \psi_1; \psi_1 = \phi_1$.

(c) When the debt/output ratio follows an AR(3) process, the IRF of the surplus output ratio is given by:

$$\begin{aligned}
\frac{\partial \frac{S_{t+j}}{Y_t}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma)d, \text{ for } j = 1, \\
&= \lambda(\phi_1 - 1)d, \text{ for } j = 2, \\
&= \lambda(\psi_2 - \psi_1)d, \text{ for } j = 3, \\
&= \lambda(\psi_{j-1} - \psi_{j-2})d, \text{ for } j > 3.
\end{aligned}$$

where $\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2} + \phi_3 \psi_{j-3}, j > 3; \psi_3 = \phi_3 + \phi_2 \psi_1 + \phi_1 \psi_2; \psi_2 = \phi_2 + \phi_1 \psi_1; \psi_1 = \phi_1$.

For an $AR(1)$, in the empirically relevant case where $\lambda > \sigma$, the initial response of the surplus is positive. That is, a negative shock to output is countered with a large enough government debt issuance that the surplus in the initial period can be negative, without jeopardizing the risk-free nature of the debt. However, the surplus response must be negative starting in the second year. That is, the deficit from year 1 reverses to a surplus already in year 2. Surpluses remain in the years that follow. As the persistence of the debt/output process ϕ_1 increases, the IRF converges to zero after year 1.

For an $AR(2)$, by choosing $\phi_1 > 1$, the government can run a deficit in the year of the shock as well as in the following year. In year 3, the IRF equals $\lambda(\psi_2 - \psi_1) = \lambda(\phi_2 + \phi_1(\phi_1 - 1))$. This expression can be positive or negative depending on parameter values but is smaller than the response in year 2. In other words, the government's ability to run a third year of deficits in response to the negative output shock is either limited or gone. The IRF flips sign in year 3 or 4. The government must revert to running surpluses as the ACFs decline: $\psi_{j-1} < \psi_{j-2}$.

With higher-order, highly persistent $AR(p)$ models for debt/output, the government is able to run deficits for longer before a reversal. For example, for an AR(3), there is an additional year of deficits possible while keeping debt risk-free. These deficits must be made up by several years of surpluses afterwards. The surplus dynamics can display more pronounced hump-shaped IRFs. However, as shown below, there is no evidence of higher-order $AR(p)$ dynamics (i.e., $p > 2$) in the observed US debt/output process.

Persistence of the Surplus The auto-covariance of the surplus/output ratio is defined as follows:

$$cov_t(s_{t+1}, s_{t+j}) = \mathbb{E}_t[s_{t+1}s_{t+j}] - \mathbb{E}_t[s_{t+1}]\mathbb{E}_t[s_{t+j}].$$

Closed-form expressions for the auto-covariances of the surplus/output ratio are given in section D of the Appendix. In the case of an $AR(1)$, we show that the conditional autocovariance declines to zero as we increase the persistence of the debt/output process. $\lim_{\phi_1 \to 1} cov_t(s_{t+1}, s_{t+j}) = 0$. This is not surprising. Recall that in the case of a constant debt/output ratio considered in the previous section, surplus/output ratios were uncorrelated at all horizons.

Predictability of the Surplus When debt is risk-free, an increase in debt today needs to be followed by higher future surpluses. For the realistic case of an AR(2) process for debt/output, those surpluses must begin 2-3 years after the initial increase in debt. In the model, the debt/output ratio should be a strong predictor of future surpluses. We study this predictability relationship in the model and contrast it to that in the data.

In sum, insisting on debt to be risk-free imposes tight constraints on how much and how long the government can run deficits in response to an adverse shock, on the persistence of the surplus/output ratio, and on the predictability of future surplus/output ratios by the current debt/output ratio.

3.2 Quantifying the Trade-Off with Counter-cyclical Debt/Output

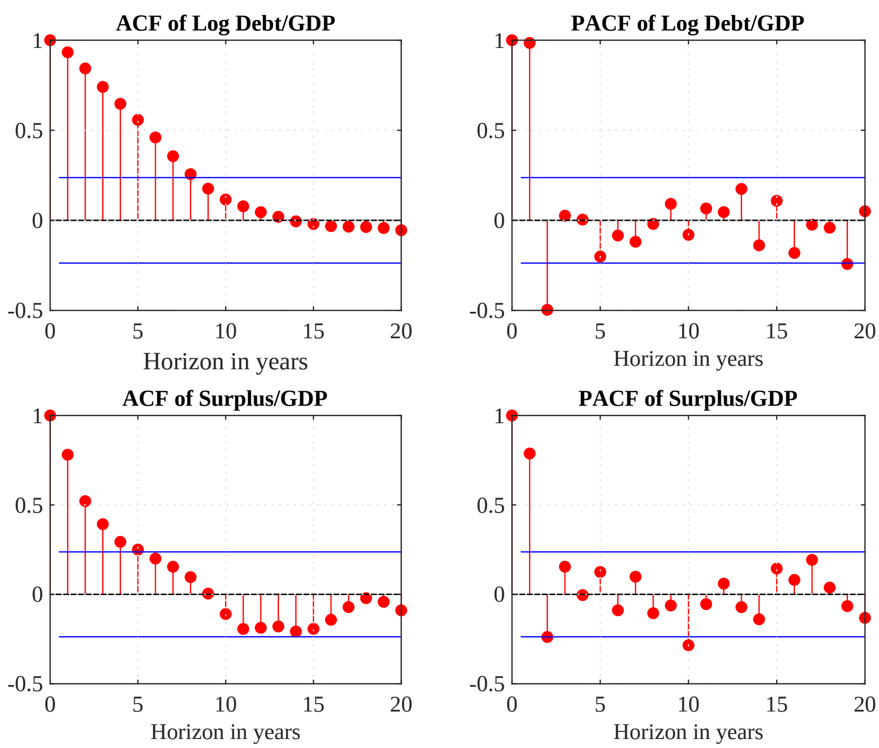
Persistence of Debt/Output and Surplus/Output in the Data The top left panel of Figure 2 plots the sample autocorrelation function (ACF) of the log government debt/output ratio as a function of the number of annual lags. The top right panel plots the partial autocorrelation function (PACF).

They are estimated on the post-war U.S. sample (1947–2019). The PACF function indicates that an AR(2) process fits the data well. Lags beyond two years in the PACF are not statistically different from zero. The point estimates for ϕ_1 and ϕ_2 are 1.40 and -0.48, respectively. Both roots lie outside the unit circle (1.66 and 1.25), so that the debt/output process is stationary. While the AR(2) is our preferred specification, if we were to fit an AR(1), the point estimate for ϕ_1 would be 0.986.

We set ϕ_0 to match the unconditional mean of the debt/output ratio of 0.43. Finally, we set $\lambda = 1.953 \times \sigma$ equal to match the slope coefficient in a regression of the debt/output ratio innovations on GDP growth in the post-war U.S. sample. A one percentage point increase in GDP growth lowers the debt/output ratio by 1.95 percentage points. We report the calibration of the spending/output ratio in Panel C of Table 1.

Figure 2: Autocorrelation in Data

The figure plots the sample autocorrelation of the U.S. log government debt/output ratio, the U.S. government surplus/output ratio, the tax/output ratio and the spending/output ratio against GDP. Sample is 1947—2019. Annual data.



Given our values of $\sigma = 0.05$, $\gamma = 1$, and $\lambda = 1.96\sigma = 0.098$, we have $\gamma\sigma - \lambda(\gamma - \sigma) < 0$. If debt/output were non-stationary (have roots inside the unit circle), then this much counter-cyclicality would result in a violation of the TVC condition. The coefficient λ would need to remain below 0.85σ, which is only half of its empirical value, for the TVC to be satisfied in this case. Once we exceed this upper bound, the value of outstanding debt explodes. To be clear, the data suggest that the debt/output ratio is stationary, in which case the TVC is satisfied irrespective of the value for λ. The parameter restriction in part (a) of Proposition 5 is satisfied. This is the case despite the risk-free interest rate being below the growth rate of output, because the risk premium $\gamma\sigma$ is large enough.

The second row of Figure 2 plots the sample ACF and PACF for the primary surplus/output ratio in the data. The dynamics of surplus/output are well described by an AR(1). The surplus is quite persistent, with an $AR(1)$ coefficient around 0.81.

We now show that the risk-free debt model cannot simultaneously match the high persistence of the debt/output ratio and that of the surplus/output ratio.

IRF for Surplus in Model The top left panel of Figure 3 plots the response of the surplus/output ratio to a negative shock to output, when debt/output follows an AR(1) process. The graph assumes that $\rho = \mu$, as in Proposition 6. Each line corresponds to a different autocorrelation ϕ_1 ranging from 0.25 to 0.99. The top right panel plots the response of the debt/output ratio. Upon impact, the debt/output ratio increases by about 4% from its mean. After that, the rate of mean-reversion is governed by ϕ_1. In the least persistent case ($\phi = 0.25$), the government runs a large surplus after the initial period deficit to bring the debt back down quickly. In the most persistent debt case ($\phi = 0.99$), the initial deficit is followed by a reversal in the next period as the surplus jumps to just above its long-run value of $\bar{s} = 0$ and then slowly converges to $\bar{s}$ from above.[8] In sum, when the debt/output ratio follows an $AR(1)$ and the debt is risk-free, there can be no S-shaped response of the surplus/output ratio to the output shock.

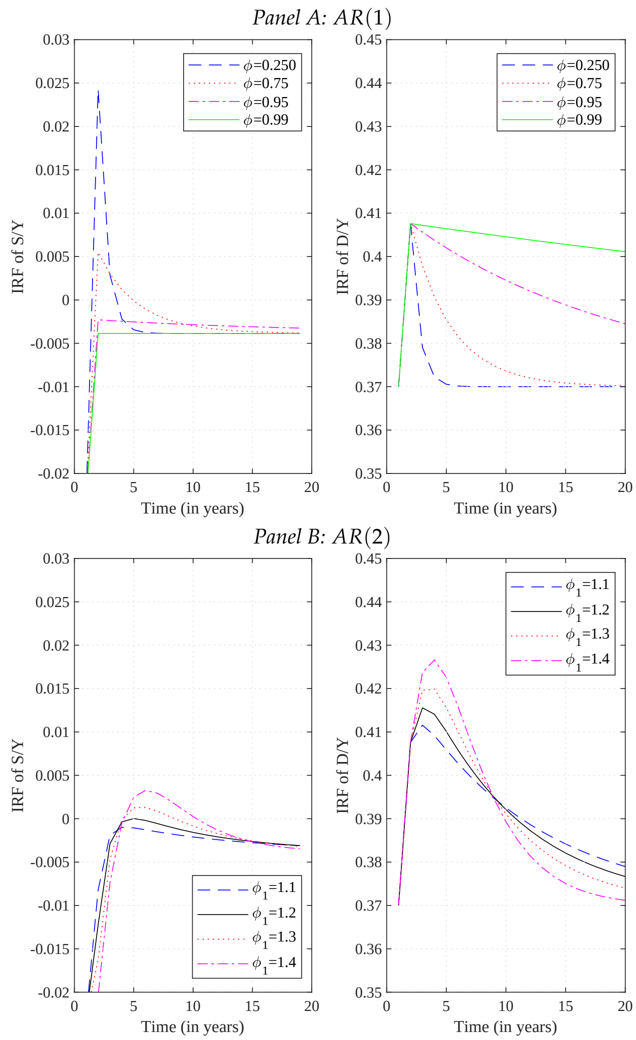
Panel B of Figure 3 plots the IRF when debt/ouput follows an AR(2), our preferred empirical specification. We vary ϕ_1 from 1.1 to 1.4 and choose ϕ_2 to match the first-order autocorrelation of debt/output. With $\phi_1 = 1.1$, the IRF looks similar to the $AR(1)$ case with ϕ_1 close to 1. However, with $\phi_1 = 1.4$ and $\phi_2 = -0.48$, the point estimates from the data, the IRF for the surplus/output ratio displays a hump-shaped pattern. Consistent with the results in Proposition 6, a state-contingent and persistent debt issuance policy enables the government to delay the fiscal adjustment. The deficit/output ratio in the year of the shock is followed by an even larger deficit in year 2. However, the deficit must shrink dramatically in year 3 and turn into a surplus starting in year 4 and beyond. The surplus eventually converges back to $\bar{s} = 0$ from above. Keeping debt risk-free still imposes severe restrictions on the size of the S-shaped surplus dynamics. Running sizeable deficits for more than two years is incompatible with risk-free debt.

Bohn (1998); Canzoneri, Cumby, and Diba (2001); Cochrane (2019, 2020) find evidence of S-shaped dynamics in the U.S. surplus/output ratios. These authors argue that such surplus dynamics are consistent with budget balance. Our results show that the S-shaped surplus dynamics in the data violate the risk-free debt condition. Governments cannot defer running a surplus for more than 2 years after output declines, if they want to keep the debt risk-free.

[8]Note that when $\rho < \mu$, the government can run a small steady-state deficit $\bar{s} = -d\,(1 - \exp(\rho - \mu)) < 0$.

Figure 3: IRF of Surplus/Output and Debt/Output in Model

The figure plots the IRF of S/Y and D/Y for an $AR(1)$ (top panel) and an $AR(2)$ (bottom panel). In Panel B, ϕ_2 is chosen to match the first-order autocorrelation. The graph assumes that $\rho = \mu$. The other parameters are given in Table 1.

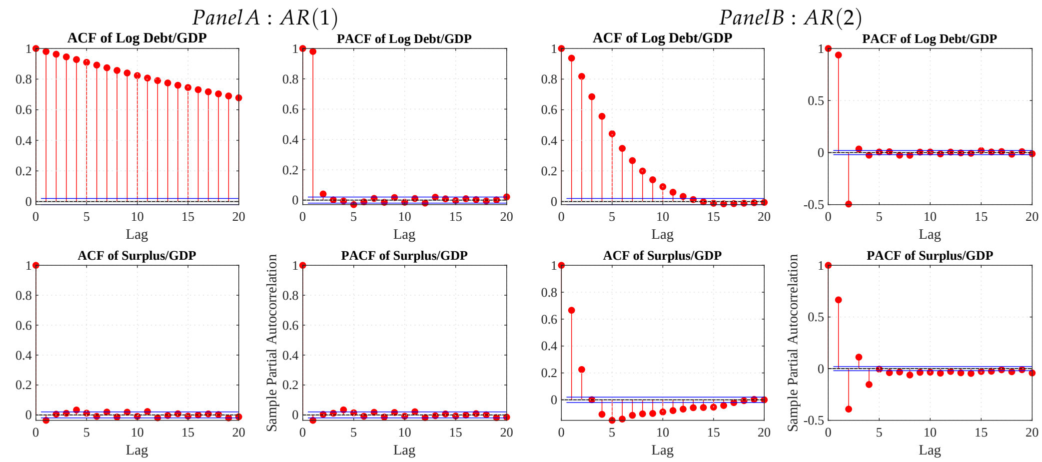


Persistence of Surplus/Output in Model Figure 4 plots the ACF and PACF of the debt/output and surplus/output ratios implied by the model of risk-free debt. Panel A is for the case where debt/output follows an AR(1) with the estimated persistence $\phi_1 = 0.985$. Panel B is for the case where debt/output follows an AR(2) with the estimated coefficients $\phi_1 = 1.4$ and $\phi_2 = -0.48$. The ACF and PACF for debt/output match the data by construction. As argued above, the AR(2) fits the ACF and PACF of the observed debt/output ratio the closest.

The key observation is that insisting on risk-free debt produces counter-factual ACF and PACF for the surplus/output ratio. In particular, the model cannot replicate the strong autocorrelation in surplus/output observed in the data. In the case of the AR(1), the ACF is zero from horizon 1 onwards. In the case of the AR(2), the ACF converges much too quickly to zero, compared to the observed one plotted in the second row of Figure 2. The ACF is no longer different from zero past two years, while in the data the ACF remains significantly positive for five years. Furthermore, the model produces a PACF(2) coefficient of -0.5, which is larger in absolute value than the one estimated in the data.

Figure 4: Autocorrelation in Model

Panel A plots the ACF and PACF of S/Y and D/Y for an $AR(1)$ with parameters $\phi_1 = 0.985$ and $\phi_2 = 0$. Panel B plots the ACF and PACF of S/Y and D/Y for an $AR(2)$. The parameters are listed in Table 1.



Predictability of Surplus/Output Table 2 reports the results for the predictability regressions;

$$s_{t+j} = a_j + b_j d_t + e_{t+j}$$
$$s_{t+j} = a_j + b_j d_t + c_j s_t + e_{t+j}$$

for horizons $j = 1$ through $j = 5$. The results for the data are in Panel A while the results for the AR(2) model are in Panel B.

In the model, the debt/GDP ratio has strong forecasting power for future surplus/output ratios, with a positive sign, even when we control for lagged surplus/output ratios. The R^2 is 25%-26% at the five-year horizon. At horizons up to 2 years, the lagged surplus/output ratio also forecasts future surplus/output ratios with a positive sign. After 2 years, the lagged surplus/output ratio has no incremental forecasting power over and above that of the lagged debt/output ratio.

In contrast, in the data, the lagged debt/output ratio has no predictive power for future surpluses. Lagged surpluses are much better predictors and substantially increase the regression R^2.

The model is unable to generate the predictability patterns in the data. It generates too much predictability by debt/output and too little predictability by the lagged surplus/output ratio. The latter is not surprising in light of our earlier finding that a model with risk-free debt generates a much faster decay in the ACF of surplus/output than what we see in the data.

Table 2: Forecasting Surplus/Output Ratios

Panel A is for post-war annual U.S. data (1947-2019). Panel B is for a 10,000 period simulation of the AR(2) model with parameters given in Table 1. We forecast the primary surplus/output ratios two to five years hence with the current debt/output ratio and the current surplus/output ratio. The table reports the regression coefficients and R^2 statistics for $s_{t+j} = a_j + b_j d_t + c_j s_t + e_{t+j}$.

Horizon j	1	2	3	4	5
Panel A: U.S. Data					
Specification 1					
b_j	-0.031	-0.0099	0.013	0.023	0.028
$[s.e.]$	[0.023]	[0.025]	[0.03]	[0.031]	[0.03]
R^2	0.043	0.0041	0.006	0.018	0.024
Specification 2					
b_j	0.0085	0.018	0.036	0.042	0.044
$[s.e.]$	[0.01]	[0.015]	[0.019]	[0.02]	[0.021]
c_j	0.81	0.57	0.47	0.37	0.33
$[s.e.]$	[0.087]	[0.13]	[0.12]	[0.11]	[0.10]
R^2	0.64	0.30	0.21	0.15	0.13
Panel B: AR(2) Model with Risk-free Debt					
Specification 1					
b_j	0.0629	0.117	0.132	0.127	0.114
R^2	0.0781	0.271	0.342	0.316	0.254
Specification 2					
b_j	0.0701	0.12	0.132	0.126	0.112
c_j	0.695	0.265	0.045	-0.055	-0.11
R^2	0.560	0.342	0.345	0.319	0.266

Covariance of Tax and Spending with GDP Finally, we compare the covariance of tax revenue/output with output growth in model and data. Given a process for spending/output, the surplus/ouput ratio implies a tax revenue/output ratio from the government's budget constraint. To make the model's implications for tax revenues as comparable to the data as possible, we posit a realistic process for spending/output. Specifically, we assume that the government commits to a policy for the spending/output ratio $x_t = G_t/Y_t$ given by:

$$\log x_t = \varphi_0^g + \varphi_1^g \log x_{t-1} - b_g \varepsilon_t - \frac{1}{2} b_g^2. \tag{8}$$

When $b_g > 0$, the spending/output ratio rises in response to a negative output shock. We estimate $(\varphi_0^g, \varphi_0^g, \beta^g)$ from the post-war U.S. data. The parameter estimates are reported in Panel C of Table 1. Spending/output is counter-cyclical in the data. A 1% point decline in output coincides with a 1.53% point increase in the spending/output ratio. The persistence of spending/output matches that in the data with an AR(1) coefficient of 0.88. With this spending process in hand, we compute the model-implied tax revenue/output.

The bottom right panel of Figure 5 plots the covariance of tax revenue/output with output growth divided by the variance of output growth, the tax revenue beta, in the model. The beta is estimated from a 10,000 period simulation of the $AR(2)$-model for the debt/output ratio. It calculates the beta over horizons ranging from 1 to 50 years. The bottom left panel plots the government spending betas. The main result is that the tax betas drop below the spending betas at longer horizons, to ensure that the debt is risk-free.

This property of tax revenues is counterfactual. In post-war U.S. data, the tax revenue beta is always above the spending beta at each horizon, as shown in the top right panel of Figure 5. The tax beta converges to 1 from above at horizons around 10 years. In the data, tax revenues are too risky at longer horizons for the debt to be risk-free.

The tax and spending betas shown here are covariances with GDP growth, in contrast with the return betas. Cash flow betas have the advantage that they do not depend on the SDF.

4 The Riskiness of the Surplus Claim Across Maturities

How much smoothing can the government achieve by issuing more debt in response to bad shocks? It depends on the horizon. This section characterizes the trade-off at different frequencies using the cash-flow betas of the surplus and tax revenues. In the presence of permanent shocks, the government can only insure taxpayers over a limited period of time. This period can be extended by imputing more persistence (higher-order dynamics) into the debt/output process.

When debt is risk-free, the cash-flow beta of the surplus for any given horizon h is fully determined by the dynamics of the debt/output ratio. We consider the case where debt/ouput follows an AR(2) with negative exposure to the current output shock. This is the process favored by the historical data.

