

NBER WORKING PAPER SERIES

LEVERAGE DYNAMICS AND FINANCIAL FLEXIBILITY

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Working Paper 26802
<http://www.nber.org/papers/w26802>

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
February 2020

We are grateful to Harry DeAngelo, Jan Eberly, Zhiguo He, Harrison Hong, Arthur Korteweg, Ye Li, Erwan Morellec, Michael Roberts, Jean-Charles Rochet, Tom Sargent, Stijn Van Nieuwerburgh, and Laura Veldkamp for helpful comments. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

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NBER Working Paper No. 26802
February 2020
JEL No. G11,G31,G32,G35

ABSTRACT

We develop a q theory of investment with endogenous leverage, payout, hedging, and risk-taking dynamics. The key frictions are costly equity issuance and incomplete markets. We show that the marginal source of external financing on an on-going basis is debt. The firm lowers its debt when making a profit, increases its debt in response to losses and induced higher interest payments, and even taps external equity markets at a cost before exhausting its endogenous debt capacity. The firm seeks to preserve its financial flexibility by prudently managing its leverage and investment. Paradoxically, it is the high cost of equity issuance that causes the firm to keep leverage low, in contrast to the predictions of static Modigliani-Miller tradeoff and Myers-Majluf pecking-order theories. Our model generates leverage and investment dynamics that are consistent with the empirical evidence.

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1 Introduction

In his AFA presidential address DeMarzo (2019) states that “Capital structure is not static, but rather evolves over time as an aggregation of sequential decisions in which shareholders have an incentive to act strategically, maximizing the share price at the potential expense of creditors.” He further notes that in his model “absent commitment a Modigliani-Miller-like value irrelevance and policy indeterminacy result holds.”

In this paper, we develop a dynamic capital structure model with no commitment, but with incomplete markets and costly equity issuance. We show that unlike in DeMarzo and He (2016) and DeMarzo (2019), a firm’s dynamic capital structure decisions affect its value despite the lack of any commitment.¹ There are three critical differences between our model.

First, we assume that issuing equity is costly, as has been emphasized by Myers and Majluf (1984) and many follow-up work, and has been confirmed in numerous empirical studies.² If that were not the case (as most dynamic capital structure models assume) the firm’s optimal dynamic financial policy would be to choose a target leverage at inception and stick to it by offsetting profits with a commensurate increase in debt and losses with a commensurate equity issue. The firm would stay at its target leverage unless it incurs an unhedgeable loss that is so large that it is ex-post optimal for shareholders to default.

Second, we assume that financial markets are incomplete. Why is this assumption important? Suppose that financial markets were (locally) complete. A firm could then avoid costly external financing and default by using fairly priced financial instruments to hedge its risk. It would be able to avoid costly default or equity issuance through dynamic hedging (e.g., via one-period-ahead Arrow securities). This is the fundamental insight of Arrow (1964), Black and Scholes (1973), Merton (1973), Harrison and Kreps (1979), Kehoe and Levine (1993), Kocherlakota (1996), and Alvarez and Jermann (2000). However, with locally incomplete financial spanning, the firm’s ability to manage its risk to avoid default is limited.³

¹Admati, DeMarzo, Hellwig, and Pfleiderer (2018) show that absent commitment shareholders choose to issue additional debt no matter how excessive the firm’s current leverage, which they refer to as the leverage ratchet effect (also see Bizer and DeMarzo, 1992).

²See e.g. Altinkilic and Hansen (2000) and Eckbo, Masulis and Norli (2007).

³The possibility of default provides a partial hedge against risks that cannot be insured because of limited spanning, as emphasized by Zame (1993) and Dubey, Geanakoplos and Shubik (2005). Unlike these papers, we show that default provides a partial insurance for the firm even under risk neutrality. Although the implications of market incompleteness for capital structure choice have long been recognized in a static setting (see e.g. Hellwig, 1981), little attention has been devoted to the consequences of market incompleteness in a dynamic setting for the evolution of a financially constrained firm’s capital structure, its debt capacity, and its risk management policies.

Third, we assume that the firm relies on short-term debt as in Abel (2016, 2018). With these assumptions we formulate a meaningful and realistic theory of leverage dynamics, whereby the firm’s leverage increases in response to losses and decreases in response to profits. When leverage exceeds the firm’s target, the firm seeks to reduce its debt whenever possible. Leverage then only continues to grow as a result of losses. These leverage dynamics in the equity inaction region are consistent with the empirical evidence that firms use realized profits to retire some of their debts and reduce leverage.⁴

The assumption of market incompleteness takes its full significance when combined with the assumption that outside equity is costly.⁵ Because it is costly to do so, the firm only issues equity episodically when necessary. The cost of issuing equity has a first-order effect on both leverage dynamics and the level of debt. Unlike most other dynamic capital structure models, which assume a cost of adjusting debt rather than equity, the firm’s payout and issuance boundaries are far apart in our model precisely because equity issuance is costly. “Target” leverage is pinned down by the firm’s optimal payout boundary and its optimal equity issuance boundary determines when the firm recapitalizes to bring down its excessively high leverage.

The building blocks of our model are: 1) a firm with an illiquid capital stock that can be increased (decreased) through investments (divestments, asset sales) subject to adjustment costs, as in the q theory of investment;⁶ 2) the firm’s operations are exposed to both continuous diffusive shocks and discrete jump shocks;⁷ 3) although some shocks can be hedged at actuarially fair terms, other (larger) shocks cannot, as hedging instruments for these contingencies are missing, costly to arrange, or difficult to describe. An important source of unhedgeable shock is a *force majeure*, catastrophic, large downward jump, as in the rare-disasters literature, e.g., Rietz (1988) and Barro (2006).

We make three simplifying assumptions. First, we assume that the firm’s expected prof-

⁴Unlike existing dynamic tradeoff theories of capital structure the leverage dynamics in our model are not vulnerable to the ‘leverage-profitability puzzle’ critique (see Titman and Wessels, 1988, Rajan and Zingales, 1995, and Frank and Goyal, 2003).

⁵Otherwise, by simply issuing equity costlessly, the firm can do away with missing state-contingent hedging contracts even when markets are incomplete.

⁶Hayashi (1982) provides conditions under which average Q is equal to marginal q . Abel and Eberly (1994) develop a unified q theory of investment in neoclassic settings. Lucas and Prescott (1971) and Abel (1983) are important early contributions. McDonald and Siegel (1986) and Dixit and Pindyck (1994) formulate investment as a real-options problem.

⁷Jump diffusion models, e.g., affine jump diffusion models, are widely used in the credit-risk literature. See Duffie and Singleton (2012).

itability is constant and that its investment opportunities are time-invariant. Second, as in DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), DeMarzo and Fishman (2007a, 2007b), DeMarzo and He (2016), and DeMarzo (2019), we also assume that shareholders are risk neutral and impatient.⁸ By impatience, we mean that shareholders discount cash flows at the rate γ , which is larger than the risk-free rate r . This is the reason why debt is the preferred source of funding, other things equal. Third, we assume that all the firm's debt is short term. The main reason for these simplifying assumptions is to allow us to use the model's homogeneity property to reduce our original two-dimensional optimization problem to a one-dimensional problem, where the only state variable is *book leverage* $x_t = X_t/K_t$ (the ratio of the firm's debt outstanding, X_t , and its capital stock, K_t).⁹

The firm can adjust its debt at any time at no cost (in the absence of default), or issue costly external equity, or make a payout to shareholders, or declare default. In the model solution we jointly pin down the firm's equilibrium credit spread and its endogenous debt capacity. Finally, the firm can manage its risk exposures, albeit to a limited extent, via available insurance and hedging contracts. In response to a loss, the firm can also adjust its investments and conduct asset sales.

The firm continuously adjusts its leverage in response to shocks and manages its risk with the goal of going back to its long-run target leverage and minimizing the volatility of its leverage. The optimal dynamic financing policy features four, mutually exclusive, endogenously determined, regions: a payout region when leverage is below target, a debt financing (equity inaction) region when the firm incurs a loss and leverage is not too high, an equity financing region to recapitalize and bring down leverage, and, a default region. Since shareholders are protected by limited liability, the firm may find it optimal to exercise its default option when hit by a sufficiently large (unhedgeable) downward shock. Therefore, the firm may have to promise interest payments at a rate exceeding the risk-free rate.

Our model generates rich leverage dynamics. The firm optimally starts at its long-term target leverage \underline{x} at $t = 0$. For any $t > 0$, any realized profit that lowers x_t below \underline{x} triggers a payout to shareholders, and any realized loss increases leverage x and negatively affects

⁸In the dynamic-contracting literature, Biais, Mariotti, Rochet, and Villeneuve (2010), DeMarzo, Fishman, He, and Wang (2012), and Piskorski and Tchistyi (2010) all make similar assumptions. Brunnermeier and Sannikov (2014) make the same preferences assumption (risk-neutrality, limited liability, and impatience) for experts in their equilibrium model.

⁹If we allowed the firm's profitability and investment opportunities to vary we would have a second dimension, and if we also allowed the firm to issue term debt, we would have a third dimension. These are clearly interesting and relevant generalizations, but they are beyond the scope of this paper.

firm value. We show that both book leverage and market leverage are highly persistent, consistent with the evidence of Fama and French (2002). For low values of leverage, the firm has sufficiently high financial slack and its leverage in expectation drifts down. That is, the firm seeks to pay out to shareholders. When leverage is low, the firm has a large spare debt capacity, so that the (endogenous) marginal servicing costs of debt and hedging costs are low. More likely than not, the firm is then able to generate a sufficiently high profit to be able to pay down its debt. In contrast, when leverage is high, (endogenous) debt servicing and hedging costs are so high that the firm tends to be drawn into a debt spiral.

Since issuing equity is costly, the firm manages leverage mostly by relying on dynamic debt adjustments (in the equity inaction region.) Only a highly levered firm may choose to issue costly equity to reduce its leverage and replenish its financial slack, but at the cost of substantially diluting existing shareholders. If the loss is so high that the existing shareholders are completely wiped out, the firm chooses to default.

These results closely align with the evidence in DeAngelo, Goncalves, and Stulz (2018) who show that firms appear to behave like households with credit-card debt: they pay down their debt when they get a positive income shock and they increase their debt when they have no choice to do otherwise. Indeed, they conclude: “Debt repayment typically plays the main direct role in deleveraging.” Our results are also consistent with the findings of Korteweg, Schwert, and Strebulaev (2019) who find that firms tend to cover operating losses by drawing down a line of credit, giving rise to similar leverage dynamics as in our model.

The highly persistent dynamics of leverage in our model are also consistent with the findings of Lemmon, Roberts, and Zender (2007) who found that “the adjusted R-square from a regression of leverage on firm fixed effects (statistical ‘stand-ins’ for the permanent component of leverage) is 60%.” Indeed, a sensible predictor of leverage in the next interval of time dt in our model is simply current leverage x_t . Over a longer time horizon our leverage dynamics are consistent with the findings of DeAngelo and Roll (2015), who emphasize that leverage is far from time invariant. In line with empirical evidence our model also generates mean-reverting leverage, as in Collin-Dufresne and Goldstein (2001).

Importantly, our results are consistent with most of the evidence on leverage dynamics even though we have not assumed any adjustment costs for debt. The preferred interpretation of the empirical literature is that the observed leverage dynamics are a reflection of the existence of debt adjustment costs (Leary and Roberts, 2005). But, in practice, what is the size of the adjustment cost of drawing down a line of credit, or retiring debt? This cost

is likely to be far too small to explain observed leverage dynamics. Rather, we point in a different direction, equity issuance costs, which create a demand for flexibility and give rise to leverage dynamics similar to those observed in the data.¹⁰

Our analysis also generates novel results on the firm’s dynamic risk management policy. The firm’s incentives to hedge or to take risk crucially depend on the degree of market incompleteness and its ability to manage risk. If the firm’s hedging opportunities are sufficiently rich so that it can effectively manage most of its risk exposures, the firm behaves prudently, seeks to self-insure against its unhedgeable shocks, and delays costly external equity financing as much as possible. In this situation, the firm’s equity value is globally concave and there are three mutually exclusive regions: an equity-payout region, a leverage roll-over (equity inaction) region, and a default region. The firm fully exploits all available hedging opportunities and hence leverage only responds to unhedgeable diffusive and jump shocks. Because equity issuance is costly, the firm postpones its equity issuance until the very last moment.¹¹ Upon equity issuance, existing shareholders are wiped out and new shareholders take over the firm.

If markets are highly incomplete the firm may become a risk-seeker when its leverage is sufficiently high. Rather than purchasing insurance to hedge its insurable jump risk, the firm may decide to sell actuarially fair insurance contracts on its own downside risk in order to collect insurance proceeds and pay down its debt. But, surprisingly, when it is close to bankruptcy, selling insurance to pay down its debt may no longer be optimal. Instead, the firm then goes to the extreme of maxing out on insurance contracts with the hope that it will be the recipient of a large lump-sum insurance payment that will enable the firm to reduce its leverage to a manageable level. Thus, excessive insurance purchases become a form of extreme risk-seeking.¹²

We illustrate these rich predictions of our model on leverage dynamics and equity issuance along a simulated path described in Figure 2. The firm begins its life at its target leverage and is subjected to diffusion and jump shocks that are partially hedgeable. The sample path comprises a sequence of four different jump shocks. The first shock is small enough that it can be perfectly hedged. The second jump is too large to be hedged. The loss it causes

¹⁰In an influential CFO survey, Graham and Harvey (2001) find that financial flexibility is among CFO’s very top considerations.

¹¹There are actually four regions, but the “equity issuance” region is a singleton in this case.

¹²Della Seta, Morellec, and Zucchi (2020) develop a model showing that short-term debt and rollover losses can foster risk-taking when firms are close to financial distress.

forces the firm to increase market leverage from 37% to 77%, as seen in Panel B. Going forward, its market leverage is so high that the firm is trapped in a debt spiral. Eventually, the firm is pushed into a first recapitalization, which almost completely dilutes the original shareholders but reduces market leverage from 97% to 47%. The third jump again is too large to be hedged and forces the firm to immediately respond with a second recapitalization. The last jump is the finishing stroke and pushes the firm to default on its debt obligations, following which it is liquidated. Figure 2 illustrates the highly non-linear nature that leverage dynamics can take in an environment with incomplete markets and external financing costs. Such dynamics are impossible to fully capture in a reduced-form regression analysis and are difficult to reconcile with the target-leverage-with-debt-adjustment-costs view of Leary and Roberts (2005) and others.

Other related literature. Our model belongs to a growing literature on dynamic corporate finance theory initiated by Leland (1994). In his seminal contribution Leland (1994) considers a dynamic tradeoff theory model, which pits the tax benefits of debt against associated financial distress costs, in a model with diffusion shocks to earnings and with no investment. In his model the firm chooses the level of perpetual risky debt at inception with no cost but infinite costs thereafter.¹³

Cooley and Quadrini (2001), Gomes (2001), Hennessy and Whited (2005, 2007), Gamba and Triantis (2008), and DeAngelo, DeAngelo, and Whited (2011) develop dynamic capital structure models with investment. An important methodological difference of our analysis is the continuous-time formulation of the firm’s problem. As in Leland (1994, 1998), DeMarzo and Sannikov (2006), Brunnermeier and Sannikov (2014), DeMarzo and He (2016), and DeMarzo (2019), our continuous-time formulation allows a sharper characterization of the underlying economic tradeoffs and of the firm’s highly nonlinear, non-monotonic, state-contingent, path-dependent policies.¹⁴ For example, the endogenous debt capacity is characterized by an economically intuitive boundary condition where the volatility of leverage is zero in equilibrium.

One empirically counterfactual implication of Leland’s model is that once the firm is running, the only source of external financing is equity, which can be raised at no cost. No

¹³Fischer, Heinkel, and Zechner (1989) is an important early contribution to this literature. Goldstein, Ju, and Leland (2001) and Strebulaev (2007) generalize Leland (1994) to allow for dynamic recapitalization.

¹⁴In their survey, Brunnermeier and Sannikov (2016) discuss the advantages of continuous-time modeling in dynamic contracting and macro-finance contexts.

new debt is allowed by assumption. A key difference of our model with respect to Leland (1994) is that the marginal source of external financing on an ongoing basis is debt. It is almost as if we put the Leland model on its head when it comes to the firm’s external financing. In practice debt is generally cheaper to issue than equity. Moreover, at the margin firms generally use debt rather than equity when they raise external financing. Seasoned equity offerings are episodic. In addition, firms tend to preserve spare debt capacity, while at the same time using financial hedging instruments to manage their risk exposures. Both features are present in our model.

The other closely related dynamic corporate finance model is DeMarzo and He (2016) and DeMarzo (2019). They generalize Leland (1994, 1998) by allowing the firm to dynamically issue new *pari passu* term debt over time.¹⁵ Unlike in our model the firm’s debt adjustment is locally deterministic (debt does not respond to profit and loss realizations) and marginal changes in leverage do not affect firm value.

Another related model is Abel (2018), who develops a dynamic tradeoff model with short-term debt and an endogenous cost of default. By assumption the firm is not allowed to issue equity nor to retain earnings. Cash-flows are assumed to follow a Markov process in which earnings before interest and taxes (EBIT) remain unchanged for a random length of time, and a new value of EBIT arrives at a date governed by a Poisson process.

Our paper is related to the literature on dynamic liquidity and risk management. Examples include Decamps, Mariotti, Rochet, and Villeneuve (2011), Bolton, Chen, and Wang (2011), Hugonnier, Malamud and Morellec (2015), and Abel and Panageas (2020), building on the seminal contributions of Baumol (1952), Tobin (1956), and Miller and Orr (1966).¹⁶ Alvarez and Lippi (2009) develop a continuous-time dynamic household cash management. Hugonnier and Morellec (2017) develop a dynamic banking model with short-term debt. Our paper jointly analyzes the firm’s dynamic capital structure, investment, and hedging as well as risk taking decisions, and further pins down the firm’s endogenous debt capacity and equilibrium credit spreads.

Our paper is also related to the recent limited-commitment-based research on dynamic liquidity and risk management. Rampini and Viswanathan (2010, 2013) develop a limited-

¹⁵The firm is also allowed to repurchase debt, but that is never optimal in their setup.

¹⁶These papers focus on dynamic cash and corporate liquidity management but not leverage dynamics. For example, in Bolton, Chen, and Wang (2011), both debt capacity and credit spread are exogenous, and there is no notion of target leverage. In our paper, both debt capacity and credit spreads are endogenously determined. And the firm has a target leverage.

commitment-based theory of risk management that focuses on the trade-off between exploiting current versus future investment opportunities. Building on the insights of Hart and Moore (1994) and using the recursive contracting methodology of Sannikov (2008), Bolton, Wang, and Yang (2019) develop a theory of dynamic liquidity and risk management based on the inalienability of risky human capital.¹⁷ These models also generate endogenous debt capacity and optimal liquidity and risk management policies, but they do not generate default in equilibrium, as financial markets are assumed to be locally complete.

