NBER WORKING PAPER SERIES

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Working Paper 27786
http://www.nber.org/papers/w27786

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
September 2020

We are very grateful to Peter Demarzo and Jonathan Berk for their feedback and advice. We also acknowledge helpful comments from Luigi Boccola, Saki Bigio, Sebastian DiTella, Francois Gourio, Patrick Kehoe, Arvind Krishnamurthy, and Matteo Maggiori. Yuan Tian provided outstanding research assistance. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

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NBER Working Paper No. 27786
September 2020
JEL No. F34,G12,H62,H63

ABSTRACT

Governments face a trade-off between insuring bondholders and taxpayers. If the government decides to fully insure bondholders by manufacturing risk-free debt, then it cannot insure taxpayers against permanent macro-economic shocks over long horizons. Instead, taxpayers will pay more in taxes in bad times. Conversely, if the government fully insures taxpayers against adverse macro shocks, then the debt becomes at least as risky as un-levered equity. Only when government debt earns convenience yields, may governments be able to insure both bondholders and taxpayers, and then only if the convenience yields are sufficiently counter-cyclical.

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The government faces a trade-off between insuring tax payers against adverse macro shocks and insuring bond holders. If the governments provides more insurance to bond investors, who then require lower risk premia, then it can provide less insurance to taxpayers. Safer debt requires more tax revenue relative to GDP when the marginal utility of the stand-in investor is high. The larger the sovereign debt burden, the steeper this trade-off becomes.

Some countries, especially the U.S., pay a low risk premium on the portfolio of outstanding Treasurys. The portfolio is priced as if it is close to risk-free. Other countries, including most of the emerging market countries, pay a large risk premium to its bond investors. The focus in the literature has been mostly on the country's willingness and ability to repay (see, e.g., Eaton and Gersovitz, 1981; Bulow and Rogoff, 1989; Aguiar and Gopinath, 2006; Arellano, 2008; DeMarzo, He, and Tourre, 2019, for examples), but the trade-off between bondholder and taxpayer insurance applies regardless of whether a country contemplates default.

Nevertheless, the U.S. insures its taxpayers against output growth risk. Panel A of Figure 1 plots the GDP growth betas of U.S. federal government spending and tax growth over longer horizons. In the post-war U.S. data, the spending cash flows are safer than the tax revenues at all horizons, even if we increase the horizon to 10 years. It appears as if the U.S. government can insure taxpayers at all horizons. The U.S. government seems to insure taxpayers by lowering tax rates in recessions, and it also insures transfer recipients by increasing its spending/output ratio in recessions. In asset pricing lingo, a claim to tax revenue, if traded, would have a high beta, while a claim to spending would have a low beta.

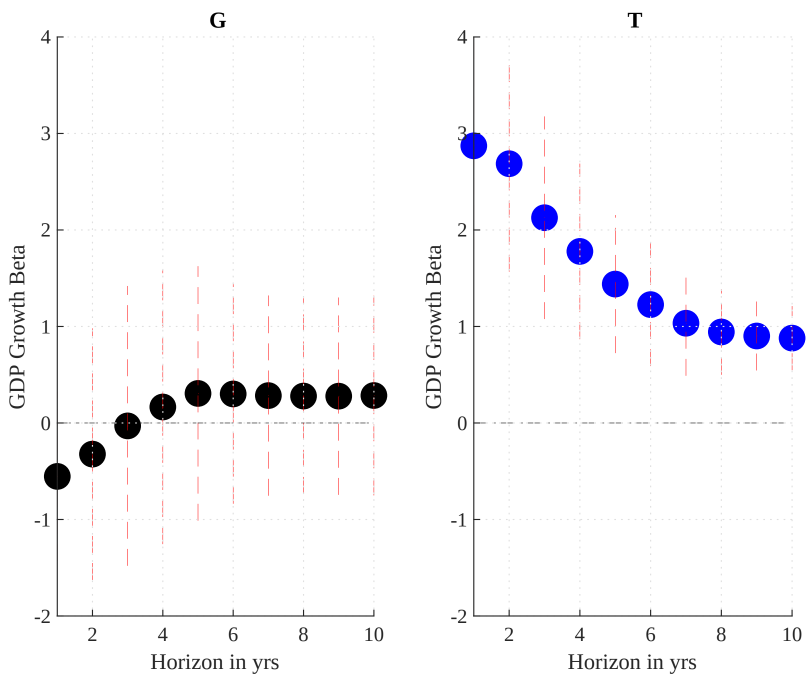
How can the Treasury manufacture debt that is completely risk-free and hence has a zero beta? That actually requires a non-trivial feat of financial engineering. The Treasury's bond portfolio is backed by a long position in a claim to tax revenue and a short position in a claim to government spending. Both are exposed to output risk. The Treasury's long position in the tax claim exceeds the short position by the value of outstanding Treasurys. To render the entire Treasury portfolio risk-free, the claim to tax revenue has to have a lower beta than the spending claim to ensure that the net beta of the Treasury portfolio is zero. Recast in Miller-Modigliani language, the tax revenue claim is the unlevered version of the spending claim. The beta of the tax claim is the weighted average of the beta of the spending claim and the beta of the debt.

The tax claim has a low beta when the PDV of future taxes increases in bad times, when the investor's marginal utility is high. The tax payer is short this claim. From the taxpayer's perspective, a low beta tax claim is a risky tax liability. As a result, the government cannot insure taxpayers when it insures bondholders by keeping the debt risk-free. The larger the debt, the larger the gap between the betas of the spending and the tax claim needs to be. As the debt grows, the beta of the tax claim has to go to zero. Conversely, if the governments insists on insuring the tax payers, then the government debt will be risky for bond holders, because they will be bearing macro-economic risk.

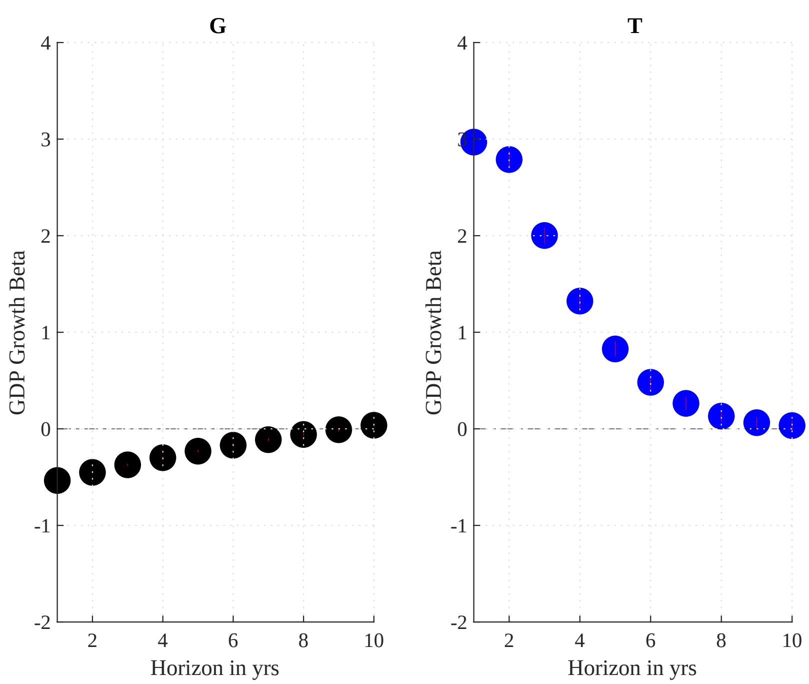
Figure 1: GDP Growth Betas of U.S. Tax Revenue and Spending

Betas in regression of log U.S. spending G growth and log tax revenue T growth over horizon j on GDP growth over horizon j. Panel A: Data. Sample is 1947—2019. Annual data. The plot shows 2 standard error bands. HAC standard errors with bandwidth equal to horizon. Panel B: Model-simulated data with Risk-free Debt. Benchmark calibration in Table 1.

Panel A: Data



Panel B: Model with Risk-free Debt



Our paper is the first one to characterize the trade-off between insuring taxpayer and debtholder at different horizons. In order to do this, we assume that the government commits to a counter-cyclical debt policy, and we derive the properties of the surplus/output ratio from the government budget constraints implied by the risk-free debt. The trade-off at different horizons only depends on the dynamics of the debt/output ratio, not of the surpluses, when the debt is risk-free.

When shocks to output are permanent, the government can only escape this trade-off over short horizons through countercyclical debt issuance: Over short horizons, the surplus can be rendered risky, meaning tax payers are insured against aggregate shocks, because this surplus risk is offset by the counter-cyclical debt issuance which neutralizes business cycle risk. Over long horizons, the tax claim has to be sufficiently safe for investors, risky for taxpayers, to offset the long-run output risk in debt issuance, as along as debt and output are co-integrated, as shown in Panel B of Figure 1, which plots the risk-free-debt-implied GDP growth betas of U.S. federal government tax growth over longer horizons, when matching counter-cyclical debt/output and spending/output ratios. If the debt is to be risk-free, then the tax revenue beta has to drop below the spending beta at longer horizons, clearly at odds with the actual betas shown in the top panel. We refer to this as the government debt risk premium puzzle: why is the portfolio of U.S. Treasurys priced as if it is close to risk-free, even though the government's cash flow fundamentals are risky?

Risk-free debt cannot match other key moments of the U.S. data. When the debt/output ratio is persistent, the implied process for the surplus/output ratio features little or no autocorrelation, clearly at odds with U.S. data. In addition, even at longer horizons, the debt/output ratio has little or no predictive power for future surpluses in U.S. data; in the model, it is the single best predictor of future surpluses at longer horizons.

There are two exceptions to this trade-off between insuring taxpayers and bond investors. First, the government imputes a (quasi-)unit root to the debt/output ratio, leading to a violation of the transversality condition in the case of counter-cyclical debt issuance and large equity risk premia. Second, the government earns large and counter-cyclical convenience yields. The U.S. government may be able to insure both taxpayers and bondholders, when the seignorage revenue from issuing Treasurys is large and counter-cyclical enough.

1 Related Literature

Our paper brings a state-of-the-art dynamic asset pricing model to bear on the valuation of public debt. To do so, we assume that surpluses are cointegrated with GDP in our analysis. If GDP growth has a permanent component, which modern macro and econometrics recognizes to be the case, then the surplus process in levels S_t inherits that permanent component from Y_t. Surpluses have long-run risk. Modern asset pricing has consistently found that permanent cash flow shocks receive a high price of risk in the market (e.g., Alvarez and Jermann, 2005; Hansen and

Scheinkman, 2009; Bansal and Yaron, 2004; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and Chernov, 2018). Hence, because of the exposure of the surplus to long-run GDP risk, the claim to current and future surpluses will typically have a substantial risk premium. Since the value of the surplus claim equals the market value of outstanding debt, the portfolio of government debt is a risky asset, except in knife-edge cases. The properties of the stationary surplus/output ratios, which the literature focuses on, are irrelevant for the long-run discount rates of surpluses.[1] For long-run discount rates, only long-run risk matters (Backus, Boyarchenko, and Chernov, 2018). Even when debt is risk-free, the risk-free rate is not the right discount rate in the presence of permanent output risk.

There is an extensive literature which tests the government's inter-temporal budget constraint.[2] These authors really test the *joint hypothesis* that the budget constraint is satisfied and that the measurability is also satisfied (to make the debt risk-free). We derive restrictions on the surplus/gdp process that are compatible with the knife-edge case of risk-free debt. The answer depends crucially on whether GDP has a permanent component or not. In the realistic case where it does, the surplus/output ratio cannot be sufficiently autocorrelated when we match the dynamics of the debt/output ratio. Further, we show analytically that the S-shaped impulse responses of the surplus/output ratio discussed by Bohn (1998); Canzoneri, Cumby, and Diba (2001); Cochrane (2019, 2020) – the government keeps running deficits for a while to be offset by surpluses in the future– are not consistent with risk-free debt, unless the debt/output ratio has higher-order dynamics not observed in the data.[3]

The U.S. government debt earns returns close to the risk-free rate, but the cash flow dynamics do not bear this out: the surpluses are too persistent, not predicted by the debt/GDP ratio and too risky. We call this the U.S. government risk premium puzzle. The U.S. government debt risk premium puzzle we document in this paper is distinct from, but related to the government debt valuation puzzle discussed by Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019), because the risk premium puzzle does not pertain to the first moments of future surpluses.

Our paper contributes to the normative literature on optimal government taxation and debt management, starting with Barro (1979)'s seminal work on tax smoothing. In the literature after Barro (1979), starting with Lucas and Stokey (1983), the risk-return tradeoff we highlight is present in the background, but is not explicitly analyzed. However, most of these models do not have plausible asset pricing implications. When markets are complete, the planner favors shifting the

[1] For example, Cochrane (2020) completely abstracts from output risk.

[2] Hansen, Roberds, and Sargent (1991); Hamilton and Flavin (1986); Trehan and Walsh (1988, 1991); Bohn (1998, 2007) derive time-series restrictions on the government revenue and spending processes that enforce the government's inter-temporal budget constraint. They use the risk-free rate as the discount rate. This is not be the right discount rate unless the risk-free rate exceeds the growth rate of output, when when debt is risk-free. Jiang (2019) derives the implications of the government's inter-temporal budget constraint for the nominal and real exchange rates, and finds support in the data.

[3] When debt is risk-free, the impulse response of the surplus/output ratio inherits the impulse response of the change in the debt/output ratio.

risk from taxpayers to bond investors (Lucas and Stokey, 1983). We do not derive the optimal tax rate, but show that, for any tax policy, the government can only truly insure taxpayers over short horizons, while keeping the debt risk-free.[4] Insuring taxpayers at all horizons against adverse macro shocks will always come at a large cost to the Treasury in a model with plausible asset prices.

By changing the maturity composition of debt, the government may be able to get closer to the optimal tax policy when when markets are incomplete, essentially by making the debt riskier (Angeletos, 2002; Buera and Nicolini, 2004; Lustig, Sleet, and Yeltekin, 2008; Arellano and Ramanarayanan, 2012; Bhandari, Evans, Golosov, and Sargent, 2017; Aguiar, Amador, Hopenhayn, and Werning, 2019), and shifting risk from taxpayers to bondholders. Our work is not focused on how the maturity choice of the government informs the riskiness of debt, but instead focuses directly on the fundamental determinants of the riskiness of the government's balance sheet.

In recent work, Mian, Straub, and Sufi (2020a,b) have examined the distributional implications of government debt issuance, pointing out that the wealthy buy a large share of government and private debt. To the extent that the Gini coefficient of debt holdings exceeds that of taxes, the government is really trading off insuring the rich vs the middle class.

Finally, convenience yields may help explain why emerging economies with more sovereign risk typically have more pro-cyclical fiscal policies (Bianchi, Ottonello, and Presno, 2019). These countries do not benefit from the convenience yields, and hence cannot escape the trade-off. It may also help to explain the government debt risk premium puzzle. In international economics, there is a growing literature that emphasizes the U.S. role as the world's safe asset supplier (see Gourinchas and Rey, 2007; Caballero, Farhi, and Gourinchas, 2008; Caballero and Krishnamurthy, 2009; Maggiori, 2017; He, Krishnamurthy, and Milbradt, 2018; Gopinath and Stein, 2018; Krishnamurthy and Lustig, 2019; Jiang, Krishnamurthy, and Lustig, 2018a, 2019; Liu, Schmid, and Yaron, 2019; Koijen and Yogo, 2019).

The paper is organized as follows. Section 2 derives the general trade-off between the insurance of bondholders and taxpayers, following standard Miller-Modigliani logic. When the government commits to plausible spending and tax revenue policies, the debt will generally be risky. We characterize these risk premia in closed form. Section 3 develops a simple version of the canonical dynamic asset pricing model with permanent shocks to output and to the investor's marginal utility. The governments commits to a spending policy and a debt policy; we back out the tax policy that keeps the debt risk-free. We start with the case of constant debt/output ratios. Section 4 introduces time-varying debt/output ratios, and Section 5 characterizes the trade-off faced by the government at different horizons. Finally, Section 6 introduces convenience yields.

Section A of the separate appendix generalizes these results in a continuous time version of

[4] When the government accumulates sufficient assets, it can implement the complete markets Ramsey allocation, as shown by Aiyagari, Marcet, Sargent, and Seppälä (2002).

the model that allows for risky debt and convenience yields. Section B provides some additional risk premium results. Section C develops a version of the model without permanent shocks. In models with only transitory shocks to output and marginal utility, the government may be able to insure taxpayers over longer horizons. However, these models have counterfactual asset pricing implications (Borovička, Hansen, and Scheinkman, 2016). The only model in which the government can insure taxpayers at all horizons is one in which the output shocks are transitory, but they are priced as if they are permanent.

2 The General Trade-off between Insuring Bondholders and Taxpayers

We use T_t to denote government revenue, and G_t to denote government spending. M_t denotes the stochastic discount factor. We assume that debt is fairly priced and does not earn any convenience yields.

Let B_t denote the market value of outstanding government debt at the beginning of period t, before expiring debt is paid off and new debt is issued. The debt can be long-term or short-term, and it can be nominal or real. In fact, it can be any contingent claim. In Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019), we show that the value of the government debt equals the sum of the expected present values of future tax revenues minus future government spending:

$$B_t = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (T_{t+j} - G_{t+j}) \right], \tag{1}$$

provided that there is no arbitrage opportunity and a transversality condition holds. This result does not rely on complete markets, and it still applies even when the government can default on its debt. See Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) for a proof. This result relies on the absence of arbitrage in bond markets and the transversality condition $\lim_{k\to\infty} \mathbb{E}_t M_{t,t+k} B_{t+k} = 0$.[5]

Let $P_t^T = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} T_{t+j} \right]$ and $P_t^G = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} G_{t+j} \right]$ denote the present values of the "cum-dividend" tax claim and spending claim. Value additivity then implies that $B_t = P_t^T - P_t^G$.

2.1 Characterizing the Government Debt Risk Premium

For notational convenience, let $D_t = B_t - S_t$ denote the difference between the market value of outstanding government debt and the government surplus. By the government budget condition, D_t is the market value of outstanding government debt at the end of period t, after expiring debt is paid off and new debt is issued.

[5] While there are equilibrium models that generate violations of the TVC (see Samuelson, 1958; Diamond, 1965; Blanchard and Watson, 1982; Brunnermeier, Merkel, and Sannikov, 2020), these violations typically show up in all long-lived assets, including stocks, not just government debt, and these models typically do not feature long-lived investors.

Let R^D_{t+1}, R^T_{t+1} and R^G_{t+1} denote the holding period returns on the bond portfolio, the tax claim, and the spending claim, respectively:

$$R^D_{t+1} = \frac{B_{t+1}}{B_t - S_t}, \qquad R^T_{t+1} = \frac{P^T_{t+1}}{P^T_t - T_t}, \qquad R^G_{t+1} = \frac{P^G_{t+1}}{P^G_t - G_t}.$$

In Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019), we also show that the government debt portfolio return is the return on a portfolio that goes long in the tax claim and short in the spending claim:

$$\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right] \;=\; \frac{P^T_t - T_t}{D_t}\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] - \frac{P^G_t - G_t}{D_t}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right]. \tag{2}$$

This result only relies on Eq. (1) and additivity. The value of a claim to surpluses equals the value of a claim to taxes minus the value of a claim to spending.

The government bond risk premium varies dramatically across countries. In some countries, such as the U.S., this risk premium $\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right]$ is small. Hall and Sargent (2011) compute a real return of 168 basis points on all U.S. Treasurys, while Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) compute a risk premium of 111 basis points for the U.S. government portfolio. The returns on debt issued by peripheral or developing countries are estimated to be much higher; Using EMBI indices on a short sample, Borri and Verdelhan (2011) estimate annual excess returns between 4% and 15%. On a much longer sample going back to the 19th century, Meyer, Reinhart, and Trebesch (2019) estimate excess returns of around 4% above U.S. and U.K bond returns, taking into account defaults.

2.2 Characterizing the Trade-Off with Return Betas

Next, we rearrange Eq. (2) and derive the following expression for the risk premium on the tax claim:

$$\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] = \frac{P^G_t - G_t}{D_t + (P^G_t - G_t)}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] + \frac{D_t}{D_t + (P^G_t - G_t)}E_t\left[R^D_{t+1} - R^f_t\right]. \tag{3}$$

Governments typically want a counter-cyclical spending claim, i.e. they want to spend more in recessions. On the other hand, they also want a risky tax claim, because they want to reduce the tax burden in recessions. As a result, the tax claim's risk premium $\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right]$ is high and the spending claim's risk premium $\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right]$ is low. When the debt value D_t is positive, the fraction $\frac{P^G_t - G_t}{D_t + (P^G_t - G_t)}$ is between 0 and 1. Then, for Eq. (3) to hold, it requires a high risk premium $\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right]$ on the government debt portfolio. As the debt risk premium is a measure of the risk premium or insurance premium charged by bondholders, the government's debt portfolio has to be very risky.

According to eqn. (3), the tax revenue claim is the unlevered version of the spending claim, or, equivalently, the spending claim is the levered version of the tax claim. This result is analogous to the Miller-Modigliani relation between the unlevered return on equity (the return on the tax claim) and the levered return on equity (the return on the spending claim).

We define the beta of an asset i as

$$\beta_t^i = \frac{-cov_t\left(M_{t+1}, R_{t+1}^i\right)}{var_t(M_{t+1})};$$

by the investor's Euler equation, β_t^i determines the conditional risk premium of this asset

$$\mathbb{E}_t\left[R_{t+1}^i - R_t^f\right] = \beta_t^i \lambda_t,$$

where the price of risk is $\lambda_t = R_t^f var_t(M_{t+1})$.

Let β_t^D, β_t^T and β_t^G denote the beta of the bond portfolio, the tax claim, and the spending claim, respectively. We assume $\beta_t^Y > 0$, so that the output claim has a positive risk premium. The following proposition characterizes the relationship of their risk exposures.

Proposition 2.1. The beta on the tax claim is a weighted average of the beta of the spending claim and the beta of the debt:

$$\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)}\beta_t^G + \frac{D_t}{D_t + (P_t^G - G_t)}\beta_t^D.$$

Governments want to provide insurance to transfer recipients by choosing $\beta_t^G < \beta^Y$, but they also want to provide insurance to taxpayers by choosing $\beta_t^T > \beta^Y$. However, the following corollary states that this is impossible if the government debt is risk-free.

This discussion implicitly assumes that taxpayers are long-lived households who value a dollar in each aggregate state in the same way as the marginal investor in Treasury markets. When markets are incomplete, agents may have different IMRS. Even when markets are incomplete, the aggregate component of the household's IMRS will be common across households, and the risk premia are identical to those in the equivalent representative agent economy, but the risk-free rate is lower (see Krueger and Lustig, 2010; Werning, 2015, for a formal derivation of this equivalence result), as long as the conditional distribution of idiosyncratic risk does not depend on the aggregate state of the economy. In section 5, we characterize this trade-off at different horizons.

Corollary 2.2. In order for debt to be risk-free ($\beta_t^D = 0$), the beta of the tax claim needs to equal the unlevered beta of the spending claim:

$$\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)} \beta_t^G.$$

If the government has a positive amount of risk-free debt $(D_t > 0)$, there is no scope to insure taxpayers. In fact, the taxpayers have to provide insurance to the rest of the economy. We start our analysis with the case in which the spending claim has a positive beta $(\beta_t^G > 0)$. Then, the government engineers risk-free debt by lowering the beta of the tax claim relative to that of the spending claim. When a taxpayer wakes up in the bad state, the news has to be worse, in present value, for the recipient of transfer payments: $\beta_t^T < \beta_t^G$. The more debt outstanding, the lower the beta of the tax claim needs to be relative to that of the spending claim.

These restrictions on the betas hold true regardless of the specific dynamics of the tax and spending process. In the next section, we will derive restrictions on the underlying cash flows by committing to a particular process for debt/output and spending.

The only way the government can provide insurance to debt holders, while keeping the debt risk-free, is by saving—choosing a negative amount of debt $(D_t < 0)$. In other words, the government can only insure taxpayers at the expense of bondholders. [6] On the other hand, if the spending claim has a negative beta $(\beta_t^G < 0)$, then the tax claim also has a negative beta: $\beta_t^T < 0$. The taxpayers have large tax payments during recessions.

2.3 Characterizing the Trade-Off with Cash Flow Betas

Thus far, we have characterized the return betas of the tax and spending claims. We can further derive some general conditions that characterize the betas of tax and spending cash flows.

When debt is risk-free, the government will only be able to provide insurance to taxpayers by choosing debt policies that depend on the entire history of shocks. Let ε_t denote the shocks to the economy. We use $\varepsilon_t^l = (\varepsilon_t, \varepsilon_{t-1}, \dots, \varepsilon_{t-l+1})$ to describe its history in the past l periods.

Proposition 2.3. When the debt is risk-free, (a) the average cash flow beta of discounted surpluses is always zero:

$$cov_t \left(M_{t+1} - \mathbb{E}_t M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{k=1}^{\infty} M_{t+1,t+k} S_{t+k} \right) = 0.$$

(b) When the government chooses a fiscal policy such that the debt/output ratio d_t is only a function of this history of past shocks and d_{t-l}: $d_t = d_t(\varepsilon_t^l; d_{t-l}) = d_t(\varepsilon_t^l)$, then the average cash

[6] Aiyagari, Marcet, Sargent, and Seppälä (2002) show that it is optimal for a government issuing only risk-free one period debt to accumulate savings $D_t \ll 0$ in the limit. This makes perfect sense, because that allows the government to choose $\beta_t^T \gg \beta_t^G$ and insure tax payers against macro shocks. In the limit, by accumulating sufficient assets, the government can implement the Lucas and Stokey (1983) complete markets allocation.

flow beta of discounted surpluses over the next $l+1$ periods is zero:

$$cov_t\left(M_{t+1}-\mathbb{E}_t M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{k=1}^{l+1} M_{t+1,t+k} S_{t+k}\right)=0. \tag{4}$$

Part (a) states that the average discounted cash flow beta for the entire surplus stream is zero. Part (b) derives tighter restrictions on the surplus cash flow betas when the government commits to a particular debt issuance policy. Note that $d_t(\varepsilon_t^l)$ implies that the surplus/output ratio $s_t(\varepsilon^{l+1})$ depends on the same history of shocks. When the government surplus is only allowed to depend on the shocks in the past l periods, the cash flow beta will be zero over shorter periods as shown in Eq. (4).

The issuance decision is the only source of state-contingency with risk-free debt. If the government only responds to the shock today in deciding issuance, then it has to pay it back next period. The issuance decision next period only responds to the shock next period. The government can issue more and lower the surplus in response to a bad shock, but it has to completely reverse that in the next period.

As a result, if the government seeks to smooth out the shocks over long periods of time, that it will have to adopt a debt issuance policy that depends on the entire history of shocks. In the limit, if we allow for arbitrary history dependence, then we end with the standard restriction for risk-free debt:

$$cov_t\left(M_{t+1}-\mathbb{E}_t M_{t+1}, S_{t+1}\right)=-cov_t\left(M_{t+1}-\mathbb{E}_t M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{k=2}^{\infty}\exp(-r^f_{t+1,t+k})S_{t+k}\right).$$

We have recovered the weakest covariance restriction. However, in this case, the debt and the $s_t(\varepsilon_t^\infty)$ the surplus/output ratio depends on the entire history. In the model that we develop next, we will allow the government to choose a debt policy that depends on the entire history of shocks, an $AR(p)$ process for the debt/output ratio, and we analyze the quantitative implications of persistent debt/output ratios.

3 Quantifying the Trade-off in a Model Economy

Next, we characterize the trade-off between insuring debtholders and taxpayers in more specific settings. We consider different spending and debt policies in order to reverse-engineer the revenue process T that always keeps the debt risk-free. For the most part of the paper, we consider an economy with permanent output shocks and homoskedastic pricing kernels:

Assumption 1. *(a) All output shocks are i.i.d. and permanent:*

$$y_{t+1} = \mu + y_t + \sigma\varepsilon_{t+1},$$

where ε_{t+1} denotes the innovation to output growth that is normally distributed and i.i.d.

(b) The log pricing kernel is given by:

$$m_{t,t+1} = -\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}.$$

(c) The government only issues one-period real risk-free debt.