Proposition 7. *Under Assumptions 1 and 3, when debt is risk-free and debt/output follows (7), the cash-flow beta of the discounted surpluses over h periods is given by beta of future debt issuance h periods from now:*

$$\begin{aligned}
\beta_t^{S,CF}(h) &= -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{j=1}^{h} M_{t+1,t+j}S_{t+j}\right) \\
&= cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)M_{t+1,t+h}D_{t+h}\right) \\
&= E_t[M_{t+1}]E_t[M_{t+1,t+h}d_{t+h}Y_{t+h}](\exp\{\gamma(\psi_{h-1}\lambda - \sigma)\} - 1). \\
sign\left(\beta_t^{S,CF}(h)\right) &= sign\left(\gamma(\psi_{h-1}\lambda - \sigma)\right)
\end{aligned}$$

where $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}, j > 2; \psi_2 = \phi_2 + \phi_1\psi_1; \psi_1 = \phi_1; \psi_0 = 1.$

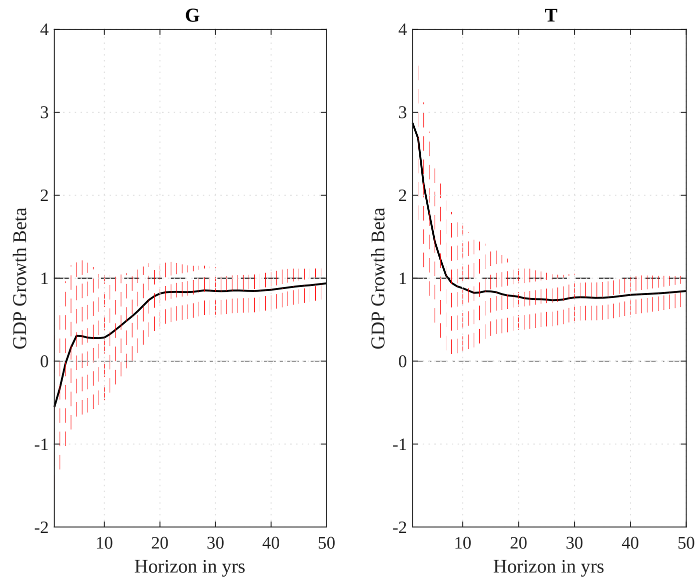
The risk properties of the government surpluses over a given horizon are completely determined by riskiness of the debt issuance process, as long as the debt is risk-free. The cash-flow beta of the surplus at various horizons does not depend on the spending and tax revenue dynamics.

Analogously, we define the cash-flow beta of discounted government spending and of tax revenues.

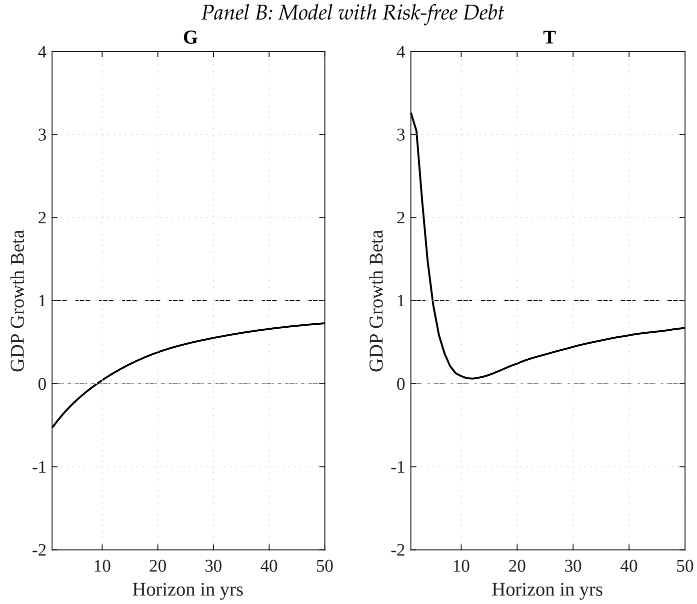
Figure 5: GDP Growth Betas of U.S. Tax Revenue and Spending

Betas in regression of log U.S. spending G growth and log tax revenue T growth over horizon j on GDP growth over horizon j. Panel A: Data. Sample is 1947—2019. Annual data. The plot shows 2 standard error bands. s.e. generated by 30,000 bootstrapped samples by drawing jointly with replacement from the 4×1 vector of innovations in the AR models for $(\log d_t, x_t, \log \tau_t, \log g_t)$. We draw vectors of innovations. Panel B: Model-simulated data with Risk-free Debt. Benchmark calibration in Table 1.

Panel A: Data



Panel B: Model with Risk-free Debt



Corollary 2. *Under Assumptions 1 and 3, and when debt is risk-free and debt/output follows an $AR(2)$, the cash flow beta of spending and taxes have to satisfy the following restrictions:*

$$\beta_t^{G,CF}(h) \quad = \quad -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{j=1}^{h} M_{t+1,t+j} G_{t+j}\right)$$

$$
\begin{aligned}
&= E_t[M_{t+1}]E_t[M_{t+1,t+h}x_{t+h}Y_{t+h}](\exp\left\{\gamma(\varphi_g^{h-1}b_g-\sigma)\right\}-1).\\
\beta_t^{T,CF}(h) &= -cov_t\left(M_{t+1},(\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{j=1}^{h}M_{t+1,t+j}T_{t+j}\right)\\
&= E_t[M_{t+1}]E_t[M_{t+1,t+h}d_{t+h}Y_{t+h}](\exp\{\gamma(\psi_{h-1}\lambda-\sigma)\}-1)\\
&+ E_t[M_{t+1}]E_t[M_{t+1,t+h}x_{t+h}Y_{t+h}](\exp\left\{\gamma(\varphi_g^{h-1}b_g-\sigma)\right\}-1).
\end{aligned}
$$

The properties of the $\beta_t^{G,CF}(h)$ depend on the persistence and cyclicality of the exogenous spending/GDP process in equation (8). The properties of $\beta_t^{T,CF}(h)$ depend on the risk properties of both the debt claim and the spending claim.

4.1 Constant debt/output

When debt/output is constant, $\lambda = 0$, and proposition 7 simplifies to:

$$\beta_t^{S,CF}(h) = \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+h}dY_{t+h}](\exp\{-\gamma\sigma\}-1).$$

The cash-flow beta of the surplus is negative at all horizons since $\gamma\sigma > 1$. In bad times, the surplus/output ratio goes up. When spending/output is constant (or also goes up), tax revenues/output must go up. The government cannot insure taxpayers against adverse output shocks. Rather, the taxpayers insure the bondholders.

The top panel of Figure 6 plots the cash-flow beta of the cumulative discounted surplus, $\beta_t^{S,CF}(h)$, divided by $\mathbb{E}_t[M_{t+1}]$. This ratio can be interpreted as the risk premium on a claim to cumulative surpluses over the next h periods. The negative risk premium indicates that surpluses are a hedge. Since taxpayers are short the surplus claim, their tax-minus-transfer liability is risky.

This risk premium on cumulative surpluses is inversely related to the risk premium on a debt strip, which is positive and constant at all horizons. This debt strip risk premium is plotted in the right panel of Figure 6. When the debt/output ratio is constant, the debt strip has the same risk as the output strip at all horizons. To offset the output risk in debt, the risk premium on the surplus has to be negative.

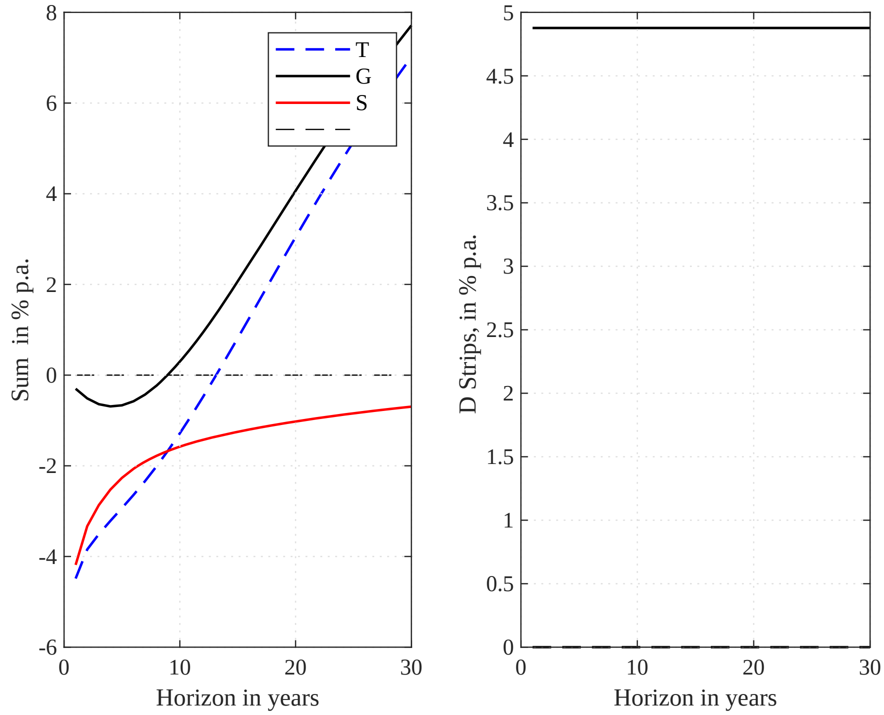
The middle plots the risk premium on each individual surplus strip,

$$-cov_t\left(M_{t+1},(\mathbb{E}_{t+1}-E_t)M_{t+1,t+h}S_{t+h}\right)/\mathbb{E}_t[M_{t+1}].$$

When debt/output is constant and there is not possibility ro raise the debt in response to an adverse shock, the surplus/output ratio must rise on impact. This makes the one-period surplus claim a hedge, with a negative risk premium of -3.75%. The year-2 surplus claim in contrast earns a small positive risk premium, reflecting the underlying output risk. The cumulative risk premium

Figure 6: Risk Premia Across Horizons – Constant Debt/Output

The figure plots the risk premium of cumulative discounted cash flows (left panel) against the horizon. The right panel plots the risk premium on the debt strips. The parameters are given in Table 1, except that here $\lambda = 0$.



in the left panel is the sum of the individual strip risk premia in the right panel.

As $h \to \infty$, the sum of discounted surpluses converges to the current value of debt D_t. Insisting on risk-free debt ($\beta_t^D = 0$) implies that $\beta_t^{S,CF}(h) \to 0$. The red line in the left panel converges to zero from below for large h.

The solid black line plot the cash-flow beta of the spending claim scaled by $\mathbb{E}_t[M_{t+1}]$. It is the risk premium on a claim to cumulative spending. Since the spending/output dynamics are exogenously given, the spending beta does not depend on the debt policy. The countercyclical nature of spending/output makes the risk premium negative at short horizons. At longer horizons, the spending risk premium turns positive reflecting the output risk in the spending claim, since the spending/output ratio is stationary.

The extent of taxpayer insurance is captured by $\beta_t^{T,CF}(h)$. The blue dashed line in Figure 6 plots $\beta_t^{T,CF}(h)$ scaled by $\mathbb{E}_t[M_{t+1}]$. It is the risk premium on a claim to the next h periods of tax revenue. When this risk premium is negative, taxpayers are providing insurance to the government rather than receiving insurance. The risk premium is negative until year 13 for our parameters. It then turns positive. The positive risk premium on longer-dated tax strips reflects cointegration between tax revenues and output and a positive risk premium for output risk.

Note that the tax beta $\beta_t^{T,CF}(h)$ in the left panel is below the spending beta $\beta_t^{G,CF}(h)$ at all horizons. As $h \to \infty$, these cash-flow betas converge to the return betas β_t^T and β_t^G. As we discussed in Corollary 1, $\beta_t^T < \beta_t^G$ was the condition to keep the debt risk-free.

4.2 AR(1) for debt/output

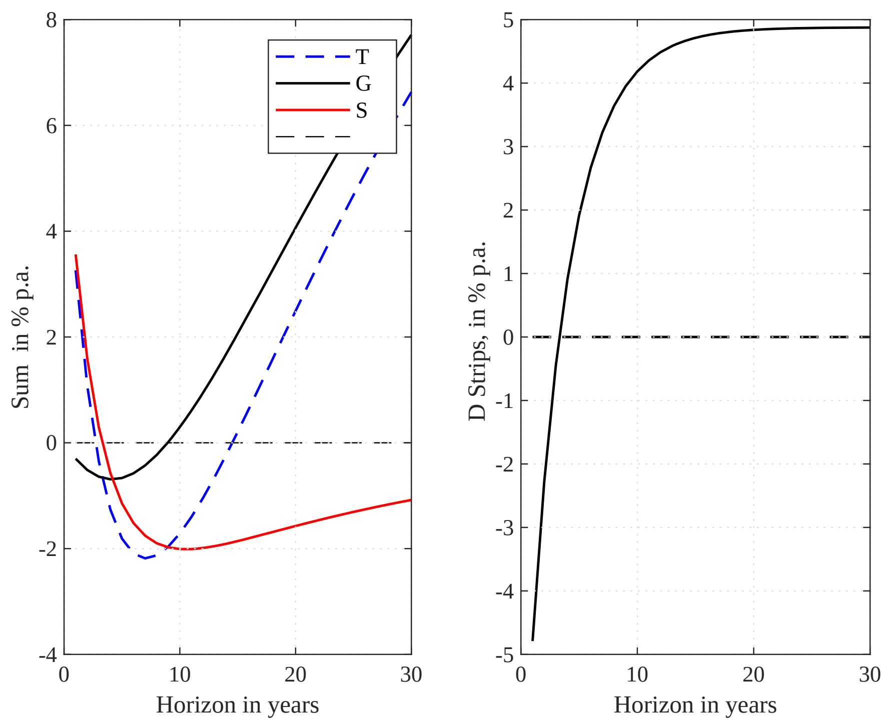
The sign and magnitude of the cash-flow beta of the surplus is now governed by $\gamma(\phi_1^{h-1}\lambda - \sigma)$. This term has a natural economic interpretation. $1 - \exp\left\{\gamma(\phi^{j-1}\lambda - \sigma)\right\}$ is the risk premium of a h-period debt strip with payoff $Y_{t+h}d_{t+h}$. The cumulative surplus can be risky over a horizon h only if this is offset by the safety of debt issuance at time $t+h$.

If $\lambda \leq 0$, and debt/output is pro-cyclical, $\beta_t^{S,CF}(h) < 0$ at all horizons. We are back in the previous case. In other words, the government cannot insure taxpayers by running deficits in bad times over any horizon.

In the empirically relevant case of $\lambda > \sigma > 0$, the initial $\beta_t^{S,CF}(1) > 0$. By issuing more debt in response to an adverse shock, the government prevents the tax rate and the surplus from going up. This provides insurance to the taxpayers $\beta_t^{T,CF}(1) > 0$. The one-period debt strip has a negative risk premium due to the counter-cyclical nature of debt issuance. The initial points ($h = 1$) in the three panels of Figure 7 show the magnitudes.

Figure 7: Risk Premia Across Horizons – Case of AR(1)

The figure plots the risk premium of cumulative discounted cash flows, $\beta_t^{i,CF}(h)/\mathbb{E}_t[M_{t+1}]$, in the left panel against the horizon h. The right panel plots the risk premium on the debt strips: $1 - \exp\left\{\gamma(\phi^{h-1}\lambda - \sigma)\right\}$. The AR(1) coefficient for debt/output is $\phi_1 = 0.75$. The other parameters are as in Table 1.



However, due to its AR(1) nature, the debt/output ratio starts to revert back to its mean the very next period. The two-period surplus risk premium depends on $\gamma(\phi_1\lambda - \sigma)$ which is still positive but not as large as the one-period risk premium since $\phi_1 < 1$. Conversely, the two-period debt strip risk premium is not as negative as the one-period debt strip risk premium. As the middle panel indicates, the risk premium on the strip that pays the annual surplus two years from now is negative. The same is true for the two-year tax strip.

The surplus beta $\beta_t^{S,CF}(h)$ inherits the dynamics of the $AR(1)$ process for the debt/output ratio and starts to decline right away. As h increases, the surplus beta eventually switches signs. This occurs at the first time h that $\phi^{h-1}\lambda < \sigma$. If the rate of mean-reversion in debt is high (ϕ_1 is small), this switch occurs sooner. If the debt/output ratio is more persistent, the sign switch occurs later.

Given the counter-cyclical nature of government spending, the tax beta $\beta_t^{T,CF}(h)$ crosses over into negative territory sooner. There is only a very limited amount of taxpayer insurance that the government can provide when debt is risk-free and follows AR(1) dynamics.

As the right panel shows, the risk premium on the debt strip increases with the horizon. As $h \to \infty$, it converges to the risk premium on a long-dated output strip. Again, this reflects the fact that debt is co-integrated with output. It is common in the literature to assume that this risk premium is zero at long horizons, because this allows discounting at the risk-free rate. In the presence of permanent shocks, this is incorrect. Similarly, the risk premia on the long-dated T-strip and G-strip also converge to risk premium on the long-dated output strip of 5% as $h \to \infty$.

When output shocks are i.i.d. and permanent, far-out surpluses are risky as they inherit the permanent output risk. Medium-term surpluses must be safe and have negative risk premia to offset both the positive risk premium of the short-run surpluses (insurance provision) and the positive risk premium of the long-run surpluses (output risk). Equivalently, the cash-flow betas of the tax strip must be below those of the spending strip at medium horizons. The pushes the cash-flow beta of the tax claim below that of the spending claim. The cash-flow beta at $h = \infty$ is the return beta, and so $\beta_t^T < \beta_t^G$ ensures that $\beta_t^D = 0$. Permanent output risk rules out insurance provision to taxpayers over long horizons.

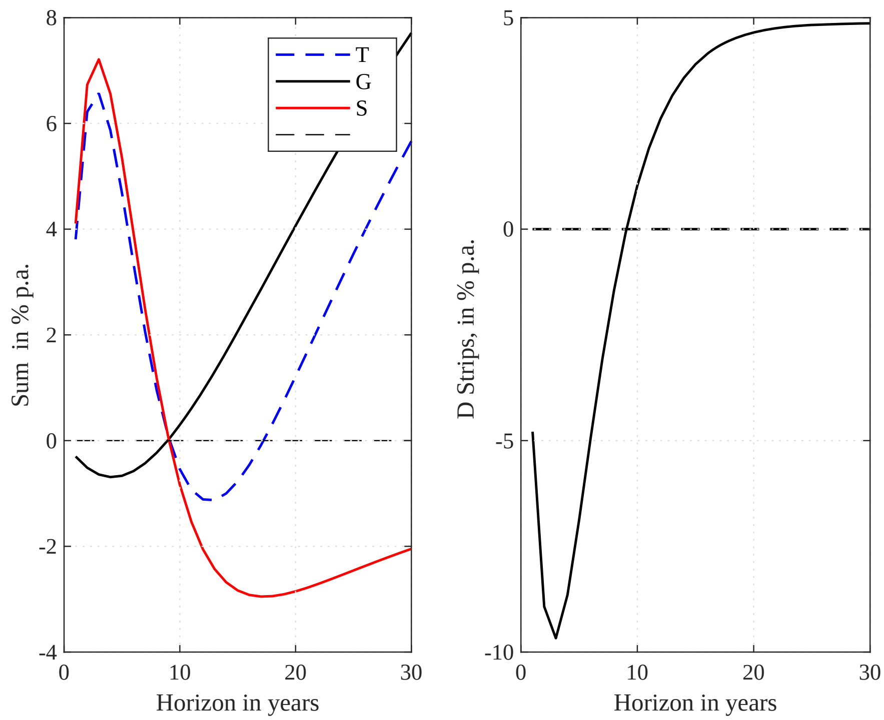
4.3 AR(2) for debt/output

In our preferred case of an AR(2) for debt/output, the sign of the cash flow beta of the surplus is determined by $\gamma(\psi_{j-1}\lambda - \sigma)$. If $\lambda > \sigma$, the initial surplus beta is positive. The second beta is larger since $\psi_1 = \phi_1 > 1$. The third beta remains positive and is larger than the second beta if $\psi_2 > \psi_1$ or $\phi_1(\phi_1 - 1) + \phi_2 > 0$. This condition is satisfied for $\phi_1 = 1.40$ and $\phi_2 = -0.48$. For these parameter values, the fourth beta is lower than the third, the fifth lower tan the fourth, etc. Eventually this beta crosses over into negative territory. The left panel of Figure 8 shows this occurs around year 9. The cash-flow beta for tax revenue follows a similar pattern. The cash-flow betas inherit the hump-shaped pattern from the debt/output ratio.

The middle panel shows that the one-year surplus strip earns a risk premium of 4% per annum. The risk premium on the two-year strip is already much lower and the three year surplus strip has a risk premium near zero. Surplus strip risk premia fall until horizon 7 and then continue to rise. They eventually turn small and positive so as to push the cumulative surplus risk premium towards zero from below. The negative surplus beta is needed to offset the permanent output risk in the debt.

Figure 8: Risk Premia Across Horizons – AR(2) for Debt/Ouput

The figure plots the risk premium contribution of cumulative discounted cash flows (left panel) against the horizon. The right panel plots the risk premium on the debt strips: $-(\exp(-\gamma(\sigma - \psi_{j-1}\lambda)) - 1)$. Benchmark calibration in Table 1.



What allows the government to provide temporary insurance to taxpayers is a debt issuance policy with more history dependence. Risk premia on debt strips, shown in the right panel, are more negative than in the AR(1) model and remain negative for longer (9 versus 3 years). The slow expansion and repayment of the debt in response to an adverse shock allows the government to postpone fiscal rectitude. But as h increases, this expression $\gamma(\sigma - \psi_{j-1}\lambda)$ turns positive and converges to $\gamma\sigma$, the risk premium on the output strip.