Finally, our model is related to the dynamic contracting and optimal dynamic security design literature which often features a combination of debt, (inside and outside) equity, and corporate liquidity as in DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), Biais, Mariotti, Rochet, and Villeneuve (2010), and DeMarzo and Fishman (2007b), and DeMarzo, Fishman, He, and Wang (2012).¹⁸

2 Model

We describe the firm's capital accumulation and production technology (the real side) in subsection 2.1, introduce the incomplete financial markets and financial policies (the financial side) that the firm make in subsection 2.2, and finally state the firm's objective and its optimization problem in subsection 2.3.

2.1 Capital Accumulation and Production

We use K and I to denote the level of the firm's capital stock and its gross investment, respectively. At each time t , the firm uses its capital stock K_t to produce cash flows proportional to its contemporaneous capital stock and equal to AK_t , where A is a constant that quantifies the productivity of capital.¹⁹ The price of capital is normalized to one, so that the firm's unlevered free cash flow net of investment is given by:

$$Y_t = AK_t - I_t. \tag{1}$$

¹⁷Ai and Li (2015) analyze an optimal contracting problem similar to the dual contracting problem in Bolton, Wang, and Yang (2019).

¹⁸Biais, Mariotti, and Rochet (2013) and Sannikov (2013) provide recent surveys on this subject. For static security design models, see Townsend (1979) and Gale and Hellwig (1985), Innes (1990), and Holmstrom and Tirole (1997, 1998).

¹⁹We can interpret this linear production function as one that features a constant-return-to-scale production function also involving other factors of production.

We assume that capital stock evolves according to the following process:²⁰

$$dK_t = \Psi(I_{t-}, K_{t-})dt + \sigma K_{t-}d\mathcal{B}_t^K - (1 - Z)K_{t-}d\mathcal{J}_t. \quad (2)$$

There are three terms contributing to the change in capital stock dK_t . The first term in (2), $\Psi(I_{t-}, K_{t-})$, corresponds to the rate of capital accumulation over time interval dt in the absence of diffusion shocks and jumps. As in the q theory of investment (Lucas and Prescott, 1971, Hayashi, 1982, Abel and Eberly, 1994, and Jermann 1998), we assume that the firm incurs capital-adjustment costs and that the capital stock depreciates over time. The function $\Psi(I_{t-}, K_{t-})$ captures both the costs of installing new capital and capital stock depreciation. As in Lucas and Prescott (1971) and Hayashi (1982), we also assume that $\Psi(I, K)$ is homogeneous of degree one in I and K , so that

$$\Psi(I, K) = \psi(i) \cdot K, \quad (3)$$

where $i = I/K$ denotes the investment-capital ratio.²¹ Given that more investment means more capital, we have $\psi'(i) > 0$. We further assume that $\psi(i)$ is concave and continuously differentiable in i .

The second term in (2) describes the Brownian shock, where σ is the diffusion-volatility parameter and \mathcal{B}^K is a standard Brownian motion. These continuous shocks can be thought of as stochastic capital depreciation shocks.

The third term in (2) describes discrete downward jumps in the capital stock, where \mathcal{J} is a jump process with a constant arrival rate λ . Let $\tau^{\mathcal{J}}$ denote the jump arrival time. If a jump does not occur at time t , so that $d\mathcal{J}_t = 0$, we have $K_t = K_{t-}$, where $K_{t-} \equiv \lim_{s \uparrow t} K_s$ is the left limit of K_t . If a jump does occur at time t , so that $d\mathcal{J}_t = 1$, the capital stock drops from K_{t-} to $K_t = ZK_{t-}$. We denote by $Z \in [0, 1]$ the *recovery fraction* of capital that survives the jump shock and assume that Z is distributed according to a well-behaved cumulative distribution function $F(Z)$. The size of a jump can be small or large. Note that a low realized value of Z means a large negative shock to the capital stock. We interpret these shocks as rare events that may put the firm into economic and financial distress.

²⁰Pindyck and Wang (2013) use this process in a general-equilibrium setting to quantify the economic cost of catastrophes. Brunnermeier and Sannikov (2014) and Barnett, Brock, and Hansen (2019) use the same capital accumulation process given in (2) with no jumps.

²¹Also see Boldrin, Christiano and Fisher (2001), Jermann (1998), and Brunnermeier and Sannikov (2014) among others for this widely-used specification.

2.2 Financial Markets and Corporate Financial Policies

We assume that investors are risk neutral.²² Let r denote the constant risk-free rate. In equilibrium investors break even by earning an expected rate of return that is equal to r . One key market imperfection in our model is that financial spanning is incomplete. In reality, a firm's ability to manage its risk exposure is often limited as certain types of risk that a firm faces are simply not hedgeable. Contractual incompleteness and/or costly financial intermediation may cause certain financial assets to be missing and markets to be incomplete. If the demand for certain hedging instruments is low, market making for these instruments are not profitable for intermediaries causing these hedging instruments to be missing.

2.2.1 Diffusion and Jump Hedging Contracts

First, we start with the diffusion hedging contracts.

Diffusion-Hedging Contracts. In practice, some diffusive shocks are easier to hedge than others in the market place.²³ To capture incomplete diffusion hedging, we assume that there exists a fully liquid diffusion-hedging contract, however, it is only partially correlated with the diffusion shock \mathcal{B}^K to capital stock. Let \mathcal{B}^S denote the standard Brownian motion that drives the payoff for this liquid hedging contract. Let ρ denote the constant correlation coefficient between the capital shock, \mathcal{B}^K , and the shock \mathcal{B}^S , which determines the payoff for the diffusion hedging contract. Therefore, we can equivalently express \mathcal{B}_t^K as follows:

$$\mathcal{B}_t^K = \rho \mathcal{B}_t^S + \sqrt{1 - \rho^2} \mathcal{B}_t^O, \quad (4)$$

where \mathcal{B}_t^O is a standard Brownian motion that is orthogonal to \mathcal{B}_t^S and captures the un-hedgeable diffusion risk.

An investor who holds one unit of the hedging contract from t to $t + dt$ receives a gain or loss equal to $\sigma d\mathcal{B}_t^S = \sigma (\mathcal{B}_{t+dt}^S - \mathcal{B}_t^S)$ at time $t + dt$ and incurs no upfront payment at t , as investors are risk neutral and markets are competitive.²⁴

²²We can generalize our model by allowing investors to be risk averse and well diversified. It is known that we can properly account for the risk-return adjustments by using the stochastic discount factor (SDF) to price the firm's free cash flows.

²³For example, diffusion shocks correlated with the stock market or commodities such as oil prices can be hedged via derivatives.

²⁴The hedging contract at inception is off-the-balance sheet but the realized payoff appears on the balance sheet. This hedging contract is analogous to a futures contract in standard no-arbitrage models, e.g., Cox, Ingersoll, and Ross (1981). We normalize the volatility of the diffusion-hedging contract so that it is equal to the capital diffusion-volatility parameter, σ , given in (2).

We denote the firm's demand for this hedging contract at time t by Θ_t . The realized instantaneous payoff for this position is then $\Theta_t \sigma d\mathcal{B}_t^S$. In general, the firm's hedging position is constrained.²⁵ Constraining the size of a firm's hedging position constitutes another form of market incompleteness. To capture this form of market imperfection, we assume that the firm's hedging demand for the hedgeable component of the diffusion risk in absolute value, $|\Theta_t|$, cannot exceed an exogenous upper bound: $\bar{\Theta}_t$.

For simplicity, we assume that $\bar{\Theta}_t$ is proportional to its capital stock K_t :

$$|\Theta_t| \leq \bar{\theta} K_t. \quad (5)$$

where $\bar{\theta}$ is a constant. The lower the value of $\bar{\theta}$, the tighter the hedging constraint. When $\bar{\theta} = 0$, the firm cannot hedge its diffusion risk at all. When $\bar{\theta} \rightarrow \infty$, this hedging constraint is removed. We allow for all values of $\bar{\theta}$.

To ease our exposition, we proceed by first analyzing the solution where the constraint (5) never binds and then proceed to the case where the constraint (5) binds under certain scenarios in Section 6. Next, we introduce the firm's options to manage jump risk.

Jump Insurance Contracts and Premium Payments. We consider the following insurance contract initiated at time $t-$ that covers the *first* stochastic arrival of a downward jump in capital stock, with a recovery fraction falling into the interval $(Z, Z + dZ)$ at the jump arrival time τ^J . The buyer of a unit of this jump insurance contract makes a premium payment per unit of time of $\lambda dF(Z)$, the product of the jump intensity, λ , and probability $dF(Z)$ that Z falls into the interval $(Z, Z + dZ)$ until the jump arrival time τ^J . When the jump event occurs at τ^J , the buyer stops making insurance payments and receives a unit lump-sum payoff. Conceptually, this jump insurance contract is analogous to a *one-step-ahead Arrow security* in discrete settings. In practice, this contract is similar to a credit default swap.²⁶

As with diffusion shocks, we assume that not all jumps can be hedged in financial markets. Specifically, we assume that jump-insurance contracts are only available for $Z \in [Z^*, 1]$. Here Z^* is a parameter that describes the level of financial spanning. The lower the value of Z^* ,

²⁵In practice, to reduce the likelihood of default, the firm may be required to post collateral and it may be required to pay hedging costs, both of which constrain the firm's hedging positions.

²⁶Pindyck and Wang (2013) and Rebelo, Wang, and Yang (2018) use similar insurance contracts in different economic applications.

the larger the available set of jump insurance contracts.²⁷

We use $\Pi_{t-}(Z)$ to denote the size of the firm's jump-risk insurance payment at time $t-$. The insurance premium payment is then equal to $\Pi_{t-}(Z)\lambda dF(Z)$ prior to the jump arrival, and at time $\tau^{\mathcal{J}}$, the firm receives the lump-sum payment $\Pi_{t-}(Z)$ if the defined event occurs.

Since the firm can purchase insurance for all levels of $Z \in [Z^*, 1]$, the maximum jump insurance premium payment per period is given by:

$$\Phi_{t-} = \int_{Z^*}^1 \Pi_{t-}(Z)\lambda dF(Z) \equiv \lambda \mathbb{E}[\Pi_{t-}(Z) \mathcal{I}_{Z \geq Z^*}] , \quad (6)$$

where the expectation, $\mathbb{E}[\cdot]$, is calculated with respect to the cumulative distribution function $F(Z)$. The indicator function $\mathcal{I}_{Z \geq Z^*}$ equals one if $Z \geq Z^*$ and zero otherwise. This indicator function imposes the restriction that jump insurance is available only for $Z \geq Z^*$.

2.2.2 Debt and Equity

We denote the firm's outstanding debt by X_t : when $X_t > 0$ it is in the debt region and when $X_t < 0$ the firm is in the savings region.

Debt. We assume that the firm issues short-term debt and it is costless to do so. After raising X_t via debt issue at t , over a small time increment dt , the firm decides at $t + dt$ whether to default on its existing debt or to roll over its short-term debt. Using the short-term debt allows us to simultaneously model the dynamics of both corporate liquidity and leverage in a parsimonious way.²⁸ Moreover, when the firm can issue short-term debt it faces a time-consistent dynamic optimization problem.²⁹

Since not all jump shocks can be hedged the firm may find it optimal to default on its debt obligations following the realization of some jump shocks, so that the firm's debt may be *risky*. The contractual rate of return for short-term debt initiated at t then exceeds the risk-free rate r by a *credit spread*, denoted by η_t , which is determined in equilibrium to

²⁷The alternative is to assume that small jump shocks are unhedgeable but large jump shocks are hedgeable. In this case, it is harder to make the firm default as large shocks are hedged. Empirically, we tend to observe that firms get into trouble with expensive equity issuance (at the cost of heavily diluting existing shareholders) or even end up with costly bankruptcy when they are exposed to large negative unhedgeable shocks. For this reason, we assume that large negative downward shocks are harder to hedge.

²⁸In the full-spanning case, introducing term debt adds no value and hence focusing on short-term debt is without loss of generality. The intuition is that for the full-spanning case, the firm can achieve any state-contingent allocation by dynamically managing its short-term debt and state-contingent risk management policies.

²⁹Rebelo, Wang, and Yang (2018) use short-term debt to model sovereign debt and default.

compensate creditors for the losses they bear if the firm defaults at t . We denote the firm's interest payment over time interval dt as

$$C_t = (r + \eta_t)X_t. \quad (7)$$

Since debt is short term, equity investors are expected to pay back both the interest and principal to creditors each period. At the next instant, the firm borrows again, rolling over its short-term debt and/or issuing equity. This process continues until the firm defaults.

Financial Distress. For simplicity, we assume that the firm is bankrupt after defaulting on its debt and is liquidated. In bankruptcy, the absolute priority rule (APR) holds, in that creditors are repaid before equity investors can collect any proceeds. As in Miller and Modigliani (1961) and Leland (1994), we assume that corporate bankruptcy causes deadweight losses. Let L_{τ^D} denote the firm's liquidation value at the moment of default τ^D . To preserve our model's homogeneity property, we assume

$$L_{\tau^D} = \ell K_{\tau^D}. \quad (8)$$

Here, ℓ is the market recovery value per unit of capital.³⁰ Default generates deadweight losses if ℓ is sufficiently low.

Costly External Equity Issuance. Because default is costly, shareholders could find it optimal to issue external equity to replenish the firm's liquidity. However, in reality, firms often face significant external financing costs due to asymmetric information and managerial incentive issues.³¹ A large empirical literature has sought to measure these costs, in particular the costs arising from the negative stock price reaction in response to the announcement of a new equity issue.³²

³⁰For essentially the same assumption, see Bolton, Chen, and Wang (2011) and DeMarzo, Fishman, He, and Wang (2012), for example.

³¹Explicitly modeling informational asymmetry would result in a substantially more involved analysis. Lucas and McDonald (1990) provides a tractable analysis by making the simplifying assumption that the informational asymmetry is short lived, i.e. it lasts one period.

³²An early study by Asquith and Mullins (1986) found that the average stock price reaction to the announcement of a common stock issue was -3% and the loss in equity value as a percentage of the size of the new equity issue was as high as -31% (See Eckbo, Masulis and Norli, 2007 for a survey.) Calomiris and Himmelberg (1997) estimate the direct *transactions costs* firms face when they issue equity and find that mean transactions costs (underwriting, management, legal, auditing and registration fees) are as high as 9% of an issue for seasoned public offerings and 15.1% for initial public offerings.

We model the firm's external equity financing as follows. Let N_t denote the firm's (undiscounted) cumulative net external equity financing up to time t and H_t to denote the corresponding (undiscounted) cumulative costs of external equity financing up to time t . Following Bolton, Chen, and Wang (2011), we assume that the firm incurs both fixed and proportional costs of issuing equity. To preserve the model's homogeneity property for tractability purposes, we further assume that these costs are proportional to capital stock K_t , so that $h_0 K_t$ denotes the fixed equity-issuance cost, and $h_1 M_t$ refers to the proportional equity-issuance cost, where M_t is the net amount raised via external equity issuance.

Finally, we turn to the firm's leverage dynamics, which also serves as the firm's law of motion connecting its sources of funds with its uses of funds.

Leverage Dynamics and Law of Motion. When the firm is solvent (i.e. when $t < \tau^D$), its debt, X_t , evolves according to the following law of motion:

$$dX_t = -[AK_{t-} - (I_{t-} + C_{t-} + \Phi_{t-})]dt - \Theta_{t-}\sigma d\mathcal{B}_t^S - \Pi_{t-}d\mathcal{J}_t - dN_t + dU_t, \quad (9)$$

where U_t denotes the firm's cumulative (nondecreasing) payout to equityholders up to time t . The first term on the right side of (9) describes the firm's operating revenues $AK_{t-}dt$ net of expenditures $(I_{t-} + C_{t-} + \Phi_{t-})dt$, which are the sum of (a.) investment outlays I_{t-} , (b.) interest payments $C_{t-} = (r + \eta_{t-})X_{t-}$, and (c.) jump-insurance premium payments Φ_{t-} . The second and third terms describe the payoffs from the diffusion hedging position and the jump hedging position, respectively. The fourth term dN_t describes the effect of net amount raised from external equity issuance on debt balance. The signs of these four terms are negative because the firm's debt is lowered when realized payoffs are positive. Finally, dU_t is the (non-negative) incremental payout to shareholders.

2.3 Optimality

Shareholders are risk-neutral with a discount rate γ . Following DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007a, 2007b), we assume that the firm's shareholders are impatient relative to other investors, $\gamma \geq r$. This impatience could be preference based or could arise indirectly because shareholders have other attractive investment opportunities. Due to this relative impatience, shareholders prefer early payouts, *ceteris paribus*.

The firm's shareholders choose investment (I_t), payout (U_t), diffusion and jump risk hedging demands (Θ_t and Π_t), short-term debt issues (if $X_t > 0$), external equity issuance

N_t , and the default timing $\tau^{\mathcal{D}}$ to maximize their equity value defined below:

$$\mathbb{E}_t \left[\int_t^{\tau^{\mathcal{D}}} e^{-\gamma(s-t)} (dU_s - dN_s - dH_s) \right], \quad (10)$$

subject to the capital accumulation equation given in (2), the law of motion given in (9), and the equilibrium pricing for all financial claims including corporate debt, diffusion-hedging contracts, and jump-insurance contracts. Because equity issuance is costly, ($dH_t > 0$ whenever $dN_t > 0$), we need to subtract the cost of equity issuance, dH_t , in (10). Because the APR is enforced, shareholders receive nothing upon default.

Let P_t and V_t denote the firm's equity value and its total market value, respectively. $V_t = P_t + X_t$. The firm's book leverage is then $x_t = X_t/K_t$. We use v_t to denote the firm's total value scaled by its capital $v_t = V_t/K_t$. Similarly, p_t denotes scaled equity value, $p_t = P_t/K_t$. Let ml_t denote the firm's market leverage, the ratio between the value of debt X_t and the firm's total value V_t :

$$ml_t \equiv \frac{X_t}{V_t} = \frac{x_t}{v_t}. \quad (11)$$

As K_t and X_t are the two state variables for the optimization problem, we use $P(K_t, X_t)$ and $V(K_t, X_t)$ to denote the firm's equity value function for the problem defined in (10) and the corresponding total firm value, respectively. We have

$$V(K_t, X_t) = P(K_t, X_t) + X_t. \quad (12)$$

The firm's average q is equal to its total market value divided by K :

$$v(x_t) = \frac{V(K_t, X_t)}{K_t} = p(x_t) + x_t. \quad (13)$$

3 Costless Equity Issuance: Classical Tradeoff Theory

Before turning to the general solution of our model it is helpful to consider the special case where equity issuance is costless ($h_0 = h_1 = 0$), as is assumed in Leland (1994), DeMarzo and He (2016), DeMarzo (2019), and most other dynamic capital structure models. Since $\gamma > r$, the firm borrows up to the point where the benefit of an additional dollar is equal to the expected incremental cost of default. In effect, the firm's optimal debt-financing calculation is similar to the classical tradeoff theory, with the tax-advantage of debt replaced

by a discounting-advantage of debt.³³ Moreover, in our model the firm faces a constant investment opportunity set, so that it is optimal for the firm to maintain a constant target leverage ratio when equity issuance is costless, as we formally establish below.