We start by considering the simplest case with a constant spending/output and debt/output ratio.

3.1 Characterizing the Trade-Off with Constant Debt/Output

To simplify the analysis, we use a stripped down version of the canonical Breeden (2005); Lucas (1978); Rubinstein (1974) endowment economy. We specifically consider the case in which the government debt is risk-free. The government commits to a spending policy and a debt issuance policy that allows for arbitrary history dependence.[7] To highlight the implications of the general trade-off between insurance of bondholders and taxpayers, we make the following assumptions for the entire section. Let Y_t and $y_t = \log Y_t$ denote output and its log.

Assumption 2. *(a) The government commits to a constant spending/output ratio $x = G_t/Y_t$.*

(b) The government commits to a constant debt/output ratio $d = D_t/Y_t$.

Then, the government budget condition implies a strongly counter-cyclical tax revenue process:

$$\frac{T_t}{Y_t} = x - d\left(1 - R^f_{t-1}\frac{Y_{t-1}}{Y_t}\right) = x - d\left(1 - R^f_{t-1}\exp[-(\mu + \sigma\varepsilon_t)]\right).$$

That is to say, to perfectly insure the bondholders by keeping the debt risk-free, the government needs to make sure that the tax revenue claim is strongly counter-cyclical: When the growth rate of output is low, the government's revenue needs to increase as a fraction of GDP. Furthermore, the magnitude of the counter-cyclical exposure is increasing in the debt-to-GDP ratio d.

Similarly, the primary surplus/output ratio is also strongly counter-cyclical:

$$s_t = \frac{S_t}{Y_t} = \frac{T_t - G_t}{Y_t} = -d\left(1 - R^f_{t-1}\exp[-(\mu + \sigma\varepsilon_t)]\right). \tag{5}$$

When the growth rate μ exceeds ρ, the government can run deficits on average, but not in every state. In fact, whenever there are negative shocks such that $\mu - \rho < -\sigma\varepsilon$, the government

[7] The debt policy is a flexible $AR(p)$ process that responds to the output innovation. This rules out other policies that satisfy the TVC: e.g., the government could simply let debt accumulate until it hits a constraint. However, when the constraint binds, the implied tax process may have to exceed output.

runs a primary surplus. In this scenario, the government does not run persistent deficits. In fact, the conditional auto-covariance of the surplus/output ratio is zero:

$$cov_t(s_t, s_{t-1}) = 0.$$

When we shrink σ to zero, then the government always runs deficits, but, in this case, $\mu > \rho$ implies a violation of TVC, which we show below. This result is more general. With risk-free debt, the autocorrelation of the surpluses tends to zero as we increase the persistence of the debt/output ratio.

Proposition 3.1. Under Assumptions 1 and 2, (a) if the transversality condition holds and the primary surplus satisfies Eq. (5), the government debt value is the sum of the values of the outstanding strips:

$$D_t = \mathbb{E}_t \left[\sum_{k=1}^{\infty} M_{t,t+k} S_{t+k} \right] = dY_t.$$

(b) Proposition 2.3 can be simplified to the following measurability constraint:

$$(\mathbb{E}_{t+1} - \mathbb{E}_t) \left[\sum_{k=1}^{\infty} M_{t,t+k} S_{t+k} \right] = 0.$$

This proposition shows that the value of outstanding debt at the end of period t is indeed a constant fraction of output, and it implies that there is no news about the present discounted value of future surpluses.[8] To see why we cannot simply discount at the risk-free rate, even when the debt is risk-free, consider the valuation equation for debt as a function of surplus/output ratios:

$$D_t = \mathbb{E}_t \left[\sum_{j=0}^{T} M_{t,t+j} Y_{t+j} s_{t+j} \right] + \mathbb{E}_t \left[M_{t,t+T} Y_{t+T} \frac{D_{t+T}}{Y_{t+T}} \right].$$

The debt/output ratio d is constant. The TVC will hold even if $\rho < \mu$ as long as the output strip price $\mathbb{E}_t \left[M_{t,t+T} Y_{t+T} \right] \to 0$. This will be the case if there is enough permanent, priced risk in output: $-\rho + \mu + \frac{1}{2}\sigma^2 < \gamma\sigma$. Note that $\rho < \mu$ implies a violation of TVC as $\sigma \to 0$. So, it is not the case that the government can always run deficits when $\rho < \mu$, at least not without violating the TVC.[9] The output risk premia matter even when debt is risk-free. The usual condition referenced in textbooks $\rho < g$ is irrelevant for the TVC. The risk-free rate is not the correct discount rate for surpluses even

[8] Hansen, Roberds, and Sargent (1991) discuss a version of this condition that uses the risk-free rate when devising an econometric approach to testing the budget constraint: $(\mathbb{E}_{t+1} - \mathbb{E}_t) \left[\sum_{k=1}^{\infty} \exp(-r^f_{t,t+k}) S_{t+k} \right] = 0$. However, this condition is equivalent to the one in the Proposition, only if the risk-free rate exceeds the growth rate of the economy. If not, this equation may fail even when the condition in Prop. 3.1 holds.

[9] See Bohn (1995) for an early reference on why discounting at the risk-free may fail. However, Bohn (1995) refers to this case as one in which the government runs persistent deficits, while the deficits really are uncorrelated over time.

when the debt is risk-free, in the presence of permanent output shocks. The correct TVC is given by: $\lim_{j\to\infty}\mathbb{E}_t\left[\exp(m_{t,t+j})D_{t+j}\right] = \lim_{j\to\infty}\exp(j(-\rho+\mu+\frac{1}{2}\sigma^2-\gamma\sigma))dY_t$, which will be satisfied even if $\rho < \mu$ iff $-\rho+\mu+\frac{1}{2}\sigma^2-\gamma\sigma < 0$.

Next, we define a k-period output strip as a claim to Y_{t+k}. The price/dividend ratios of the strips are denoted by ξ_k, where $\xi_1 = \exp(-\rho-\gamma\sigma+\mu+0.5\sigma^2)$. The expected return on an output strip is given by $E_t\left[R^Y_{t+1}\right] = \frac{\exp(\mu+0.5\sigma^2)}{\exp(-\rho-\gamma\sigma+\mu+0.5\sigma^2)} = \exp(\rho+\gamma\sigma)$. Hence, the log of the multiplicative equity risk premium is $\gamma\sigma$.

Corollary 3.2. (a) The value of the spending and the revenue claim is given by:

$$P^G_t - G_t = x\frac{\xi_1}{1-\xi_1}Y_t, P^T_t - T_t = \left(d + x\frac{\xi_1}{1-\xi_1}\right)Y_t.$$

(b) The risk premia on the tax claim and the spending claim satisfy

$$\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right],$$

where $\beta^T = \frac{x\frac{\xi_1}{1-\xi_1}}{d+x\frac{\xi_1}{1-\xi_1}}\beta^G$.

The Treasury investor is long in a government revenue claim and short in a spending claim. To make the debt risk-free, as long as the debt/output ratio d is positive, we need to render the government revenue process much safer. More precisely, since the government spending is a constant ratio of the output level, $\beta^G = \beta^Y > 0$. Then, a positive d implies the fraction $\frac{x\frac{\xi_1}{1-\xi_1}}{d+x\frac{\xi_1}{1-\xi_1}}$ is between 0 and 1, which requires the return on the tax claim to be less risky than the return on the output claim: $0 < \beta^T < \beta^Y$. As a result, there is no scope to insure taxpayers at any positive debt level. As the debt/output ratio d increases, the government needs to make the tax revenue increasingly safe. The tax claim is really a portfolio of a claim to government spending and risk-free debt. The larger the debt/output ratio d, the safer the tax claim needs to be. As the debt/output ratio approaches infinity, the beta of the tax claim tends to 0.

3.2 Quantitative Model Implications for Trade-off

Our calibration of the static part of the model in Panel A of Table 1 matches post-war U.S. data. The maximum Sharpe ratio γ is 1. The standard deviation of output σ is 0.05. The growth rate of the economy μ is 3.1%. The risk-free rate ρ is 2%. Spending accounts for 10% of output ($x = 0.10$). We analyze a calibrated economy in which the risk-free rate is lower than the growth rate of output. However, the TVC is satisfied in this economy, because $\log\xi_1 = -0.0145 < 0$. The government cannot simply roll over the debt. The surpluses and debt issuance need to satisfy tight restrictions.

Figure 2 plots the risk premia on the tax and the spending claim as we vary the debt/output ratio d. The risk premium on the spending claim is 5.43% per annum. This is the unlevered equity premium. By Corollary 3.2, the risk premium on the tax claim satisfies

$$\mathbb{E}_t\left[R^T_{t+1}-R^f_t\right] = \frac{x\frac{\xi_1}{1-\xi_1}}{d+x\frac{\xi_1}{1-\xi_1}}\mathbb{E}_t\left[R^G_{t+1}-R^f_t\right], \tag{6}$$

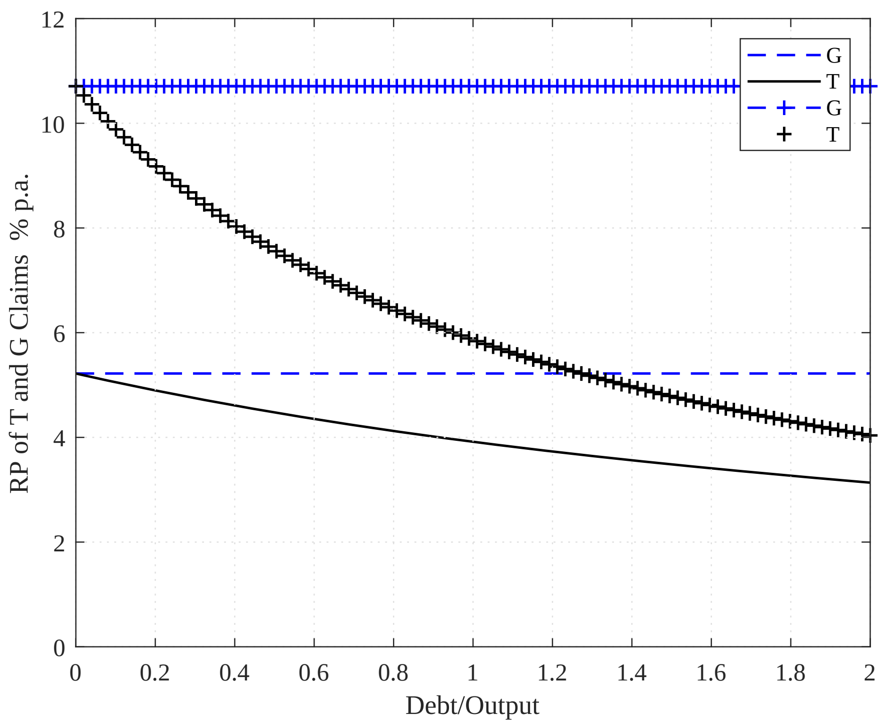
the risk premium on the tax claim falls to 4% when $d=1$, and close to 3% when $d=2$. As the government becomes more levered, the tax claims needs to be safer, and the scope for taxpayer insurance disappears. This trade-off steepens when we increase the maximum Sharpe ratio γ from 1 to 2 (+ line). The risk premium on the spending claim is 10.86% per annum. This is the unlevered equity premium. The risk premium on the tax claim falls to 6% when $d=1$, and close to 4% when $d=2$.

Table 1: Benchmark Calibration

Panel A: Preferences and Output Dynamics		
γ	1	maximum Sharpe ratio
ρ	2%	risk-free rate
μ	3%	mean of growth rate of output
σ	5%	std. of growth rate of output
Panel B: Spending/Output Ratio Dynamics: $\log x_t=\varphi^g_1\log x_{t-1}+\varphi^g_0-\beta^g\varepsilon_t-\frac{1}{2}(\beta^g)^2$.		
β^g	$1.53\times\sigma$	sensitivity of spending/output to output innovations
φ^g_1	0.88	AR(1) coeff of spending/output
$\varphi^g_0/(1-\varphi^g_1)$	10%	mean of spending/output
Panel C: Debt/Output Ratio Dynamics: $\log d_t=\sum_{j=1}^p\phi_j\log d_{t-j}+\phi_0-\lambda\varepsilon_t-\frac{1}{2}\lambda^2$.		
λ	$1.94\times\sigma$	sensitivity of debt/output to output innovations
$\phi_0/(1-\phi_1-\phi_2)$	0.43	mean of debt/output
ϕ_1	1.4	AR(1) coeff of debt/output
ϕ_2	-0.48	AR(2) coeff of debt/output

Figure 2: Risk Premium of T and G Claims with $\gamma = 1, 2$

The figure plots the implied risk premium of the T and G claims when the debt/output ratio and spending/output ratio are constant. λ and β^g are set to 0, for γ of 1 (−) and γ of 2 (+). Other Parameters–Benchmark calibration in Table 1.



4 Dynamics of Debt and Surpluses in a Model Economy with State-Contingent Debt/Output

When the debt/output ratio is constant, there is no scope for insuring taxpayers at any horizon. Next, we allow the government to introduce state-contingent variation in the debt/output ratio. This will create limited opportunities for the government to temporarily insure taxpayers over short horizons.

4.1 Persistence of Debt and Surpluses in U.S. Post-war Data

Figure 3 plots the sample annual autocorrelation of the log government debt/output ratio and the government surplus/output ratio as functions of lags. In the post-war U.S. sample (1947—2019), the $AR(1)$ process for the log debt/output ratio fits the data rather well. The estimated $AR(1)$ coefficient ϕ in annual data is 0.986. We also estimated an $AR(2)$-process. This yields estimates of ϕ_1 of 1.04 and ϕ_2 of -.48. The unconditional mean of the debt/output ratio is 0.43. The federal government's primary surplus is also quite persistent, with an $AR(1)$ coefficient around 0.81. We will show that the risk-free debt model cannot match the high persistence of the debt/output ratio and the surplus/output ratios. Finally, we set $\lambda = 1.953 \times \sigma$ equal to match the slope coefficient in a regression of the debt/output ratio innovations on GDP growth in the post-war U.S. sample (1947—2019). A one pp. increase in GDP growth lowers the debt/output ratio by 1.95 pps. We report the calibration of the spending/output ratio in Panel C of Table 1.

When the debt is risk-free, returning to the valuation equation for debt, and assuming the TVC is satisfied, the debt to GDP ratio is the single best predictor of future discounted surpluses:

$$\frac{D_t}{Y_t} = \mathbb{E}_t \left[\sum_{j=0}^{T} M_{t,t+j} \frac{Y_{t+j}}{Y_t} s_{t+j} \right].$$

To check this, we ran the following regression on annual U.S. data in the post-war sample:

$$\frac{S_{t+k}}{GDP_{t+k}} = c_k + b_k \frac{D_t}{GDP_t} + e_{t+k}.$$

The results are reported in Table 2. The debt/GDP ratio has not forecasting power for future surplus/output ratios. If anything, it forecasts at short horizons with wrong sign. There is no predictability at longer horizons. Lagged surpluses are better predictors.

Figure 3: Autocorrelation of U.S. Government Log Debt/Output and Surplus/Output Ratios

The figure plots the sample autocorrelation of the U.S. log government debt/output ratio, the U.S. government surplus/output ratio, the tax/output ratio and the spending/output ratio against GDP. Sample is 1947—2019. Annual data.

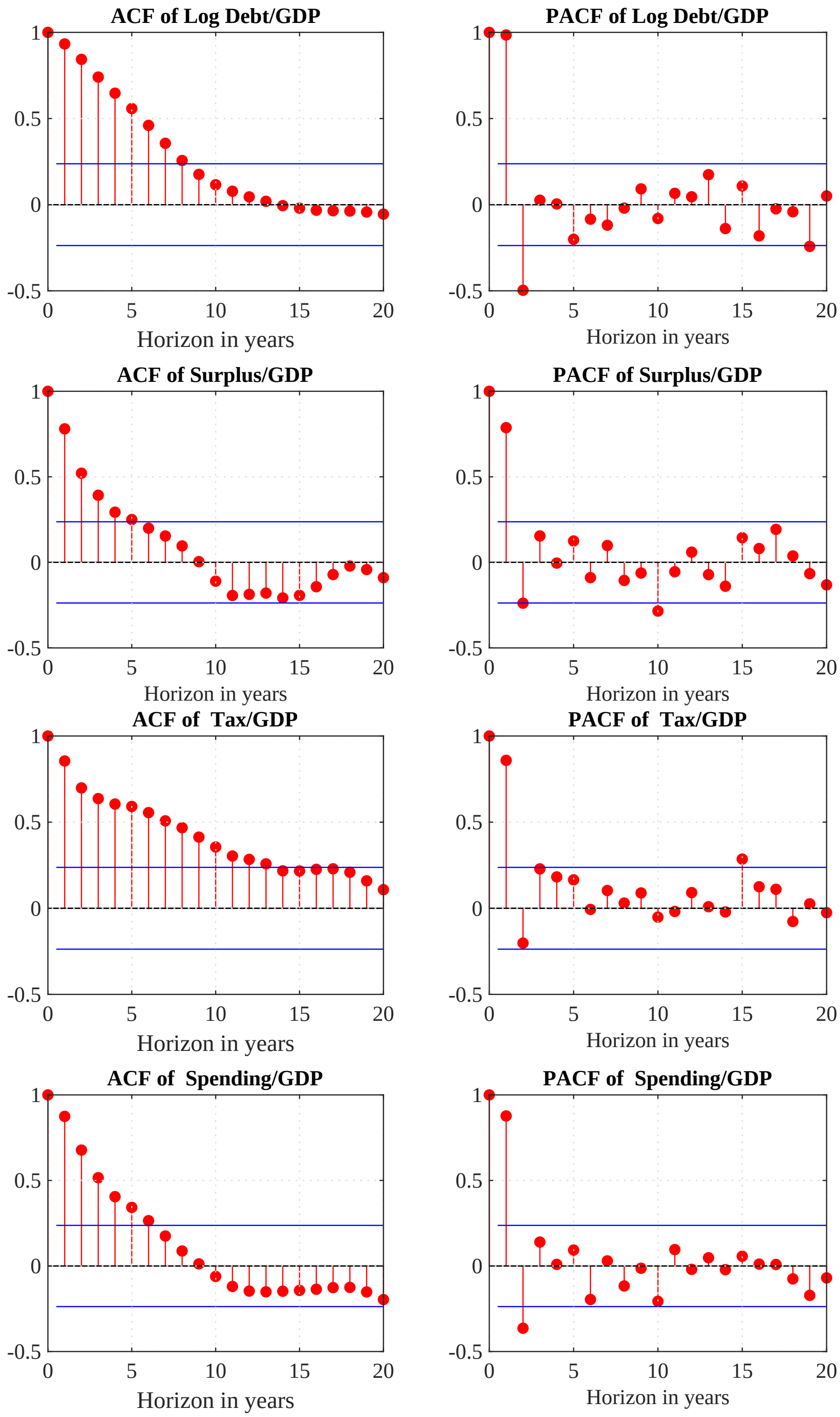


Table 2: Forecasting Surplus/Output Ratios

Panel I: We forecast the primary surplus/output ratios in post-war annual U.S. data (1947-2019). In Panel A, we report the results for $\frac{S_{t+k}}{GDP_{t+k}} = c_k + b_k \frac{D_t}{GDP_t} + e_{t+k}$. In Panel B, we report results for $\frac{S_{t+k}}{GDP_{t+k}} = c_k + b_k \frac{D_t}{GDP_t} + d_k \frac{S_t}{GDP_t} + e_{t+k}$. Panel II: Benchmark Model (10,000 sims). Benchmark calibration in Table 1.

Horizon k	1	2	3	4	5
Panel I: U.S. Data					
$\frac{S_{t+k}}{GDP_{t+k}} = c_k + b_k \frac{D_t}{GDP_t} + e_{t+k}$					
b_k	-0.031	-0.0099	0.013	0.023	0.028
$[s.e.]$	[0.023]	[0.025]	[0.03]	[0.031]	[0.03]
R^2	0.043	0.0041	0.006	0.018	0.024
$\frac{S_{t+k}}{GDP_{t+k}} = c_k + b_k \frac{D_t}{GDP_t} + d_k \frac{S_t}{GDP_t} + e_{t+k}$					
b_k	0.0085	0.018	0.036	0.042	0.044
$[s.e.]$	[0.01]	[0.015]	[0.019]	[0.02]	[0.021]
d_k	0.81	0.57	0.47	0.37	0.33
$[s.e.]$	[0.087]	[0.13]	[0.12]	[0.11]	[0.10]
R^2	0.64	0.30	0.21	0.15	0.13
Panel II: Benchmark Model with Risk-free Debt					
$\frac{S_{t+k}}{GDP_{t+k}} = c_k + b_k \frac{D_t}{GDP_t} + e_{t+k}$					
b_k	0.0629	0.117	0.132	0.127	0.114
R^2	0.0781	0.271	0.342	0.316	0.254
$\frac{S_{t+k}}{GDP_{t+k}} = c_k + b_k \frac{D_t}{GDP_t} + d_k \frac{S_t}{GDP_t} + e_{t+k}$					
b_k	0.0701	0.12	0.132	0.126	0.112
d_k	0.695	0.265	0.045	-0.055	-0.11
R^2	0.560	0.342	0.345	0.319	0.266

4.2 Characterizing the Trade-Off with Counter-cyclical Debt/Output

We allow the government to vary the debt/output ratio counter-cyclically.

Assumption 3. *(a) The government commits to a policy for the spending/output ratio* $x_t = G_t/Y_t$ *given by:*

$$\log x_t = \varphi_1^g \log x_{t-1} + \varphi_0^g - \beta^g \varepsilon_t - \frac{1}{2}(\beta^g)^2.$$

(b) The government commits to a policy for the debt/output ratio $d_t = D_t/Y_t$ *given by:*

$$\log d_t = \phi \log d_{t-1} + \phi_0 - \lambda \varepsilon_t - \frac{1}{2}\lambda^2,$$

where $\lambda > 0$ *so that the debt-output ratio increases in response to a negative output shock* ε_t.

The government will only be able to insure taxpayers over short horizons, when the shocks are permanent. We also counter-cyclical variation in the spending/output ratio ($\beta^g > 0$). The results in Section 2 still apply. The value of the spending and the revenue claim is given by:

$$P_t^G - G_t = x\frac{\xi_1}{1-\xi_1}Y_t, \qquad P_t^T - T_t = \left(d_t + x\frac{\xi_1}{1-\xi_1}\right)Y_t.$$

The tax claim's conditional beta satisfies

$$\beta_t^T = \frac{x\frac{\xi_1}{1-\xi_1}}{d_t + x\frac{\xi_1}{1-\xi_1}}\beta_t^G.$$

Can the government systematically issue more risk-free debt, instead of raising taxes, when the economy is hit by a permanent, adverse shock, in order to break the restriction on insurance of taxpayers? To start analysis, we assume that the debt/output ratio evolves according to an $AR(1)$-process:

$$\log d_t = \phi \log d_{t-1} + \phi_0 - \lambda \varepsilon_t - \frac{1}{2}\lambda^2.$$

This assumption encompasses two cases. First, when $0 < \phi < 1$, the debt/output process is mean-reverting. Second, when $\phi = 1$ and $\phi_0 = 0$, the debt/output process is a martingale. In both cases, a positive λ means that the debt/output ratio increases when the shock ε_t is negative—implying a counter-cyclical debt policy. First, we need to make sure the TVC is satisfied. How persistent can debt be without violating TVC?

Proposition 4.1. Under Assumptions 1 and 3, (a) when $0 < \phi < 1$, the transversality condition is satisfied if

$$\log(\xi_1) = -\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) < 0.$$

(b) When $\phi = 1$ and $\phi_0 = 0$, then the transversality condition is satisfied if

$$\log(\xi_1) + \lambda(\gamma - \sigma) = -\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) + \lambda(\gamma - \sigma) < 0.$$

When the government does not pursue counter-cyclical stabilization ($\lambda = 0$), then the TVC is (trivially) satisfied as long as the discount rate on an output strip is positive ($\log(\xi_1) < 0$). When the government does pursue counter-cyclical stabilization ($\lambda > 0$) and the risk premium γ is large enough:

$$(\lambda - \sigma)(\gamma - \sigma) > \rho - \mu + \frac{1}{2}\sigma^2,$$

the TVC is violated for the case of $\phi = 1$ and $\phi_0 = 0$. In comparison, the value of λ does not affect if the transversality condition is violated for the case of $0 < \phi < 1$. The counter-cyclical insurance $\lambda > 0$ provided by the debt issuance policy is so valuable to risk-averse investors (measured by $(\gamma - \sigma)\lambda$) that the price of a claim to the debt outstanding in the distant future $d_{t+T}Y_{t+T}$ fails to converge to zero, because this claim is a terrific hedge. This is the first important insight contributed by asset pricing theory. If we want to rule out arbitrage opportunities and output is subject to permanent, priced risk, then there have to be limits to the government's ability to pursue counter-cyclical debt issuance.

The PACF for the debt/output process suggests an $AR(2)$ process might be a better fit. To capture these higher-order dynamics, we introduce an $AR(2)$ process for the debt/output ratio:

$$\log d_t = \phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} + \phi_0 - \lambda\varepsilon_t - \frac{1}{2}\lambda^2.$$

The parameter estimates are reported in Panel C of Table 1.

When the roots lie outside the unit circle, the debt/output process is mean-reverting. As before, a positive λ means that the debt/output ratio increases when the shock ε_t is negative—implying a counter-cyclical debt policy. When the roots of the $AR(2)$ lie outside of the unit circle, the results in (a) of Prop. 4.1 apply. If not, the results in (b) of Prop. 4.1 apply.

Quantitative Implications This limits how much counter-cyclical debt issuance is feasible without violating the transversality condition when the debt/output ratio has a unit root ($\phi = 1$). In our calibrated economy, the upper bound for λ is 0.30σ, thus severly limiting the scope for counter-cyclical policy. Once we exceed this upper bound, the value of outstanding debt explodes. Hence, risk premiarisk premium in financial markets constrain counter-cyclical fiscal policy. The intuition is simple. When the government exceeds this bound, it has granted itself an arbitrage opportunity. However, as long as $\phi < 1$, the TVC is satisfied even though the risk-free of 2% is lower than the growth rate of the economy (3%).

4.2.1 Persistence of Surpluses

We can compute the autocorrelation (ACF) and impulse response functions (IRF) of the surpluses in closed form when the government issues only risk-free debt. These moments are particularly informative because these do not depend on the properties of the pricing kernel. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process.

We start from the following expressions for the surplus/output ratios in $t+1$ and $t+j$ respectively.

$$\begin{aligned} \frac{S_{t+1}}{Y_{t+1}} &= d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1}) - d_{t+1}, \\ \frac{S_{t+j}}{Y_{t+j}} &= d_{t+j-1} \exp(r_{t+j-1}^f - \mu - \sigma\varepsilon_{t+j}) - d_{t+j}. \end{aligned}$$

We use s_{t+1} to denote $\frac{S_{t+1}}{Y_{t+1}}$. We assume that the risk-free rate equals the growth rate of the economy ($g = \rho$) to derive a closed-form expression for the IRF of the surplus.

Proposition 4.2. Under Assumptions 1 and 3, (a) when the debt/output ratio follows an $AR(1)$ process, the debt is risk-free and the TVC is satisfied, then the IRF of the surplus output ratio is given by:

$$\begin{aligned} \frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma)\exp(\bar{d}), \textit{ for } j = 1, \\ &= \lambda\phi^{j-1}(\phi - 1)\exp(\bar{d}), \textit{ for } j > 1. \end{aligned}$$

(b) When the debt/output ratio follows an $AR(2)$ process, the debt is risk-free and the TVC is satisfied, then the IRF of the surplus output ratio is given by:

$$\begin{aligned} \frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma)\exp(\bar{d}), \textit{ for } j = 1, \\ &= \lambda(\psi_1 - 1)\exp(\bar{d}), \textit{ for } j = 2, \\ &= \lambda(\psi_{j-1} - \psi_{j-2})\exp(\bar{d}), \textit{ for } j > 2. \end{aligned}$$

where $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}, j > 2; \psi_2 = \phi_2 + \phi_1\psi_1; \psi_1 = \phi_1$.

