NBER WORKING PAPER SERIES

PORTFOLIO CHOICE AND ASSET PRICING WITH NONTRADED ASSETS

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Working Paper No. 2774

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
November 1988

I am grateful for comments by Bernard Dumas, Elhanan Helpman, Peter Sellin, Philippe Weil, Ingrid Werner, and participants in seminars at IIES and NBER Summer Institute. I thank Maria Gil for secretarial and editorial assistance. This research is part of NBER's research program in International Studies. Any opinions expressed are those of the author not those of the National Bureau of Economic Research.

NBER Working Paper #2774
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ABSTRACT

This paper examines portfolio choice and asset pricing when some assets are nontraded, for instance when a country cannot trade claims to its output on world capital markets, when a government cannot trade claims to future tax revenues, or when an individual cannot trade claims to his future wages. The close relation between portfolio choice with and implicit pricing of nontraded assets is emphasized. A variant of Cox, Ingersoll and Ross's Fundamental Valuation Equation is derived and used to interpret the optimal portfolio. Explicit solutions are presented to the portfolio and pricing problem for some special cases, including when income from the nontraded assets is a diffusion process, not spanned by traded assets, and affected by a state variable.

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I. INTRODUCTION

The literature on portfolio choice in finance and international finance generally assumes that all income arises from traded assets, and that there are no nonmarketable or nontraded assets. In the real world, however, the existence of income from nontraded assets seems to be the rule rather than exception. Nontraded assets would not present a problem if traded assets provided perfect hedges against income from nontraded assets, but that seems again to be the exception. Income from nontraded assets that can only be imperfectly hedged by traded assets will affect portfolio choice as well as consumption and saving decisions.

The existence of nontraded assets could be a result of asset market imperfections, which in turn are caused by the usual reasons: transactions cost, moral hazard, legal restrictions like capital controls, etc. As examples we can think of an individual who cannot trade claims to his future wages (his human capital) for obvious moral hazard reasons, a government which cannot trade claims to future tax receipts, or a country which cannot trade claims to its GDP in world capital markets.

The literature has mostly avoided the problem of income from nontraded assets, with a few exceptions. The literature on segmented markets (see for instance the survey by Adler and Dumas (1983)) may come to mind, but that deals with assets traded in markets separate from each other and not with the effect of nontraded assets on traded assets portfolios. I have found few references on nontraded assets proper. Mayers (1972, 1973) deals with nonmarketable assets and portfolio choice in the static Capital Asset Pricing Model.[1] Merton (1971) solves an individuals portfolio problem in a continuous time model when the income from nontraded assets is a Poisson process, but not for the more interesting case when the income is a diffusion process. Fischer's (1975) article on the demand for indexed bonds in a continuous-time model also includes the case when there is

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1 Mayers (1972) emphasizes that the notion of investors including other than just marketable assets in their portfolio decision is not a new one and gives references to Markowitz (1967) and Hirshleifer (1970) where the possibility of income from nonmarketable assets are mentioned.

a nontraded wage income that follows a diffusion process. A discussion and interpretation of the case is provided but no analytical solution is presented.[2] Modest (1984) uses a model similar to Fischer's and presents an analytic solution, but only under the assumption that traded assets provide a perfect hedge against wages (that is, wages are spanned by traded assets).[3] Losq (1978) solves the consumption and portfolio problem with nontraded wage income both in a two-period CAPM and in a continuous time model for some special cases, and that paper is a major source of inspiration for the current work.[4] Breeden (1979) in his derivation of a single beta Capital Asset Pricing Model in continuous time includes a riskless wage income in the portfolio problem but no explicit solutions are presented. Persson and Svensson (1987), in discussing capital flows and exchange rate variability, solves the general equilibrium in a two-country two-period CAPM where the countries in some cases cannot trade claims to their own output.[5]

Two papers by Stulz (1984) and Adler and Detemple (1988) consider firms' optimal hedging against a nontraded asset (a claim to uncertain cash payments at a given future date). They assume that the implicit value of, and hence the implicit rate of return on, the nontraded assets is a known exogenous stochastic process, and then solve the portfolio problem. In contrast, we determine the implicit value of and rate of return on the nontraded asset endogenously. In general, as we shall see, the implicit value of the nontraded asset depends on both the available set of nontraded assets and the investor's preferences, and hence the value will in general differ for different investors. Only when the return from the nontraded asset is spanned by traded assets is the value of the

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2 Fischer (1974) does present an analytic solution for the case with constant relative risk aversion. As far as I can see, the solution is correct only under the additional assumption of the wage being spanned by the traded assets (see footnote 13).

3 Richard (1975) examines portfolio choice and life insurance in a model with nontraded human capital and uncertain length of life, but income from human capital is assumed to be sure if the investor is alive.

4 I am grateful to Bernard Dumas who showed me Losq's unpublished paper.

5 Dumas (1988) discusses pricing of physical assets and the Law of One Price in a situation when physical capital can be moved internationally only at a cost, but where international financial markets are complete. Since all relevant assets (including claims on physical capital) are internationally traded, there are no nontraded assets.

nontraded asset independent of preferences and may be taken to be exogenously given to the individual investor. Duffie and Jackson (1988) present explicit solutions in several special cases to the finite-horizon portfolio problem of a single agent maximizing the expected utility of the terminal value of a fixed portfolio of nontraded assets and a portfolio of traded assets. As noted below, the solution to one of their cases corresponds to one of our solutions, although the solution from the finite horizon terminal wealth maximization cannot directly be meaningfully extended to an infinite horizon.

This paper, then, will discuss and present solutions to the portfolio problem in a continuous time model when there is some income from nontraded assets. It can be seen as an extension of Losq (1978). The model used is the standard one, from Merton (1971), except that there is an additional exogenous stochastic income from some nontraded assets. The model has several interpretations. It can be interpreted as a model of an individual investor/consumer who can trade in a given set of assets with given stochastic rates of return. The investor/consumer has a stochastic wage income, and claims on future wages are nontradeable (due to moral hazard problems, for instance); or other income from nontraded assets (which are nontraded due to regulation, legal restrictions, transactions costs, etc.). The model can also be interpreted as a model of a closed economy with representative consumers and firms in general equilibrium. Then the set of traded assets corresponds to a set of constant-returns-to-scale no-adjustment-cost physical investment activities with given stochastic rates of return. In addition there is, corresponding to income from nontraded assets, stochastic income from some primary factors of production in fixed supply (labor or land, for instance).

The model can also be interpreted as one of a small open economy, with a set of internationally traded assets and possibly some domestic constant-returns-to-scale no-adjustment-cost investment activities. In addition some or all domestic investment production activities are <u>not</u> constant-returns-to-scale or have significant adjustment costs (due to some factors in relatively fixed supply, for instance). In particular, the set of internationally traded assets does not, for one reason or another, include claims on domestic GDP, or all domestic no constant-returns-to-scale no-adjustment-cost investment activities; also the internationally traded assets are only imperfect hedge against the above domestic sources of income. (This is actually the interpretation I first had in mind when I got interested in the problem, because of dissatisfaction with the existing work on actual and optimal international portfolio composition, especially currency composition, where income from internationally nontraded assets is consistently disregarded. (See for instance the survey by Adler and Dumas (1983).)

There are, I believe, several new elements in the present paper. First, we emphasize the close relation between the problem of portfolio choice with income from nontraded assets (nontraded income, for short) to the problem of finding the implicit price of and rates of return on the nontraded assets, that is, the price of the nontraded asset at which it would be willingly held in case trade in that asset was possible. We shall see that it is indeed very clarifying to solve for the implicit price and interpret its components. We shall apply a modified form of the "fundamental valuation equation" of Cox, Ingersoll and Ross (1985) to find the implicit price and rates of return. Second, we extend Losq's (1978) analysis by solving the problem not only when the nontraded income is an ordinary Brownian motion but also when its drift and standard deviation is affected by a stochastic state variable. In the latter case it is helpful to distinguish nontraded income from from the traditional state variables that affect traded assets rates of return and generate a "stochastic investment opportunity set" in the usual terminology, since nontraded income enter separately in the wealth equation. In line with this, it is also helpful to distinguish a separate "nontraded-income hedge" portfolio in addition to the traditional "state-variable hedge" portfolios.

We shall present explicit solutions to the portfolio problem in three different special cases, each with one specific simplification. The three simplifications are spanning, independence of a stochastic state variable, and instantaneous risklessness, each considered one at a time. Hence we consider the cases: (A) The nontraded income is restricted to be spanned by the traded assets, whereas the nontraded income may depend on a state variable and be instantaneously risky. Then the implicit price of the nontraded asset can be found by a simple no-arbitrage relation, and the solution to the portfolio problem is easy to get from the solution to the problem when all assets are freely traded. (B) The nontraded income is restricted not to depend on any state variable, whereas the nontraded income need not be spanned and may be instantaneously risky. (This is the case Losq (1978) solves.) Then the implicit price of the traded asset includes a term corresponding to the value of unhedgeable variance of the nontraded income. (C) The nontraded income is restricted to be instantaneously riskless, whereas it need not be spanned and may be a function of a stochastic state variable. (In Fischer (1974, 1975) and Breeden (1979) the exogenous income is actually instantaneously riskless.) Then there is no nontraded income hedge portfolio, only a state hedge portfolio. The implicit price of the nontraded assets in this case then includes a term corresponding to the value of the unhedgeable state variable variance. In particular, even if the nontraded income is riskless, the nontraded asset is not riskless since there are implicit stochastic capital gains. In cases (B) and (C) the utility function is restricted to have constant absolute risk aversion.