3.1 Solution

Let \bar{x} denote the firm's endogenous maximal book leverage beyond which it defaults. Let \underline{x} denote the firm's target leverage and $v(\underline{x})$ denote the firm's corresponding average q at the target leverage. By definition, $\underline{x} \leq \bar{x}$. Moreover, for all $x \in [\underline{x}, \bar{x}]$, we have $v(x) = v(\underline{x}) = \underline{x} + p(\underline{x})$, as equity issuance is costless. Since equity value at the default boundary \bar{x} is zero ($p(\bar{x}) = 0$), it follows that

$$\bar{x} = v(\underline{x}). \quad (14)$$

That is, market leverage at \bar{x} is 100% and hence the firm declares default when $x_t \geq \bar{x}$.

At its target leverage, the firm's average q , $v(\underline{x})$, satisfies the Gordon-growth formula:

$$v(\underline{x}) = \frac{1}{\gamma - g(\underline{i})} \left[(A - \underline{i}) + (\gamma - r)\underline{x} - \lambda(v(\underline{x}) - \ell) \left(\int_0^{\bar{Z}} Z dF(Z) \right) \right], \quad (15)$$

where $g(\underline{i})$ is the endogenously chosen constant growth rate and the constant investment-capital ratio \underline{i} satisfies:

$$\frac{1}{\psi'(\underline{i})} = v(\underline{x}) = \bar{x}. \quad (16)$$

Equation (16) is the standard first-order condition (FOC) for investment in q theory models. The firm equates its marginal cost of investing, $1/\psi'(\underline{i})$, with its marginal q , $v(\underline{x}) = p(\underline{x}) + \underline{x}$. Under costless equity issuance, the marginal cost of equity financing is one, which explains why the marginal cost of financing does not appear in the FOC. It also explains why the firm's marginal q is equal to its average q , as in Hayashi (1982).

The term inside the square brackets on the right side of (15) is the total expected cash-flow to the firm's (debt and equity) investors (scaled by K). This payoff is equal to the sum of three terms: the unlevered scaled FCF ($A - \underline{i}$), the net flow benefit due to the cheaper cost of debt financing $(\gamma - r)\underline{x}$, and the expected firm-value loss due to costly default. The denominator on the right side of (15) is equal to the difference between the equity holders'

³³Our main argument does not depend on the details of the funding advantage of debt. We could formulate our model with a tax benefit of debt instead of a discounting-benefit. DeMarzo and He (2016) and DeMarzo (2019) discuss the similarities between the two formulations.

discount rate γ and the expected growth rate $g(\cdot)$ (evaluated at the optimal investment-capital ratio \underline{i}), as in the standard Gordon growth model.

Note that the expected growth rate of the firm's free cash-flow $g(\cdot)$ includes a jump term:

$$g(i) = \psi(i) - \lambda(1 - \mathbb{E}(Z)). \quad (17)$$

As we model jumps as negative shocks, any realization of a jump lowers the expected growth rate from $\psi(i)$ by $\lambda(1 - \mathbb{E}(Z))$. Indeed, the second term in (17) is equal to the product of the jump arrival probability per unit of time, λ , and the expected percentage loss, $(1 - \mathbb{E}(Z))$. Note also that the relevant discount rate is that of shareholders, γ , who are the residual claimants of the firm's free cash flows.

The endogenous default boundary can also be expressed in terms the recovery boundary value upon the arrival of a jump, \bar{Z} , which satisfies

$$\bar{Z} = \min \{ \underline{x}/\bar{x}, Z^* \}. \quad (18)$$

Equation (18) connects the firm's recovery boundary default value \bar{Z} to its leverage default threshold, \bar{x} , target leverage \underline{x} , and the maximum insurable jump loss $(1 - Z^*)$. Given that the firm can avoid default by insuring any loss $(1 - Z) \in (0, 1 - Z^*]$, we must have $\bar{Z} \leq Z^*$. Also by definition, as the firm is indifferent between defaulting or not when $Z = \bar{Z}$, we have $\bar{Z} = \underline{x}/\bar{x}$, provided that $\bar{Z} = \underline{x}/\bar{x} < Z^*$. Combining these two inequalities, we obtain (18).

By choosing \underline{x} and \underline{i} to maximize (15) subject to (14), (16), and (18), we characterize the solution: \bar{x} , \underline{i} , \underline{x} , $v(\underline{x})$, and \bar{Z} . We provide details for the derivation in Appendix A.1.

There are two possible solutions, depending on parameter values:

- A. The optimal default boundary is such that $\bar{Z} = Z^*$ and $v(\underline{x}) = \underline{x} = \bar{x}$. That is, the optimal policy is to set market leverage at 100% at \underline{x} , so that $p(\bar{x}) = p(\underline{x}) = 0$.
- B. Target leverage and the optimal default boundary are such that $\underline{x} = \bar{Z}\bar{x}$ and $\bar{Z} < Z^*$. The firm's optimal market leverage is then equal to $\underline{x}/v(\underline{x}) = \bar{Z}\bar{x}/\bar{x} = \bar{Z}$, which is less than 100%. Under this solution the firm optimally self-insures any unhedgeable downward jump shock such that $\bar{Z} \leq Z < Z^*$ by issuing equity to make up this loss.

The standard tradeoff theory, balancing distress costs and the funding advantage of debt, implicitly assumes that financial markets are incomplete. Otherwise, the firm can use fairly priced financial securities to perfectly hedge its risk, thereby fully capturing the funding advantage of debt and achieving 100% market leverage without incurring any costly default. Next, we summarize the first-best result.

3.2 First Best

Under complete spanning, we obtain the first-best solution.³⁴ Note that even when equity issuance is costly, we can attain the first-best solution, by purely relying on fairly priced hedging contract and avoiding any equity issuance.

The firm then never defaults and attains the value $\underline{x} = q^{FB}$ by setting its investment at $\underline{i} = i^{FB}$ for all $t > 0$. Here, q^{FB} is Tobin's average q given by

$$q^{FB} = \max_i \frac{A - i}{r - g(i)}, \quad (19)$$

where $g(i)$ is given in (17),³⁵ and investment, i^{FB} satisfies the FOC: $\psi'(i^{FB})q^{FB} = 1$.

As shareholders are more impatient than debtholders ($\gamma > r$), it is efficient to make a lump-sum payment, $q^{FB}K_0$ at $t = 0$ to shareholders, in effect “selling” all future cash flows to creditors, so that the firm's market leverage is 100% for all $t > 0$. This resource allocation is efficient, as it brings forward all future cash flows for impatient shareholders.

4 Solution: Costly Equity Issuance

In this section, we characterize the firm's optimal policies. We focus on the case where the hedging constraint given in (5) never binds. In Section 6, we turn to the other important case where the constraint given in (5) may bind. We show in Appendix C how these results are derived from a recursive formulation of the firm's dynamic optimization problem.

Because our model is homogeneous of degree one in X and K , our analysis is considerably simplified by characterizing the solution in one dimension. For convenience, we define the following scaled variables: $\bar{x}_t = \bar{X}_t/K_t$, $\hat{x}_t = \hat{X}_t/K_t$, $\underline{x}_t = \underline{X}_t/K_t$, $c_t = C_t/K_t$, $m_t = M_t/K_t$, $\theta_t = \Theta_t/K_t$, $\pi_t = \Pi_t/K_t$, $\phi_t = \Phi_t/K_t$, and $x_t^{\mathcal{J}} = X_t^{\mathcal{J}}/K_t^{\mathcal{J}}$. The scaled jump insurance-premium $\phi_t = \Phi_t/K_t$ is then given by $\phi_t = \phi(x_{t-}, Z^*)$, where

$$\phi(x_{t-}, Z^*) = \lambda \mathbb{E}[\pi(x_{t-}, Z) \mathcal{I}_{Z \geq Z^*}]. \quad (20)$$

4.1 Debt Financing (Equity Inaction) Region

Consider first the firm's optimal investment policy.

³⁴Specifically, by incorporating fairly priced diffusion-hedging contracts for the shock $d\mathcal{B}_t^O$ and setting $Z^* = 0$, we ensure that all (diffusion and jump) shocks are fully hedged. To obtain full spanning we must, of course, also remove the hedging constraint given in (5).

³⁵See Bolton, Chen, and Wang (2011), DeMarzo, Fishman, He, and Wang (2012), and Brunnermeier and Sannikov (2014), among others for essentially the same FB solution.

Investment. We show in Appendix C.1 that the optimal investment is given by:

$$\psi'(i(x)) = \frac{-p'(x)}{p(x) - xp'(x)}. \quad (21)$$

Because $\psi(\cdot)$ is concave, $\psi'(\cdot)$ is decreasing and hence i is a monotonically increasing function of the ratio of marginal q (that is, $P_K(K, X) = p(x) - xp'(x)$) and the marginal cost of debt, $-p'(x)$. Importantly, investment not only depends on marginal q , but also on the marginal cost of debt financing.³⁶ Next, we analyze the firm's leverage dynamics.

Leverage Dynamics. By using Ito's Lemma, we have the following law of motion for x_t in the equity-inaction region:

$$dx_t = \mu_x(x_{t-}) dt - \sigma \sqrt{1 - \rho^2} x_{t-} d\mathcal{B}_t^O - \sigma (\theta_{t-} + \rho x_{t-}) d\mathcal{B}_t^S + (x_t^J - x_{t-}) d\mathcal{J}_t, \quad (22)$$

where $\mu_x(\cdot)$ is given by:

$$\mu_x(x_{t-}) = -\underbrace{(A - [i(x_{t-}) + \phi(x_{t-}) + c(x_{t-})])}_{\text{scaled free cash-flow}} - \underbrace{x_{t-}\psi(i(x_{t-}))}_{\text{capital growth}} + \underbrace{\sigma^2(\rho\theta_{t-} + x_{t-})}_{\text{quadratic covariation}}. \quad (23)$$

The first term in (23) corresponds to the negative of free cash-flow and shows how capital expenditures, jump insurance premia, and debt interest payments increase the firm's leverage. The second term in (23) shows how capital accumulation reduces leverage $x_t = X_t/K_t$ by increasing K_t . The last term in (23) shows how unhedgeable diffusion shocks to the capital stock increase leverage (due to Jensen's inequality.) We derive similar dynamics for the firm's market leverage, ml_t , defined in (11) (see Appendix B).

Diffusion and Jump-Risk Hedging. Again, in Appendix C.1 we show that the diffusion-hedging demand, θ_{t-} , is linear in x_{t-} , in that $\theta_{t-} = \theta(x_{t-})$, where

$$\theta(x_{t-}) = -\rho x_{t-}. \quad (24)$$

Since only the risk spanned by \mathcal{B}^S is hedgeable, the firm optimally sets $\theta(x)$ to $-\rho x$ so that its remaining exposure to diffusion shocks is purely via \mathcal{B}^O which is orthogonal to \mathcal{B}^S .

For hedgeable jumps, the hedging demand, $\pi(x, Z)$, is given by

$$\pi(x_{t-}, Z) = x_{t-}(1 - Z), \text{ if } Z \in [Z^*, 1]. \quad (25)$$

³⁶The marginal q and $-p'(x)$ are correlated in our model. Also, as we show, the marginal cost of debt financing is greater than one, $-p'(x) \geq 1$.

What is the total cost for the firm to purchasing these hedgeable jump-insurance contracts? Substituting (25) into (20) and integrating over all hedgeable jumps $Z \in [Z^*, 1]$ we obtain the following explicit expression for the firm's total insurance premium payment:

$$\phi(x_{t-}) = x_{t-} \cdot \left(\lambda \int_{Z^*}^1 (1 - Z) dF(Z) \right). \quad (26)$$

In words, $\phi(x_{t-})K_{t-}$ is the firm's jump-hedging cost per unit of time.

Substituting (24) into (22), we obtain the following expression for the book leverage dynamics:

$$dx_t = \mu_x(x_{t-}) dt - \sigma \sqrt{1 - \rho^2} x_{t-} d\mathcal{B}_t^O + (x_t^{\mathcal{J}} - x_{t-}) d\mathcal{J}_t. \quad (27)$$

We see that only the unhedgeable diffusion shocks to capital, \mathcal{B}_t^O , appear in (27). The firm completely neutralizes the effect of hedgeable diffusion shocks, \mathcal{B}_t^S , on leverage x at actuarially fair terms. For the same reason, the diffusion volatility of x , $\sigma \sqrt{1 - \rho^2} x_{t-}$, only reflects the unhedgeable diffusion shocks.³⁷

The last term in (27) describes the effect of jumps on leverage x . First, obviously, when there is no jump, $d\mathcal{J}_t = 0$ so that this term is equal to zero. Second, when a hedgeable jump arrives, i.e., $d\mathcal{J}_t = 1$ and $Z \in [Z^*, 1]$, the jump-insurance contract seller compensates the firm via a contingent repayment, $\Pi_{t-} = X_{t-}(1 - Z)$, so that the firm's debt decreases from its pre-jump level X_{t-} to $X_t^{\mathcal{J}} = X_{t-} - \Pi_{t-} = ZX_{t-}$. Therefore,

$$x_t^{\mathcal{J}} = \frac{X_t^{\mathcal{J}}}{K_t^{\mathcal{J}}} = \frac{ZX_{t-}}{ZX_{t-}} = x_{t-}, \text{ if } Z \in [Z^*, 1]. \quad (28)$$

The hedgeable jump has no effect on leverage and the jump term in (27) disappears for all values of Z in the region of $[Z^*, 1]$.

In sum, for both diffusion and jump shocks that are hedgeable at actuarially fair terms, it is optimal for the firm to choose state-contingent hedging policies θ_{t-} and π_{t-} , given in (24) and (25), to fully insulate its leverage $x_t = X_t/K_t$ from these hedgeable shocks. Hedging allows the firm to effectively manage its leverage policy.

For unhedgeable jumps $Z \in [0, Z^*)$, by definition, we have $X_t^{\mathcal{J}} = X_{t-}$, $K_t^{\mathcal{J}} = ZK_{t-}$, and

$$x_t^{\mathcal{J}} = \frac{X_t^{\mathcal{J}}}{K_t^{\mathcal{J}}} = \frac{X_{t-}}{ZK_{t-}} = \frac{x_{t-}}{Z}, \text{ if } Z \in [0, Z^*). \quad (29)$$

³⁷The minus sign for the volatility term reflects the fact that a positive unhedgeable diffusion shock $d\mathcal{B}_t^O$ to K decreases x .

That is, in the absence of default, unhedgeable jump arrivals automatically increase the firm's leverage, as $x_t^{\mathcal{J}} > x_{t-} > 0$ and decrease equity value from $P(K_{t-}, X_{t-})$ to $P(ZK_{t-}, X_{t-})$, as

$$P(K_t^{\mathcal{J}}, X_t^{\mathcal{J}}) = p(x_t^{\mathcal{J}})K_t^{\mathcal{J}} = p(x_{t-}/Z)ZK_{t-} < p(x_{t-})K_{t-} = P(K_{t-}, X_{t-}). \quad (30)$$

The inequality in (30) follows from the concavity of $p(\cdot)$. That is, provided that the firm remains solvent, both stock price $P(K, X)$ and leverage $x = X/K$ respond passively to unhedgeable shocks.

4.2 Default and Payout Regions

Default or Not? The firm defaults whenever a jump causes its leverage x_t to exceed the endogenous leverage capacity, \bar{x} , where $p(\bar{x}) = 0$, as implied by the limited liability protection for equity investors.

We can also describe the firm's optimal default strategy via the recovery-threshold \bar{Z}_t . When a jump arrives at t and $Z_t = \bar{Z}_t$, the firm is indifferent between defaulting or not. Therefore, $x_t^{\mathcal{J}} = x_{t-}/\bar{Z}_t = \bar{x}$. Solving this equation yields $\bar{Z}_t = x_{t-}/\bar{x}$. By incorporating the firm's option of using actuarially fairly priced insurance contracts to manage hedgeable risk, we can write \bar{Z}_t as a function of its pre-jump level of x_{t-} as follows:

$$\bar{Z}_t = \bar{Z}(x_{t-}) \equiv \min\{x_{t-}/\bar{x}, Z^*\}, \quad (31)$$

where $0 < x_{t-}/\bar{x} \leq 1$. In sum, when a moderate unhedgeable jump occurs at t such that $Z \in [x_{t-}/\bar{x}, Z^*)$, the firm does not default and fully repays its outstanding debt. Doing so preserves the firm's option value of using default to hedge even larger downward shocks. When the jump is so large that $Z < x_{t-}/\bar{x}$ it is optimal for shareholders to default.

Equilibrium Credit Spreads. By describing the firm's default policy with the threshold \bar{Z}_t we obtain the following expression for the expected liquidation value upon default:

$$\mathbb{E}_{t-}(L_t \lambda \mathcal{I}_{X_t > \bar{X}_t}) = \mathbb{E}_{t-}(\ell K_t \lambda \mathcal{I}_{Z < \bar{Z}_t}) = \lambda \ell K_{t-} \mathbb{E}_{t-}(Z \mathcal{I}_{Z < \bar{Z}_t}) = \lambda \ell K_{t-} \int_0^{\bar{Z}_t} Z dF(Z). \quad (32)$$

Combining equation (32) with the zero-profit condition for debt-investors as given in equation (C.7) in Appendix C.2, we obtain the following pricing equation for the credit spread η_{t-} :

$$\eta(x_{t-}) = \lambda \left[F(\bar{Z}(x_{t-})) - \left(\frac{\ell}{x_{t-}} \right) \int_0^{\bar{Z}(x_{t-})} Z dF(Z) \right]. \quad (33)$$

This equation ties the equilibrium credit spread to the firm's default strategy.

For the special case where creditors recover nothing upon the firm's default, $\ell = 0$, the equilibrium η is then simply given by $\lambda F(\bar{Z}(x_{t-}))$, the probability of default. When creditors' recovery in default is positive, $\ell > 0$, the credit spread is lower than $\lambda F(\bar{Z}(x_{t-}))$ as creditors recover a fraction, $L_t/X_t = \ell Z/x_{t-}$ of the firm's outstanding debt X_t .

Next, we provide additional conditions to pin down \bar{x} .

Equity Issuance versus Liquidation. When the cost of a seasoned equity offering is not too high, the firm optimally issues equity when $x = \hat{x}$. In Appendix C.3 we show that \hat{x} is given by the following value-matching condition:

$$p(\hat{x}) = p(\hat{x} - m) - (h_0 + m + h_1 m). \quad (34)$$

Differentiating with respect to m , it follows from condition (34) that the net optimal amount raised m is given by:

$$-p'(\hat{x} - m) = 1 + h_1. \quad (35)$$

Finally, the debt capacity \bar{x} is the same as the equity issuance boundary \hat{x} , where $p(\hat{x}) = 0$.

When the cost of issuing equity is too high the firm prefers liquidating its capital over issuing equity. The firm's debt is then limited by the liquidation value of capital: $\bar{x} = \ell$.