(c) When the debt/output ratio follows an AR(3) process, the debt is risk-free and the TVC is satisfied, then the IRF of the surplus output ratio is given by:

$$\begin{aligned}\frac{\partial \frac{S_{t+j}}{Y_t}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma)\exp(\overline{d}), \textit{ for } j = 1, \\ &= \lambda(\psi_1 - 1)\exp(\overline{d}), \textit{ for } j = 2, \\ &= \lambda(\psi_2 - \psi_1)\exp(\overline{d}), \textit{ for } j = 3, \\ &= \lambda(\psi_{j-1} - \psi_{j-2})\exp(\overline{d}), \textit{ for } j > 3.\end{aligned}$$

where $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2} + \phi_3\psi_{j-3}, j > 3; \psi_3 = \phi_3 + \phi_2\psi_1 + \phi_1\psi_2; \psi_2 = \phi_2 + \phi_1\psi_1; \psi_1 = \phi_1$.

For an $AR(1)$, when $\lambda > \sigma$, the initial response is positive, but is negative starting in the 2nd year. As the persistence increases, the IRF converges to zero after year 1. In the case of an $AR(2)$, by choosing $\phi_1 > 1$, the government can run a deficit for 2 years in response to a negative output shock, but after that it reverts to running surpluses, as the ACFs decline: for $j > 2 : \psi_{j-1} < \psi_{j-2}$.

With higher-order, highly persistent $AR(p)$ models, the government may be able to larger hump-shaped IRFs. However, there is no evidence of higher-order $AR(p)$ dynamics (i.e., $p > 2$) in the US debt process (see Figure 3).

The auto-covariance of the surplus/output ratio is defined as follows:

$$cov_t(s_{t+1}, s_{t+j}) = \mathbb{E}_t[s_{t+1}s_{t+j}] - \mathbb{E}_t[s_{t+1}]\mathbb{E}_t[s_{t+j}].$$

The closed-form expressions for the autocovariances of the surplus/output ratio are given in section D of the Appendix. In the case of an $AR(1)$, we show that the conditional autocovariance declines to zero as we increase the persistence of the debt/output process. $\lim_{\phi \to 1} cov_t(s_{t+1}, s_{t+j}) = 0$. This is not surprising. In the case of a constant debt/output ratio, the surplus/output ratios are uncorrelated.

4.3 Quantitative Model Implications for Surplus Dynamics

AR(1) We report the persistence of the surplus in the calibrated version of the model.

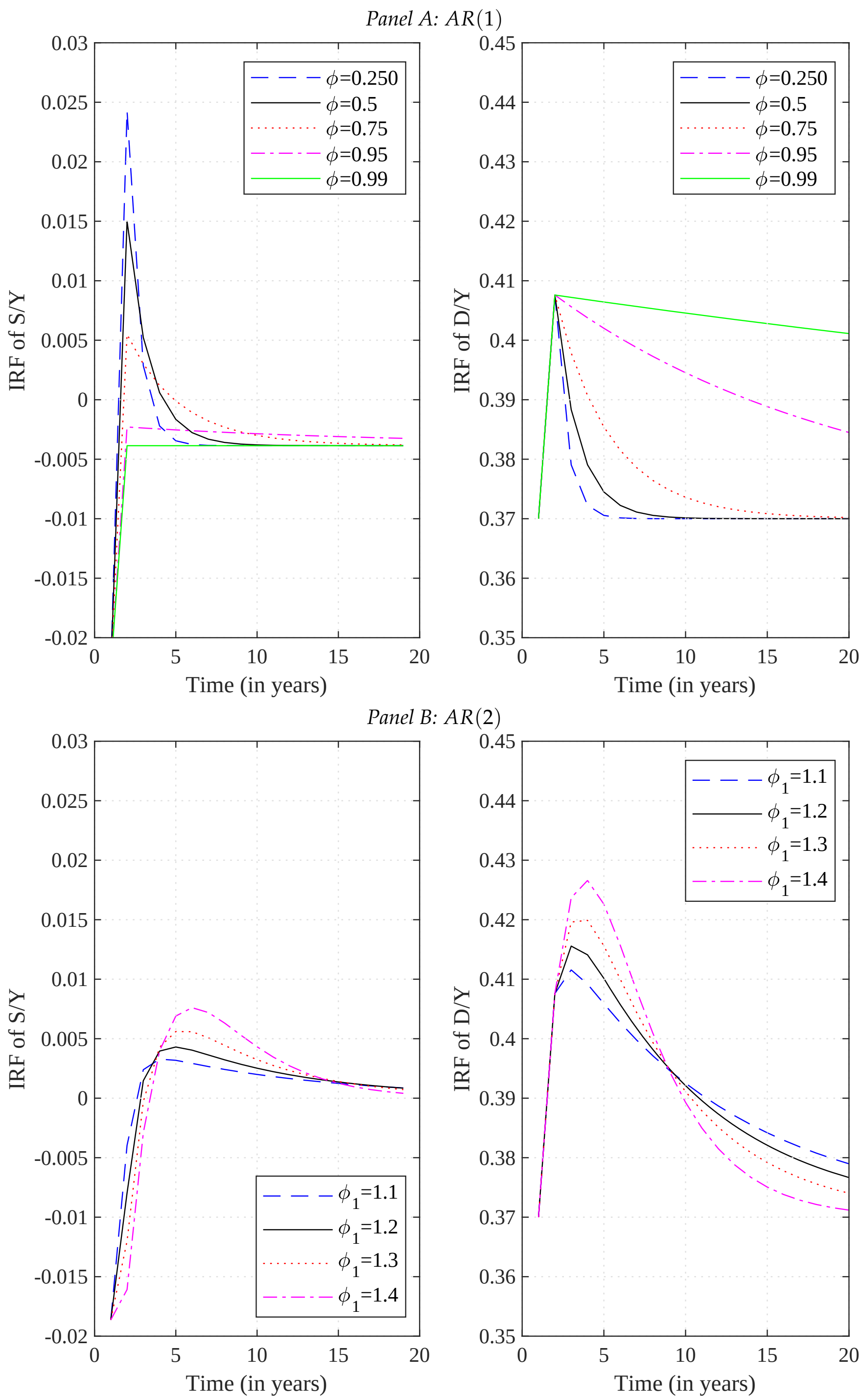
Panel A in Figure 4 plots the IRF to a one standard deviation negative innovation to output growth for a range of values of ϕ. We vary ϕ from 0.25 to 0.99. Upon impact, the debt/output ratio increases from its mean by about 8.9%. After that, the rate of mean-reversion is governed by ϕ. We choose $\rho = \mu = 2\%$ so that the long-run response of the surplus is zero. In the least persistent case ($\phi = 0.25$), the government immediately runs large surpluses after period 1. In the most persistent case, the ($\phi = 0.99$), the government runs a balanced budget starting in period 2. In all case, the long run surplus converges to a small deficit given by:

$$\frac{S_t}{Y_t} = \frac{T_t - G_t}{Y_t} = -d\left(1 - \exp(\rho - g)\right),$$

because $\rho < g$. Nevertheless, the TVC is satisfied. When the debt/output ratio follows an $AR(1)$, and the debt is risk-free, there can be no S-shaped responses to shocks.

Figure 4: IRF of Surplus/Output Ratios and Debt/Output Ratios ($AR(1)$)

The figure plots the IRF of S/Y and D/Y for an $AR(1)$ (top panel) and an $AR(2)$ (bottom panel). In Panel B, ϕ_2 is chosen to match 1st-order autocorrelation. We choose $\rho = \mu = 2\%$. Other parameters–Benchmark calibration in Table 1.



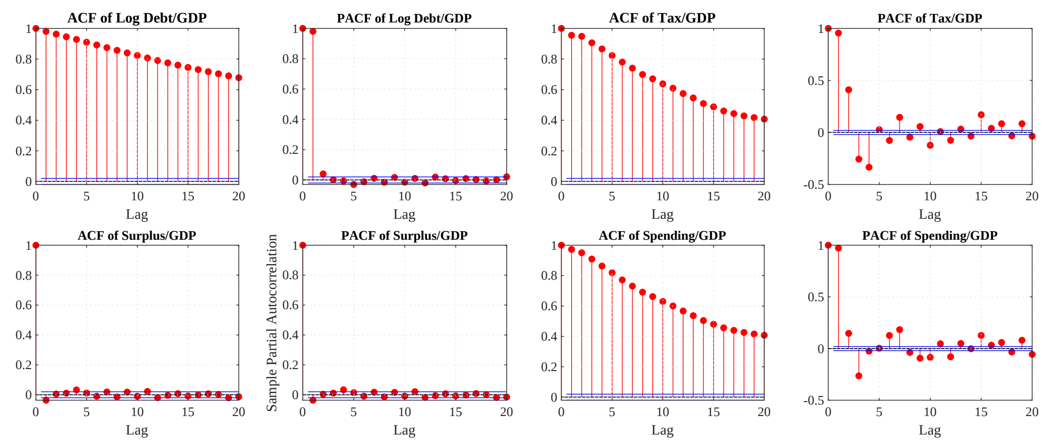
Panel A in Figure 5 plots the ACF of the debt/output ratio and the surplus/output ratio against the horizon for different values of ϕ. We evaluate these by simulating a path of $T = 1000$ observation. As explained, the autocorrelations are mostly non-positive.

These predictions do not depend on the properties of the SDF. But they are at odds with the data. As discussed, increasing the persistence of the debt process pushes the conditional autocorrelations of the surplus/output ratio to zero. When the government issues risk-free debt, the surpluses cannot feature significant autocorrelation if the surplus/output ratio is persistent. The only way around is to choose a sensitivity λ that is much larger than σ, which is empirically implausible.

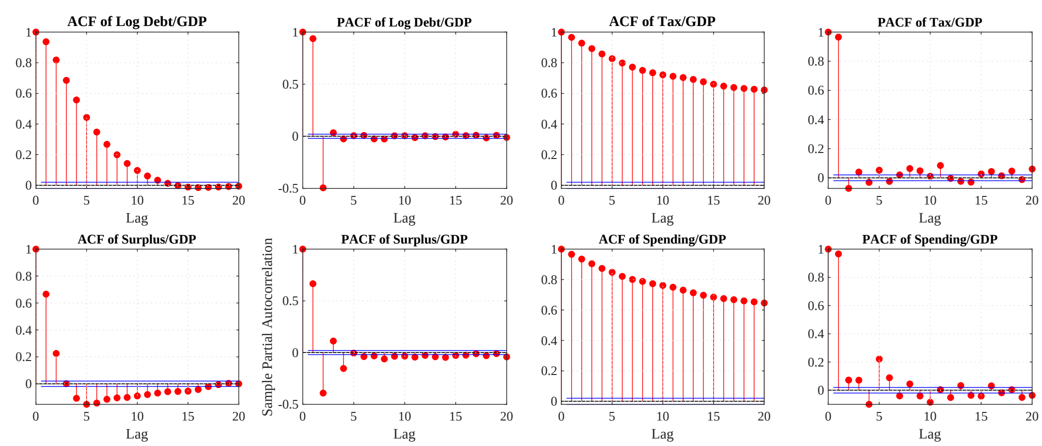
Figure 5: Autocorrelation of Surplus/Output Ratios and Debt/Output Ratios

Panel A plots the ACF and PACF of S/Y and D/Y for an $AR(1)$ with parameters $\phi_1 = 0.985$ and $\phi_2 = 0$. Other parameters–Benchmark calibration in Table 1. Panel B plots the ACF and PACF of S/Y and D/Y for an $AR(2)$: Benchmark calibration in Table 1.

$Panel A : AR(1)$



$Panel A : AR(2)$



AR(2) We also consider an $AR(2)$ process. Panel B in Figure 4 plots the IRF to a one standard deviation negative innovation to output growth for a range of values of ϕ_1. We vary ϕ_1 from 1 to 1.5. We choose ϕ_2 to match the first-order autocorrelation of 0.94. Upon impact, the debt/output ratio increases from its mean by about 8.9%. With $\phi_1 = 1.1$, the IRF looks essentially like the one obtained with an $AR(1)$ with ϕ close to 1. However, with $\phi_1 = 1.4$, and $\phi_2 = -0.48$, the IRF for the debt/output ratio displays a hump-shaped pattern.[10] Consistent with the results in Proposition 4.2, this hump-shaped pattern in the IRF of debt essentially delays the fiscal adjustment in surpluses by one year. The government runs an even larger deficit in the 2nd period. However, starting in year 3, the government runs surpluses. There is no significant S-shaped pattern; the government cannot run large deficits for more than 2 periods. Similarly, the model produces an AR(3) of 0, compared to 0.6 in the data. The model with risk-free debt cannot match the persistence in surpluses and taxes we see in the data.

Panel B in Figure 5 plots the ACF for S/Y and D/Y for $\phi_1 = 1.4$ and $\phi_2 = -0.48$ –the benchmark calibration of the debt/output ratio process. While this $AR(2)$ model produces more persistence in the surplus/output ratio, the ACF declines much faster than in the data. In the model, the AC(3) is essentially zero. Furthermore, the model produces a large, negative PACF(2) coefficient of -0.5, inconsistent with the estimated PACF for the surplus/output ratio.

The surplus forecasting results are reported in Panel II of Table 2, to be compared to the results in the data, listed in Panel I of Table 2. In the model, the debt/GDP ratio has strong forecasting power for future surplus/output ratios, with a positive sign, even when we control for lagged surplus/out ratios. At horizons up to 2 years, the lagged surplus/output ratio also forecasts future surpluses with a positive sign. After 2 years, the sign flips, and the surplus/output ratios have no incremental forecasting power. Given our results for the persistence of the surplus, this is not surprising. The model with risk-free debt generates a much faster decay in the slope coefficients on the lagged surplus than we see in the data.

Bohn (1998); Canzoneri, Cumby, and Diba (2001); Cochrane (2019, 2020) find evidence of S-shaped dynamics in the U.S. surplus/GDP ratios: Surplus initially declines after a negative shock, but then subsequently the government runs larger surpluses. The authors argue that these dynamics are consistent with budget balance. However, the S-shaped surplus dynamics in the data violate the risk-free debt conditions. Governments cannot defer the increase in the tax rate when output declines for more than 1 or 2 years, if they want to keep the debt risk-free. That would require $AR(p)$ dynamics with $p > 3$.

Finally, we consider the implied tax revenue betas inside the model. These are generated from 10,000 simulations from the $AR(2)$-model for the debt/output ratio. The results are plotted in Panel B of Figure 1. In the model, even in the $AR(2)$ case, the tax betas drop below the spending betas at longer horizons, to ensure that the debt is risk-free. This is counterfactual. In post-war

[10] We stop here because ϕ_1 of 1.5 produces complex roots.

U.S. data, the cash flow betas of the tax revenue claim converge to 1 at horizons between 5 and 10 years, as shown in Panel A of Figure 1. The cash flows are too risky even at longer horizons for the debt to be risk-free.

5 Frequency Decomposition of the Trade-off in a Model Economy with State-Contingent Debt/Output

How much smoothing can the government achieve by issuing more debt in response to bad shocks? It all depends on the horizon. This section characterizes the trade-off at different frequencies. In the presence of permanent shocks, the government can only insure taxpayers over a limited period of time. This period can be extended by imputing higher-order dynamics to the debt/output process.

5.1 Cash Flow Betas at Different Frequencies

At any given horizon, the trade-off is fully determined by the dynamics of the debt/output ratio, as long as the debt is risk-free.

Proposition 5.1. Under Assumptions 1 and 3, when debt is risk-free and debt/output follows an $AR(2)$, the cash flow beta of the discounted surpluses over j periods is given by beta of future debt issuance j periods from now:

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}\right) \\ = \; & -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)M_{t+1,t+j}D_{t+j}\right) \\ = \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\psi_{j-1}\lambda))-1). \end{aligned}$$

The sign of the cash flow covariance is $sign\left(\gamma(\sigma-\psi_{j-1}\lambda)\right)$, where ψ_j denotes the ACF: $\psi_j = \phi_1\psi_{j-1}+\phi_2\psi_{j-2}$, and where $\psi_1 = 1$.

The risk properties of the government surpluses over a given horizon are completely determined by riskiness of the debt issuance process, as long as the debt is risk-free, because consecutive surpluses cancel out when properly discounted. So, we do not need any information on the spending and tax revenue dynamics to do this decomposition at different horizons.

AR(1) Let us start by setting $\phi_2 = 0$, which means the debt/output follows an $AR(1)$ process. When debt is risk-free, the cash flow beta of the surpluses over j periods is given by minus the beta of future debt issuance:

$$
\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k} S_{t+k}\right) \\
= \;& -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) M_{t+1,t+j} D_{t+j}\right), \\
= \;& -E_t[M_{t+1}] E_t[M_{t+1,t+j} d_{t+j} Y_{t+j}](\exp(-\gamma(\sigma - \phi^{j-1}\lambda)) - 1).
\end{aligned}
$$

The sign of the cash flow covariance is $sign\left(\gamma(\sigma - \phi^{j-1}\lambda)\right)$. The sign of this covariance determines the horizon over which the government can provide insurance to taxpayers. Hence, the sign of the cash flow covariance is determined by the sign of $\gamma(\sigma - \phi^{j-1}\lambda)$, which has a natural economic interpretation: It is the risk premium of a debt strip that pays $Y_{t+k} d_{t+k}$. Hence, the discounted surplus earn the negative of the risk premia on the weighted baskets of debt strips. The surplus can be risky over horizon j only if this offset by safety of future debt issuance.

If $\lambda \leq 0$, all cash flow covariances for the discounted surpluses are positive. In other words, the government cannot insure taxpayers at any horizon. In figure 6, we discussed the case of $\lambda = 0$. The intuition is simple. Because debt and output are co-integrated, debt strips are as risky as claims to output at all horizons. Surpluses have to be safe enough to offset this risk, so that the total debt is risk-free.

However, if $\lambda > \sigma$, the initial covariance is negative. In the short run, debt strips are less risky than output. The government is insuring taxpayers who pay the next surpluses. As j increases the covariance declines and switches signs. If the rate of mean-reversion is high and ϕ is small, this switch occurs sooner. If the debt/output ratio is more persistent, the switch occurs later. As j increases, this expression $\gamma(\sigma - \phi^{j-1}\lambda)$ converges to $\gamma\sigma$, the risk premium on the output strip, because debt is co-integrated with output. In the long run, the entire cash flow covariance is positive but converges towards 0. Note that the covariance inherits the dynamics of the $AR(1)$-process for the debt/output ratio and starts to decline right away. The shocks are i.i.d. and permanent. Hence, in the long run, an adverse shock to output has to lead to permanently higher surpluses.

In the case of permanent output shocks, this covariance always approaches 0 from above as $j \to \infty$, as $E_t[M_{t+1,t+j} d_{t+j} Y_{t+j}]$ approaches 0. This means at some finite horizon, the surplus process is risky from the perspective of the taxpayer, who is providing these cash flows, and hence is short this claim. The only way to escape this is to impute a unit root to the debt/output ratio by pushing ϕ to 1, but that would violate the TVC, unless we are close enough to risk neutrality.

AR(2) In the more case of an AR(2), the sign of the cash flow covariance is determined by the sign of $\gamma(\sigma - \psi_{j-1}\lambda)$. If $\lambda > \sigma$, the initial covariance is negative. The government is insuring taxpayers who pay the next surpluses by issuing more debt in response to adverse shocks. As j increases, the covariance declines and switches signs. If the rate of mean-reversion is high and ψ_j declines quickly, this switch occurs sooner. If the debt/output ratio is more persistent, the switch occurs later. $\gamma(\sigma - \psi_{j-1}\lambda)$ has a natural economic interpretation: It is the risk premium of a debt strip that pays $Y_{t+k}d_{t+k}$. As j increases, this expression $\gamma(\sigma - \psi_{j-1}\lambda)$ converges to $\gamma\sigma$, the risk premium on the output strip. In the long run, the entire cash flow covariance is positive but converges towards 0. Note that the covariance inherits the dynamics of the $AR(1)$-process for the debt/output ratio and starts to decline right away.

Next, we can look at the tax liability itself. When the debt is risk-free, government surpluses only depend on the debt process, whereas tax revenues depend on both the debt process and the government spending process. In particular, the risk profile of the tax process over horizon j is determined by both the risk profiles of debt and spending.

Corollary 5.2. Under Assumptions 1 and 3, and when debt is risk-free and debt/output follows an $AR(2)$, the cash flow beta of taxes have to satisfy the following restriction.

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\ = & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma - \psi_{j-1}\lambda)) - 1) \\ + & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}G_{t+k}\right). \end{aligned}$$

In the AR(1) case, ψ_{j-1} is simply ϕ^{j-1}. When this covariance is negative over horizon j, the government provides insurance to taxpayers over horizon j. Note that the government is fighting the accumulation of output strip risk premium $\gamma\sigma$ because of the debt-claim exposure to output. As we take $j \to \infty$, we recover the return betas in section 2.

Section C of the appendix develops a version of the model without permanent shocks. This model produces radically different implications, but has counterfactual asset pricing implications.

5.2 Quantitative Model Implications for Trade-off at Different Frequencies

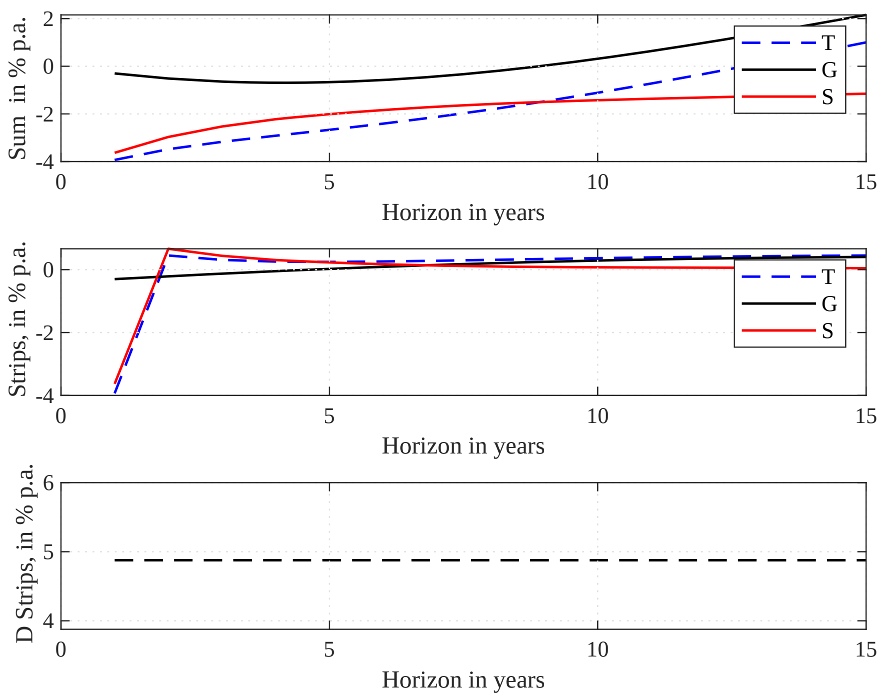
Dynamics of Spending/Output Ratio: Implications for Trade-off To quantify the trade-off, we need to calibrate the process for government spending. The U.S. spending/output ratio varies counter-cyclically.

We report the calibration of the spending/output ratio dynamics in Panel B of Table 1. In the 1947—2019 sample, we estimate the persistence of the spending/output ratio: $\varphi = 0.88$, and we estimate $\beta^g = 1.53 \times \sigma$. This is the slope coefficient in a regression of the spending/output ratio innovations on GDP growth; Spending/output increases by 1.53 pps. per pp decrease in output growth.

Constant Debt/Output. Figure 6 plots the risk premium (in % per annum) contributions of cumulative discounted cash flows (top panel) and the strips (middle panel) against the horizon. We

Figure 6: Horizon Decomp. of Risk Premium on Govt. Cash Flows with Constant Debt/Output and Spending/Output

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (middle panel) against the horizon. The bottom panel plots the risk premium on the debt strips: $-(\exp(-\gamma\sigma) - 1)$. Benchmark calibration: $\lambda = 0$. Other Parameters–Benchmark calibration in Table 1.



plot the risk premium contribution at each horizon j given by

$$
\begin{aligned}
&-cov_t\left(M_{t+1},(\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{k=1}^{j}M_{t+1,t+k}S_{t+k}\right)/E_t[M_{t+1}],\\
&=-dE_t[M_{t+1,t+j}Y_{t+j}](\exp(-\gamma\sigma)-1).
\end{aligned}
$$

against the horizon j in the top panel. This is a special case of Proposition 5.1: $\lambda = 0$. This measure is directly informative about the insurance provided by the government to taxpayers with a horizon j. This expression is *negative at all horizons* in this case: The government cannot insure citizens who pay the primary surplus at any horizon when the debt/output ratio is constant, because debt issuance has the same exposure to output risk as a claim to GDP. As a result, to offset this, the risk premium contributions have to be negative at all horizons. For large j, this expression converges to zero, because the debt is risk-free.

The taxpayer cares about the following risk measure:

$$
-cov_t\left(M_{t+1},(\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{k=1}^{j}M_{t+1,t+k}T_{t+k}\right)/E_t[M_{t+1}]. \tag{7}
$$

When this risk premium is negative, the taxpayers are instead providing insurance to the government. The risk premium contributions are negative until year 15.

As shown in the top panel of Figure 6, the risk premium on a claim to the discounted future surpluses is negative everywhere and converges to zero as we increase the horizon j in eqn. (7). This follows because the debt is risk-free. Risk-free debt is achieved by keeping the contributions of the risk premium on the tax claim below those on the spending claim at all horizons. In the bottom panel of Figure 6, we plot the contribution of each strip:

$$
-cov_t\left(M_{t+1},(\mathbb{E}_{t+1}-\mathbb{E}_t)M_{t+1,t+j}S_{t+j}\right)/E_t[M_{t+1}]
$$

against the horizon j. The 1-year strip on the surplus earns a risk premium of -3.75% per annum. It is safe because the surplus decreases in bad times, when the investor's marginal utility is high.

We start by considering moderately persistent AR1 process for the debt/output ratio: ϕ is 0.75.[11] All the other parameters are given in Table 1.

AR(1) Figure 7 plots the risk premium (in % per annum) on cumulative discounted cash flows (top panel) and the strips (bottom panel) against the horizon. Instead of plotting cash flow betas, we plot the risk premium computed by

$$
\begin{aligned}
&-cov_t\left(M_{t+1},(\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{k=1}^{j}M_{t+1,t+k}S_{t+k}\right)/E_t[M_{t+1}] && (8)\\
=\;&E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda))-1). && (9)
\end{aligned}
$$

[11] When we use more persistent processes, we get a quasi-violation of the TVC.

against the horizon j in the top panel. This measure is directly informative about the insurance provided by the government to taxpayers with a horizon j. In particular, this taxpayer cares about the following risk measure:

$$-cov_t\left(M_{t+1},(\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{k=1}^{j}M_{t+1,t+k}T_{t+k}\right)/E_t[M_{t+1}]. \tag{10}$$

When this risk premium is positive, this means that the government is insuring taxpayers. When it is negative, the taxpayers are instead providing insurance to the government.

As shown in the top panel of Figure 7, the risk premium on a claim to the discounted future surpluses converges to zero as we increase the horizon j in eqn. (12). This follows because the debt is risk-free. Risk-free debt is achieved by keeping the contributions of the risk premium on the tax claim below those on the spending claim, as we increase the horizon beyond 2 years. The risk premium on the cumulative surpluses crosses zero when $\sigma > \phi^{j-1}\lambda$.

In the middle panel of Figure 7, we plot the contribution of each strip:

$$-cov_t\left(M_{t+1},(\mathbb{E}_{t+1}-\mathbb{E}_t)M_{t+1,t+j}S_{t+j}\right)/E_t[M_{t+1}]$$

against the horizon j. The 1-year strip on the surplus earns a risk premium of 3% per annum. It is risky because the surplus decreases in bad times ($\lambda > 0$), when marginal utility is high. In order to make the debt risk-free, the risk premium on the 2-year strip is close to -2%. And these strips earn negative risk premium until they revert to zero after 15 years. Hence, the government has to commit to increasing the surplus 1 year after the negative shock. This applies to all the surpluses that follow. This result illustrates the limits to smoothing shocks with risk-free debt. If the debt/output ratio follows an $AR(1)$ process, then you can really not smooth across multiple periods. The cumulative risk premium on the surplus start to decline right away. They inherit the dynamics of the debt/output ratio.