Section II lays out the model, presents the general problem, and interprets the optimal portfolio. Section III derives the partial differential equation for the implicit price of the nontraded asset. Section IV presents explicit solutions to the portfolio problem and the pricing problem for the special cases (A), (B) and (C). Section V contains conclusions and a discussion of some limitations of the analysis and possible directions for future research. An appendix includes some technical details.

For a more detailed presentation of the analytical methods used in this paper, see Merton (1971), Fischer (1975), Chow (1979) or Ingersoll (1987).

II. THE GENERAL PROBLEM

Consider an investor in an economy with one consumption good. Time is continuous. The investor has preferences at time t over uncertain consumption paths $\{C(\tau)\}_{\tau=t}^{\infty}$ represented by the expected utility integral

(2.1) $\quad E_t \int_t^{\infty} U(C(\tau),\tau)d\tau,$

where E_t denotes the expected value operator conditional upon information available at time t, and $U(C,\tau)$ is a standard instantaneous utility function.

The investor can continuously trade n+1 different assets, denoted $i = 0, 1, .., n$. Asset 0 has an instantaneously riskless real rate of return, $r(s,t)dt$, during an interval dt. The real rate of return may be a function of a stochastic state variable, $s(t)$, to be specified below, and of time. Assets $i = 1, ..,n$ have (instantaneously) risky rates of return. Let $q_i(t)$ denote the price of asset i in terms of goods, and let the returns occur in the form of capital gains. The price of asset i is generated by an Ito process,

(2.2) $\quad dq_i(t)/q_i(t) = \mu_{qi}(s(t),t)dt + S_{qi}(s(t),t)d\omega(t).$

Here $\mu_{qi}(s,t)$, which may be a function of the state variable and time, is the expected instantaneous rate of return on asset i. Furthermore, $S_{qi}(s,t)$ is a row m-vector, and the (column) m-vector $\omega(t) = (\omega_j(t))_1^m$ denotes a vector of m independent Wiener processes (the number m of Wiener processes must be at least as large as the number of risky assets plus the number of state variables in order to allow for linearly independent risky assets and state variables). That is, the increments $d\omega_j(t)$, $j = 1, .., m$, are serially independent no matter how short the interval dt, and they fulfill

(2.3) $\quad E[d\omega_j(t)] = 0,\ Var[d\omega_j(t)] = dt,\ Cov[d\omega_j(t),d\omega_k(t)] = 0\ (j \neq k),\ j, k = 1, .., m.$

Hence, the stochastic component of the rate of return, the second term on the right-hand side in (2.2), is a weighted sum of the increments of m independent Wiener processes. It follows that the instantaneous variance of the rate of return of asset i is given by the inner product $S_{qi}S_{qi}' = \Sigma_j(S_{qij})^2$, where ' denotes transpose. For convenience, we shall call S_{qi} the standard deviation vector of the rate of return on asset i. (The (instantaneous) standard deviation of the rate of return on asset i is the scalar $(S_{qi}S_{qi}')^{1/2}$.)

For the case when the expected rate of return and the standard deviation vector are constant, (2.2) is a geometric Brownian motion and it implies that (i) the rate of return on asset i is normally distributed and (ii) the price of asset i is lognormally distributed.[6]

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6 Equation (2.2) is an equivalent way of writing the integral

With more compact notation, we shall write

(2.4) $\quad D_q^{-1}dq(t) = \mu_q(s(t),t)dt + S_q(s(t),t)d\omega(t),$

where D_q denotes the diagonal n×n-matrix with the n-vector $q(t) = (q_i(t))_1^n$ as the diagonal, $dq(t)$ and $\mu_q(s,t)$ are column n-vectors, and $S_q(s,t)$ is an n×m-matrix. Hence $D_q^{-1}dq(t)$ denotes the column n-vector with components $dq_i/q_i(t)$, etc.

At each instant t the investor chooses the rate of consumption $C(t)$ and the composition of his wealth, $W(t)$, into the different assets. Let $X_i(t) \gtrless 0$ be the amount of wealth held in asset i, i = 0, 1, .., n. The investor's portfolio $X_0(t)$ of the riskless asset and $X(t) = (X_i(t))_{i=1}^{\infty}$ of the risky assets fulfills

(2.5) $\quad X_0(t) + 1'X(t) = W(t),$

where 1 denotes an n-vector of ones and $1'X(t)$ denotes the inner product (and sum) $\Sigma_{i=1}^n X_i(t)$.

The present setup differs from the standard continuous-time portfolio problem in that the investor is assumed to have an exogenous stochastic income , in addition to the return on his portfolio. This exogenous income corresponds to an implicit nontraded asset, the claim to the exogenous income. We shall refer to the exogenous income simply by (nontraded) income. The stochastic income during an interval dt is denoted $dy(t)$ and follows the Ito process

(2.6) $\quad dy(t) = \mu_y(s(t),t)dt + S_y(s(t),t)d\omega(t).$

The drift coefficient (the instantaneous expected value) $\mu_y(s,t)$ and the (instantaneous) standard deviation m-vector $S_y(s,t)$ may be functions of the state variable and time.

During an interval dt the change in wealth, $dW(t)$, for given $X_0(t) = X_0$, $X(t) =$

$$q_i(t) = q_i(t_0) + \int_{t_0}^t q_i(\tau)\mu_{qi}(s(\tau),\tau)d\tau + \int_{t_0}^t q_i(s(\tau),\tau)S_{qi}(s(\tau),\tau)d\omega(\tau).$$

To see that the price of asset i are lognormally distributed when μ_{qi} and S_{qi} are constant, let $z_i = \ln q_i$. From Ito's lemma it follows that $dz_i = (\mu_{qi} + \sigma_{q_iq_i}/2)dt + S_{yi}d\omega$, that is, z_i is generated by a Brownian motion. Then $z_i(t)$ is normally distributed with mean $z_i(t_0) + (\mu_{qi} + \sigma_{q_iq_i}/2)(t - t_0)$ and variance $\sigma_{q_iq_i}(t - t_0)$.

X, and $C(t) = C$, is given by[7]

(2.7) $$dW(t) = X_0 r(s,t)dt + X'D_q^{-1}dq(t) + dy(t) - Cdt.$$

The change in wealth consists of four terms, the return on the riskless asset, the return on the risky assets, the exogenous income, and (with a negative sign) consumption. With (2.4)-(2.6), equation (2.7) can be written as

(2.8) $$dW(t) = [X'\nu_q(s,t) + W(t)r(s,t) + \mu_y(s,t) - C]dt + [X'S_q(s,t) + S_y(s,t)]d\omega(t),$$

where we have introduced the n-vector of expected excess rates of return for the risky assets, $\nu_q(s,t)$, defined by

(2.9) $$\nu_q(s,t) = \mu_q(s,t) - r(s,t)1.$$

In the general case the expected returns and the standard deviation vectors of the assets and the exogenous income may be affected by a set of exogenous stochastic state variables. To simplify the notation we assume that there is only one state variable, $s(t)$. It is also generated by an Ito process, namely

(2.10) $$ds(t) = \mu_s(s(t),t)dt + S_s(s(t),t)d\omega(t).$$

We note that (2.6) and (2.10) imply that the special case when nontraded income is a geometric Brownian motion, $dy/y = \bar{\mu}_y dt + \bar{S}_y d\omega$, is included, since we may choose $\mu_y(s,t) = \mu_s(s,t) = \bar{\mu}_y s$, $S_y(s,t) = S_s(s,t) = \bar{S}_y s$, and indentify y with s.

The optimization problem of the investor is now to choose, for given current levels of wealth and the state variable, $W(t) = W$ and $s(t) = s$, and contingent upon future realizations of wealth and the state variable, the consumption path $\{C(\tau)\}_{\tau=t}^{\infty}$ and portfolio path $\{X(\tau)_{\tau=t}^{\infty}\}$ for risky assets so as to maximize (2.1) subject to (2.8) and (2.10). The portfolio path for the riskless asset $\{X_0(\tau)\}_{\tau=t}^{\infty}$ then follows from (2.5). We let $J(W,s,t)$ denote the corresponding indirect utility function, and $C^*(W,s,t)$ and $X^*(W,s,t)$ the corresponding optimal solution.[8]

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7 See Merton (1971) for a detailed derivation.

8 We assume that a solution exists and is unique, and that we may disregard bankruptcy (that is, we assume either that consumption is always positive and nonnegativity constraints on consumption are never binding or that negative consumption is allowed). Karatzas, Lehoczky, Sethi and Shreve (1986) rigorously demonstrate existence and discuss bankruptcy in the standard model without nontraded assets and state variables.

The Bellman equation is

$$(2.11)\qquad 0 = \max_{(C,X)} \{U(C,t) + J_t + J_W[X'\nu_q(s,t) + Wr(s,t) + \mu_y(s,t) - C] + \tfrac{1}{2}J_{WW}[X'\sigma_{qq}(s,t)X + 2X'\sigma_{qy}(s,t) + \sigma_{yy}(s,t)] + J_s\mu_s(s,t) + J_{Ws}X'\sigma_{qs}(s,t) + \tfrac{1}{2}J_{ss}\sigma_{ss}(s,t)\}.$$

Here σ_{qq}, σ_{qy}, σ_{yy}, σ_{qs}, and σ_{ss} are the (instantaneous) covariances matrices/vectors/scalars for and between q, y and s, given by S_qS_q', S_qS_y', S_yS_y', S_qS_s', and S_sS_s', respectively.