In both equity-issue and liquidation cases, the firm postpones its default decision until the last moment, when it exhausts its debt capacity, and equity is completely wiped out: $p(\bar{x}) = 0$. Obviously, when the firm's debt exceeds \bar{x} , i.e., $x \geq \bar{x}$ the firm immediately defaults and equity is worthless:

$$p(x) = 0, \text{ when } x \geq \bar{x}. \quad (36)$$

Although default generates deadweight costs, it is also value-enhancing for the firm's shareholders, as it provides a partial hedge against risks that cannot be insured otherwise. In contrast, when financial spanning is complete and hedging can be achieved at actuarially fair terms, the firm will instead use the state-contingent hedging instruments to manage risk and default is no longer optimal. In this case, there is no default and debt is risk-free.

Next, we turn to the region where the firm pays out to its shareholders.

Equity Payout. When $x < \underline{x}$, the firm makes a scaled lump-sum payment, $\underline{x} - x$, to equityholders, so that $p(x) = p(\underline{x}) + \underline{x} - x$, where the endogenous payout boundary \underline{x} can be shown to satisfy the following smooth-pasting and super-contact conditions: ³⁸

$$p'(\underline{x}) = -1 \quad \text{and} \quad p''(\underline{x}) = 0. \quad (37)$$

4.3 Summary

We show in Appendix C that in the equity-inaction region where $x \in (\underline{x}, \bar{x})$, the equity value function $p(x)$ satisfies the following ODE:

$$\begin{aligned} \gamma p(x) = & [i(x) + \phi(x) + c(x) - A] p'(x) + \psi(i(x))(p(x) - x p'(x)) + \frac{(1 - \rho^2) \sigma^2 x^2}{2} p''(x) \\ & + \lambda \mathbb{E} \left[p(x) \left(\int_{Z^*}^1 Z dF(Z) \right) + \int_{\bar{Z}(x)}^{Z^*} Z p(x/Z) dF(Z) - p(x) \right], \end{aligned} \quad (38)$$

where: (i) $c(x) = (r + \eta(x))x$; (ii) the equilibrium credit spread $\eta(x)$ is given in (33); (iii) $\bar{Z}(x)$ is given in (31); (iv) the jump-insurance premium payment $\phi(x)$ is given by (26); and, (v) $i(x)$ is given by (21). This ODE is solved subject to the payout and default boundary conditions (encompassing both equity-issuance and liquidation decisions) described above. The optimal diffusion- and jump- hedging policies are given by (24) and (25), respectively.

We briefly discuss the last term in (38), which reflects three possible outcomes as a result of a jump shock. First, if $Z \in [Z^*, 1]$ the firm's value remains unchanged at $p(x)$ as $x_t^J = x_{t-}$. Neither leverage nor firm value change in response to insurable jump shocks. This case corresponds to the (local) complete-hedging solution, captured by the first term inside the square brackets. Second, if $Z \in [\bar{Z}(x), Z^*)$, the jump is unhedgeable, but the firm does not default. As a result, its leverage increases to $x_t^J = x_{t-}/Z > x_{t-}$. In this case, the equity value decreases to $p(x/Z)$. Finally, if $Z \in [0, \bar{Z}(x))$, the unhedgeable jump-induced loss is so large that the firm defaults and equity is entirely wiped out.

In sum, the firm optimally manages its risk and leverage dynamics as follows. First, for hedgeable shocks, it is optimal to fully insulate leverage x from these shocks. Second, for unhedgeable shocks that cause sufficiently large losses, it is optimal for the firm to default on its debt obligations. Third, for all other unhedgeable shocks, it is optimal for the firm to roll over its existing debt and let leverage randomly drift in response to these unhedgeable shocks.

³⁸Similar value-matching and smooth-pasting conditions also appear in optimal contracting models, e.g., DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), Biais, Mariotti, Rochet, and Villeneuve (2010), DeMarzo, Fishman, He, and Wang (2012), and others.

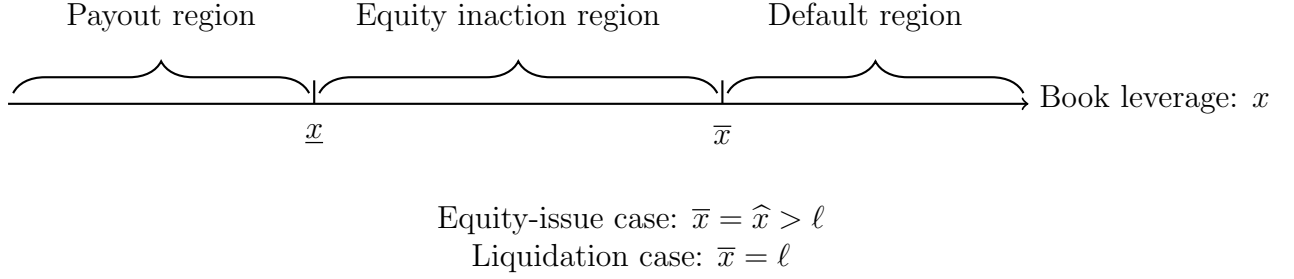


Figure 1: This figure demonstrates the mutually exclusive regions for the case when the firm's equity value is globally concave. The upper boundary \bar{x} is the equilibrium debt capacity. The lower boundary \underline{x} is the payout boundary. When the equity-issuance cost is not too high, the firm issues equity when $\bar{x} = \hat{x} > \ell$. When the equity-issuance cost is too high, the firm never issues equity and $\bar{x} = \ell$. Finally, due to incomplete financial spanning, the firm defaults in response to sufficiently large downward jumps, i.e., when $x > \bar{x}$.

In equilibrium, a financially distressed firm should use costly external equity financing or asset liquidation as the last defense: As long as the firm can achieve an equilibrium debt capacity through equity issuance that is larger than the liquidation value of capital, it is optimal for the firm to issue equity rather than liquidate its asset.

Figure 1 illustrates the mutually exclusive regions for the solution when the firm's equity value function is globally concave. We have two special cases. First, when the cost of equity issuance is not too high, we have $\bar{x} = \hat{x} > \ell$, a singleton. Second, when the cost of equity issuance is too high, the firm never issues any equity and we have $\bar{x} = \ell$, also a singleton. These two special cases are illustrated in Figure 1.

4.4 Leverage Dynamics and Equity Dilution: A Simulated Path

In this section we illustrate the rich predictions of our model on leverage dynamics and equity issuance along a simulated path described in Figure 2. We show how the firm dynamically manages its leverage and equity issuance along a given sample path for the case with $Z^* = 0.95$, which allows us to illustrate all aspects of corporate financial policy. The other parameter values are reported in Table 1 in the next section. For brevity, we do not display the simulated (hedgeable and unhedgeable) Brownian motion paths and only plot the four realized jumps in Panel A.

The first jump occurs at $t = 1.40$ and results in a fractional capital loss of $1 - Z = 4\%$. Neither market leverage ml_t nor equity ownership change as a result of this shock because

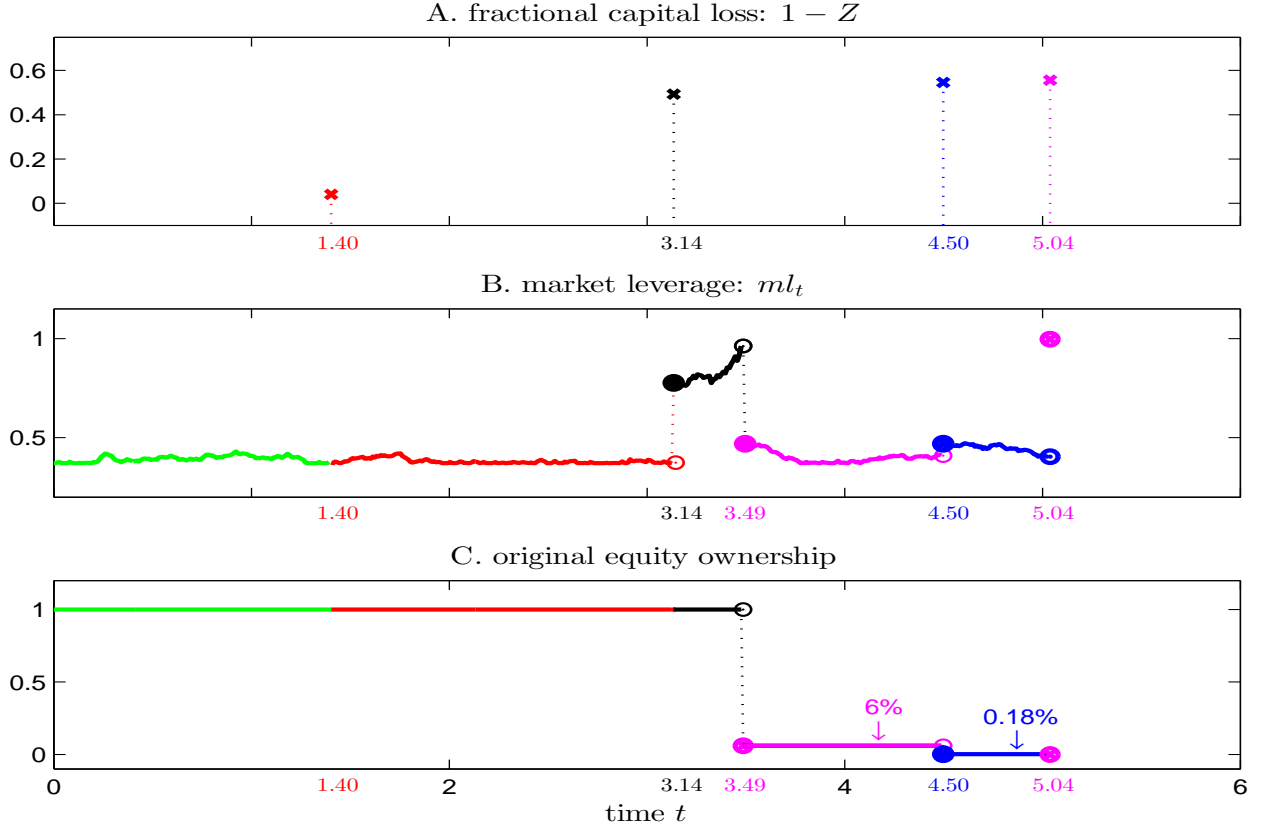


Figure 2: This figure uses a simulated sample path for the case where $Z^* = 0.95$ to illustrate how the firm uses risk management, equity issuance, debt rollover, and default in response to shocks.

Panel A highlights the four jump-shocks resulting in fractional capital losses $1 - Z$ of respectively 4%, 49%, 55%, 56% at respectively $t = 1.40, 3.14, 4.50, 5.04$. The first jump causes no change in market leverage as it is perfectly hedged ex ante. The second jump is not hedgeable ex ante, and causes a substantial increase in market leverage from 0.37 to 0.77; the firm optimally rolls over its debt, so that leverage jumps following the capital loss. The third jump is also not hedgeable. It triggers a recapitalization, as a result of which there is only a moderate increase in market leverage from 0.41 to 0.47, but the price is a hefty dilution of ownership of incumbent shareholders. The fourth jump triggers a default. Note finally that a negative (unhedgeable) diffusion shock at $t = 3.49$ triggers another recapitalization so that leverage is reduced from 0.97 to 0.47. This is an illustration of a situation where a small (continuous, diffusive) shock can trigger a fundamental change in the firm's financial policy, with a more than half reduction in leverage.

shocks of such small size could be fully hedged ex ante ($1 - Z \leq 1 - Z^* = 0.05$).

The second jump occurs at $t = 3.14$ and causes a loss that is too large to be hedged ($1 - Z = 49\%$). Still, the firm is able to absorb this loss and to roll over its debt. As a

result of the shock its market leverage increases from 37% to 77%, as seen in Panel B.³⁹ The firm does not respond by immediately issuing equity and retiring some of its outstanding debt because external equity issuance is costly. But, once its market leverage is very high it becomes very difficult for the firm to reduce its leverage going forward. Indeed, the high debt servicing costs and other payments due on its hedging positions trap the firm in a debt spiral, as in Brunnermeier and Sannikov (2014).⁴⁰

At $t = 3.49$ the firm's leverage hits the equity-issuance boundary where market leverage equals 97%. At that point a negative unhedgeable diffusion shock pushes the firm to raise external equity and retire some of its debt. The firm raises 0.54 percent of its pre-jump level of capital in return for giving up 94% of the firm's equity to the new shareholders. As a result of this recapitalization, the original equity investors are almost wiped out, holding on to only 6% of the firm's equity. However, the firm's market leverage is substantially reduced from 97% to 47% after this recapitalization.⁴¹ This is an illustration of a situation where a small (continuous, diffusive) negative shock can trigger a fundamental change in a firm's financial policy; here the firm responds by decreasing its leverage by more than half.

The third jump occurs at $t = 4.50$ and causes such a large loss ($1 - Z = 55\%$), that the firm has to respond by issuing immediately a large amount of external equity so as to maintain its leverage at a sustainable level. As a result, the market leverage barely increases from 0.41 to 0.47. (Without the recapitalization, the firm's leverage would have reached an excessively high level.) This second recapitalization again heavily dilutes the firm's existing shareholders. The firm's original shareholders from $t = 0$ see their ownership share decrease from 6% (following the previous recapitalization) to barely 0.18%, a tiny stake.⁴²

³⁹The post-jump debt level X_t , remains the same as the pre-jump level X_{t-} , but $K_t = ZK_{t-} = 0.51K_{t-}$. Therefore, book leverage increases from its pre-jump level $x_{t-} = 0.49$ to $x_t = x_{t-}/Z = 0.96$. Correspondingly, the firm's average q decreases from $v(x_{t-}) = 1.30$ to $v(x_t) = 1.24$. As a result, market leverage increases from $ml_{t-} = x_{t-}/v(x_{t-}) = 0.49/1.30 = 0.37$ to $ml_t = x_t/v(x_t) = 0.96/1.24 = 0.77$.

⁴⁰The debt-spiral mechanism in our model is a partial-equilibrium mechanism that is somewhat different from the general-equilibrium effect in Brunnermeier and Sannikov (2014). Both models are illustrations of the conceptual and quantitative importance of non-linearities in leverage dynamics.

⁴¹At $t = 3.49$, book leverage is $x_t = \hat{x} = 1.15$, equity value is $p(x_t) = p(\hat{x}) = 0.04$, and average q is $v(x_t) = p(x_t) + x_t = 1.19$, implying a very high market leverage: $ml_t = x_t/v(x_t) = 1.15/1.19 = 97\%$. A negative unhedgeable diffusion shock triggers the firm to recapitalize its balance sheet by raising external equity in the net amount of $m_t = 0.54$. Hence, right after equity issuance, the book leverage decreases significantly to $x_t - m_t = 1.15 - 0.54 = 0.61$ and its equity value (including new equity) increases substantially from $p(1.15) = 0.04$ to $p(0.61) = 0.69$, which decreases market leverage from 97% to $0.61/(0.61 + 0.69) = 47\%$. The original equity investors now only own 6% of the firm.

⁴²At $t = 4.50$, the pre-jump book leverage is $x_{t-} = 0.53$, the average q is $v(x_{t-}) = 1.30$, and market leverage is $ml_{t-} = x_{t-}/v(x_{t-}) = 0.53/1.30 = 0.41$. Without equity issuance, the post-jump book leverage

Finally, at $t = 5.04$, the firm suffers another capital loss of $1 - Z = 56\%$, which pushes leverage beyond the default boundary $1 - \bar{Z}(x_t) = 0.55$ (as $x_t = 0.54$ and $\bar{Z}(x_t) = x_t/\bar{x} = 0.54/1.19 = 0.45$), so that the firm's shareholders respond by defaulting on the firm's debt obligations, and creditors in turn respond by liquidating the firm's assets.

4.5 Special Case with Exogenous Earnings

In this subsection, we briefly outline the special case where the firm's earnings are exogenous with subject to permanent shocks in levels. This special case is obtained by setting $I_t = \bar{i}K_t$, with \bar{i} a constant rather than an endogenously chosen level. That is, we replace (1), (2), and (3) with the following exogenous process for earnings Y :

$$dY_t = \mu_Y Y_{t-} dt + \sigma Y_{t-} d\mathcal{B}_t^Y - (1 - Z)Y_{t-} d\mathcal{J}_t. \quad (39)$$

where $\mu_Y = \psi(\bar{i})$ and $\mathcal{B}_t^Y = \mathcal{B}_t^K$. This earnings process, and its special case with no jumps, may be more familiar to readers of the dynamic corporate finance literature, which uses this setup.⁴³ We can straightforwardly reinterpret all our previous results by shutting down the investment margin and replacing $(A - \bar{i})K_t$ with Y_t . But, our analysis also applies more generally to firms with endogenous capital accumulation.

5 Quantitative Analysis

In this section, we explore our model's conceptual and quantitative results in detail.

Parameter Values. First, we specify the scaled investment efficiency function as:

$$\psi(i) = i - \frac{\xi}{2}i^2 - \delta. \quad (40)$$

would have been $x_{t-}/Z = 1.17$, average q would have been $v(1.17) = 1.19$, and market leverage would have been $1.17/1.19 = 98\%$, which is too high. Therefore the firm immediately recapitalizes to adjust its leverage. By again issuing a net amount of equity, $m_t = 0.56$, the firm's book leverage significantly decreases from $x_t = 1.17$ to $x_t - m_t = 0.61$, its equity value (including new equity) substantially increases from 0.02 to $p(0.61) = 0.69$, and its average q increases from $0.02 + 1.17 = 1.19$ to $v(0.61) = 0.69 + 0.61 = 1.30$. As a result, market leverage changes modestly from 41% to $0.61/(0.61 + 0.69) = 47\%$ at the cost of substantial equity dilution. The pre-jump shareholders' stake decreases from 100% (by definition) to $0.02/0.69 = 3\%$ and the ownership stake of the original shareholders decreases from 6% to $6\% \times 3\% = 0.18\%$.

⁴³See Leland (1994, 1998), Leland and Toft (1996), Goldstein, Ju, and Leland (2001), DeMarzo and He (2016), and DeMarzo (2019) for a partial list in the contingent-claim capital-structure literature.

We set the investment adjustment cost parameter at $\xi = 1.5$ and the annual depreciation rate δ at 10%, the liquidation value at $\ell = 20\%$, and the annual productivity A at 21% in line with Bolton, Chen, and Wang (2011).

Following the rare-disasters literature,⁴⁴ we assume that the cumulative distribution function, $F(Z)$, for the capital recovery fraction, $Z \in [0, 1]$, is given by the following power law:

$$F(Z) = Z^\beta. \quad (41)$$

We choose the annual arrival rate of a jump at $\lambda = 0.734$, $\beta = 23.17$, and the annual diffusion volatility of capital shocks at $\sigma = 13.55\%$, as in Barro (2006) and Pindyck and Wang (2013).

Although the entrepreneur (shareholders) and the debt investors are risk neutral, whether shocks are hedgeable or not plays a critical role in our model. For the continuous and diffusive shocks, we set the correlation coefficient ρ between the shock to the firm's capital stock, \mathcal{B}^K , and the hedgeable component of the capital diffusion shock, \mathcal{B}^S , i.e., $\rho = 0.2$.