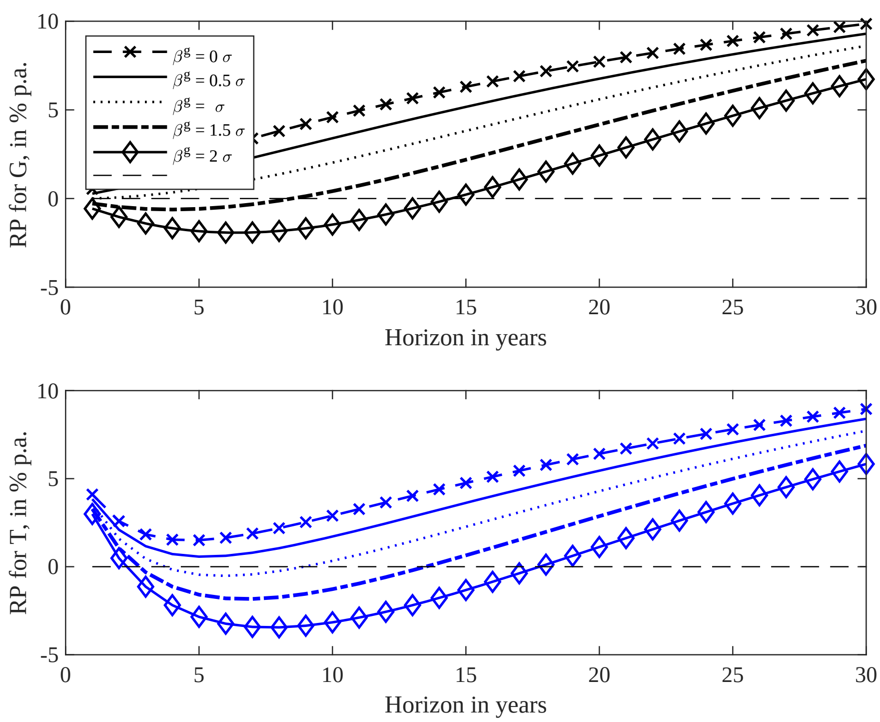
Section C of the appendix develops a version of the model without permanent shocks. This model produces radically different implications, but has counterfactual asset pricing implications.

4.4 Counter-cyclical Spending

The government insures transfer recipients by spending a larger fraction of GDP in recession. The counter-cyclical nature of spending further constraints the government in navigating the trade-off between insurance of bondholders and taxpayers. Figure 9 plots scaled cash-flow betas for spending, $\beta_t^{G,CF}(h)/\mathbb{E}_t[M_{t+1}]$, in the top panel, and the implied tax revenue betas, $\beta_t^{T,CF}(h)/\mathbb{E}_t[M_{t+1}]$, in the right panel for a range of values of the cyclicality of spending b_g. As government spending becomes more counter-cyclical, the risk premium on the spending claim declines. The risk premium on the tax claim has to decline as well in order to keep the government debt risk-free. As the tax claim becomes safer, taxpayers face a riskier tax liability proposition. As the governments provides more insurance to transfer recipients, this reduces the scope for insurance of taxpayers. When spending is acyclical (blue line), the tax claim inherits the risk properties of the surplus claim.

Figure 9: Varying the Counter-cyclicality of Spending

This figure plots the scaled cash-flow beta of spending $\beta_t^{G,CF}(h)/\mathbb{E}_t[M_{t+1}]$ in the top panel and the implied tax revenue betas, $\beta_t^{T,CF}(h)/\mathbb{E}_t[M_{t+1}]$, in the bottom panel for a range of values of the cyclicality of spending b_g.



4.5 Debt Persistence

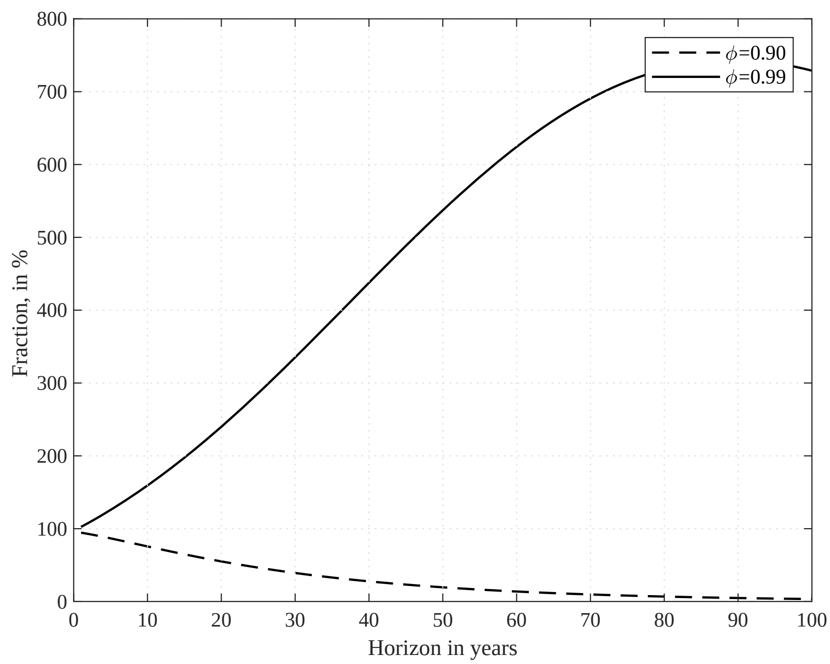
To provide more intertemporal smoothing, the government can increase the persistence of the debt/output process. This allows the government to spread out the adjustment further over time.

When $\phi = 0.90$ (dash-dotted line) , we have to increase the horizon beyond 40 years to see the risk premium on the total surplus go to zero, as shown in the top panel of Figure 10. This allows for a riskier surplus in the first year, and a smaller downward adjustment in the risk premium in the following years (see bottom panel). However, even in the case of $\phi = 0.90$, the risk premium flips signs in year 2.

As the government increases the persistence of the debt/output process to 0.99, The government almost imputes a unit root to the debt/output ratios and seems to escape the trade-off between insuring taxpayers and bondholders. As a result of the near-unit-root, the TVC is quasi violated in our calibration, given that the market price of risk γ is large. To visualize this, we plot the following fraction: $E_t[M_{t+k}D_{t+k}]/D_t$, the tail value of debt as a percentage of the debt outstanding today. For $j = 50$, the fraction is 150%. Under the risk-neutral measure, investors expect the debt to increase faster than the risk-free rate; the government increases the debt/output ratio along paths characterized by adverse aggregate histories, because $\lambda > 0$. For $j = 100$, the fraction is 100%. This means that the expected value of debt 100 years from now accounts for the entire value of the debt (and the value of the first 100 years of surpluses for 0%).

Figure 10: Horizon Decomp. of Risk Premium on Govt. Surpluses and Debt Persistence

The figure plots the tail value at t of the debt expected at $t + j$ as a fraction of debt today. AR(1) with ϕ between 0.5 and 0.99. Other Parameters–Benchmark calibration in Table 1.



5 Revisiting the Trade-off when Debt Earns Convenience Yields

When the transversality conditions holds, and there are no arbitrage opportunities in debt markets, there is only one way to relax this trade-off between insurance of bondholders and taxpayers. Some governments are endowed with the ability to see Treasurys at prices that exceed their fair market value. In other words, investors earn convenience yields on their debt holdings. Typically, the debt then serves the role of a special, safe assets for domestic or foreign investors.

Our analysis begins with a reduced-form characterization of the convenience yield.[9] In discrete time, the convenience yield λ_t is defined as a wedge in the investors' Euler equation:

$$\mathbb{E}_t\left[M_{t,t+1}R_t\right] = \exp(-\lambda_t). \tag{9}$$

The following proposition shows that the convenience yield can be interpreted as an additional seigniorage revenue to the government.

Proposition 8. *In the absence of arbitrage opportunities, the value of the government debt equals:*

$$B_t = \sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} + (1 - e^{-\lambda_{t+j}})D_{t+j} - G_{t+j})\right] = P_t^T + P_t^\lambda - P_t^G,$$

provided that a transversality condition holds.

The seigniorage revenue is $(1 - e^{-\lambda_{t+j}})D_{t+j}$, which is exactly the amount of interest the government does not need to pay due to the convenience yield. The value of government debt reflects the value of all future convenience yields earned on future debt. We refer to this value as the Treasury's seignorage revenue:

$$P_t^\lambda \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(1 - e^{-\lambda_{t+j}})D_{t+j}\right].$$

The government debt portfolio return is the return on a portfolio that goes long in the tax claim and short in the spending claim:

$$\begin{aligned}\mathbb{E}_t\left[R_{t+1}^D - R_t^f\right] \quad &= \quad \frac{P_t^T - T_t}{B_t - S_t}\mathbb{E}_t\left[R_{t+1}^T - R_t^f\right] + \frac{P_t^\lambda - T_t}{B_t - S_t}\mathbb{E}_t\left[R_{t+1}^\lambda - R_t^f\right] \\ &- \quad \frac{P_t^G - G_t}{B_t - S_t}\mathbb{E}_t\left[R_{t+1}^G - R_t^f\right],\end{aligned}$$

where R_{t+1}^D, R_{t+1}^T,R_{t+1}^λ and R_{t+1}^G are the holding period returns on the bond portfolio, the tax claim, the seignorage claim, and the spending claim, respectively. We take government spending

[9] See Liu, Schmid, and Yaron (2019) for a structural model of convenience yields and fiscal policy.

process, and the debt return process as exogenously given, and we explore the implications for the properties of the tax claim.

Constant Spending/Output Ratio Let's take a simple benchmark. If we assume that the spending/output ratio is constant and $\beta_t^Y = \beta_t^G$. We define $K_t = (1 - e^{-\lambda_t})D_t$ to be seignorage revenue. Suppose that the (convenience yield) seignorage process has a zero beta. If the government wants risk-free debt, then the implied beta of the tax revenue process

$$\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)} \gg \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)},$$

which exceeds the beta of the tax revenue without seignorage. If the seignorage revenue is sufficiently counter-cyclical, then the government can insure both taxpayers and bondholders at the same time.

Krishnamurthy and Vissing-Jorgensen (2012) estimate convenience yields on U.S. Treasurys of around 75 bps. These convenience yields are counter-cylical. Using the deviations from CIP in Treasury markets, Jiang, Krishnamurthy, and Lustig (2018a,b); Koijen and Yogo (2019) estimate convenience yields that foreign investors derive from their holdings of dollar safe assets; these estimates exceed 200 bps.

We can characterize the sensitivity of the average tax rate to aggregate output growth in closed form.

Assumption 4. *The government commits to a constant spending/output ratio $x = G_t/Y_t$, and a mean-reverting process for the log tax/output ratio $\tau_t = \log(T_t/Y_t)$ with a constant sensitivity to output innovations β^τ:*

$$\Delta\tau_{t+1} = \theta(\bar{\tau} - \tau_t) + \beta_\tau \sigma \varepsilon_{t+1}.$$

Corollary 3. *Under Assumptions 1 and 4, for the debt to be conditionally risk-free, the sensitivity of the average tax rate needs to satisfy:*

$$\beta_\tau = \frac{1}{1+q_\tau}\left(\frac{P_t^G - G_t - (1 + (1+q_\kappa)\beta_\kappa)(P_t^\lambda - K_t)}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)} - 1\right).$$

If $\beta_\kappa \ll -\frac{1}{1+q_\kappa}$, then the counter-cyclical convenience yields increase the sensitivity of tax rates to output innovations. For example, we can have a constant average tax rate and risk-free debt when:

$$\beta_\kappa = -\frac{1}{1+q_\kappa}\frac{D_t}{P_t^\lambda - K_t}$$

Consider the case in which the government runs zero primary surpluses in all future states of the world: $D_t = P_t^{\lambda} - K_t$. In this case, the average tax rate is constant if $\beta_\kappa = -\frac{1}{1+q_\kappa}$. This is -1 in the random walk case with $\theta = 0$. Please see section A of the Appendix for details.

6 Conclusion

The government engineers risk-free debt by choosing the exposure of the tax claim to output risk judiciously. The more debt outstanding, the lower this exposure must become, the more output risk must be borne by the taxpayer. There is no scope for insurance of taxpayers over long horizons in the presence of permanent, priced shocks to output. The only way the government can provide insurance to tax payers over long horizons while keeping the debt risk-free is by saving or by earning a large convenience yield on its debt.

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A Risky Debt

In general, when we specify exogenous processes for taxes and spending, the implied debt is risky. This section derives more general characterizations of the risk-return trade-off, by specifying exogenous processes for taxes and spending, and allowing for arbitrary mean-reversion in the tax rate, and risky debt. This approach is more common in the literature. We will do this in a continuous time version of our model.

Let y_t denote log of real GDP. Let τ_t denote the log tax-to-gdp ratio and let g_t denote the log spending-to-gdp ratio. We specify exogenous processes for *both spending and taxes*:

$$\begin{aligned} dy_t &= \mu dt + \sigma dZ_t, \\ d\tau_t &= \theta(\bar{\tau} - \tau_t)dt + \beta_\tau \sigma dZ_t, \\ dg_t &= \theta(\bar{g} - g_t)dt + \beta_g \sigma dZ_t, \end{aligned}$$

where θ governs the degree of persistence in τ and g. Importantly, this specification does not allow the government to choose a tax process that is more risky in the short run, but less risky at intermediate horizons (See for example Figure 8.)

Then $T_t = \exp(\tau_t + y_t)$ and $G_t = \exp(g_t + y_t)$. Let B_t denote the real value of debt. Let P_t^τ denote the present value of the claim on tax and P_t^g denote the present value of the claim on spending. Let M_t denote the SDF. The asset pricing equations are

$$\begin{aligned} 0 &= \mathcal{A}[M_t T_t dt + d(M_t P_t^\tau)], \\ 0 &= \mathcal{A}[M_t G_t dt + d(M_t P_t^g)], \\ 0 &= \mathcal{A}[M_t (T_t - G_t) dt + d(M_t B_t)]. \end{aligned}$$

Note: The last equation can be thought of as the continuous-time version of the government budget condition.

Proposition 9. *When the TVC holds, the value of the debt equals the price of a claim to tax revenue minus the price of a claim to spending:*

$$B_t = P_t^\tau - P_t^g.$$

Let M_t denote the cumulative SDF, and let m_t denote its log. We assume

$$\begin{aligned} dm_t &= -(r + \frac{1}{2}\gamma^2)dt - \gamma dZ_t, \\ dM_t &= -M_t r dt - M_t \gamma dZ_t. \end{aligned}$$

We conjecture that the tax claim and the spending claim are priced according to:

$$\begin{aligned} P_t^\tau &= f_\tau(\tau_t) T_t \\ P_t^g &= f_g(g_t) G_t. \end{aligned}$$

The debt/GDP ratio is given by: $\frac{B_t}{Y_t} = f_\tau(\tau_t)\tau_t - f_g(g_t)g_t$. Then, we conjecture $f_\tau(\tau_t) = \exp(p_\tau + q_\tau \tau_t)$ and $f_g(g_t) = \exp(p_g + q_g g_t)$.

Proposition 10. *When $\theta > 0$, the risk exposure of the debt return is*

$$\begin{aligned} [r_t^B, dM_t] &= -\frac{M_t \gamma \sigma}{B_t}\left(T_t f_\tau \left(1+(1+q_\tau)\beta_\tau\right) - G_t f_g \left(1+(1+q_g)\beta_g\right)\right), \\ \textit{where} \quad q_\tau &= -\frac{\theta}{\kappa_1^\tau \theta + (1-\kappa_1^\tau)}, \\ q_g &= -\frac{\theta}{\kappa_1^g \theta + (1-\kappa_1^g)}. \end{aligned}$$

κ_1^τ and κ_1^g are the constants that are very close to 1 from the Campbell-Shiller approximation:

$$\begin{aligned} \kappa_1^\tau &= \frac{1}{1+\exp(\overline{P_t^\tau / T_t})} \\ \kappa_1^g &= \frac{1}{1+\exp(\overline{P_t^g / G_t})} \end{aligned}$$

Random Walk Cash Flows We start with the simplest case in which spending and tax revenue follow a random walk ($\theta = 0$). In this case $q_\tau = q_g = 0$, and the debt/output ratio is non-stationary. The risk exposure of the debt claim is

$$\begin{aligned} [r_t^B, dM_t] &= -M_t \gamma \frac{1}{B_t}(f_\tau T_t (1+\beta_\tau) - f_g G_t (1+\beta_g))\sigma, \\ \textit{where} \quad f_\tau &= (r - \mu - \frac{1}{2}(1+\beta_\tau)^2 \sigma^2 + \gamma(1+\beta_\tau)\sigma)^{-1}, \\ f_g &= (r - \mu - \frac{1}{2}(1+\beta_g)^2 \sigma^2 + \gamma(1+\beta_g)\sigma)^{-1}. \end{aligned}$$

Risk-free debt is a knife-edge case. The debt is risk-free if and only if

$$(B_t + P_t^g)(1+\beta_\tau) = P_t^\tau (1+\beta_\tau) \quad = \quad P_t^g (1+\beta_g).$$

Even when allowing for a non-stationary debt/output ratio, the government has to implement a counter-cyclical tax policy if it wants to keep the debt risk-free. For example, when $\beta_g = 0$, i.e. spending is a constant fraction of GDP, this equation requires that the loading of the average tax rate on the output shock satisfy:

$$\beta_\tau = \frac{f_g g_t}{d_t + f_g g_t} - 1,$$

which is negative as long as $d_t = f_\tau \tau_t - f_g g_t > 0$. So, risk-free debt implies countercyclical taxation. This result confirms Barro (1979)'s conjecture that tax rates inherit the random walk property of output and spending if the debt is to be risk-free. As the debt/output ratio increases, the β_τ converges to -1. When the government insures transfer recipients by spending more in recessions, and hence choosing $\beta_g < 0$, then β_τ will have to be even more negative.

Even when debt is risky, there may still be a random walk component in the tax rates. The only way to eliminate this random walk component is to set $\beta_\tau = 0$, which would imply that the instantaneous covariance equals that of the output claim:

$$-\frac{M_t \gamma \sigma}{B_t}\left(T_t f_\tau \left(1+(1+q_\tau)\beta_\tau\right) - G_t f_g \left(1+(1+q_g)\beta_g\right)\right) = -M_t \gamma \sigma$$

Hence, if we want to completely eliminate the random walk component in taxes, then the debt becomes an unlevered equity claim.

More generally, $[r_t^B, dM_t]$ is decreasing in β_τ. This formula highlights the trade-off between insuring the taxpayers and insuring the debtholders. If the government wants to smooth the tax burden by increasing β_τ, the debt will be riskier because the instantaneous covariance $[r_t^B, dM_t]$ decreases.

Mean-Reverting Cash Flows We consider the case in which $\theta > 0$. Since $1 + q_\tau > 0$ and $1 + q_g > 0$, the same intuition applies: $[r_t^B, dM_t]$ is decreasing in β_τ, implying a trade-off between insuring the taxpayers and insuring the debtholders. The debt is risk-free if and only if the following condition is satisfied:

$$(f_g g_t + d_t)\,(1 + (1+q_\tau)\beta_\tau) \quad = \quad g_t f_g\left(1 + (1+q_g)\beta_g\right).$$

For example, when $\beta_g = 0$, i.e. spending is a constant fraction of GDP, the sensitivity of the average tax rate to the output shock is given by:

$$\beta_\tau \quad = \quad \frac{1}{1+q_\tau}\left(\frac{f_g g_t}{d_t + f_g g_t} - 1\right),$$

which is negative as long as $d_t = f_\tau \tau_t - f_g g_t > 0$. All else equal, mean reversion renders the tax rate even more countercylical, because $\frac{1}{1+q_\tau} > 1$, when $\theta > 0$. The higher θ, the larger this ratio. To get the tax rate revert back to its mean faster, the tax rate has to be more counter-cylical. So, the government can eliminate the random walk in taxes but only by forcing tax payers to insure the rest of the economy even more against aggregate shocks. So, risk-free debt implies countercyclical taxation.

General Model with Convenience Yield Now, move on to a general model with convenience yield. The Euler equation is

$$0 \quad = \quad \mathcal{A}[M_t B_t \lambda_t dt + M_t(T_t - G_t)dt + d(M_t B_t)],$$

We define $K_t = B_t \lambda_t$ as the flow benefit of convenience yield generated by the government debt, define $\kappa_t = K_t / Y_t$ as the conv yield-to-gdp ratio, and assume

$$d\kappa_t \quad = \quad \theta(\bar{\kappa} - \kappa_t)dt + \beta_\kappa \gamma dZ_t.$$

Then the debt value can be solved from

$$0 \quad = \quad \mathcal{A}[M_t(T_t + K_t - G_t)dt + d(M_t B_t)].$$

Similarly, we let P_t^τ denote the present value of the claim on convenience yield. Then

$$P_t^\tau \quad = \quad f_\kappa(\kappa_t) K_t,$$

where $f_\kappa(\kappa_t) = \exp(p_\kappa + q_\kappa \kappa_t)$.

Proposition 11. *When $\theta > 0$, the risk exposure of the debt return is*

$$[r_t^B, dM_t] \quad = \quad -\frac{M_t \gamma \gamma}{B_t}\left(f_\tau T_t(1 + (1+q_\tau)\beta_\tau) + f_\kappa K_t(1 + (1+q_\kappa)\beta_\kappa) - f_g G_t(1 + (1+q_g)\beta_g)\right).$$

To produce risk-free debt (i.e. $[r_t^B, dM_t] = 0$), we need

$$f_\tau T_t(1 + (1+q_\tau)\beta_\tau) + f_\kappa K_t(1 + (1+q_\kappa)\beta_\kappa) \quad = \quad f_g G_t(1 + (1+q_g)\beta_g)$$

A countercyclical convenience yield stream (negative β_κ) helps generate a countercyclical spending stream (negative β_g), thereby partially alleviating the pressure for tax to be countercyclical.