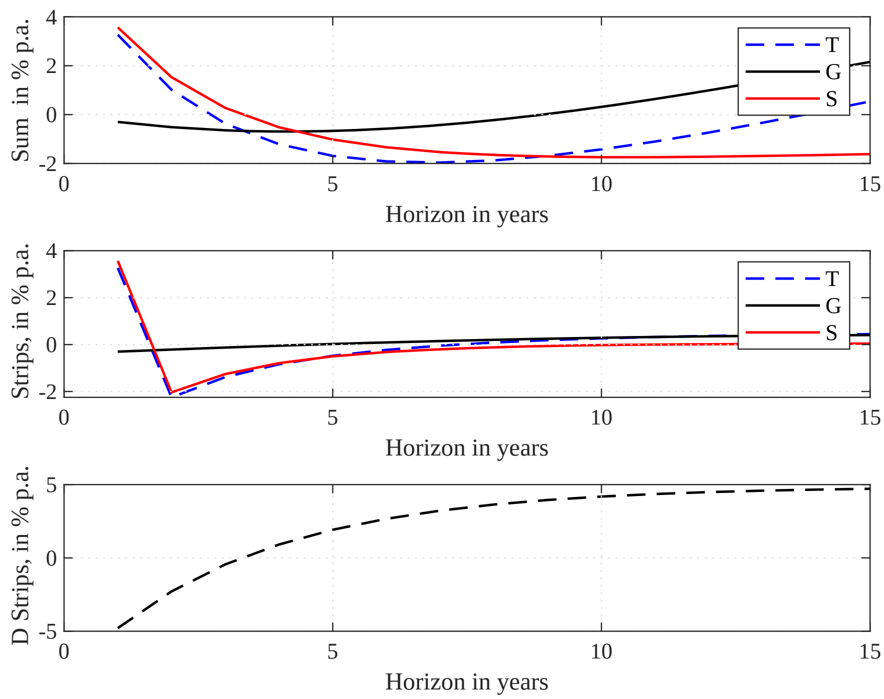
The bottom panel of Figure 7 plots the risk premium on the debt strips, which pay off $d_{t+k}Y_{t+k}$, given by

$$\gamma(\sigma-\phi^{k-1}\lambda)\approx-(\exp(-\gamma(\sigma-\phi^{k-1}\lambda))-1).$$

As we have shown, when debt is risk-free, the risk premium for the cumulative surplus claim inherits the negative of the sign of this debt strip risk premium. As $j\to\infty$, this risk premium converges to the risk premium on the output strips, given by $\gamma\sigma\approx-(\exp(-\gamma\sigma)-1)$ of 5%, because the output innovations are permanent. It is common in the literature to assume that this risk premium is zero at long horizons, because this allows discounting at the risk-free rate. Of course, in the presence of permanent shocks, this is wrong. This positive risk premium on the debt strip explains why the surplus claim in the top panel approaches 0 zero from below. Permanent output risk rules out insurance provided to taxpayers over long horizons.

Figure 7: Horizon Decomp. of Risk Premium on Govt. Cash Flows. Case of AR(1).

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (middle panel) against the horizon. The bottom panel plots the risk premium on the debt strips: $-(\exp(-\gamma(\sigma - \phi^{j-1}\lambda)) - 1)$. AR(1) with ϕ_1 is 0.75. Other Parameters – Benchmark calibration in Table 1.



AR(2) We use the estimated AR(2) parameters that provide the best fit for the data. Figure 8 plots the risk premium (in % per annum) on cumulative discounted cash flows (top panel) and the strips (bottom panel) against the horizon. Instead of plotting cash flow betas, we plot the risk premium computed by

$$\begin{aligned} & -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum\nolimits_{k=1}^{j} M_{t+1,t+k}S_{t+k}\right)/E_t[M_{t+1}] & (11) \\ = \quad & E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\psi_{j-1}\lambda))-1). & (12) \end{aligned}$$

against the horizon j in the top panel. This measure is directly informative about the insurance provided by the government to taxpayers with a horizon j. In particular, this taxpayer cares about the following risk measure:

$$-cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum\nolimits_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right)/E_t[M_{t+1}]. \quad (13)$$

When this risk premium is positive, this means that the government is insuring taxpayers. When it is negative, the taxpayers are instead providing insurance to the government.

As shown in the top panel of Figure 8, the risk premium on a claim to the discounted future surpluses converges to zero as we increase the horizon j in eqn. (12). The risk premium on the cumulative surpluses crosses zero when $\sigma > \psi_{j-1}\lambda$.

In the middle panel of Figure 8, we plot the contribution of each strip:

$$-cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)M_{t+1,t+j}S_{t+j}\right)/E_t[M_{t+1}]$$

against the horizon j. The 1-year strip on the surplus earns a risk premium of 3% per annum. It is risky because the surplus decreases in bad times ($\lambda > 0$), when marginal utility is high. In order to make the debt risk-free, the risk premium on the 2-year strip is close to -1.5%. And these strips earn negative risk premium until they revert to zero after 15 years. Hence, the government has to commit to increasing the surplus 1 year after the negative shock. This applies to all the surpluses that follow.

If the debt/output ratio follows an $AR(2)$ process, then you can do some limited smoothing across multiple periods, because the cumulative risk premium on the surplus inherit the hump-shaped dynamics of the debt/output ratio.

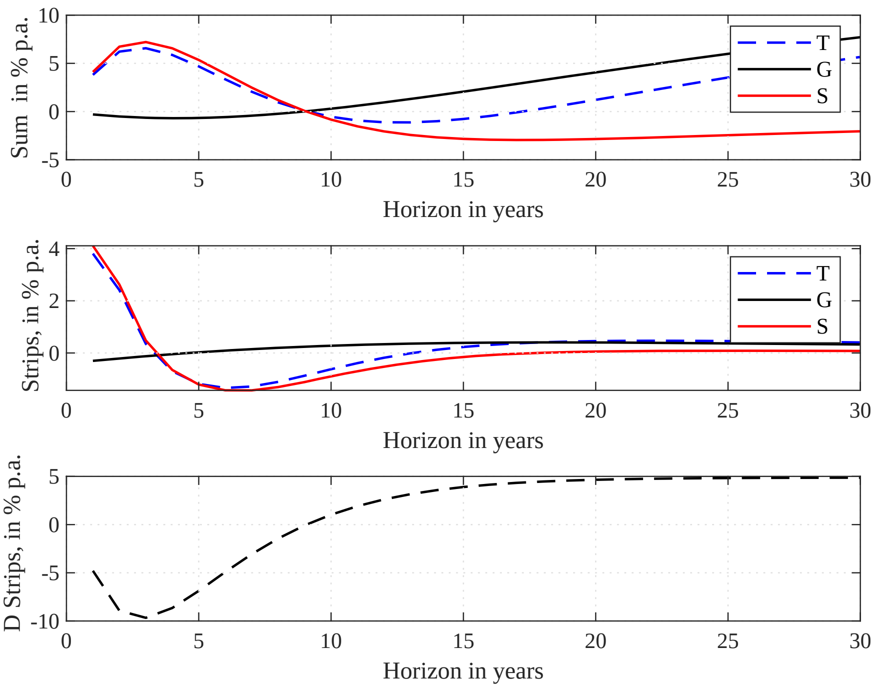
The bottom panel of Figure 8 plots the risk premium on the debt strips, which pay off $d_{t+k}Y_{t+k}$, given by

$$\gamma(\sigma-\psi_{k-1}\lambda) \approx -(\exp(-\gamma(\sigma-\psi_{k-1}\lambda))-1).$$

As we have shown, when debt is risk-free, the risk premium for the cumulative surplus claim inherits the negative of the sign of this debt strip risk premium. As $j \to \infty$, this risk premium

Figure 8: Horizon Decomp. of Risk Premium on Govt Cash Flows: Benchmark Calibration.

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (middle panel) against the horizon. The bottom panel plots the risk premium on the debt strips: $-(\exp(-\gamma(\sigma - \psi_{j-1}\lambda)) - 1)$. Benchmark calibration in Table 1.



converges to the risk premium on the output strips, given by $\gamma\sigma \approx -(\exp(-\gamma\sigma) - 1)$ of 5%, because the output innovations are permanent. It is common in the literature to assume that this risk premium is zero at long horizons, because this allows discounting at the risk-free rate. Of course, in the presence of permanent shocks, this is wrong. This positive risk premium on the debt strip explains why the surplus claim in the top panel approaches 0 zero from below. Permanent output risk rules out insurance provided to taxpayers over long horizons.

5.3 Counter-cyclical Spending

The government insures transfer recipients by spending a larger fraction of GDP in recession. This further constraints the government in navigating the trade-off between insurance of bondholders and taxpayers.

We consider the implications of varying β^g which governs the response to GDP growth shocks. In the post-war sample, when regressing the innovation in log spending on the log change in output, we get a a slope coefficient of 0.28. By contrast, when we run the same regression for tax revenue, we get a slope coefficient of 1.86.

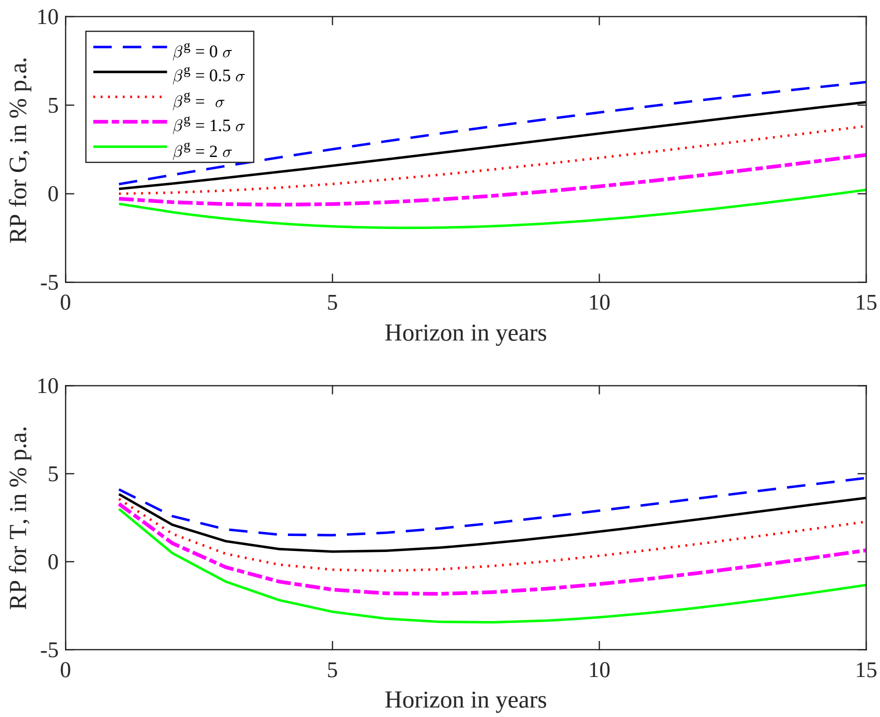
Taxpayers with a horizon j care about the riskiness of the tax process over horizon j.

Corollary 5.3. Under Assumptions 1 and 3, the cash flow beta of taxes have to satisfy the following restriction.

$$
\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k} T_{t+k}\right) \\
= \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j} d_{t+j} Y_{t+j}](\exp(-\gamma(\sigma - \phi^{j-1}\lambda)) - 1) \\
+ \; & \sum_{k=1}^{j} E_t[M_{t+1}]E_t[M_{t+1,t+j} x_{t+j} Y_{t+j}](\exp(-\gamma(\sigma - \phi_G^{j-1}\beta^g)) - 1).
\end{aligned}
$$

Figure 9: Horizon Decomp. of Risk Premium on Govt. Cash Flows and Counter-cyclical Spending

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (bottom panel) against the horizon. We vary β^g between 0 and $2 \times \sigma$.AR(1) with ϕ_1 is 0.75. Other parameters–Benchmark calibration in Table 1.



In particular, this taxpayer with horizon j cares about the following risk measure:

$$
-cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k} T_{t+k}\right) / E_t[M_{t+1}]. \tag{14}
$$

When this risk premium is positive, this means that the government is insuring taxpayers. When it is negative, the taxpayers are instead providing insurance to the government. Figure 9 plots the implied spending (top panel) and tax claim risk (bottom panel) risk premium contributions. As government spending becomes more counter-cyclical, the risk premium on the tax claim has to decline as well, in order to keep the government debt risk-free. The empirically relevant line is the case of $\beta^g = 0.031\%$. In that case, the one-period vol of spending is only 25% of the output vol (σ).

As the tax claim becomes safer, taxpayers face a riskier tax liability proposition. When spending is risk-free (blue line), the tax claim inherits the risk properties of the surplus claim:

$$-E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda))-1)$$

As the governments provides more insurance to transfer recipients, this reduces the scope for insurance of taxpayers one-for-one.

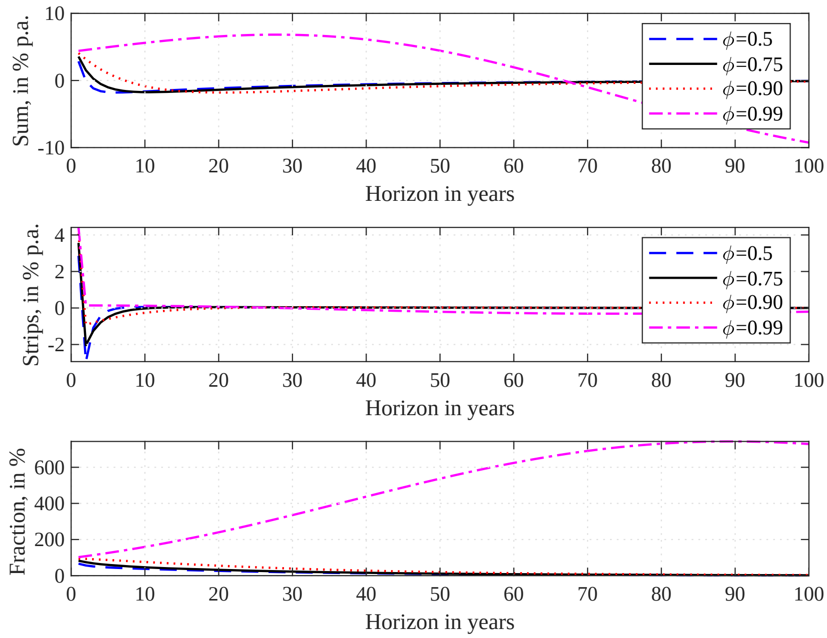
5.4 Debt Persistence

To provide more intertemporal smoothing, the government can increase the persistence of the debt/output process. This allows the government to spread out the adjustment further over time. When $\phi = 0.90$ (dash-dotted line) , we have to increase the horizon beyond 40 years to see the risk premium on the total surplus go to zero, as shown in the top panel of Figure 10. This allows for a riskier surplus in the first year, and a smaller downward adjustment in the risk premium in the following years (see bottom panel). However, even in the case of $\phi = 0.90$, the risk premium flips signs in year 2.

As the government increases the persistence of the debt/output process to 0.99, The government almost imputes a unit root to the debt/output ratios and seems to escape the trade-off between insuring taxpayers and bondholders. As a result of the near-unit-root, the TVC is quasi violated in our calibration, given that the market price of risk γ is large. To visualize this, we plot the following fraction: $E_t[M_{t+k}D_{t+k}]/D_t$, the tail value of debt as a percentage of the debt outstanding today. For $j = 50$, the fraction is 150%. Under the risk-neutral measure, investors expect the debt to increase faster than the risk-free rate; the government increases the debt/output ratio along paths characterized by adverse aggregate histories, because $\lambda > 0$. For $j = 100$, the fraction is 100%. This means that the expected value of debt 100 years from now accounts for the entire value of the debt (and the value of the first 100 years of surpluses for 0%).

Figure 10: Horizon Decomp. of Risk Premium on Govt. Surpluses and Debt Persistence

The figure plots the risk premium contribution of cumulative discounted surpluses (top panel) and the surplus strips (middle panel) against the horizon. The bottom panel plots the tail value at t of the debt expected at $t + j$ as a fraction of debt today. AR(1) with ϕ between 0.5 and 0.99. Other Parameters–Benchmark calibration in Table 1.



6 Revisiting the Trade-off when Debt Earns Convenience Yields

When the transversality conditions holds, and there are no arbitrage opportunities in debt markets, there is only one way to relax this trade-off between insurance of bondholders and taxpayers. Some governments are endowed with the ability to see Treasurys at prices that exceed their fair market value. In other words, investors earn convenience yields on their debt holdings. Typically, the debt then serves the role of a special, safe assets for domestic or foreign investors.

Our analysis begins with a reduced-form characterization of the convenience yield.[12] In discrete time, the convenience yield λ_t is defined as a wedge in the investors' Euler equation:

$$\mathbb{E}_t \left[M_{t,t+1} R_t \right] = \exp(-\lambda_t). \tag{15}$$

The following proposition shows that the convenience yield can be interpreted as an additional seigniorage revenue to the government.

Proposition 6.1. In the absence of arbitrage opportunities, the value of the government debt equals:

$$B_t = \sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (T_{t+j} + (1 - e^{-\lambda_{t+j}}) D_{t+j} - G_{t+j}) \right] = P_t^T + P_t^\lambda - P_t^G,$$

provided that a transversality condition holds.

The seigniorage revenue is $(1 - e^{-\lambda_{t+j}}) D_{t+j}$, which is exactly the amount of interest the government does not need to pay due to the convenience yield. The value of government debt reflects the value of all future convenience yields earned on future debt. We refer to this value as the Treasury's seignorage revenue:

$$P_t^\lambda \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (1 - e^{-\lambda_{t+j}}) D_{t+j} \right].$$

The government debt portfolio return is the return on a portfolio that goes long in the tax claim and short in the spending claim:

$$\begin{aligned} \mathbb{E}_t \left[R_{t+1}^D - R_t^f \right] \quad &= \quad \frac{P_t^T - T_t}{B_t - S_t} \mathbb{E}_t \left[R_{t+1}^T - R_t^f \right] + \frac{P_t^\lambda - T_t}{B_t - S_t} \mathbb{E}_t \left[R_{t+1}^\lambda - R_t^f \right] \\ &- \quad \frac{P_t^G - G_t}{B_t - S_t} \mathbb{E}_t \left[R_{t+1}^G - R_t^f \right], \end{aligned}$$

where R_{t+1}^D, R_{t+1}^T,R_{t+1}^λ and R_{t+1}^G are the holding period returns on the bond portfolio, the tax claim, the seignorage claim, and the spending claim, respectively. We take government spending

[12]See Liu, Schmid, and Yaron (2019) for a structural model of convenience yields and fiscal policy.

process, and the debt return process as exogenously given, and we explore the implications for the properties of the tax claim.

Constant Spending/Output Ratio Let's take a simple benchmark. If we assume that the spending/output ratio is constant and $\beta_t^Y = \beta_t^G$. We define $K_t = (1 - e^{-\lambda_t})D_t$ to be seignorage revenue. Suppose that the (convenience yield) seignorage process has a zero beta. If the government wants risk-free debt, then the implied beta of the tax revenue process

$$\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)} \gg \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)},$$

which exceeds the beta of the tax revenue without seignorage. If the seignorage revenue is sufficiently counter-cyclical, then the government can insure both taxpayers and bondholders at the same time.

Krishnamurthy and Vissing-Jorgensen (2012) estimate convenience yields on U.S. Treasurys of around 75 bps. These convenience yields are counter-cylical. Using the deviations from CIP in Treasury markets, Jiang, Krishnamurthy, and Lustig (2018a,b); Koijen and Yogo (2019) estimate convenience yields that foreign investors derive from their holdings of dollar safe assets; these estimates exceed 200 bps.

We can characterize the sensitivity of the average tax rate to aggregate output growth in closed form.

Assumption 4. *The government commits to a constant spending/output ratio $x = G_t/Y_t$, and a mean-reverting process for the log tax/output ratio $\tau_t = \log(T_t/Y_t)$ with a constant sensitivity to output innovations β^τ:*

$$\Delta\tau_{t+1} = \theta(\bar{\tau} - \tau_t) + \beta_\tau \sigma \varepsilon_{t+1}.$$

Corollary 6.2. Under Assumptions 1 and 4, for the debt to be conditionally risk-free, the sensitivity of the average tax rate needs to satisfy:

$$\beta_\tau = \frac{1}{1+q_\tau}\left(\frac{P_t^G - G_t - (1 + (1+q_\kappa)\beta_\kappa)(P_t^\lambda - K_t)}{D_t + (P_t^G - G_t) - (P_t^\lambda - K_t)} - 1\right).$$

If $\beta_\kappa \ll -\frac{1}{1+q_\kappa}$, then the counter-cyclical convenience yields increase the sensitivity of tax rates to output innovations. For example, we can have a constant average tax rate and risk-free debt when:

$$\beta_\kappa = -\frac{1}{1+q_\kappa}\frac{D_t}{P_t^\lambda - K_t}$$

Consider the case in which the government runs zero primary surpluses in all future states of the world: $D_t = P_t^\lambda - K_t$. In this case, the average tax rate is constant if $\beta_\kappa = -\frac{1}{1+q_\kappa}$. This is -1 in the random walk case with $\theta = 0$. Please see section A of the Appendix for details.

7 Conclusion

The government engineers risk-free debt by choosing the beta of the tax claim judiciously. The more debt outstanding, the lower the beta of the tax claim needs to be. There is no scope for insurance of taxpayers over long horizons in the presence of permanent and priced shocks to output. The only way the government can provide insurance to tax payers over long horizons while keeping the debt risk-free is by saving.

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A Risky Debt

In general, when we specify exogenous processes for taxes and spending, the implied debt is risky. This section derives more general characterizations of the risk-return trade-off, by specifying exogenous processes for taxes and spending, and allowing for arbitrary mean-reversion in the tax rate, and risky debt. This approach is more common in the literature. We will do this in a continuous time version of our model.

Let y_t denote log of real GDP. Let τ_t denote the log tax-to-gdp ratio and let g_t denote the log spending-to-gdp ratio. We specify exogenous processes for *both spending and taxes*:

$$\begin{aligned} dy_t &= \mu dt + \sigma dZ_t, \\ d\tau_t &= \theta(\bar{\tau} - \tau_t)dt + \beta_\tau \sigma dZ_t, \\ dg_t &= \theta(\bar{g} - g_t)dt + \beta_g \sigma dZ_t, \end{aligned}$$

where θ governs the degree of persistence in τ and g. Importantly, this specification does not allow the government to choose a tax process that is more risky in the short run, but less risky at intermediate horizons (See for example Figure 8.)

Then $T_t = \exp(\tau_t + y_t)$ and $G_t = \exp(g_t + y_t)$. Let B_t denote the real value of debt. Let P_t^τ denote the present value of the claim on tax and P_t^g denote the present value of the claim on spending. Let M_t denote the SDF. The asset pricing equations are

$$\begin{aligned} 0 &= \mathcal{A}[M_t T_t dt + d(M_t P_t^\tau)], \\ 0 &= \mathcal{A}[M_t G_t dt + d(M_t P_t^g)], \\ 0 &= \mathcal{A}[M_t (T_t - G_t) dt + d(M_t B_t)]. \end{aligned}$$

Note: The last equation can be thought of as the continuous-time version of the government budget condition.

Proposition A.1. When the TVC holds, the value of the debt equals the price of a claim to tax revenue minus the price of a claim to spending:

$$B_t = P_t^\tau - P_t^g.$$

Let M_t denote the cumulative SDF, and let m_t denote its log. We assume

$$\begin{aligned} dm_t &= -(r + \frac{1}{2}\gamma^2)dt - \gamma dZ_t, \\ dM_t &= -M_t r dt - M_t \gamma dZ_t. \end{aligned}$$

We conjecture that the tax claim and the spending claim are priced according to:

$$\begin{aligned} P_t^\tau &= f_\tau(\tau_t) T_t \\ P_t^g &= f_g(g_t) G_t. \end{aligned}$$

The debt/GDP ratio is given by: $\frac{B_t}{Y_t} = f_\tau(\tau_t)\tau_t - f_g(g_t)g_t$. Then, we conjecture $f_\tau(\tau_t) = \exp(p_\tau + q_\tau \tau_t)$ and $f_g(g_t) = \exp(p_g + q_g g_t)$.

Proposition A.2. When $\theta > 0$, the risk exposure of the debt return is

$$\begin{aligned}
[r_t^B, dM_t] &= -\frac{M_t \gamma \sigma}{B_t} \left(T_t f_\tau \left(1 + (1+q_\tau)\beta_\tau\right) - G_t f_g \left(1 + (1+q_g)\beta_g\right)\right), \\
\textit{where} \quad q_\tau &= -\frac{\theta}{\kappa_1^\tau \theta + (1-\kappa_1^\tau)}, \\
q_g &= -\frac{\theta}{\kappa_1^g \theta + (1-\kappa_1^g)}.
\end{aligned}$$

κ_1^τ and κ_1^g are the constants that are very close to 1 from the Campbell-Shiller approximation:

$$\begin{aligned}
\kappa_1^\tau &= \frac{1}{1+\exp(\overline{P_t^\tau / T_t})} \\
\kappa_1^g &= \frac{1}{1+\exp(\overline{P_t^g / G_t})}
\end{aligned}$$

Random Walk Cash Flows We start with the simplest case in which spending and tax revenue follow a random walk ($\theta = 0$). In this case $q_\tau = q_g = 0$, and the debt/output ratio is non-stationary. The risk exposure of the debt claim is

$$\begin{aligned}
[r_t^B, dM_t] &= -M_t \gamma \frac{1}{B_t} (f_\tau T_t (1+\beta_\tau) - f_g G_t (1+\beta_g))\sigma, \\
\textit{where} \quad f_\tau &= (r - \mu - \frac{1}{2}(1+\beta_\tau)^2 \sigma^2 + \gamma(1+\beta_\tau)\sigma)^{-1}, \\
f_g &= (r - \mu - \frac{1}{2}(1+\beta_g)^2 \sigma^2 + \gamma(1+\beta_g)\sigma)^{-1}.
\end{aligned}$$

Risk-free debt is a knife-edge case. The debt is risk-free if and only if

$$(B_t + P_t^g)(1+\beta_\tau) = P_t^\tau (1+\beta_\tau) \quad = \quad P_t^g (1+\beta_g).$$

Even when allowing for a non-stationary debt/output ratio, the government has to implement a counter-cyclical tax policy if it wants to keep the debt risk-free. For example, when $\beta_g = 0$, i.e. spending is a constant fraction of GDP, this equation requires that the loading of the average tax rate on the output shock satisfy:

$$\beta_\tau = \frac{f_g g_t}{d_t + f_g g_t} - 1,$$

which is negative as long as $d_t = f_\tau \tau_t - f_g g_t > 0$. So, risk-free debt implies countercyclical taxation. This result confirms Barro (1979)'s conjecture that tax rates inherit the random walk property of output and spending if the debt is to be risk-free. As the debt/output ratio increases, the β_τ converges to -1. When the government insures transfer recipients by spending more in recessions, and hence choosing $\beta_g < 0$, then β_τ will have to be even more negative.

Even when debt is risky, there may still be a random walk component in the tax rates. The only way to eliminate this random walk component is to set $\beta_\tau = 0$, which would imply that the instantaneous covariance equals that of the output claim:

$$-\frac{M_t \gamma \sigma}{B_t} \left(T_t f_\tau \left(1 + (1+q_\tau)\beta_\tau\right) - G_t f_g \left(1 + (1+q_g)\beta_g\right)\right) = -M_t \gamma \sigma$$

Hence, if we want to completely eliminate the random walk component in taxes, then the debt becomes an unlevered equity claim.

More generally, $[r_t^B, dM_t]$ is decreasing in β_τ. This formula highlights the trade-off between insuring the taxpayers and insuring the debtholders. If the government wants to smooth the tax burden by increasing β_τ, the debt will be riskier because the instantaneous covariance $[r_t^B, dM_t]$ decreases.