The first-order conditions are

$$(2.12)\qquad U_C - J_W = 0 \text{ and}$$

$$(2.13)\qquad J_W\nu_q + J_{WW}[\sigma_{qq}X + \sigma_{qy}] + J_{Ws}\sigma_{qs} = 0.$$

The first-order conditions can be solved for the consumption function and optimal portfolio,

$$(2.14a)\qquad C^*(W,s,t) = U_C^{-1}(J_W) \text{ and}$$

$$(2.14b)\qquad X^*(W,s,t) = (-J_W/J_{WW})\sigma_{qq}^{-1}\nu_q - \sigma_{qq}^{-1}\sigma_{qy} - (J_{Ws}/J_{WW})\sigma_{qq}^{-1}\sigma_{qs} = X^t + X^{hy} + X^{hs},$$

The optimal portfolio is the sum of three terms and may be interpreted as the combination of three different portfolios, the "tangency" portfolio

$$(2.15a)\qquad X^t = (-J_{WW}/J_W)\sigma_{qq}^{-1}\nu_q,$$

the "income hedge" portfolio

$$(2.15b)\qquad X^{hy} = -\sigma_{qq}^{-1}\sigma_{qy},$$

and the "state variable hedge" portfolio

$$(2.15c)\qquad X^{hs} = -(J_{Ws}/J_{WW})\sigma_{qq}^{-1}\sigma_{qs}.$$

The tangency portfolio is the portfolio that corresponds to the point of tangency of the mean-variance efficient locus and the borrowing-lending line. A hedge portfolio against a random variable is the portfolio (i) whose return has the maximum negative correlation with the variable, or, equivalently, (ii) that minimizes the variance of the sum of the random variable and the return on the portfolio.[9] (See Ingersoll (1987) for further

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[9] The income hedge portfolio (2.15b) is the solution to

discussion.) The term $(-J_W/J_{WW})$ in the tangency portfolio is the absolute tolerance to wealth risk (the reciprocal of the absolute aversion to wealth risk). Similarly, we can call the term $(-J_W/J_{Ws})$ the absolute tolerance to the state variable risk (the reciprocal of the absolute aversion to the state variable risk). Then the term (J_{Ws}/J_{WW}) in the state variable hedge portfolio can be interpreted as the ratio of the absolute tolerance to wealth risk to the absolute tolerance to state variable risk (or the ratio of the absolute aversion to state variable risk to the absolute aversion to wealth risk).

The consumption function and the optimal portfolio in (2.14) are expressed in terms of the partial derivatives of the indirect utility function. In order to get a complete solution, equations (2.14a,b) are substituted into the Bellman equation (2.11) which results in a partial differential equation for the indirect utility function (see (A.1) in the appendix). Solution of this partial differential equation gives the explicit indirect utility function which can then be substituted back into (2.14) to get the explicit consumption function $C^*(W,s,t)$ and optimal portfolio $X^*(W,s,t)$.

We note that if nontraded income and the state variable are uncorrelated with the returns on traded assets, the corresponding hedging portfolios will be zero. Even in this case the existence of the nontraded asset affects the solution. We will see this in detail below, but we can realize this already at this stage since the variances of nontraded income and the state variable still enter the partial differential equation (A.1) for the indirect utility function.

III. PRICING A CLAIM TO INCOME

Let us consider the (implicit) value of a claim to the exogenous stochastic income dy. The value of the claim is the price at which the claim would be willingly held by the investor if the claim was freely traded. The problem of finding the value of a claim to a given stochastic income is a problem of independent interest. The problem is however of

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(i) $\min_X[X'\sigma_{qy}/(X'\sigma_{qq}X\sigma_{yy})^{1/2}]$ as well as to (ii) $\min_X \mathrm{Var}[X'D_q^{-1}dq + dy]/dt = \min_X \mathrm{Var}[X'\sigma_{qq}X + 2X'\sigma_{qy} + \sigma_{yy}]$.

particular interest here, since it is closely related to the problem of finding the optimal portfolio of traded assets when the claim to income is nontraded. Indeed, knowing the implicit value of the claim facilitates very much the interpretation of the optimal portfolio of traded assets.

The value of the claim, F, will in general be a function of the level of wealth, the state variable, and time. It is practical to consider it a function of the investor's comprehensive (implicit) wealth, $\hat{W}$, including the value of the claim and therefore given by

$$\hat{W} = W + F. \tag{3.1}$$

Let us hence denote the value of the claim by $F(\hat{W},s,t)$.

Finding the implicit price at which the claim is willingly held also involves finding its implicit rate of return. To see this, we shall write the rate of return in two ways. First, in analogy with (2.2) we let $dq_F(t)/q_F(t)$ denote the (implicit) rate of return on the claim during an interval dt, we let $\mu_F(\hat{W},s,t)$ denote the expected rate of return, and we let $S_F(\hat{W},s,t)$ denote the standard deviation vector for the rate of return. Then we can write the rate of return as the Ito process

$$dq_F(t)/q_F(t) = \mu_F(\hat{W}(t),s(t),t)dt + S_F(\hat{W}(t),s(t),t)d\omega(t), \tag{3.2}$$

where $\mu_F(\cdot)$ and $S_F(\cdot)$ remain to be determined. Second, we note that return on the claim during an interval dt consists of a stochastic (explicit) dividend, $dy(t)$, and a stochastic (implicit) capital gain, $dF(t) = dF(\hat{W}(t),s(t),t)$. Therefore, we have the identity

$$dq_F(t)/q_F(t) \equiv (dy(t) + dF(t))/F(\hat{W}(t),s(t),t), \tag{3.3}$$

where $F(\cdot)$ and dF remain to be determined.

The expected rate of return $\mu_F(\hat{W},s,t)$ and the standard deviation of the rate of return $S_F(\hat{W},s,t)$ can now be identified from (2.6), (3.2) and (3.3). More precisely, Ito's lemma is used to develop the differential dF in (3.3) (see appendix B for details). Then we can identify the terms multiplying dt to get the expected return in terms of the partials of F,

$$F\mu_F \equiv \mu_y + F_t + F_{\hat{W}}(\mu_{\hat{W}} - C^*) + F_s\mu_s + \tfrac{1}{2}F_{\hat{W}\hat{W}}\sigma_{\hat{W}\hat{W}} + F_{\hat{W}s}\sigma_{\hat{W}s} \tag{3.4}$$

$$+ \frac{1}{2}F_{ss}\sigma_{ss},$$

where we use the notation

(3.5a) $\mu_{\hat{W}} \equiv X^{*\prime}\nu_q + F\nu_F + \hat{W}r$

for the expected change in comprehensive wealth, and

(3.5b) $S_{\hat{W}} \equiv X^{*\prime}S_q + FS_F,$

for the standard deviation vector of comprehensive wealth. Furthermore, $\sigma_{\hat{W}\hat{W}}$ and $\sigma_{\hat{W}s}$ denote, respectively, the variance of comprehensive wealth and the covariance between comprehensive wealth and the state variable. Similarly, by identifying the terms multiplying $d\omega$ we get the standard deviation vector in terms of partials of F,

(3.6) $FS_F \equiv S_y + F_{\hat{W}}S_{\hat{W}} + F_sS_s.$

Combining these identities with the relevant equilibrium conditions, either a no-arbitrage condition (when the claim to income is spanned by the traded assets) or an implicit first-order condition with respect to the amount of implicit wealth invested in the claim (when the claim to income is not spanned by the traded assets), gives a partial differential equation in $F(\hat{W},s,t)$, the nontraded asset's Fundamental Valuation Equation, to which a solution will have to be found.

IV. SPECIAL CASES

We shall present solutions to the portfolio problem and the pricing problem only for selected special cases. First, we make some simplifying assumptions throughout the analysis. We assume that the rates of return on the traded assets are independent of the state variable and time (that is, they are constant),

(4.1a) $r(s,t) = r,\ \mu_q(s,t) = \mu_q,$ and $S_q(s,t) = S_q.$

We also assume that the drift and standard deviation of the state variable are independent of time,

(4.1b) $\mu_s(s,t) = \mu_s(s)$ and $S_s(s,t) = S_s(s).$

We moreover assume that the instantaneous utility function has an exponential discount factor, (4.1c) $\quad U(C,t) = e^{-\delta t}V(C), \ \ \delta > 0.$

Under these assumptions it is easy to show that the indirect utility function also has an exponential discount factor and can be written

(4.2) $\quad J(W,s,t) = e^{-\delta t}I(W,s).$

Then the optimal solution and the value of the claim to income will be independent of time.

Second, we consider some specific simplifications. In section IV.A we assume that both income and the state variable are spanned by the traded assets. In section IV.B we relax the assumption of income being spanned, but assume that there is no state variable. In section IV.C we assume that income is instantaneously riskless but allow for a non-spanned state variable. Throughout section IV.B and IV.C the instantaneous utility function is restricted to have constant absolute risk aversion.