B Return Betas and Cash Flows

What is the relation between the return betas and the cash flow betas? Well, in this simple case, with constant debt/output and constant spending/output ratios, there is a one-to-one mapping:

Corollary 4. *The expected returns can be expressed as a function of the cash flow betas:*

$$\begin{aligned}
\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] &= \frac{x}{d(1-\xi_1)+x\xi_1}\frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})}, \\
&= \frac{x}{d(1-\xi_1)+x\xi_1}\exp(\mu + \frac{1}{2}\sigma^2)(1-\exp(-\gamma\sigma)) \\
\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] &= \frac{1}{\xi_1}\frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})} \\
&= \frac{1}{\xi_1}\exp(\mu + \frac{1}{2}\sigma^2)(1-\exp(-\gamma\sigma)),
\end{aligned}$$

where $\xi_1 = \exp(-\rho - \gamma\sigma + \mu + 0.5\sigma^2)$.

C Quantifying the Trade-off in Model with Transitory Output Shocks

Next, we consider the impact of transitory shocks to the level of output, but we, in a first pass, we keep our original pricing kernel with permanent shocks to the level of marginal utility. We call this the goldilocks economy. In this setting, the government can insure taxpayers at all horizons while keeping the debt risk-free.

C.1 Permanent Shocks to Marginal Utility

Assumption 5. *(a) The shocks to output are transitory:*

$$y_{t+1} = \xi_0 + \xi y_t + \sigma\varepsilon_{t+1}$$

where ε_{t+1} *still denotes the innovation to output growth that is normally distributed and i.i.d.*

(b) The log pricing kernel is

$$m_{t,t+1} = -\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}.$$

This asset pricing model is fundamentally misspecified. This pricing kernel does not reflect the mean-reversion in output and hence cannot be micro-founded. However, we use this model as an expositional device. In this setting, the government faces no trade-off between insuring taxpayers and bondholders. When there are no permanent shocks to output, but the pricing kernel does not reflect this, then the government can insure taxpayers over all horizons.

Proposition 12. *The cash flow beta of the surpluses over j periods is given by:*

$$\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}\right) \\
= \;& -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)) - 1)
\end{aligned}$$

when $j \geq 2$*. The sign of the cash flow covariance is sign* $\left(\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda\right)$.

Hence, the sign of the cash flow covariance is determined by the sign of $\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda)$. As before, this is the risk premium on a debt strip, and compensates investors for output risk. Because the innovations are temporary, the output component of this risk premium converges to zero. The transitory nature of output risk broadens the scope for insurance of taxpayers. As we consider $\xi \to 1$, we revert back to the expression derived in the benchmark model: $\gamma(\sigma - \phi^{k-1}\lambda)$. If $\lambda > \sigma$, the initial covariance is negative. If the rate of mean-reversion in output is higher than in the debt/output ratio, $\phi > \xi$, the covariance stays negative for all j. As a result, the government can now insure taxpayers at all horizons. This was not feasible in the case of permanent innovations.

Corollary 5. *The cash flow beta of taxes have to satisfy the following restriction.*

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k} T_{t+k}\right) \\ = & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)) - 1) \\ + & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k} G_{t+k}\right). \end{aligned}$$

Quantitative Implications We return to our calibrated economy. Figure 11 plots the risk premium contributions of the surpluses over different horizons for the benchmark calibration:

$$\begin{aligned} & -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k} S_{t+k}\right) / \mathbb{E}_t[M_{t+1}] \\ = & E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)) - 1) \end{aligned}$$

However, the output process no longer has a unit root. We start by considering the case in which $\phi = \xi$. At all horizons, the tax claim is risky, contributing positive risk premium across all horizons, because λ exceeds σ. The tax claim is also risky across all horizons. In this goldilocks scenario, the government can insure taxpayers at all horizons. $\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda)$ is positive across all horizons.

Figure 11: Risk Premium on Govt. Cash Flows with Transitory Shocks

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (middle panel) against the horizon. The bottom panel plots the risk premium on the debt strips: $-(\exp(-\gamma(\sigma\xi^{j-1} - \phi^{j-1}\lambda)) - 1)$. Calibration: ϕ is 0.75 and ξ is 0.75. Other parameters–Benchmark calibration in Table 1.

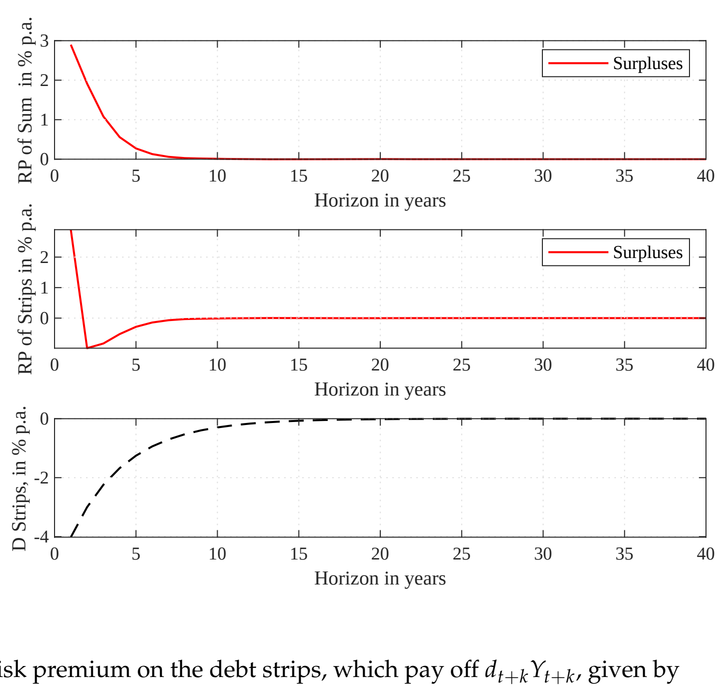


Figure 11 plots the risk premium on the debt strips, which pay off $d_{t+k}Y_{t+k}$, given by

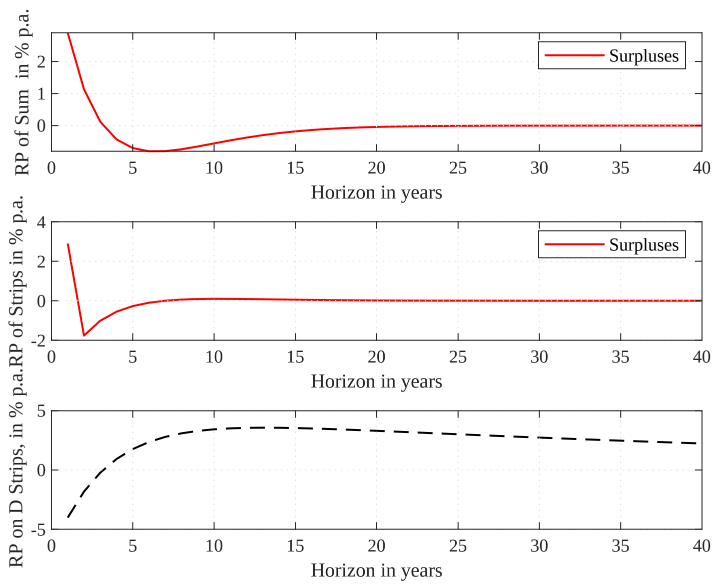
$$\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda) \approx -(\exp(-\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda)) - 1).$$

Given that λ exceeds σ, the risk premium on the debt strips are uniformly negative. These are the mirror image of the surplus risk premium in the top panel of Figure 11. As $j \to \infty$, this debt strip risk premium converges to the risk premium on the output strips, 0%, because the output innovations are transitory, and the pricing kernel does not have a transitory component which contributes interest rate risk. Why can the government insure taxpayers over long horizons (by delivering a risky tax claim)? Because the debt strip risk premium are negative at all horizons.

Of course, insurance of taxpayers only works if the governments commits to a debt policy that is at least as persistent as the output process ($\phi > \xi$). Figure 12 plots the risk premia contributions when the output shocks are close to a unit root, but the debt/output ratio reverts back to the mean at a faster rate. In this case, the government has to produce safer surplus claims over longer horizons.

Figure 12: Risk Premium on Govt. Cash Flows with Transitory Shocks

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (bottom panel) against the horizon. Calibration: ϕ is 0.75 and ξ is 0.98. Other parameters–Benchmark calibration in Table 1.



C.2 Transitory Shocks to Marginal Utility

Next, we consider an internally consistent model: we shut down permanent shocks to the level of output, as well as to marginal utility.

Assumption 6. *(a) The shocks to output are transitory:*

$$y_{t+1} \quad = \quad \xi_0 + \xi y_t + \sigma \varepsilon_{t+1}$$

where ε_{t+1} still denotes the innovation to output growth that is normally distributed and i.i.d.

(b) The log pricing kernel is

$$m_{t,t+1} \quad = \quad -\rho - \frac{1}{2}\gamma^2 - \gamma \frac{\sigma \varepsilon_{t+1} + (\xi - 1) y_t}{\sigma}.$$

When shocks to output are transitory, most asset pricing models predict that there are no permanent shocks to the marginal utility of wealth. This specific modification of the pricing kernel is motivated by the fact that if the agent's consumption is equal to the output and has CRRA preference with a relative risk aversion of γ/σ, the marginal utility growth is $m_{t,t+1} = -\tilde{\rho} - \gamma/\sigma(\xi_0 + (\xi - 1)y_t + \sigma\varepsilon_{t+1})$. In this case, the marginal utility of wealth can be written as:

$$\Lambda_{t+1} = \exp(-\tilde{\rho}(t+1) - (\gamma/\sigma) y_{t+1}).$$

There are no permanent shocks to the marginal utility of wealth. Given this pricing kernel, the log of the risk-free rate is given by:

$$r_t^f \quad = \quad \rho + \gamma \frac{(\xi - 1) y_t}{\sigma}.$$

Note that this model has counterfactual asset pricing implications. In the model, the interest rate risk will make the long bond the riskiest asset in the economy. Modern asset pricing has consistently found that permanent cash flow shocks receive a high price of risk in the market (e.g., Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Bansal and Yaron, 2004; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and Chernov, 2018). This model has no permanent priced risk, except when $\xi = 1$. In that case, we recover the pricing kernel in our benchmark model.

When there are no permanent shocks to output and the pricing kernel, then the government can insure taxpayers over longer horizons.

Proposition 13. *The cash flow beta of the surpluses over j periods is given by:*

$$\begin{aligned} & cov_t \left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) \textstyle\sum_{k=1}^{j} M_{t+1,t+k} S_{t+k} \right) \\ = \quad & -E_t[M_{t+1}] E_t[M_{t+1,t+j} d_{t+j} Y_{t+j}] (\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1 - \xi^{j-1}))) - 1) \end{aligned}$$

when $j \geq 2$. The sign of the cash flow covariance is sign $\left(\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1 - \xi^{j-1}))\right)$.

Hence, the sign of the cash flow covariance is determined by the sign of $\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda + \frac{\gamma}{\sigma}(1 - \xi^{k-1}))$. As before, this is the risk premium on a debt strip. The first component, $\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)$, compensates for output risk. The second component, $\frac{\gamma}{\sigma}(1 - \xi^{j-1})$, compensates for interest rate risk. Because the innovations are temporary, the output component of this risk premium converges to zero. The interest rate risk does not converge to zero; the long bond is the riskiest asset in an economy with only transitory risk. The transitory nature of output risk broadens the scope for insurance of taxpayers, but this is counteracted by interest rate risk. As we consider $\xi \to 1$, we revert back to the expression derived in the benchmark model: $\gamma(\sigma - \phi^{k-1}\lambda)$. The interest rate risk term disappears.

Corollary 6. *The cash flow beta of taxes have to satisfy the following restriction.*

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\ = \quad & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma-\phi^{j-1}\lambda+\frac{\gamma}{\sigma}(1-\xi^{j-1})))-1) \\ + \quad & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}G_{t+k}\right). \end{aligned}$$

which can be restated as:

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\ = \quad & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_{t+1}[M_{t+1,t+j}Y_{t+j}]\exp(\frac{1-\phi^j}{1-\phi}(\phi_0-\frac{1}{2}\lambda^2)+\sum_{k=1}^{j-1}((\gamma-\sigma)\xi^{k-1}\phi^{k-1}\lambda+\frac{1}{2}(\phi^{k-1}\lambda)^2) \\ + \quad & \phi^j\log d_t-\phi^{j-1}\lambda((\sigma-\gamma)\xi^{j-1}+\gamma)+\frac{1}{2}(\phi^{j-1}\lambda)^2)(\exp(-\gamma(\xi^{j-1}\sigma-\phi^{j-1}\lambda+\frac{\gamma}{\sigma}(1-\xi^{j-1})))-1) \\ + \quad & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}G_{t+k}\right). \end{aligned}$$

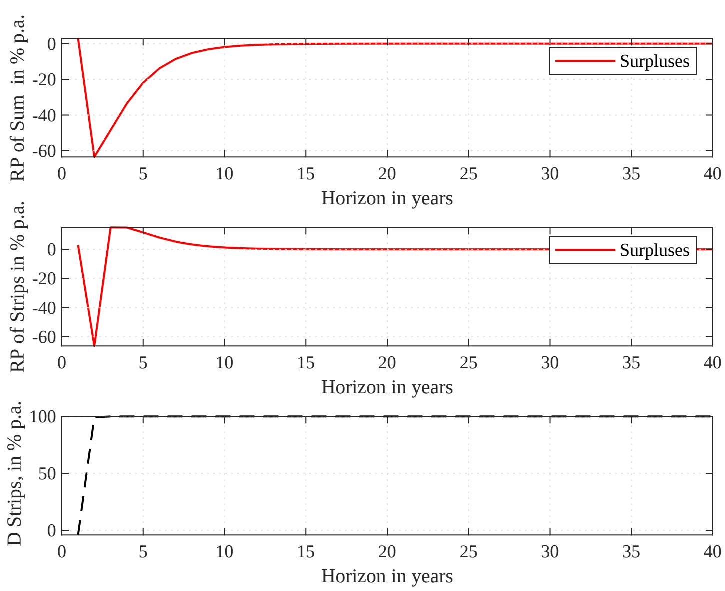
Quantitative Model Implications We return to our calibrated economy. Figure 13 plots the risk premium contributions of the surpluses over different horizons j for the benchmark calibration:

$$\begin{aligned} & -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}\right)/\mathbb{E}_t[M_{t+1}] \\ = \quad & E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma-\phi^{j-1}\lambda+\frac{\gamma}{\sigma}(1-\xi^{j-1})))-1) \end{aligned}$$

However, the output process no longer has a unit root. At short horizons, the tax claim is safe, contributing negative risk premium, but the tax claim turns risky over horizons that exceed 10 years.

Figure 13: Risk Premium on Govt. Cash Flows with Transitory Shocks

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (middle panel) against the horizon. Calibration: ϕ is 0.75. Other parameters–Benchmark calibration in Table 1.



D Autocovariances

D.1 Permanent Shocks

Corollary 7. *The conditional autocovariances of the surplus/output ratios are*

$$\begin{aligned}
& cov_t(s_{t+1}, s_{t+j}) \\
= \quad & \exp(2\rho - 2\mu + \sigma^2)\exp\left((1+\phi^{j-1})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2(j-1)}}{1-\phi^2}\right) \\
\times \quad & (\exp(\sigma\lambda\phi^{j-2}) - 1) \\
- \quad & \exp(\rho - \mu + .5\sigma^2)\exp\left((\phi+\phi^{j-1})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2(j-1)}}{1-\phi^2} + \frac{1}{2}\lambda^2\right) \\
\times \quad & (\exp(\lambda^2\phi^{j-2}) - 1) \\
- \quad & \exp(\rho - \mu + .5\sigma^2)\exp\left((1+\phi^{j})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2j}}{1-\phi^2}\right) \\
\times \quad & (\exp(\sigma\lambda\phi^{j-1}) - 1) \\
+ \quad & \exp\left((\phi+\phi^{j})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2j}}{1-\phi^2} + \frac{1}{2}\lambda^2\right)(\exp(\lambda^2\phi^{j-1}) - 1),
\end{aligned}$$

and the conditional variance of the surplus/output ratio is

$$\begin{aligned}
var_t(s_{t+1}) \quad = \quad & \exp(2\rho - 2\mu + \sigma^2)\exp(2\log d_t)(\exp(\sigma^2) - 1) \\
- \quad & 2\exp(\rho - \mu + .5\sigma^2)\exp\left((1+\phi)(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\right) \\
\times \quad & (\exp(\lambda\sigma) - 1) \\
+ \quad & \exp\left(2\phi(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \lambda^2\right)(\exp(\lambda^2) - 1)
\end{aligned}$$

D.2 Transitory Shocks

Corollary 8. *In the presence of transitory shocks, (a) the conditional autocovariances of the surplus/output ratios are*

$$\begin{aligned}
cov_t(s_{t+1}, s_{t+j}) \quad = \quad & \exp(2\rho - 2\psi_0 + \sigma^2)d_t\exp(-(\psi-1)y_t)\mathbb{E}_t[d_{t+j-1}\exp(-(\psi-1)y_{t+j-1})] \\
\times \quad & (\exp(\sigma\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
- \quad & \exp(\rho - \psi_0 + .5\sigma^2)\mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j-1}\exp(-(\psi-1)y_{t+j-1})] \\
\times \quad & (\exp(\lambda\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
- \quad & \exp(\rho - \psi_0 + .5\sigma^2)d_t\exp(-(\psi-1)y_t)\mathbb{E}_t[d_{t+j}](\exp(\sigma\lambda\phi^{j-1}) - 1) \\
+ \quad & \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j}](\exp(\lambda^2\phi^{j-1}) - 1)
\end{aligned}$$

where the conditional forecasts are

$$\mathbb{E}_t[d_{t+j}] \quad = \quad \exp\left(\phi^j(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2j}}{1-\phi^2}\right)$$

and

$$
\begin{aligned}
& \mathbb{E}_t[\exp(\log d_{t+j-1} - (\psi - 1) y_{t+j-1})] \\
= \; & \exp\left(\phi^{j-1}(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} - (\psi-1)\psi^{j-1}(y_t - \frac{\psi_0}{1-\psi}) - (\psi-1)\frac{\psi_0}{1-\psi}\right. \\
+ \; & \left. \frac{1}{2}\sum_{k=0}^{j-2}(\phi^k \lambda + \psi^k(\psi-1)\sigma)^2\right)
\end{aligned}
$$

(b) The conditional variance of the surplus/output ratio is

$$
\begin{aligned}
var_t(s_{t+1}) &= \exp(2\rho - 2\psi_0 + \sigma^2)\exp(2\log d_t - 2(\psi-1)y_t)(\exp(\sigma^2) - 1) \\
&- 2\exp(\rho - \psi_0 + .5\sigma^2)\exp\left((1+\phi)(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 - (\psi-1)y_t\right) \\
&\times (\exp(\lambda\sigma) - 1) \\
&+ \exp\left(2\phi(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \lambda^2\right)(\exp(\lambda^2) - 1)
\end{aligned}
$$

E Proofs

E.1 Proof of Eq. (1) in Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019)

Proof. All objects in this appendix are in nominal terms but we drop the superscript $\$$ for ease of notation. The government faces the following one-period budget constraint:

$$
G_t - T_t + Q_{t-1}^1 = \sum_{h=1}^{H}(Q_t^h - Q_{t-1}^{h+1})P_t^h,
$$

where G_t is total nominal government spending, T_t is total nominal government revenue, Q_t^h is the number of nominal zero-coupon bonds of maturity h outstanding in period t each promising to pay back \$1 at time $t+h$, and P_t^h is today's price for a h-period zero-coupon bond with \$1 face value. A unit of $h+1$-period bonds issued at $t-1$ becomes a unit of h-period bonds in period t. That is, the stock of bonds evolves of each maturity evolves according to $Q_t^h = Q_{t-1}^{h+1} + \Delta Q_t^h$. Note that this notation can easily handle coupon-bearing bonds. For any bond with deterministic cash-flow sequence, we can write the price (present value) of the bond as the sum of the present values of each of its coupons.

The left-hand side of the budget constraint denotes new financing needs in the current period, due to primary deficit $G-T$ and one-period debt from last period that is now maturing. The right hand side shows that the money is raised by issuing new bonds of various maturities. Alternatively, we can write the budget constraint as total expenses equalling total income:

$$
G_t + Q_{t-1}^1 + \sum_{h=1}^{H} Q_{t-1}^{h+1}P_t^h = T_t + \sum_{h=1}^{H} Q_t^h P_t^h,
$$

We can now iterate the budget constraint forward. The period t constraint is given by:

$$
\begin{aligned}
T_t - G_t &= Q_{t-1}^1 - Q_t^1 P_t^1 + Q_{t-1}^2 P_t^1 - Q_t^2 P_t^2 + Q_{t-1}^3 P_t^2 - Q_t^3 P_t^3 \\
&+ \cdots - Q_t^H P_t^H + Q_{t-1}^{H+1} P_t^H.
\end{aligned}
$$

Consider the period-$t+1$ constraint,

$$
\begin{aligned}
T_{t+1} - G_{t+1} &= Q_t^1 - Q_{t+1}^1 P_{t+1}^1 + Q_t^2 P_{t+1}^1 - Q_{t+1}^2 P_{t+1}^2 + Q_t^3 P_{t+1}^2 - Q_{t+1}^3 P_{t+1}^3 \\
&+ \cdots - Q_{t+1}^H P_{t+1}^H + Q_t^{H+1} P_{t+1}^H.
\end{aligned}
$$

multiply both sides by M_{t+1}, and take expectations conditional on time t:

$$\begin{aligned}\mathbb{E}_t\left[M_{t+1}(T_{t+1}-G_{t+1})\right] &= Q_t^1P_t^1-\mathbb{E}_t[Q_{t+1}^1M_{t+1}P_{t+1}^1]+Q_t^2P_t^2-\mathbb{E}_t[Q_{t+1}^2M_{t+1}P_{t+1}^2]+Q_t^3P_t^3\\ &\quad -\mathbb{E}_t[Q_{t+1}^3M_{t+1}P_{t+1}^3]+\cdots+Q_t^HP_t^H\\ &\quad -\mathbb{E}_t[Q_{t+1}^HM_{t+1}P_{t+1}^H]+Q_t^{H+1}P_t^{H+1},\end{aligned}$$

where we use the asset pricing equations $\mathbb{E}_t[M_{t+1}] = P_t^1$, $\mathbb{E}_t[M_{t+1}P_{t+1}^1] = P_t^2$, $\cdots$, $\mathbb{E}_t[M_{t+1}P_{t+1}^{H-1}] = P_t^H$, and $\mathbb{E}_t[M_{t+1}P_{t+1}^H] = P_t^{H+1}$.