Mean-Reverting Cash Flows We consider the case in which $\theta > 0$. Since $1 + q_\tau > 0$ and $1 + q_g > 0$, the same intuition applies: $[r_t^B, dM_t]$ is decreasing in β_τ, implying a trade-off between insuring the taxpayers and insuring the debtholders. The debt is risk-free if and only if the following condition is satisfied:

$$(f_g g_t + d_t)(1 + (1 + q_\tau)\beta_\tau) \quad = \quad g_t f_g \left(1 + (1 + q_g)\beta_g\right).$$

For example, when $\beta_g = 0$, i.e. spending is a constant fraction of GDP, the sensitivity of the average tax rate to the output shock is given by:

$$\beta_\tau \quad = \quad \frac{1}{1 + q_\tau}\left(\frac{f_g g_t}{d_t + f_g g_t} - 1\right),$$

which is negative as long as $d_t = f_\tau \tau_t - f_g g_t > 0$. All else equal, mean reversion renders the tax rate even more countercylical, because $\frac{1}{1+q_\tau} > 1$, when $\theta > 0$. The higher θ, the larger this ratio. To get the tax rate revert back to its mean faster, the tax rate has to be more counter-cylical. So, the government can eliminate the random walk in taxes but only by forcing tax payers to insure the rest of the economy even more against aggregate shocks. So, risk-free debt implies countercyclical taxation.

General Model with Convenience Yield Now, move on to a general model with convenience yield. The Euler equation is

$$0 \quad = \quad \mathcal{A}[M_t B_t \lambda_t dt + M_t(T_t - G_t)dt + d(M_t B_t)],$$

We define $K_t = B_t \lambda_t$ as the flow benefit of convenience yield generated by the government debt, define $\kappa_t = K_t / Y_t$ as the conv yield-to-gdp ratio, and assume

$$d\kappa_t \quad = \quad \theta(\bar{\kappa} - \kappa_t)dt + \beta_\kappa \gamma dZ_t.$$

Then the debt value can be solved from

$$0 \quad = \quad \mathcal{A}[M_t(T_t + K_t - G_t)dt + d(M_t B_t)].$$

Similarly, we let P_t^τ denote the present value of the claim on convenience yield. Then

$$P_t^\tau \quad = \quad f_\kappa(\kappa_t)K_t,$$

where $f_\kappa(\kappa_t) = \exp(p_\kappa + q_\kappa \kappa_t)$.

Proposition A.3. When $\theta > 0$, the risk exposure of the debt return is

$$[r_t^B, dM_t] \quad = \quad -\frac{M_t \gamma \gamma}{B_t}\left(f_\tau T_t(1 + (1 + q_\tau)\beta_\tau) + f_\kappa K_t(1 + (1 + q_\kappa)\beta_\kappa) - f_g G_t(1 + (1 + q_g)\beta_g)\right).$$

To produce risk-free debt (i.e. $[r_t^B, dM_t] = 0$), we need

$$f_\tau T_t(1 + (1 + q_\tau)\beta_\tau) + f_\kappa K_t(1 + (1 + q_\kappa)\beta_\kappa) \quad = \quad f_g G_t(1 + (1 + q_g)\beta_g)$$

A countercyclical convenience yield stream (negative β_κ) helps generate a countercyclical spending stream (negative β_g), thereby partially alleviating the pressure for tax to be countercyclical.

B Return Betas and Cash Flows

What is the relation between the return betas and the cash flow betas? Well, in this simple case, with constant debt/output and constant spending/output ratios, there is a one-to-one mapping:

Corollary B.1. The expected returns can be expressed as a function of the cash flow betas:

$$\begin{aligned}
\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] &= \frac{x}{d(1-\xi_1)+x\xi_1}\frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})}, \\
&= \frac{x}{d(1-\xi_1)+x\xi_1}\exp(\mu+\frac{1}{2}\sigma^2)(1-\exp(-\gamma\sigma)) \\
\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] &= \frac{1}{\xi_1}\frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})} \\
&= \frac{1}{\xi_1}\exp(\mu+\frac{1}{2}\sigma^2)(1-\exp(-\gamma\sigma)),
\end{aligned}$$

where $\xi_1 = \exp(-\rho-\gamma\sigma+\mu+0.5\sigma^2)$.

C Quantifying the Trade-off in Model with Transitory Output Shocks

Next, we consider the impact of transitory shocks to the level of output, but we, in a first pass, we keep our original pricing kernel with permanent shocks to the level of marginal utility. We call this the goldilocks economy. In this setting, the government can insure taxpayers at all horizons while keeping the debt risk-free.

C.1 Permanent Shocks to Marginal Utility

Assumption 5. *(a) The shocks to output are transitory:*

$$y_{t+1} = \xi_0 + \xi y_t + \sigma\varepsilon_{t+1}$$

where ε_{t+1} still denotes the innovation to output growth that is normally distributed and i.i.d.

(b) The log pricing kernel is

$$m_{t,t+1} = -\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}.$$

This asset pricing model is fundamentally misspecified. This pricing kernel does not reflect the mean-reversion in output and hence cannot be micro-founded. However, we use this model as an expositional device. In this setting, the government faces no trade-off between insuring taxpayers and bondholders. When there are no permanent shocks to output, but the pricing kernel does not reflect this, then the government can insure taxpayers over all horizons.

Proposition C.1. The cash flow beta of the surpluses over j periods is given by:

$$\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}\right) \\
=\; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma-\phi^{j-1}\lambda))-1)
\end{aligned}$$

when $j \geq 2$. The sign of the cash flow covariance is $sign\left(\gamma(\xi^{j-1}\sigma-\phi^{j-1}\lambda\right)$.

Hence, the sign of the cash flow covariance is determined by the sign of $\gamma(\sigma\xi^{k-1}-\phi^{k-1}\lambda)$. As before, this is the risk premium on a debt strip, and compensates investors for output risk. Because the innovations are temporary, the output component of this risk premium converges to zero. The transitory nature of output risk broadens the scope for insurance of taxpayers. As we consider $\xi \to 1$, we revert back to the expression derived in the benchmark model: $\gamma(\sigma - \phi^{k-1}\lambda)$. If $\lambda > \sigma$, the initial covariance is negative. If the rate of mean-reversion in output is higher than in the debt/output ratio, $\phi > \xi$, the covariance stays negative for all j. As a result, the government can now insure taxpayers at all horizons. This was not feasible in the case of permanent innovations.

Corollary C.2. The cash flow beta of taxes have to satisfy the following restriction.

$$
\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\
= & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)) - 1) \\
+ & \; cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}G_{t+k}\right).
\end{aligned}
$$

Quantitative Implications We return to our calibrated economy. Figure 11 plots the risk premium contributions of the surpluses over different horizons for the benchmark calibration:

$$
\begin{aligned}
& -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}\right)/\mathbb{E}_t[M_{t+1}] \\
= & \; E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)) - 1)
\end{aligned}
$$

However, the output process no longer has a unit root. We start by considering the case in which $\phi = \xi$. At all horizons, the tax claim is risky, contributing positive risk premium across all horizons, because λ exceeds σ. The tax claim is also risky across all horizons. In this goldilocks scenario, the government can insure taxpayers at all horizons. $\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda)$ is positive across all horizons.

Figure 11: Risk Premium on Govt. Cash Flows with Transitory Shocks

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (middle panel) against the horizon. The bottom panel plots the risk premium on the debt strips: $-(\exp(-\gamma(\sigma\xi^{j-1} - \phi^{j-1}\lambda)) - 1)$. Calibration: ϕ is 0.75 and ξ is 0.75. Other parameters–Benchmark calibration in Table 1.

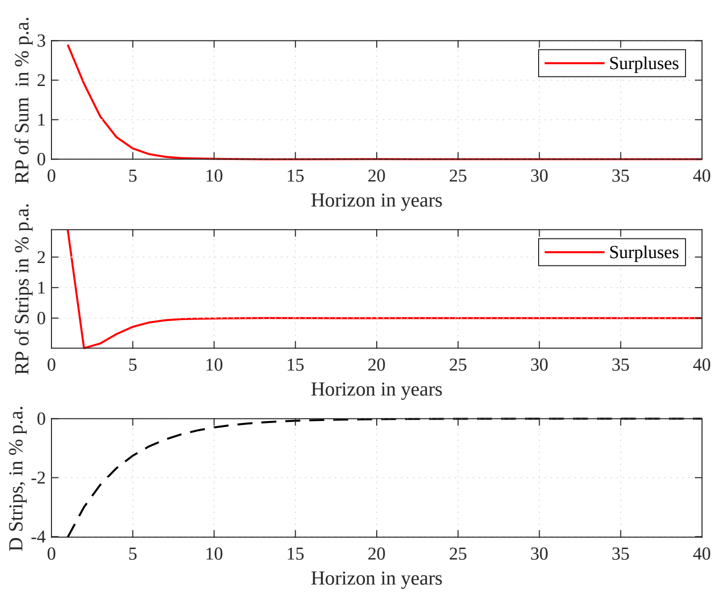


Figure 11 plots the risk premium on the debt strips, which pay off $d_{t+k}Y_{t+k}$, given by

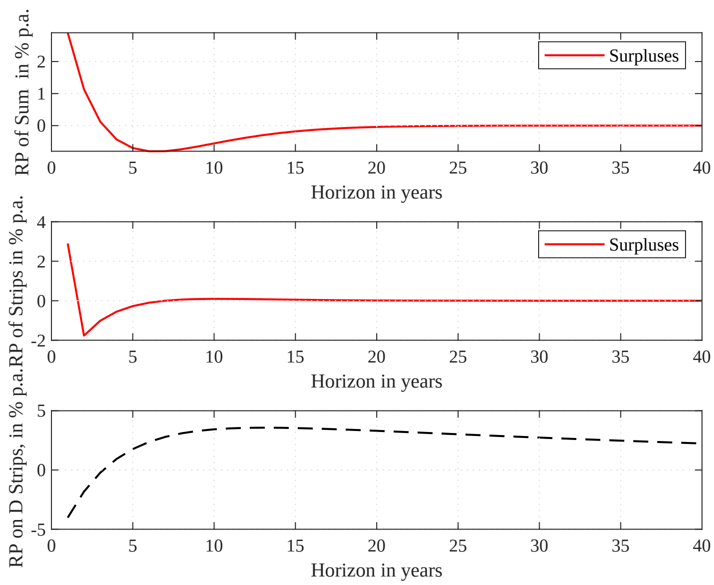
$$\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda) \approx -(\exp(-\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda)) - 1).$$

Given that λ exceeds σ, the risk premium on the debt strips are uniformly negative. These are the mirror image of the surplus risk premium in the top panel of Figure 11. As $j \to \infty$, this debt strip risk premium converges to the risk premium on the output strips, 0%, because the output innovations are transitory, and the pricing kernel does not have a transitory component which contributes interest rate risk. Why can the government insure taxpayers over long horizons (by delivering a risky tax claim)? Because the debt strip risk premium are negative at all horizons.

Of course, insurance of taxpayers only works if the governments commits to a debt policy that is at least as persistent as the output process ($\phi > \xi$). Figure 12 plots the risk premia contributions when the output shocks are close to a unit root, but the debt/output ratio reverts back to the mean at a faster rate. In this case, the government has to produce safer surplus claims over longer horizons.

Figure 12: Risk Premium on Govt. Cash Flows with Transitory Shocks

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (bottom panel) against the horizon. Calibration: ϕ is 0.75 and ξ is 0.98. Other parameters–Benchmark calibration in Table 1.



C.2 Transitory Shocks to Marginal Utility

Next, we consider an internally consistent model: we shut down permanent shocks to the level of output, as well as to marginal utility.

Assumption 6. *(a) The shocks to output are transitory:*

$$y_{t+1} = \xi_0 + \xi y_t + \sigma \varepsilon_{t+1}$$

where ε_{t+1} still denotes the innovation to output growth that is normally distributed and i.i.d.

(b) The log pricing kernel is

$$m_{t,t+1} = -\rho - \frac{1}{2}\gamma^2 - \gamma \frac{\sigma\varepsilon_{t+1} + (\xi - 1)y_t}{\sigma}.$$

When shocks to output are transitory, most asset pricing models predict that there are no permanent shocks to the marginal utility of wealth. This specific modification of the pricing kernel is motivated by the fact that if the agent's consumption is equal to the output and has CRRA preference with a relative risk aversion of γ/σ, the marginal utility growth is $m_{t,t+1} = -\tilde{\rho} - \gamma/\sigma(\xi_0 + (\xi - 1)y_t + \sigma\varepsilon_{t+1})$. In this case, the marginal utility of wealth can be written as:

$$\Lambda_{t+1} = \exp(-\tilde{\rho}(t+1) - (\gamma/\sigma)y_{t+1}).$$

There are no permanent shocks to the marginal utility of wealth. Given this pricing kernel, the log of the risk-free rate is given by:

$$r_t^f = \rho + \gamma \frac{(\xi - 1)y_t}{\sigma}.$$

Note that this model has counterfactual asset pricing implications. In the model, the interest rate risk will make the long bond the riskiest asset in the economy. Modern asset pricing has consistently found that permanent cash flow shocks receive a high price of risk in the market (e.g., Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Bansal and Yaron, 2004; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and Chernov, 2018). This model has no permanent priced risk, except when $\xi = 1$. In that case, we recover the pricing kernel in our benchmark model.

When there are no permanent shocks to output and the pricing kernel, then the government can insure taxpayers over longer horizons.

Proposition C.3. The cash flow beta of the surpluses over j periods is given by:

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k} S_{t+k}\right) \\ = \quad & -E_t[M_{t+1}]E_t[M_{t+1,t+j} d_{t+j} Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1 - \xi^{j-1}))) - 1) \end{aligned}$$

when $j \geq 2$. The sign of the cash flow covariance is $sign\left(\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1 - \xi^{j-1}))\right)$.

Hence, the sign of the cash flow covariance is determined by the sign of $\gamma(\sigma\xi^{k-1} - \phi^{k-1}\lambda + \frac{\gamma}{\sigma}(1 - \xi^{k-1}))$. As before, this is the risk premium on a debt strip. The first component, $\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)$, compensates for output risk. The second component, $\frac{\gamma}{\sigma}(1 - \xi^{j-1})$, compensates for interest rate risk. Because the innovations are temporary, the output component of this risk premium converges to zero. The interest rate risk does not converge to zero; the long bond is the riskiest asset in an economy with only transitory risk. The transitory nature of output risk broadens the scope for insurance of taxpayers, but this is counteracted by interest rate risk. As we consider $\xi \to 1$, we revert back to the expression derived in the benchmark model: $\gamma(\sigma - \phi^{k-1}\lambda)$. The interest rate risk term disappears.

Corollary C.4. The cash flow beta of taxes have to satisfy the following restriction.

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\ = & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma-\phi^{j-1}\lambda+\frac{\gamma}{\sigma}(1-\xi^{j-1})))-1) \\ + & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}G_{t+k}\right). \end{aligned}$$

which can be restated as:

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\ = & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_{t+1}[M_{t+1,t+j}Y_{t+j}]\exp(\frac{1-\phi^j}{1-\phi}(\phi_0-\frac{1}{2}\lambda^2)+\sum_{k=1}^{j-1}((\gamma-\sigma)\xi^{k-1}\phi^{k-1}\lambda+\frac{1}{2}(\phi^{k-1}\lambda)^2) \\ + & \phi^j\log d_t-\phi^{j-1}\lambda((\sigma-\gamma)\xi^{j-1}+\gamma)+\frac{1}{2}(\phi^{j-1}\lambda)^2)(\exp(-\gamma(\xi^{j-1}\sigma-\phi^{j-1}\lambda+\frac{\gamma}{\sigma}(1-\xi^{j-1})))-1) \\ + & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}G_{t+k}\right). \end{aligned}$$

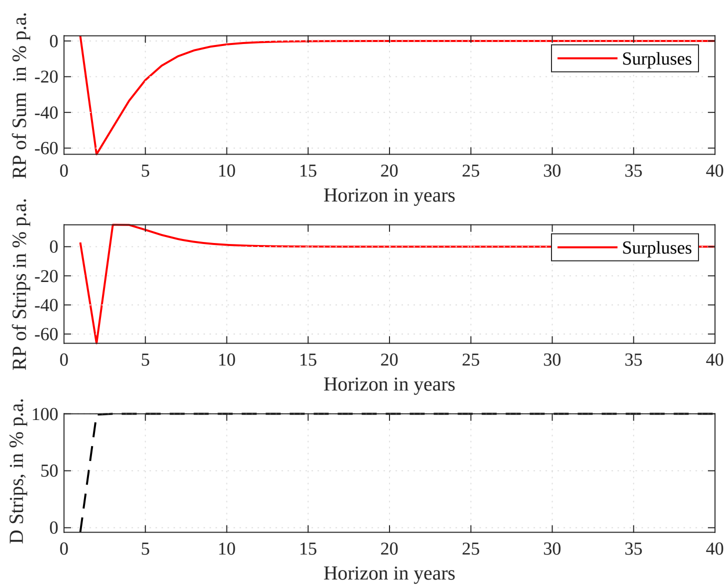
Quantitative Model Implications We return to our calibrated economy. Figure 13 plots the risk premium contributions of the surpluses over different horizons j for the benchmark calibration:

$$\begin{aligned} & -cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}\right)/\mathbb{E}_t[M_{t+1}] \\ = & E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma-\phi^{j-1}\lambda+\frac{\gamma}{\sigma}(1-\xi^{j-1})))-1) \end{aligned}$$

However, the output process no longer has a unit root. At short horizons, the tax claim is safe, contributing negative risk premium, but the tax claim turns risky over horizons that exceed 10 years.

Figure 13: Risk Premium on Govt. Cash Flows with Transitory Shocks

The figure plots the risk premium contribution of cumulative discounted cash flows (top panel) and the strips (middle panel) against the horizon. Calibration: ϕ is 0.75. Other parameters–Benchmark calibration in Table 1.



D Autocovariances

D.1 Permanent Shocks

Corollary D.1. The conditional autocovariances of the surplus/output ratios are

$$
\begin{aligned}
& cov_t(s_{t+1}, s_{t+j}) \\
= \; & \exp(2\rho - 2\mu + \sigma^2) \exp\left((1+\phi^{j-1})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 \frac{1-\phi^{2(j-1)}}{1-\phi^2} \right) \\
\times \; & (\exp(\sigma\lambda\phi^{j-2}) - 1) \\
- \; & \exp(\rho - \mu + .5\sigma^2) \exp\left((\phi+\phi^{j-1})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 \frac{1-\phi^{2(j-1)}}{1-\phi^2} + \frac{1}{2}\lambda^2 \right) \\
\times \; & (\exp(\lambda^2\phi^{j-2}) - 1) \\
- \; & \exp(\rho - \mu + .5\sigma^2) \exp\left((1+\phi^{j})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 \frac{1-\phi^{2j}}{1-\phi^2} \right) \\
\times \; & (\exp(\sigma\lambda\phi^{j-1}) - 1) \\
+ \; & \exp\left((\phi+\phi^{j})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 \frac{1-\phi^{2j}}{1-\phi^2} + \frac{1}{2}\lambda^2 \right) (\exp(\lambda^2\phi^{j-1}) - 1),
\end{aligned}
$$

and the conditional variance of the surplus/output ratio is

$$
\begin{aligned}
var_t(s_{t+1}) = \; & \exp(2\rho - 2\mu + \sigma^2) \exp(2\log d_t)(\exp(\sigma^2) - 1) \\
- \; & 2\exp(\rho - \mu + .5\sigma^2) \exp\left((1+\phi)(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 \right) \\
\times \; & (\exp(\lambda\sigma) - 1) \\
+ \; & \exp\left(2\phi(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \lambda^2 \right) (\exp(\lambda^2) - 1)
\end{aligned}
$$

D.2 Transitory Shocks

Corollary D.2. In the presence of transitory shocks, (a) the conditional autocovariances of the surplus/output ratios are

$$
\begin{aligned}
cov_t(s_{t+1}, s_{t+j}) = \; & \exp(2\rho - 2\psi_0 + \sigma^2) d_t \exp(-(\psi-1)y_t) \mathbb{E}_t[d_{t+j-1} \exp(-(\psi-1)y_{t+j-1})] \\
\times \; & (\exp(\sigma\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
- \; & \exp(\rho - \psi_0 + .5\sigma^2) \mathbb{E}_t[d_{t+1}] \mathbb{E}_t[d_{t+j-1} \exp(-(\psi-1)y_{t+j-1})] \\
\times \; & (\exp(\lambda\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
- \; & \exp(\rho - \psi_0 + .5\sigma^2) d_t \exp(-(\psi-1)y_t) \mathbb{E}_t[d_{t+j}](\exp(\sigma\lambda\phi^{j-1}) - 1) \\
+ \; & \mathbb{E}_t[d_{t+1}] \mathbb{E}_t[d_{t+j}](\exp(\lambda^2\phi^{j-1}) - 1)
\end{aligned}
$$

where the conditional forecasts are

$$
\mathbb{E}_t[d_{t+j}] = \exp\left(\phi^j(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 \frac{1-\phi^{2j}}{1-\phi^2} \right)
$$

and

$$\begin{aligned}
& \mathbb{E}_t[\exp(\log d_{t+j-1} - (\psi - 1) y_{t+j-1})] \\
= \; & \exp\left(\phi^{j-1}(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} - (\psi-1)\psi^{j-1}(y_t - \frac{\psi_0}{1-\psi}) - (\psi-1)\frac{\psi_0}{1-\psi} \right. \\
+ \; & \left. \frac{1}{2}\sum_{k=0}^{j-2}(\phi^k \lambda + \psi^k(\psi-1)\sigma)^2 \right)
\end{aligned}$$

(b) The conditional variance of the surplus/output ratio is

$$\begin{aligned}
var_t(s_{t+1}) \; = \; & \exp(2\rho - 2\psi_0 + \sigma^2)\exp(2\log d_t - 2(\psi-1)y_t)(\exp(\sigma^2) - 1) \\
- \; & 2\exp(\rho - \psi_0 + .5\sigma^2)\exp\left((1+\phi)(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 - (\psi-1)y_t\right) \\
\times \; & (\exp(\lambda\sigma) - 1) \\
+ \; & \exp\left(2\phi(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \lambda^2\right)(\exp(\lambda^2) - 1)
\end{aligned}$$

E Proofs

E.1 Proof of Eq. (1) in Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019)

Proof. All objects in this appendix are in nominal terms but we drop the superscript $\$$ for ease of notation. The government faces the following one-period budget constraint:

$$G_t - T_t + Q_{t-1}^1 = \sum_{h=1}^{H}(Q_t^h - Q_{t-1}^{h+1})P_t^h,$$

where G_t is total nominal government spending, T_t is total nominal government revenue, Q_t^h is the number of nominal zero-coupon bonds of maturity h outstanding in period t each promising to pay back \$1 at time $t+h$, and P_t^h is today's price for a h-period zero-coupon bond with \$1 face value. A unit of $h+1$-period bonds issued at $t-1$ becomes a unit of h-period bonds in period t. That is, the stock of bonds evolves of each maturity evolves according to $Q_t^h = Q_{t-1}^{h+1} + \Delta Q_t^h$. Note that this notation can easily handle coupon-bearing bonds. For any bond with deterministic cash-flow sequence, we can write the price (present value) of the bond as the sum of the present values of each of its coupons.

The left-hand side of the budget constraint denotes new financing needs in the current period, due to primary deficit $G-T$ and one-period debt from last period that is now maturing. The right hand side shows that the money is raised by issuing new bonds of various maturities. Alternatively, we can write the budget constraint as total expenses equalling total income:

$$G_t + Q_{t-1}^1 + \sum_{h=1}^{H} Q_{t-1}^{h+1}P_t^h = T_t + \sum_{h=1}^{H} Q_t^h P_t^h,$$

We can now iterate the budget constraint forward. The period t constraint is given by:

$$\begin{aligned}
T_t - G_t \; = \; & Q_{t-1}^1 - Q_t^1 P_t^1 + Q_{t-1}^2 P_t^1 - Q_t^2 P_t^2 + Q_{t-1}^3 P_t^2 - Q_t^3 P_t^3 \\
& + \cdots - Q_t^H P_t^H + Q_{t-1}^{H+1} P_t^H.
\end{aligned}$$

Consider the period-$t+1$ constraint,

$$\begin{aligned}
T_{t+1} - G_{t+1} \; = \; & Q_t^1 - Q_{t+1}^1 P_{t+1}^1 + Q_t^2 P_{t+1}^1 - Q_{t+1}^2 P_{t+1}^2 + Q_t^3 P_{t+1}^2 - Q_{t+1}^3 P_{t+1}^3 \\
& + \cdots - Q_{t+1}^H P_{t+1}^H + Q_t^{H+1} P_{t+1}^H.
\end{aligned}$$

multiply both sides by M_{t+1}, and take expectations conditional on time t:

$$\begin{aligned}\mathbb{E}_t\left[M_{t+1}(T_{t+1}-G_{t+1})\right] &= Q_t^1P_t^1-\mathbb{E}_t[Q_{t+1}^1M_{t+1}P_{t+1}^1]+Q_t^2P_t^2-\mathbb{E}_t[Q_{t+1}^2M_{t+1}P_{t+1}^2]+Q_t^3P_t^3\\ &\quad -\mathbb{E}_t[Q_{t+1}^3M_{t+1}P_{t+1}^3]+\cdots+Q_t^HP_t^H\\ &\quad -\mathbb{E}_t[Q_{t+1}^HM_{t+1}P_{t+1}^H]+Q_t^{H+1}P_t^{H+1},\end{aligned}$$

where we use the asset pricing equations $\mathbb{E}_t[M_{t+1}]=P_t^1$, $\mathbb{E}_t[M_{t+1}P_{t+1}^1]=P_t^2$, $\cdots$, $\mathbb{E}_t[M_{t+1}P_{t+1}^{H-1}]=P_t^H$, and $\mathbb{E}_t[M_{t+1}P_{t+1}^H]=P_t^{H+1}$.

Consider the period $t+2$ constraint, multiplied by $M_{t+1}M_{t+2}$ and take time-t expectations:

$$\begin{aligned}\mathbb{E}_t\left[M_{t+1}M_{t+2}(T_{t+2}-G_{t+2})\right] &= \mathbb{E}_t[Q_{t+1}^1M_{t+1}P_{t+1}^1]-\mathbb{E}_t[Q_{t+2}^1M_{t+1}M_{t+2}P_{t+2}^1]+\mathbb{E}_t[Q_{t+1}^2M_{t+1}P_{t+1}^2]\\ &\quad -\mathbb{E}_t[Q_{t+2}^2M_{t+1}M_{t+2}P_{t+2}^2]+\mathbb{E}_t[Q_{t+1}^3M_{t+1}P_{t+1}^3]-\cdots\\ &\quad +\mathbb{E}_t[Q_{t+1}^HM_{t+1}P_{t+1}^H]-\mathbb{E}_t[Q_{t+2}^HM_{t+1}M_{t+2}P_{t+2}^H]\\ &\quad +\mathbb{E}_t[Q_{t+1}^{H+1}M_{t+1}P_{t+1}^{H+1}],\end{aligned}$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M_{t+2}]=P_{t+1}^1$, $\mathbb{E}_{t+1}[M_{t+2}P_{t+2}^1]=P_{t+1}^2$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected discounted surpluses at t, $t+1$, and $t+2$ we get:

$$\begin{aligned}T_t-G_t+\mathbb{E}_t\left[M_{t+1}(T_{t+1}-G_{t+1})\right]+\mathbb{E}_t\left[M_{t+1}M_{t+2}(T_{t+2}-G_{t+2})\right]=\textstyle\sum_{h=0}^H Q_{t-1}^{h+1}P_t^h+\\ -\mathbb{E}_t[Q_{t+2}^1M_{t+1}M_{t+2}P_{t+2}^1]-\mathbb{E}_t[Q_{t+2}^2M_{t+1}M_{t+2}P_{t+2}^2]-\cdots--\mathbb{E}_t[Q_{t+2}^HM_{t+1}M_{t+2}P_{t+2}^H].\end{aligned}$$

Similarly consider the one-period government budget constraints at times $t+3$, $t+4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t+J$, we get:

$$\sum_{h=0}^H Q_{t-1}^{h+1}P_t^h = \mathbb{E}_t\left[\sum_{j=0}^J M_{t,t+j}(T_{t+j}-G_{t+j})\right]+\mathbb{E}_t\left[M_{t,t+J}\sum_{h=1}^H Q_{t+J}^hP_{t+J}^h\right]$$

where we used the cumulate SDF notation $M_{t,t+j}=\prod_{i=0}^j M_{t+i}$ and by convention $M_{t,t}=M_t=1$ and $P_t^0=1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next J years plus the present value of the government bond portfolio that will be outstanding at time $t+J$. The latter is the cost the government will face at time $t+J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J\to\infty$:

$$\sum_{h=0}^H Q_{t-1}^{h+1}P_t^h = \mathbb{E}_t\left[\sum_{j=0}^\infty M_{t,t+j}(T_{t+j}-G_{t+j})\right]+\lim_{J\to\infty}\mathbb{E}_t\left[M_{t,t+J}\sum_{h=1}^H Q_{t+J}^hP_{t+J}^h\right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j}-G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the TVC:

$$\lim_{J\to\infty}\mathbb{E}_t\left[M_{t,t+J}\sum_{h=1}^H Q_{t+J}^hP_{t+J}^h\right]=0.$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise there is a government debt bubble.

