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RISK SHARING EXTERNALITIES

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ABSTRACT

Financial crises typically arise because firms and financial institutions choose balance sheets that expose them to aggregate risk. We propose a theory to explain these risk exposures. We study a financial accelerator model where entrepreneurs can issue state-contingent claims to consumers. Even though entrepreneurs could use these contingent claims to hedge negative shocks, we show that they tend not to do so. This is because it is costly to buy insurance against these shocks as consumers are also harmed by them. This effect is self-reinforcing, as the fact that entrepreneurs are unhedged amplifies the negative effects of shocks on consumers' incomes. We show that this feedback can be quantitatively important and lead to inefficiently high risk exposure for entrepreneurs.

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1 Introduction

The exposure of financial institutions to risks from the subprime mortgage market is widely seen as a root cause of the financial crisis of 2008-2009. This exposure created the potential for shocks in the housing market to be heavily amplified, as recognized early on by Greenlaw, Hatzius, Kashyap, and Shin (2008). Why did banks not do more to protect their balance sheet from these risks, say by shedding some of their riskier positions or by choosing a safer funding structure? More generally, why were these risks not better spread across the economy?

Spurred by the global financial crisis, economists have developed models in which balance sheet losses of financial institutions can negatively affect firms' hiring and investment decisions—for example, Gertler and Kiyotaki (2010), Jermann and Quadrini (2012), He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014). These contributions provide the framework now commonly used to quantify the importance of financial factors over the business cycle, and to design appropriate policy responses. However, these models sidestep the questions raised above, by assuming that the "specialists"—the agents that represent financial institutions—have very limited risk-management tools. In particular, a common assumption in these models is that specialists hold only one risky asset and issue non-state-contingent debt, so that their risk exposure is mechanically linked to their leverage. In this paper, we break this tight link by allowing specialists to issue fully state-contingent debt and we study why they choose to be exposed to aggregate risk, whether this exposure is socially efficient, and if not what is the appropriate policy response.

Our paper makes two contributions. First, we offer a theory of why specialists are exposed to aggregate risk. Our mechanism builds on a general equilibrium effect: when the net worth of specialists falls and the economy experiences a financial crisis, the income of all other agents contracts as well. Due to this feature, insuring these states of the world ex-ante is costly, and this reduces the specialists' incentive to hedge. Second, we show that equilibrium risk-management is sub-optimal from the point of view of social welfare, and we study optimal corrective policies. In our model, specialists issue too much debt indexed to crisis states because they fail to internalize the general equilibrium effect of their choices on aggregate volatility. Optimal policy taxes debt indexed to crisis states, so as to reduce the exposure of specialists to aggregate risk. Interestingly, we show that in our framework simple taxes on leverage—such as those commonly considered in the literature—are not effective in reducing the risk exposure of specialists.

We develop these arguments in the context of a model with two groups of agents, consumers and entrepreneurs. Entrepreneurs are the specialists, and we can think of them

as representing a sector that consolidates financial institutions and the non-financial firms that borrow from them. Entrepreneurs borrow from consumers to finance their purchases of factors of production, capital and labor. The source of risk in the economy is a shock that affects the "quality" of capital held by the entrepreneurs, as in Gertler and Karadi (2011) and Brunnermeier and Sannikov (2014). Due to limited commitment, the entrepreneurs face an upper bound on their ability to raise debt from consumers. This implies that reductions in the aggregate net worth of the entrepreneurs can lead to a contraction in economic activity and in the labor income of consumers. This is the general equilibrium effect, or "macro spillover" at the core of our positive and normative results.

Facing the limited commitment constraint mentioned above, the entrepreneurs issue a full set of state-contingent claims. This assumption is meant to capture a variety of ways in which financial institutions can make their balance sheet less exposed to aggregate shocks (for example, by choosing between debt and equity financing, by choosing debt of different maturities, debt denominated in different currencies, taking derivative positions, etc.). By appropriately using state-contingent claims, the entrepreneurs can hedge fluctuations in their net worth. For example, they can promise smaller payments to consumers when the economy is hit by a negative shock. This would imply that consumers bear more aggregate risk, and would stabilize entrepreneurs' net worth. A more stable net worth would then reduce the process of financial amplification in the economy.

We start by studying the positive implications of the model, focusing on the equilibrium allocation of risk between consumers and entrepreneurs and the effects that it has on the stability of entrepreneurs' net worth. We show that the elasticity of entrepreneurs' net worth to aggregate shocks depends on two key model ingredients: the strength of the macro spillover described above, and the risk aversion of consumers. The macro spillover implies that states of the world in which the entrepreneurs have low net worth are also states in which the consumers have low labor income. Risk aversion implies that consumers demand a premium for bearing risk in these states of the world. These two ingredients, combined, make it costly for entrepreneurs to hedge.

We first show this result theoretically, in a special case of our model that is analytically tractable. Next, we show that this mechanism can be quantitatively strong and produce a large exposure of entrepreneurs to aggregate risk. Specifically, under plausible calibrations our economy with state-contingent debt produces an elasticity of entrepreneurial net worth to aggregate shocks and a degree of financial amplification that is quantitatively comparable to those obtained in an economy where entrepreneurs can only issue non-state contingent debt. These results are not driven by the type of aggregate shocks we consider, as we obtain very similar results when the aggregate shock affects the pledgeability of

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capital as in Jermann and Quadrini (2012), rather than the capital stock.

The presence of the macro spillover not only hinders risk sharing between consumers and entrepreneurs, but also generates a pecuniary externality that makes the privately optimal portfolio choices of the agents socially inefficient. To understand the source of this externality, consider the problem of consumers. When choosing their financial assets, they do not understand that any payment received in a given state of the world increases the debt of entrepreneurs, reduces entrepreneurs' net worth and negatively affects the current and future wages of consumers when the collateral constraint binds. Because consumers fail to internalize these negative spillovers, they tend to overvalue payments received in these states relative to what a social planner would do. This pushes down the interest rate for debt instruments indexed to these states, and induces entrepreneurs to take on excessive risk relative to the social optimum.

In the last part of the paper, we study numerically the optimal policy of a social planner that can impose Pigouvian taxes on the state-contingent claims issued by the entrepreneurs. We show that the optimal policy does not tax debt uniformly. Rather, it levies higher taxes on debt instruments indexed to states in which collateral constraints are tighter. These policies are successful in reducing the risk exposure of the entrepreneurs, and the resulting equilibrium features a substantially weaker financial amplification.

We finally contrast the optimal policy with a policy that taxes all debt instruments uniformly. These taxes could reduce the risk exposure of entrepreneurs because they reduce their incentives to issue debt. However, entrepreneurs respond by cutting mostly debt indexed to good states of the world, so their overall risk exposure changes little. That is, these tools are effective in reducing leverage, but they generate an incentive to substitute toward riskier types of debt. These substitution effects provide a cautionary tale for macroprudential tools that target leverage uniformly.

Literature. This paper is related to the large literature on the role of financial factors in the amplification and propagation of aggregate shocks. This literature goes back to the seminal contributions of Bernanke and Gertler (1986), Kiyotaki and Moore (1997) and Bernanke, Gertler, and Gilchrist (1999), and has been very active following the global financial crisis. The logic of financial amplification in these models builds on two main assumptions: the presence of a financial constraint and incomplete financial markets. The first assumption implies that aggregate shocks affecting the net worth of specialists propagate to the rest of the economy, while the second assumption restricts the ability of the specialists to hedge aggregate shocks ex ante.

Important contributions in this literature show that the assumed incompleteness of fi-

nancial markets is critical for financial amplification. Krishnamurthy (2003) introduces state-contingent claims in a three period version of Kiyotaki and Moore (1997) and shows that the amplification mechanism disappears, as specialists perfectly hedge their net worth. Di Tella (2017) shows an analogous result in the context of a dynamic model similar to Brunnermeier and Sannikov (2014). Our incomplete hedging results may appear surprising in light of these contributions. However, as argued above, our results require two ingredients: risk averse consumers and an active macro spillover. One or both of these ingredients are muted in these papers.[1] Other papers that find limited amplification in more quantitative models are Carlstrom, Fuerst, and Paustian (2016), Dmitriev and Hoddenbagh (2017) and Cao and Nie (2017). The mechanism identified in our paper is potentially at work in those models, but—as we discuss in Section 4—their calibrations make it quantitatively weak.

Some papers have proposed other mechanisms to explain why specialists are exposed to aggregate risk even when they can hedge it. Di Tella (2017) shows that the incentive to hedge depends on the source of shocks: aggregate shocks to idiosyncratic volatility lead to imperfect hedging and amplification in his model. Another type of aggregate shock that produces a related result is studied in Dávila and Philippon (2017), who introduce shocks to the degree of financial market completeness. Rampini and Viswanathan (2010) show that specialists do not hedge their net worth if their motive for investing are so strong that their financial constraint binds for every state-contingent claim and if the amount that specialists can pledge does not vary much with the states of the world. Asriyan (2018) introduces information and trading frictions in a three period version of Kiyotaki and Moore (1997) with risk neutral agents and state-contingent claims. He shows that these frictions can distort state prices away from their actuarially fair values so that specialists do not perfectly hedge their net worth in equilibrium.

Our welfare analysis is related to the large literature on inefficiencies and pecuniary externalities in models with financial market imperfections, going back to Geanakoplos and Polemarchakis (1986) and Kehoe and Levine (1993). The pecuniary externality that matters in our model is "distributive"—using the language introduced by Dávila and Korinek (2018)—and works through wages and labor income. This connects our paper to Caballero and Lorenzoni (2014), Bianchi (2016) and Itskhoki and Moll (2019), although we are the first to explore the implications that this type of pecuniary externality has on risk sharing.

A number of papers study models in which constrained inefficiency takes the form of excessive leverage (e.g., Lorenzoni (2008) and Bianchi (2011)) and derive implications for macroprudential policy. See Bianchi and Mendoza (2018) and reference therein. Unlike those papers, we study an environment where the specialists can issue multiple types of

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[1]See also the discussion in Section 3.

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debt rather than just a non-contingent bond. In an open economy setting, Korinek (2018) performs a similar exercise and finds that the optimal policy targets specific types of debt. Our contribution to this debate is to show that, in presence of state-contingency, some simple policies, like a restriction on total leverage, may be ineffective in reducing risk taking or can even backfire and lead to increased risk.

Finally, the macro spillover that plays a central role in this paper was also present in our previous work on self-fulfilling currency crises (Bocola and Lorenzoni, 2019). However, the analysis of how that spillover affects amplification and efficiency is entirely novel to this paper.

Layout. The paper is organized as follows. Section 2 introduces the model. Section 3 studies a special case that is analytically tractable, and characterizes the risk sharing problem between consumers and entrepreneurs. Section 4 presents numerical results in a calibrated version of the full model, while Section 5 discusses the welfare analysis and its implications for the design of macro-prudential policies. Section 6 concludes.

2 Model

We consider an economy populated by two groups of agents of equal size: consumers and entrepreneurs. Entrepreneurs accumulate capital that is used together with labor to produce the final good, and they issue financial claims. Consumers earn labor income and buy financial claims from entrepreneurs. Financial claims are state-contingent promises to repay one unit of consumption in the next period. We now describe the details of the environment and define an equilibrium.

2.1 Environment

Technology and shocks. Time is discrete and indexed by $t = 0, 1, 2 \ldots$. Uncertainty is described by a Markov process that takes finite values in the set $\mathcal{S}$. We denote by s_t the state of the process at time t, and by $s^t = (s_0, s_1, \ldots, s_t)$ the history of states up to period t. The process for s_t is given by the transition matrix $\pi(s_{t+1}|s_t)$.

The capital stock is subject to random depreciation captured by the stochastic parameter u_t. Namely, k_{t-1} units of capital accumulated at the end of time $t-1$ yield $u_t k_{t-1}$ units of capital that can be used in production at time t and a residual stock of $(1-\delta)u_t k_{t-1}$ units of capital after production. The parameter u_t depends on the state of the Markov process

according to the function $u_t = u(s_t)$, and is the only exogenous source of uncertainty in the model. The variable u_t is similar to the capital-quality shock used in Gertler and Karadi (2011) and Brunnermeier and Sannikov (2014).

Entrepreneurs have exclusive access to the technology that allows capital accumulated in period $t - 1$ to be productive in period t, so all capital is held by entrepreneurs in equilibrium. The entrepreneurs use capital and labor services provided by consumers to produce final goods according to the Cobb-Douglas production function:

$$y_t = (u_t k_{t-1})^\alpha l_t^{1-\alpha}.$$

The labor market is perfectly competitive, and the wage rate is w_t. We assume that entrepreneurs need to pay a fraction γ of the wage bill before their revenues are realized. This assumption ensures that the financial conditions of entrepreneurs can have a contemporaneous effect on labor demand, see Jermann and Quadrini (2012).

All equilibrium variables are in general functions of the history s^t, but whenever no confusion is possible we leave this dependence implicit in the subscript t.

Preferences. Entrepreneurs have log-preferences over consumption streams $\{c_{e,t}\}$, so they maximize

$$\mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta_e^t \log(c_{e,t}) \right].$$

Consumers have Epstein-Zin preferences, so their utility is defined recursively as

$$V_t = \left\{ (1 - \beta) x_t^{1-\rho} + \beta \left[\mathbb{E}_t (V_{t+1}^{1-\sigma}) \right]^{\frac{1-\rho}{1-\sigma}} \right\}^{\frac{1}{1-\rho}},$$

where x_t is given by

$$x_t = c_t - \chi \frac{l_t^{1+\psi}}{1 + \psi}.$$

This specification of the consumers' utility eliminates the wealth effect on labor supply as in Greenwood, Hercowitz, and Huffman (1988).