Consider the period $t+2$ constraint, multiplied by $M_{t+1}M_{t+2}$ and take time-t expectations:

$$\begin{aligned}\mathbb{E}_t\left[M_{t+1}M_{t+2}(T_{t+2}-G_{t+2})\right] &= \mathbb{E}_t[Q_{t+1}^1M_{t+1}P_{t+1}^1]-\mathbb{E}_t[Q_{t+2}^1M_{t+1}M_{t+2}P_{t+2}^1]+\mathbb{E}_t[Q_{t+1}^2M_{t+1}P_{t+1}^2]\\ &\quad -\mathbb{E}_t[Q_{t+2}^2M_{t+1}M_{t+2}P_{t+2}^2]+\mathbb{E}_t[Q_{t+1}^3M_{t+1}P_{t+1}^3]-\cdots\\ &\quad +\mathbb{E}_t[Q_{t+1}^HM_{t+1}P_{t+1}^H]-\mathbb{E}_t[Q_{t+2}^HM_{t+1}M_{t+2}P_{t+2}^H]\\ &\quad +\mathbb{E}_t[Q_{t+1}^{H+1}M_{t+1}P_{t+1}^{H+1}],\end{aligned}$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M_{t+2}] = P_{t+1}^1$, $\mathbb{E}_{t+1}[M_{t+2}P_{t+2}^1] = P_{t+1}^2$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected discounted surpluses at t, $t+1$, and $t+2$ we get:

$$\begin{aligned}&T_t-G_t+\mathbb{E}_t\left[M_{t+1}(T_{t+1}-G_{t+1})\right]+\mathbb{E}_t\left[M_{t+1}M_{t+2}(T_{t+2}-G_{t+2})\right]=\textstyle\sum_{h=0}^H Q_{t-1}^{h+1}P_t^h+\\ &-\mathbb{E}_t[Q_{t+2}^1M_{t+1}M_{t+2}P_{t+2}^1]-\mathbb{E}_t[Q_{t+2}^2M_{t+1}M_{t+2}P_{t+2}^2]-\cdots--\mathbb{E}_t[Q_{t+2}^HM_{t+1}M_{t+2}P_{t+2}^H].\end{aligned}$$

Similarly consider the one-period government budget constraints at times $t+3$, $t+4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t+J$, we get:

$$\sum_{h=0}^H Q_{t-1}^{h+1}P_t^h = \mathbb{E}_t\left[\sum_{j=0}^J M_{t,t+j}(T_{t+j}-G_{t+j})\right]+\mathbb{E}_t\left[M_{t,t+J}\sum_{h=1}^H Q_{t+J}^hP_{t+J}^h\right]$$

where we used the cumulate SDF notation $M_{t,t+j} = \prod_{i=0}^j M_{t+i}$ and by convention $M_{t,t} = M_t = 1$ and $P_t^0 = 1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next J years plus the present value of the government bond portfolio that will be outstanding at time $t+J$. The latter is the cost the government will face at time $t+J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J\to\infty$:

$$\sum_{h=0}^H Q_{t-1}^{h+1}P_t^h = \mathbb{E}_t\left[\sum_{j=0}^\infty M_{t,t+j}(T_{t+j}-G_{t+j})\right]+\lim_{J\to\infty}\mathbb{E}_t\left[M_{t,t+J}\sum_{h=1}^H Q_{t+J}^hP_{t+J}^h\right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j}-G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the TVC:

$$\lim_{J\to\infty}\mathbb{E}_t\left[M_{t,t+J}\sum_{h=1}^H Q_{t+J}^hP_{t+J}^h\right]=0.$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise there is a government debt bubble.

If the TVC is satisfied, the outstanding debt today, B_t, reflects the expected present-discounted value of the current and all future primary surpluses:

$$B_t = \sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j}) \right].$$

This is equation (1) in the main text. □

E.2 Proof of Proposition 1

Proof. From the investor's Euler equation $\mathbb{E}_t[M_{t+1}(R_{t+1}^i - R_t^f)]$=0, we know that the expected excess return on the tax claim, spending claim, and debt claims are given by

$$\begin{aligned}
\mathbb{E}_t \left[R_{t+1}^T - R_t^f \right] &= \frac{-cov_t \left(M_{t+1}, R_{t+1}^T \right)}{\mathbb{E}_t[M_{t+1}]} = \frac{-cov_t \left(M_{t+1}, R_{t+1}^T \right)}{var_t[M_{t+1}]} \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]} = \beta_t^T \lambda_t, \\
\mathbb{E}_t \left[R_{t+1}^G - R_t^f \right] &= \frac{-cov_t \left(M_{t+1}, R_{t+1}^G \right)}{\mathbb{E}_t[M_{t+1}]} = \frac{-cov_t \left(M_{t+1}, R_{t+1}^G \right)}{var_t[M_{t+1}]} \frac{var_t[M_{t+1}]}{E_t[M_{t+1}]} = \beta_t^G \lambda_t, \\
\mathbb{E}_t \left[R_{t+1}^D - R_t^f \right] &= \frac{-cov_t \left(M_{t+1}, R_{t+1}^D \right)}{\mathbb{E}_t[M_{t+1}]} = \frac{-cov_t \left(M_{t+1}, R_{t+1}^D \right)}{var_t[M_{t+1}]} \frac{var_t[M_{t+1}]}{E_t[M_{t+1}]} = \beta_t^D \lambda_t.
\end{aligned}$$

□

E.3 Proof of Proposition 2

Proof. Part (a) is proven in the text. What follows is the proof for part (b). Consider a government that only issues risk-free debt. Note that the surplus at $t+1$ is given by the government budget constraint:

$$S_{t+1} \quad = \quad d_t Y_t R_t^f - d_{t+1} Y_{t+1},$$

and the surplus at $t+2$ is given by:

$$S_{t+2} \quad = \quad d_{t+1} Y_{t+1} R_{t+1}^f - d_{t+2} Y_{t+2}.$$

Debt Policy depends on current shock Suppose the government commits to an arbitrary perturbation of the debt/output ratio at each date t+k, d_{t+k}, by an amount $\Delta_{t+k}(\varepsilon_{t+k})$ which only depends on the shock realization in the current period. Then the new surplus at $t+1$ is:

$$\tilde{S}_{t+1} \quad = \quad d_t Y_t R_t^f - (d_{t+1} + \Delta_{t+1}(\varepsilon_{t+1})) Y_{t+1},$$

and the new surplus at $t+2$ is given by:

$$\tilde{S}_{t+2} \quad = \quad R_{t+1}^f (d_{t+1} + \Delta_{t+1}(\varepsilon_{t+1})) Y_{t+1} - (d_{t+2} + \Delta_{t+2}(\varepsilon_{t+2})) Y_{t+2}.$$

The sum of the discounted perturbed surpluses is:

$$\begin{aligned}
\tilde{S}_{t+1} + \mathbb{E}_{t+1}[M_{t+1,t+2} \tilde{S}_{t+2}] &= d_t Y_t R_t^f - \mathbb{E}_{t+1} \left[M_{t+1,t+2} (d_{t+2} + \Delta_{t+2}(\varepsilon_{t+2})) Y_{t+2} \right] \\
&= S_{t+1} + \mathbb{E}_{t+1}[M_{t+1,t+2} S_{t+2}] - \mathbb{E}_{t+1}[M_{t+1,t+2} \Delta_{t+2}(\varepsilon_{t+2}) Y_{t+2}]
\end{aligned}$$

The innovation in the discounted surpluses is unchanged

$$(\mathbb{E}_{t+1}-\mathbb{E}_t)(\tilde{S}_{t+1}+\mathbb{E}_{t+1}[M_{t+1,t+2}\tilde{S}_{t+2}])=(\mathbb{E}_{t+1}-\mathbb{E}_t)(S_{t+1}+\mathbb{E}_{t+1}[M_{t+1,t+2}S_{t+2}])$$

because

$$(\mathbb{E}_{t+1}-\mathbb{E}_t)[M_{t+1,t+2}\Delta_{t+2}(\varepsilon_{t+2})Y_{t+2}]=0.$$

The last statement follows from the fact that Δ_{t+2} only depends on ε_{t+2} on not on ε_{t+1}. Moreover, the discounted surplus summed over periods $t+1$ and $t+2$ does not change in response to a shock at time $t+1$:

$$(\mathbb{E}_{t+1}-\mathbb{E}_t)(S_{t+1}+\mathbb{E}_{t+1}[M_{t+1,t+2}S_{t+2}])=0$$

because

$$(\mathbb{E}_{t+1}-\mathbb{E}_t)[M_{t+1,t+2}d_{t+2}(\varepsilon_{t+2})Y_{t+2}]=0.$$

The debt issuance policy at time $t+2$ only depends on ε_{t+2} on not on ε_{t+1}. This argument readily extends to the sum of any number of future discounted surplus terms.

Debt Policy depends on current and one-period lagged shock Next, suppose the government commits to an arbitrary perturbation of d_{t+k} by $\Delta_{t+k}(\varepsilon^2_{t+k})$, where $\varepsilon^2_{t+k}\equiv(\varepsilon_{t+k},\varepsilon_{t+k-1})$. Then we know that the new surpluses at $t+1$, $t+2$, and $t+3$ are given by:

$$\begin{aligned}\tilde{S}_{t+1} &= R^f_t d_t Y_t-(d_{t+1}+\Delta_{t+1}(\varepsilon^2_{t+1}))Y_{t+1},\\ \tilde{S}_{t+2} &= R^f_{t+1}(d_{t+1}+\Delta_{t+1}(\varepsilon^2_{t+1}))Y_{t+1}-(d_{t+2}+\Delta_{t+2}(\varepsilon^2_{t+2}))Y_{t+2},\\ \tilde{S}_{t+3} &= R^f_{t+2}(d_{t+2}+\Delta_{t+2}(\varepsilon^2_{t+2}))Y_{t+2})-(d_{t+3}+\Delta_{t+3}(\varepsilon^2_{t+3}))Y_{t+3}.\end{aligned}$$

The discounted sum of surpluses under the new and the old debt issuance policy are:

$$\begin{aligned}\tilde{S}_{t+1}+\mathbb{E}_{t+1}[M_{t+1,t+2}\tilde{S}_{t+2}]+\mathbb{E}_{t+2}[M_{t+1,t+3}\tilde{S}_{t+3}] &= S_{t+1}+\mathbb{E}_{t+1}[M_{t+1,t+2}S_{t+2}]+\mathbb{E}_{t+2}[M_{t+2,t+3}S_{t+3}]]-\\ &\quad \mathbb{E}_t[M_{t+1,t+3}\Delta_{t+3}(\varepsilon_{t+3},\varepsilon_{t+2})Y_{t+3}]\end{aligned}$$

The last term does not depend on ε_{t+1}

$$(\mathbb{E}_{t+1}-\mathbb{E}_t)[M_{t+1,t+3}\Delta_{t+3}(\varepsilon_{t+3},\varepsilon_{t+2})Y_{t+3}]=0.$$

Hence, the innovation in the discounted surpluses summed over three years is unchanged:

$$(\mathbb{E}_{t+1}-\mathbb{E}_t)(\tilde{S}_{t+1}+\mathbb{E}_{t+1}[M_{t+1,t+2}\tilde{S}_{t+2}])+\mathbb{E}_{t+1}[M_{t+1,t+3}\tilde{S}_{t+3}])=(\mathbb{E}_{t+1}-\mathbb{E}_t)(S_{t+1}+\mathbb{E}_{t+1}[M_{t+1,t+2}S_{t+2}]+\mathbb{E}_{t+1}[M_{t+1,t+3}S_{t+3}]).$$

Furthermore, the news in this discounted sum of surpluses is zero

$$(\mathbb{E}_{t+1}-\mathbb{E}_t)(S_{t+1}+\mathbb{E}_{t+1}[M_{t+1,t+2}S_{t+2}]+\mathbb{E}_{t+1}[M_{t+1,t+3}S_{t+3}])=0.$$

because

$$(\mathbb{E}_{t+1}-\mathbb{E}_t)[M_{t+1,t+3}d_{t+3}(\varepsilon_{t+3},\varepsilon_{t+2})Y_{t+3}]=0.$$

In other words, if there is a shock at time $t+1$, then the debt issuance policy and the surplus dynamics respond to this shock but the cumulative discounted surplus measured over periods $t+1$ to $t+3$ does not.

The general result for a debt policy with limited history dependence, $\Delta_{t+k}(\varepsilon^h_{t+k}))$, follows by induction. □

E.4 Proof of Proposition 3

Proof. We start from the one-period budget constraint:

$$T_t = G_t - (D_t - R^f_{t-1} D_{t-1}).$$

With TVC,

$$\begin{aligned} R^f_{t-1} D_{t-1} &= S_t + D_t = S_t + \frac{R^f_t D_t}{R^f_t} = S_t + \mathbb{E}_t[\exp(m_{t,t+1}) R^f_t D_t], \\ &= S_t + \mathbb{E}_t[\exp(m_{t,t+1})(S_{t+1} + \exp(m_{t+1,t+2}) R^f_{t+1} D_{t+1})] = \mathbb{E}_t[\sum_{k=0}^{\infty} M_{t,t+k} S_{t+k}]. \end{aligned}$$

So, this implies that we can state the value of outstanding debt at t:

$$\begin{aligned} R^f_t D_t &= \mathbb{E}_{t+1}[\sum_{k=0}^{\infty} \exp(m_{t+1,t+1+k}) S_{t+1+k}] = \mathbb{E}_{t+1}[\sum_{k=1}^{\infty} \exp(m_{t+1,t+k}) S_{t+k}] \\ D_t &= \mathbb{E}_t[\exp(m_{t,t+1})] \mathbb{E}_{t+1}[\sum_{k=1}^{\infty} \exp(m_{t+1,t+k}) S_{t+k}] \end{aligned}$$

Note that D_t is t-measurable,

$$D_t = \mathbb{E}_t[\sum_{k=1}^{\infty} M_{t,t+k} S_{t+k}],$$

and the measurability condition for risk-free debt is satisfied:

$$(\mathbb{E}_{t+1} - \mathbb{E}_t)[\sum_{k=1}^{\infty} M_{t,t+k} S_{t+k}] = 0.$$

Conjecture the pricing of the surplus strip is

$$\mathbb{E}_t\left[M_{t,t+k} Y_{t+k}\right] = \xi_k Y_t \tag{10}$$

for $k \geq 0$. Then the pricing of the first spending strip is

$$\begin{aligned} \mathbb{E}_t\left[\exp(m_{t,t+1}) Y_{t+1}\right] &= \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - 1)^2) Y_t, \\ \xi_1 Y_t &= \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2) Y_t. \end{aligned}$$

Similarly, the pricing of the second spending strip is

$$\begin{aligned} \mathbb{E}_t\left[\exp(m_{t,t+2}) Y_{t+2}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1}) \mathbb{E}_{t+1}[\exp(m_{t+1,t+2}) Y_{t+2}]\right], \\ &= \mathbb{E}_t\left[\exp(m_{t,t+1}) \xi_1 Y_{t+1}\right], \\ \xi_2 Y_t &= \xi_1 \mathbb{E}_t\left[\exp(m_{t,t+1} + \mu + \varepsilon_{t+1})\right] Y_t, \\ &= \xi_1 \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2) Y_t. \end{aligned}$$

The price of the output strips is given by

$$\begin{aligned} \mathbb{E}_t\left[M_{t,t+k} Y_{t+k}\right] &= \xi_k Y_t, \textit{ where} \\ \xi_k &= \xi_{k-1} \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2), k \geq 1 \end{aligned}$$

$$\xi_1 = \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2).$$

We define a k-period surplus strip as a claim to S_{t+k}. The price of the surplus strips is given by

$$\begin{aligned}
\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] &= \chi_k Y_t, \textit{ where} \\
\chi_k &= \chi_{k-1}\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2), \\
\chi_1 &= d\left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\right].
\end{aligned}$$

To replicate safe debt, we short dY_t risky strips to output next period, and we take a similarly sized long position in the risk-free. We implement the same strategy for all future output strips. Note that we cannot simply price these strips off the risk-free yield curve, even though the entire debt is risk-free.

The pricing of the first surplus strip is

$$\begin{aligned}
\mathbb{E}_t\left[\exp(m_{t,t+1})S_{t+1}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1})\{-dY_{t+1}\left(1 - R_t^f\exp[-(\mu+\varepsilon_{t+1})]\right)\}\right], \\
&= -d\mathbb{E}_t\left[\exp(m_{t,t+1})Y_{t+1}\right] + dY_tR_t^f\mathbb{E}_t\left[\exp(m_{t,t+1})\right], \\
&= -d\mathbb{E}_t\left[\exp(m_{t,t+1})Y_{t+1}\right] + dY_t, \\
&= -d\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)Y_t + dY_t, \\
&= \left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\right]dY_t. \\
\chi_1 Y_t &= \left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\right]dY_t.
\end{aligned}$$

Similarly, the pricing of the second surplus strip is

$$\begin{aligned}
\mathbb{E}_t\left[\exp(m_{t,t+2})S_{t+2}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1})\mathbb{E}_{t+1}[\exp(m_{t+1,t+2})S_{t+2}]\right], \\
&= \mathbb{E}_t\left[\exp(m_{t,t+1})\chi_1 Y_{t+1}\right], \\
\chi_2 Y_t &= \chi_1\mathbb{E}_t\left[\exp(m_{t,t+1} + \mu + \sigma\varepsilon_{t+1})\right]Y_t, \\
&= \chi_1\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)Y_t.
\end{aligned}$$

We short a risky strip to output 2 periods from now, and go long in the risk-free. The problem then becomes solving the fixed-point problem for the sequence z_k:

$$\begin{aligned}
\chi_2 &= \chi_1\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2), \\
\chi_k &= \chi_{k-1}\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2).
\end{aligned}$$

This fixed-point problem has a unique solution:

$$\sum_{k=1}^{\infty}\chi_k = \chi_1(1 + K + K^2 + \ldots) = \frac{1}{1-K}\chi_1 = d,$$

where $K = \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)$. We also have the following TVC:

$$\lim_{j\to\infty}\mathbb{E}_t\left[m_{t,t+j}D_{t+j}\right] = \lim_{j\to\infty}d\mathbb{E}_t\left[m_{t,t+j}Y_{t+j}\right] = 0.$$

□

E.5 Proof of Corollary 4

Proof. From the gross risk-free rate xpression $R^f_{t+1} = \exp(\rho)$ and the one-period government budget constraint, we get that:

$$\frac{T_t}{Y_t} = x - d\left(1 - R^f_{t-1}\frac{Y_{t-1}}{Y_t}\right),$$

we have that the return on the tax claim can be stated as:

$$\begin{aligned} R^T_{t+1} &= \frac{P^T_{t+1}}{P^T_t - T_t} = \frac{(d + x\frac{\xi_1}{1-\xi_1})Y_{t+1} + (x - d\left(1 - R^f_t \frac{Y_t}{Y_{t+1}}\right))Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t}, \\ &= \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t} + \frac{d\exp(\rho)}{(d + x\frac{\xi_1}{1-\xi_1})}. \end{aligned}$$

Similarly, the return on the spending claim can be stated as:

$$\begin{aligned} R^G_{t+1} &= \frac{P^G_{t+1}}{P^G_t - G_t} = \frac{x\frac{\xi_1}{1-\xi_1}Y_{t+1} + xY_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t}, \\ &= \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t}. \end{aligned}$$

Armed with these expressions, we get the following expression for the covariance:

$$cov(R^T_{t+1}, M_{t,t+1}) = \frac{x\frac{\xi_1}{1-\xi_1}}{(d + x\frac{\xi_1}{1-\xi_1})}cov(R^G_{t+1}, M_{t,t+1}),$$

which also translates to

$$\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}\mathbb{E}_t\left[R^Y_{t+1} - R^f_t\right].$$

□

E.6 Proof of Proposition 5: Case of $AR(1)$

Proof. From

$$T_t = G_t - (D_t - R^f_{t-1}D_{t-1}).$$

This implies that:

$$\begin{aligned} \frac{T_t}{Y_t} &= x - \left(d_t - R^f_{t-1}d_{t-1}\frac{Y_{t-1}}{Y_t}\right) \\ &= x - \left(d_t - R^f_{t-1}d_{t-1}\exp[-(\mu + \sigma\varepsilon_t)]\right). \end{aligned}$$

Assume that the debt/output ratio evolves according to a martingale process: $d_t = d_{t-1}\exp(-\lambda\varepsilon_t - (1/2)\lambda^2)$. To guarantee risk-free debt, the tax process has to satisfy

$$\frac{T_t}{Y_t} = x - d_{t-1}\left(d^{\phi-1}_{t-1}\exp(\phi_0 - \lambda\varepsilon_t - (1/2)\lambda^2) - R^f_{t-1}\frac{Y_{t-1}}{Y_t}\right),$$

$$= \quad x - d_{t-1}\left(d_{t-1}^{\phi-1}\exp(\phi_0 - \lambda\varepsilon_t - (1/2)\lambda^2) - R_{t-1}^f \exp[-(\mu + \sigma\varepsilon_t)]\right).$$

The surplus process that results is given by:

$$\frac{S_t}{Y_t} = d_{t-1}R_{t-1}^f \exp[-(\mu + \sigma\varepsilon_t)] - d_{t-1}^{\phi}\exp(\phi_0 - \lambda\varepsilon_t - \frac{1}{2}\lambda^2).$$