For jumps, we choose $Z^* = 0.5$ so that only jump shocks that cause the fraction of capital losses, $(1 - Z)$, to exceed $1 - Z^* = 50\%$ are not hedgeable. These unhedgeable catastrophic jump shocks have first-order effects on corporate policies and valuation.

Turning to the preferences, we set the entrepreneur's annual subjective discount rate γ to 5% and the annual risk-free rate to $r = 4.6\%$ as in DeMarzo and Sannikov (2006) and DeMarzo, Fishman, He and Wang (2012). The key is to require $\gamma > r$ so that the firm has incentives to pay out to shareholders when it has sufficiently large slack. Finally, for the equity-issuance costs, we set $h_0 = 0.1$ and $h_1 = 0.02$, broadly in line with estimates and numbers used in the literature.⁴⁵

Table 1 summarizes the parameter values. Whenever applicable, the parameter values are on an annualized basis.

Classical Tradeoff Theory: Costless Equity Issuance. For the parameter values in Table 1 we obtain that the firm's target leverage is extremely high in special case where there are no equity issuance costs. For the case where $Z^* = 0.5$, the firm is all debt financed with 100% market leverage. Additional, the firm's scaled enterprise value is $v(\underline{x}) = \underline{x} = \bar{x} = 1.416$. This corresponds to Case A summarized in Section 3.1.

⁴⁴See Barro (2006) and Pindyck and Wang (2013) among others.

⁴⁵Among others, see Altinkilic and Hansen (2000), Riddick and Whited (2009), and Bolton, Chen, and Wang (2011).

Table 1: PARAMETER VALUES

This table summarizes the parameter values for our baseline analysis in Section 4. Whenever applicable, parameter values are annualized.

Parameters	Symbol	Value
subjective discount rate	γ	5%
risk-free rate	r	4.6%
productivity	A	21%
capital diffusion volatility	σ	13.55%
jump arrival rate	λ	0.734
jump recovery parameter	β	23.17
capital liquidation recovery	ℓ	20%
adjustment cost parameter	ξ	1.5
depreciation rate	δ	10.07%
correlation coefficient	ρ	0.2
financial spanning parameter	Z^*	0.5
equity issue fixed cost	h_0	0.1
equity issue propositional cost	h_1	0.02
hedging constraint	$\bar{\theta}$	1

When $Z^* = 0.95$, the firm's market leverage is equal to $\underline{x}/v(\underline{x}) = \bar{Z} = 70.3\% < Z^*$, which is still high.⁴⁶ It is well known that the dynamic tradeoff models (e.g., Leland, 1994), predict excessively high leverage.

We point out that this empirically counter-factual prediction is primarily due to the assumption that equity issuance is costless. Without equity issuance costs, the firm's marginal cost of issuing equity is one and hence does not value financial flexibility. The firm's payout boundary is the same as its equity-issuance boundary. As we show, once we incorporate costly equity issuance, our model generates substantial lower leverage, as its target leverage is directly tied to its payout boundary but its equity issuance boundary can be quite far from its payout boundary.

Figure 3 shows how to determine the firm's target leverage in our generalized tradeoff theory by plotting its enterprise value and investment as we exogenously vary market leverage, ml . It is common to represent the capital structure solution under the static tradeoff theory with a plot where firm value is on the vertical axis and market leverage is exogenously varied

⁴⁶The solution features a target book leverage of $\underline{x} = 0.936$, a target enterprise value of $v(\underline{x}) = v(\bar{x}) = 1.332$, and an implied target equity value of $p(\underline{x}) = v(\underline{x}) - \underline{x} = 0.396$.

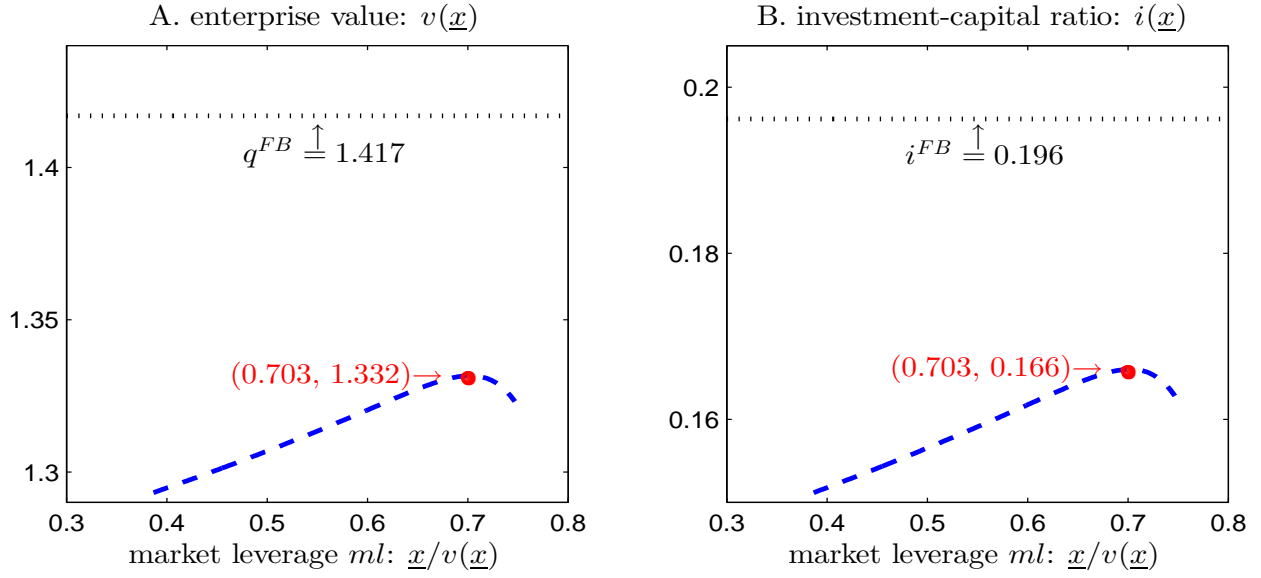


Figure 3: **Optimal capital structure and investment under classic tradeoff theory with $Z^* = 0.95$ and without external financing costs, $h_0 = h_1 = 0$.** Panels A and B plot the average q and investment as a function of market leverage. For both panels, (14), (15), (16), and (18) are satisfied. The firm targets its market leverage at $\underline{x}/v(\underline{x}) = 70.3\%$, where $v(\underline{x})$ attains the maximal value at 1.33, and sets investment $i(\underline{x})$ at 0.166. The first-best enterprise value and investment are $q^{FB} = 1.417$ and $i^{FB} = 0.196$, respectively. Parameter values other than $h_0 = h_1 = 0$ and $Z^* = 0.95$ are given in Table 1.

along the horizontal axis. The optimal leverage is then given by the point where firm value is maximized.⁴⁷ In this figure, the firm has optimized over all decision margins other than its market leverage choice, i.e., (14), (15), (16), and (18) are satisfied. Panel A shows the firm's enterprise value $v(\underline{x})$ increases with market leverage and reaches its maximum $v(\underline{x}) = 1.332$ when market leverage, $\underline{x}/v(\underline{x})$, is equal to 70.3%. Increasing market leverage beyond 70.3% lowers the firm's enterprise value. Therefore, 70.3% is the optimal target market leverage.

Panel B shows that the firm's investment $i(\underline{x})$ essentially follows the same pattern as $v(\underline{x})$ does in Panel A: $i(\underline{x})$ first increases with leverage, reaches its maximum value, 0.165 at the firm's target market leverage, 70.3%, and then decreases once its market leverage exceeds its target level. Due to financial distress costs and $\gamma > r$, even when issuing equity is costless, the firm under-invests, which destroys a substantial fraction of its value: $i(\underline{x}) = 0.166 < i^{FB} = 0.196$ and $v(\underline{x}) = 1.332 < q^{FB} = 1.417$.

⁴⁷See e.g. Figure 1 in both Myers' 1984 AFA presidential address and Shyam-Sunder and Myers (1999)

As we show next, however, when incorporating external equity issuance costs into the classical tradeoff theory we obtain much more plausible predictions about leverage dynamics. Indeed, the firm barely spends any time at its target.

Average and Marginal q , Investment, and Marginal Cost of Debt Financing.

The firm's average q is given by (13). The net marginal value of debt financing for the firm as a whole is then

$$V_X(K_t, X_t) = v'(x_t) = p'(x_t) + 1, \quad (42)$$

where the first equality follows from the homogeneity property and $p'(x_t)$ is the marginal equity value of debt financing. As the firm is financially constrained, $p'(x_t) \leq -1$. Therefore, the net marginal cost of debt financing to the whole firm is weakly positive: $-v'(x_t) \geq 0$.

The firm's marginal q is given by

$$q_m(x_t) = \frac{\partial V(K_t, X_t)}{\partial K_t} = p(x_t) - x_t p'(x_t). \quad (43)$$

Therefore, the firm's marginal q is larger than its average q , as

$$q_m(x_t) - v(x_t) = -x_t [1 + p'(x_t)] \geq 0, \quad (44)$$

provided that the firm is levered, $x_t \geq 0$. At the endogenous payout boundary \underline{x} , the firm is indifferent between paying out its profit to shareholders and retaining it inside the firm, i.e., $-p'(\underline{x}) = 1$ and $v'(\underline{x}) = 0$. That is, the firm's average q is equal to its marginal q .

Panel A in Figure 4 plots Tobin's average q , $v(x)$, as a function of leverage x . It shows that Tobin's average q is lower than the first-best level, $q^{FB} = 1.417$, for all levels of leverage x . As the firm is unable to completely hedge its exposure to idiosyncratic risk, it is thus financially constrained and hence its value is lowered, i.e., $v(x) < q^{FB}$. Limited risk management opportunity also lowers its debt capacity from the FB level, $q^{FB} = 1.417$, to $\bar{x} = 1.19$. That is, the firm is only able to borrow up to 84% of its capital stock without triggering costly equity issue or inefficient liquidation.

In addition, the higher is the firm's leverage x , the greater the loss in value, measured by $q^{FB} - v(x)$. The intuition is as follows. The more levered the firm is, the closer it is to costly equity issuance and/or inefficient liquidation, the more distorted its investment, the more valuable its risk management opportunity, and the larger its value loss.

Panel B shows that the more levered the firm is, the higher the firm's marginal q . This is because $q'_m(x) = -xp''(x) > 0$ as the firm is levered, $x > 0$. This result may appear at first

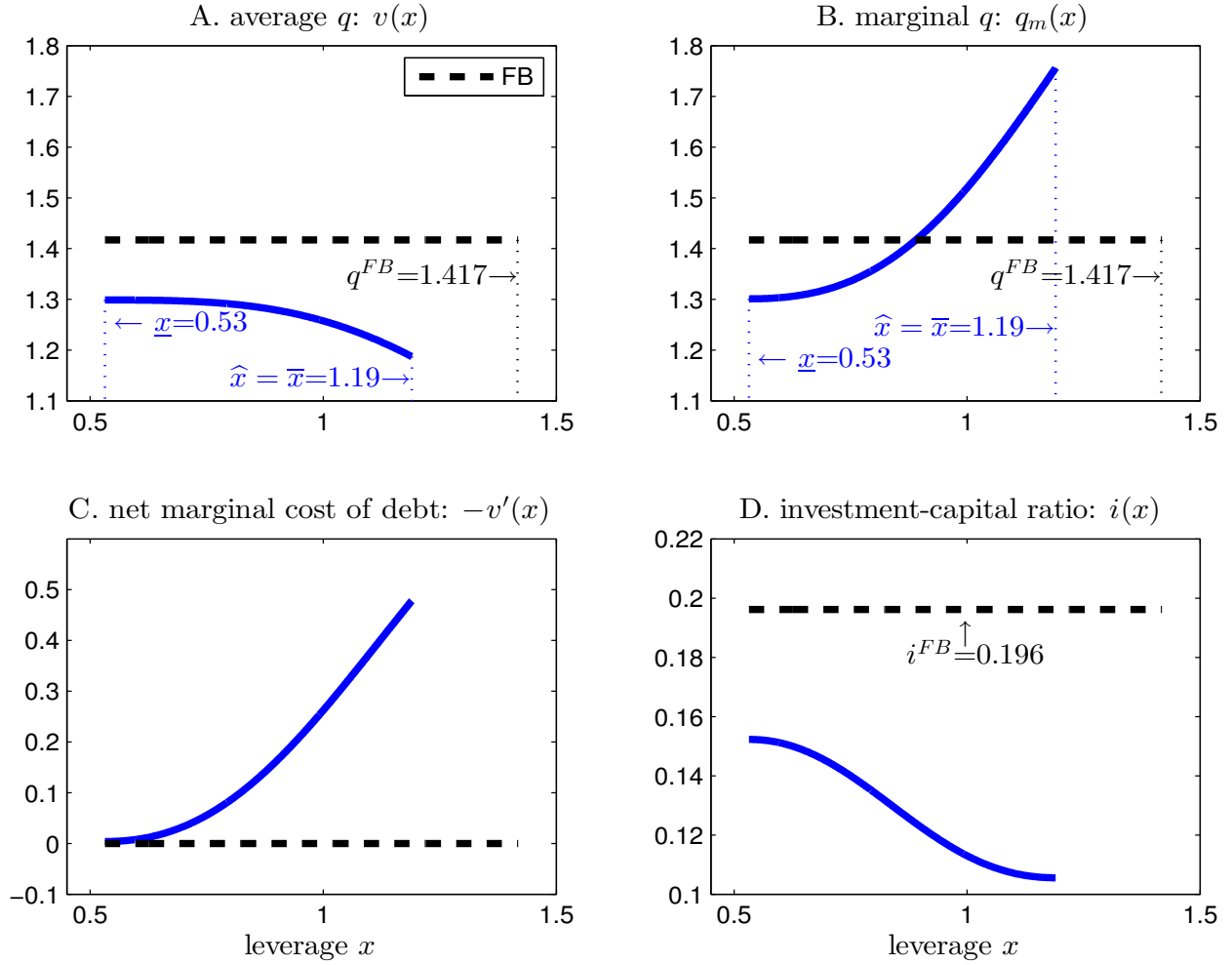


Figure 4: **Tobin's average q** , $v(x) = p(x) + x$, **the marginal q** , $q_m(x) = v(x) - xv'(x)$, **the net marginal cost of debt financing**, $-v'(x) = -p'(x) - 1$, and **the investment-capital ratio $i(x)$** . The first-best investment-capital ratio is $i^{FB} = 0.196$ and the first-best average q is $q^{FB} = 1.417$. The endogenous payout boundary is $\underline{x} = 0.53$ and the endogenous upper boundary, at which the firm either issues equity or liquidates, is $\hat{x} = \bar{x} = 1.19$. All parameter values other than $Z^* = 0.95$ are given in Table 1.

counter-intuitive, as it is often said that a firm's marginal q increases with its investment opportunity. However, this conventional wisdom is incorrect as the firm's investment opportunity by construction is constant over time. The reason that the marginal q increases with leverage is as follows. The higher the firm's capital stock K_t , the larger the firm's borrowing capacity as $\bar{X}_t = \bar{x}K_t$ increases with K_t , and hence the higher the marginal value of capital. Interestingly, for sufficient high leverage, i.e., $x \geq 0.87$, the marginal q is greater than q^{FB} .

Comparing Panels A and B shows that the firm's marginal q is always higher than its average q . This result is shown in (13), as the firm's marginal source of financing in the region where $\underline{x} < x < \bar{x}$. Moreover, while the marginal q increases with leverage x , the average q decreases in leverage x . Bolton, Chen, and Wang (2011) report a similar result for the case when the firm is financed by credit line.

Panel C shows that the firm's marginal cost of debt financing, $-v'(x)$, increases with its leverage x . When the firm is highly levered and is near its equity-issuance/liquidation boundary $\hat{x} = \bar{x} = 1.19$, reducing the firm's debt by one dollar generates an extra benefit of forty-eight cents at the margin in addition to reducing its liability by one dollar.

Panel D shows that the firm under-invests relative to the first best. The degree of under-investment, measured by the gap between the two lines in Panel D, increases with leverage. This is the standard debt-overhang result in Myers (1977). What is new here is that the cause of debt overhang is the unhedgeable shocks due to incomplete financial market spanning. An increase in debt reduces financial slack and gets the firm closer to costly equity issuance or default.

Even when paying dividends, i.e., $x_t = \underline{x}$, the firm still underinvests. At $\hat{x} = \bar{x} = 1.19$, the firm's investment is $i(\bar{x}) = 0.106$, which is about half of the first-best level $i^{FB} = 0.196$. As a result, the average q , $q(\bar{x}) = 1.19$, which is much lower than $q^{FB} = 1.417$. These results illustrate how the quantitative effects of limited risk hedging opportunities can be large.⁴⁸

6 Risk-seeking Incentives

So far, we have focused on the case where the firm's equity value is globally concave. But, with a strictly positive probability of default under limited financial spanning, shareholders may have risk-shifting incentives when debt is risky. This is indeed the case when the firm's hedging opportunities are sufficiently constrained, as we show in this section.

6.1 Model Solution

Risk seeking may be optimal when the firm's hedging constraint (5) is tight and binding, so that $\theta(x) = \bar{\theta}$ for some levels of x , or when the firm's ability to hedge jump shocks is

⁴⁸In Appendix D we further illustrate the firm's optimal hedging policy. Figure 7 describes the firm's diffusion and jump hedging demand when the equity value function is globally concave, and Figure 9 describes the firm's diffusion and jump hedging demands, as well as the firm's insurance premium payments for the risk-seeking case.

sufficiently limited (when Z^* is sufficiently high).

Diffusion Risk Taking. Consider the case where $\rho \geq 0$. The optimal $\theta(x)$ is given by

$$\begin{cases} \theta(x) = \max \{ -\rho x, -\bar{\theta} \} , & p''(x) < 0 , \\ \theta(x) = \bar{\theta} , & p''(x) \geq 0 . \end{cases} \quad (45)$$

That is, the optimal position $\theta(x)$ critically depends on whether the firm is endogenously risk averse or risk loving. First, when $p''(x) < 0$, the firm's hedging position is equal to the larger of $-\rho x$ (as in the baseline model of Section 4), or $-\bar{\theta}$, (when the hedging constraint binds.) Second, when $p''(x) > 0$, the firm is endogenously risk seeking and hence *maximizes* its risk exposure by going long on the diffusion-contingent hedging contract, $\theta(x) = \bar{\theta}$. In general, the firm may change $\theta(x)$ and even switch its sign (that is, switch between speculating and hedging) depending on its leverage.

Jump Risk Taking. When the firm chooses to seek risk, $\pi(x, Z)$ no longer satisfies (25). We thus need to revise the optimization problem with respect to Π for the general formulation given in (C.4), and, allowing for convexity, rewrite the firm's problem for π , as:⁴⁹

$$\max_{\pi(x, Z)} \left[Z p \left(\frac{x - \pi(x, Z)}{Z} \right) - \pi(x, Z) \cdot (-p'(x)) \right] . \quad (46)$$

The scaled insurance premium payment per unit of time is equal to $\pi(x, Z) \lambda dF(Z)$. Since debt is the marginal source of financing, the marginal cost of debt financing is $(-p'(x)) \geq 1$ and the economic cost of purchasing $\pi(x, Z)$ unit is equal to $\pi(x, Z) \cdot (-p'(x))$ multiplied by the probability $\lambda dF(Z) > 0$ that this contingency occurs. The first term in (46) gives the firm's equity value (scaled by pre-jump capital stock) if this jump contingency occurs.