Financial markets and limited commitment. Each period agents trade a full set of one-period state-contingent claims. We let $q(s_{t+1}|s^t)$ be the price at time t of a claim that pays one unit of consumption at $t + 1$, conditional on history $s^{t+1} = (s^t, s_{t+1})$. We denote by $a(s^t)$ the claims held by consumers at the beginning of period t. Similarly, $b(s^t)$ denote the

claims owed by entrpreneurs at the beginning of the period. Market clearing requires that

$$a(s^t) = b(s^t)$$

for every history s^t.

Entrepreneurs enter period t with $u_t k_{t-1}$ units of capital (in efficiency units) and with debt b_t. Each period t is divided in three stages. In the first stage, entrepreneurs hire workers and issue within-period debt to pay for a fraction γ of their wage bill $w_t l_t$. In the second stage, production takes place, goods are sold, entrpreneurs pay the remaining fraction of the wage bill $(1 - \gamma) w_t l_t$ and decide whether to repay their total liabilities $b_t + \gamma w_t l_t$ or to default. If they default, entrepreneurs can hide the firms' profits and a fraction $1 - \theta$ of the undepreciated capital stock and start anew with initial wealth

$$y_t - (1 - \gamma) w_t l_t + (1 - \theta)(1 - \delta) u_t k_{t-1}.$$

In the third and last stage, entrepreneurs issue new liabilities $b(s^{t+1})$ and use these resources along with their net worth to buy capital goods.

Notice that we assume that an entrepreneur who defaults is not excluded from financial markets.[2] It follows that the entrepreneur chooses repayment if and only if

$$y_t - w_t l_t - b_t + (1 - \delta) u_t k_{t-1} \geq y_t - (1 - \gamma) w_t l_t + (1 - \theta)(1 - \delta) u_t k_{t-1}.$$

Making explicit the dependence on the state of the world, this constraint is equivalent to the state-contingent collateral constraint

$$b(s^t) + \gamma w(s^t) l(s^t) \leq \theta(1 - \delta) u(s_t) k(s^{t-1}). \tag{1}$$

2.2 Competitive equilibrium

In a competitive equilibrium, consumers choose sequences for consumption, labor supply and state-contingent claims to maximize their utility subject to the budget constraint

$$c(s^t) + \sum_{s_{t+1}} q(s_{t+1}|s^t) a(s^{t+1}) = w(s^t) l(s^t) + a(s^t)$$

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[2]A similar assumption is made in Rampini and Viswanathan (2010) and Cao, Lorenzoni, and Walentin (2019).

for each history s^t and a no-Ponzi-game condition. The first order condition for $a(s^{t+1})$ takes the form

$$q(s_{t+1}|s^t) = \beta\pi(s_{t+1}|s_t) \left(\frac{x(s^t)}{x(s^{t+1})}\right)^\rho \left(\frac{RW(s^t)}{V(s^{t+1})}\right)^{\sigma-\rho}, \tag{2}$$

where $RW(s^t) = \mathbb{E}_t \left[V(s^t, s_{t+1})^{1-\sigma}\right]^{1/(1-\sigma)}$. Optimal labor supply requires

$$\chi l(s^t)^\psi = w(s^t). \tag{3}$$

Entrepreneurs choose sequences for consumption, capital, labor demand and state-contingent claims to maximize their utility subject to the collateral constraints (1) and their budget constraint

$$c_e(s^t) + k(s^t) = n(s^t) + \sum_{s_{t+1}} q(s_{t+1}|s^t)b(s^{t+1}),$$

where $n(s^t)$ denotes the net worth

$$n(s^t) = y(s^t) - w(s^t)l(s^t) + (1-\delta)u(s_t)k(s^{t-1}) - b(s^t). \tag{4}$$

Denoting by $\mu(s^{t+1})$ the Lagrange multiplier on the collateral constraint in state s^{t+1}, we can write the entrepreneurs' first-order conditions for $b(s^{t+1})$ as

$$q(s_{t+1}|s^t)\frac{1}{c_e(s^t)} = \pi(s_{t+1}|s_t)\left(\beta_e\frac{1}{c_e(s^{t+1})} + \mu(s^{t+1})\right). \tag{5}$$

This condition is a standard intertemporal Euler equation with state-contingent debt and a collateral constraint that limits the amount of claims that an entrepreneur can issue, state by state.

The optimality conditions for labor and capital take the following form:

$$\frac{1}{c_e(s^t)}\left[(1-\alpha)[u(s_t)k(s^{t-1})]^\alpha l(s^t)^{-\alpha} - w(s^t)\right] = \gamma w_t\mu(s^t) \tag{6}$$

$$\frac{1}{c_e(s^t)} = \mathbb{E}_t\left\{\beta_e\frac{1}{c_e(s^{t+1})}\left[\alpha u(s_{t+1})^\alpha\left(\frac{l(s^{t+1})}{k(s^t)}\right)^{1-\alpha} + (1-\delta)u(s_{t+1})\right]\right\} +$$
$$+\mathbb{E}_t[\theta(1-\delta)u(s_{t+1})\mu(s^{t+1})]. \tag{7}$$

The first condition shows that there is a wedge between the marginal product of labor and the real wage if the collateral constraint is binding, because hiring labor requires some

capacity to borrow. The second condition is a standard intertemporal condition for capital accumulation. Relative to a frictionless economy, investing in capital has the additional benefit of relaxing the collateral constraints, which is captured by the last term on the right-hand side.[3]

The advantage of assuming log preferences for entrepreneurs is that their consumption function is linear in net worth, $c_e(s^t) = (1 - \beta_e)n(s^t)$, irrespective of whether the collateral constraint is binding or not. This property, proved in Online Appendix A, simplifies the analysis of the equilibrium.

An equilibrium is given by sequences of quantities $\{c(s^t), c_e(s^t), k(s^t), l(s^t), a(s^t), b(s^t)\}$ and prices $\{w(s^t), q(s_{t+1}|s^t)\}$ such that the quantities solve the individual optimization problems above and markets clear. In Online Appendix A we give a formal definition of a recursive competitive equilibrium and describe the global numerical algorithm that we use to compute it.

2.3 Discussion

Before moving on, let us discuss some of the simplifying assumptions we made.

First, our model does not feature endogenous asset prices, as the price of capital is always 1. This mutes a canonical feed-back between asset prices and entrepreneurial net worth, which may lead to inefficiently high levels of risk taking, as shown for example in Lorenzoni (2008). In the current paper, we abstract from this channel in order to isolate the novel mechanism that works via the endogeneity of labor income. We do not expect endogenous asset prices to substantially change the mechanism investigated here.

Second, the main driving force in the model is a shock to the quality of capital. In our framework, this shock substitutes for the missing volatility of asset prices and allow us to generate sizable movements in the value of assets held by entrepreneurs. As we will see in Section 4, our mechanism does not rely on this specific source of risk, and is still present with different aggregate shocks.

Finally, entrepreneurs and consumers are assumed to be distinct agents, a fairly common assumption in the literature. There are different ways to interpret this assumption. One is to view the entrepreneurs as the controlling shareholders of the financial firms they represent and to interpret all equity financing they raise as part of the state-contingent claims issued. The other one is to interpret the entrepreneurs as all the shareholders of these firms, with consumers being barred from holding shares. In the second interpretation, it would be

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[3]This does not mean that more capital is invested relative to an economy without the collateral constraints because, in equilibrium, the collateral constraints also affect the internal rate of return of entrepreneurs.

interesting to allow for the possibility of issuing shares to all agents, subject to some friction (as for example in Gertler and Kiyotaki (2010)), something we leave to future work.

3   Equilibrium risk sharing and financial amplification

In this section and the next we characterize the risk sharing problem of consumers and entrepreneurs, and show how it affects the economy's response to aggregate shocks. In particular, we study to what extent the effects of the capital quality shocks are amplified due to the presence of the collateral constraint, and how this "financial amplification" depends on the equilibrium allocation of aggregate risk between consumers and entrepreneurs. In this section we consider a simplified version of the model and focus on analytical results. In the next section we go back to the full model and derive numerical results.

Consider a special case of our economy in which all uncertainty is resolved in one period. The economy starts at date 0 with $u_0 = 1$. At $t = 1$ the capital quality u_1 is drawn from a continuous distribution with density $f(u_1)$. From $t = 2$ on, the capital quality is deterministic and equal to $u_t = 1$. We make some additional simplifying assumptions: entrepreneurs and consumers have the same discount factor, $\beta_e = \beta$, the elasticity of intertemporal substitution is infinite, $\rho = 0$, there is no working capital requirement, $\gamma = 0$, and labor supply is inelastic at $l_t = 1$.

Given the assumptions above, we can characterize an equilibrium in two steps. First, we study the equilibrium from date 1 on. Next, we go back to date 0 and study the equilibrium in the market for contingent claims.

3.1   Continuation equilibrium

From date 1 on, the economy follows a deterministic path. Since there is no uncertainty and $\rho = 0$, the interest rate is constant and equal to $1/\beta - 1$. Given that there is no working capital constraint, firms are unconstrained in hiring labor and wages are equal to the marginal product of labor

$$w_t = (1 - \alpha)(u_t k_{t-1})^\alpha.$$

The dynamics of k_t and n_t are characterized as follows. For a finite number of periods J, the collateral constraint is binding and the dynamics of investment and net worth are:

$$k_t = \frac{\beta n_t}{1 - \beta\theta(1 - \delta)}, \qquad n_{t+1} = \alpha k_t^\alpha + (1 - \delta)(1 - \theta)k_t. \tag{8}$$

The first equation comes from the fact that entrepreneurs save a fraction β of their wealth and can lever it at most by the factor $1/[1 - \beta\theta(1 - \delta)]$. The second is obtained by combining the definition of net worth in equation (4), the wage derived above, and the binding collateral constraint (1). After J periods the collateral constraint is slack, n_t stays constant in all following period, and the capital stock reaches the first-best level, that is, the level it would achieve if entrepreneurs did not face financial constraints,

$$k^* = \left(\frac{\alpha\beta}{1 - \beta(1 - \delta)} \right)^{1/(1-\alpha)}. \tag{9}$$

The number of periods J that the economy spends in the constrained region depends on the initial value of n_1: if n_1 is above the threshold $n^* = k^*[1 - \beta\theta(1 - \delta)]/\beta$, then the entrepreneur has enough resources to finance k^*. Thus, $J = 0$ and the economy reaches the first-best allocation in period 1. Else, $J > 0$, and the economy evolves according to (8) until net worth exceeds n^*.

Given $\rho = 0$, we can use the consumers' intertemporal budget constraint to compute the expected utility of consumers at $t = 1$:

$$V_1 = (1 - \beta) \left(a_1 + \sum_{t=1}^{\infty} \beta^{t-1} w_t \right).$$

The expected present value of labor income in this expression can be split in two parts: $w_1 = (1 - \alpha)(u_1 k_0)^\alpha$, and $W \equiv \sum_{t=2}^{\infty} \beta^{t-1}(1 - \alpha)k_{t-1}^\alpha$. In the continuation equilibrium described above, W is only a function of n_1. In particular, a higher value of initial net worth n_1 implies a (weakly) higher path for the capital stock and, therefore, a (weakly) higher path of wages for consumers.

The next lemma summarizes these properties of the continuation equilibrium.

Lemma 1 (Continuation equilibrium). *There is a unique continuation equilibrium that only depends on the state variables k_0, u_1, n_1 and does not depend on the parameter σ. In the continuation equilibrium the collateral constraint is binding for a finite number of periods J, with $J = 0$ iff $n_1 \geq n^*$. The present value of wages is given by*

$$w_1 + W(n_1)$$

where $W(n_1)$ is strictly increasing for $n_1 < n^*$ and constant for $n_1 \geq n^*$.

3.2 Risk sharing at date 0

The continuation equilibrium depends on entrepreneurs' net worth at date $t = 1$. This, in turn, is determined by the portfolio choices of consumers and entrepreneurs at date $t = 0$ which, for the purposes of this section, we denote by $a_1(u_1)$ and $b_1(u_1)$.

Let us focus on the special case in which the collateral constraint $b_1(u_1) \leq \theta(1 - \delta)u_1 k_0$ is not binding at date 0, so $\mu_1(u_1) = 0$ for all u_1. The optimality conditions for state-contingent claims (2) and (5) can then be combined with the market clearing condition $a_1(u_1) = b_1(u_1)$ to obtain

$$\left(\frac{RW_0}{(1 - \beta)[b_1(u_1) + w_1(u_1) + W(n_1(u_1))]} \right)^\sigma = \frac{n_0}{n_1(u_1)} \qquad \text{for all } u_1, \tag{10}$$

where we use the proportionality between consumption and net worth for entrepreneurs and the expression for net worth at date 1,

$$n_1(u_1) = \alpha(u_1 k_0)^\alpha + (1 - \delta)u_1 k_0 - b_1(u_1).$$

Equation (10) is a standard risk sharing condition that equalizes the ratio of the marginal utilities of the two agents across states of the world.