If the TVC is satisfied, the outstanding debt today, D_t, reflects the expected present-discounted value of the current and all future primary surpluses:

$$D_t = \sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j}) \right].$$

This is Eq. (1) in the main text.

□

E.2 Proof of Proposition 2.1

Proof. From the investor's Euler equation, we know that the expected excess return on the tax claim is given by

$$E_t \left[R_{t+1}^T - R_t^f \right] = \frac{-cov\left(M_{t+1}, R_{t+1}^T\right)}{E_t M_{t+1}} = \frac{-cov\left(M_{t+1}, R_{t+1}^T\right)}{var_t M_{t+1}} \frac{var_t M_{t+1}}{E_t M_{t+1}} = \beta_t^T \lambda_t,$$

and we know that the expected excess return on the spending claim is given by:

$$E_t \left[R_{t+1}^G - R_t^f \right] = \frac{-cov\left(M_{t+1}, R_{t+1}^G\right)}{E_t M_{t+1}} = \frac{-cov\left(M_{t+1}, R_{t+1}^G\right)}{var_t M_{t+1}} \frac{var_t M_{t+1}}{E_t M_{t+1}} = \beta_t^G \lambda_t.$$

Finally, the expected excess return on the debt is also given by:

$$E_t \left[R_{t+1}^D - R_t^f \right] = \frac{-cov\left(M_{t+1}, R_{t+1}^D\right)}{E_t M_{t+1}} = \frac{-cov\left(M_{t+1}, R_{t+1}^D\right)}{var_t M_{t+1}} \frac{var_t M_{t+1}}{E_t M_{t+1}} = \beta_t^D \lambda_t.$$

□

E.3 Proof of Proposition 2.3

Proof. Consider a government that only issues risk-free debt. Note that the surplus at $t+1$ is given by:

$$S_{t+1} \quad = \quad d_t Y_t \exp(r_t^f) - d_{t+1} Y_{t+1}, \tag{16}$$

and the surplus at $t+2$ is given by:

$$S_{t+2} \quad = \quad d_{t+1} Y_{t+1} \exp(r_{t+1}^f) - d_{t+2} Y_{t+2}. \tag{17}$$

We assume that $\{S_t\}$ satisfies the government budget constraint. Next, suppose the government commits to an arbitrary perturbation of d_{t+k} by $\Delta_{t+k}(\varepsilon_{t+k})$. Then we know that the new surplus at $t+1$ is:

$$\tilde{S}_{t+1} \quad = \quad \exp(r_t^f) d_t Y_t - (d_{t+1} + \Delta_{t+1}(\varepsilon_{t+1})) Y_{t+1},$$

and the new surplus at $t+2$ is given by:

$$\tilde{S}_{t+2} \quad = \quad \exp(r_{t+1}^f)(d_{t+1} + \Delta_{t+1}(\varepsilon_{t+1})) Y_{t+1} - (d_{t+2} + \Delta_{t+2}(\varepsilon_{t+2})) Y_{t+2}.$$

Hence, the sum of the discounted perturbed surpluses $\tilde{S}_{t+1} + E_{t+1}[M_{t+1,t+2}\tilde{S}_{t+2}] = S_{t+1} + E_{t+1}[M_{t+1,t+2} S_{t+2}] = -E_{t+1}[M_{t+1,t+2} d_{t+2} + \Delta_{t+2}(\varepsilon_{t+2})) Y_{t+2}]$ is unchanged, because Δ_{t+2} only depends on ε_{t+2}.

We also know that the future surpluses cannot respond to the shock ε_{t+1}:

$$\tilde{S}_{t+2} \quad = \quad \exp(r_{t+1}^f)(d_{t+1} + \Delta_{t+1}(\varepsilon_{t+1})) Y_{t+1} - (d_{t+2} + \Delta_{t+2}(\varepsilon_{t+2})) Y_{t+2},$$

and the surplus at $t+3$ is given by:

$$\tilde{S}_{t+3} = \exp(r^f_{t+2})(d_{t+2} + \Delta_{t+2}(\varepsilon_{t+2}))Y_{t+2}) - d_{t+3}Y_{t+3}.$$

So, this rule only allows for a state-contingent shock to the surplus in one period, but it zeros out over two periods. $\tilde{S}_{t+1} + \exp(-r^f_{t+1})\tilde{S}_{t+2} = S_{t+1} + \exp(-r^f_{t+1})S_{t+2}$ does not depend on ε_{t+1}). Hence:

$$(\mathbb{E}_{t+1} - \mathbb{E}_t)(S_{t+1} + E_{t+1}[M_{t+1,t+2}S_{t+2}]) = 0$$

Next, suppose the government commits to an arbitrary perturbation of d_{t+k} by $\Delta_{t+k}(\varepsilon^2_{t+k})$. Then we know that the new surplus at $t+1$ is:

$$\tilde{S}_{t+1} = \exp(r^f_t)d_tY_t - (d_{t+1} + \Delta_{t+1}(\varepsilon^2_{t+1}))Y_{t+1},$$

and the new surplus at $t+2$ is given by:

$$\tilde{S}_{t+2} = \exp(r^f_{t+1})(d_{t+1} + \Delta_{t+1}(\varepsilon^2_{t+1}))Y_{t+1} - (d_{t+2} + \Delta_{t+2}(\varepsilon^2_{t+2}))Y_{t+2},$$

The surplus at $t+3$ is given by:

$$\tilde{S}_{t+3} = \exp(r^f_{t+2})(d_{t+2} + \Delta_{t+2}(\varepsilon^2_{t+2}))Y_{t+2}) - (d_{t+3} + \Delta_{t+3}(\varepsilon^2_{t+3}))Y_{t+3}.$$

So, this rule only allows for a state-contingent shock to the surplus in one period, but it zeros out over three periods. $\tilde{S}_{t+1} + E_{t+1}[M_{t+1,t+2}(\tilde{S}_{t+2} + E_{t+2}[M_{t+2,t+3}\tilde{S}_{t+3}])] = S_{t+1} + E_{t+1}[M_{t+1,t+2}(S_{t+2} + E_{t+2}[M_{t+2,t+3}S_{t+3}])]$ does not depend on ε_{t+1}.

Hence, we obtain the following result:

$$(\mathbb{E}_{t+1} - \mathbb{E}_t)\left(S_{t+1} + E_{t+1}[M_{t+1,t+2}\left(S_{t+2} + E_{t+2}[M_{t+2,t+3}S_{t+3}]\right)]\right) = 0.$$

The result follows by induction. □

E.4 Proof of Proposition 3.1

Proof. We start from the one-period budget constraint:

$$T_t = G_t - (D_t - R^f_{t-1}D_{t-1}).$$

With TVC,

$$\begin{aligned} R^f_{t-1}D_{t-1} &= S_t + D_t = S_t + \frac{R^f_t D_t}{R^f_t} = S_t + \mathbb{E}_t[\exp(m_{t,t+1})R^f_t D_t], \\ &= S_t + \mathbb{E}_t[\exp(m_{t,t+1})(S_{t+1} + \exp(m_{t+1,t+2})R^f_{t+1}D_{t+1})] = \mathbb{E}_t[\sum_{k=0}^{\infty} M_{t,t+k}S_{t+k}]. \end{aligned}$$

So, this implies that we can state the value of outstanding debt at t:

$$\begin{aligned} R^f_t D_t &= \mathbb{E}_{t+1}[\sum_{k=0}^{\infty} \exp(m_{t+1,t+1+k})S_{t+1+k}] = \mathbb{E}_{t+1}[\sum_{k=1}^{\infty} \exp(m_{t+1,t+k})S_{t+k}] \\ D_t &= \mathbb{E}_t[\exp(m_{t,t+1})]\mathbb{E}_{t+1}[\sum_{k=1}^{\infty} \exp(m_{t+1,t+k})S_{t+k}] \end{aligned}$$

Note that D_t is t-measurable,

$$D_t = \mathbb{E}_t[\sum_{k=1}^{\infty} M_{t,t+k} S_{t+k}],$$

and the measurability condition for risk-free debt is satisfied:

$$(\mathbb{E}_{t+1} - \mathbb{E}_t)[\sum_{k=1}^{\infty} M_{t,t+k} S_{t+k}] = 0.$$

Conjecture the pricing of the surplus strip is

$$\mathbb{E}_t \left[M_{t,t+k} Y_{t+k} \right] = \xi_k Y_t \tag{18}$$

for $k \geq 0$. Then the pricing of the first spending strip is

$$\begin{aligned}
\mathbb{E}_t \left[\exp(m_{t,t+1}) Y_{t+1} \right] &= \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - 1)^2) Y_t, \\
\xi_1 Y_t &= \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2) Y_t.
\end{aligned}$$

Similarly, the pricing of the second spending strip is

$$\begin{aligned}
\mathbb{E}_t \left[\exp(m_{t,t+2}) Y_{t+2} \right] &= \mathbb{E}_t \left[\exp(m_{t,t+1}) \mathbb{E}_{t+1} [\exp(m_{t+1,t+2}) Y_{t+2}] \right], \\
&= \mathbb{E}_t \left[\exp(m_{t,t+1}) \xi_1 Y_{t+1} \right], \\
\xi_2 Y_t &= \xi_1 \mathbb{E}_t \left[\exp(m_{t,t+1} + \mu + \varepsilon_{t+1}) \right] Y_t, \\
&= \xi_1 \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2) Y_t.
\end{aligned}$$

The price of the output strips is given by

$$\begin{aligned}
\mathbb{E}_t \left[M_{t,t+k} Y_{t+k} \right] &= \xi_k Y_t, \textit{ where} \\
\xi_k &= \xi_{k-1} \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2), k \geq 1 \\
\xi_1 &= \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2).
\end{aligned}$$

We define a k-period surplus strip as a claim to S_{t+k}. The price of the surplus strips is given by

$$\begin{aligned}
\mathbb{E}_t \left[M_{t,t+k} S_{t+k} \right] &= \chi_k Y_t, \textit{ where} \\
\chi_k &= \chi_{k-1} \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2), \\
\chi_1 &= d \left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2) \right].
\end{aligned}$$

To replicate safe debt, we short dY_t risky strips to output next period, and we take a similarly sized long position in the risk-free. We implement the same strategy for all future output strips. Note that we cannot simply price these strips off the risk-free yield curve, even though the entire debt is risk-free.

The pricing of the first surplus strip is

$$\begin{aligned}
\mathbb{E}_t \left[\exp(m_{t,t+1}) S_{t+1} \right] &= \mathbb{E}_t \left[\exp(m_{t,t+1}) \{ -dY_{t+1} \left(1 - R_t^f \exp[-(\mu + \varepsilon_{t+1})] \right) \} \right], \\
&= -d\mathbb{E}_t \left[\exp(m_{t,t+1}) Y_{t+1} \right] + dY_t R_t^f \mathbb{E}_t \left[\exp(m_{t,t+1}) \right], \\
&= -d\mathbb{E}_t \left[\exp(m_{t,t+1}) Y_{t+1} \right] + dY_t,
\end{aligned}$$

$$
\begin{aligned}
&= -d\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)Y_t + dY_t, \\
&= \left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\right] dY_t. \\
\chi_1 Y_t &= \left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\right] dY_t.
\end{aligned}
$$

Similarly, the pricing of the second surplus strip is

$$
\begin{aligned}
\mathbb{E}_t\left[\exp(m_{t,t+2})S_{t+2}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1})\mathbb{E}_{t+1}[\exp(m_{t+1,t+2})S_{t+2}]\right], \\
&= \mathbb{E}_t\left[\exp(m_{t,t+1})\chi_1 Y_{t+1}\right], \\
\chi_2 Y_t &= \chi_1 \mathbb{E}_t\left[\exp(m_{t,t+1} + \mu + \sigma\varepsilon_{t+1})\right] Y_t, \\
&= \chi_1 \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)Y_t.
\end{aligned}
$$

We short a risky strip to output 2 periods from now, and go long in the risk-free. The problem then becomes solving the fixed-point problem for the sequence z_k:

$$
\begin{aligned}
\chi_2 &= \chi_1 \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2), \\
\chi_k &= \chi_{k-1} \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2).
\end{aligned}
$$

This fixed-point problem has a unique solution:

$$
\sum_{k=1}^{\infty} \chi_k = \chi_1(1 + K + K^2 + \ldots) = \frac{1}{1-K}\chi_1 = d,
$$

where $K = \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)$. We also have the following TVC:

$$
\lim_{j\to\infty} \mathbb{E}_t\left[m_{t,t+j}D_{t+j}\right] = \lim_{j\to\infty} d\mathbb{E}_t\left[m_{t,t+j}Y_{t+j}\right] = 0.
$$

□

E.5 Proof of Corollary 3.2

Proof. From the gross risk-free rate xpression $R^f_{t+1} = \exp(\rho)$ and the one-period government budget constraint, we get that:

$$
\frac{T_t}{Y_t} = x - d\left(1 - R^f_{t-1}\frac{Y_{t-1}}{Y_t}\right),
$$

we have that the return on the tax claim can be stated as:

$$
\begin{aligned}
R^T_{t+1} &= \frac{P^T_{t+1}}{P^T_t - T_t} = \frac{(d + x\frac{\xi_1}{1-\xi_1})Y_{t+1} + (x - d\left(1 - R^f_t\frac{Y_t}{Y_{t+1}}\right))Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t}, \\
&= \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t} + \frac{d\exp(\rho)}{(d + x\frac{\xi_1}{1-\xi_1})}.
\end{aligned}
$$

Similarly, the return on the spending claim can be stated as:

$$
R^G_{t+1} = \frac{P^G_{t+1}}{P^G_t - G_t} = \frac{x\frac{\xi_1}{1-\xi_1}Y_{t+1} + xY_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t},
$$

$$= \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t}.$$

Armed with these expressions, we get the following expression for the covariance:

$$cov(R^T_{t+1}, M_{t,t+1}) = \frac{x\frac{\xi_1}{1-\xi_1}}{(d + x\frac{\xi_1}{1-\xi_1})}cov(R^G_{t+1}, M_{t,t+1}),$$

which also translates to

$$\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}\mathbb{E}_t\left[R^Y_{t+1} - R^f_t\right].$$

□

E.6 Proof of Proposition 4.1: Case of $AR(1)$

Proof. From

$$T_t = G_t - (D_t - R^f_{t-1}D_{t-1}).$$

This implies that:

$$\begin{aligned}
\frac{T_t}{Y_t} &= x - \left(d_t - R^f_{t-1}d_{t-1}\frac{Y_{t-1}}{Y_t}\right) \\
&= x - \left(d_t - R^f_{t-1}d_{t-1}\exp[-(\mu + \sigma\varepsilon_t)]\right).
\end{aligned}$$

Assume that the debt/output ratio evolves according to a martingale process: $d_t = d_{t-1}\exp(-\lambda\varepsilon_t - (1/2)\lambda^2)$. To guarantee risk-free debt, the tax process has to satisfy

$$\begin{aligned}
\frac{T_t}{Y_t} &= x - d_{t-1}\left(d^{\phi-1}_{t-1}\exp(\phi_0 - \lambda\varepsilon_t - (1/2)\lambda^2) - R^f_{t-1}\frac{Y_{t-1}}{Y_t}\right), \\
&= x - d_{t-1}\left(d^{\phi-1}_{t-1}\exp(\phi_0 - \lambda\varepsilon_t - (1/2)\lambda^2) - R^f_{t-1}\exp[-(\mu + \sigma\varepsilon_t)]\right).
\end{aligned}$$

The surplus process that results is given by:

$$\frac{S_t}{Y_t} = d_{t-1}R^f_{t-1}\exp[-(\mu + \sigma\varepsilon_t)] - d^{\phi}_{t-1}\exp(\phi_0 - \lambda\varepsilon_t - \frac{1}{2}\lambda^2).$$

We conjecture that the price of the surplus strips is given by:

$$\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = (\chi_{k,t} - \psi_{k,t})Y_t.$$

The pricing of the first surplus strip is

$$\begin{aligned}
\mathbb{E}_t\left[\exp(m_{t,t+1})S_{t+1}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1})\{-Y_{t+1}\left(d_{t+1} - R^f_t d_t\exp[-(\mu + \sigma\varepsilon_{t+1})]\right)\}\right], \\
&= \mathbb{E}_t\left[-\exp(\phi\log d_t + m_{t,t+1} + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}\right] + d_tY_t, \\
&= -\exp(\phi\log d_t + \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)Y_t + d_tY_t, \\
(\chi_{1,t} - \psi_{1,t})Y_t &= \left[d_t - \exp(\phi_0 + \phi\log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)\right]Y_t.
\end{aligned}$$

So, we define:

$$
\begin{aligned}
(\chi_{1,t})Y_t &= d_t Y_t, \\
(\psi_{1,t})Y_t &= \exp(\phi_0 + \phi \log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)Y_t.
\end{aligned}
$$

Similarly the pricing of the k-th surplus strip is

$$
\begin{aligned}
\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1})\mathbb{E}_{t+1}[\exp(m_{t+1,t+k})S_{t+k}]\right], \\
(\chi_{k,t} - \psi_{k,t})Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})(\chi_{k-1,t+1} - \psi_{k-1,t+1})Y_{t+1}\right],
\end{aligned}
$$

where the χ's are defined by the following recursion:

$$
\begin{aligned}
\chi_{2,t}Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})\chi_{1,t+1}Y_{t+1}\right], \\
\chi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(-\lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)\exp(\mu + \sigma\varepsilon_{t+1})\right]\exp(\phi \log d_t + \phi_0), \\
&= \exp(\phi_0 + \phi \log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2),
\end{aligned}
$$

and where the ψ's are defined by the following recursion:

$$
\begin{aligned}
\psi_{2,t}Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})\psi_{1,t+1}Y_{t+1}\right], \\
\psi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1} + \phi_0 + \phi \log d_{t+1} - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + g + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \mu + \sigma\varepsilon_{t+1})\right], \\
\psi_{2,t} &= \exp(-2\rho + \phi_0 + \phi\phi_0 + \phi^2 \log d_t - \frac{1}{2}(\gamma^2 + \phi\lambda^2), \\
- & \frac{1}{2}(\gamma^2 + \lambda^2) + 2g + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \frac{1}{2}(\gamma + \lambda\phi - \sigma)^2), \\
&= \psi_{1,t}\exp(-\rho + \phi\phi_0 + (\phi^2 - \phi)\log d_t - \frac{1}{2}(\gamma^2 + \phi\lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda\phi - \sigma)^2).
\end{aligned}
$$

Finally, we note that $\chi_{k+1,t} = \psi_{k,t}$, so that implies that:

$$
\sum_{k=1}^{\infty} \mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = \chi_{1,t}Y_t = D_t,
$$

For some $0 < \phi < 1$, we have that

$$
\begin{aligned}
\mathbb{E}_t[\exp(m_{t,t+1})D_{t+1}] &= \mathbb{E}_t[\exp(m_{t,t+1})Y_{t+1}d_{t+1}], \\
&= d_t^{\phi}\mathbb{E}_t[\exp(m_{t,t+1} - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}], \\
&= d_t^{\phi}\exp(\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)Y_t, \\
&= \exp(\kappa_1)\exp(\phi \log d_t)Y_t,
\end{aligned}
$$

where we used the debt/output dynamics. Define $\kappa_1 = \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2$.

$$
\begin{aligned}
\mathbb{E}_t[\exp(m_{t,t+2})D_{t+2}] &= \mathbb{E}_t[\exp(m_{t,t+1})\mathbb{E}_{t+1}[\exp(m_{t+1,t+2})D_{t+2}]], \\
&= \mathbb{E}_t[\exp(m_{t,t+1})\exp(\kappa_1)\exp(\phi \log d_{t+1})Y_{t+1}], \\
&= \mathbb{E}_t[\exp(m_{t,t+1})\exp(\kappa_1)\exp(\phi^2 \log d_t + \phi\phi_0 - \phi\lambda\varepsilon_{t+1} - \frac{1}{2}\phi\lambda^2)\exp(\mu + \sigma\varepsilon_{t+1})]Y_t,
\end{aligned}
$$

$$= \exp(\kappa_1 + \kappa_2) \exp(\phi^2 \log d_t) Y_t.$$

Define $\kappa_2 = \phi\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \phi\lambda^2) + g + \frac{1}{2}(\gamma + \phi\lambda - \sigma)^2$. Then:

$$\begin{aligned}
\lim_{j\to\infty} \mathbb{E}_t[\exp(m_{t,t+j})D_{t+j}] &= \lim_{j\to\infty} \exp(\sum_{k=1}^{j} \kappa_k) \exp(\phi^j \log d_t) Y_t, \\
&= \lim_{j\to\infty} \exp(\frac{\phi_0}{1-\phi} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi}) + \mu j + \sum_{k=1}^{j} \frac{1}{2}(\gamma + \lambda\phi^{k-1} - \sigma)^2) Y_t, \\
&= \lim_{j\to\infty} \exp(\frac{\phi_0}{1-\phi} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi}) + \mu j + j\frac{1}{2}(\gamma - \sigma)^2 + C) Y_t,
\end{aligned}$$

which is 0 if and only if $-\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) < 0$. This equality does not depend on ϕ and λ. So, this case is similar to the i.i.d. debt case $\phi = 0$. When $\lambda = 0$, $d_t = \exp(\phi_0)$ is a constant. Now, assume $\phi = 1$. Then $\kappa_j = \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2$, and $\lim_{j\to\infty} \mathbb{E}_t[\exp(m_{t,t+j})D_{t+j}] = \lim_{j\to\infty} \exp(\sum_{k=1}^{j} \kappa_k) \exp(\log d_t) Y_t$, which is 0 if and only if $\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2 < 0$.

□

E.7 Proof of Proposition 4.1: Case of $AR(2)$

Proof. From

$$T_t = G_t - (D_t - R^f_{t-1} D_{t-1}).$$

This implies that:

$$\begin{aligned}
S_t &= -\left(d_t Y_t - R^f_{t-1} d_{t-1} Y_{t-1}\right), \\
&= d_{t-1} R^f_{t-1} Y_{t-1} - \exp(\phi_0 + \phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} - \lambda \varepsilon_t - \frac{1}{2}\lambda^2) Y_t.
\end{aligned}$$

Conjecture the price of the surplus strips is given by

$$\mathbb{E}_t\left[M_{t,t+k} S_{t+k}\right] = (\chi_{k,t} - \psi_{k,t}) Y_t.$$

The pricing of the first surplus strip is

$$\begin{aligned}
\mathbb{E}_t\left[\exp(m_{t,t+1}) S_{t+1}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1})\{-Y_{t+1}\left(d_{t+1} - R^f_t d_t \exp[-(\mu + \sigma\varepsilon_{t+1})]\right)\}\right], \\
&= \mathbb{E}_t\left[-\exp(\phi_1 \log d_t + \phi_2 \log d_{t-1} + m_{t,t+1} + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) Y_{t+1}\right] + d_t Y_t, \\
&= -\exp(\phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2) Y_t + d_t Y_t, \\
(\chi_{1,t} - \psi_{1,t}) Y_t &= \left[d_t - \exp(\phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)\right] Y_t.
\end{aligned}$$

We define

$$\begin{aligned}
(\chi_{1,t}) Y_t &= d_t Y_t, \\
(\psi_{1,t}) Y_t &= \exp(\phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2) Y_t.
\end{aligned}$$

Similarly the pricing of the k-th surplus strip is

$$
\begin{aligned}
\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] &= \mathbb{E}_t\left[\exp(m_{t,t+1})\mathbb{E}_{t+1}[\exp(m_{t+1,t+k})S_{t+k}]\right], \\
(\chi_{k,t}-\psi_{k,t})Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})(\chi_{k-1,t+1}-\psi_{k-1,t+1})Y_{t+1}\right],
\end{aligned}
$$

where the χ's are defined by the following recursion:

$$
\begin{aligned}
\chi_{2,t}Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})\chi_{1,t+1}Y_{t+1}\right], \\
\chi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho-\frac{1}{2}\gamma^2-\gamma\varepsilon_{t+1})\exp(-\lambda\varepsilon_{t+1}-\frac{1}{2}\lambda^2)\exp(\mu+\sigma\varepsilon_{t+1})\right]\exp(\phi_1\log d_t+\phi_2\log d_{t-1}+\phi_0), \\
&= \exp(\phi_0+\phi_1\log d_t+\phi_2\log d_{t-1}-\rho-\frac{1}{2}(\gamma^2+\lambda^2)+\mu+\frac{1}{2}(\gamma+\lambda-\sigma)^2).
\end{aligned}
$$

and the ψ's are defined by the following recursion:

$$
\begin{aligned}
\psi_{2,t}Y_t &= \mathbb{E}_t\left[\exp(m_{t,t+1})\psi_{1,t+1}Y_{t+1}\right], \\
\psi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho-\frac{1}{2}\gamma^2-\gamma\varepsilon_{t+1}+\phi_0+\phi_1\log d_{t+1}+\phi_2\log d_t-\rho-\frac{1}{2}(\gamma^2+\lambda^2)+\mu+\frac{1}{2}(\gamma+\lambda-\sigma)^2\ +\mu+\sigma\varepsilon_{t+1})\right], \\
\psi_{2,t} &= \exp(-2\rho+\phi_0+\phi_1\phi_0+(\phi_1^2+\phi_2)\log d_t+\phi_1\phi_2\log d_{t-1}-\frac{1}{2}(\gamma^2+\phi_1\lambda^2), \\
&- \frac{1}{2}(\gamma^2+\lambda^2)+2\mu+\frac{1}{2}(\gamma+\lambda-\sigma)^2+\frac{1}{2}(\gamma+\lambda\phi_1-\sigma)^2).
\end{aligned}
$$

We note that $\chi_{k+1,t}=\psi_{k,t}$, so this expression can be simplified as follows:

$$
\sum_{k=1}^{\infty}\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = \chi_{1,t}Y_t = D_t
$$

$$
d_t = \exp(\phi_1\log d_{t-1}+\phi_2\log d_{t-2}+\phi_0-\lambda\varepsilon_t-\frac{1}{2}\lambda^2).
$$

$$
\begin{aligned}
\mathbb{E}_t[\exp(m_{t,t+1})D_{t+1}] &= \mathbb{E}_t[\exp(m_{t,t+1})Y_{t+1}d_{t+1}], \\
&= d_t^{\phi}\mathbb{E}_t[\exp(m_{t,t+1}-\lambda\varepsilon_{t+1}-\frac{1}{2}\lambda^2)Y_{t+1}], \\
&= d_t^{\phi}\exp(\phi_0-\rho-\frac{1}{2}(\gamma^2+\lambda^2)+\mu+\frac{1}{2}(\gamma+\lambda-\sigma)^2)Y_t, \\
&= \exp(\kappa_1)\exp(\phi_1\log d_t+\phi_2\log d_{t-1})Y_t,
\end{aligned}
$$

Define $\kappa_1=\phi_0-\rho+-\frac{1}{2}(\gamma^2+\lambda^2)+\mu+\frac{1}{2}(\gamma+\lambda-\sigma)^2$.