A. Income and the State Variable Spanned By Traded Assets

We assume now that both income and the state variable are spanned by the $n+1$ traded assets. Spanning here means that there exist linear combinations of the traded assets which have the same risk characteristics as income and the state variable. This can be precisely defined in several equivalent ways: First, the hedge portfolios X^{hy} and X^{hs} in (2.15) are perfect hedges, that is, the returns on them are perfectly negatively correlated with income and the state variable, respectively. Second, the standard deviation vectors $S_y(s)$ and $S_s(s)$ are linear combinations of the rows of the standard deviation matrix S_q. The weights in the linear combination are given by the hedge portfolios. More precisely,

(4.3a) $\quad S_y(s) + X^{hy}(s)'S_q = 0$ and

(4.3b) $\quad S_s(s) + (-\sigma_{qq}^{-1}\sigma_{qs}(s))'S_q = 0.$

Third, the "unhedgeable" variances of income and the state variable conditional upon the set of traded assets, $\sigma_{y|q}(s)$ and $\sigma_{s|q}(s)$, defined by

(4.4a) $\quad \sigma_{y|q}(s) = \sigma_{yy}(s) - \sigma_{yq}(s)\sigma_{qq}^{-1}\sigma_{qy}(s)$ and

(4.4b) $\quad \sigma_{s|q}(s) = \sigma_{ss}(s) - \sigma_{sq}(s)\sigma_{qq}^{-1}\sigma_{qs}(s),$

are both zero,[10]

$$(4.5) \qquad \sigma_{y|q}(s) = \sigma_{s|q}(s) = 0.$$

Spanning obviously excludes the possibility that income and the state variable are uncorrelated with the rates of return on traded assets.

One way to find the optimal portfolio is to directly solve the partial differential equation (A.1) in the appendix, taking (4.5) into account. Another way is to first find the implicit price of a claim to income. We shall use the latter way, which we find useful and illuminating. Then we have to derive a partial differential equation for the value of the claim to income, $F(\hat{W},s,t)$, and solve that.

First, it is easily seen that if income and the state variable are spanned by the traded assets, so is the claim to income (see appendix C). If the claim to income is spanned, the "F hedge" portfolio X^{hF} of traded assets, given by

$$(4.6) \qquad X^{hF} = - \sigma_{qq}^{-1}\sigma_{qF}F,$$

has its return perfectly negatively correlated with the return on the claim to income. This means that the combination of the F hedge portfolio and the claim to income has an instantaneously riskless return. Absent arbitrage, the combination must pay the riskless rate of return. That is, we have the no-arbitrage relation

$$(4.7) \qquad F\mu_F + X^{hF\prime}\mu_q = (F + X^{hF\prime}1)r.$$

On the left side we have the expected return on the combination of the claim and the hedge portfolio; on the right hand side we have the riskless return on the investment.

Let us now combine this no-arbitrage relation with (3.4), the expression for expected returns on the claim in terms of the derivatives of F. This gives us a partial differential equation in F, the <u>fundamental valuation equation with a no-arbitrage condition</u>, which after some manipulations (see appendix D) can be written

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[10] The unhedgeable variance of income is the minimum variance of the sum of income and the return on a portfolio of the trade assets, that is, the variance of the sum of income and the return on the income hedge portfolio.

A fourth equivalent definition of that income is spanned by the risky assets is that the variance/covariance matrix $\begin{bmatrix} \sigma_{qq} & \sigma_{qy} \\ \sigma_{yq} & \sigma_{yy} \end{bmatrix}$ is singular.

$$(4.8) \qquad \tfrac{1}{2}F_{\hat{W}\hat{W}}\sigma_{\hat{W}\hat{W}} + F_{\hat{W}s}\sigma_{\hat{W}s} + \tfrac{1}{2}F_{ss}\sigma_{ss} + F_{\hat{W}}(\mu_{\hat{W}} - C^* - \sigma_{\hat{W}q}\sigma_{qq}^{-1}\nu_q) + F_s(\mu_s - \sigma_{sq}\sigma_{qq}^{-1}\nu_q) + F_t - Fr + \mu_y - \sigma_{yq}\sigma_{qq}^{-1}\nu_q = 0.$$

This is the partial differential equation we need to solve. Under the assumption of spanning, we may guess that the value of nontraded income is independent of wealth. Then all terms in (4.8) involving derivatives with respect to wealth drop out. We then note that parameters of the utility function do not enter, neither directly via derivatives of the indirect utility function nor indirectly via wealth and consumption. This is an example of "preference-free" pricing of assets, which results when the asset in question is spanned and a no-arbitrage relation like (4.7) can be used instead of a first-order condition.

Let us restrict the drift and standard deviation of income to be linear in the state variable,

$$(4.9a) \qquad \mu_y(s) = \bar{\mu}_y s \text{ and } S_y(s) = \bar{S}_y s,$$

where $\bar{\mu}_y$ and $\bar{S}_y$ are constant. Furthermore, we restrict the state variable to have constant drift and standard deviation,

$$(4.9b) \qquad \mu_s(s) = \mu_s \text{ and } S_s(s) = S_s.$$

This formulation implies that the state variable is normally distributed and can take negative values, making income negative. Hence our analysis is symmetrical in the sense that it also covers the case when the nontraded asset is a liability with a negative value. Indeed, this symmetry is used below when "extraneous" solutions to the fundamental valuation are eliminated.

As mentioned, under the assumption of spanning we may guess that F is independent of $\hat{W}$. Also, under the assumptions (4.1a)-(4.1c) we may guess that F is independent of t. Therefore all terms in (4.8) involving derivatives of F with respect to $\hat{W}$ and t vanish, and (4.8) becomes a second order ordinary differential equation. Finally, we guess that F is linear in s. Then it is easy to see that a solution to F is

$$(4.10a) \qquad F(s) = (\bar{\mu}_y s - \bar{\sigma}_{yq}\sigma_{qq}^{-1}\nu_q s)/r + \{[(\bar{\mu}_y - \bar{\sigma}_{yq}\sigma_{qq}^{-1}\nu_q)/r]\mu_s - [(\bar{\mu}_y - \bar{\sigma}_{yq}\sigma_{qq}^{-1}\nu_q)/r](\sigma_{sq}\sigma_{qq}^{-1})\nu_q\}/r,$$

where $\bar{\sigma}_{yq} = \bar{S}_y S_q'$.

The solution (4.10a) is a particular solution to the second order linear ordinary differential equation that results after restricting F to be independent of wealth and time. The general solution the that differential equation is the sum of the particular solution and the homogeneous solution, the solution to the homogeneous part of the equation. With reference to reasonable asymptotic properties of F, namely that F should be roughly proportional to the state variable for large values of the state variable, the extraneous homogeneous solution can be eliminated. The reason why F should be asymptotically proportional to the state variable is that for large values of the state variable, the relative variance of the state variable becomes insignificant, since the variance of the state variable is constant (see appendix E).

Before interpreting the expression for F, we shall see what the rest of the solution looks like. Let $\hat{J}(\hat{W},t)$ denote the indirect utility function to the portfolio problem when the claim to income is freely traded. The indirect utility function $\hat{J}(\hat{W},t)$ will not depend on the state variable for given levels of comprehensive wealth $\hat{W}$ since the state variable is spanned by the traded assets and all assets, including the claim to income, are in that case freely traded. Then we may conjecture that the indirect utility function $J(W,s,t)$ for the problem when the claim to income is not traded will fulfill

(4.10b) $\quad J(W,s,t) = \hat{J}(W + F(s),t)$

That is, the effect on utility of the state variable enters only through the effect on comprehensive wealth $\hat{W} = W + F(s)$. (Solving the partial differential equation (A.1) in the appendix confirms that (4.10b) holds.)

It follows from (4.10b) that the term in the state variable hedge portfolio X^{hs} in (2.15c), the ratio of absolute tolerances to wealth risk and the state variable risk, fulfills

(4.11) $\quad J_{Ws}/J_{WW} = \hat{J}_{\hat{W}\hat{W}}F_s/\hat{J}_{\hat{W}\hat{W}} = F_s,$

where, by (4.10a),

(4.12) $\quad F_s = (\bar{\mu}_y - \bar{\sigma}_{yq}\sigma_{qq}^{-1}\nu_q)/r.$

Hence the optimal portfolio (2.14b) is

(4.13) $\quad X^* = (-J_W/J_{WW})\sigma_{qq}^{-1}\nu_q - \sigma_{qq}^{-1}\bar{\sigma}_{qy}s - F_s\sigma_{qq}^{-1}\sigma_{qs}$

$$= X^t + X^{hy}(s) + X^{hs}.$$

The income hedge portfolio $X^{hy}(s)$, the second term on the right-hand side of (4.13) is linear in the state variable. This of course follows directly from the assumption that the drift and standard derivation of income is linear in s. The state variable hedge portfolio X^{hs}, the third term on the right-hand side of (4.13) is independent of the state variable. Furthermore, from (4.6), (D.2) in the appendix, and (4.11) we have

(4.14) $\quad X^{hF}(s) = -\sigma_{qq}^{-1}\bar{\sigma}_{qy}s - F_s\sigma_{qq}^{-1}\sigma_{qs} = X^{hy}(s) + X^{hs}.$

The F hedge portfolio is simply the combination of the income hedge portfolio and the state variable hedge portfolio, Thus, the optimal portfolio can be written

(4.15) $\quad X^* = X^t + X^{hF},$

the sum of the tangency portfolio and the F hedge portfolio.