The shareholder's optimal exposure to the contingency of a jump arrival with recovery Z boils to choose $\pi(x, Z)$ by maximizing (46). Unlike the risk-taking decisions involving diffusion shocks, the jump risk-taking decisions requires a global search of $\pi(x, Z)$ as jumps are discrete rather than continuous.

Equity Issuance and Risk Taking. The firm no longer postpones its equity issuance to the point when it has exhausted all its (endogenous) debt capacity, which would be optimal

⁴⁹The more general formulation for the optimal choice of $\pi(x, Z)$, stated in (46), boils down to the explicit formula given in (25) when $p''(x) < 0$.

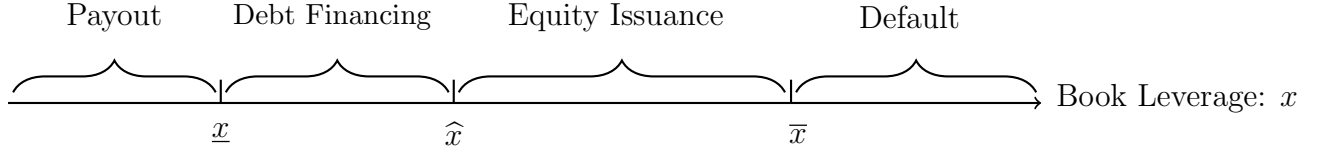


Figure 5: This figure demonstrates the four possible regions for the general case. In general, when the firm's value is not globally concave and the cost of equity issue is not too high, the firm finds it optimal to issue equity for a range of x , i.e., the firm also has an equity-issuance region, where $\hat{x} \leq x < \bar{x}$.

if $p(x)$ is globally concave as we have shown earlier. When $p(x)$ is convex for a certain range of leverage, issuing equity before exhausting its debt capacity can be optimal.

We show that there is an equity-issuance region characterized by $\hat{x} \leq x < \bar{x}$, where \hat{x} and \bar{x} are endogenous scaled equity issuance and liquidation boundaries. The following value-matching and smooth-pasting conditions hold in this region:

$$p(x) = p(x - m) - (h_0 + m + h_1 m), \quad (47)$$

$$-p'(x - m) = 1 + h_1. \quad (48)$$

Additionally, the firm's optimality with regard to equity issuance implies that $p(x)$ is continuous and differentiable at its endogenous equity-issuance boundary \hat{x} . That is, the following value-matching and smoothing pasting conditions must hold:

$$p(\hat{x}+) = p(\hat{x}-) \quad \text{and} \quad p'(\hat{x}+) = p'(\hat{x}-), \quad (49)$$

where $\hat{x}+$ and $\hat{x}-$ denote the right and left limits of \hat{x} .

The mechanism for issuing equity proactively (before the firm exhausts its debt capacity) is very different from that in Bolton, Chen, and Wang (2013), where the firm issues equity sooner than exhausting its financing capacity because the firm anticipates that the firm's financing cost may increase in the future. Equity issuance in our model arises from the firm's anticipation that paying the equity issuance cost to replenish its liquidity before fully exhausting its financing capacity is optimal.

Figure 5 demonstrates the four possible regions for the general case where $\hat{x} < \bar{x}$. For the special case that we have analyzed earlier, i.e., when the firm has no incentives to seek risk, $\hat{x} = \bar{x}$ and conditions (47) and (48) specialize to (34) and (35).

Solution Summary. Having analyzed the firm's decisions, we next characterize the shareholders' scaled value, $p(x)$, via the following ODE in the region where $\underline{x} < x < \hat{x}$:

$$\begin{aligned} \gamma p(x) = & [i(x) + \phi(x) + c(x) - A] p'(x) + \psi(i(x))(p(x) - xp'(x)) \\ & + \frac{x^2 + 2x\rho\theta(x) + \theta(x)^2}{2} \sigma^2 p''(x) + \lambda \mathbb{E} [Zp(x^J) - p(x)] . \end{aligned} \quad (50)$$

The post-jump leverage, x_t^J , is given by

$$x_t^J = \frac{x_{t-} - \pi(x_{t-}, Z)}{Z} \mathcal{I}_{Z \geq Z^*} + \frac{x_{t-}}{Z} \mathcal{I}_{\bar{Z}(x_{t-}) \leq Z < Z^*} + \frac{x_{t-}}{Z} \mathcal{I}_{Z < \bar{Z}(x_{t-})} \quad (51)$$

and $\pi(x_{t-}, Z)$ is given by (46).

In addition, $c(x) = (r + \eta(x))x$ is the equilibrium interest payment, the scaled jump-insurance premium payment $\phi(x)$ is given by (20), and $i(x)$ is given by (21). The conditions for the payout boundary \underline{x} are still given by (37). The conditions for equity issuance boundary, \hat{x} , are given by (49). When $x > \bar{x}$, the firm defaults, $p(x) = 0$, i.e., (36) holds.

Finally, the dynamics of x_t in the equity-inaction region (\underline{x}, \hat{x}) is given by (22). Importantly, the firm's leverage is now exposed to the hedgeable risk, $d\mathcal{B}_t^S$, in addition to the unhedgeable risk, $d\mathcal{B}_t^O$. This is because the firm may be a risk seeker rather than a hedger.

In summary, the firm may find it optimal to engage in risk seeking when risk management opportunities are limited. The intuition is that when risk management opportunities are limited, by managing risk prudently locally, the firm may not be able to pull itself out of a highly levered (financially distressed) situation. In that case, the firm may find it optimal to take more aggressive financial policies, e.g., speculation and equity issuance, with the hope of significantly deleveraging itself within a short period of time. Of course, doing so comes with a greater risk of going bankruptcy as well.

6.2 Quantitative Results

Figure 6 illustrates the effect of changing the firm's (jump) hedging opportunity on its investment and value. We compare the case where $Z^* = 0.95$ to our baseline model where $Z^* = 0.5$. If $Z^* = 0.95$, hedging contracts for jumps causing the fractional loss $1 - Z$ to exceed $1 - Z^* = 5\%$ are not available in the markets.

Average q , Marginal q , Marginal Cost of Debt Financing, and Investment. Panel A shows that reducing the firm's hedging opportunity set (by increasing Z^* from 0.5 to 0.95)

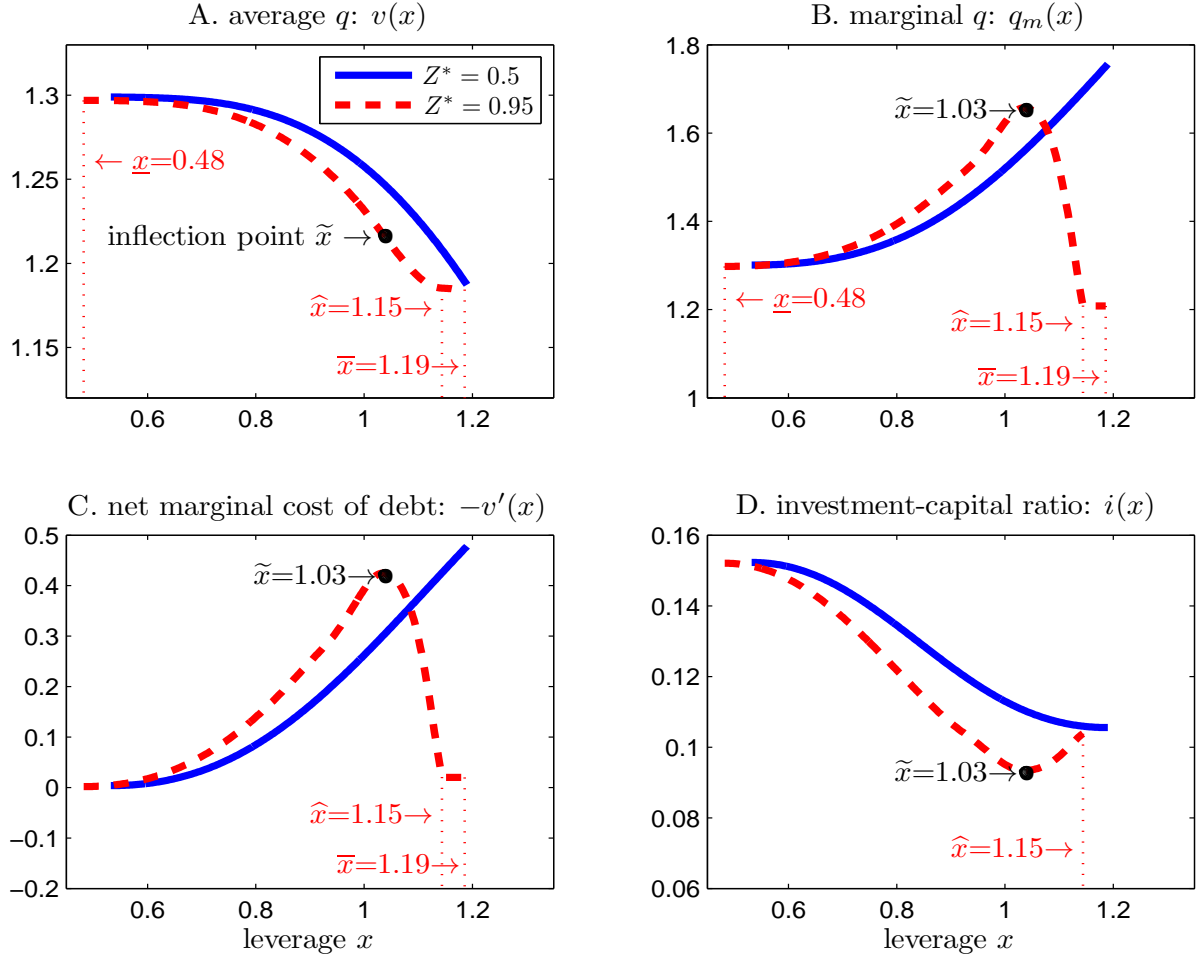


Figure 6: **Tobin's average q** , $v(x) = p(x) + x$, **marginal q** , $q_m(x) = v(x) - xv'(x)$, **net marginal cost of debt**, $-v'(x) = -p'(x) - 1$, and **investment-capital ratio** $i(x)$. The FB investment-capital ratio is $i^{FB} = 0.196$ and the FB average q is $q^{FB} = 1.417$. For the baseline case where $Z^* = 0.5$, the payout boundary is $\underline{x} = 0.53$ and the endogenous upper (equity-issuance and liquidation) boundary is $\bar{x} = 1.19$. When $Z^* = 0.95$, the payout boundary is $\underline{x} = 0.48$, the liquidation boundary is $\bar{x} = 1.19$, and the equity-issuance boundary is $\hat{x} = 1.15$. All parameter values other than Z^* are given in Table 1.

lowers its average q . This value-destruction result (measured by the corresponding reduction of the firm's average q) is intuitive and holds for all levels of leverage x .

A less obvious result is that reducing the firm's hedging opportunity causes it to delay its payout to shareholders: The payout boundary \underline{x} decreases from 0.53 to 0.48 as we increase Z^* from 0.5 to 0.95. The intuition is as follows. When the firm is near its endogenous payout

boundary, the firm naturally has a strong balance sheet. In this case, the more limited the firm's hedging opportunity, the greater the marginal benefit of using the marginal unit of its profit to reduce its leverage, and therefore, the more valuable it is for shareholders to postpone the firm's payout to themselves.

A key result for the case with $Z^* = 0.95$ is that the firm's average q is no longer globally concave in leverage x . Let \tilde{x} denote the inflection point for the firm's average q , i.e., the point at which $v''(\tilde{x}) = p''(\tilde{x}) = 0$. In our numerical example, $\tilde{x} = 1.03$. To the left of \tilde{x} , i.e., when $\underline{x} < x < \tilde{x} = 1.03$, the average q , $v(x)$, is decreasing and concave in leverage x . This is the region where leverage is sufficiently low so that the firm is locally risk-averse to uncertainty.

Panels B and C confirm this result by showing that the firm's marginal q , $q_m(x)$, and its net marginal cost of debt, $-v'(x)$, increase with leverage x , reach their respective maximum values at the inflection point $\tilde{x} = 1.03$, and then decrease with x in the region where $x > \tilde{x} = 1.03$. It is immediate to see that at the inflection point, i.e., when $x_t = \tilde{x} = 1.03$, we have $q'_m(\tilde{x}) = -\tilde{x}p''(\tilde{x}) = 0$ and $v''(\tilde{x}) = p''(\tilde{x}) = 0$.

For a financially distressed firm, the more limited its hedging opportunity set, the riskier the firm's earnings, the more likely and sooner the firm uses its costly external financing or bankruptcy option.⁵⁰ Indeed, increasing Z^* from 0.5 to 0.95 causes the firm to issue costly external equity sooner by lowering its equity-issuance boundary from 1.19 to $\hat{x} = 1.15$. The liquidation boundary is $\bar{x} = 1.19$, which is almost the same for the two levels of Z^* . That is, while $\bar{x} = \hat{x} = 1.19$ for the case with $Z^* = 0.5$, we now have a region of leverage, $1.15 = \hat{x} < x < \bar{x} = 1.19$, where it is optimal for the firm to issue equity.

Panel D shows that the more limited the firm's hedging opportunity, the more significant the firm's under-investment (compared with the FB.) However, the degree of under-investment is not monotonic in leverage. Investment decreases with leverage x in the region where $x < \tilde{x} = 1.03$, reaches the lowest level at $\tilde{x} = 1.03$, and then increases with leverage x .

By differentiating the FOC for investment, (21), with respect to x , we obtain:

$$\psi''(i(x))i'(x) = \frac{-p(x)p''(x)}{(p(x) - xp'(x))^2}. \quad (52)$$

Because $\psi(\cdot)$ is concave by assumption, the sign of $i'(x)$ is the opposite of that of $p''(x)$. Therefore, $i(x)$ is decreasing in x in the region $x < \tilde{x}$, where $p(x)$ is concave, and increasing

⁵⁰We illustrate the credit spread as a function of book leverage in Figure 10 of Appendix D. As can be seen credit spreads sharply increase when the firm has risk-seeking incentives.

in the region $x > \tilde{x}$, where $p(x)$ is convex. At the inflection point $\tilde{x} = 1.03$, the investment-capital ratio attains its minimal value of 0.093.

The intuition for the behavior in the region to the left of the inflection point, i.e., $x < \tilde{x} = 1.03$, is the standard Myers' debt-overhang result: the higher the level of the firm's risky debt the lower its investment.

In contrast, when the firm is financially distressed, the firm's investment becomes less distorted as its leverage increases. The intuition is that when the firm is highly levered and very close to bankruptcy, increasing the firm's survival likelihood by cutting investment to increase its financial slack is unlikely to work. Instead, the firm relies on more aggressive strategies, such as costly lumpy equity issuance and speculation, to replenish the firm's liquidity buffer and preserve the firm's going-concern value.⁵¹ These risk-seeking policies also induce the firm to reduce its under-investment when $x > \tilde{x}$.

In Appendix D we also illustrate the stationary distribution for firm leverage (Figure 11) and the comparative statics of the firm's policies with respect to changes in equity issuance costs (Figure 12). The main observation from Figure 11 is that leverage is mostly bunched around the firm's target level; higher levels of leverage are less and less likely, the higher is leverage. This is consistent with the findings in Figure 2 of DeAngelo and Roll (2015). The key insight from Figure 12 is that target leverage and the recapitalization boundary are lower when equity issuance costs rise, so that leverage decreases with equity issuance costs.

7 Conclusion

Our model predicts that corporate leverage decreases when its profits go up and decreases otherwise. This financing behavior is driven by corporate demand for financial flexibility when facing equity issuance costs and incomplete markets. Our model also explains why leverage is persistent, why at the same time it is time varying and stochastic, and why it is mean-reverting in the long run. Paradoxically, the explanation for leverage persistence and other dynamics, far from being that debt is costly to adjust, is that it is costly to raise funds by issuing equity. One would think that when equity is costly firms would want to rely more on debt. But that is a static intuition. From a dynamic perspective firms seek to avoid costly equity issuance in the future and therefore maintain financial slack today. Somewhat counter-intuitively it is the cost of equity that causes leverage to be low on average and

⁵¹We illustrate these risk-seeking policies in Figure 9 of Appendix D.

explains the average level and dynamics of leverage.⁵²

Additional insights from our analysis are as follows. First, when the firm issues equity in order to delever, existing equityholders are highly diluted, consistent with the evidence in DeAngelo, DeAngelo, and Stulz (2010), and moreover, the firm taps equity markets at a cost before exhausting its endogenous debt capacity, consistent with findings of Fama and French (2005). Equity holders are generally conservative with corporate leverage choices even in normal times but they can be aggressive with both risk seeking and deleveraging in bad times. Second, most of the time the firm is endogenously risk averse, in which case it is optimal for the firm to manage its liabilities and risk so as to minimize the volatility of leverage. This is a very simple and natural objective.

Another important conceptual take-away from our analysis is that capital structure theory relies on either contractual incompleteness or market incompleteness. If firms had access to a complete set of one-step ahead Arrow securities to manage their risk exposures, they would never have to issue either risky debt or costly external equity, as they could manage all their risk exposures and achieve a constrained efficient production outcome by dynamically trading the complete set of one-period-ahead Arrow securities.

For parsimony and tractability, we have left out term debt. In practice, however, firms tend to use long-term debt to avoid rollover risk and interest rate risk, or to finance lumpy capital expenditures as Korteweg, Schwert and Strebulaev (2019) have shown. Introducing term debt significantly complicates our analysis. We leave it for future research.

There has been a long-running and still unresolved empirical debate between the tradeoff and pecking order theories of capital structure ever since Myer's 1984 presidential address.⁵³ We develop a generalized tradeoff theory by building on the core insights from both the tradeoff and pecking order theories. As in the standard tradeoff theory, there is a benefit of debt over equity and a cost of financial distress. In the spirit of the pecking-order theory, our model accounts for the fact that issuing equity is costly, giving rise to a demand for financial flexibility. But importantly, our model's predictions differ from both static theories in very important ways. Our dynamic model generates highly nonlinear, non-monotonic, state-contingent, path-dependent dynamics of leverage, investment, hedging, risk-taking, and payout policies, which is broadly consistent with the empirical evidence.

⁵²A common explanation is that firms face adjustment costs in changing their leverage. But adjustment costs of debt in practice seem too small to explain these dynamics.

⁵³See Fama and French (2002) and Frank and Goyal (2008) for example.

Appendices

A Proofs for Section 3

A.1 Costless Equity Issuance

The solution for this case can be derived from the general case with costly equity issuance by setting $h_0 = h_1 = 0$.