$$
\begin{aligned}
\mathbb{E}_t[\exp(m_{t,t+2})D_{t+2}] &= \mathbb{E}_t[\exp(m_{t,t+1})\mathbb{E}_{t+1}[\exp(m_{t+1,t+2})D_{t+2}]], \\
&= \mathbb{E}_t[\exp(m_{t,t+1})\exp(\kappa_1)\exp(\phi_1\log d_{t+1}+\phi_2\log d_t)Y_{t+1}], \\
&= \mathbb{E}_t[\exp(m_{t,t+1})\exp(\kappa_1)\exp((\phi_1^2+\phi_1\phi_2+\phi_2)\log d_t+\phi_1\phi_0-\phi_1\lambda\varepsilon_{t+1}-\frac{1}{2}\phi_1\lambda^2)\exp(\mu+\sigma\varepsilon_{t+1})]Y_t, \\
&= \exp(\kappa_1+\kappa_2)\exp((\phi_1^2+\phi_1\phi_2+\phi_2)\exp(\log d_t)Y_t.
\end{aligned}
$$

Define $\kappa_2=\phi_1\phi_0-\rho-\frac{1}{2}(\gamma^2+\phi_1\lambda^2)+\mu+\frac{1}{2}(\gamma+\phi_1\lambda-\sigma)^2$.

$$
\lim_{j\to\infty}\mathbb{E}_t[\exp(m_{t,t+j})D_{t+j}] = \lim_{j\to\infty}\exp(\sum_{k=1}^{j}\kappa_k)\exp(\psi_j\log d_t)Y_t,
$$

$$= \lim_{j\to\infty} \exp(\frac{\phi_0}{1-\phi_1-\phi_2} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi_1-\phi_2}) + \mu j + \sum_{k=1}^{j} \frac{1}{2}(\gamma + \lambda\psi_{k-1} - \sigma)^2)Y_t,$$
$$= \lim_{j\to\infty} \exp(\frac{\phi_0}{1-\phi_1-\phi_2} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi_1-\phi_2}) + \mu j + j\frac{1}{2}(\gamma - \sigma)^2 + C)Y_t,$$

which is 0 if and only if $-\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) < 0$. This equality does not depend on ϕ and λ. So this case is similar to the i.i.d. debt case $\phi = 0$. More extremely, when $\lambda = 0$, $d_t = \exp(\phi_0)$ is a constant. Now, assume $\phi = 1$. Then

$$\kappa_j = \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2,$$

and $\lim_{j\to\infty} \mathbb{E}_t[\exp(m_{t,t+j})D_{t+j}] = \lim_{j\to\infty} \exp(\sum_{k=1}^{j} \kappa_k) \exp(\log d_t)Y_t$, which is 0 if and only if $\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2 < 0$. □

E.8 Proof of Proposition 4.2 : Case of $AR(1)$

Proof. When the log of the debt/output process follows an $AR(1)$, the surplus/output ratio is given by:

$$\frac{S_{t+1}}{Y_{t+1}} = \exp(r_t^f - g - \sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \phi^j \lambda\varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) - \exp(+\phi(-\sum_{j=0}^{\infty} \phi^j \lambda\varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2).$$

We assume that $r_t^f = g$. This expression for the surplus/output ratio can be restated as:

$$\frac{S_{t+1}}{Y_{t+1}} = \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \phi^j \lambda\varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) - \exp(-\sum_{j=0}^{\infty} \phi^j \lambda\varepsilon_{t+1-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

Next, we compute the derivative of the surplus/output ratio at $t+1$:

$$\frac{\partial \frac{S_{t+1}}{Y_{t+1}}}{\partial \varepsilon_{t+1}} = (\lambda) \exp(g + \sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \phi^j \lambda\varepsilon_{t+1-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) - \sigma \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \phi^j \lambda\varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+1}}{Y_{t+1}}}{\partial \varepsilon_{t+1}} = + (\lambda - \sigma) \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

Next, we compute the derivative of the surplus/output ratio at $t+2$. The surplus/output ratio at $t+2$ is given by:

$$\frac{S_{t+2}}{Y_{t+2}} = -\lambda \exp(-\sum_{j=0}^{\infty} \phi^j \lambda\varepsilon_{t+1-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) + \lambda\phi \exp(\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \phi^j \lambda\varepsilon_{t+2-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+2}}{Y_{t+2}}}{\partial \varepsilon_{t+1}} = -\lambda \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) + \lambda\phi \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

This generalizes to the following expression. For $j \geq 2$, we obtain:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = -\lambda\phi^{j-1} \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}) + \lambda\phi^j \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}).$$

Assume $r^f = g$. Then we obtain the IRF:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = \lambda \phi^{j-1}(\phi - 1)\exp(\bar{d}), j > 1,$$
$$\frac{\partial \frac{S_{t+1}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = (\lambda - \sigma)\exp(\bar{d}), j = 1.$$

□

E.9 Proof of Proposition 4.2: Case of $AR(2)$

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. We assume that $r_t^f = g$. When the log of the debt/output process follows an $AR(2)$, the surplus/output ratio is given by:

$$\frac{S_{t+1}}{Y_{t+1}} = \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \rho^j \lambda \varepsilon_{t-j} + \bar{d}) - \exp(+\bar{d} + \phi_1(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j}) + \phi_2(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-1-j}) - \lambda \varepsilon_{t+1} - \frac{1}{2}\lambda^2).$$

Next, we compute the derivative of the surplus/output ratio at $t+1$, and we evaluate this derivative at $\varepsilon_{t+j} = 0$:

$$\frac{\partial \frac{S_{t+1}}{Y_{t+1}}}{\partial \varepsilon_{t+1}} = + \quad (\lambda - \sigma)\exp(\bar{d})).$$

The surplus/output ratio at $t+2$ is given by:

$$\frac{S_{t+2}}{Y_{t+2}} = \exp(-\sigma\varepsilon_{t+2} - \sum_{j=0}^{\infty} \rho^j \lambda \varepsilon_{t+1-j} + \bar{d}) - \exp(+\bar{d} + \phi_1(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+1-j}) + \phi_2(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j}) - \lambda \varepsilon_{t+2} - \frac{1}{2}\lambda^2).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+2}}{Y_{t+2}}}{\partial \varepsilon_{t+1}} = -\lambda \exp(\bar{d}) + \lambda(\phi_1)\exp(\bar{d})).$$

The surplus/output ratio at $t+3$ is given by:

$$\frac{S_{t+3}}{Y_{t+3}} = \exp(-\sigma\varepsilon_{t+3} - \sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+2-j} + \bar{d}) - \exp(\bar{d} + \phi_1(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+2-j}) + \phi_2(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+1-j}) - \lambda \varepsilon_{t+3} - \frac{1}{2}\lambda^2).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+3}}{Y_{t+3}}}{\partial \varepsilon_{t+1}} = -\psi_1 \lambda \exp(\bar{d}) + \lambda(\phi_1 \psi_1 + \phi_2)\exp(\mu + \bar{d}).$$

This generalizes to the following expression. For $j > 2$, we obtain:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = -\lambda \psi_{j-1} \exp(+\bar{d})) + \lambda \psi_j \exp(\bar{d})).$$

Assume $r^f = \mu$. Then we obtain the IRF:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = (\lambda - \sigma)\exp(\bar{d}), \textit{ for } j = 1,$$
$$= \lambda(\phi_1 - 1)\exp(\bar{d}), \textit{ for } j = 2,$$

$$= \quad \lambda(\psi_{j-1} - \psi_{j-2}) \exp(\bar{d}), \textit{ for } j > 2.$$

□

E.10 Proof of Proposition 4.2: Case of AR(3)

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. We assume that the risk-free rate equals the growth rate of the economy. When the log of the debt/output process follows an AR(3), the surplus/output ratio is given by:

$$\begin{aligned}
\frac{S_{t+1}}{Y_{t+1}} &= \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\rho^j\lambda\varepsilon_{t-j} + \bar{d}) \\
&- \exp(+\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-1-j} + \phi_3(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-2-j}) - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2).
\end{aligned}$$

Next, we compute the derivative of the surplus/output ratio at $t+1$, and we evaluate this derivative at $\varepsilon_{t+j} = 0$:

$$\frac{\partial \frac{S_{t+1}}{Y_{t+1}}}{\partial \varepsilon_{t+1}} = + \quad (\lambda - \sigma)\exp(\bar{d})).$$

The surplus/output ratio at $t+2$ is given by:

$$\begin{aligned}
\frac{S_{t+2}}{Y_{t+2}} &= \exp(-\sigma\varepsilon_{t+2} - \sum_{j=0}^{\infty}\rho^j\lambda\varepsilon_{t+1-j} + \bar{d}) \\
&- \exp(\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+1-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-j}) + \phi_3(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-1-j}) - \lambda\varepsilon_{t+2} - \frac{1}{2}\lambda^2).
\end{aligned}$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+2}}{Y_{t+2}}}{\partial \varepsilon_{t+1}} = -\lambda\exp(\bar{d})) + \lambda(\phi_1)\exp(\bar{d})).$$

The surplus/output ratio at $t+3$ is given by:

$$\begin{aligned}
\frac{S_{t+3}}{Y_{t+3}} &= \exp(-\sigma\varepsilon_{t+3} - \sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+2-j} + \bar{d}) \\
&- \exp(\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+2-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+1-j} + \phi_3(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-j}) - \lambda\varepsilon_{t+3} - \frac{1}{2}\lambda^2).
\end{aligned}$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+3}}{Y_{t+3}}}{\partial \varepsilon_{t+1}} = -\psi_1\lambda\exp(\bar{d}) + \lambda(\phi_1\psi_1 + \phi_2)\exp(\mu + \bar{d}).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+4}}{Y_{t+4}}}{\partial \varepsilon_{t+1}} = -\rho_2\lambda\exp(\bar{d}) + \lambda(\phi_1\rho_2 + \phi_2\psi_1 + \phi_3)\exp(\mu + \bar{d}).$$

This generalizes to the following expression. For $j > 2$, we obtain:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = -\lambda \psi_{j-1} \exp(\bar{d}) + \lambda \psi_j \exp(\mu + \bar{d}).$$

Assume $r^f = \mu$. Then we obtain the IRF:

$$\begin{aligned}
\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= (\lambda - \sigma) \exp(\bar{d}), \textit{ for } j = 1, \\
&= \lambda(\phi_1 - 1) \exp(\bar{d}), \textit{ for } j = 2, \\
&= \lambda(\phi_1 \psi_1 + \phi_2 - \psi_1) \exp(\bar{d}), \textit{ for } j = 3, \\
&= \lambda(\psi_{j-1} - \psi_{j-2}) \exp(\bar{d}), \textit{ for } j > 3.
\end{aligned}$$

□

E.11 Proof of Proposition 5.1: Case of *AR(1)*

Proof. As a result, we can solve for an expression of the log debt/output ratio as a function of the past shocks:

$$\log d_t = -\sum_{j=0}^{\infty} \phi^j \lambda \varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1 - \phi}.$$

Consider a government that only issues risk-free debt. Note that the surplus at $t+1$ is given by:

$$S_{t+1} = d_t Y_t \exp(r_t^f) - \exp(\phi \log d_t + \phi_0 - \lambda \varepsilon_{t+1} - \frac{1}{2}\lambda^2) Y_{t+1}.$$

We get the following expression for the covariance:

$$\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1} Y_{t+1}) \\
&= -E_t[M_{t+1} d_{t+1} Y_{t+1}] + E_t[M_{t+1}] E_t[d_{t+1} Y_{t+1}] \\
&= -\exp(-\rho - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \mu + y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho) \exp(\frac{1}{2}(\lambda - \sigma)^2 + \mu + y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 - \frac{1}{2}(\lambda - \sigma)^2) - 1) E_t[M_{t+1}] E_t[d_{t+1} Y_{t+1}] \\
&= -E_t[M_{t+1}] E_t[d_{t+1} Y_{t+1}](\exp(-\gamma(\sigma - \lambda)) - 1).
\end{aligned}$$

By the same token, we get the following expression for the covariance of the discounted surpluses over two periods:

$$\begin{aligned}
& cov_t(M_{t+1}, S_{t+1} + E_{t+1}[M_{t+1,t+2} S_{t+2})] \\
= \; & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+2} d_{t+2} Y_{t+2}]) \\
= \; & -E_t[M_{t+1}] E_{t+1}[M_{t+1,t+2} d_{t+2} Y_{t+2}](\exp(-\gamma(\sigma - \phi\lambda)) - 1)
\end{aligned}$$

Check the proof of Prop. 2.3 to see why the sum of the discounted surpluses drop out, and only the debt issuance term remains. We get the following expression for the covariance of the discounted surpluses over j periods:

$$\begin{aligned}
& cov_t(M_{t+1}, \sum_{k=1}^{j} E_{t+1}[M_{t+1,t+j} S_{t+j}]) \\
= \; & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j} d_{t+j} Y_{t+j}])
\end{aligned}$$

$$= \quad -E_t[M_{t+1}]E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda))-1).$$

□

E.12 Proof of Proposition 5.1: case of *AR(2)*

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. As a result, we can solve for an expression of the log debt/output ratio as a function of the past shocks:

$$\log d_t = -\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1-\phi_2}.$$

where $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}$. Consider a government that only issues risk-free debt. Note that the surplus at $t+1$ is given by:

$$S_{t+1} \quad = \quad d_t Y_t \exp(r_t^f) - \exp(+\phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}.$$

As a result, we get the following expression for the covariance:

$$\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}) \\
&= -E_t[M_{t+1}d_{t+1}Y_{t+1}] + E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -\exp(-\rho - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma+\lambda-\sigma)^2 + \mu + y_t + \phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho)\exp(\frac{1}{2}(\lambda-\sigma)^2 + \mu + y_t + \phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma+\lambda-\sigma)^2 - \frac{1}{2}(\lambda-\sigma)^2) - 1)E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma-\lambda)) - 1).
\end{aligned}$$

By the same token, we get the following expression for the covariance of the discounted surpluses over two periods:

$$\begin{aligned}
& cov_t(M_{t+1}, S_{t+1} + E_{t+1}[M_{t+1,t+2}S_{t+2})] \\
=\ & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}]) \\
=\ & -E_t[M_{t+1}]E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}](\exp(-\gamma(\sigma-\psi_1\lambda)) - 1)
\end{aligned}$$

Check the proof of Prop. 2.3 to see why the sum of the discounted surpluses drop out, and only the debt issuance term remains. And we get the following expression for the covariance of the discounted surpluses over j periods:

$$\begin{aligned}
& cov_t(M_{t+1}, \sum_{k=1}^{j} E_{t+1}[M_{t+1,t+j}S_{t+j}]) \\
=\ & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]) \\
=\ & -E_t[M_{t+1}]E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\psi_{j-1}\lambda)) - 1).
\end{aligned}$$

□

E.13 Proof of Corollary ??: Case of *AR(1)*

Proof. Start from the restriction:

$$cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right)$$

$$
\begin{aligned}
&= -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda))-1) \\
&+ xcov_t\left(M_{t+1},(\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j}M_{t+1,t+k}Y_{t+k}\right) \\
&= -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda))-1) \\
&+ x\sum_{k=1}^{j}E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}](\exp(-\gamma\sigma)-1)
\end{aligned}
$$

We substitute for the price of debt strips:

$$
\begin{aligned}
&\mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] \\
=\ &\exp(\sum_{k=1}^{j}\kappa_k)\exp(\phi^j\log d_t)Y_t \\
=\ &\exp(\frac{\phi_0(1-\phi^j)}{1-\phi}-\rho j-\frac{1}{2}(\gamma^2 j+\frac{\lambda^2(1-\phi^j)}{1-\phi})+\mu j+\sum_{k=1}^{j}\frac{1}{2}(\gamma+\lambda\phi^{k-1}-\sigma)^2)\exp(\phi^j\log d_t)Y_t
\end{aligned}
$$

For $j>1$, we obtain the following expression:

$$
\begin{aligned}
&\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
=\ &\exp(\frac{\phi_0(1-\phi^{j-1})}{1-\phi}-\rho(j-1)-\frac{1}{2}(\gamma^2(j-1)+\frac{\lambda^2(1-\phi^{j-1})}{1-\phi})+\mu(j-1)+\sum_{k=1}^{j-1}\frac{1}{2}(\gamma+\lambda\phi^{k-1}-\sigma)^2) \\
&\exp(\phi^{j-1}\log d_{t+1})Y_{t+1},
\end{aligned}
$$

and, for $j=1$, we get that:

$$
\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]=\exp(\frac{\phi_0}{1-\phi})\exp(\log d_{t+1})Y_{t+1}.
$$

For $j>1$, this simplifies to the following expression:

$$
\begin{aligned}
&\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
=\ &\exp(\frac{1-\phi^j}{1-\phi}(\phi_0-\frac{1}{2}\lambda^2)-\rho(j-1)-\frac{1}{2}\gamma^2(j-1)+\mu j+\sum_{k=1}^{j-1}\frac{1}{2}(\gamma+\lambda\phi^{k-1}-\sigma)^2) \\
&\exp(\phi^j\log d_t+\frac{1}{2}(-\phi^{j-1}\lambda+\sigma)^2)Y_t.
\end{aligned}
$$

Note that by a similar logic, the price of the output strips is given by:

$$
\begin{aligned}
&\mathbb{E}_t[M_{t+1,t+j}Y_{t+j}] \\
=\ &\exp(-\rho(j-1)-\frac{1}{2}\gamma^2(j-1)+\mu j+(j-1)\frac{1}{2}(\gamma-\sigma)^2+\frac{1}{2}(\sigma)^2)Y_t
\end{aligned}
$$

To summarize, for $j>1$, this implies that we have the following expression:

$$
\begin{aligned}
&\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
=\ &\mathbb{E}_t[M_{t+1,t+j}Y_{t+j}]\exp(\frac{1-\phi^j}{1-\phi}(\phi_0-\frac{1}{2}\lambda^2)+\sum_{k=1}^{j-1}\frac{1}{2}((\lambda\phi^{k-1})^2+2(\gamma-\sigma)\lambda\phi^{k-1})) \\
&\exp(\phi^j\log d_t+\frac{1}{2}((\phi^{j-1}\lambda)^2-2\sigma\phi^{j-1}\lambda)).
\end{aligned}
$$

and for $j = 1$, we get that:

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] = \exp(\frac{\phi_0}{1-\phi})\exp(\phi\log d_t)\exp(\mu+\frac{1}{2}\sigma^2)Y_t.$$

□

E.14 Proof of Corollary 5.2: Case of *AR(2)*

Proof. We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. Start from the restriction:

$$\begin{aligned}
& cov_t\left(M_{t+1},(\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j}M_{t+1,t+k}T_{t+k}\right) \\
= & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\psi_{j-1}\lambda))-1) \\
+ & \; x cov_t\left(M_{t+1},(\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j}M_{t+1,t+k}Y_{t+k}\right) \\
= & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\psi_{j-1}\lambda))-1) \\
+ & \; x\sum_{k=1}^{j}E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}](\exp(-\gamma\sigma)-1)
\end{aligned}$$

We substitute for the price of debt strips:

$$\begin{aligned}
& \mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] \\
= & \exp(\sum_{k=1}^{j}\kappa_k)\exp(\psi_j\log d_t)Y_t \\
= & \exp(\sum_{k=1}^{j}\psi_{k-1}\phi_0-\rho j-\frac{1}{2}(\gamma^2 j+\sum_{k=1}^{j}\psi_{k-1}\lambda^2)+gj+\sum_{k=1}^{j}\frac{1}{2}(\gamma+\lambda\psi_{k-1}-\sigma)^2)\exp(\psi_j\log d_t)Y_t
\end{aligned}$$

For $j > 1$, we obtain the following expression:

$$\begin{aligned}
& \mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= & \exp(\sum_{k=1}^{j-1}\psi_{k-1}\phi_0-\rho(j-1)-\frac{1}{2}(\gamma^2(j-1)+\sum_{k=1}^{j-1}\psi_{k-1}\lambda^2)+\mu(j-1)+\sum_{k=1}^{j-1}\frac{1}{2}(\gamma+\lambda\psi_{k-1}-\sigma)^2) \\
& \exp(\psi_{j-1}\log d_{t+1})Y_{t+1},
\end{aligned}$$

and, for $j = 1$, we get that:

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] = \exp(\frac{\phi_0}{1-\phi_1-\phi_2})\exp(\log d_{t+1})Y_{t+1}.$$

For $j > 1$, this simplifies to the following expression:

$$\begin{aligned}
& \mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= & \exp(\sum_{k=1}^{j}\psi_{k-1}(\phi_0-\frac{1}{2}\lambda^2)-\rho(j-1)-\frac{1}{2}\gamma^2(j-1)+\mu j+\sum_{k=1}^{j-1}\frac{1}{2}(\gamma+\lambda\psi_{k-1}-\sigma)^2) \\
& \exp(\rho_j\log d_t+\frac{1}{2}(-\psi_{j-1}\lambda+\sigma)^2)Y_t.
\end{aligned}$$

Note that by a similar logic, the price of the output strips is given by:

$$\mathbb{E}_t[M_{t+1,t+j}Y_{t+j}]$$

$$= \exp(-\rho(j-1) - \frac{1}{2}\gamma^2(j-1) + \mu j + (j-1)\frac{1}{2}(\gamma-\sigma)^2 + \frac{1}{2}(\sigma)^2)Y_t.$$

To summarize, for $j > 1$, this implies that we have the following expression:

$$\begin{aligned}
& \mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \; & \mathbb{E}_t[M_{t+1,t+j}Y_{t+j}] \exp(\sum_{k=1}^{j} \psi_{k-1}(\phi_0 - \frac{1}{2}\lambda^2) + \sum_{k=1}^{j-1} \frac{1}{2}((\lambda\psi_{k-1})^2 + 2(\gamma-\sigma)\lambda\psi_{k-1})) \\
& \exp(\psi_j \log d_t + \frac{1}{2}((\psi_{j-1}\lambda)^2 - 2\sigma\psi_{j-1}\lambda)),
\end{aligned}$$

and for $j = 1$, we get that:

$$\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] = \exp((\frac{\phi_0}{1-\phi_1-\phi_2}) \exp(\phi_1 \log d_t + \phi_2 \log d_{t-1}) \exp(g + \frac{1}{2}\sigma^2)Y_t.$$

□

E.15 Proof of Proposition A.2

Proof. Notice

$$\begin{aligned}
dT_t &= d\exp(y_t)\exp(\tau_t) + \exp(y_t)d\exp(\tau_t) + [d\exp(y_t), d\exp(\tau_t)]dt \\
&= T_t((\mu dt + \frac{1}{2}\gamma^2 dt + \gamma dZ_t) + (\theta(\bar{\tau}-\tau_t)dt + \frac{1}{2}(\beta_\tau\gamma)^2 dt + \beta_\tau\gamma dZ_t) + \beta_\tau\gamma^2 dt) \\
&= T_t((\mu + \theta(\bar{\tau}-\tau_t) + \frac{1}{2}(1+\beta_\tau)^2\gamma^2)dt + (1+\beta_\tau)\gamma dZ_t)
\end{aligned}$$

Conjecture

$$\begin{aligned}
P_t^\tau &= f_\tau(\tau_t)T_t \\
P_t^g &= f_g(g_t)G_t
\end{aligned}$$

then

$$\begin{aligned}
dP_t^\tau &= df_\tau T_t + f_\tau dT_t + [df_\tau, dT_t]dt \\
&= T_t(f_\tau' d\tau_t + \frac{1}{2}f_\tau''\beta_\tau^2\gamma^2 dt) + f_\tau'\beta_\tau\gamma T_t(1+\beta_\tau)\gamma dt \\
&+ f_\tau T_t((\mu + \theta(\bar{\tau}-\tau_t) + \frac{1}{2}(1+\beta_\tau)^2\gamma^2)dt + (1+\beta_\tau)\gamma dZ_t) \\
&= T_t\left(f_\tau'\theta(\bar{\tau}-\tau_t) + \frac{1}{2}f_\tau''\beta_\tau^2\gamma^2 + f_\tau'\beta_\tau(1+\beta_\tau)\gamma^2 + f_\tau(\mu + \theta(\bar{\tau}-\tau_t) + \frac{1}{2}(1+\beta_\tau)^2\gamma^2)\right)dt \\
&+ T_t\left(f_\tau(1+\beta_\tau) + f_\tau'\beta_\tau\right)\gamma dZ_t
\end{aligned}$$

Substitute into the Euler equation,

$$\begin{aligned}
0 &= \mathcal{A}[M_t T_t dt + dM_t P_t^\tau + M_t dP_t^\tau + [dM_t, dP_t^\tau]dt] \\
-1 &= -rf_\tau + f_\tau'\theta(\bar{\tau}-\tau_t) + \frac{1}{2}f_\tau''\beta_\tau^2\gamma^2 + f_\tau'\beta_\tau(1+\beta_\tau)\gamma^2 + f_\tau(\mu + \theta(\bar{\tau}-\tau_t) + \frac{1}{2}(1+\beta_\tau)^2\gamma^2) \\
&- \gamma f_\tau(1+\beta_\tau)\gamma - \gamma f_\tau'\beta_\tau\gamma
\end{aligned}$$

We take a continuous time version of Campbell-Shiller approximation (Eraker Shaliastovich (2008)):

$$\begin{aligned} dr_t^\tau &= \log \frac{P_{t+dt}^\tau + T_{t+dt}}{P_t^\tau} \\ &\approx \kappa_0^\tau dt + \kappa_1^\tau d\log f_\tau - (1-\kappa_1^\tau)\log f_\tau dt + d\log T_t \end{aligned}$$

The Euler equation is

$$\begin{aligned} 0 &= \mathcal{A}[d\exp(m_t + \int_0^t dr_k^\tau)] \\ &= \mathcal{A}[dm_t + dr_t^\tau + \frac{1}{2}[dm_t + dr_t^\tau, dm_t + dr_t^\tau]dt] \end{aligned}$$

Then, we conjecture $f_\tau(\tau_t) = \exp(p_\tau + q_\tau \tau_t)$,

$$\begin{aligned} 0 &= \mathcal{A}[dm_t + dr_t^\tau + \frac{1}{2}[dm_t + dr_t^\tau, dm_t + dr_t^\tau]dt] \\ &= \mathcal{A}[-(r+\frac{1}{2}\gamma^2)dt - \gamma dZ_t + \kappa_0^\tau dt + \kappa_1^\tau d\log f_\tau - (1-\kappa_1^\tau)\log f_\tau dt + d\log T_t \\ &+ \frac{1}{2}[-\gamma dZ_t + dr_t^\tau, -\gamma dZ_t + dr_t^\tau]dt] \\ &= -(r+\frac{1}{2}\gamma^2) + \kappa_0^\tau + \kappa_1^\tau q_\tau \theta(\bar{\tau}-\tau_t) - (1-\kappa_1^\tau)(p_\tau + q_\tau\tau_t) + \mu + \theta(\bar{\tau}-\tau_t) \\ &+ \frac{1}{2}((1+\beta_\tau + \kappa_1^\tau q_\tau \beta_\tau)\gamma - \gamma)^2 \end{aligned}$$

which implies

$$\begin{aligned} r &= \mu - \frac{1}{2}\gamma^2 + \kappa_0^\tau + \kappa_1^\tau q_\tau \theta\bar{\tau} - (1-\kappa_1^\tau)p_\tau + \theta\bar{\tau} + \frac{1}{2}((1+\beta_\tau+\kappa_1^\tau q_\tau\beta_\tau)\gamma - \gamma)^2 \\ q_\tau &= -\frac{\theta}{\kappa_1^\tau\theta + (1-\kappa_1^\tau)} \end{aligned}$$

Since κ_1^τ is a constant close to but lower than 1, and $0 < \theta < 1$, $-1 < q_\tau < 0$. To see this, note

$$q_\tau - (-1) = \frac{(1-\kappa_1^\tau)(1-\theta)}{\kappa_1^\tau\theta + (1-\kappa_1^\tau)} > 0.$$

Similarly, $f_g(g_t) = \exp(p_g + q_g g_t)$, where

$$\begin{aligned} r &= \mu - \frac{1}{2}\gamma^2 + \kappa_0^g + \kappa_1^g q_g \theta\bar{g} - (1-\kappa_1^g)p_g + \theta\bar{g} + \frac{1}{2}((1+\beta_g+\kappa_1^g q_g\beta_g)\gamma - \gamma)^2 \\ q_g &= -\frac{\theta}{\kappa_1^g\theta + (1-\kappa_1^g)} \end{aligned}$$