From (10) we can derive the equilibrium sensitivity of entrepreneurs' net worth to the capital quality shock, as shown in the next proposition. Let ω denote the period 1 ratio of entrepreneurs' wealth to total wealth in the economy, including the human wealth of consumers, that is,

$$\omega \equiv \frac{n_1}{n_1 + a_1 + \sum_{t=1}^\infty \beta^{t-1} w_t}.$$

Proposition 1. *If in equilibrium $\mu_1(u_1) = 0$ for all u_1, then the sensitivity of entrepreneurial net worth to the u_1 shock is*

$$n_1'(u_1) = \frac{\omega}{\omega + (1 - \omega)\frac{1}{\sigma} - \omega W'(n_1)} \left(\alpha u_1^{\alpha-1} k_0^\alpha + (1 - \delta)k_0 \right). \tag{11}$$

Proof. Using the definition of net worth, we can write equation (10) as follows

$$n_1(u_1) = \xi \left[(u_1 k_0)^\alpha + (1 - \delta)u_1 k_0 - n_1(u_1) + W(n_1(u_1)) \right]^\sigma,$$

where $\xi > 0$ is a constant, independent of u_1. Differentiating with respect to u_1 and rearranging, gives (11). □
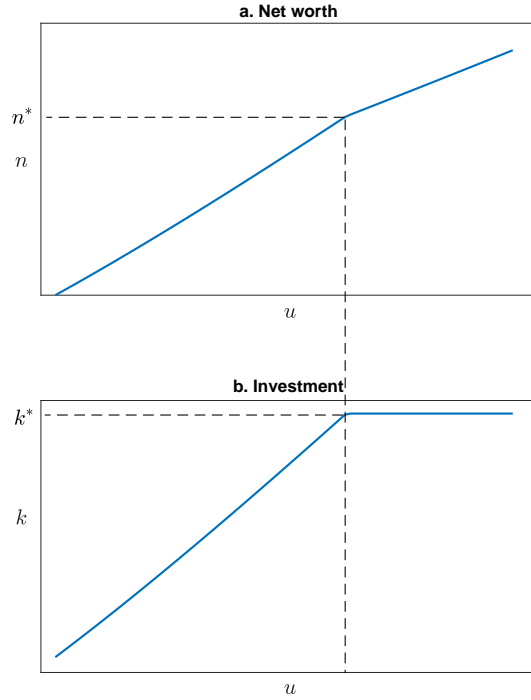
Figure 1: Net worth and investment as functions of the capital quality shock u_1

Equation (11) and the analysis in Section 3.1 can be used to identify the forces that determine financial amplification. As a benchmark, notice that in the first-best case with no collateral constraints we have $k_t = k^*$ for all $t \geq 1$, implying that the shocks to capital quality would not affect aggregate investment.[4] Therefore, any response of k_1 to the u_1 shock is a form of financial amplification.

Suppose first that consumers are risk neutral, so $\sigma = 0$. In this case, (11) implies that n_1 is independent of u_1. That is, all aggregate risk is absorbed by the consumers, so the shock u_1 has no effects on entrepreneurs' net worth. By implication, we know from the characterization of the continuation equilibrium that also k_1 is independent of u_1. Therefore, the presence of state-contingent claims eliminates financial amplification, in the sense that the response of investment to the capital quality shock is equal to zero as in the economy without the collateral constraint.[5] This echoes the baseline result in Krishnamurthy (2003).

Figure 1 shows the relation between entrepreneurs' net worth and the capital quality shock (top panel) and the choice of capital (bottom panel), when consumers are risk averse

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[4]This is due to the assumption that consumers have linear preferences after $t = 1$ and the capital quality shock is iid.

[5]The *level* of k_1 may be different from the first-best, because entrepreneurs may still be constrained if their initial level of net worth n_0 is small enough.

($\sigma > 0$). The first relation comes from equation (11), the second from Lemma 1. With risk averse consumers, the aggregate shock is shared by both agents, so the net worth of entrepreneurs is positively related to u_1. If $n_1 \geq n^*$, k_1 is still independent from u_1. However, a sufficiently negative capital quality shock at $t = 1$ leads net worth to fall below the threshold n^* and reduces k_1.

This discussion emphasizes that the degree of financial amplification depends on the equilibrium sensitivity of net worth to u_1. Equation (11) identifies two key determinants of this elasticity: σ and W'. We now discuss the role of these two elements in detail.

The expression in parentheses on the right-hand side of (11) represents the effect of u_1 on the economy's resources. How much of that effect is borne by the entrepreneurs depends on the ratio

$$\frac{\omega}{\omega + (1 - \omega)\frac{1}{\sigma} - \omega W'(n_1)}.$$

To interpret this ratio, let us consider separately the cases $n_1 \geq n^*$ and $n_1 < n^*$.

If $n_1 \geq n^*$ then $W'(n_1)$ is zero, and the ratio above is just

$$\omega \frac{1}{\omega + (1 - \omega)\frac{1}{\sigma}}.$$

Define the risk tolerance as the inverse of the coefficient of relative risk aversion. Then the risk tolerance of the entrepreneurs is 1—due to log preferences—and the average risk tolerance in the economy, weighted by the agents' wealth shares, is $\omega + (1 - \omega)1/\sigma$. Therefore, we obtain the standard result that agents share aggregate risk in proportion to their risk tolerance: the less risk tolerant are consumers, the higher the sensitivity of entrepreneurial net worth to the aggregate shock in equilibrium. See Gârleanu and Panageas (2015) for example.

Equation (11) highlights a second determinant of the equilibrium risk taking behavior of entrepreneurs, which operates only when the collateral constraint in the continuation equilibrium binds, $n_1 < n^*$. Because $W'(n_1) > 0$ in this constrained region, we can see from equation (11) that the share of the shock borne by entrepreneurs is larger.[6] The intuition for the last result is that a reduction in n_1 in the constrained region reduces consumers' lifetime labor income, making them more willing to purchase state-contingent claims that pay off in that contingency. In equilibrium, this makes it harder for the entrepreneurs to smooth their net worth in those states of the world, increasing the sensitivity of n_1 to u_1. In other words, the response of n_1 increases the background risk perceived by consumers endogenously, making it costlier for the entrepreneurs to insure against the aggregate shock.

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[6]This is the reason why in Figure 1 the relation between n_1 and u_1 is steeper when $n_1 < n^*$.

The importance of endogenous labor income in the results above can also be seen comparing our model to a different environment with an "AK" technology. With this production function, consumers do not earn labor income and their consumption is only financed by holdings of financial assets. In Online Appendix B we show that such model features no financial amplification relative to the first-best economy even when consumers are risk-averse, as long as they have the same CRRA preferences as entrepreneurs. This case is closely related to the no-amplification result in Di Tella (2017), who also considers an economy with an "AK" technology.

4 Quantitative analysis

In the previous section, we have seen that the presence of state-contingent claims may or may not dampen the degree of financial amplification in the economy, depending on the relative risk tolerance of consumers and entrepreneurs and on the size of the general equilibrium spillovers of entrepreneurial net worth on consumers' labor income. We now go back to the fully fledged stochastic model and ask how sizable is financial amplification in our economy under plausible calibrations of the model parameters.

We compare our benchmark model with two other economies: a *first best* economy, equivalent in all respects to the benchmark with the exception that entrepreneurs do not face the collateral constraints (1); and an *incomplete markets* economy, in which entrepreneurs can only issue non-state-contingent bonds, so the following additional constraint is present:

$$b((s^t, s_{t+1})) = \bar{b}(s^t) \qquad \forall (s^t, s_{t+1}).$$

In the incomplete markets economy, the limited enforcement friction implies the financial constraints

$$\bar{b}(s^t) + \gamma w(s^{t+1}) l(s^{t+1}) \leq \theta(1-\delta) u(s_{t+1}) k(s^t) \tag{12}$$

for all (s^t, s_{t+1}).

4.1 Calibration

Table 1 reports the model parameters that we use in our simulations. A period in the model corresponds to a quarter. We set the following parameters to standard values: the capital income share α is 0.33, the depreciation of capital is 2.5%, the discount factor of consumers β is 0.99, and the Frish elasticity of labor supply ψ is 1. In addition, we choose χ so that worked hours equal 1 in the deterministic steady state of the model. We further set ρ to 1,

so that consumers have a unitary elasticity of intertemporal substitution as entrepreneurs. The parameter γ represents the fraction of the wage bills that needs to be paid in advanced by entrepreneurs. We set it to 0.50, in the mid range of values considered in the literature.[7] Conditional on the above parameters, β_e and θ control the steady state level of the capital to net worth ratio (k_{ss}/n_{ss}) and the equilibrium return to capital. We choose β_e and θ so that the former equals 4 and the latter is 50 annualized basis points above the risk-free rate.[8] This gives us β_e equal to 0.984 and θ equal to 0.818. We assume that the capital quality shock takes two possible values, $u_t = \{u_H, u_L\}$ with $u_H = 1$. Thus, the calibration of this process consists in choosing values for u_L and for transition probabilities. In line with Gertler and Karadi (2011), we set $u_L = 0.925$ and $P(u_{t+1} = u_L | u_t = u_L) = 0.66$. We further set $P(u_{t+1} = u_H | u_t = u_H) = 0.99$, so that financial crises in the model are rare events. The remaining parameter is the risk aversion of consumers, σ. Rather than choosing a specific value, we will vary it across numerical experiments in the plausible range of $[1, 10]$.

4.2 Results

In Table 2 we report statistics computed on model simulated data using three different values of the consumers' coefficient of relative risk aversion: $\sigma = 1$, $\sigma = 5$, and $\sigma = 10$. In each case, we report results for the first best economy (FB), the economy with incomplete financial markets (IM), and the baseline economy with state-contingent claims (CM).

For each specification we simulate the model for $T = 200,000$ periods and select the periods in which the capital quality shock switches from H to L between $t-1$ and t. Panel A in the table reports the average percentage change in entrepreneurial net worth when the switch occurs, and the average percentage change in

$$\tilde{n}_t = \theta(1 - \delta)u_t k_{t-1} - b_{t-1}(u_t),$$

a variable that measures the entrepreneurs' maximum capacity to issue intra-period loans to finance working capital. Both variables are relevant to understand how financial factors affect the demand for capital and labor by entrepreneurs. Panel A also reports the average

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[7]For example, Jermann and Quadrini (2012) set this parameter to 1 in their sensitivity analysis, while Bianchi and Mendoza (2018) set it to 0.16. The key results presented in this section survive when using smaller or larger values for γ within this range.

[8]The entrepreneurs in our model consolidate financial and non-financial firms. Using US data, Gertler and Karadi (2011) target an average leverage ratio of 4 for the consolidated financial and non-financial corporate sector. The excess returns to capital that arise in the deterministic steady state reflect deviations from arbitrage induced by the presence of binding collateral constraints. Garleanu and Pedersen (2011) and Bocola (2016) document that these arbitrage rents were sizable during the global financial crisis of 2008-2009, but they typically average few basis points in advanced economies in normal times. We chose 50 basis points to be consistent with this evidence.

Table 1: Model parameters

Parameter	Concept	Value
α	Capital income share	0.330
δ	Capital depreciation	0.025
β	Discount factor, consumers	0.990
ψ	Frisch elasticity	1.000
χ	Disutility of labor	1.980
ρ	Inverse IES, consumers	1.000
γ	Fraction of wages paid in advance	0.500
β_e	Discount factor, entrepreneurs	0.984
θ	Fraction of pleadgeable assets	0.818
u_L	Capital quality in low state	0.925
$\Pr\left(u' = u_L \mid u = u_L\right)$	Transition probability	0.660
$\Pr\left(u' = u_H \mid u = u_H\right)$	Transition probability	0.990

percentage change in labor, investment and output. Panel B reports indicators for the entrepreneurs' balance sheet in the period immediately preceding the L shock: the average net worth, the average value of $\tilde{n}_{t-1}$, the average leverage ratio, and the average ratio between bonds issued in period $t-1$ contingent on the L state realizing at time t and those contingent on the H state, denoted respectively $b_{L,t}$ and $b_{H,t}$. In the incomplete market economy this ratio is always equal to 1 by construction.

Let us start with the case $\sigma = 1$ and look at the differences between the three economies.

In the first best economy, a negative capital quality shock lowers the marginal product of labor, leading to a reduction in labor demand and a fall in hours worked. The direct effect of the shock, coupled with the reduction in labor input, leads to a fall in output. The persistence of u_t implies that capital quality is expected to be low also in future periods, reducing the incentive to accumulate capital and leading to a fall in investment.

In the incomplete market economy, the shock has larger effects on labor, investment, and output. The differences are due to the financial amplification mechanism. In the incomplete market economy, entrepreneurs issue non-state-contingent claims and face the collateral constraint (12). The first ingredient implies that their balance sheet is exposed to aggregate risk: a negative capital quality shock reduces the value of the capital held but not the value of entrepreneurs' liabilities. So, both n_t and $\tilde{n}_t$ fall (on average by 25% and 97% respectively). The second ingredient implies that these balance sheet effects depress

Table 2: Entrepreneurs' balance sheet and financial amplification

| | $\sigma = 1$ | | | $\sigma = 5$ | | | $\sigma = 10$ | | |
	FB	IM	CM	FB	IM	CM	FB	IM	CM
	Panel A: Quantities								
$\Delta(\log n_t)$		-25.21	-3.12		-25.23	-6.75		-25.29	-16.71
$\Delta(\log \tilde{n}_t)$		-97.98	16.26		-97.56	-14.35		-97.73	-73.31
$\Delta(\log l_t)$	-1.89	-4.81	-1.66	-1.89	-4.60	-6.37	-1.89	-4.17	-11.10
$\Delta(\log i_t)$	-6.60	-16.26	-6.00	-7.60	-17.09	-10.30	-9.16	-18.36	-17.69
$\Delta(\log y_t)$	-3.77	-5.73	-3.62	-3.77	-5.59	-6.78	-3.77	-5.30	-9.94
	Panel B: Entrepreneurs' balance sheet								
n_{t-1}		7.89	6.32		7.86	6.28		7.80	6.69
$\tilde{n}_{t-1}$		2.29	0.95		2.29	0.97		2.28	1.57
k_{t-1}/n_{t-1}		3.09	3.95		3.09	3.94		3.08	3.55
$b_{L,t}/b_{H,t}$		1.00	0.91		1.00	0.93		1.00	0.97

Notes: *Each economy is simulated for $T = 200,000$ periods. For each simulation, we select every j such that $u_{j-1} = u_H$ and $u_j = u_L$. We then compute a given statistic x_j and average across j. In panel A the changes in the variables are multiplied by 100, so to obtain percentage changes.*

the demand for capital and labor by entrepreneurs. The combination of these two forces leads to a deeper recession relative to the first best.