Recall that in the equity-issuance region characterized by $\hat{x} \leq x < \bar{x}$, the value-matching and smooth-pasting conditions given in (47) and (48), respectively, can be simplified as:

$$p(x) = p(x - m) - m, \quad (\text{A.1})$$

$$p'(x - m) = -1. \quad (\text{A.2})$$

Also, as $p'(\underline{x}) = -1$ at the payout boundary \underline{x} , we must have $p'(x) = -1$ for $\underline{x} \leq x \leq \hat{x}$. Otherwise if $-p'(x) > 1$ for some x in this region, the firm will issue equity immediately. Hence, equity value takes the following form:

$$p(x) = p(\underline{x}) + (\underline{x} - x), \quad (\text{A.3})$$

which implies $v(x) = v(\underline{x}) = v(\hat{x}) = v(\bar{x})$ for $x \leq \bar{x}$. Note that (A.3) holds in both the payout region when $x \leq \underline{x}$ and $\underline{x} \leq x \leq \bar{x}$.

Next, we rewrite the ODE given in (50) for the general case in the region $\underline{x} \leq x \leq \hat{x}$ as:

$$\gamma p(x) = -[i(x) + \phi(x) + c(x) - A] + \psi(i(x))(p(x) + x) + \lambda \mathbb{E}[Zp(x^{\mathcal{J}}) - p(x)]. \quad (\text{A.4})$$

Conjecturing and verifying, we obtain that the firm completely hedges its hedgeable jump risk. Hence, the post-jump leverage, $x_t^{\mathcal{J}}$, is given by

$$x_t^{\mathcal{J}} = x_{t-} \mathcal{I}_{Z \geq Z^*} + \frac{x_{t-}}{Z} \mathcal{I}_{\bar{Z}(x_{t-}) \leq Z < Z^*} + \frac{x_{t-}}{Z} \mathcal{I}_{Z < \bar{Z}(x_{t-})}. \quad (\text{A.5})$$

The FOC for investment is $1 = \psi'(i)v(x)$. As $v(x) = p(x) + x$, we can rewrite (A.4) as

$$\begin{aligned} [\gamma + \lambda - \psi(i(x))]v(x) &= (\gamma + \lambda)x - [i(x) + \phi(x) + c(x) - A] \\ &+ \lambda \left[\int_{\bar{Z}(x)}^{Z^*} (Zv(x/Z) - x) dF(Z) + \int_{Z^*}^1 Z(v(x) - x) dF(Z) \right]. \end{aligned} \quad (\text{A.6})$$

By using (26) and the pricing equation (33) for the credit spread η , we obtain

$$\begin{aligned} [\gamma + \lambda - \psi(i(x))]v(x) &= (\gamma - r)x + A - i(x) \\ &+ \lambda \left[\int_0^{\bar{Z}(x)} Z \ell dF(Z) + \int_{\bar{Z}(x)}^{Z^*} Z v(x/Z) dF(Z) + \int_{Z^*}^1 Z v(x) dF(Z) \right] \end{aligned} \quad (\text{A.7})$$

in the region where $\underline{x} \leq x \leq \hat{x}$. By substituting $v(x) = v(\underline{x}) = v(\hat{x}) = v(\bar{x})$ for $x \leq \bar{x}$, which we have shown, into (A.7), we obtain:

$$\begin{aligned} [\gamma + \lambda - \psi(i(\underline{x}))]v(\underline{x}) &= (\gamma - r)x + A - i(\underline{x}) + v(\underline{x})\lambda \mathbb{E}(Z) \\ &- \lambda(v(\underline{x}) - \ell) \left[\int_0^{\bar{Z}(x)} Z dF(Z) \right] \end{aligned} \quad (\text{A.8})$$

Rearranging (A.8), and since the preceding equation holds for $x = \underline{x}$, we have

$$[\gamma - g(i(\underline{x}))] v(\underline{x}) = A - i(\underline{x}) + (\gamma - r)\underline{x} - \lambda(v(\underline{x}) - \ell) \left[\int_0^{\bar{Z}(\underline{x})} Z dF(Z) \right],$$

where $g(\cdot)$ is given by (17).

The optimization problem is now reduced to a single control variable problem \underline{x} , so that we can just choose \underline{x} to maximize $v(\underline{x})$. In summary, we now have shown the solution for the costless equity-issuance case is characterized by (14), (15), (16), and (18) in Section 3.1. Our results are valid for any concave $\phi(\cdot)$ and admissible distribution $F(\cdot)$ for Z . In our numerical analysis, we focus on the case where $F(Z)$ is given by a power law as in (41).

A.2 First Best

Under the first-best, the firm chooses its investment policy to maximize its market value:

$$V^{FB}(K_t) = \max_I \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} (AK_s - I_s) ds \right], \quad (\text{A.9})$$

where the firm's capital stock, K , follows (2).

By using the dynamic programming method, we obtain

$$rV^{FB}(K) = \max_I AK - I + \Psi(I, K)V_K^{FB}(K) + \frac{\sigma^2 K^2}{2} V_{KK}^{FB}(K) + \lambda \mathbb{E} [V^{FB}(ZK) - V^{FB}(K)]. \quad (\text{A.10})$$

The homogeneity property implies that $V^{FB}(K) = q^{FB}K$, where q^{FB} satisfies

$$rq^{FB} = \max_i A - i + [\psi(i) - \lambda(1 - \mathbb{E}(Z))]q^{FB}. \quad (\text{A.11})$$

Let i^{FB} denote the firm's first-best investment. Equation (A.11) implies that $\psi'(i^{FB})q^{FB} = 1$, the FOC for i^{FB} . Substituting $I_t = i^{FB}K_t$ into (2), we obtain:

$$dK_t = K_{t-} [\psi(i^{FB})dt + \sigma d\mathcal{B}_t^K - (1 - Z)d\mathcal{J}_t]. \quad (\text{A.12})$$

Substituting $I_t = i^{FB}K_t$ and (A.12) into (A.9) and simplifying the expression, we obtain that $V^{FB}(K) = q^{FB}K$, where q^{FB} satisfies the present-value formula, (19).

B Market Leverage

Applying Ito's Lemma to market leverage ml_t defined in (11) and using the homogeneity property, we obtain the following process for ml_t :

$$\begin{aligned} d\left(\frac{x_t}{v(x_t)}\right) &= \frac{v(x_{t-}) - x_{t-}v'(x_{t-})}{v(x_{t-})^2} dx_t + \left(\frac{v''(x_{t-})}{v(x_{t-})^3} - \frac{v'(x_{t-})}{v(x_{t-})^2}\right) (dx_t)^2 + \left(\frac{x_t^{\mathcal{J}}}{v(x_t^{\mathcal{J}})} - \frac{x_{t-}}{v(x_{t-})}\right) d\mathcal{J}_t \\ &= \mu_{ml}(x_{t-}) dt - \sigma_{ml}^O(x_{t-}) d\mathcal{B}_t^O - \sigma_{ml}^S(x_{t-}) d\mathcal{B}_t^S + \left(\frac{x_t^{\mathcal{J}}}{v(x_t^{\mathcal{J}})} - \frac{x_{t-}}{v(x_{t-})}\right) d\mathcal{J}_t, \end{aligned} \quad (\text{B.1})$$

where

$$\mu_{ml}(x) = \frac{v(x) - xv'(x)}{v(x)^2} \mu_x(x) + \frac{v''(x) - v'(x)v(x)}{v(x)^3} \sigma_x(x)^2, \quad (\text{B.2})$$

$$\sigma_{ml}^O(x) = \frac{v(x) - xv'(x)}{v(x)^2} \sigma \sqrt{1 - \rho^2} x, \quad (\text{B.3})$$

$$\sigma_{ml}^S(x) = \frac{v(x) - xv'(x)}{v(x)^2} \sigma (\theta(x) + \rho x), \quad (\text{B.4})$$

$$\sigma_x(x) = \sigma \sqrt{x^2 + 2\rho x\theta(x) + \theta(x)^2}. \quad (\text{B.5})$$

and $\mu_x(x)$ is given by (22).

C Solution: Costly Equity Issuance

In this section, we first characterize the firm's risk management, investment, and debt financing decisions in the equity-inaction region, where the marginal source of financing is debt. We then solve for the firm's optimal default, equity issuance, and payout decisions. We focus on the case where the hedging constraint given in (5) never binds. In Section 6, we turn to the other important case where the constraint given in (5) may bind. Decomposing our analysis into these two cases substantially facilitates the exposition of the economic mechanism and allows us to better understand the economic insights.

C.1 Debt Financing Region: Risk Management and Investment

In this region, the marginal source of financing is debt. We use dynamic programming to solve for the firm's equity value $P(K, X)$ and to derive the firm's optimal risk-management and investment policies.

Hamilton-Jacobi-Bellman (HJB) Equation. The equity-value function $P(K, X)$ satisfies the following HJB equation in the equity-inaction region:

$$\begin{aligned} \gamma P(K, X) = & \max_{\Theta, I, \Pi} \Psi(I, K) P_K(K, X) + [I - AK + C + \Phi] P_X(K, X) \\ & + \frac{\sigma^2 K^2}{2} P_{KK}(K, X) + \frac{\sigma^2 \Theta^2}{2} P_{XX}(K, X) + \rho \sigma^2 \Theta K P_{KX}(K, X) \\ & + \lambda \mathbb{E} [P(ZK, X - \Pi) - P(K, X)], \end{aligned} \quad (\text{C.1})$$

where $\mathbb{E}[\cdot]$ in (C.1) is evaluated with respect to the cumulative distribution function $F(Z)$ for the capital-stock recovery fraction Z , and includes both default and no-default events.

The first term on the right side of (C.1) describes how expected capital accumulation changes equity value $P(K, X)$; the second term captures how expected changes in debt X affect $P(K, X)$; the third term reflects the effects of capital-stock diffusion shocks on $P(K, X)$; the fourth and fifth terms capture the effects of diffusion-hedging demand Θ on $P(K, X)$; and finally the last term (appearing on the third line) of (C.1) describes the effects of jumps on the expected change in $P(K, X)$. When a jump arrives at time t ($d\mathcal{J}_t = 1$), the firm's capital stock falls from K_{t-} to $K_t = ZK_{t-}$ and debt is reduced from X_{t-} to the level of $X_t = X_{t-} - \Pi_{t-}$. By definition, $\Pi_{t-} \neq 0$ for hedgeable jump shocks, $Z \in [Z^*, 1]$ and $\Pi_{t-} = 0$ for unhedgeable jump shocks such that $Z \in [0, Z^*)$. In addition, a jump shock may also trigger a default on the firm's debt.

First-Order Conditions (FOCs). Consider first the diffusion-hedging demand Θ . When the hedging constraint given in (5) does not bind, the value function $P(K, X)$ is concave in X and the following FOC characterizes the optimal diffusion-hedging demand:

$$\Theta = -\frac{\rho K P_{KX}(K, X)}{P_{XX}(K, X)}. \quad (\text{C.2})$$

This equation resembles the intertemporal hedging demand in standard intertemporal portfolio choice models à la Merton (1971). Moreover, following the homogeneity property and simplifying the FOC for (C.2), we obtain the diffusion- hedging demand given by (24).

Next, consider the jump-insurance demand Π . We use \mathcal{J} as the superscript to indicate the arrival of a jump at time t ($d\mathcal{J}_t = 1$) so that $K_t^{\mathcal{J}}$ and $X_t^{\mathcal{J}}$ denote the firm's post-jump capital stock and debt at t , respectively. When a jump occurs at t and $Z \in [0, Z^*)$, $\Pi(Z) = 0$ by definition. When a jump occurs at t and the recovery fraction Z lies in the insurable range of $[Z^*, 1]$, the firm's capital stock decreases from K_{t-} to $K_t^{\mathcal{J}} = ZK_{t-}$ and its debt decreases from X_{t-} to $X_t^{\mathcal{J}} = X_{t-} - \Pi_{t-}$. By substituting the formula for Φ_{t-} given in (6) into the HJB equation (C.1), we obtain that the firm maximizes the following expression by choosing Π_{t-} :

$$\lambda \mathbb{E} \left[(P(K_t^{\mathcal{J}}, X_t^{\mathcal{J}}) + \Pi_{t-} P_X(K_{t-}, X_{t-})) \mathcal{I}_{Z \geq Z^*} \right]. \quad (\text{C.3})$$

This problem can be simplified to the choosing Π for each $Z \in [Z^*, 1]$:

$$\max_{\Pi} [P(ZK, X - \Pi) - (-P_X(K, X)) \cdot \Pi] \quad (\text{C.4})$$

The intuition for (C.4) is as follows: The insurance premium payment per unit of time is equal to $\Pi \lambda dF(Z)$. Since debt is the marginal source of financing to purchase jump insurance, and the marginal cost of debt financing is $-P_X(K, X) \geq 1$, the economic cost of the jump-hedging demand Π is equal to $-P_X(K, X) \Pi \lambda dF(Z) > 0$. The expected benefit of this jump insurance is that when there is a jump in the capital stock from K to ZK (with probability $\lambda dF(Z)$ per unit of time), the firm is able to reduce its debt from X to $X - \Pi$ via the insurance payment Π . Therefore, the firm chooses Π for each $Z \in [Z^*, 1]$ to maximize the difference between the hedging benefit $(P(ZK, X - \Pi) - P(K, X)) \lambda dF(Z)$ and the cost $(-P_X(K, X)) \Pi \lambda dF(Z)$. Factoring out $\lambda dF(Z)$ and dropping the pre-jump-equity-value term $P(K, X)$ which is independent of Π , we obtain (C.4).

Finally, the FOC for investment I is:

$$-P_X(K, X) = \Psi_I(I, K) P_K(K, X). \quad (\text{C.5})$$

Because $\Psi(I, K)$ is increasing and concave in I , the SOC is satisfied for I . The left side of (C.5) is the marginal cost of debt $-P_X(K, X)$ and the right side of (C.5) is given by the product of (a.) the expected marginal increase of capital stock, $\Psi_I(I, K)$, and (b.) the marginal value of capital, $P_K(K, X)$, also known as the marginal q . The firm optimally chooses investment I to equate the two sides at the margin. Due to costly external financing, investment depends on the (endogenous) stochastic marginal value of debt. This is in contrast to the neoclassic q -theory of investment, where $-P_X(K, X) = 1$. Finally, using the homogeneity property and simplifying the FOC for (C.5), we obtain the investment equation given by (21).

C.2 Default Region: Debt Capacity, Equity Issue, and Liquidation

Having described the firm's policies in the equity-inaction region, we now turn to the questions of: 1.) when should the firm raise external financing, and in what form? 2.) How much debt can the firm raise? and 3.) under what conditions should the firm liquidate itself?

Endogenous Debt Capacity and Default Boundary. As the firm cannot manage all its risk exposures via actuarially fair insurance contracts, declaring default is optimal under certain scenarios, for example upon the arrival of a large downward unhedgeable jump in earnings. Put differently, there exists a range of values of debt for which it is optimal for the firm to default. In this default region, the APR is applied, giving creditors the right to seize the firm's assets and be paid before shareholders. Because shareholders are protected by limited liability and because default is inefficient, a firm that maximizes its equity value will default if and only if equity value is zero.⁵⁴ Let \bar{X}_t denote the endogenous default boundary and equivalently debt capacity. The default region is defined by $X_t \geq \bar{X}_t$ and

$$P(K_t, X_t) = 0, \text{ when } X_t \geq \bar{X}_t. \quad (\text{C.6})$$

We characterize the firm's default boundary as a function $\bar{X}(\cdot)$ of its capital stock K_t by evaluating (C.6) at \bar{X}_t . Because $\bar{X}(\cdot)$ is a free boundary, we need an additional condition, to which we return.

Debt Pricing: Credit Spreads. The equilibrium credit spread η_{t-} can be derived from the following zero-profit condition for debt-investors given the firm's default policy \bar{X}_t :

$$X_{t-}(1 + rdt) = X_{t-}(1 + (r + \eta_{t-})dt) \mathbb{P}_{t-}(X_t \leq \bar{X}_t) + \mathbb{E}_{t-}(L_t \lambda \mathcal{I}_{X_t > \bar{X}_t})dt. \quad (\text{C.7})$$

The first term on the right side of (C.7) describes the full debt repayment, $X_{t-} + C_{t-}dt$, where $C_{t-} = (r + \eta_{t-})X_{t-}$ is the interest payment, to creditors multiplied by $\mathbb{P}_{t-}(X_t \leq \bar{X}_t)$, the probability at time $t-$ that the firm won't default at time t . The second term gives the creditors' expected payoff upon default, which is equal to the firm's stochastic liquidation value, L_t , given by (8). In sum, the equilibrium condition (C.7) states that creditors' expected rate of return is equal to the risk-free rate, r .

Equation (C.7) shows that jumps are necessary to generate default in our model. To see this, suppose that there are only diffusion shocks, then (C.7) implies that the credit spread η_t must be zero. The intuition that is as follows. For a small time increment dt , diffusions shocks can cause losses of order \sqrt{dt} with strictly positive probability. These losses cannot be compensated with any finite credit spread η_t , as this compensation is only of order $\eta_t dt$, which is much lower than \sqrt{dt} . In contrast, in Leland (1994) and other diffusion-shock-based models default is possible because the firm issues term debt, which cannot be adjusted.

Substituting the expected liquidation value upon default given in (32) into (C.7), we obtain the pricing equation for the credit spread given by (33).

Next, we analyze the equity issuance decision and link it to the firm's debt capacity, \bar{X}_t .

⁵⁴If equity value is strictly positive after creditors liquidate the firm, shareholders would have defaulted too soon. A dominant strategy for shareholders is to keep on rolling over the firm's debt as long as equity value is strictly positive, thereby preserving the firm as a going-concern. This argument implies that (C.6) must hold for the debt default region.

External Equity Issuance. The firm's option to issue equity is valuable as doing so allows it to reduce its debt level and lower the likelihood of a costly default. As $P(K, X)$ must be continuous when the firm issues equity, the following value-matching condition holds:

$$P(K, X) = P(K, X - M) - (h_0 K + M + h_1 M). \quad (\text{C.8})$$

The left and right sides of (C.8) are the firm's equity value before and after equity issuance, respectively. On the right side of (C.8), $P(K, X - M)$ is the firm's equity value after it raises external funds with the net amount of M . The firm's debt after the equity issue is reduced to $X - M$. The second term on the right side of (C.8) is the total cost of the equity issue, which includes the fixed cost $h_0 K$ and the variable cost $h_1 M$. The proceeds from the equity issue are used to repay a fraction of the firm's outstanding debt. Doing so reduces the likelihood of costly default and inefficient liquidation.

What is the optimal amount M the firm should raise when it goes to public equity markets? The optimal level of M must be chosen so that the marginal equity value of reducing leverage at $(X - M)$ is equal to the marginal cost of raising external equity. This tradeoff is described by the following smooth-pasting condition (Dumas, 1991):

$$-P_X(K, X - M) = 1 + h_1. \quad (\text{C.9})$$

Obviously, the fixed cost $h_0 K_t$ does not appear in (C.9).