Let us interpret the results derived so far. The easiest case is when there is no state variable, or, equivalently, when the state variable is deterministic and constant. This case corresponds to

(4.16) $\quad \mu_s = 0$ and $S_s = 0.$

Then

(4.17a) $\quad F(s) = (\bar{\mu}_y s - \bar{\sigma}_{yq}\sigma_{qq}^{-1}\nu_q s)/r$ and

(4.17b) $\quad X^{hF}(s) = X^{hy}(s) = -\sigma_{qq}^{-1}\bar{\sigma}_{qy}s.$

The value of the claim is simply the expected income, $\bar{\mu}_y s$, plus the expected excess return of the perfect income hedge portfolio, $X^{hy}(s)'\nu_q$, discounted by the riskless rate of return r. This is the first term in (4.10a).

When there is a state variable, the value of the claim has a second component as well, the second term in (4.10a). The second component is the expected return on the state variable, the value of which is $[(\bar{\mu}_y - \bar{\sigma}_{yq}\sigma_{qq}^{-1}\nu_q)/r]\mu_s$, plus the expected return on the perfect state variable hedge, the value of which is $[(\bar{\mu}_y - \bar{\sigma}_{yq}\sigma_{qq}^{-1}\nu_q)/r]X^{hs\prime}\nu_q$, again discounted by the riskless rate of return. Hence, owning the claim is equivalent to also owning an asset, the state variable, with expected return $[(\bar{\mu}_y - \bar{\sigma}_{yq}\sigma_{qq}^{-1}\nu_q)/r]\mu_s$ and standard derivation vector $[(\bar{\mu}_y - \bar{\sigma}_{yq}\sigma_{qq}^{-1}\nu_q)/r]S_s$.

The results above are independent of the precise form of the utility function $V(C)$. (As mentioned, we have an example of preference-free pricing.) For specific utility functions, we can in addition find the explicit consumption function and tangency portfolio. Let us therefore see what the consumption function and tangency portfolio looks like for two different instantaneous utility function.

First, with constant absolute aversion to consumption risk, we have

$$(4.18) \qquad V(C) = e^{-\Gamma C}/(-\Gamma),$$

with the coefficient of absolute risk aversion $\Gamma = -V_{CC}V_C > 0$. For that case it is known that the indirect utility function $\hat{J}(\hat{W},t)$ to the standard problem when all assets are traded is of the form (see Merton (1969))

$$(4.19a) \qquad \hat{J}(\hat{W},t) = e^{-\delta t}e^{A-\Gamma r\hat{W}}/(-\Gamma r),$$

where A is a constant. Hence, by (4.10b) and (4.19a) we can conclude that for the problem when the claim to income is not traded, the indirect utility function is

$$(4.19b) \qquad J(W,t) = e^{-\delta t}e^{A-\Gamma r[W + F(s)]}/(-\Gamma r).$$

Substitution in the Bellman equation (A.1) in the appendix gives

$$(4.19c) \qquad A = (r - \delta - \nu_q{}'\sigma_{qq}^{-1}\nu_q/2)/r.$$

From (4.19b) the absolute tolerance to wealth risk is

$$(4.20) \qquad -J_W/J_{WW} = 1/\Gamma r.$$

and the tangency portfolio is

$$(4.21) \qquad X^t = \sigma_{qq}^{-1}\nu_q/\Gamma r,$$

independent of wealth and the state variable. From (2.14a) and (4.18) it follows that the consumption function is

$$(4.22) \qquad C^*(W,s) = V_C^{-1}(e^{\delta t}J_W) = -\log(e^{\delta t}J_W)/\Gamma = -A/\Gamma + r[W + F(s)].$$

Second, let the instantaneous utility function have constant relative aversion to consumption risk,

$$(4.23) \qquad V(C) = C^{1-\gamma}/(1 - \gamma).$$

with the coefficient of relative risk aversion $\gamma = -CV_{CC}/V_C > 0$. The indirect utility function for the standard problem with all assets traded and no state variable is of the form (see Merton (1969))

(4.24a) $\hat{J}(\hat{W},t) = e^{-\delta t} A \hat{W}^{1-\gamma}/(1-\gamma)$.

Hence, the indirect utility function for the problem when the claim to income is not traded is

(4.24b) $J(W,t) = e^{-\delta t} A[W + F(s)]^{1-\gamma}/(1 - \gamma)$.

Substitution in the Bellman equation (A.1) reveals that

(4.24c) $A = \{[\delta - (1 - \gamma)(r + \nu'\sigma_{qq}^{-1}\nu/2\gamma)]/\gamma\}^{-\gamma}$.

This solution is meaningful only if A is positive. Hence the rate of time preference must fulfill the condition

(4.24d) $\delta > \max[0,\ (1-\gamma)(r + \nu'\sigma_{qq}^{-1}\nu/2\gamma)]$.

From (4.24a) the absolute tolerance to wealth risk is

(4.25) $-J_W/J_{WW} = [W + F(s)]/\gamma$,

and the tangency portfolio is

(4.26) $X^t(W,s) = \sigma_{qq}^{-1}\nu_q[W + F(s)]/\gamma$,

proportional to comprehensive wealth. The consumption function is

(4.27) $C^*(W,s) = V_C^{-1}(e^{\delta t}J_W) = (e^{\delta t}J_W)^{-1/\gamma} = A^{-1/\gamma}[W + F(s)]$.

The solution presented in Modest (1984) is a special case of this.

The discussion above has been conducted in terms of the implicit value F of the claim to nontraded income. Given that F has been calculated, we can then use (3.4) and (3.6) to calculate the drift and standard deviation for the implicit rate of return on the claim to nontraded income.

B. No State Variable

We shall now consider the situation when nontraded income is not spanned by the traded assets. We assume throughout this section that there is no state variable. That is, the drift and standard deviation vector of income are constant,

(4.28) $\mu_y(s) = \mu_y$ and $S_y(s) = S_y$.

That income is not spanned means that the income hedge portfolio is no longer a perfect hedge against income, that the standard deviation vector S_y is no longer a linear combination of the standard deviation matrix S_q, and that the unhedgeable variance (4.4a) of income conditional upon the set of traded assets is positive,

$$\sigma_{y|q} > 0. \tag{4.29}$$

If there is no perfect hedge to income, we cannot use an arbitrage relation like (4.7) to derive a partial differential equation for the value of a claim to income. Instead we will use the implicit first-order condition for F, that is, the first-order condition when the relevant Bellman equation for the problem when also claims to income are traded is maximized with respect to the amount of comprehensive wealth held in claims to income. For completeness we first derive the partial differential equation for the case when there is a state variable. The first-order condition for F is then (see the derivation of (F.5) in appendix F)

$$\hat{J}_{\hat{W}}\nu_F + \hat{J}_{\hat{W}\hat{W}}(\sigma_{Fq}X + \sigma_{FF}F) + \hat{J}_{\hat{W}s}\sigma_{Fs} = 0, \tag{4.30}$$

where $\nu_F = \mu_F - r$ is the expected excess return on claims to income.

We note that the term in parenthesis can be written $\sigma_{F\hat{W}}$, we use (3.6) to express $\sigma_{F\hat{W}}$ and σ_{Fs} in (4.30), and we combine the result with (3.4) to eliminate $F\mu_F$ (see appendix F). This gives the Fundamental Valuation Equation[11]

$$\begin{aligned}
&\tfrac{1}{2}F_{\hat{W}\hat{W}}\sigma_{\hat{W}\hat{W}} + F_{\hat{W}s}\sigma_{\hat{W}s} + \tfrac{1}{2}F_{ss}\sigma_{ss} \\
&\quad + F_{\hat{W}}[\mu_{\hat{W}} - C^* - (-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}})\sigma_{\hat{W}\hat{W}} - (-\hat{J}_{\hat{W}s}/\hat{J}_{\hat{W}})\sigma_{\hat{W}s}] \\
&\quad + F_s[\mu_s - (-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}})\sigma_{\hat{W}s} - (-\hat{J}_{\hat{W}s}/\hat{J}_{\hat{W}})\sigma_{ss}] \\
&\quad + F_t - Fr \\
&\quad + \mu_y - (-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}})\sigma_{y\hat{W}} - (-\hat{J}_{\hat{W}s}/\hat{J}_{\hat{W}})\sigma_{ys} = 0.
\end{aligned} \tag{4.31}$$

This is the partial differential equation that needs to be solved. Since we assume that

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[11] See Cox, Ingersoll and Ross (1985). Equation (4.31) differs from equation (31) in Cox, Ingersoll and Ross since here dividends dy are instantaneously risky instead of instantaneously riskless, which adds the last two terms in (4.31). Also, Cox, Ingersoll and Ross assume that there is no net supply of the riskless asset and that the riskless rate of return is endogenous. Assuming $X_0 = 0$ implies $\hat{W}r(\hat{W},s,t) = \mu_{\hat{W}} - (-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}})\sigma_{\hat{W}\hat{W}} - (-\hat{J}_{\hat{W}s}/\hat{J}_{\hat{W}})\sigma_{\hat{W}s}$ which makes the term multiplying $F_{\hat{W}}$ equal to $\hat{W}r - C^*$, as in Cox, Ingersoll and Ross's formulation.

there is no state variable, all terms involving s drop out. Let us also assume that the utility function has constant absolute aversion to consumption risk, (4.18). Then we can guess that F is independent of $\hat{W}$ and hence constant. That gives