When entrepreneurs can issue state-contingent claims, the fall in labor, investment, and output are comparable to those of the first best economy. That is, the financial amplification mechanism is muted. Unlike in the incomplete market case, entrepreneurs can now insure against the capital quality shock by reducing their contingent liabilities in state L. Panel B of Table 2 shows that this is precisely what they do in equilibrium: the ratio $b_{L,t}/b_{H,t}$ is on average 0.91, meaning that entrepreneurs promise to pay less in the L state. This liability structure implies that both n_t and $\tilde{n}_t$ are less affected by the negative capital quality shock, eliminating the first step of the amplification mechanism described above. These results mirror the findings in Carlstrom, Fuerst, and Paustian (2016), Dmitriev and Hoddenbagh (2017) and Cao and Nie (2017). They study financial accelerator models with endogenous labor income and log utility for consumers, and show that in their economies financial amplification is muted when debt contracts can be indexed to aggregate shocks.

Financial amplification, however, increases substantially if consumers have a coefficient of relative risk aversion greater than one. Table 2 shows that the behavior of the first best and of the incomplete market economies does not change much once we increase σ, a
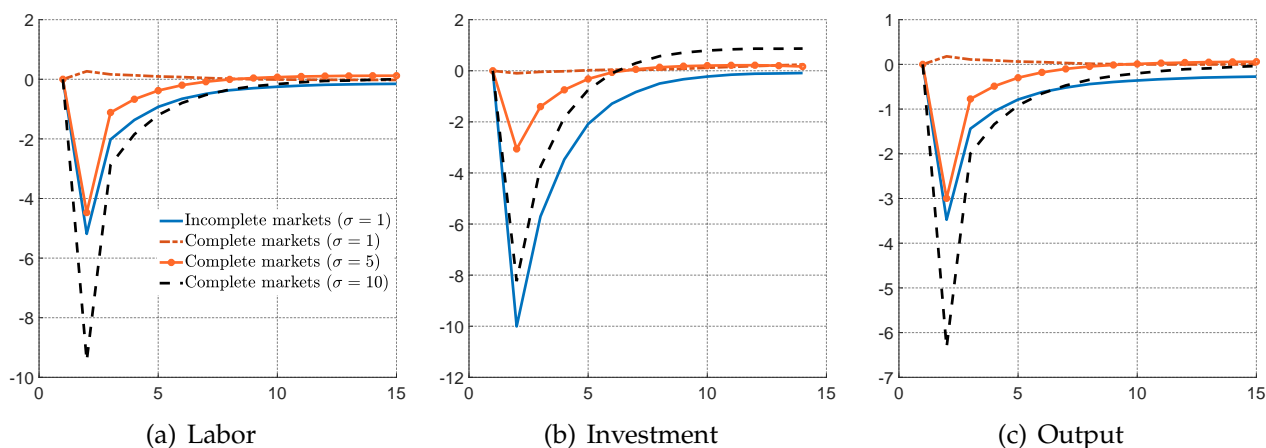
Figure 2: Impulse response functions

Notes: *We compute $2 \times M$ simulations of length T. We initialize the simulations at $t = 0$ setting each state variable at the mean of the ergodic distribution. In the first M simulations, we set $u_1 = u_L$, in the others we set $u_1 = u_H$. To obtain impulse response functions, we take the difference between the first and second set of simulations, averaging across M. We set $M = 5000$ and $T = 15$. We plot impulse response functions in percentage deviations from the first best economy.*

result related to the findings in Tallarini (2000). The economy with state-contingent claims behaves differently: the average ratio $b_{L,t}/b_{H,t}$ increases to 0.93 when $\sigma = 5$. Entrepreneurs use state contingent debt less to protect their net worth against a negative shock and, as a result, the sensitivity of n_t and $\tilde{n}_t$ to the shock increases. The larger fall in these two variables constrains entrepreneurs' demand of labor and capital, leading to a deeper recession. With $\sigma = 5$ the fall in labor and output in the economy with complete markets is comparable to that of the economy with incomplete markets.[9] Increasing σ further leads to more risk taking by entrepreneurs and to stronger financial amplification.

Figure 2 gives a more complete representation of the dynamics following the shock, plotting impulse response functions for labor, investment, and output. To isolate the financial amplification mechanism, we plot impulse response functions in deviations from the first best. When $\sigma = 1$, the complete market economy features essentially no financial amplification. As we increase σ, all three series respond by more than in the first best. Quantitatively, the responses are comparable to those of the economy with incomplete markets for plausible levels of σ.

Why do entrepreneurs choose a riskier balance sheet as σ increases? To provide some intuition, in Figure 3 we plot how asset prices and entrepreneurs' borrowing choices change as we vary σ. The top left panel reports the average value of the ratio $q(s_{t+1}|s^t)/(\beta\pi(s_{t+1}|s^t))$,

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[9]This happens despite the fact that with complete markets the fall in net worth is smaller than with incomplete markets. The reason is that the economy with incomplete markets starts from a higher level of net worth in equilibrium, so the post-shock levels of net worth in the two economies are quantitatively similar.
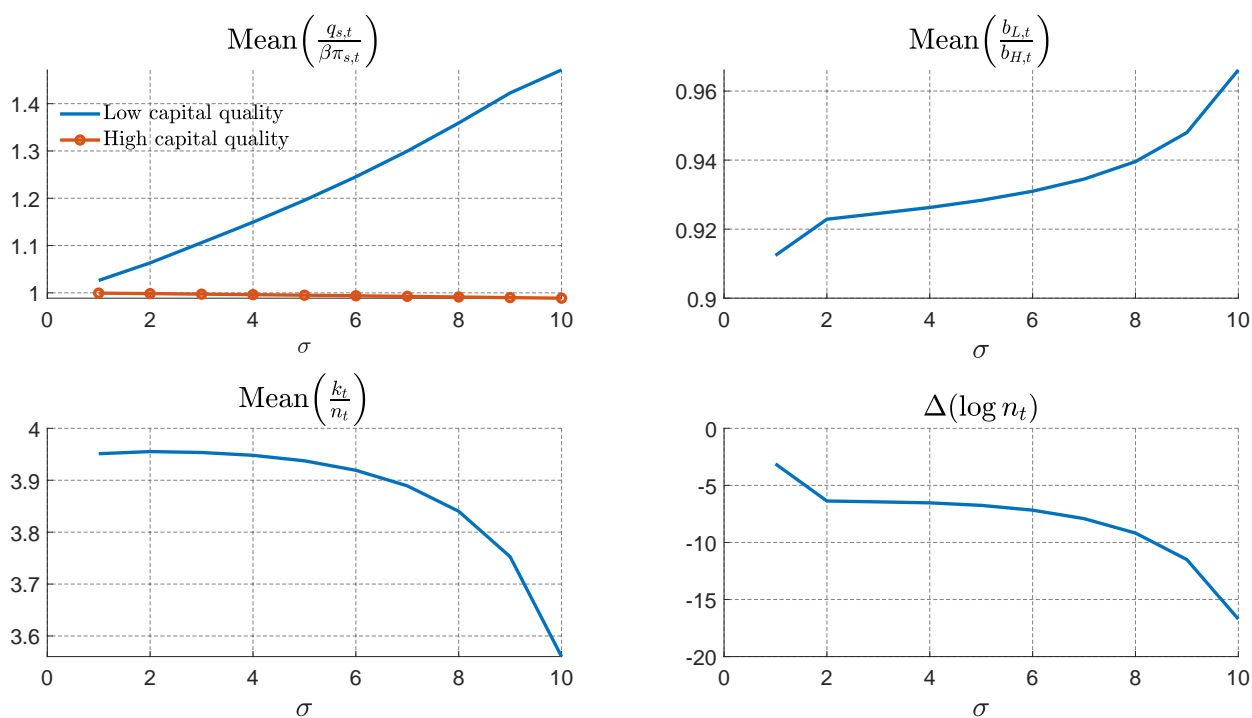
Figure 3: Asset prices and entrepreneurs' balance sheet

Notes: *For each value of σ, we simulate the complete market economy for $T = 200,000$ periods, and we compute average values of $q(s_{t+1}|s^t)/(\beta\pi(s_{t+1}|s^t))$ (panel a), of $b_{L,t}/b_{H,t}$ (panel b), of k_t/n_t (panel c), and of the percentage change in net worth after a negative capital quality shock (panel d).*

for $s_{t+1} = H$ and for $s_{t+1} = L$. The ratio measures how much it costs to buy insurance against state s_{t+1}, relative to the actuarially fair price that would be charged by a risk neutral insurer. A high value of the ratio reflects a high insurance premium on state s_{t+1}, or, equivalently, a low risk premium, as agents are willing to receive a low rate of return on assets that pay in that state. The remaining panels report the average $b_{L,t}/b_{H,t}$ ratio, the average entrepreneurs' leverage, and the average percentage change in net worth after a negative capital quality shock.

After a low capital quality shock, consumers' current and future labor incomes decline, increasing their marginal utility. When σ is close to one, consumers are not concerned about the negative shock, and the insurance premium is small—$q(s_{t+1}|s^t)/(\beta\pi(s_{t+1}|s^t))$ is on average close to 1. For higher values of σ, consumers are willing to pay a higher premium to insure against this shock. Facing this higher premium, entrepreneurs have a stronger incentive to sell debt contingent on the low shock. This endogenously makes the entrepreneurs' net worth more sensitive to the shock and, in general equilibrium, makes consumers' incomes even more procyclical, reinforcing the initial effect on risk premia.[10]

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[10]The self-reinforcing nature of the mechanism suggests that multiple equilibria may be possible. We have

Notice that there is also a countervailing force at work. As entrepreneurs take on more aggregate risk, they partly adjust by reducing their investment in capital, thus reducing their leverage k_t/n_t, as seen in the bottom-left panel. This force however only partly offsets the mechanism described above.

It is useful to remark that high values of σ in the range considered are more realistic choices in terms of the risk premia they produce. For instance, the expected excess return on the capital stock in the model is 0.3% when consumers have log preferences ($\sigma = 1$), while it equals 0.7% when $\sigma = 10$.

4.3 Isolating the general equilibrium spillover on labor income

The mechanism just described contains two steps: first, risk averse consumers are willing to pay high insurance premia for insuring a bad realization of the capital quality shock; second, high insurance premia endogenously make consumers' incomes more sensitive to the shock, reinforcing the first step.

The results in Table 2 and Figure 2 show that the combined effect of these two steps can be quantitatively relevant. We now attempt a decomposition to evaluate the importance of the second step, that is, to evaluate how much the macro spillover in our model reinforces the direct effect of consumers' risk aversion.

We consider an economy that is identical to that of Section 2, except that consumers earn the counterfactual wage that would arise in the first best economy.[11] Wages still respond to the capital quality shock—as they do in the first best—but they are not affected by the changes in investment and labor demand that are due to the presence of the collateral constraint. By construction, in this economy there is no spillover from entrepreneurs' net worth to consumers' labor income. For brevity, we call it the "no spillover" economy.

Table 3 reports the average response of key variables to the low capital-quality shock in the first best economy, in the benchmark economy with state-contingent claims, and in the economy with no spillover. In all cases, we set $\sigma = 5$. The first two columns reproduce results in Table 2. The third column shows that the amplification mechanism is substantially reduced if we shut down the macro spillover. Net worth falls by 3.24% instead of 6.75% and the responses of labor, investment, and output are comparable to those of the first best economy.

Panel B of the Table helps understand this result. Absent the spillover, labor income falls

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constructed examples where this happens, but they require very large responses of wages to the net worth or very large levels of risk aversion.

[11]See Online Appendix C for a detailed description of this version of the model.

Table 3: Quantifying the general equilibrium spillover

	First best	Complete markets	No spillover
	Panel A: Quantities		
$\Delta(\log n_t)$		-6.75	-3.24
$\Delta(\log l_t)$	-1.89	-6.37	-1.66
$\Delta(\log i_t)$	-7.60	-10.30	-9.89
$\Delta(\log y_t)$	-3.77	-6.78	-3.61
	Panel B: Prices and entrepreneurs' balance sheet		
$\Delta(\log LI_t)$	-3.77	-12.75	-3.77
$q_{L,t}/\pi_{L,t}$		1.20	1.03
$q_{H,t}/\pi_{H,t}$		1.00	1.00
k_{t-1}/n_{t-1}		3.94	3.95
$b_{L,t}/b_{H,t}$		0.93	0.91

Notes: *See Table 2.*

by 3.2% after the L shock, substantially less than the 12% of the benchmark model. Thus, even if consumers are more risk averse than entrepreneurs, they do not bid up as much the price for insuring a low realization of the capital quality shock: $q_{L,t}/(\beta\pi_{L,t})$ is 1.03 in the no spillover economy, compared to 1.20 in our benchmark economy. Given these state prices, entrepreneurs have a better incentive to stabilize their net worth by reducing their contingent debt in the L state.

In summary, to generate quantitatively meaningful financial amplification in our model, we need both consumers to be more risk averse than entrepreneurs and labor income to be sufficiently responsive to entrepreneurs' net worth.

4.4 A shock to the pleadgeability of capital

So far, we have focused on capital-quality shocks. To conclude this section, we ask whether the mechanism in this paper also operates for other types of shocks. In particular, we look at a shock that is also widely used in models of financial amplification: a shock to the financial constraint.

Following Jermann and Quadrini (2012), we consider shocks to the parameter θ that determines the fraction of the capital stock that can be pledged in financial contracts. We assume that θ can take two values, $\theta(s_t) \in \{\theta_L, \theta_H\}$, with transition matrix $\pi(s_t|s_{t-1})$.

We parametrize the transition probabilities following Jermann and Quadrini (2012), and consider a symmetric Markov chain with $\Pr(\theta' = \theta_s | \theta = \theta_s) = 0.97$. We set $\theta_H = 0.82$, as in the previous sections, while θ_L is set to 0.72.[12] All remaining parameters are the same as in Table 1.