Then,

$$\begin{aligned} dB_t &= dP_t^\tau - dP_t^g \\ &= T_t\left(f_\tau'\theta(\bar{\tau}-\tau_t) + \frac{1}{2}f_\tau''\beta_\tau^2\gamma^2 + f_\tau'\beta_\tau(1+\beta_\tau)\gamma^2 + f_\tau(\mu + \theta(\bar{\tau}-\tau_t) + \frac{1}{2}(1+\beta_\tau)^2\gamma^2)\right)dt \\ &+ T_t\left(f_\tau(1+\beta_\tau) + f_\tau'\beta_\tau\right)\gamma dZ_t \\ &- G_t\left(f_g'\theta(\bar{g}-g_t) + \frac{1}{2}f_g''\beta_g^2\gamma^2 + f_g'\beta_g(1+\beta_g)\gamma^2 + f_g(\mu + \theta(\bar{g}-g_t) + \frac{1}{2}(1+\beta_g)^2\gamma^2)\right)dt \\ &- G_t\left(f_g(1+\beta_g) + f_g'\beta_g\right)\gamma dZ_t \end{aligned}$$

The risk exposure of the debt return is

$$[r_t^B, dM_t] = -\frac{M_t\gamma\gamma}{B_t}\left(T_t\left(f_\tau(1+\beta_\tau)+f_\tau'\beta_\tau\right)-G_t\left(f_g(1+\beta_g)+f_g'\beta_g\right)\right)$$
$$= -\frac{M_t\gamma\gamma}{B_t}\left(T_t f_\tau\left(1+(1+q_\tau)\beta_\tau\right)-G_t f_g\left(1+(1+q_g)\beta_g\right)\right)$$

□

E.16 Proof of Proposition A.3

In this case,

$$\begin{aligned} dB_t &= dP_t^\tau + dP_t^\kappa - dP_t^g \\ &= (...)dt \\ &+ (f_\tau T_t(1+(1+q_\tau)\beta_\tau)+f_\kappa K_t(1+(1+q_\kappa)\beta_\kappa)-f_g G_t(1+(1+q_g)\beta_g))\gamma dZ_t \end{aligned}$$

and the return on the government debt is

$$r_t^B = \frac{(T_t+K_t-G_t)dt+dB_t}{B_t}$$

The risk exposure of the debt return is

$$[r_t^B, dM_t] = -\frac{M_t\gamma\gamma}{B_t}\left(f_\tau T_t(1+(1+q_\tau)\beta_\tau)+f_\kappa K_t(1+(1+q_\kappa)\beta_\kappa)-f_g G_t(1+(1+q_g)\beta_g)\right)$$

E.17 Proof of Proposition A.1

Proof. Iterate the debt valuation equation,

$$\lim_{u\to\infty}\mathbb{E}_0 M_u B_u = M_0 B_0 + \lim_{u\to\infty}\mathbb{E}_0[\int_0^u d(M_t B_t)] \tag{19}$$

If the following TVC,

$$\lim_{u\to\infty}\mathbb{E}_0 M_u B_u = 0 \tag{20}$$

is satisfied, then

$$M_0 B_0 = -\lim_{u\to\infty}\mathbb{E}_0[\int_0^u d(M_t B_t)] = \mathbb{E}_0[\int_0^\infty M_t(T_t-G_t)dt] \tag{21}$$

or

$$B_t = P_t^\tau - P_t^g \tag{22}$$

□

E.18 Proof of Corollary B.1

Proof. From $R_{t+1}^f = \rho\exp(\rho)$ and $\frac{T_t}{Y_t} = x - d\left(1-R_{t-1}^f\frac{Y_{t-1}}{Y_t}\right)$, we have that the return on the tax claim can be stated as:

$$R_{t+1}^T = \frac{P_{t+1}^T}{P_t^T - T_t} = \frac{(d+x\frac{\xi_1}{1-\xi_1})Y_{t+1}+(x-d\left(1-R_t^f\frac{Y_t}{Y_{t+1}}\right))Y_{t+1}}{(d+x\frac{\xi_1}{1-\xi_1})Y_t}$$

$$= \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{(d+x\frac{\xi_1}{1-\xi_1})Y_t} + \frac{d\exp(\rho)}{(d+x\frac{\xi_1}{1-\xi_1})}.$$

Similarly, we have an expression for the return on the spending claim:

$$R^G_{t+1} = \frac{P^G_{t+1}}{P^G_t - G_t} = \frac{x\frac{\xi_1}{1-\xi_1}Y_{t+1} + xY_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t} = \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t}.$$

As a result, we can state the risk premium as follows:

$$\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] = -\frac{cov\left(M_{t+1}, R^T_{t+1}\right)}{E_t(M_{t+1})} = \frac{x}{d(1-\xi_1)+x\xi_1}\frac{-cov\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})},$$
$$\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] = -\frac{cov\left(M_{t+1}, R^G_{t+1}\right)}{E_t(M_{t+1})} = \frac{1}{\xi_1}\frac{-cov\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})},$$

where we have used that $\xi_1 = \exp(-\rho - \frac{1}{2}\gamma^2 + g + \frac{1}{2}(\gamma-\sigma)^2) = \exp(-\rho - \gamma\sigma + g + \frac{1}{2}\sigma^2)$.

Then plug in

$$\begin{aligned}
\frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{E_t(M_{t+1})} &= \frac{-cov_t\left(\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}), \exp(g+\sigma\varepsilon_{t+1})\right)}{E_t(\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}))} \\
&= \frac{-cov_t\left(\exp(-\gamma\varepsilon_{t+1}), \exp(\sigma\varepsilon_{t+1})\right)}{\exp(-\rho)}\exp(-\rho - \frac{1}{2}\gamma^2 + g) \\
&= -(\exp(\frac{1}{2}(\gamma^2+\sigma^2))(\exp(-\gamma\sigma)-1))\exp(-\frac{1}{2}\gamma^2+g) \\
&= \exp(g + \frac{1}{2}\sigma^2)(1-\exp(-\gamma\sigma))
\end{aligned}$$

□

E.19 Proof of Proposition C.3

Proof. Since

$$S_{t+1} = d_t Y_t \exp(r^f_t) - d_{t+1}Y_{t+1},$$

we get the following expression for the covariance:

$$\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}) \\
&= -E_t[M_{t+1}d_{t+1}Y_{t+1}] + E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -\exp(-\rho - \frac{\gamma}{\sigma}(\psi-1)y_t - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma+\lambda-\sigma)^2 + \xi_0 + \xi y_t + \phi\log d_t \\
&+ \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho - \frac{\gamma}{\sigma}(\xi-1)y_t)\exp(\frac{1}{2}(\lambda-\sigma)^2 + \xi_0 + \xi y_t + \phi\log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma+\lambda-\sigma)^2 - \frac{1}{2}(\lambda-\sigma)^2) - 1)E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}] \\
&= -E_t[M_{t+1}]E_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma-\lambda)) - 1).
\end{aligned}$$

By the same token, we get the following expression for the covariance of the discounted surpluses over $j \geq 2$

periods:

$$
\begin{aligned}
& cov_t(M_{t+1}, E_{t+1}[\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}]) \\
= \; & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]) \\
= \; & -E_t[M_{t+1}M_{t+1,t+j}d_{t+j}Y_{t+j}] + E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \; & -E_t[\exp(-\rho - \frac{\gamma}{\sigma}(\xi-1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(\ldots - \frac{\gamma(\xi-1)}{\sigma}(1+\xi+\ldots+\xi^{j-2})y_{t+1}) \\
& \exp(\phi^j \log d_t - \phi^{j-1}\lambda\varepsilon_{t+1} + \ldots)\exp(\xi^j y_t + \xi^{j-1}\sigma\varepsilon_{t+1} + \ldots)] + E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda - \frac{\gamma(\xi-1)}{\sigma}\frac{1-\xi^{j-1}}{1-\xi})) - 1) \\
= \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1-\xi^{j-1}))) - 1)
\end{aligned}
$$

□

E.20 Proof of Corollary C.4

Proof. Start from the restriction:

$$
\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\
= \; & -E_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \frac{\gamma}{\sigma}(1-\xi^{j-1}))) - 1) \\
+ \; & x cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}Y_{t+k}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)M_{t+1,t+k}Y_{t+k}\right) \\
= \; & E_t[M_{t+1}M_{t+1,t+k}Y_{t+k}] - E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}] \\
= \; & E_t[\exp(-\rho - \frac{\gamma}{\sigma}(\xi-1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})M_{t+1,t+k}\exp(\xi^k y_t + \xi^{k-1}\sigma\varepsilon_{t+1} + \ldots)] \\
- \; & E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}] \\
= \; & -E_t[M_{t+1}]E_t[M_{t+1,t+k}Y_{t+k}](\exp(-\gamma(\xi^{k-1}\sigma + \frac{\gamma}{\sigma}(1-\xi^{k-1}))) - 1).
\end{aligned}
$$

Next, we conjecture

$$
\mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] \quad = \quad \exp(\sum_{k=1}^{j}\tilde{\kappa}_k)\exp(\phi^j \log d_t + f_j y_t)
$$

Note

$$
\begin{aligned}
\mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] \quad = \quad & \mathbb{E}_t[M_{t,t+1}\exp(\sum_{k=1}^{j-1}\kappa_k)\exp(\phi^{j-1}\log d_{t+1} + f_{j-1}y_{t+1})] \\
= \quad & \mathbb{E}_t[\exp(-\rho - \frac{\gamma}{\sigma}(\xi-1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(\sum_{k=1}^{j-1}\tilde{\kappa}_k) \\
& \exp(\phi^{j-1}(\phi\log d_t + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) + f_{j-1}(\xi_0 + \xi y_t + \sigma\varepsilon_{t+1}))]
\end{aligned}
$$

So we confirm the conjecture,

$$\begin{aligned}\exp(\tilde{\kappa}_j) &= \mathbb{E}_t[\exp(-\rho - \frac{\gamma}{\sigma}(\xi-1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1} + \phi^{j-1}(\phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) + f_{j-1}(\xi_0 + \sigma\varepsilon_{t+1}))] \\ \tilde{\kappa}_j &= -\rho - \frac{1}{2}\gamma^2 + \phi^{j-1}(\phi_0 - \frac{1}{2}\lambda^2) + f_{j-1}\xi_0 + \frac{1}{2}(-\gamma - \phi^{j-1}\lambda + f_{j-1}\sigma)^2\end{aligned}$$

and

$$\begin{aligned}f_j &= -\frac{\gamma}{\sigma}(\xi-1) + f_{j-1}\xi \\ &= \xi^j + \frac{\gamma}{\sigma}(1-\xi^j) = \frac{\sigma-\gamma}{\sigma}\xi^j + \frac{\gamma}{\sigma}\end{aligned}$$

So, for $j > 1$,

$$\begin{aligned}& \mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\ = \; & \mathbb{E}_t[\exp(\sum_{k=1}^{j-1}\tilde{\kappa}_k)\exp(\phi^{j-1}\log d_{t+1} + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma})y_{t+1})] \\ = \; & \exp((-\rho - \frac{1}{2}\gamma^2)(j-1) + \frac{1-\phi^{j-1}}{1-\phi}(\phi_0 - \frac{1}{2}\lambda^2) + \left(\frac{1-\xi^{j-1}}{1-\xi}\frac{\sigma-\gamma}{\sigma} + \frac{\gamma}{\sigma}(j-1)\right)\xi_0 \\ + \; & \sum_{k=1}^{j-1}\frac{1}{2}(-\gamma - \phi^{k-1}\lambda + ((\sigma-\gamma)\xi^{k-1} + \gamma))^2 \\ + \; & \phi^{j-1}(\phi\log d_t + \phi_0 - \frac{1}{2}\lambda^2) + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma})(\xi_0 + \xi y_t) + \frac{1}{2}(-\phi^{j-1}\lambda + ((\sigma-\gamma)\xi^{j-1} + \gamma))^2)\end{aligned}$$

By a similar logic,

$$\begin{aligned}& \mathbb{E}_{t+1}[M_{t+1,t+j}Y_{t+j}] \\ = \; & \exp((-\rho - \frac{1}{2}\gamma^2)(j-1) + \left(\frac{1-\xi^{j-1}}{1-\xi}\frac{\sigma-\gamma}{\sigma} + \frac{\gamma}{\sigma}(j-1)\right)\xi_0 \\ + \; & \sum_{k=1}^{j-1}\frac{1}{2}(-\gamma + ((\sigma-\gamma)\xi^{k-1} + \gamma))^2 + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma})(\xi_0 + \xi y_t) + \frac{1}{2}(((\sigma-\gamma)\xi^{j-1} + \gamma))^2)\end{aligned}$$

So

$$\begin{aligned}& \mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\ = \; & \mathbb{E}_{t+1}[M_{t+1,t+j}Y_{t+j}]\exp(\frac{1-\phi^j}{1-\phi}(\phi_0 - \frac{1}{2}\lambda^2) + \sum_{k=1}^{j-1}((\gamma-\sigma)\xi^{k-1}\phi^{k-1}\lambda + \frac{1}{2}(\phi^{k-1}\lambda)^2) \\ + \; & \phi^j\log d_t - \phi^{j-1}\lambda((\sigma-\gamma)\xi^{j-1} + \gamma) + \frac{1}{2}(\phi^{j-1}\lambda)^2)\end{aligned}$$

□

E.21 Proof of Corollary D.1

Proof. We plug in the expressions for the respective surpluses:

$$\begin{aligned}\frac{S_{t+1}}{Y_{t+1}} &= d_t\exp(r_t^f - \mu - \sigma\varepsilon_{t+1}) - d_{t+1}, \\ \frac{S_{t+j}}{Y_{t+j}} &= d_{t+j-1}\exp(r_{t+j-1}^f - \mu - \sigma\varepsilon_{t+j}) - d_{t+j},\end{aligned}$$

into the expression for the conditional covariances:

$$\begin{aligned}
cov_t(s_{t+1}, s_{t+j}) &= \mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1}) d_{t+j-1} \exp(r_{t+j-1}^f - \mu - \sigma\varepsilon_{t+j})] \\
&- \mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_{t+j-1} \exp(r_{t+j-1}^f - \mu - \sigma\varepsilon_{t+j})] \\
&+ \mathbb{E}_t[-d_{t+1} d_{t+j-1} \exp(r_t^f - \mu - \sigma\varepsilon_{t+j})] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[d_{t+j-1} \exp(r_t^f - \mu - \sigma\varepsilon_{t+j})] \\
&+ \mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1}) \times -d_{t+j}] - \mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[-d_{t+j}] \\
&+ \mathbb{E}_t[-d_{t+1} \times -d_{t+j}] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[-d_{t+j}].
\end{aligned}$$

This expression can be restated as:

$$\begin{aligned}
cov_t(s_{t+1}, s_{t+j}) &= d_t \mathbb{E}_t[\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_{t+j-1} \exp(r_{t+j-1}^f - \mu - \sigma\varepsilon_{t+j})] \\
&\times (\exp(\sigma\lambda\phi^{j-2}) - 1) \\
&- \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j-1} \exp(r_t^f - \mu - \sigma\varepsilon_{t+j})](\exp(\lambda^2\phi^{j-2}) - 1) \\
&- d_t \mathbb{E}_t[\exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_{t+j}](\exp(\sigma\lambda\phi^{j-1}) - 1) \\
&+ \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j}](\exp(\lambda^2\phi^{j-1}) - 1)
\end{aligned}$$

which implies

$$\begin{aligned}
cov_t(s_{t+1}, s_{t+j}) &= \exp(2\rho - 2\mu + \sigma^2) d_t \mathbb{E}_t[d_{t+j-1}](\exp(\sigma\lambda\phi^{j-2}) - 1) \\
&- \exp(\rho - \mu + .5\sigma^2)\mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j-1}](\exp(\lambda^2\phi^{j-2}) - 1) \\
&- \exp(\rho - \mu + .5\sigma^2) d_t \mathbb{E}_t[d_{t+j}](\exp(\sigma\lambda\phi^{j-1}) - 1) \\
&+ \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j}](\exp(\lambda^2\phi^{j-1}) - 1)
\end{aligned}$$

We have the following expressions for the conditional forecasts:

$$\mathbb{E}_t[\log d_{t+j}] = \exp\left(\phi^j(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi}\right)$$

and

$$\begin{aligned}
\mathbb{E}_t[d_{t+j}] &= \exp\left(\phi^j(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2(1 + \phi^2 + \ldots + \phi^{2(j-1)})\right) \\
&= \exp\left(\phi^j(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2j}}{1-\phi^2}\right)
\end{aligned}$$

We plug these conditional forecasts into the conditional covariances: For $j > 1$,

$$\begin{aligned}
& cov_t(s_{t+1}, s_{t+j}) \\
= \;& \exp(2\rho - 2\mu + \sigma^2)\exp\left((1+\phi^{j-1})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2(j-1)}}{1-\phi^2}\right)(\exp(\sigma\lambda\phi^{j-2}) - 1) \\
- \;& \exp(\rho - \mu + .5\sigma^2)\exp\left((\phi+\phi^{j-1})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2(j-1)}}{1-\phi^2} + \frac{1}{2}\lambda^2\right) \\
\times \;& (\exp(\lambda^2\phi^{j-2}) - 1) \\
- \;& \exp(\rho - \mu + .5\sigma^2)\exp\left((1+\phi^{j})(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2j}}{1-\phi^2}\right)(\exp(\sigma\lambda\phi^{j-1}) - 1)
\end{aligned}$$

$$+ \quad \exp\left((\phi+\phi^j)(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 \frac{1-\phi^{2j}}{1-\phi^2} + \frac{1}{2}\lambda^2 \right) (\exp(\lambda^2\phi^{j-1}) - 1)$$

Also, when $j = 1$,

$$\begin{aligned}
var_t(s_{t+1}) &= \mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1}) d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1})] \\
&- \mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1})] \\
&+ \mathbb{E}_t[-d_{t+1} d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1})] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1})] \\
&+ \mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1}) \times -d_{t+1}] - \mathbb{E}_t[d_t \exp(r_t^f - \mu - \sigma\varepsilon_{t+1})]\mathbb{E}_t[-d_{t+1}] \\
&+ \mathbb{E}_t[-d_{t+1} \times -d_{t+1}] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[-d_{t+1}] \\
&= \exp(2\rho - 2\mu + \sigma^2)\exp(2\log d_t)(\exp(\sigma^2) - 1) \\
&- 2\exp(\rho - \mu + .5\sigma^2) d_t \mathbb{E}_t[d_{t+1}](\exp(\lambda\sigma) - 1) \\
&+ \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+1}](\exp(\lambda^2) - 1)
\end{aligned}$$

which implies

$$\begin{aligned}
var_t(s_{t+1}) &= \exp(2\rho - 2\mu + \sigma^2)\exp(2\log d_t)(\exp(\sigma^2) - 1) \\
&- 2\exp(\rho - \mu + .5\sigma^2)\exp\left((1+\phi)(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 \right)(\exp(\lambda\sigma) - 1) \\
&+ \exp\left(2\phi(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \lambda^2 \right)(\exp(\lambda^2) - 1)
\end{aligned}$$

□

E.22 Proof of Corollary D.2

Proof. We plug in the expressions for the respective surpluses:

$$\begin{aligned}
\frac{S_{t+1}}{Y_{t+1}} &= d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1}) - d_{t+1}, \\
\frac{S_{t+j}}{Y_{t+j}} &= d_{t+j-1} \exp(r_{t+j-1}^f - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j}) - d_{t+j},
\end{aligned}$$

into the expression for the conditional covariances:

$$\begin{aligned}
& cov_t(s_{t+1}, s_{t+j}) \\
= \;& \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1}) d_{t+j-1} \exp(r_{t+j-1}^f - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
- \;& \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_{t+j-1} \exp(r_{t+j-1}^f - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
+ \;& \mathbb{E}_t[-d_{t+1} d_{t+j-1} \exp(r_t^f - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
- \;& \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[d_{t+j-1} \exp(r_t^f - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
+ \;& \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1}) \times -d_{t+j}] \\
- \;& \mathbb{E}_t[d_t \exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})]\mathbb{E}_t[-d_{t+j}] \\
+ \;& \mathbb{E}_t[-d_{t+1} \times -d_{t+j}] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[-d_{t+j}].
\end{aligned}$$

This expression can be restated as:

$$cov_t(s_{t+1}, s_{t+j}) = d_t \mathbb{E}_t[\exp(r_t^f - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})]$$

$$
\begin{aligned}
&\times \quad \mathbb{E}_t[d_{t+j-1}\exp(r^f_{t+j-1} - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
&\times \quad (\exp(\sigma\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
&- \quad \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j-1}\exp(r^f_t - \psi_0 - (\psi-1)y_{t+j-1} - \sigma\varepsilon_{t+j})] \\
&\times \quad (\exp(\lambda\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
&- \quad d_t\mathbb{E}_t[\exp(r^f_t - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_{t+j}](\exp(\sigma\lambda\phi^{j-1}) - 1) \\
&+ \quad \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j}](\exp(\lambda^2\phi^{j-1}) - 1)
\end{aligned}
$$

which implies

$$
\begin{aligned}
cov_t(s_{t+1}, s_{t+j}) \quad = \quad & \exp(2\rho - 2\psi_0 + \sigma^2)d_t\exp(-(\psi-1)y_t)\mathbb{E}_t[d_{t+j-1}\exp(-(\psi-1)y_{t+j-1})] \\
\times \quad & (\exp(\sigma\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
- \quad & \exp(\rho - \psi_0 + .5\sigma^2)\mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j-1}\exp(-(\psi-1)y_{t+j-1})] \\
\times \quad & (\exp(\lambda\phi^{j-2}(\lambda + (\psi-1)\sigma)) - 1) \\
- \quad & \exp(\rho - \psi_0 + .5\sigma^2)d_t\exp(-(\psi-1)y_t)\mathbb{E}_t[d_{t+j}](\exp(\sigma\lambda\phi^{j-1}) - 1) \\
+ \quad & \mathbb{E}_t[d_{t+1}]\mathbb{E}_t[d_{t+j}](\exp(\lambda^2\phi^{j-1}) - 1)
\end{aligned}
$$

where the conditional forecasts are

$$
\mathbb{E}_t[d_{t+j}] \quad = \quad \exp\left(\phi^j(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2\frac{1-\phi^{2j}}{1-\phi^2}\right)
$$

and

$$
\begin{aligned}
& \mathbb{E}_t[\exp(\log d_{t+j-1} - (\psi-1)y_{t+j-1})] \\
= \quad & \exp\left(\phi^{j-1}(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + \frac{\phi_0 - .5\lambda^2}{1-\phi} - (\psi-1)\psi^{j-1}(y_t - \frac{\psi_0}{1-\psi}) - (\psi-1)\frac{\psi_0}{1-\psi}\right. \\
+ \quad & \left.\frac{1}{2}\sum_{k=0}^{j-2}(\phi^k\lambda + \psi^k(\psi-1)\sigma)^2\right)
\end{aligned}
$$

Also, when $j = 1$,

$$
\begin{aligned}
var_t(s_{t+1}) \quad = \quad & \mathbb{E}_t[d_t\exp(r^f_t - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})d_t\exp(r^f_t - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})] \\
- \quad & \mathbb{E}_t[d_t\exp(r^f_t - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})]\mathbb{E}_t[d_t\exp(r^f_t - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})] \\
+ \quad & \mathbb{E}_t[-d_{t+1}d_t\exp(r^f_t - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})] \\
- \quad & \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[d_t\exp(r^f_t - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})] \\
+ \quad & \mathbb{E}_t[d_t\exp(r^f_t - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1}) \times -d_{t+1}] \\
- \quad & \mathbb{E}_t[d_t\exp(r^f_t - \psi_0 - (\psi-1)y_t - \sigma\varepsilon_{t+1})]\mathbb{E}_t[-d_{t+1}] \\
+ \quad & \mathbb{E}_t[-d_{t+1} \times -d_{t+1}] - \mathbb{E}_t[-d_{t+1}]\mathbb{E}_t[-d_{t+1}]
\end{aligned}
$$

which implies

$$
\begin{aligned}
var_t(s_{t+1}) \quad = \quad & \exp(2\rho - 2\psi_0 + \sigma^2)\exp(2\log d_t - 2(\psi-1)y_t)(\exp(\sigma^2) - 1) \\
- \quad & 2\exp(\rho - \psi_0 + .5\sigma^2)\exp\left((1+\phi)(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \frac{1}{2}\lambda^2 - (\psi-1)y_t\right) \\
\times \quad & (\exp(\lambda\sigma) - 1)
\end{aligned}
$$

$$+ \quad \exp\left(2\phi(\log d_t - \frac{\phi_0 - .5\lambda^2}{1-\phi}) + 2\frac{\phi_0 - .5\lambda^2}{1-\phi} + \lambda^2\right)(\exp(\lambda^2) - 1)$$

□

F Notes about Convenience Yields

The government debt portfolio return is the return on a portfolio that goes long in the tax claim and short in the spending claim:

$$\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right] = \frac{P^T_t - T_t}{B_t - S_t}\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] + \frac{P^\lambda_t - T_t}{B_t - S_t}\mathbb{E}_t\left[R^\lambda_{t+1} - R^f_t\right] - \frac{P^G_t - G_t}{B_t - S_t}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right],$$

where R^D_{t+1}, R^T_{t+1},R^λ_{t+1} and R^G_{t+1} are the holding period returns on the bond portfolio, the tax claim, and the spending claim, respectively. We take government spending process, and the debt return process as exogenously given, and we explore the implications for the properties of the tax claim.

Proposition F.1. In the absence of arbitrage opportunities, if the TVC holds, the expected excess return on the tax claim is the unlevered return on the spending claim and the debt claim:

$$\begin{aligned}
\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] &= \frac{P^G_t - G_t}{D_t + (P^G_t - G_t) - (P^\lambda_t - K_t)}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] \\
&+ \frac{D_t}{D_t + (P^G_t - G_t) - (P^\lambda_t - K_t)}E_t\left[R^D_{t+1} - R^f_t\right] \\
&- \frac{P^\lambda_t - K_t}{D_t + (P^G_t - G_t) - (P^\lambda_t - K_t)}\mathbb{E}_t\left[R^\lambda_{t+1} - R^f_t\right]
\end{aligned}$$

If we want the debt to be risk-free, then the following equation holds for expected returns:

$$\begin{aligned}
\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] &= \frac{P^G_t - G_t}{D_t + (P^G_t - G_t) - (P^\lambda_t - K_t)}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] \\
&- \frac{P^\lambda_t - K_t}{D_t + (P^G_t - G_t) - (P^\lambda_t - K_t)}\mathbb{E}_t\left[R^\lambda_{t+1} - R^f_t\right]
\end{aligned}$$

$$\begin{aligned}
\beta^T_t &= \frac{P^G_t - G_t}{D_t + (P^G_t - G_t) - (P^\lambda_t - K_t)}\beta^G_t \\
&- \frac{P^\lambda_t - K_t}{D_t + (P^G_t - G_t) - (P^\lambda_t - K_t)}\beta^\lambda_t.
\end{aligned}$$

Suppose we consider the case of a constant spending ratio and a constant convenience yield ratio. Then this implies that the beta of the tax revenue process is given by:

$$\beta^T_t = \frac{(P^G_t - G_t) - (P^\lambda_t - K_t}{D_t + (P^G_t - G_t) - (P^\lambda_t - K_t)}$$

On the other hand, suppose that the convenience yield seignorage process has a zero beta. Then the implied beta of the tax revenue process

$$\beta^T_t = \frac{P^G_t - G_t}{D_t + (P^G_t - G_t) - (P^\lambda_t - K_t)},$$

which exceeds the beta of the tax revenue without seignorage: $\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)}$. If the seignorage revenue is sufficiently counter-cyclical, then the government can insure both taxpayers and bondholders at the same time. For example, consider the case in which the government runs zero primary surpluses in all future states of the world. Then the beta of the tax revenue is one $\beta_t^T = 1$, where $D_t = P_t^\lambda - K_t$. In this case, the average tax rate is constant: $\Delta \log \tau_{t+1} = 0$.