(4.32) $$F = [\mu_y - (-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}})\sigma_{y\hat{W}}]/r.$$

With constant absolute aversion to consumption risk, the absolute aversion to wealth risk is

(4.33) $$(-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}}) = \Gamma r.$$

From this and from the rewriting $\sigma_{y\hat{W}}$ (see appendix G) it follows that the value of a claim to income fulfills

(4.34) $$F = [\mu_y - \sigma_{yq}\sigma_{qq}^{-1}\nu_q - \Gamma r\sigma_{y|q}]/r.$$

From (4.33) and (F.2) the optimal portfolio will fulfill

(4.35) $$X^* = \sigma_{qq}^{-1}\nu_q/\Gamma r - \sigma_{qq}^{-1}\sigma_{qy} = X^t + X^{hy}.$$

The indirect utility function is of the same form as (4.19a), namely

(4.36a) $$J(W,t) = e^{-\delta t}e^{A - \Gamma r(W + F)}/(-\Gamma r).$$

but the constant A now reflects also the unhedgeable variance in income,

(4.36b) $$A = [r - \delta - \nu_q'\sigma_{qq}^{-1}\nu_q/2 - (\Gamma r)^2\sigma_{y|q}/2]/r.$$

The consumption function is then

(4.36c) $$C^*(W) = -A/\Gamma + r(W + F).$$

Both (4.32) and (4.34) are intuitive. We see in (4.32) that F is the expected income less the covariance between income and comprehensive wealth weighed with the absolute aversion to wealth risk, everything discounted by the riskless rate of return. In (4.34) we see that F is the expected income plus the expected excess return on the income hedge portfolio (the first two terms in the bracket) less the unhedgeable variance $\sigma_{y|q}$ weighted with the absolute aversion to wealth risk, everything discounted by the riskless rate. This latter expression is also derived by Losq (1978), by directly solving the Bellman equation. We see that if income is spanned, and hence the unhedgeable variance $\sigma_{y|q}$ is zero, (4.34) collapses to (4.17a).

If income is uncorrelated with the rates of return on traded assets, the income hedge portfolio is zero. There is no effect on the portfolio of traded risky assets of the existence on nontraded income. There is a wealth effect though, and consumption and holdings of the riskless traded asset are affected.

Given that the value of the claim to nontraded income has been calculated, we can use (3.4) and (3.6) to compute the drift and standard deviation of the implicit rate of return on the claim.

For the finite horizon problem when utility of terminal wealth is maximized, and for the special case with no state variable and constant absolute risk aversion, Duffie and Jackson (1988, Case 2) derive a portfolio of risky assets which is essentially the portfolio X^* in (4.35) multiplied with the term $e^{-r(T-t)}$, where T is the horizon. Hence, that portfolio of risky assets approaches zero when the horizon T becomes large. Obviously, the results from finite horizon terminal wealth maximization cannot directly be meaningfully extended to the case with an infinite horizon and consumption at each date.

C. Income Instantaneously Riskless

We now assume that income is instantaneously riskless, that is,[12]

(4.37) $\quad S_y(s) = 0.$

We assume that the drift in income is linear in the state variable, and that the drift and standard deviation vector of the state variable are constant, that is (4.9a) and (4.9b). When income is instantaneously riskless, the Bellman equations (2.11) and (A.1) are simplified since all terms involving variances and covariances of income vanish. The optimal portfolio is simplified since there is no income-hedge portfolio. In the fundamental valuation equation (4.31) the last two terms vanish.[13]

Even if nontraded income is instantaneously riskless, the nontraded claim to income is instantaneously risky, since implicit capital gains will be instantaneously risky.

[12] Fischer (1974, 1975) and Breeden (1979) implicitly assume that the exogenous income is instantaneously rikless.

[13] As mentioned, Cox, Ingersoll and Ross (1985) derive the fundamental valuation equation for an asset with an instantaneously riskless dividend.

Let us restrict the utility function to have constant absolute aversion to consumption risk, (4.18). Then we may assume that the value of the claim to income is independent of comprehensive wealth, which simplifies the fundamental valuation equation. After some manipulations (see appendix H) the valuation equation can be written

(4.38) $\quad \frac{1}{2}F_{ss}\sigma_{ss} - F_s^{\,2}(-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}})\sigma_{s|q} + F_s[\mu_s - \sigma_{sq}\sigma_{qq}^{-1}\nu_q] - Fr + \bar{\mu}_y s = 0.$

We guess that F is linear in s, use (4.33), and get the solution

(4.39) $\quad F(s) = \bar{\mu}_y s/r + [(\bar{\mu}_y/r)\mu_s - (\bar{\mu}_y/r)\sigma_{sq}\sigma_{qq}^{-1}\nu_q - \Gamma r(\bar{\mu}_y/r)^2\sigma_{s|q}]/r.$

Again it makes sense that F should be roughly proportional to the state variable for large positive and negative values of the state variable, since the relative variance of the state variable is then small. I conjecture, although I have not been able to provide a formal argument, that this asymptotic property is enough to exclude other solutions to (4.38).[14]

Given the solution (4.39), the indirect utility function will be

(4.40a) $\quad J(W,s,t) = e^{-\delta t}e^{A - \Gamma r[W + F(s)]}/(-\Gamma r)$, with

(4.40b) $\quad A = [r - \delta - \nu_q{}'\sigma_{qq}^{-1}\nu_q/2 - (\Gamma\bar{\mu}_y)^2\sigma_{s|q}/2]/r.$

The consumption function is

(4.40c) $\quad C^*(W,s) = -A/\Gamma + r[W + F(s)].$

Since the ratio of the absolute tolerance to wealth risk to the absolute tolerance to state variable risk is

(4.41) $\quad J_{Ws}/J_{WW} = F_s = \bar{\mu}_y/r,$

from (2.14b) the optimal portfolio is

(4.42) $\quad X^* = \sigma_{qq}^{-1}\nu_q/\Gamma r - (\bar{\mu}_y/r)\sigma_{qq}^{-1}\sigma_{qs} = X^t + X^{hs} = X^t + X^{hF}.$

From (4.39) we see that the value of the claim again has two components. The first component is the value of a riskless income stream $\bar{\mu}_y s$, which equals $\bar{\mu}_y s/r$, the first term in (4.39). The second component is the value of a risky state variable, equivalent to

[14] We want to exclude solutions with a non-zero second-order derivative F_{ss}. It is easy to show that any polynomial solution will be linear. I have been unable to find a general solution to (4.38), which has the special property of having a term which is quadratic in the first derivative.

an asset with expected return $(\bar{\mu}_y/r)\mu_s$ and standard deviation $(\bar{\mu}_y/r)S_s$, which value is the present value of the expected return on the state variable and on the state-variable hedge portfolio, $(\bar{\mu}_y/r)(\bar{\mu}_s - \sigma_{sq}\sigma_{qq}^{-1}\nu_q)/r$, less the present value of the unhedgeable return variance of s, $(\bar{\mu}_y/r)^2\sigma_{s|q}/r$, weighed by the absolute aversion to wealth risk, Γr.

From (4.42) we see that the state variable hedge portfolio and the F hedge portfolio equal, as they should, the negative of the product of σ_{qq}^{-1} and the covariance between traded asset returns and the return $(\bar{\mu}_y/r)ds$ on the state variable. There is, as mentioned, no income hedge portfolio when income is instantaneously riskless.

If the state variable is uncorrelated with the rates of return on traded assets, there is no state variable and F hedge portfolio. There is no effect of the existence of nontraded income on the portfolio of traded risky assets, but there is a wealth effect and an effect on consumption and holdings of the riskless traded asset.

Again, the drift and standard deviation of the implicit return on the claim to nontraded income can be calculated from (3.4) and (3.6).

V. CONCLUSIONS

Let us first summarize the results. We have extended the small previous literature on portfolio choice with income from nontraded assets (nontraded income, for short) by relating the portfolio problem with nontraded asset to the problem of pricing a nontraded asset, and by including a state variable that affects the nontraded income. We have seen that the existence of nontraded assets clearly affects the optimal portfolio of traded assets as well as the indirect utility and consumption functions.

In the general formulation of the problem we have found it helpful to distinguish a nontraded-income hedge portfolio from the ordinary state-variable hedge portfolios. This nontraded-income hedge portfolio is the portfolio whose return has the maximum negative correlation with the nontraded income. We have seen that the optimal portfolio of traded assets can be written as a linear combination of the riskless asset, the tangency portfolio, the income hedge portfolio, and the state-variable hedge portfolio corresponding to the state variable that effects the nontraded income.

We have also derived a variant of the fundamental valuation equation of Cox, Ingersoll and Ross (1985) for a claim to an instantaneously risky nontraded income with and without spanning. This valuation equation is of independent interest, but it is also helpful in interpreting the solution to the optimal portfolio problem. With nontraded assets the indirect utility and consumption functions are modified by the addition of the implicit value of the non traded asset to value of traded asset. When the nontraded income is not spanned by traded asset, the "unhedgeable" variance of the nontraded income also enters these functions.

Our analysis has also covered the symmetrical case when the claim to nontraded income has a negative value and is actually a liability, since income and the state variable may become negative in the model.