A shock to θ_t, unlike a shock to u_t, has no effects on the first-best allocation. Therefore, we focus on comparing the complete market and the incomplete market economies. Figure 4 reports impulse response functions to a negative shock to θ_t. Let us start with the incomplete market economy, with $\sigma = 1$.[13] A fall in θ_t does not directly affect the balance sheet of the entrepreneur: neither the value of the assets, nor debt payments change. Thus, on impact, net worth does not change. However, the shock affects the borrowing capacity of the entrepreneur leading to a contemporaneous fall in the demand of capital and to a fall in the demand of labor in future periods. The net worth of entrepreneurs increases in the periods following the shock, because of the higher profits made by the entrepreneurs when the collateral constraint binds. The increase in net worth mitigates the reduction in θ_t, so the effects of the shock on output essentially go away after the first two periods.

Consider now the economy with state-contingent claims. When $\sigma = 0$, entrepreneurs use contingent claims to partly insure against the shock to θ_t: entrepreneurial net worth increases on impact after the shock because contingent debt payments are lower. The increase in net worth on impact immediately dampens the contraction in θ_t, so the reduction in investment and labor are less pronounced relative to the incomplete market economy. As in the model with capital-quality shocks, the combination of complete markets and low consumers' risk aversion dampens financial amplification.[14]

As we increase σ, consumers are less willing to bear risk and the degree of financial amplification in the economy increases. With $\sigma = 10$ (dotted line in the figure), net worth actually falls after the θ_t: consumers are willing to pay a premium for hedging the fall in their future labor income when $\theta_t = \theta_L$, and entrepreneurs provide this insurance by issuing more debt that pays in those states of the world. The associated fall in net worth implies a stronger decline in the demand of labor and capital, larger even than in the economy with incomplete markets.

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[12]Jermann and Quadrini (2012) construct a time series for θ_t using US data. A fall of 0.1 is in line with the fall observed in this time series during the Great Recession.

[13]As in the case of u shocks, the responses to θ shocks in the incomplete market economy are minimally affected by σ.

[14]It is worth noting that the economy still features a substantial degree of financial amplification even with risk neutral consumers. Unlike for the capital quality shock, a shock to θ_t does not have a negative effect on entrepreneurs' net worth, so entrepreneurs have weaker incentives to insure against it. This is related to results in Di Tella (2017) and Dávila and Philippon (2017), that show that the nature of the shock matters for hedging incentives.
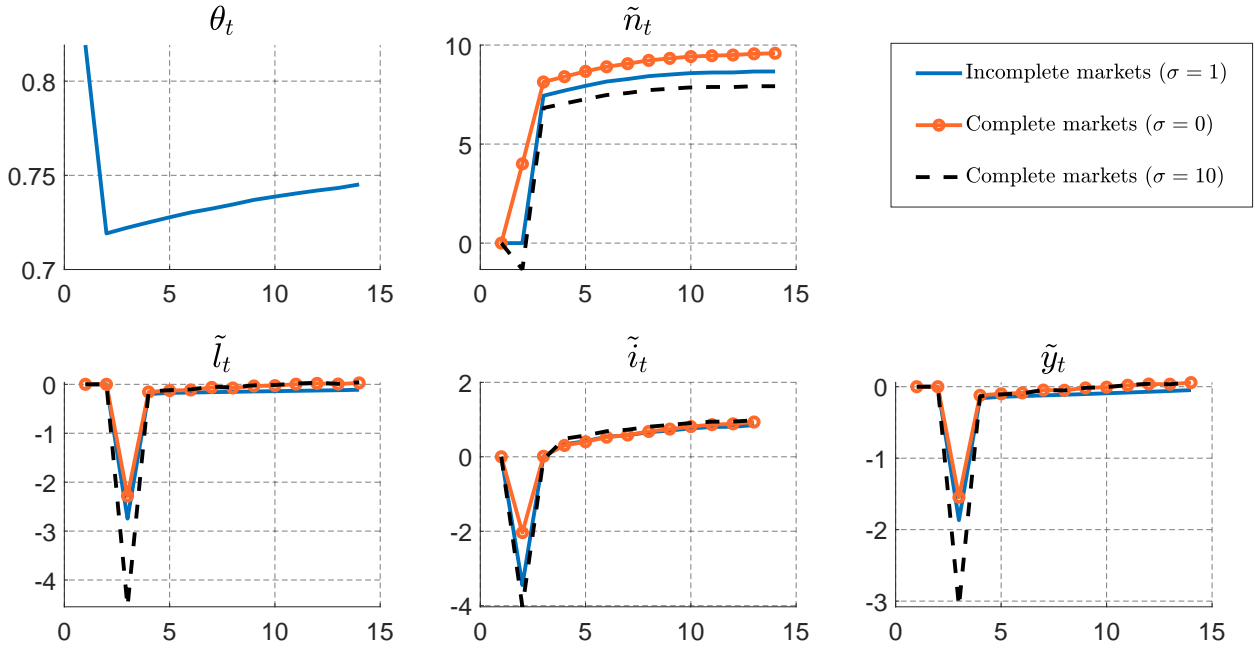
Figure 4: Impulse response functions: θ shocks

Notes: *See the note to Figure 2 for the calculation of impulse response functions. Net-worth, labor, investment, and output are in percentage deviations from pre-shock values.*

Summing up, the mechanism emphasized in our paper seems to play a relevant role also in models with shocks to the collateral constraint.

5 Welfare analysis

We now turn to the welfare implications of the model, proceeding in two steps. In Section 5.1 we use the special case of Section 3 to show that the competitive equilibrium can be constrained inefficient. In particular, in equilibrium entrepreneurs may use state contingent claims less than it is socially efficient, because they do not internalize the stabilizing effects of their risk mitigation strategies on consumers' incomes. In Section 5.2 we go back to the general model—calibrated as in Section 4—and study numerically an optimal policy aimed at correcting this externality.

5.1 Inefficient risk sharing

Consider the simplified model of Section 3 with a binary distribution for u_1, with values u_H and u_L and probabilities, respectively, π_H and π_L. Let us focus on an economy that satisfies the following two properties in equilibrium:

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a. In period 0, the collateral constraint is slack for both H-contingent and L-contingent debt, that is, $\mu_1(u_1) = 0$ for $u_1 = \{u_H, u_L\}$;

b. In period 1, the collateral constraint is binding in state L and slack in state H, $\mu_2(u_L) > 0$ and $\mu_2(u_H) = 0$.

Consider a social planner who can manipulate the entrepreneurs' choices of $b_1^H = b_1(u_H)$ and $b_1^L = b_1(u_L)$ at date 0. The planner has no other way to intervene and is subject to the same limited enforcement constraints (1) faced by the agents in the economy. Given that the collateral constraint is slack at $t = 0$ the planner can increase b_1^H and reduce b_1^L at the margin. Suppose the planner chooses small changes $db_1^H > 0$ and $db_1^L < 0$, what are the effects on the expected utility of consumers and entrepreneurs, V_0 and $V_{e,0}$?

To make the analysis more transparent, we assume that the planner intervenes at the end of date 0, after agents have already traded the laissez-faire equilibrium state contingent claims and have already made their consumption and investment choices, without expecting any government intervention.[15] At the end of the period, unexpectedly, the planner intervenes by re-opening the market for financial claims, changing the entrepreneurs' portfolio choice and implementing a unilateral transfer dT from consumers to entrepreneurs. This intervention, by construction, only affects allocations from period 1 onward. Moreover, the changes in state prices $dq_{L,1}$ and $dq_{H,1}$ caused by the intervention only have second order effect, because they only affect the additional trades db_1^H and db_1^L made after the planner's intervention. So the feasible set of interventions, that satisfy both the entrepreneurs' and the consumers' budget constraints at $t = 0$, is given by the following equation:

$$q_{H,1}db_1^H + q_{L,1}db_1^L + dT = 0, \tag{13}$$

where $q_{s,1}$ are the laissez-faire equilibrium state prices.

From period 1 onward, the changes db_1^H and db_1^L have general equilibrium consequences. In particular, they affect the net worth of entrepreneurs at date 1 and, via this channel, they affect the equilibrium path of capital and wages if the collateral constraint binds. Given that the collateral constraint is not binding following the H shock, the wage path is only affected following the L shock. Moreover, thanks to the assumptions that $\rho = 0$ and that there is no uncertainty after $t = 1$, the interest rate is unaffected by the intervention and is equal to $1/\beta - 1$. These observations imply that the effect of the intervention on the

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[15] Alternatively, we could assume that the planner has sufficient instruments to keep the consumption and investment choices at date 0 at their competitive equilibrium value.

expected utility of a consumer is:

$$dV_0 = \pi_H \Xi \left(V_1^H\right)^{-\sigma} db_1^H + \pi_L \Xi \left(V_1^L\right)^{-\sigma} \left[1 - \sum_{t=1}^{\infty} \beta^{t-1} \frac{dw_t^L}{dn_1^L}\right] db_1^L, \tag{14}$$

where Ξ is a constant factor, not relevant for our derivations below. The direct welfare effects of changing bond holdings in state s are given by $\pi_s \Xi \left(V_1^s\right)^{-\sigma} db_1^s$, these are the effects that are internalized by the consumers' private portfolio decisions. The presence of the second term in square brackets captures a pecuniary externality. If entrepreneurs make larger payments to consumers in the L state, entrepreneurs' net worth is lower, and the capital stock and labor income are lower in all future periods. This reallocates resources from consumers to entrepreneurs in proportion to the wage changes.[16]

The pecuniary externality just described implies that consumers in the competitive equilibrium overvalue payments received in state L relative to the planner, because they do not take into account that receiving a payment from entrepreneurs in state L depresses their own future labor income. If consumers internalized these effects they would be willing to trade the L-contingent bond for the H-contingent bond at a rate lower than the equilibrium relative price, since

$$\frac{\pi_L \left(V_1^L\right)^{-\sigma} \left[1 - \sum_{t=1}^{\infty} \beta^{t-1} \frac{dw_t^L}{dn_1^L}\right]}{\pi_H \left(V_1^H\right)^{-\sigma}} < \frac{\pi_L \left(V_1^L\right)^{-\sigma}}{\pi_H \left(V_1^H\right)^{-\sigma}} = \frac{q_{L,1}}{q_{H,1}}.$$

Similar calculations give the effect of (db_1^H, db_1^L) on the entrepreneurs' expected utility:

$$dV_{e,0} = -\pi_H \beta \frac{1}{c_{e,1}^H} db_1^H - \pi_L \beta \frac{1}{c_{e,1}^L} \left[1 - \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_{e,1}^L}{c_{e,t}^L} \frac{dw_t^L}{dn_1^L}\right] db_1^L. \tag{15}$$

Also this expression includes a pecuniary externality, for the same reason as for equation (14).[17] Therefore, entrepreneurs also tend to overvalue payments in state L. However, the crucial observation is that the terms in square brackets in equation (14) and (15) are different. In particular, the fact that entrepreneurs are constrained at least for a period following the L shock implies that $c_{e,1}^L < c_{e,t}^L$ for $t = 2, 3, ...$. As a consequence, entrepreneurs

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[16]In equation (14), the expression dw_t^L / dn_1^L denotes the general equilibrium effect on wages of changing the initial net worth of entrepreneurs at $t = 1$. The effect of db_1^L on dn_1^L is mechanically equal to -1. It is useful to note that this expression is identical to the macro spillover $W'(n_1)$ identified in Section 3. Here it's useful to express it as a sum, for comparison to equation (15) below.

[17]Apart from this term, the formula contains no additional terms due to changes in the paths of capital and debt from period 1 onward. This follows because debt and investment choices are individually optimal and an envelope argument applies.

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discount more than consumers the effects on future wages, so the degree of overvaluation of L claims is relatively weaker for them:

$$\sum_{t=1}^{\infty} \beta^{t-1} \frac{c_{e,1}^L}{c_{e,t}^L} \frac{dw_t^L}{dn_1^L} < \sum_{t=1}^{\infty} \beta^{t-1} \frac{dw_t^L}{dn_1^L}.$$

Given the inequality above, there exists a pair $db_1^L < 0 < db_1^H$ such that both consumers and entrepreneurs are better off, i.e., $dV_0 > 0$ and $dV_{e,0} > 0$. The logic of this result is that by shifting the debt composition away from the L state and towards the H state, the benefit for the entrepreneurs dominates the cost for the consumers, due to the fact that consumers are partly compensated by higher future wages in state L, and that higher future wages are paid in periods in which the entrepreneurs are, in relative terms, less constrained.

Given a pair (db_1^H, db_1^L) that makes both agents better off, we can use the budget constraint (13) to obtain dT. Because $\frac{\pi_L / c_{e,1}^L}{\pi_H / c_{e,1}^H} = \frac{q_{L,1}}{q_{H,1}}$, any pair $db_1^H > 0 > db_1^L$ that makes the entrepreneur better off must satisfy $q_{H,1} db_1^H + q_{L,1} db_1^L < 0$. Therefore the transfer dT needs to be positive.

We have proved the following result.

Proposition 2 (Constrained inefficiency). *Consider a competitive equilibrium of the simplified economy that satisfies conditions (a) and (b). There exists a small perturbation to the state contingent bonds, db_1^L, db_1^H, and a transfer dT from consumers to entrepreneurs at date 0, that yield a Pareto improvement.*

5.2   Optimal policy

Having established that the competitive equilibrium may be inefficient, we now study the optimal policy of a planner that can tax borrowing and investment choices of entrepreneurs. Here we have two objectives. First, to give a quantitative assessment of the taxes needed to correct the externality and of their effects on the equilibrium allocation. Second, to compare the policy that uses state contingency tools with a blunter policy that taxes borrowing equally in all states of the world. The latter policy is a simple leverage constraint of the type usually studied in existing models of macroprudential policy.