We conjecture that the price of the surplus strips is given by:

$$\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = (\chi_{k,t} - \psi_{k,t})Y_t.$$

The pricing of the first surplus strip is

$$\begin{aligned}
\mathbb{E}_t\left[\exp(m_{t,t+1})S_{t+1}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1})\{-Y_{t+1}\left(d_{t+1} - R_t^f d_t \exp[-(\mu + \sigma\varepsilon_{t+1})]\right)\}\right], \\
&= \mathbb{E}_t\left[-\exp(\phi\log d_t + m_{t,t+1} + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}\right] + d_t Y_t, \\
&= -\exp(\phi\log d_t + \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)Y_t + d_t Y_t, \\
(\chi_{1,t} - \psi_{1,t})Y_t &= \left[d_t - \exp(\phi_0 + \phi\log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)\right] Y_t.
\end{aligned}$$

So, we define:

$$\begin{aligned}
(\chi_{1,t})Y_t &= d_t Y_t, \\
(\psi_{1,t})Y_t &= \exp(\phi_0 + \phi\log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)Y_t.
\end{aligned}$$

Similarly the pricing of the k-th surplus strip is

$$\begin{aligned}
\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1})\mathbb{E}_{t+1}[\exp(m_{t+1,t+k})S_{t+k}]\right], \\
(\chi_{k,t} - \psi_{k,t})Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})(\chi_{k-1,t+1} - \psi_{k-1,t+1})Y_{t+1}\right],
\end{aligned}$$

where the χ's are defined by the following recursion:

$$\begin{aligned}
\chi_{2,t}Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})\chi_{1,t+1}Y_{t+1}\right], \\
\chi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(-\lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)\exp(\mu + \sigma\varepsilon_{t+1})\right]\exp(\phi\log d_t + \phi_0), \\
&= \exp(\phi_0 + \phi\log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2),
\end{aligned}$$

and where the ψ's are defined by the following recursion:

$$\begin{aligned}
\psi_{2,t}Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})\psi_{1,t+1}Y_{t+1}\right], \\
\psi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1} + \phi_0 + \phi\log d_{t+1} - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + g + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \mu + \sigma\varepsilon_{t+1})\right], \\
\psi_{2,t} &= \exp(-2\rho + \phi_0 + \phi\phi_0 + \phi^2\log d_t - \frac{1}{2}(\gamma^2 + \phi\lambda^2), \\
&- \frac{1}{2}(\gamma^2 + \lambda^2) + 2g + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \frac{1}{2}(\gamma + \lambda\phi - \sigma)^2), \\
&= \psi_{1,t}\exp(-\rho + \phi\phi_0 + (\phi^2 - \phi)\log d_t - \frac{1}{2}(\gamma^2 + \phi\lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda\phi - \sigma)^2).
\end{aligned}$$

Finally, we note that $\chi_{k+1,t} = \psi_{k,t}$, so that implies that:

$$\sum_{k=1}^{\infty} \mathbb{E}_t \left[M_{t,t+k} S_{t+k} \right] = \chi_{1,t} Y_t = D_t,$$

For some $0 < \phi < 1$, we have that

$$\begin{aligned}
\mathbb{E}_t[\exp(m_{t,t+1})D_{t+1}] &= \mathbb{E}_t[\exp(m_{t,t+1})Y_{t+1}d_{t+1}],\\
&= d_t^{\phi}\mathbb{E}_t[\exp(m_{t,t+1} - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}],\\
&= d_t^{\phi}\exp(\phi_0 - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2)Y_t,\\
&= \exp(\kappa_1)\exp(\phi\log d_t)Y_t,
\end{aligned}$$

where we used the debt/output dynamics. Define $\kappa_1 = \phi_0 - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2$.

$$\begin{aligned}
\mathbb{E}_t[\exp(m_{t,t+2})D_{t+2}] &= \mathbb{E}_t[\exp(m_{t,t+1})\mathbb{E}_{t+1}[\exp(m_{t+1,t+2})D_{t+2}]],\\
&= \mathbb{E}_t[\exp(m_{t,t+1})\exp(\kappa_1)\exp(\phi\log d_{t+1})Y_{t+1}],\\
&= \mathbb{E}_t[\exp(m_{t,t+1})\exp(\kappa_1)\exp(\phi^2\log d_t + \phi\phi_0 - \phi\lambda\varepsilon_{t+1} - \frac{1}{2}\phi\lambda^2)\exp(\mu+\sigma\varepsilon_{t+1})]Y_t,\\
&= \exp(\kappa_1+\kappa_2)\exp(\phi^2\log d_t)Y_t.
\end{aligned}$$

Define $\kappa_2 = \phi\phi_0 - \rho - \frac{1}{2}(\gamma^2+\phi\lambda^2) + g + \frac{1}{2}(\gamma+\phi\lambda-\sigma)^2$. Then:

$$\begin{aligned}
\lim_{j\to\infty}\mathbb{E}_t[\exp(m_{t,t+j})D_{t+j}] &= \lim_{j\to\infty}\exp(\sum_{k=1}^{j}\kappa_k)\exp(\phi^j\log d_t)Y_t,\\
&= \lim_{j\to\infty}\exp(\frac{\phi_0}{1-\phi} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi}) + \mu j + \sum_{k=1}^{j}\frac{1}{2}(\gamma+\lambda\phi^{k-1}-\sigma)^2)Y_t,\\
&= \lim_{j\to\infty}\exp(\frac{\phi_0}{1-\phi} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi}) + \mu j + j\frac{1}{2}(\gamma-\sigma)^2 + C)Y_t,
\end{aligned}$$

which is 0 if and only if $-\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) < 0$. This equality does not depend on ϕ and λ. So, this case is similar to the i.i.d. debt case $\phi = 0$. When $\lambda = 0$, $d_t = \exp(\phi_0)$ is a constant. Now, assume $\phi = 1$. Then $\kappa_j = \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2$, and $\lim_{j\to\infty}\mathbb{E}_t[\exp(m_{t,t+j})D_{t+j}] = \lim_{j\to\infty}\exp(\sum_{k=1}^{j}\kappa_k)\exp(\log d_t)Y_t$, which is 0 if and only if $\phi_0 - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2 < 0$.

□

E.7 Proof of Proposition 5: Case of $AR(2)$

Proof. From

$$T_t = G_t - (D_t - R^f_{t-1}D_{t-1}).$$

This implies that:

$$\begin{aligned}
S_t &= -\left(d_t Y_t - R^f_{t-1}d_{t-1}Y_{t-1}\right),\\
&= d_{t-1}R^f_{t-1}Y_{t-1} - \exp(\phi_0 + \phi_1\log d_{t-1} + \phi_2\log d_{t-2} - \lambda\varepsilon_t - \frac{1}{2}\lambda^2)Y_t.
\end{aligned}$$

Conjecture the price of the surplus strips is given by

$$\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = (\chi_{k,t} - \psi_{k,t})Y_t.$$

The pricing of the first surplus strip is

$$\begin{aligned}
\mathbb{E}_t\left[\exp(m_{t,t+1})S_{t+1}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1})\{-Y_{t+1}\left(d_{t+1} - R_t^f d_t \exp[-(\mu+\sigma\varepsilon_{t+1})]\right)\}\right], \\
&= \mathbb{E}_t\left[-\exp(\phi_1 \log d_t + \phi_2 \log d_{t-1} + m_{t,t+1} + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}\right] + d_t Y_t, \\
&= -\exp(\phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2)Y_t + d_t Y_t, \\
(\chi_{1,t} - \psi_{1,t})Y_t &= \left[d_t - \exp(\phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2)\right]Y_t.
\end{aligned}$$

We define

$$\begin{aligned}
(\chi_{1,t})Y_t &= d_t Y_t, \\
(\psi_{1,t})Y_t &= \exp(\phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2)Y_t.
\end{aligned}$$

Similarly the pricing of the k-th surplus strip is

$$\begin{aligned}
\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1})\mathbb{E}_{t+1}[\exp(m_{t+1,t+k})S_{t+k}]\right], \\
(\chi_{k,t} - \psi_{k,t})Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})(\chi_{k-1,t+1} - \psi_{k-1,t+1})Y_{t+1}\right],
\end{aligned}$$

where the χ's are defined by the following recursion:

$$\begin{aligned}
\chi_{2,t}Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})\chi_{1,t+1}Y_{t+1}\right], \\
\chi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(-\lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)\exp(\mu+\sigma\varepsilon_{t+1})\right]\exp(\phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0), \\
&= \exp(\phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2).
\end{aligned}$$

and the ψ's are defined by the following recursion:

$$\begin{aligned}
\psi_{2,t}Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})\psi_{1,t+1}Y_{t+1}\right], \\
\psi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1} + \phi_0 + \phi_1 \log d_{t+1} + \phi_2 \log d_t - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2 + \mu + \sigma\varepsilon_{t+1})\right], \\
\psi_{2,t} &= \exp(-2\rho + \phi_0 + \phi_1\phi_0 + (\phi_1^2+\phi_2)\log d_t + \phi_1\phi_2 \log d_{t-1} - \frac{1}{2}(\gamma^2+\phi_1\lambda^2), \\
&- \frac{1}{2}(\gamma^2+\lambda^2) + 2\mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2 + \frac{1}{2}(\gamma+\lambda\phi_1-\sigma)^2).
\end{aligned}$$

We note that $\chi_{k+1,t} = \psi_{k,t}$, so this expression can be simplified as follows:

$$\sum_{k=1}^{\infty}\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = \chi_{1,t}Y_t = D_t$$

$$d_t = \exp(\phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} + \phi_0 - \lambda\varepsilon_t - \frac{1}{2}\lambda^2).$$

$$\mathbb{E}_t[\exp(m_{t,t+1})D_{t+1}] = \mathbb{E}_t[\exp(m_{t,t+1})Y_{t+1}d_{t+1}],$$

$$= d_t^{\phi} \mathbb{E}_t[\exp(m_{t,t+1} - \lambda \varepsilon_{t+1} - \frac{1}{2}\lambda^2) Y_{t+1}],$$
$$= d_t^{\phi} \exp(\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2) Y_t,$$
$$= \exp(\kappa_1) \exp(\phi_1 \log d_t + \phi_2 \log d_{t-1}) Y_t,$$

Define $\kappa_1 = \phi_0 - \rho + -\frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2$.

$$\begin{aligned}
\mathbb{E}_t[\exp(m_{t,t+2}) D_{t+2}] &= \mathbb{E}_t[\exp(m_{t,t+1}) \mathbb{E}_{t+1}[\exp(m_{t+1,t+2}) D_{t+2}]], \\
&= \mathbb{E}_t[\exp(m_{t,t+1}) \exp(\kappa_1) \exp(\phi_1 \log d_{t+1} + \phi_2 \log d_t) Y_{t+1}], \\
&= \mathbb{E}_t[\exp(m_{t,t+1}) \exp(\kappa_1) \exp((\phi_1^2 + \phi_1\phi_2 + \phi_2) \log d_t + \phi_1\phi_0 - \phi_1 \lambda \varepsilon_{t+1} - \frac{1}{2}\phi_1 \lambda^2) \exp(\mu + \sigma \varepsilon_{t+1})] Y_t, \\
&= \exp(\kappa_1 + \kappa_2) \exp((\phi_1^2 + \phi_1\phi_2 + \phi_2) \exp(\log d_t) Y_t.
\end{aligned}$$

Define $\kappa_2 = \phi_1\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \phi_1\lambda^2) + \mu + \frac{1}{2}(\gamma + \phi_1\lambda - \sigma)^2$.

$$\begin{aligned}
\lim_{j\to\infty} \mathbb{E}_t[\exp(m_{t,t+j}) D_{t+j}] &= \lim_{j\to\infty} \exp(\sum_{k=1}^{j} \kappa_k) \exp(\psi_j \log d_t) Y_t, \\
&= \lim_{j\to\infty} \exp(\frac{\phi_0}{1-\phi_1-\phi_2} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi_1-\phi_2}) + \mu j + \sum_{k=1}^{j} \frac{1}{2}(\gamma + \lambda \psi_{k-1} - \sigma)^2) Y_t, \\
&= \lim_{j\to\infty} \exp(\frac{\phi_0}{1-\phi_1-\phi_2} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi_1-\phi_2}) + \mu j + j\frac{1}{2}(\gamma - \sigma)^2 + C) Y_t,
\end{aligned}$$

which is 0 if and only if $-\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) < 0$. This equality does not depend on ϕ and λ. So this case is similar to the i.i.d. debt case $\phi = 0$. More extremely, when $\lambda = 0$, $d_t = \exp(\phi_0)$ is a constant. Now, assume $\phi = 1$. Then

$$\kappa_j = \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2,$$

and $\lim_{j\to\infty} \mathbb{E}_t[\exp(m_{t,t+j}) D_{t+j}] = \lim_{j\to\infty} \exp(\sum_{k=1}^{j} \kappa_k) \exp(\log d_t) Y_t$, which is 0 if and only if $\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2 < 0$. □

E.8 Proof of Proposition 6 : Case of $AR(1)$

Proof. When the log of the debt/output process follows an $AR(1)$, the surplus/output ratio is given by:

$$\frac{S_{t+1}}{Y_{t+1}} = \exp(r_t^f - g - \sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \phi^j \lambda \varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) - \exp(+\phi(-\sum_{j=0}^{\infty} \phi^j \lambda \varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) + \phi_0 - \lambda \varepsilon_{t+1} - \frac{1}{2}\lambda^2).$$

We assume that $r_t^f = g$. This expression for the surplus/output ratio can be restated as:

$$\frac{S_{t+1}}{Y_{t+1}} = \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \phi^j \lambda \varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) - \exp(-\sum_{j=0}^{\infty} \phi^j \lambda \varepsilon_{t+1-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

Next, we compute the derivative of the surplus/output ratio at $t+1$:

$$\frac{\partial \frac{S_{t+1}}{Y_{t+1}}}{\partial \varepsilon_{t+1}} = (\lambda) \exp(g + \sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \phi^j \lambda \varepsilon_{t+1-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) - \sigma \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \phi^j \lambda \varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+1}}{Y_{t+1}}}{\partial \varepsilon_{t+1}} = + \quad (\lambda - \sigma)\exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

Next, we compute the derivative of the surplus/output ratio at $t+2$. The surplus/output ratio at $t+2$ is given by:

$$\frac{S_{t+2}}{Y_{t+2}} = -\lambda \exp(-\sum_{j=0}^{\infty} \phi^j \lambda \varepsilon_{t+1-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) + \lambda\phi \exp(\sigma \varepsilon_{t+1} - \sum_{j=0}^{\infty} \phi^j \lambda \varepsilon_{t+2-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+2}}{Y_{t+2}}}{\partial \varepsilon_{t+1}} = -\lambda \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) + \lambda\phi \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

This generalizes to the following expression. For $j \geq 2$, we obtain:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = -\lambda\phi^{j-1} \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) + \lambda\phi^j \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

Assume $r^f = g$. Then we obtain the IRF:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = \lambda\phi^{j-1}(\phi - 1)d, j > 1,$$
$$\frac{\partial \frac{S_{t+1}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = (\lambda - \sigma)d, j = 1.$$

□

E.9 Proof of Proposition 6: Case of $AR(2)$

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. We assume that $r_t^f = g$. When the log of the debt/output process follows an $AR(2)$, the surplus/output ratio is given by:

$$\frac{S_{t+1}}{Y_{t+1}} = \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \rho^j \lambda \varepsilon_{t-j} + \bar{d}) - \exp(+\bar{d} + \phi_1(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j}) + \phi_2(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-1-j}) - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2).$$

Next, we compute the derivative of the surplus/output ratio at $t+1$, and we evaluate this derivative at $\varepsilon_{t+j} = 0$:

$$\frac{\partial \frac{S_{t+1}}{Y_{t+1}}}{\partial \varepsilon_{t+1}} = + \quad (\lambda - \sigma)\exp(\bar{d})).$$

The surplus/output ratio at $t+2$ is given by:

$$\frac{S_{t+2}}{Y_{t+2}} = \exp(-\sigma\varepsilon_{t+2} - \sum_{j=0}^{\infty} \rho^j \lambda \varepsilon_{t+1-j} + \bar{d}) - \exp(+\bar{d} + \phi_1(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+1-j}) + \phi_2(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j}) - \lambda\varepsilon_{t+2} - \frac{1}{2}\lambda^2).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+2}}{Y_{t+2}}}{\partial \varepsilon_{t+1}} = -\lambda \exp(\bar{d}) + \lambda(\phi_1)\exp(\bar{d})).$$

The surplus/output ratio at $t+3$ is given by:

$$\frac{S_{t+3}}{Y_{t+3}} = \exp(-\sigma\varepsilon_{t+3} - \sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+2-j} + \bar{d}) - \exp(\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+2-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+1-j}) - \lambda\varepsilon_{t+3} - \frac{1}{2}\lambda^2).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial\frac{S_{t+3}}{Y_{t+3}}}{\partial\varepsilon_{t+1}} = -\psi_1\lambda\exp(\bar{d}) + \lambda(\phi_1\psi_1 + \phi_2)\exp(\mu + \bar{d}).$$

This generalizes to the following expression. For $j > 2$, we obtain:

$$\frac{\partial\frac{S_{t+j}}{Y_{t+j}}}{\partial\varepsilon_{t+1}} = -\lambda\psi_{j-1}\exp(+\bar{d})) + \lambda\psi_j\exp(\bar{d})).$$

Assume $r^f = \mu$. Then we obtain the IRF:

$$\begin{aligned}\frac{\partial\frac{S_{t+j}}{Y_{t+j}}}{\partial\varepsilon_{t+1}} &= (\lambda - \sigma)\exp(\bar{d}),\ \textit{for}\ j = 1,\\ &= \lambda(\phi_1 - 1)\exp(\bar{d}),\ \textit{for}\ j = 2,\\ &= \lambda(\psi_{j-1} - \psi_{j-2})\exp(\bar{d}),\ \textit{for}\ j > 2.\end{aligned}$$

□

E.10 Proof of Proposition 6: Case of AR(3)

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. We assume that the risk-free rate equals the growth rate of the economy. When the log of the debt/output process follows an AR(3), the surplus/output ratio is given by:

$$\begin{aligned}\frac{S_{t+1}}{Y_{t+1}} &= \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\rho^j\lambda\varepsilon_{t-j} + \bar{d})\\ &- \exp(+\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-1-j} + \phi_3(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-2-j}) - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2).\end{aligned}$$

Next, we compute the derivative of the surplus/output ratio at $t+1$, and we evaluate this derivative at $\varepsilon_{t+j} = 0$:

$$\frac{\partial\frac{S_{t+1}}{Y_{t+1}}}{\partial\varepsilon_{t+1}} = + \quad (\lambda - \sigma)\exp(\bar{d})).$$

The surplus/output ratio at $t+2$ is given by:

$$\begin{aligned}\frac{S_{t+2}}{Y_{t+2}} &= \exp(-\sigma\varepsilon_{t+2} - \sum_{j=0}^{\infty}\rho^j\lambda\varepsilon_{t+1-j} + \bar{d})\\ &- \exp(\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+1-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-j}) + \phi_3(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-1-j}) - \lambda\varepsilon_{t+2} - \frac{1}{2}\lambda^2).\end{aligned}$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial\frac{S_{t+2}}{Y_{t+2}}}{\partial\varepsilon_{t+1}} = -\lambda\exp(\bar{d})) + \lambda(\phi_1)\exp(\bar{d})).$$

The surplus/output ratio at $t+3$ is given by:

$$\begin{aligned}\frac{S_{t+3}}{Y_{t+3}} &= \exp(-\sigma\varepsilon_{t+3} - \sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+2-j} + \bar{d}) \\ &- \exp(\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+2-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+1-j} + \phi_3(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-j}) - \lambda\varepsilon_{t+3} - \frac{1}{2}\lambda^2).\end{aligned}$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial\frac{S_{t+3}}{Y_{t+3}}}{\partial\varepsilon_{t+1}} = -\psi_1\lambda\exp(\bar{d}) + \lambda(\phi_1\psi_1 + \phi_2)\exp(\mu + \bar{d}).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial\frac{S_{t+4}}{Y_{t+4}}}{\partial\varepsilon_{t+1}} = -\rho_2\lambda\exp(\bar{d}) + \lambda(\phi_1\rho_2 + \phi_2\psi_1 + \phi_3)\exp(\mu + \bar{d}).$$

This generalizes to the following expression. For $j > 2$, we obtain:

$$\frac{\partial\frac{S_{t+j}}{Y_{t+j}}}{\partial\varepsilon_{t+1}} = -\lambda\psi_{j-1}\exp(\bar{d}) + \lambda\psi_j\exp(\mu + \bar{d}).$$

Assume $r^f = \mu$. Then we obtain the IRF:

$$\begin{aligned}\frac{\partial\frac{S_{t+j}}{Y_{t+j}}}{\partial\varepsilon_{t+1}} &= (\lambda - \sigma)\exp(\bar{d}), \textit{ for } j = 1, \\ &= \lambda(\phi_1 - 1)\exp(\bar{d}), \textit{ for } j = 2, \\ &= \lambda(\phi_1\psi_1 + \phi_2 - \psi_1)\exp(\bar{d}), \textit{ for } j = 3, \\ &= \lambda(\psi_{j-1} - \psi_{j-2})\exp(\bar{d}), \textit{ for } j > 3.\end{aligned}$$

□

E.11 Proof of Proposition 7: Case of *AR(1)*

Proof. As a result, we can solve for an expression of the log debt/output ratio as a function of the past shocks:

$$\log d_t = -\sum_{j=0}^{\infty}\phi^j\lambda\varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}.$$