The implicit value of a claim to a given exogenous stochastic nontraded income is the value of the claim for the owner of the claim. Hence, except under spanning the value differs between investors/consumers and depends on the preferences of the owner, his holdings of other nontraded assets, etc. Except under spanning, the implicit value of the claim is not the value of the claim that would result under free trade in the claim, since other investors/consumers generally value the claim differently. As mentioned above, in the literature on optimal hedging against future cash payments Stulz (1984) and Adler and Detemple (1988) take the stochastic process for the value of a nontraded claim to that future cash payment to be exogenously given to the holder of the claim, which as far as I can see makes sense only if the claim is spanned by traded assets.

We have reported explicit analytical solutions to the portfolio problem and the pricing problem for three special cases, namely (A) the nontraded income is spanned by the risky assets and there is a state variable affecting it, (B) the nontraded income is not spanned, there is no state variable, and the instantaneous utility function has constant absolute risk aversion, and (C) the nontraded income is instantaneously riskless, there is a state variable, and the instantaneous utility function has constant absolute risk aversion.

There are some obvious limitations in the analysis. These limitations also provide natural directions for further research. Throughout we have assumed the existence of a riskless asset with an exogenously given constant rate of return, which gives very simple expressions for the implicit price of the nontraded asset. Without this assumption, there is an endogenously determined instantaneously riskless rate of return which equilibrates the market at zero net supply of the instantaneously riskless asset, as shown in Cox, Ingersoll and Ross (1985). With this endogenous no longer constant riskless rate of return it seems that the solutions to the fundamental valuation equation for the implicit price of the nontraded asset would be a more complicated integral.

The restriction to constant absolute risk aversion in cases (B) and (C) implies that the price of the nontraded asset is independent of wealth. This is because with constant absolute risk aversion, the demand for risky assets is independent of wealth. One way to understand why it is difficult to find a solution for other utility functions, for instance with constant relative risk aversion, is that then the price of the nontraded asset will depend on wealth, since then demand for risky assets is proportional to wealth (which, of course, seems more plausible than demand for risky assets independent of wealth). Then the terms in the fundamental valuation equation involving derivatives with respect to wealth are not zero, and the equation is much more difficult to solve.[15] These difficulties do not arise in the spanning case, since then the price of the nontraded asset can be assumed to be independent of wealth anyhow.[16]

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[15] Equivalently, the implicit rate of return on the nontraded asset is a complicated stochastic process with wealth as a state variable and far from a simple Brownian motion.

Cox and Huang (1987) demonstrate a new and more powerful method of solving continuous-time optimum portfolio problems, which involve formulating a linear partial differential equation instead of the nonlinear partial differential equation that results with the traditional method used by Merton (1971). Their method depends critically upon an assumption that the number of risky assets is equal to the number of underlying independent Wiener processes that describe the uncertain environment. That assumption is definitely violated in the case when a nontraded asset is not spanned by traded assets, and it therefore seems that their method cannot be applied in that case.

[16] Losq (1978) does present an analytic solution to the case when there is constant relative risk aversion but only under the awkward assumption that the drift and standard deviation of the nontraded income are proportional to wealth.

Fischer (1974) presents an analytic solution to the case with constant relative risk aversion when the nontraded income is instantaneously riskless but proportional to a state

As to the empirical importance of the effects we have studied, it is of course in principle possible to estimate the crucial variances and covariances. Of particular interest would be the covariance between nontraded income and rates of return on traded assets. Fama and Schwert (1977) examine how the existence of nontraded human capital empirically affects the security market line in Mayers's CAPM model. Their finding for US data is that the effect on the security market line is small and that the correlation between measures of returns on human capital and returns on an aggregate market portfolio is weak, but that this does not exclude that the correlation may be strong for particular assets. As they emphasize there are numerous measurement problems involved. We can easily think of examples of returns on specific human capital strongly correlated with returns on specific assets, for instance wages and stock prices in a given industry. I am not aware of attempts to estimate correlations between nontraded income and returns on traded assets for countries, governments, and other agents. It would clearly be very interesting to see such estimates, in particular of to what extent nontraded income is spanned by traded assets.

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variable that follows a geometric Brownian motion. In our notation, $S_y = 0$, $\mu_y(s) = s$, $\mu_s(s) = \bar{\mu}_s s$ and $S_s(s) = \bar{S}_s s$. One can infer that the suggested solution is of the form (in our notation) $J(W,s,t) = e^{-\delta t}A(W + s/B)^{1-\gamma}/(1-\gamma)$. As far as I can see this solution works only under the assumption that the nontraded income is spanned by the traded assets. Then B is a constant, which gives the optimal portfolio $X^* = X^t + X^{hs} = \sigma_{qq}^{-1}\nu_q(W + s/B)/\gamma - (1/B)\sigma_{qq}^{-1}\bar{\sigma}_{qs}s$ and the consumption function $C^* = (e^{\delta t}J_W)^{-1/\gamma} = A^{-1/\gamma}(W + s/B)$. This corresponds to Fischer's equations (61), (62) and (66). Without spanning, B is <u>not</u> a constant but a function of s/W, according to Fischer's equation (67). Then the derivatives J_W and J_{WW} have more terms, and the suggested solution will not work.

APPENDIX

A. The Partial Differential Equation for J(W,s,t)

Substitution of (2.14a) and (2.14b) into (2.11) gives

$$
\begin{aligned}
\text{(A.1)} \qquad 0 = {} & U(U_C^{-1}(J_W)) + J_t + J_W[Wr + \mu_y - \sigma_{yq}\sigma_{qq}^{-1}\nu_q] - J_W U_C^{-1}(J_W) \\
& - \tfrac{1}{2}(J_W{}^2/J_{WW})\nu_q{}'\sigma_{qq}^{-1}\nu_q + \tfrac{1}{2}J_{WW}[\sigma_{yy} - \sigma_{yq}\sigma_{qq}^{-1}\sigma_{qy}] \\
& + J_s\mu_s \\
& + J_{Ws}(\sigma_{ys} - \sigma_{yq}\sigma_{qq}^{-1}\sigma_{qs}) \\
& - (J_W J_{Ws}/J_{WW})\sigma_{sq}\sigma_{qq}^{-1}\nu_q - \tfrac{1}{2}(J_{Ws}{}^2/J_{WW})\sigma_{sq}\sigma_{qq}^{-1}\sigma_{qs} \\
& + \tfrac{1}{2}J_{ss}\sigma_{ss}.
\end{aligned}
$$

B. Differentiating $F(\hat{W},s,t)$

Applying Ito's lemma to express the differential dF in (3.3) gives

$$
\begin{aligned}
\text{(B.1)} \qquad dF(\hat{W},s,t) &= F_t dt + F_{\hat{W}} d\hat{W} + F_s ds \\
&\qquad + \tfrac{1}{2}F_{\hat{W}\hat{W}}(d\hat{W})^2 + F_{\hat{W}s}d\hat{W}ds + \tfrac{1}{2}F_{ss}(ds)^2 \\
&= [F_t + F_{\hat{W}}(\mu_{\hat{W}} - C^*) + F_s\mu_s + \\
&\qquad + \tfrac{1}{2}F_{\hat{W}\hat{W}}\sigma_{\hat{W}\hat{W}} + F_{\hat{W}s}\sigma_{\hat{W}s} + \tfrac{1}{2}F_{ss}\sigma_{ss}]dt \\
&\qquad + [F_{\hat{W}}S_{\hat{W}} + F_s S_s]d\omega,
\end{aligned}
$$

where we use the notation

(B.2) $\quad d\hat{W} = (\mu_{\hat{W}} - C^*)dt + S_{\hat{W}}d\omega,$ that is,

(B.3a) $\quad \mu_{\hat{W}} = X^{*\prime}\nu_q + FS_F + \hat{W}r$ and

(B.3b) $\quad S_{\hat{W}} = X^{*\prime}S_q + FS_F.$

C. Demonstration in IV.A that F is spanned by traded assets

From (3.2), (3.3) and (B.1) we see that the standard deviation vector of the rate of return on the claim will fulfill

(C.1) $\quad FS_F = S_y + F_{\hat{W}}S_{\hat{W}} + F_sS_s = S_y + F_{\hat{W}}(X^{*\prime}S_q + FS_F) + F_sS_s,$ hence,

(C.2) $\quad FS_F = (S_y + F_{\hat{W}}X^{*\prime}S_q + F_sS_s)/(1 - F_{\hat{W}}).$

If $F_{\hat{W}} \neq 1$ (we shall indeed see that $F_{\hat{W}} = 0$ under spanning), it follows from (4.6) that

S_F is a linear combination of S_q, since S_y and S_s are linear combinations of S_q. Hence, the claim to income is spanned by the traded assets.

D. Derivation of (4.8)

Combining (3.4) and (4.7) gives

(D.1) $$\tfrac{1}{2}F_{\hat{W}\hat{W}}\sigma_{\hat{W}\hat{W}} + F_{\hat{W}s}\sigma_{\hat{W}s} + \tfrac{1}{2}F_{ss}\sigma_{ss} + F_{\hat{W}}(\mu_{\hat{W}} - C^*) + F_s\mu_s + F_t - Fr$$
$$+ \mu_y + X^{hF}{}'\nu_q = 0.$$

The hedge portfolio X^{hF} depends on F and its derivatives, however, via $\sigma_{qF}F$ in (4.6). Thus we use (3.6) to write

(D.2) $$\sigma_{qF}F = S_qS_F{}'F = S_q(S_y + F_{\hat{W}}S_{\hat{W}} + F_sS_s)' = \sigma_{qy} + F_{\hat{W}}\sigma_{q\hat{W}} + F_s\sigma_{qs}.$$

Substituting in (4.6) and then in (D.1) gives (4.8).