We start from the laissez-faire equilibrium studied in Section 4 and consider a planner who intervenes for one period only: the planner sets proportional taxes or subsidies on capital purchases and on state-contingent claims issued by entrepreneurs at time t. In addition, the planner can make a lump-sum transfer at date t to redistribute the efficiency gains between consumers and entrepreneurs. We assume the planner intervenes in the last

stage of period t, after production has taken place, at the moment in which entrepreneurs choose their capital investment and consumers and entrepreneurs trade state-contingent claims (see the model timing on page 7). In this way, the planner is not allowed to relax the working capital constraint in period t and must satisfy the working capital and collateral constraints in period $t+1$. So all welfare gains are solely due to the planner inducing different choices of capital and of state-contingent debt.[18]

Let $\mathbf{s} = [u, K, B]$ denote the aggregate state variables, in line with the definition of recursive equilibrium in Online Appendix A. In recursive notation, the problem of the entrepreneur is to choose consumption c_e, capital k', labor demand l, and state contingent claims $b'(\mathbf{s}')$ to maximize

$$\log(c_e) + \beta_e \mathbb{E}_{\mathbf{s}} \left[V^e \left(b'(\mathbf{s}'), k'; \mathbf{s}' \right) \right],$$

subject to

$$
\begin{aligned}
n &= (uk)^\alpha l^{1-\alpha} - w(\mathbf{s})l + (1-\delta)uk - b \\
c_e + [1+\tau_k(\mathbf{s})]\, k' &\leq n + \sum_{\mathbf{s}'} \left[1 - \tau_b \left(\mathbf{s}'|\mathbf{s} \right) \right] q \left(\mathbf{s}'|\mathbf{s} \right) b'(\mathbf{s}') + T_e(\mathbf{s}) \\
b'(\mathbf{s}') + \gamma w(\mathbf{s}')l' &\leq \theta(1-\delta)u'k',
\end{aligned}
$$

where τ_k is a proportional tax on capital, τ_b is a tax on sales of state-contingent claims, T_e is a lump-sum transfer, and $V^e(.)$ is the value function of entrepreneurs, expressed as function of the individual state variables (b, k) and of the aggregate state variables $[u, K, B]$.[19]

The problem of the consumer is to choose consumption c, labor l, and state contingent claims $a'(\mathbf{s}')$, to maximize

$$(1-\beta) \left(c - \chi \frac{l^{1+\psi}}{1+\psi} \right)^{1-\rho} + \beta \left[\mathbb{E}_{\mathbf{s}} \left(V \left(a'(\mathbf{s}'); \mathbf{s}' \right) \right)^{1-\sigma} \right]^{\frac{1-\rho}{1-\sigma}},$$

subject to the constraint

$$\sum_{\mathbf{s}'} q \left(\mathbf{s}'|\mathbf{s} \right) a'(\mathbf{s}') + c \leq w(\mathbf{s})l + a + T_c(\mathbf{s}),$$

[18]The assumption that the planner only intervenes for one period means that we look at the effect of policies for values of the state variables in the ergodic distribution of our calibrated equilibrium. Clearly, this exercise provides a lower bound to the welfare gains that optimal policy can achieve in our environment. We find it useful to focus on one period interventions because the fact that they do not alter the ergodic state distribution make the results easier to interpret.

[19]For ease of exposition, here we use entrepreneurs' debt B as a state variable, while in Online Appendix A we use the transformed variable $\tilde{N}$, which is more convenient for the numerical solution of the model.

where T_c is a lump sum transfer, and $V(.)$ is the value function of consumers, expressed as a function of the individual state variable a and of the aggregate state variables. Note that V and V^e are the value functions in the laissez-faire competitive equilibrium because the planner only intervenes for one period.

A competitive equilibrium with one-period government intervention at date t is given by taxes, prices, and allocations such that consumers and entrepreneurs solve the optimization problems just defined, $a'(\mathbf{s}') = b'(\mathbf{s}') = B'$, $k' = K'$, the labor market clears and the government budget constraint holds.

To evaluate the benefits of state-contingent borrowing taxes, we also consider the case of a planner that must tax all contingent claims at the same rate, that is, a planner subject to the constraint that $\tau_b(\mathbf{s}'|\mathbf{s})$ be the same for all $\mathbf{s}'$.

We consider a planner that maximizes the utility of the consumers subject to keeping the entrepreneurs at their laissez-faire utility level. We solve the planner's problem numerically, for all states in our grid, and report the response of the economy to a negative capital quality shock in Table 4, which is analogous to Table 2.[20] Specifically, we simulate the economy for many periods, select all the periods in which the shock switches from u_H to u_L between $t-1$ and t, and report statistics regarding the entrepreneurs' balance sheet and the behavior of macroeconomic variables, assuming that the planner intervened at $t-1$. We compare four different cases: the first best (FB), the laissez-faire equilibrium (CE), the equilibrium under optimal policy (PL), and the equilibrium under the constrained policy (PL-c). For this illustration, we set σ to 10.

First, let us consider the behavior of quantities in Panel A. The lassaiz-faire competitive equilibrium features on average a much larger fall in labor, investment and output compared to the first best economy because of the financial amplification mechanism discussed in Section 4. Under the optimal policy, the planner is able to substantially reduce financial amplification, as the fall in labor and output are closer to those in the first-best. In addition, comparing columns PL and PL-c, we see that the ability to tax differently different types of debt is critical for this result: the optimal policy of a planner that is restricted to impose a uniform tax on debt does not curtail financial amplification.

Panels B and C report the average taxes set by the planner and average statistics on the entrepreneurs' balance sheet. At the optimal policy, the planner subsidizes investment, and the size of the subsidy is similar in PL and PL-c. This result is not related to a macro-prudential motive and arises because the collateral constraint distorts down the level of capital in this economy relative to the first best. So, the planner can achieve efficiency gains

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[20]See Online Appendix D for a description of the planner's problem.

Table 4: Optimal policy

	FB	CE	PL	PL-c
Panel A: Quantities				
$\Delta(\log n_t)$		-16.71	-16.22	-16.52
$\Delta(\log \tilde{n}_t)$		-73.31	-54.47	-72.86
$\Delta(\log l_t)$	-1.89	-11.10	-2.29	-11.13
$\Delta(\log i_t)$	-9.16	-17.69	-19.30	-21.27
$\Delta(\log y_t)$	-3.77	-9.94	-4.01	-9.93
Panel B: Taxes				
$1 - \tau_b(u_L)$		1.00	0.80	1.01
$1 - \tau_b(u_H)$		1.00	1.00	1.01
$1 + \tau_k$		1.00	0.98	0.99
Panel C: Entrepreneurs' balance sheet				
n_t		5.66	5.68	5.98
$\tilde{n}_t$		0.75	0.90	0.75
k_t / n_t		3.55	3.56	3.56
$b_{L,t} / b_{H,t}$		0.97	0.96	0.97

Notes: *See note to Table 2.*

with a subsidy on capital.[21] The PL and PL-c economy, however, differ substantially on the optimal debt taxes.

In the PL economy, the planner imposes on average a tax of 20% on sales of *L*-contingent bonds and a zero tax on *H*-contingent bonds. As a result, entrepreneurs reduce their reliance on *L*-contingent debt, so the ratio $b_{L,t}/b_{H,t}$ is lower relative to the CE case. This implies that entrepreneurs' borrowing capacity at the beginning of period t, $\tilde{n}_t$, is higher, reducing the effect of the shock on labor demand and output. Turning to the case in which the planner can only impose a non-state-contingent tax on borrowing, the table shows that the optimal tax is close to zero on average. Consistently with this, balance sheets and aggregate effects in the PL-c economy are similar to those in the CE economy.

The result of a near zero tax in the PL-c economy may appear surprising in light of many papers in the literature that report sizable optimal debt taxes in similar models with

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[21]Incidentally, this explains why investment falls more in the planner solution than in the laissez-faire equilibrium in panel A. Since the planner can only intervene at $t - 1$, the investment subsidy is only present at date $t - 1$, driving down investment between $t - 1$ and t. In the laissez-faire equilibrium this policy effect is absent.
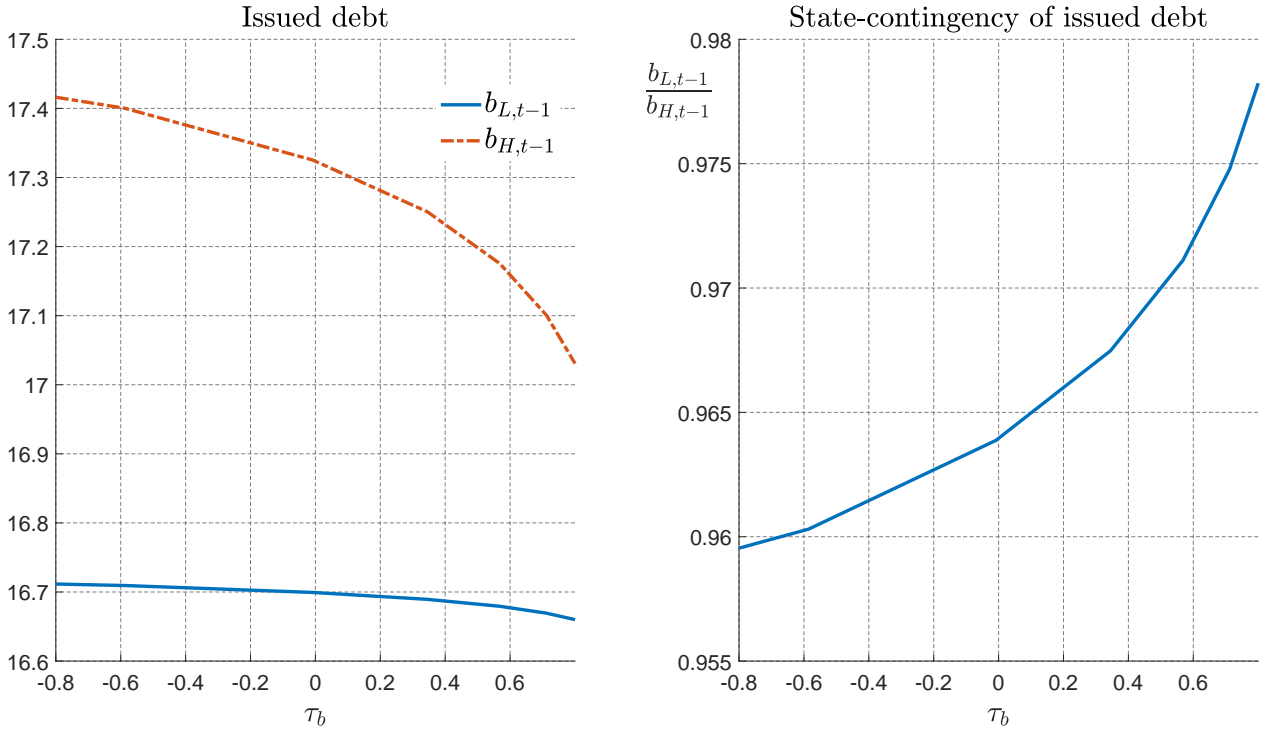
Figure 5: Tax on debt, leverage and risk taking

Notes: *The left panel reports the equilibrium levels of $b_{L,t-1}$ and $b_{H,t-1}$ when varying τ_b. When constructing the figure, we set $u_{t-1} = 1$ and set the other state variables at $t-1$ at the ergodic mean. In addition, the tax on capital and the transfers are set so that the level of capital remains at its optimal level in the constrained planner problem and the entrepreneur achieves the same utility as in the laissez-faire competitive equilibrium. The right panel is constructed in a similar fashion.*

non-state-contingent debt. To better understand the intuition behind the result, Figure 5 reports entrepreneurs' debt in the PL-c economy when the planner varies τ_b. The left panel shows that as τ_b increases entrepreneurs reduce their contingent debt in both states of the world, but much more in state H, so the ratio of L-contingent to H-contingent debt increases (right panel). Thus, a uniform tax on borrowing is not particularly effective in reducing the risk taking of entrepreneurs, because entrepreneurs substitute a lower level of leverage with less state-contingency in future debt payments. Because of this feature, the planner chooses essentially not to engage in macroprudential policy. In models without state-contingent-debt the private sector cannot respond to a tax by lowering leverage and at the same time reducing the degree of state contingency, so a uniform tax on debt is more effective.

We summarize the discussion above in two observations. First, when borrowers have means to adjust the state-contingency of their liabilities, the welfare benefits of a uniform tax on leverage may be overstated. Second, the ability of regulation to reduce financial amplification and improve welfare rests crucially on the ability to discourage the riskier

forms of borrowing with targeted instruments.

In this section, we have developed these arguments in terms of Pigouvian taxes. However, it is possible to derive similar results by considering a planner that imposes quantitative restrictions on leverage. A restriction on debt state-by-state corresponds to the PL case, while a restriction on the total value of debt to the PL-c case. A restriction on the total value of debt that includes different weights on *L*-contingent and *H*-contingent debt can mimic the PL case. So, in this sense, our result points to the importance of choosing appropriate risk-weights when regulating the liability side of balance sheets.

6   Conclusion

In this paper we have asked why financial institutions tend to be exposed to aggregate risk despite the availability of several instruments to hedge this exposure. To answer this question, we have used a canonical financial accelerator model in which agents trade fully state-contingent claims. We have obtained two main results.

First, we showed that entrepreneurs may not hedge negative aggregate shocks in equilibrium because insuring these states can be too costly for them. We have isolated the importance of two factors for this result: the general equilibrium spillover of entrepreneurs' net worth on consumers' labor income and the risk aversion of consumers. Under plausible calibrations of our model, these two effects are strong enough to make the productive sector as exposed to aggregate risk as it would be in a corresponding economy where only a non-state-contingent bond can be used for risk-management. These results show that it is feasible to introduce risk-management considerations in this class of models without compromising their ability to generate financial amplification.