Consider a government that only issues risk-free debt. Note that the surplus at $t+1$ is given by:

$$S_{t+1} = d_t Y_t \exp(r_t^f) - \exp(\phi\log d_t + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}.$$

We get the following expression for the covariance:

$$\begin{aligned}cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}) \\ &= -E_t[M_{t+1}d_{t+1}Y_{t+1}] + E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\ &= -\exp(-\rho - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \mu + y_t + \phi\log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\ &+ \exp(-\rho)\exp(\frac{1}{2}(\lambda - \sigma)^2 + \mu + y_t + \phi\log d_t + \phi_0 - \frac{1}{2}\lambda^2)\end{aligned}$$

$$
\begin{aligned}
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma+\lambda-\sigma)^2 - \frac{1}{2}(\lambda-\sigma)^2) - 1)E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma-\lambda)) - 1).
\end{aligned}
$$

By the same token, we get the following expression for the covariance of the discounted surpluses over two periods:

$$
\begin{aligned}
& cov_t(M_{t+1}, S_{t+1} + E_{t+1}[M_{t+1,t+2}S_{t+2})] \\
= \;& cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}]) \\
= \;& -E_t[M_{t+1}]E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}](\exp(-\gamma(\sigma-\phi\lambda)) - 1)
\end{aligned}
$$

Check the proof of Prop. 2 to see why the sum of the discounted surpluses drop out, and only the debt issuance term remains. We get the following expression for the covariance of the discounted surpluses over j periods:

$$
\begin{aligned}
& cov_t(M_{t+1}, \sum_{k=1}^{j} E_{t+1}[M_{t+1,t+j}S_{t+j}]) \\
= \;& cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]) \\
= \;& -E_t[M_{t+1}]E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda)) - 1).
\end{aligned}
$$

□

E.12 Proof of Proposition 7: case of *AR(2)*

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. As a result, we can solve for an expression of the log debt/output ratio as a function of the past shocks:

$$
\log d_t = -\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1-\phi_2}.
$$

where $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}$. Consider a government that only issues risk-free debt. Note that the surplus at $t+1$ is given by:

$$
S_{t+1} = d_t Y_t \exp(r_t^f) - \exp(+\phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}.
$$

As a result, we get the following expression for the covariance:

$$
\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}) \\
&= -E_t[M_{t+1}d_{t+1}Y_{t+1}] + E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -\exp(-\rho - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma+\lambda-\sigma)^2 + \mu + y_t + \phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho)\exp(\frac{1}{2}(\lambda-\sigma)^2 + \mu + y_t + \phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma+\lambda-\sigma)^2 - \frac{1}{2}(\lambda-\sigma)^2) - 1)E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma-\lambda)) - 1).
\end{aligned}
$$

By the same token, we get the following expression for the covariance of the discounted surpluses over two periods:

$$
\begin{aligned}
& cov_t(M_{t+1}, S_{t+1} + E_{t+1}[M_{t+1,t+2}S_{t+2})] \\
= \;& cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}]) \\
= \;& -E_t[M_{t+1}]E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}](\exp(-\gamma(\sigma-\psi_1\lambda)) - 1)
\end{aligned}
$$

Check the proof of Prop. 2 to see why the sum of the discounted surpluses drop out, and only the debt issuance term remains. And we get the following expression for the covariance of the discounted surpluses over j periods:

$$\begin{aligned} & cov_t(M_{t+1}, \sum_{k=1}^{j} E_{t+1}[M_{t+1,t+j}S_{t+j}]) \\ = \; & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]) \\ = \; & -E_t[M_{t+1}]E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\psi_{j-1}\lambda))-1). \end{aligned}$$

□

E.13 Proof of Corollary ??: Case of *AR(1)*

Proof. Start from the restriction:

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\ = \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda))-1) \\ + \; & x cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}Y_{t+k}\right) \\ = \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda))-1) \\ + \; & x\sum_{k=1}^{j} E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}](\exp(-\gamma\sigma)-1) \end{aligned}$$

We substitute for the price of debt strips:

$$\begin{aligned} & \mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] \\ = \; & \exp(\sum_{k=1}^{j}\kappa_k)\exp(\phi^j\log d_t)Y_t \\ = \; & \exp(\frac{\phi_0(1-\phi^j)}{1-\phi}-\rho j-\frac{1}{2}(\gamma^2 j+\frac{\lambda^2(1-\phi^j)}{1-\phi})+\mu j+\sum_{k=1}^{j}\frac{1}{2}(\gamma+\lambda\phi^{k-1}-\sigma)^2)\exp(\phi^j\log d_t)Y_t \end{aligned}$$

For $j>1$, we obtain the following expression:

$$\begin{aligned} & \mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\ = \; & \exp(\frac{\phi_0(1-\phi^{j-1})}{1-\phi}-\rho(j-1)-\frac{1}{2}(\gamma^2(j-1)+\frac{\lambda^2(1-\phi^{j-1})}{1-\phi})+\mu(j-1)+\sum_{k=1}^{j-1}\frac{1}{2}(\gamma+\lambda\phi^{k-1}-\sigma)^2) \\ & \exp(\phi^{j-1}\log d_{t+1})Y_{t+1}, \end{aligned}$$

and, for $j=1$, we get that:

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] = \exp(\frac{\phi_0}{1-\phi})\exp(\log d_{t+1})Y_{t+1}.$$

For $j>1$, this simplifies to the following expression:

$$\begin{aligned} & \mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\ = \; & \exp(\frac{1-\phi^j}{1-\phi}(\phi_0-\frac{1}{2}\lambda^2)-\rho(j-1)-\frac{1}{2}\gamma^2(j-1)+\mu j+\sum_{k=1}^{j-1}\frac{1}{2}(\gamma+\lambda\phi^{k-1}-\sigma)^2) \\ & \exp(\phi^j\log d_t+\frac{1}{2}(-\phi^{j-1}\lambda+\sigma)^2)Y_t. \end{aligned}$$

Note that by a similar logic, the price of the output strips is given by:

$$\begin{aligned}
& \mathbb{E}_t[M_{t+1,t+j}Y_{t+j}] \\
= \quad & \exp(-\rho(j-1) - \frac{1}{2}\gamma^2(j-1) + \mu j + (j-1)\frac{1}{2}(\gamma-\sigma)^2 + \frac{1}{2}(\sigma)^2)Y_t
\end{aligned}$$

To summarize, for $j > 1$, this implies that we have the following expression:

$$\begin{aligned}
& \mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \quad & \mathbb{E}_t[M_{t+1,t+j}Y_{t+j}]\exp(\frac{1-\phi^j}{1-\phi}(\phi_0 - \frac{1}{2}\lambda^2) + \sum_{k=1}^{j-1}\frac{1}{2}((\lambda\phi^{k-1})^2 + 2(\gamma-\sigma)\lambda\phi^{k-1})) \\
& \exp(\phi^j \log d_t + \frac{1}{2}((\phi^{j-1}\lambda)^2 - 2\sigma\phi^{j-1}\lambda)).
\end{aligned}$$

and for $j = 1$, we get that:

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] = \exp(\frac{\phi_0}{1-\phi})\exp(\phi\log d_t)\exp(\mu + \frac{1}{2}\sigma^2)Y_t.$$

□

E.14 Proof of Corollary 2: Case of *AR(2)*

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. Start from the restriction:

$$\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\
= \quad & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma - \psi_{j-1}\lambda)) - 1) \\
+ \quad & x cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}Y_{t+k}\right) \\
= \quad & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma - \psi_{j-1}\lambda)) - 1) \\
+ \quad & x\sum_{k=1}^{j} E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}](\exp(-\gamma\sigma) - 1)
\end{aligned}$$

We substitute for the price of debt strips:

$$\begin{aligned}
& \mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] \\
= \quad & \exp(\sum_{k=1}^{j}\kappa_k)\exp(\psi_j\log d_t)Y_t \\
= \quad & \exp(\sum_{k=1}^{j}\psi_{k-1}\phi_0 - \rho j - \frac{1}{2}(\gamma^2 j + \sum_{k=1}^{j}\psi_{k-1}\lambda^2) + gj + \sum_{k=1}^{j}\frac{1}{2}(\gamma + \lambda\psi_{k-1} - \sigma)^2)\exp(\psi_j\log d_t)Y_t
\end{aligned}$$

For $j > 1$, we obtain the following expression:

$$\begin{aligned}
& \mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \quad & \exp(\sum_{k=1}^{j-1}\psi_{k-1}\phi_0 - \rho(j-1) - \frac{1}{2}(\gamma^2(j-1) + \sum_{k=1}^{j-1}\psi_{k-1}\lambda^2) + \mu(j-1) + \sum_{k=1}^{j-1}\frac{1}{2}(\gamma + \lambda\psi_{k-1} - \sigma)^2) \\
& \exp(\psi_{j-1}\log d_{t+1})Y_{t+1},
\end{aligned}$$

and, for $j = 1$, we get that:

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] = \exp(\frac{\phi_0}{1-\phi_1-\phi_2})\exp(\log d_{t+1})Y_{t+1}.$$

For $j > 1$, this simplifies to the following expression:

$$\begin{aligned}
& \mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \quad & \exp(\sum_{k=1}^{j}\psi_{k-1}(\phi_0 - \frac{1}{2}\lambda^2) - \rho(j-1) - \frac{1}{2}\gamma^2(j-1) + \mu j + \sum_{k=1}^{j-1}\frac{1}{2}(\gamma+\lambda\psi_{k-1}-\sigma)^2) \\
& \exp(\rho_j \log d_t + \frac{1}{2}(-\psi_{j-1}\lambda+\sigma)^2)Y_t.
\end{aligned}$$

Note that by a similar logic, the price of the output strips is given by:

$$\begin{aligned}
& \mathbb{E}_t[M_{t+1,t+j}Y_{t+j}] \\
= \quad & \exp(-\rho(j-1) - \frac{1}{2}\gamma^2(j-1) + \mu j + (j-1)\frac{1}{2}(\gamma-\sigma)^2 + \frac{1}{2}(\sigma)^2)Y_t.
\end{aligned}$$

To summarize, for $j > 1$, this implies that we have the following expression:

$$\begin{aligned}
& \mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \quad & \mathbb{E}_t[M_{t+1,t+j}Y_{t+j}]\exp(\sum_{k=1}^{j}\psi_{k-1}(\phi_0 - \frac{1}{2}\lambda^2) + \sum_{k=1}^{j-1}\frac{1}{2}((\lambda\psi_{k-1})^2 + 2(\gamma-\sigma)\lambda\psi_{k-1})) \\
& \exp(\psi_j \log d_t + \frac{1}{2}((\psi_{j-1}\lambda)^2 - 2\sigma\psi_{j-1}\lambda)),
\end{aligned}$$

and for $j = 1$, we get that:

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] = \exp((\frac{\phi_0}{1-\phi_1-\phi_2})\exp(\phi_1 \log d_t + \phi_2 \log d_{t-1})\exp(g + \frac{1}{2}\sigma^2)Y_t.$$

□

E.15 Proof of Proposition 10

Proof. Notice

$$\begin{aligned}
dT_t &= d\exp(y_t)\exp(\tau_t) + \exp(y_t)d\exp(\tau_t) + [d\exp(y_t), d\exp(\tau_t)]dt \\
&= T_t((\mu dt + \frac{1}{2}\gamma^2 dt + \gamma dZ_t) + (\theta(\bar{\tau}-\tau_t)dt + \frac{1}{2}(\beta_\tau\gamma)^2 dt + \beta_\tau\gamma dZ_t) + \beta_\tau\gamma^2 dt) \\
&= T_t((\mu + \theta(\bar{\tau}-\tau_t) + \frac{1}{2}(1+\beta_\tau)^2\gamma^2)dt + (1+\beta_\tau)\gamma dZ_t)
\end{aligned}$$

Conjecture

$$\begin{aligned}
P_t^\tau &= f_\tau(\tau_t)T_t \\
P_t^g &= f_g(g_t)G_t
\end{aligned}$$

then

$$\begin{aligned}
dP_t^\tau &= df_\tau T_t + f_\tau dT_t + [df_\tau, dT_t]dt \\
&= T_t(f_\tau' d\tau_t + \frac{1}{2}f_\tau'' \beta_\tau^2\gamma^2 dt) + f_\tau'\beta_\tau\gamma T_t(1+\beta_\tau)\gamma dt
\end{aligned}$$

$$\begin{aligned}
&+ \quad f_\tau T_t((\mu+\theta(\bar\tau-\tau_t))+\frac{1}{2}(1+\beta_\tau)^2\gamma^2)dt+(1+\beta_\tau)\gamma dZ_t) \\
&= \quad T_t\left(f'_\tau\theta(\bar\tau-\tau_t)+\frac{1}{2}f''_\tau\beta_\tau^2\gamma^2+f'_\tau\beta_\tau(1+\beta_\tau)\gamma^2+f_\tau(\mu+\theta(\bar\tau-\tau_t)+\frac{1}{2}(1+\beta_\tau)^2\gamma^2)\right)dt \\
&+ \quad T_t\left(f_\tau(1+\beta_\tau)+f'_\tau\beta_\tau\right)\gamma dZ_t
\end{aligned}$$

Substitute into the Euler equation,

$$\begin{aligned}
0 &= \mathcal{A}[M_tT_tdt+dM_tP_t^\tau+M_tdP_t^\tau+[dM_t,dP_t^\tau]dt] \\
-1 &= -rf_\tau+f'_\tau\theta(\bar\tau-\tau_t)+\frac{1}{2}f''_\tau\beta_\tau^2\gamma^2+f'_\tau\beta_\tau(1+\beta_\tau)\gamma^2+f_\tau(\mu+\theta(\bar\tau-\tau_t)+\frac{1}{2}(1+\beta_\tau)^2\gamma^2) \\
&- \gamma f_\tau(1+\beta_\tau)\gamma-\gamma f'_\tau\beta_\tau\gamma
\end{aligned}$$

We take a continuous time version of Campbell-Shiller approximation (Eraker Shaliastovich (2008)):

$$\begin{aligned}
dr_t^\tau &= \log\frac{P_{t+dt}^\tau+T_{t+dt}}{P_t^\tau} \\
&\approx \kappa_0^\tau dt+\kappa_1^\tau d\log f_\tau-(1-\kappa_1^\tau)\log f_\tau dt+d\log T_t
\end{aligned}$$

The Euler equation is

$$\begin{aligned}
0 &= \mathcal{A}[d\exp(m_t+\int_0^t dr_k^\tau)] \\
&= \mathcal{A}[dm_t+dr_t^\tau+\frac{1}{2}[dm_t+dr_t^\tau,dm_t+dr_t^\tau]dt]
\end{aligned}$$

Then, we conjecture $f_\tau(\tau_t)=\exp(p_\tau+q_\tau\tau_t)$,

$$\begin{aligned}
0 &= \mathcal{A}[dm_t+dr_t^\tau+\frac{1}{2}[dm_t+dr_t^\tau,dm_t+dr_t^\tau]dt] \\
&= \mathcal{A}[-(r+\frac{1}{2}\gamma^2)dt-\gamma dZ_t+\kappa_0^\tau dt+\kappa_1^\tau d\log f_\tau-(1-\kappa_1^\tau)\log f_\tau dt+d\log T_t \\
&+ \frac{1}{2}[-\gamma dZ_t+dr_t^\tau,-\gamma dZ_t+dr_t^\tau]dt] \\
&= -(r+\frac{1}{2}\gamma^2)+\kappa_0^\tau+\kappa_1^\tau q_\tau\theta(\bar\tau-\tau_t)-(1-\kappa_1^\tau)(p_\tau+q_\tau\tau_t)+\mu+\theta(\bar\tau-\tau_t) \\
&+ \frac{1}{2}((1+\beta_\tau+\kappa_1^\tau q_\tau\beta_\tau)\gamma-\gamma)^2
\end{aligned}$$

which implies

$$\begin{aligned}
r &= \mu-\frac{1}{2}\gamma^2+\kappa_0^\tau+\kappa_1^\tau q_\tau\theta\bar\tau-(1-\kappa_1^\tau)p_\tau+\theta\bar\tau+\frac{1}{2}((1+\beta_\tau+\kappa_1^\tau q_\tau\beta_\tau)\gamma-\gamma)^2 \\
q_\tau &= -\frac{\theta}{\kappa_1^\tau\theta+(1-\kappa_1^\tau)}
\end{aligned}$$

Since κ_1^τ is a constant close to but lower than 1, and $0<\theta<1$, $-1<q_\tau<0$. To see this, note

$$q_\tau-(-1) = \frac{(1-\kappa_1^\tau)(1-\theta)}{\kappa_1^\tau\theta+(1-\kappa_1^\tau)}>0.$$

Similarly, $f_g(g_t)=\exp(p_g+q_g g_t)$, where

$$\begin{aligned}
r &= \mu-\frac{1}{2}\gamma^2+\kappa_0^g+\kappa_1^g q_g\theta\bar g-(1-\kappa_1^g)p_g+\theta\bar g+\frac{1}{2}((1+\beta_g+\kappa_1^g q_g\beta_g)\gamma-\gamma)^2 \\
q_g &= -\frac{\theta}{\kappa_1^g\theta+(1-\kappa_1^g)}
\end{aligned}$$

Then,

$$\begin{aligned}
dB_t &= dP_t^\tau - dP_t^g \\
&= T_t\left(f'_\tau\theta(\bar{\tau}-\tau_t)+\frac{1}{2}f''_\tau\beta_\tau^2\gamma^2+f'_\tau\beta_\tau(1+\beta_\tau)\gamma^2+f_\tau(\mu+\theta(\bar{\tau}-\tau_t)+\frac{1}{2}(1+\beta_\tau)^2\gamma^2)\right)dt \\
&+ T_t\left(f_\tau(1+\beta_\tau)+f'_\tau\beta_\tau\right)\gamma dZ_t \\
&- G_t\left(f'_g\theta(\bar{g}-g_t)+\frac{1}{2}f''_g\beta_g^2\gamma^2+f'_g\beta_g(1+\beta_g)\gamma^2+f_g(\mu+\theta(\bar{g}-g_t)+\frac{1}{2}(1+\beta_g)^2\gamma^2)\right)dt \\
&- G_t\left(f_g(1+\beta_g)+f'_g\beta_g\right)\gamma dZ_t
\end{aligned}$$

The risk exposure of the debt return is

$$\begin{aligned}
[r_t^B, dM_t] &= -\frac{M_t\gamma\gamma}{B_t}\left(T_t\left(f_\tau(1+\beta_\tau)+f'_\tau\beta_\tau\right)-G_t\left(f_g(1+\beta_g)+f'_g\beta_g\right)\right) \\
&= -\frac{M_t\gamma\gamma}{B_t}\left(T_tf_\tau\left(1+(1+q_\tau)\beta_\tau\right)-G_tf_g\left(1+(1+q_g)\beta_g\right)\right)
\end{aligned}$$

□

E.16 Proof of Proposition 11

In this case,

$$\begin{aligned}
dB_t &= dP_t^\tau + dP_t^\kappa - dP_t^g \\
&= (...)dt \\
&+ (f_\tau T_t(1+(1+q_\tau)\beta_\tau)+f_\kappa K_t(1+(1+q_\kappa)\beta_\kappa)-f_gG_t(1+(1+q_g)\beta_g))\gamma dZ_t
\end{aligned}$$

and the return on the government debt is

$$r_t^B = \frac{(T_t+K_t-G_t)dt+dB_t}{B_t}$$

The risk exposure of the debt return is

$$[r_t^B, dM_t] = -\frac{M_t\gamma\gamma}{B_t}\left(f_\tau T_t(1+(1+q_\tau)\beta_\tau)+f_\kappa K_t(1+(1+q_\kappa)\beta_\kappa)-f_gG_t(1+(1+q_g)\beta_g)\right)$$

E.17 Proof of Proposition 9

Proof. Iterate the debt valuation equation,

$$\lim_{u\to\infty}\mathbb{E}_0M_uB_u = M_0B_0+\lim_{u\to\infty}\mathbb{E}_0[\int_0^u d(M_tB_t)] \tag{11}$$

If the following TVC,

$$\lim_{u\to\infty}\mathbb{E}_0M_uB_u = 0 \tag{12}$$

is satisfied, then

$$M_0B_0 = -\lim_{u\to\infty}\mathbb{E}_0[\int_0^u d(M_tB_t)] = \mathbb{E}_0[\int_0^\infty M_t(T_t-G_t)dt] \tag{13}$$

or

$$B_t = P_t^\tau - P_t^g \tag{14}$$

□

E.18 Proof of Corollary 4

Proof. From $R^f_{t+1} = \rho\exp(\rho)$ and $\frac{T_t}{Y_t} = x - d\left(1 - R^f_{t-1}\frac{Y_{t-1}}{Y_t}\right)$, we have that the return on the tax claim can be stated as:

$$\begin{aligned} R^T_{t+1} &= \frac{P^T_{t+1}}{P^T_t - T_t} = \frac{(d + x\frac{\xi_1}{1-\xi_1})Y_{t+1} + (x - d\left(1 - R^f_t\frac{Y_t}{Y_{t+1}}\right))Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t} \\ &= \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t} + \frac{d\exp(\rho)}{(d + x\frac{\xi_1}{1-\xi_1})}. \end{aligned}$$

Similarly, we have an expression for the return on the spending claim:

$$R^G_{t+1} = \frac{P^G_{t+1}}{P^G_t - G_t} = \frac{x\frac{\xi_1}{1-\xi_1}Y_{t+1} + xY_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t} = \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t}.$$

As a result, we can state the risk premium as follows:

$$\begin{aligned} \mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] &= -\frac{cov\left(M_{t+1}, R^T_{t+1}\right)}{E_t(M_{t+1})} = \frac{x}{d(1-\xi_1) + x\xi_1}\frac{-cov\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})}, \\ \mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] &= -\frac{cov\left(M_{t+1}, R^G_{t+1}\right)}{E_t(M_{t+1})} = \frac{1}{\xi_1}\frac{-cov\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})}, \end{aligned}$$

where we have used that $\xi_1 = \exp(-\rho - \frac{1}{2}\gamma^2 + g + \frac{1}{2}(\gamma - \sigma)^2) = \exp(-\rho - \gamma\sigma + g + \frac{1}{2}\sigma^2)$.