E. Eliminating "extraneous" solutions to the Fundamental Valuation Equation

The homogeneous solution to the second order linear ordinary differential equation resulting from (4.8) after elimination of the derivatives with respect to wealth and time is of the form

(E.1) $$F^H = A_1\exp(\lambda_1 s) + A_2\exp(\lambda_2 s),$$

where A_1 and A_2 are constants, and λ_1 and λ_2 are the roots (assumed distinct) of the characteristic equation. It seems that an argument along the following lines implies that both A_1 and A_2 are zero. Suppose at least one of the roots, λ_1, say, has a positive real part. Under (4.9) it makes economic sense that for large values of the state variable, F should be approximately proportional to the state variable, since the variance of the state variable then is small relative to the value of the state variable. Then we must have A_1 equal to zero. Suppose at least one of the roots, λ_2, say, has negative real part. In the model, the state variable has not been restricted to be nonnegative. Actually, under (4.9b) it is normally distributed and can take negative values. Neither have we restricted the nontraded income to be nonnegative (or allowed for free disposal if it becomes negative). Hence, our treatment of nontraded income is symmetrical in that it may be positive or negative, and the nontraded asset may actually be a liability, corresponding to negative values of F. Given this, we can consider what happens to F when the state variable takes large negative values. It then again makes economic sense that F should be roughly proportional to the state variable also for large negative values of the state variable. Then A_2 must be equal to zero as well.

F. Derivation of the Fundamental Valuation Equation (4.31)

When claims to income are freely traded, let F denote the amount of comprehensive wealth $\hat{W}$ held in claims to income. That is, the budget constraint is

$$X_0 + 1'X + F = \hat{W}. \tag{F.1}$$

Comprehensive wealth will develop according to

$$d\hat{W} = [X'\nu_q + F\nu_F + \hat{W}r + \mu_y - C]dt + [X'S_q + FS_F + S_y(s,t)]d\omega. \tag{F.2}$$

The first-order conditions with respect to C, X, and F of the relevant Bellman equation will be

$$U_C = J_{\hat{W}} = 0 \text{ and} \tag{F.3}$$

$$J_{\hat{W}}\nu_q + J_{\hat{W}\hat{W}}(\sigma_{qq}X + \sigma_{qF}F) + J_{\hat{W}s}\sigma_{qs} = 0. \tag{F.4}$$

$$J_{\hat{W}}\nu_F + J_{\hat{W}\hat{W}}(\sigma_{Fq}X + \sigma_{FF}F) + J_{\hat{W}s}\sigma_{Fs} = 0. \tag{F.5}$$

First, it follows from (F.4) that the optimal portfolio X^* can be written

$$X^* = (-J_{\hat{W}}/J_{\hat{W}\hat{W}})\sigma_{qq}^{-1}\nu_q - \sigma_{qq}^{-1}\sigma_{qF}F - (J_{\hat{W}s}/J_{\hat{W}\hat{W}})\sigma_{qq}^{-1}\sigma_{qs} \tag{F.6}$$

Second, we note that the term in parenthesis in (F.5) is $\sigma_{F\hat{W}}$. Using (3.6) to rewrite $\sigma_{F\hat{W}}$ and σ_{Fs} in (F.5), and by identifying $\sigma_{\hat{W}\hat{W}}$, $\sigma_{\hat{W}s}$ and $\sigma_{\hat{W}y}$, we can manipulate (F.5) to read

$$\begin{aligned} F\nu_F = F(\mu_F - r) = \\ &= F_W[((-J_{\hat{W}\hat{W}}/J_{\hat{W}})\sigma_{\hat{W}\hat{W}} + (-J_{\hat{W}s}/J_{\hat{W}})\sigma_{\hat{W}s}] \\ &\quad + F_s[(-J_{\hat{W}\hat{W}}/J_{\hat{W}})\sigma_{\hat{W}s} + (-J_{\hat{W}s}/J_{\hat{W}})\sigma_{ss} \\ &\quad + (-J_{\hat{W}\hat{W}}/J_{\hat{W}})\sigma_{\hat{W}y} + (-J_{\hat{W}s}/J_{\hat{W}})\sigma_{sy}. \end{aligned} \tag{F.7}$$

Combining (F.7) with (3.4) to eliminate $F\mu_F$ gives (4.31).

We note in passing that (F.7), of course, is fully consistent with the usual equation for the security market line. For when there is no state variable, for an asset whose price is independent of wealth (F.7) implies

$$\text{(F.8)} \qquad \nu_F = (-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}})\sigma_{\hat{W}y}/F.$$

Similarly, for $F = \hat{W}$ we get

$$\text{(F.9)} \qquad \nu_{\hat{W}} = (-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}})\sigma_{\hat{W}\hat{W}}/\hat{W}.$$

Eliminating $(-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}})$ from (F.7) and (F.8) we get

$$\text{(F.10)} \qquad \nu_F = \beta_F \nu_{\hat{W}}, \text{ with}$$

$$\text{(F.11)} \qquad \beta_F = \mathrm{Cov}[dy/F, d\hat{W}/\hat{W}]/\mathrm{Var}[d\hat{W}/\hat{W}],$$

the equation for the security market line.

G. Rewriting $\sigma_{y\hat{W}}$ in (4.32)

We have

$$\text{(G.1)} \qquad \sigma_{y\hat{W}} = \sigma_{yq}X^* + \sigma_{yF}F = (-\hat{J}_{\hat{W}}/\hat{J}_{\hat{W}\hat{W}})\sigma_{yq}\sigma_{qq}^{-1}\nu_q + \sigma_{y|q},$$

where we have used

$$\text{(G.2)} \qquad X^* = (-\hat{J}_{\hat{W}}/\hat{J}_{\hat{W}\hat{W}})\sigma_{qq}^{-1}\nu_q - \sigma_{qq}^{-1}\sigma_{qy}, \text{ and}$$

$$\text{(G.3)} \qquad \sigma_{yF}F = S_y S_F' F = S_y(S_y' + F_{\hat{W}}S_{\hat{W}}' + F_s S_s') = \sigma_{yy}$$

(since there is no state variable and F does not depend on $\hat{W}$).

H. Derivation of (4.38)

When the utility function has constant absolute aversion to consumption risk, (4.18), we may assume that the value of the claim to income is independent of comprehensive wealth. Then the fundamental valuation equation (4.31) simplifies to

$$\text{(H.1)} \qquad \tfrac{1}{2}F_{ss}\sigma_{ss} + F_s[\mu_s - (-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}})\sigma_{s\hat{W}} - (-\hat{J}_{\hat{W}s}/\hat{J}_{\hat{W}})\sigma_{ss}] - Fr + \bar{\mu}_y s = 0.$$

The covariance $\sigma_{s\hat{W}}$ depends on F_s, though. Therefore we write it as

$$\text{(H.2)} \qquad \sigma_{s\hat{W}} = \sigma_{sq}X^* + \sigma_{sF}F = (-\hat{J}_{\hat{W}}/\hat{J}_{\hat{W}\hat{W}})\sigma_{sq}\sigma_{qq}^{-1}\nu_q - \sigma_{sq}\sigma_{qq}^{-1}\sigma_{qF}F$$
$$- (\hat{J}_{\hat{W}s}/\hat{J}_{\hat{W}\hat{W}})\sigma_{sq}\sigma_{qq}^{-1}\sigma_{qs} + F_s\sigma_{ss}',$$

where we have used that the optimal portfolio fulfills

$$\text{(H.3)} \qquad X^* = (-\hat{J}_{\hat{W}}/\hat{J}_{\hat{W}\hat{W}})\sigma_{qq}^{-1}\nu_q - \sigma_{qq}^{-1}\sigma_{qF}F - (\hat{J}_{\hat{W}s}/\hat{J}_{\hat{W}\hat{W}})\sigma_{qq}^{-1}\sigma_{qs}$$

and that

(H.4) $\sigma_{sF}F = F_s\sigma_{ss}$.

By incorporating (H.2) in (H.1) the valuation equation can be written

(H.5) $$\tfrac{1}{2}F_{ss}\sigma_{ss} - F_s^2(-\hat{J}_{\hat{W}\hat{W}}/\hat{J}_{\hat{W}})\sigma_{s|q} + F_s[\mu_s - \sigma_{sq}\sigma_{qq}^{-1}\nu_q - (-\hat{J}_{\hat{W}s}/\hat{J}_{\hat{W}})\sigma_{s|q}]$$
$$- Fr + \bar{\mu}_y s = 0.$$

We can simplify (H.5) further by realizing that $\hat{J}_{\hat{W}s} = 0$. For we know from (3.6) that

(H.6) $FS_F = F_sS_s$,

since $S_y = 0$ and $F_{\hat{W}} = 0$. This means that when the claim to income is traded, it is a perfect hedge against the state variable. That is, the state variable is spanned by the claim to income, and the indirect utility function $\hat{J}(\hat{W},t)$ when claims to income are traded will therefore not depend separately on the state variable, that is, $\hat{J}_{\hat{W}s} = 0$. This gives (4.38).

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