Second, we showed that the resulting competitive equilibrium is constrained inefficient and it features too much exposure of entrepreneurs to aggregate risk. In the optimal policy, a planner reduces this exposure by taxing only certain debt instruments, specifically those whose payments are indexed to the negative aggregate shocks. On the contrary, uniform taxes on all debt instruments, despite reducing overall leverage, are not effective in limiting the entrepreneurs' risk exposure because they incentivize a substitution toward riskier debt instruments. More generally, our results emphasize that macroprudential policies targeted toward certain debt instruments can be substantially more effective than policies that discourage leverage *tout court*—a common prescription of the incomplete market models used in the literature.

These policy prescriptions are obtained in an environment where a full set of state-

contingent claims is available. In future research on macroprudential policy, it may be useful to consider models in between the two extremes of no state contingency and full state contingency, to capture more realistically the set of risk-management tools available to financial institutions.

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by Luigi Bocola and Guido Lorenzoni

A Recursive equilibrium and numerical solution

The aggregate state vector is $\mathbf{s} = [u, K, \tilde{N}]$, where K denotes the aggregate capital stock and $\tilde{N}$ is defined by

$$\tilde{N} = \theta(1 - \delta)uK - B, \tag{A.1}$$

with B being the total claims entrepreneurs need to pay to consumers. The state follows the law of motion $\mathbf{\Gamma}(.)$ with transition matrix $\pi(\mathbf{s}'|\mathbf{s})$ and both consumers and entrepreneurs observe this. By a slight abuse of notation, $w(\mathbf{s})$ denotes the wage as a function of the state $\mathbf{s}$ and $q(\mathbf{s}'|\mathbf{s})$ the price of an Arrow-Debreu security that pays one unit of consumption next period if the state is $\mathbf{s}'$.

The representative consumer's problem is then

$$V(a; \mathbf{s}) = \max_{c,l,a'(\mathbf{s}')} \left\{ (1 - \beta) \left(c - \chi \frac{l^{1+\psi}}{1+\psi} \right)^{1-\rho} + \beta \left[\mathbb{E}_{\mathbf{s}} \left(V(a'(\mathbf{s}'); \mathbf{s}')^{1-\sigma} \right) \right]^{\frac{1-\rho}{1-\sigma}} \right\}^{\frac{1}{1-\rho}} \quad \text{s.t.}$$

$$c + \sum_{\mathbf{s}'} q(\mathbf{s}'|\mathbf{s})a'(\mathbf{s}') \leq w(\mathbf{s})l + a.$$

The representative entrepreneur's problem is

$$V^e(b, k; \mathbf{s}) = \max_{c_e, k', l, b'(\mathbf{s}')} \left\{ \log(c_e) + \beta_e \mathbb{E}_{\mathbf{S}} \left[V^e(b'(\mathbf{s}'), k'; \mathbf{s}') \right] \right\} \quad \text{s.t.}$$

$$n = (uk)^\alpha l^{1-\alpha} - w(\mathbf{s})l + (1-\delta)uk - b$$

$$c_e + k' \leq n + \sum_{\mathbf{s}'} q(\mathbf{s}'|\mathbf{s})b'(\mathbf{s}')$$

$$b'(\mathbf{s}') + \gamma w(\mathbf{s}')l' \leq \theta(1-\delta)u'k'.$$

We can now define a recursive competitive equilibrium.

Definition A-1. *A recursive competitive equilibrium is given by value functions and policy functions for consumers and for entrepreneurs and pricing functions $\{q(\mathbf{s}'|.), w(.)\}$ such that (i) consumers' and entrepreneurs' policies solve their decision problems taking prices as given; (ii) the labor market and the markets for contingent claims clear; (ii) the law of motion $\mathbf{\Gamma}(.)$ is consistent with agents' optimization.*

The following result simplifies the numerical computation of the equilibrium

Lemma A-1. *The consumption function of entrepreneurs' is linear in net worth,*

$$c_e(b, k; \mathbf{s}) = (1 - \beta_e)n(\mathbf{s}).$$

Proof. Consider the problem of entrepreneurs, and suppose that we know the optimal policy for the ratios $\tilde{b} = b/(uk)$ and $\tilde{l} = l/(uk)$. Under the optimal policy, these ratios must satisfy the collateral constraint in the entrepreneur's problem. So, given $\tilde{b}'(\mathbf{s}')$ and $\tilde{l}(\mathbf{s})$, we can solve for the optimal consumption/investment problem of the entrepreneur by solving

$$V^e(k; \mathbf{s}) = \max_{c_e, k'} \left\{ \log(c_e) + \beta_e \mathbb{E}_{\mathbf{S}} \left[V^e(b'(\mathbf{s}'), k'; \mathbf{s}') \right] \right\} \quad \text{s.t.}$$

$$c_e + \left[1 - \sum_{\mathbf{s}'} q(\mathbf{s}'|\mathbf{s}) b'(\mathbf{s}') u' \tilde{b}'(\mathbf{s}') \right] k' \leq \left[\tilde{l}(\mathbf{s})^{1-\alpha} - w(\mathbf{s}) \tilde{l}(\mathbf{s}) + (1 - \delta) - \tilde{b} \right] uk,$$

This is an optimal saving problem with log utility and a single asset that pays the stochastic return

$$\frac{\left[\tilde{l}(\mathbf{s}')^{1-\alpha} - w(\mathbf{s}') \tilde{l}(\mathbf{s}') + (1 - \delta) - \tilde{b}'(\mathbf{s}') \right] u'}{\left[1 - \sum_{\mathbf{s}'} q(\mathbf{s}'|\mathbf{s}) b'(\mathbf{s}') u' \tilde{b}'(\mathbf{s}') \right]}.$$

Following standard arguments, we can then show that the optimal consumption policy satisfies

$$c_e = (1 - \beta_e)n(\mathbf{s}),$$

where

$$n(\mathbf{s}) = \left[\tilde{l}(\mathbf{s})^{1-\alpha} - w(\mathbf{s}) \tilde{l}(\mathbf{s}) + (1 - \delta) - \tilde{b} \right] uk.$$

$\square$

We solve for a competitive equilibrium using a global solution algorithm. Let $\{\mathbf{s}_i\}$ be a set of points in the state space, and let $\{K'_i, V_i, q_{Li}, q_{Hi}\}$ be an initial guess for each collocation point i for the next period capital stock, the consumers' value function and the price of Arrow-Debreu securities paying one unit of consumption next period if $u' = u_L$ and $u' = u_H$ respectively. Our algorithm updates this guess until a convergence criterion is satisfied.

Before discussing the details of the algorithm, let us first explain how we can update an initial guess.

Updating the initial guess. In order to update K_i' we can use the Euler equation for capital (7), which we report here for convenience

$$1 = \beta_e \sum_s \pi_{si} \left\{ \frac{N_i}{N_{si}'} \left[\alpha u_s^\alpha (\hat{K}_i')^{\alpha-1} (L_{si}')^{1-\alpha} + (1-\delta) u_s \right] \right\} +$$
$$+ (1-\beta_e) N_i \theta (1-\delta) \sum_s \pi_{si} u_s \mu_{si}'. \tag{A.2}$$

In the above notation, π_{si} is the conditional probability of $u' = u_s$ given that today we are in state $\mathbf{s}_i$ and $\hat{K}_i$ is the updated choice of capital. Note that we have made use of Lemma A-1 and substituted for entrepreneurs' consumption.

In order to obtain $\hat{K}_i$ from this expression we need $(N_i, N_{si}', L_{si}', \mu_{si}')$ at each collocation point $\mathbf{s}_i$ and each realization of the of the capital quality shock tomorrow $u' = u_s$. Below we explain how we can obtain these variables using the initial guess and the equilibrium conditions of the model.

To compute N_i, we can first compute aggregate labor at $\mathbf{s}_i$ using the expression

$$L_i = \min \left\{ \left[\frac{(1-\alpha)(u_i K_i)^\alpha}{\chi} \right]^{\frac{1}{\alpha+\eta}}, \left(\frac{\tilde{N}_i}{\chi \gamma} \right)^{\frac{1}{1+\eta}} \right\}. \tag{A.3}$$

The first expression in the "min" is aggregate labor when the collateral constraint does not bind, and it is obtained by equating (3) and (6) for $\mu_t = 0$. The second expression is labor when the constraint binds, and it follows from the definition of $\tilde{N}$, the collateral constraint (1), and the fact that $W_i L_i$ equals $\chi L_i^{1+\psi}$ from the consumers' optimal labor supply (3).

Given this expression for labor, we have that aggregate net worth is given by

$$N_i = (u_i K_i)^\alpha L_i^{1-\alpha} - \chi L_i^{1+\psi} + (1-\delta)(1-\theta) u_i K_i + \tilde{N}_i, \tag{A.4}$$

where we have used the definition of $\tilde{N}$.

The values for $(N_{si}', L_{si}', \mu_{si}')$ depends on whether the collateral constraints bind or not at $u' = u_s$. First, we guess that both constraints do not bind. Setting $\mu_{si}' = 0$, we can compute N_{si}' for $s = \{L, H\}$ using the risk-sharing conditions (5), the value for N_i obtained earlier and the guess for the price of state-contingent claims,

$$N_{si}' = \pi_{si} \frac{\beta_e N_i}{q_{si}}.$$

3

In addition, if the constraints do not bind, we have that

$$L'_{si} = \left[\frac{(1-\alpha)(u_s K'_i)^\alpha}{\chi} \right]^{\frac{1}{\alpha+\eta}}.$$

To verify our guess, we can compute B'_{si} from the definition of net worth,

$$B'_{si} = (u_s K'_i)^\alpha (L'_{si})^{1-\alpha} - \chi (L'_{si})^{1+\psi} + (1-\delta) u_s K'_i - N'_{si},$$

and our guess is verified if $B'_{si} \leq \theta(1-\delta) u_s K'_i - \gamma \chi (L'_{si})^{1+\psi}$. If the guess is verified, we can update the policy function of capital $\hat{K}'_i$ using the unconstrained Euler equation of entrepreneurs,

$$\beta_e \sum_s \pi_{si} \frac{N_{si}}{N'_{si}} \left(\alpha u_s^\alpha (\tilde{K}'_i)^{\alpha-1} (L'_{si})^{1-\alpha} + (1-\delta) u_s \right) = 1.$$

If $B'_{si} > \theta(1-\delta) u_s K'_i - \gamma \chi (L'_{si})^{1+\psi}$ for some s, we need to solve for the next period constrained allocation conditional on $u' = u_s$. This is done by solving a fixed-point problem in N'_{si}. To do so, we can write

$$(1-\beta_e) N'_{si} \mu'_{si} = \frac{q_{si} N'_{si}}{\beta_e \pi_{si} N_i} - 1 \tag{A.5}$$

using equation (5). Using the demand and supply of labor, we can express L'_{si} as a function of N'_{si} and of the initial guess for capital K'_i,

$$L'(N'_{si}) = \left[\frac{(1-\alpha)(u_s K'_i)^\alpha}{\chi \left[1 + \gamma \left(\frac{q_{si} N'_{si}}{\pi_{si} \beta_e N_i} - 1 \right) \right]} \right]^{\frac{1}{\alpha+\eta}}. \tag{A.6}$$

The fixed-point problem consists in choosing N'_{si} so that the following equation is satisfied

$$N'_{si} = (u_s K'_i)^\alpha L'(N'_{si})^{1-\alpha} - (1-\gamma)\chi L'(N'_{si})^{1+\psi} + (1-\theta)(1-\delta) u_s K'_i.$$

Once we have found N'_{si}, we can compute (μ'_{si}, L'_{si}) using equations (A.5) and (A.6). We can then use equation (A.2) to obtain an update of next period capital, $\hat{K}'_i$.

Note that as a by-product of this previous step we also obtain an expression for B'_{si} at each collocation point and each state $s = \{L, H\}$ given the initial guess $\{K'_i, q_{si}\}$. Given B'_{si} and the initial guess K'_i, we can compute $\tilde{N}'_{si}$ using equation (A.1).

Let's now move to the update of $\{V_i, q_{Li}, q_{Hi}\}$. Using the fact that entrepreneurs' consumption is linear in net worth, we can get the aggregate consumption of consumers from

the resource constraint

$$C_i = (u_i K_i)^\alpha L_i^{1-\alpha} + (1-\delta) u_i K_i - (1-\beta_e) N_i - K_i'. \qquad (A.7)$$

The update for the value function of consumers at $\mathbf{s}_i$, $\tilde{V}_i$, is then given by

$$\hat{V}_i = \left\{ (1-\beta) \left(C_i - \chi \frac{L_i^{1+\psi}}{1+\psi} \right)^{1-\rho} + \beta \left[\sum \pi_{si} (V_{si}')^{1-\sigma} \right]^{\frac{1-\rho}{1-\sigma}} \right\}^{\frac{1}{1-\rho}},$$

where V_{si}', the value function at the point $(u_s, K_i', \tilde{N}_{si}')$, is obtained by interpolation using the initial guess V_i.

The update for the price of Arrow-Debreu securities, $\{\hat{q}_{Li}, \hat{q}_{Hi}\}$ is obtained similarly using equation (2) in the main text. This step requires interpolating the policy function for capital in order to obtain C_{si}'.

Numerical algorithm. We can now describe the numerical algorithm for the obtaining a competitive equilibrium:

> **Step 0: Defining the grid.** First, let $\mathcal{U} = [u_L, u_H]$. Set upper and lower bounds on the state variables $(K, \tilde{N})$, and construct for each of these a set of points $\mathcal{K} = [K_1, \ldots, K_{N_K}]$, $\tilde{\mathcal{N}} = [\tilde{N}_1, \ldots, \tilde{N}_{N_{\tilde{N}}}]$. The grid $\mathcal{S}$ is constructed by taking the Cartesian product of $\mathcal{U}, \mathcal{K}, \tilde{\mathcal{N}}$.