Then plug in

$$\begin{aligned} \frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})} &= \frac{-cov_t\left(\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}), \exp(g + \sigma\varepsilon_{t+1})\right)}{E_t(\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}))} \\ &= \frac{-cov_t\left(\exp(-\gamma\varepsilon_{t+1}), \exp(\sigma\varepsilon_{t+1})\right)}{\exp(-\rho)}\exp(-\rho - \frac{1}{2}\gamma^2 + g) \\ &= -(\exp(\frac{1}{2}(\gamma^2 + \sigma^2))(\exp(-\gamma\sigma) - 1))\exp(-\frac{1}{2}\gamma^2 + g) \\ &= \exp(g + \frac{1}{2}\sigma^2)(1 - \exp(-\gamma\sigma)) \end{aligned}$$

□

E.19 Proof of Proposition 13

Proof. Since

$$S_{t+1} = d_t Y_t \exp(r^f_t) - d_{t+1}Y_{t+1},$$

we get the following expression for the covariance:

$$\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}) \\
&= -E_t[M_{t+1}d_{t+1}Y_{t+1}] + E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -\exp(-\rho - \frac{\gamma}{\sigma}(\psi-1)y_t - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma+\lambda-\sigma)^2 + \xi_0 + \xi y_t + \phi \log d_t \\
&+ \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho - \frac{\gamma}{\sigma}(\xi-1)y_t)\exp(\frac{1}{2}(\lambda-\sigma)^2 + \xi_0 + \xi y_t + \phi\log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma+\lambda-\sigma)^2 - \frac{1}{2}(\lambda-\sigma)^2) - 1)E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma-\lambda)) - 1).
\end{aligned}$$

By the same token, we get the following expression for the covariance of the discounted surpluses over $j \geq 2$ periods:

$$\begin{aligned}
& cov_t(M_{t+1}, E_{t+1}[\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}]) \\
= \; & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]) \\
= \; & -E_t[M_{t+1}M_{t+1,t+j}d_{t+j}Y_{t+j}] + E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \; & -E_t[\exp(-\rho - \frac{\gamma}{\sigma}(\xi-1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(\ldots - \frac{\gamma(\xi-1)}{\sigma}(1+\xi+\ldots+\xi^{j-2})y_{t+1}) \\
& \exp(\phi^j \log d_t - \phi^{j-1}\lambda\varepsilon_{t+1} + \ldots)\exp(\xi^j y_t + \xi^{j-1}\sigma\varepsilon_{t+1} + \ldots)] + E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda - \frac{\gamma(\xi-1)}{\sigma}\frac{1-\xi^{j-1}}{1-\xi})) - 1) \\
= \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1-\xi^{j-1}))) - 1)
\end{aligned}$$

□

E.20 Proof of Corollary 6

Proof. Start from the restriction:

$$\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\
= \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1-\xi^{j-1}))) - 1) \\
+ \; & x cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}Y_{t+k}\right)
\end{aligned}$$

where

$$\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)M_{t+1,t+k}Y_{t+k}\right) \\
= \; & E_t[M_{t+1}M_{t+1,t+k}Y_{t+k}] - E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}] \\
= \; & E_t[\exp(-\rho - \frac{\gamma}{\sigma}(\xi-1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})M_{t+1,t+k}\exp(\xi^k y_t + \xi^{k-1}\sigma\varepsilon_{t+1} + \ldots)] \\
- \; & E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}] \\
= \; & -E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}](\exp(-\gamma(\xi^{k-1}\sigma + \frac{\gamma}{\sigma}(1-\xi^{k-1}))) - 1).
\end{aligned}$$

Next, we conjecture

$$\mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] \quad = \quad \exp(\sum_{k=1}^{j}\tilde{\kappa}_k)\exp(\phi^j \log d_t + f_j y_t)$$

Note

$$\begin{aligned}
\mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] &= \mathbb{E}_t[M_{t,t+1}\exp(\sum_{k=1}^{j-1}\kappa_k)\exp(\phi^{j-1}\log d_{t+1} + f_{j-1}y_{t+1})] \\
&= \mathbb{E}_t[\exp(-\rho - \frac{\gamma}{\sigma}(\xi-1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(\sum_{k=1}^{j-1}\tilde{\kappa}_k) \\
&\quad \exp(\phi^{j-1}(\phi\log d_t + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) + f_{j-1}(\xi_0 + \xi y_t + \sigma\varepsilon_{t+1}))]
\end{aligned}$$

So we confirm the conjecture,

$$\begin{aligned}
\exp(\tilde{\kappa}_j) &= \mathbb{E}_t[\exp(-\rho - \frac{\gamma}{\sigma}(\xi-1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1} + \phi^{j-1}(\phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) + f_{j-1}(\xi_0 + \sigma\varepsilon_{t+1}))] \\
\tilde{\kappa}_j &= -\rho - \frac{1}{2}\gamma^2 + \phi^{j-1}(\phi_0 - \frac{1}{2}\lambda^2) + f_{j-1}\xi_0 + \frac{1}{2}(-\gamma - \phi^{j-1}\lambda + f_{j-1}\sigma)^2
\end{aligned}$$

and

$$\begin{aligned}
f_j &= -\frac{\gamma}{\sigma}(\xi - 1) + f_{j-1}\xi \\
&= \xi^j + \frac{\gamma}{\sigma}(1-\xi^j) = \frac{\sigma-\gamma}{\sigma}\xi^j + \frac{\gamma}{\sigma}
\end{aligned}$$

So, for $j > 1$,

$$\begin{aligned}
&\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \ & \mathbb{E}_t[\exp(\sum_{k=1}^{j-1}\tilde{\kappa}_k)\exp(\phi^{j-1}\log d_{t+1} + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma})y_{t+1})] \\
= \ & \exp((-\rho - \frac{1}{2}\gamma^2)(j-1) + \frac{1-\phi^{j-1}}{1-\phi}(\phi_0 - \frac{1}{2}\lambda^2) + \left(\frac{1-\xi^{j-1}}{1-\xi}\frac{\sigma-\gamma}{\sigma} + \frac{\gamma}{\sigma}(j-1)\right)\xi_0 \\
+ \ & \sum_{k=1}^{j-1}\frac{1}{2}(-\gamma - \phi^{k-1}\lambda + ((\sigma-\gamma)\xi^{k-1} + \gamma))^2 \\
+ \ & \phi^{j-1}(\phi\log d_t + \phi_0 - \frac{1}{2}\lambda^2) + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma})(\xi_0 + \xi y_t) + \frac{1}{2}(-\phi^{j-1}\lambda + ((\sigma-\gamma)\xi^{j-1} + \gamma))^2)
\end{aligned}$$

By a similar logic,

$$\begin{aligned}
&\mathbb{E}_{t+1}[M_{t+1,t+j}Y_{t+j}] \\
= \ & \exp((-\rho - \frac{1}{2}\gamma^2)(j-1) + \left(\frac{1-\xi^{j-1}}{1-\xi}\frac{\sigma-\gamma}{\sigma} + \frac{\gamma}{\sigma}(j-1)\right)\xi_0 \\
+ \ & \sum_{k=1}^{j-1}\frac{1}{2}(-\gamma + ((\sigma-\gamma)\xi^{k-1} + \gamma))^2 + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma})(\xi_0 + \xi y_t) + \frac{1}{2}(((\sigma-\gamma)\xi^{j-1} + \gamma))^2)
\end{aligned}$$

So

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]$$

$$= \mathbb{E}_{t+1}[M_{t+1,t+j}Y_{t+j}]\exp(\frac{1-\phi^j}{1-\phi}(\phi_0 - \frac{1}{2}\lambda^2) + \sum_{k=1}^{j-1}((\gamma-\sigma)\xi^{k-1}\phi^{k-1}\lambda + \frac{1}{2}(\phi^{k-1}\lambda)^2)$$
$$+ \phi^j \log d_t - \phi^{j-1}\lambda((\sigma-\gamma)\xi^{j-1} + \gamma) + \frac{1}{2}(\phi^{j-1}\lambda)^2)$$

□

E.21 Proof of Corollary 7

Proof. We plug in the expressions for the respective surpluses:

$$\begin{aligned} \frac{S_{t+1}}{Y_{t+1}} &= d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1}) - d_{t+1}, \\ \frac{S_{t+j}}{Y_{t+j}} &= d_{t+j-1}\exp(r_{t+j-1}^f - \mu - \sigma\varepsilon_{t+j}) - d_{t+j}, \end{aligned}$$

into the expression for the conditional covariances:

$$\begin{aligned} cov_t(s_{t+1}, s_{t+j}) &= \mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1}) d_{t+j-1}\exp(r_{t+j-1}^f - \mu - \sigma\varepsilon_{t+j})] \\ &- \mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_{t+j-1}\exp(r_{t+j-1}^f - \mu - \sigma\varepsilon_{t+j})] \\ &+ \mathbb{E}_t[-d_{t+1}d_{t+j-1}\exp(r_t^f - \mu - \sigma\varepsilon_{t+j})] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[d_{t+j-1}\exp(r_t^f - \mu - \sigma\varepsilon_{t+j})] \\ &+ \mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1}) \times -d_{t+j}] - \mathbb{E}_t[d_t\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[-d_{t+j}] \\ &+ \mathbb{E}_t[-d_{t+1} \times -d_{t+j}] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[-d_{t+j}]. \end{aligned}$$

This expression can be restated as:

$$\begin{aligned} cov_t(s_{t+1}, s_{t+j}) &= d_t\mathbb{E}_t[\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_{t+j-1}\exp(r_{t+j-1}^f - \mu - \sigma\varepsilon_{t+j})] \\ &\times (\exp(\sigma\lambda\phi^{j-2}) - 1) \\ &- \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j-1}\exp(r_t^f - \mu - \sigma\varepsilon_{t+j})](\exp(\lambda^2\phi^{j-2}) - 1) \\ &- d_t\mathbb{E}_t[\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_{t+j}](\exp(\sigma\lambda\phi^{j-1}) - 1) \\ &+ \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j}](\exp(\lambda^2\phi^{j-1}) - 1) \end{aligned}$$

which implies

$$\begin{aligned} cov_t(s_{t+1}, s_{t+j}) &= \exp(2\rho - 2\mu + \sigma^2)d_t\mathbb{E}_t[d_{t+j-1}](\exp(\sigma\lambda\phi^{j-2}) - 1) \\ &- \exp(\rho - \mu + .5\sigma^2)\mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j-1}](\exp(\lambda^2\phi^{j-2}) - 1) \\ &- \exp(\rho - \mu + .5\sigma^2)d_t\mathbb{E}_t[d_{t+j}](\exp(\sigma\lambda\phi^{j-1}) - 1) \\ &+ \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j}](\exp(\lambda^2\phi^{j-1}) - 1) \end{aligned}$$

We have the following expressions for the conditional forecasts:

$$\mathbb{E}_t[\log d_{t+j}] = \exp\left(\phi^j(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi}\right)$$

and

$$\mathbb{E}_t[d_{t+j}] = \exp\left(\phi^j(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2(1 + \phi^2 + \ldots + \phi^{2(j-1)})\right)$$

$$= \exp\left(\phi^j(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2j}}{1-\phi^2}\right)$$

We plug these conditional forecasts into the conditional covariances: For $j > 1$,

$$\begin{aligned}
& cov_t(s_{t+1}, s_{t+j}) \\
= \; & \exp(2\rho - 2\mu + \sigma^2)\exp\left((1+\phi^{j-1})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2(j-1)}}{1-\phi^2}\right)(\exp(\sigma\lambda\phi^{j-2}) - 1) \\
- \; & \exp(\rho - \mu + .5\sigma^2)\exp\left((\phi+\phi^{j-1})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2(j-1)}}{1-\phi^2} + \frac{1}{2}\lambda^2\right) \\
\times \; & (\exp(\lambda^2\phi^{j-2}) - 1) \\
- \; & \exp(\rho - \mu + .5\sigma^2)\exp\left((1+\phi^{j})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2j}}{1-\phi^2}\right)(\exp(\sigma\lambda\phi^{j-1}) - 1) \\
+ \; & \exp\left((\phi+\phi^{j})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2j}}{1-\phi^2} + \frac{1}{2}\lambda^2\right)(\exp(\lambda^2\phi^{j-1}) - 1)
\end{aligned}$$

Also, when $j = 1$,

$$\begin{aligned}
var_t(s_{t+1}) \; = \; & \mathbb{E}_t[d_t\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})d_t\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})] \\
- \; & \mathbb{E}_t[d_t\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_t\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})] \\
+ \; & \mathbb{E}_t[-d_{t+1}d_t\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[d_t\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})] \\
+ \; & \mathbb{E}_t[d_t\exp(r_t^f - \mu - \sigma\varepsilon_{t+1}) \times -d_{t+1}] - \mathbb{E}_t[d_t\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[-d_{t+1}] \\
+ \; & \mathbb{E}_t[-d_{t+1} \times -d_{t+1}] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[-d_{t+1}] \\
= \; & \exp(2\rho - 2\mu + \sigma^2)\exp(2\log d_t)(\exp(\sigma^2) - 1) \\
- \; & 2\exp(\rho - \mu + .5\sigma^2)d_t\mathbb{E}_t[d_{t+1}](\exp(\lambda\sigma) - 1) \\
+ \; & \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+1}](\exp(\lambda^2) - 1)
\end{aligned}$$

which implies

$$\begin{aligned}
var_t(s_{t+1}) \; = \; & \exp(2\rho - 2\mu + \sigma^2)\exp(2\log d_t)(\exp(\sigma^2) - 1) \\
- \; & 2\exp(\rho - \mu + .5\sigma^2)\exp\left((1+\phi)(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\right)(\exp(\lambda\sigma) - 1) \\
+ \; & \exp\left(2\phi(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \lambda^2\right)(\exp(\lambda^2) - 1)
\end{aligned}$$

□

E.22 Proof of Corollary 8

Proof. We plug in the expressions for the respective surpluses:

$$\begin{aligned}
\frac{S_{t+1}}{Y_{t+1}} &= d_t\exp(r_t^f - \psi_0 - (\psi - 1)y_t - \sigma\varepsilon_{t+1}) - d_{t+1}, \\
\frac{S_{t+j}}{Y_{t+j}} &= d_{t+j-1}\exp(r_{t+j-1}^f - \psi_0 - (\psi - 1)y_{t+j-1} - \sigma\varepsilon_{t+j}) - d_{t+j},
\end{aligned}$$

into the expression for the conditional covariances:

$$
\begin{aligned}
& cov_t(s_{t+1}, s_{t+j}) \\
= \quad & \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1}) d_{t+j-1} \exp(r_{t+j-1}^f - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
- \quad & \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_{t+j-1} \exp(r_{t+j-1}^f - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
+ \quad & \mathbb{E}_t[-d_{t+1} d_{t+j-1} \exp(r_t^f - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
- \quad & \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[d_{t+j-1} \exp(r_t^f - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
+ \quad & \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1}) \times -d_{t+j}] \\
- \quad & \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})]\mathbb{E}_t[-d_{t+j}] \\
+ \quad & \mathbb{E}_t[-d_{t+1} \times -d_{t+j}] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[-d_{t+j}].
\end{aligned}
$$

This expression can be restated as:

$$
\begin{aligned}
cov_t(s_{t+1}, s_{t+j}) \quad = \quad & d_t \mathbb{E}_t[\exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})] \\
\times \quad & \mathbb{E}_t[d_{t+j-1} \exp(r_{t+j-1}^f - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
\times \quad & (\exp(\sigma\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
- \quad & \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j-1} \exp(r_t^f - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
\times \quad & (\exp(\lambda\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
- \quad & d_t \mathbb{E}_t[\exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_{t+j}](\exp(\sigma\lambda\phi^{j-1}) - 1) \\
+ \quad & \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j}](\exp(\lambda^2\phi^{j-1}) - 1)
\end{aligned}
$$

which implies

$$
\begin{aligned}
cov_t(s_{t+1}, s_{t+j}) \quad = \quad & \exp(2\rho - 2\psi_0 + \sigma^2) d_t \exp(-(\psi-1)y_t)\mathbb{E}_t[d_{t+j-1}\exp(-(\psi-1)y_{t+j-1})] \\
\times \quad & (\exp(\sigma\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
- \quad & \exp(\rho - \psi_0 + .5\sigma^2)\mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j-1}\exp(-(\psi-1)y_{t+j-1})] \\
\times \quad & (\exp(\lambda\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
- \quad & \exp(\rho - \psi_0 + .5\sigma^2) d_t \exp(-(\psi-1)y_t)\mathbb{E}_t[d_{t+j}](\exp(\sigma\lambda\phi^{j-1}) - 1) \\
+ \quad & \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j}](\exp(\lambda^2\phi^{j-1}) - 1)
\end{aligned}
$$

where the conditional forecasts are

$$
\mathbb{E}_t[d_{t+j}] \quad = \quad \exp\left(\phi^j(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2j}}{1-\phi^2}\right)
$$

and

$$
\begin{aligned}
& \mathbb{E}_t[\exp(\log d_{t+j-1} - (\psi-1)y_{t+j-1})] \\
= \quad & \exp\left(\phi^{j-1}(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} - (\psi-1)\psi^{j-1}(y_t - \frac{\psi_0}{1-\psi}) - (\psi-1)\frac{\psi_0}{1-\psi}\right. \\
+ \quad & \left.\frac{1}{2}\sum_{k=0}^{j-2}(\phi^k\lambda + \psi^k(\psi-1)\sigma)^2\right)
\end{aligned}
$$

Also, when $j = 1$,

$$\begin{aligned} var_t(s_{t+1}) &= \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1}) d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})] \\ &- \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})] \\ &+ \mathbb{E}_t[-d_{t+1} d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})] \\ &- \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})] \\ &+ \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1}) \times -d_{t+1}] \\ &- \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})]\mathbb{E}_t[-d_{t+1}] \\ &+ \mathbb{E}_t[-d_{t+1} \times -d_{t+1}] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[-d_{t+1}] \end{aligned}$$

which implies

$$\begin{aligned} var_t(s_{t+1}) &= \exp(2\rho - 2\psi_0 + \sigma^2)\exp(2\log d_t - 2(\psi-1)y_t)(\exp(\sigma^2) - 1) \\ &- 2\exp(\rho - \psi_0 + .5\sigma^2)\exp\left((1+\phi)(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 - (\psi-1)y_t\right) \\ &\times (\exp(\lambda\sigma) - 1) \\ &+ \exp\left(2\phi(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \lambda^2\right)(\exp(\lambda^2) - 1) \end{aligned}$$

□

F Notes about Convenience Yields

The government debt portfolio return is the return on a portfolio that goes long in the tax claim and short in the spending claim:

$$\mathbb{E}_t\left[R_{t+1}^D - R_t^f\right] = \frac{P_t^T - T_t}{B_t - S_t}\mathbb{E}_t\left[R_{t+1}^T - R_t^f\right] + \frac{P_t^\lambda - T_t}{B_t - S_t}\mathbb{E}_t\left[R_{t+1}^\lambda - R_t^f\right] - \frac{P_t^G - G_t}{B_t - S_t}\mathbb{E}_t\left[R_{t+1}^G - R_t^f\right],$$

where R_{t+1}^D, R_{t+1}^T,R_{t+1}^λ and R_{t+1}^G are the holding period returns on the bond portfolio, the tax claim, and the spending claim, respectively. We take government spending process, and the debt return process as exogenously given, and we explore the implications for the properties of the tax claim.

Proposition 14. *In the absence of arbitrage opportunities, if the TVC holds, the expected excess return on the tax claim is the unlevered return on the spending claim and the debt claim:*

$$\begin{aligned} \mathbb{E}_t\left[R_{t+1}^T - R_t^f\right] &= \frac{P_t^G - G_t}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)}\mathbb{E}_t\left[R_{t+1}^G - R_t^f\right] \\ &\quad + \frac{D_t}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)}E_t\left[R_{t+1}^D - R_t^f\right] \\ &\quad - \frac{P_t^\lambda - K_t}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)}\mathbb{E}_t\left[R_{t+1}^\lambda - R_t^f\right] \end{aligned}$$

If we want the debt to be risk-free, then the following equation holds for expected returns:

$$\begin{aligned} \mathbb{E}_t\left[R_{t+1}^T - R_t^f\right] &= \frac{P_t^G - G_t}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)}\mathbb{E}_t\left[R_{t+1}^G - R_t^f\right] \\ &\quad - \frac{P_t^\lambda - K_t}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)}\mathbb{E}_t\left[R_{t+1}^\lambda - R_t^f\right] \end{aligned}$$

$$\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)}\beta_t^G - \frac{P_t^\lambda - K_t}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)}\beta_t^\lambda.$$

Suppose we consider the case of a constant spending ratio and a constant convenience yield ratio. Then this implies that the beta of the tax revenue process is given by:

$$\beta_t^T = \frac{(P_t^G - G_t) - (P_t^\lambda - K_t}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)}$$

On the other hand, suppose that the convenience yield seignorage process has a zero beta. Then the implied beta of the tax revenue process

$$\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)},$$

which exceeds the beta of the tax revenue without seignorage: $\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)}$. If the seignorage revenue is sufficiently counter-cyclical, then the government can insure both taxpayers and bondholders at the same time. For example, consider the case in which the government runs zero primary surpluses in all future states of the world. Then the beta of the tax revenue is one $\beta_t^T = 1$, where $D_t = P_t^\lambda - K_t$. In this case, the average tax rate is constant: $\Delta \log \tau_{t+1} = 0$.