> **Step 1: Equilibrium conditions at the candidate solution.** Start with a guess at each collocation point $\{K_i', V_i, q_{Li}, q_{Hi}\}$. Following the steps outlined earlier, update the guess $\{\hat{K}_i', \hat{V}_i, \hat{q}_{Li}, \hat{q}_{Hi}\}$.

> **Step 2: Iteration.** Compute the Euclidean distance between the initial and updated guess at every collocation point, and let r to be the maximum distance. If $r \leq 10^{-6}$, stop the algorithm. If not, update the guess and repeat Step 1-2. $\square$

The specifics for the algorithm are as follows. The upper bound on K is 15% above its value in a deterministic steady state while the lower bound is 200% below this value. The points for $\tilde{N}$ are between $[0, 5]$. We let $N_K = 61$ and $N_{\tilde{N}} = 41$. So, we have a total of 5002 collocation points. The initial guess for $\{K_i', V_i, q_{Li}, q_{Hi}\}$ is obtained from the solution of the first-best economy. After every iteration, the new guess for variable x_i is

$$x_i = \alpha x_i + (1-\alpha)\hat{x}_i,$$

5

where $\alpha = 0.8$ and $\hat{x}_i$ is computed as described above.

B The "AK" economy

We consider two modifications to the model of Section 2. First, we assume that the production function is linear in capital,

$$y_t = u_t k_{t-1},$$

with u_t being an iid stochastic process. Because labor is not a factor of production, consumers do not earn labor income and the collateral constraint is as in equation (1) with $\gamma = 0$. Second, we assume that consumers and entrepreneurs have the same CRRA preferences,

$$u(c(s^t)) = \frac{c(s^t)^{1-\sigma}}{1 - \sigma}.$$

Let us denote

$$A_t = u_t[1 + (1 - \delta)].$$

We make the following restrictions on the distribution of A_t,

$$\beta \mathbb{E}\left[A_{t+1}^{1-\sigma}\right] \leq 1, \qquad \left\{\theta\beta + (1-\theta)\beta_e \mathbb{E}\left[A_{t+1}^{1-\sigma}\right]\right\}^{\frac{1}{\sigma}} \frac{\beta}{\theta\beta + (1-\theta)\beta_e} \leq 1. \qquad \text{(A.8)}$$

These assumptions are satisfied for any distribution of u when $\sigma = 1$.

The next proposition characterizes the competitive equilibrium of this economy.

Proposition A-1. *Suppose that the restrictions in (A.8) are satisfied. Then, there is a stationary equilibrium in which*

$$b_{t+1} = \theta A_t k_t \qquad k_t = \kappa A_t k_{t-1} \qquad c_{e,t} = h A_t k_{t-1} \qquad c_t = f A_t k_{t-1}. \qquad \text{(A.9)}$$

Proof. To prove the proposition, we verify that the allocation in (A.9) satisfies the necessary and sufficient conditions for an equilibrium for some (κ, h, f).

First, given the allocation in (A.9), we can write the resource constraint as

$$f + \kappa + h = 1.$$

Thus, we need to show that there exists $\kappa > 0$ and $h > 0$ that satisfy the optimality conditions of consumers and entrepreneurs, and such that $\kappa + h < 1$.

Given the allocation in (A.9) the growth rate of consumption of entrepreneurs and consumers is given by κA_{t+1}. Because $\beta > \beta_e$, we have that the risk-sharing conditions are satisfied with the collateral constraint binding in every state of the world,

$$\mu_{t+1} c_{e,t} = (\beta - \beta_e)(A_{t+1}\kappa)^{-\sigma}.$$

Using this expression, we can write the Euler equation for capital, (7), as follows

$$1 = \mathbb{E}_t \left\{ \left[\theta\beta (A_{t+1}\kappa)^{-\sigma} + (1-\theta)\beta_e (A_{t+1}\kappa)^{-\sigma} \right] A_{t+1} \right\}.$$

Solving for κ, we obtain,

$$\kappa^\sigma = (\theta\beta + (1-\theta)\beta_e)\mathbb{E}[A_{t+1}^{1-\sigma}] < 1 \tag{A.10}$$

The budget constraint of entrepreneurs can be written as

$$k_t + c_{e,t} = (1-\theta)A_t k_{t-1} + \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \theta A_{t+1} k_t \right].$$

Substituting the allocation in (A.9) in the above equation and rearranging terms we obtain

$$h = (1-\theta)\left(1 - \kappa \frac{\beta_e}{\theta\beta + (1-\theta)\beta_e} \right) < 1. \tag{A.11}$$

Given the restrictions in (A.8), we can easily verify from equations (A.10) and (A.11) that $\kappa + h < 1$. □

Given the allocation in (A.9), we can obtain the impulse response function of capital to a percentage increase in A_t,

$$\frac{\partial \mathbb{E}_t[\log k_{t+j}]}{\partial \log A_t} = 1 \quad \forall j \geq 0.$$

It is straightforward to verify that this is the same impulse response function that would arise in a version of the model without the collateral constraint. In other words, the "AK" economy with state-contingent claims features no financial amplification relative to the first-best economy.

C Model without general equilibrium spillover

In this section we explain how we eliminate the general equilibrium spillover on wages. In the first best economy, labor income can be written as

$$LI(u, K^{fb}) = \chi \left[\frac{(1-\alpha)(uK^{fb})^{\alpha}}{\chi} \right]^{\frac{1+\eta}{\alpha+\eta}}. \tag{A.12}$$

To eliminate the general equilibrium spillover of entrepreneurs' net worth on consumers' labor income we proceed as follows. We assume that consumers solve the same decision problem as in Section 2, with the exception that their wages are the ones of the first best. That is, consumers supply labor as if they were working for firms operating in an economy without the collateral constraints (1), and their labor income is given by (A.12).

Letting the aggregate state vector be $\mathbf{s} = [u, \tilde{N}, K, K^{fb}]$, we can write the consumers' problem as

$$V(a; \mathbf{s}) = \max_{c, a'_{\mathbf{s}'}} \left\{ (1-\beta) \left(c - \frac{LI(u, K^{fb})}{1+\psi} \right)^{1-\rho} + \beta \left[\mathbb{E}_{\mathbf{s}} \left(V(a'_{\mathbf{s}'}; \mathbf{s}')^{1-\sigma} \right)^{\frac{1-\rho}{1-\sigma}} \right] \right\}^{\frac{1}{1-\rho}} \quad \text{s.t.}$$

$$c + \sum_{\mathbf{s}'} q(\mathbf{s}'|\mathbf{s}) a'_{\mathbf{s}'} \leq LI(u, K^{fb}) + a,$$

where we have substituted for the optimal labor supply of consumers if they were to face the wage process of the first best economy.

The entrepreneurs, instead, solve the same decision problem as before. One way of interpreting this extension is that there are two consumers in the economy: hand to mouth consumers that work for entrepreneurs and supply labor optimally, and Ricardian consumers that work for a sector that does not face the collateral constraints (1) and that trade contingent claims with entrepreneurs.

In equilibrium, we require that *i)* the supply of labor by the hand to mouth consumers to equal the demand of labor by entrepreneurs; and *ii)* the supply of bonds by entrepreneurs equal the demand of bonds by the Ricardian consumers. Note that the resource constraint in this economy will not be satisfied because the labor income earned by Ricardian consumers differ from the payments for labor services by entrepreneurs.

To solve numerically for an equilibrium, we proceed in two steps. In the first step, we solve the equivalent first best economy and obtain the policy function $K'^{fb}(u, K^{fb})$. This policy function is relevant because it allows to forecasts the future labor income of

Ricardian consumers according to equation (A.12). In the second step, we solve for the decision problem of consumers and entrepreneurs and make sure that the entrepreneurs' labor market and the market for contingent claims clear.

D   Solving the planner's problem

In this section we explain how we solve numerically for the competitive equilibrium with taxes. Let $\mathbf{s} = [u, K, \tilde{N}]$ be a point in the state space, and consider the following problem

$$
\max_{C, C_e, K', \tilde{N}'(\mathbf{s}')} (1 - \beta) \left(C - \chi \frac{L^{1+\psi}}{1+\psi} \right)^{1-\rho} + \beta \left[\mathbb{E}_\mathbf{s} \left[V \left(A'(\mathbf{s}'); u', \tilde{N}(\mathbf{s}'), K' \right) \right]^{1-\sigma} \right]^{\frac{1-\rho}{1-\sigma}}
$$

subject to

$$
\begin{aligned}
C + C_e + K' &= (uK)^\alpha L^{1-\alpha} + (1-\delta)uK \\
\log(C_e) + \beta_e \mathbb{E}_s \left[V^e \left(B'(\mathbf{s}'), K'; u', \tilde{N}'(\mathbf{s}'), K' \right) \right] &\geq V^e(B, K; \mathbf{s}) \\
L &= \min \left\{ \left[\frac{(1-\alpha)(uK)^\alpha}{\chi} \right]^{\frac{1}{\alpha+\eta}}, \left(\frac{\tilde{N}}{\chi\gamma} \right)^{\frac{1}{1+\eta}} \right\} \\
B'(\mathbf{s}') = A'(\mathbf{s}') &= \theta(1-\delta)u'K' - \tilde{N}'(\mathbf{s}'),
\end{aligned}
$$

where V and V^e are the value function of the competitive equilibrium without taxes.

The solution to this problem delivers an upper bound to the utility that consumers achieve in the competitive equilibrium with taxes defined in Section 5.2, because the latter must satisfy all the above constraints. It is also possible to show that a solution to the above problem can be implemented as a competitive equilibrium with an appropriate choices of taxes and transfers. Thus, we can find the optimal policy by solving the social planner problem above.

In what follows, we first explain how we solve this problem numerically. Next, we show how to construct the taxes and transfers that implement the optimal allocation as a competitive equilibrium.

Solving the planner's problem numerically.   To solve the planner's problem, let's fix a point in the state space $\mathbf{s}_i$, and let V_i^e be the value of entrepreneurs in the competitive equilibrium with taxes at $\mathbf{s}_i$. Construct a grid of feasible values for $[K', \tilde{N}'_H, \tilde{N}'_L]$. For each point in the grid, we perform the following steps in order to evaluate the objective function:

i Given a point in the grid, $[K', \tilde{N}'_H, \tilde{N}'_L]$, compute the continuation values of entrepreneurs by interpolating V^e. Choose C_e so that entrepreneurs get as much utility as in the competitive equilibrium without taxes, V^e_i.

ii Given $[K', \tilde{N}'_H, \tilde{N}'_L]$ and C_e, compute C using the resource constraint and the expression for labor in the constraint set.

iii Given $[K', \tilde{N}'_H, \tilde{N}'_L]$, use V to interpolate the continuation values for consumers.

Given these three steps, we can evaluate the welfare of consumers at any point in the grid. We then choose the point that maximizes consumers' welfare. We repeat this procedure for all the points in the state space $\{\mathbf{s}_i\}$.

Implementing the allocation as a competitive equilibrium with taxes and transfers. Let $\{C^p, C^p_e, (K')^p, (\tilde{N}'_H)^p, (\tilde{N}'_L)^p\}$ be the allocation that achieves the optimum in the planner's problem at $\mathbf{s}_i$. Using the policy functions of the laissez-faire competitive equilibrium and $(K'^p, (\tilde{N}'_s)^p)$, we can compute the next period consumption, labor, and the value functions of consumers for any $u' = u_s$. Let us label those (C'_s, L'_s, V'_s). For consumers to be willing to choose (C^p, L^p, C'_s, L'_s) in the competitive equilibrium, the price of state-contingent claims that pays if $u' = u_s$ must satisfy

$$
q_s = \beta \pi_{si} \left(\frac{C^p - \chi \frac{L^{1+\psi}}{1+\psi}}{C'_s - \chi \frac{(L'_s)^{1+\psi}}{1+\psi}} \right)^\rho \left(\frac{RW}{V'_s} \right)^{\sigma - \rho},
$$

where π_{si} is the conditional probability of $u' = u_s$ given $\mathbf{s}_i$.

Let us now consider entrepreneurs. Using $\{C^p, C^p_e, (K')^p, (\tilde{N}'_H)^p, (\tilde{N}'_L)^p\}$ and the policy functions of the competitive equilibrium, we can calculate the implied future net worth for any $u' = u_s$, N'_s, and the implied value for the Lagrange multiplier, μ'_s. A necessary conditions for $\{C^p_e, (K')^p, (\tilde{N}'_s)^p\}$ to be chosen by the entrepreneurs is that the taxes on debt and capital satisfy

$$
1 - \tau_b(s) = \frac{\beta_e \pi_{si}}{q_s} \left[\frac{C^p_e}{(1 - \beta_e) N'_s} + C_e \mu'_s \right] \qquad \text{for } s = \{H, L\} \tag{A.13}
$$

$$
1 + \tau_k = \beta_e \sum_{s=\{H,L\}} \pi_{si} \left\{ \frac{C^p_e}{(1 - \beta_e) N'_s} \left[\alpha u^\alpha_s ((K')^p)^{\alpha - 1} (L'_s)^{1-\alpha} + (1 - \delta) u_s \right] \right\} +
$$

$$
+ \ C^p_e \theta (1 - \delta) \sum_s \pi_{si} u_s \mu'_{si}. \tag{A.14}
$$

The taxes defined in (A.13) and (A.14) guarantee that the entrepreneurs' first order con-

ditions are satisfied at the planner's allocation. The transfers (T_b, T_e) can then be chosen so that the budget constraint of entrepreneurs, consumers and the government is satisfied.