Finally, because external equity financing is costly, it is optimal for the firm to postpone its equity issue as much as possible, as doing so reduces the present discounted costs of tapping external equity markets.⁵⁵ The firm issues equity only at the very last moment before reaching the bankruptcy point when its equity value is zero. Let \hat{X}_t denote the firm's debt level at and above which the firm chooses to issue costly external equity, then the equity-issue and default boundaries are the same, $\hat{X}_t = \bar{X}_t$, and $P(K_t, \hat{X}_t) = 0$.

Again using the homogeneity property, we can transform the value-matching condition (C.8) into (34). The FOC (C.9) for the net amount raised m then becomes (35), and $p(\hat{x}) = P(K_t, \hat{X}_t)/K_t = 0$.

Firm Liquidation. When equity issuance is too costly, the firm chooses liquidation over equity issue and the firm's debt capacity \bar{X}_t must equal the liquidation value of its capital stock: $\bar{X}_t = L_t$. Suppose this were not true. If $\bar{X}_t < L_t$, then the firm can always make itself better off by increasing debt capacity, which is feasible. If $\bar{X}_t > L_t$, then the creditors cannot break even in response to diffusion shocks.⁵⁶

Summary. We can calculate the debt capacity by using the following two-step procedure. First, suppose that the firm uses costly equity issue to reduce its debt. We can solve for the equity issue boundary \hat{X}_t by using (C.8), (C.9), and $P(K_t, \hat{X}_t) = 0$. If $\hat{X}_t > L_t$, we have obtained the debt capacity and $\bar{X}_t = \hat{X}_t$. However, if $\hat{X}_t < L_t$, the firm is better off never using its equity issuance option as it is too costly. As a result, the equilibrium debt capacity is then $\bar{X}_t = L_t$, which is supported by the firm's liquidation policy.

⁵⁵This intuition is analogous to that in Bolton, Chen, and Wang (2011) who show that the firm should only issue equity after running out of its cash balances. Unlike in Bolton, Chen, and Wang (2011), however, here debt capacity is endogenous and debt is risky.

⁵⁶See the discussion for the intuition in the paragraph following the debt pricing equation (C.7).

C.3 Payout Region

When the firm's debt is very low the firm is better off paying out the firm's excess liquidity, as shareholders are impatient ($\gamma > r$). A basic question is how low the firm's debt needs to be before it starts paying out. We denote by \underline{X}_t the firm's endogenous payout boundary: When $X_t > \underline{X}_t$ the firm retains all its earnings and when $X_t < \underline{X}_t$ the firm pays out the lump-sum amount $\underline{X}_t - X_t$ to its shareholders.

Since $P(K, X)$ must be continuous before and after the payout decision, the value $P(K, X)$ for $X < \underline{X}$ is given by

$$P(K, X) = P(K, \underline{X}) + (\underline{X} - X), \quad \text{for } X < \underline{X}. \quad (\text{C.10})$$

Moreover, since the above equation holds for X close to \underline{X} , we may take the limit and obtain the following condition for the endogenous lower boundary \underline{X} :

$$P_X(K, \underline{X}) = -1, \quad (\text{C.11})$$

which states that the marginal cost of debt is equal to one at \underline{X} , as the firm is indifferent between distributing and retaining one dollar. Equation (C.11) defines \underline{X} as a function of K , which we denote by $\underline{X}(K)$. As the payout boundary $\underline{X}(K)$ is optimally chosen by the firm, we also have the following super-contact condition (Dumas, 1991):

$$P_{XX}(K, \underline{X}(K)) = 0. \quad (\text{C.12})$$

Finally, substituting $P(K, X) = p(x)K$ in (C.11) and (C.12), we obtain the scaled equity value which satisfies the smooth-pasting and super-contact conditions given in (37).

D Hedging and Risk Taking

In this Appendix, we report hedging and risk-taking results for both the cases where $v(x)$ is globally concave and the general case.

D.1 Globally concave case

Diffusion and Jump Hedging Demand. Panel A of Figure 7 shows that the firm optimally exploits all available hedging opportunities by setting $\theta(x_t) = -\rho x_t = -0.2x_t$, so that its leverage x_t is independent of the hedgeable diffusion shock, $d\mathcal{B}_t^S$. Panel B of Figure 7 describes the optimal jump hedging strategy. As for the hedgeable diffusion shocks, the firm optimally chooses $\pi(x_t, Z) = (1 - Z)x_t = 0.5x_t$, where $Z = 0.5$, so that its leverage x_t does not respond to hedgeable jump shocks. That is, the arrival of a jump whose Z satisfies $Z \in [Z^*, 1)$ does not change x_t at all.

In sum, the firm is endogenously averse to fluctuations in leverage x and through its risk management policy the firm seems to minimize the volatility of leverage.

Leverage Drift and Costly External Equity Financing. When $Z^* = 0.5$ firm value $v(x)$ is globally concave and the firm's equity-issuance and bankruptcy boundaries are the same: $\bar{x} = \hat{x} = 1.19$. When leverage reaches $\bar{x} = 1.19$ the firm engages in a recapitalization by issuing costly external equity, raising a net amount $m = 0.53$, and using these proceeds to bring down its leverage to $\bar{x} - m = 1.19 - 0.53 = 0.66$, where the firm's net marginal benefit of reducing debt is equal to its marginal cost of issuing equity, $-v'(0.66) = 0.02$.

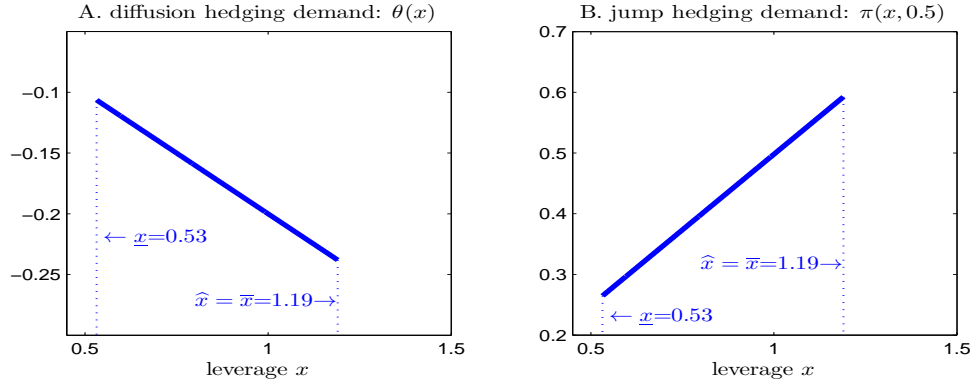


Figure 7: **Diffusion hedging demand, $\theta(x)$, and jump hedging demand for $Z = 0.5$, $\pi(x, 0.5)$.** All parameter values are given in Table 1.

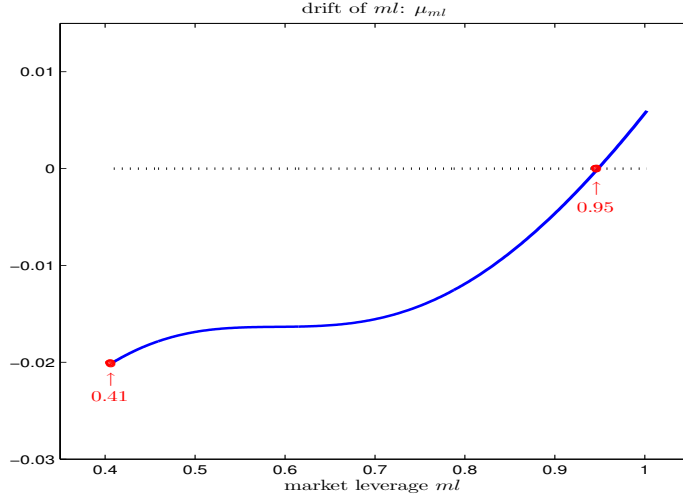


Figure 8: **The drift of market leverage for the case $Z^* = 0.5$.** Other parameter values are given in Table 1.

Figure 8 plots the drift of market leverage ml_t (excluding jumps) as a function of its current level ml_t . This figure captures the firm's leverage dynamics in a succinct way under its optimal financial policy. Note first that the firm's target market leverage is equal to 41% at its the left boundary. Second, when current leverage is very high the drift of leverage is positive, indicating that it is difficult for the firm to reduce its leverage going forward. The reason is that when leverage is high the firm faces large interest payments and hedging costs, causing leverage to drift upward (positive μ_{ml} .) This generates a self-reinforcing debt spiral. Although the firm cuts investment to get out of this spiral, it is difficult to do so because of the positive drift and debt overhang. That being said, for most levels of market leverage the drift is negative. The firm is able on average to bring leverage down from these levels, get closer to its target leverage, so as to be able to resume

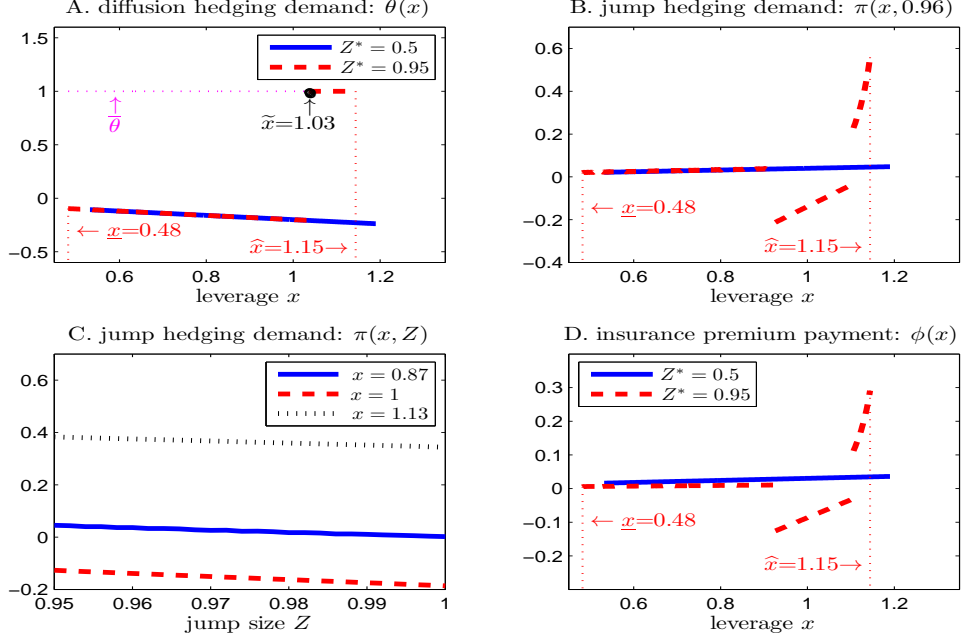


Figure 9: **Diffusion hedging demand, $\theta(x)$, jump hedging demand for $Z = 0.95$, $\pi(x, 0.95)$ and $\pi(x, Z)$ for $x = 0.87, x = 1, 1.13$, and the insurance premium payment, $\phi(x)$.** All parameter values other than Z^* are given in Table 1.

payouts to its shareholders.

D.2 Risk-seeking case

Diffusion and Jump Hedging Demands and Insurance Premium Payments. Figure 9 focuses on the impact of changing the firm's hedging opportunity set on its hedging strategies and the cost of hedging/insurance payments. Panel A shows that the firm completely hedges all its hedgeable idiosyncratic risk exposure by setting $\theta(x) = -\rho x = -0.2x$ in the region where $p(x)$ is concave. In contrast, in the convex region, the firm takes the maximally allowed diffusion risk exposure by setting $\theta(x) = \bar{\theta} = 1$ for $x \in (\tilde{x}, \hat{x}) = (1.03, 1.15)$. For $Z^* = 0.95$, $\theta(x)$ jumps precisely at the inflection point $\tilde{x} = 1.03$.

Panels B and C demonstrate that the firm's jump hedging demand, $\pi(x_t, Z)$, drastically changes with the firm's hedging opportunity. Consider $\pi(x_t, 0.96)$, its demand for the jump contingency with $Z = 0.96$. When the firm's balance sheet is sufficiently strong, i.e., $x \in (0.48, 0.92)$, it chooses to completely hedge this jump exposure by setting $\pi(x_t, 0.96) = (1 - 0.96)x_t = 0.04x_t$, as it does for the case with $Z^* = 0.5$. The solid line in Panel C plots the function, $\pi(0.87, Z) = (1 - Z) \times 0.87$ making the same complete-hedging point, as $0.87 \in (0.48, 0.92)$.

Second, when the firm is sufficiently highly levered, i.e., $x \in (0.92, 1.11)$, the firm's net marginal cost of debt, $-v'(x)$ is greater than 0.28 and reaches 0.42 at $x = \hat{x} = 1.03$, as can be seen from Panel C of Figure 9. In this case, it is optimal for the firm to attempt reducing its leverage by taking on actuarially fairly priced bets before tapping costly external equity financing: Rather than

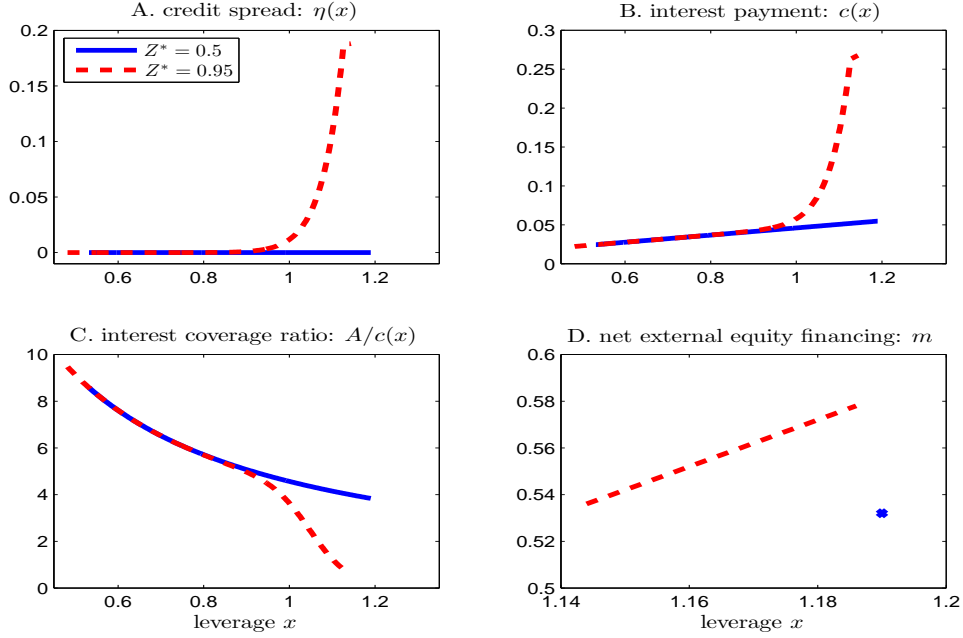


Figure 10: **The equilibrium credit spread, $\eta(x)$, interest payment, $c(x)$, interest coverage ratio, $A/c(x)$, and (net) equity issuance, m .** All parameter values other than Z^* are given in Table 1.

hedging out this jump contingency, the firm *sells* jump-insurance contracts and uses the insurance proceeds to pay down debt. For example, $\pi(1, 0.96)$ switches the sign and changes from 0.04 to -0.14, as we increase Z^* from 0.5 to 0.95. The dashed line in Panel C plots the function, $\pi(1, Z)$, which is negative and decreasing with Z , confirming that it is optimal to sell jump insurances, as $1 \in (0.92, 1.11)$.

Finally, for a firm that is very highly levered and close to exhausting its debt capacity, i.e., when $x > 1.11$, rather than speculating on the potential jump losses of its own capital stock as we have just discussed, the firm chooses a very large (positive) jump hedging demand. For example, $\pi(x, 0.96) = 0.23$ for $x = 1.11$ and $\pi(x, 0.96) = 0.56$ for $x = \hat{x} = 1.15$. In this region, even by selling jump insurance is unlikely to save the firm from financial distress. The firm concludes that buying (seemingly excessive) jump insurances is the best bet in the interest of shareholders. The dotted line in Panel C plots the function, $\pi(1.13, Z)$, which is positive and very high, as $1.13 \in (1.11, 1.15)$. This result is in sharp contrast with the other two cases. In this case, receiving a large negative capital stock shock saves the firm from bankruptcy.

Panel D shows that the insurance premium payment, $\phi(x)$, approximately tracks the shape of the jump hedging demand, $\pi(x, Z)$, as shown in Panel B. This follows from the fact that the jump insurance premium payment $\phi(x)$ is equal to the integration of $\pi(x, Z)$ over all hedgeable jump exposures, i.e., $Z \geq Z^*$.

Credit Spreads, Interest Payments, Coverage Ratio, and Equity Financing. Figure 10 displays the impact of changing the firm's hedging opportunity set on its debt and equity

financing strategies and terms.

Panel A shows that the equilibrium credit spread $\eta(x)$ increases x . The higher the firm's leverage the higher default risk. Unlike for the case with $Z^* = 0.5$, where the credit spread is essentially equal to zero regardless of the firm's leverage, the effect of leverage on $\eta(x)$ is very large: The sensitivity of credit spread with respect to leverage increases significantly as leverage exceeds 0.9, reaching 19% at the equity-issuance boundary $x = \hat{x} = 1.15$. This panel shows the large impact of lacking hedging opportunity on the firm's cost of borrowing.

Panel B shows that the interest payment is also sensitive to leverage. This is expected as the coupon, $c(x)$, is equal to the product of the interest rate, $(r + \eta(x))$, and leverage x . As the firm's book leverage x increases from 0.9 to 1.15, the coupon payment $c(x)$ increases from four percent of its capital stock to 27 percent causing the firm's interest coverage ratio, $A/c(x)$, to drop substantially from a very prudent level, 4.96 to quite a risky level, 0.78, as seen from Panel C.

Panel D plots the net external equity financing amount, $m(x)$, as a function of x in the equity issuance region, where $x \in (\hat{x}, \bar{x}) = (1.15, 1.19)$ for the case of $Z^* = 0.95$. As the firm's shareholders use the entire net amount raised via equity issuance to pay down the firm's debt, the firm's book leverage after its equity issuance is $x_t - m_t$. Additionally, the FOC for equity issuance implies that the net marginal cost of debt, $-v'(\cdot)$, at the post-equity-issue leverage of $(x_t - m_t)$, must equal to the marginal cost of equity financing, h_1 . In our example, $h_1 = 0.02$. By using the plot of $-v'(x_t)$ in Panel C of Figure 6, we conclude that the firm's book leverage after equity issuance is equal to 0.61.

Finally, the stand-alone dot in Panel D depicts the net equity financing amount for the case of $Z^* = 0.5$. Recall that the equity issuance and the bankruptcy boundaries are the same: $\bar{x} = \hat{x} = 1.19$, the post-equity-issue leverage then must equal to $0.66 = 1.19 - 0.53$.

As we improve the firm's hedging opportunity by decreasing Z^* from 0.95 to 0.5, the firm delays issuing equity (by increasing \hat{x} from 1.15 to 1.19), raises less equity (by decreasing m), but achieves a higher post-equity-issue leverage x (an increase from 0.61 to 0.66.) The intuition is that improving the firm's hedging opportunity (by decreasing Z^*) improves the firm's ability to take on debt, enhancing debt capacity.

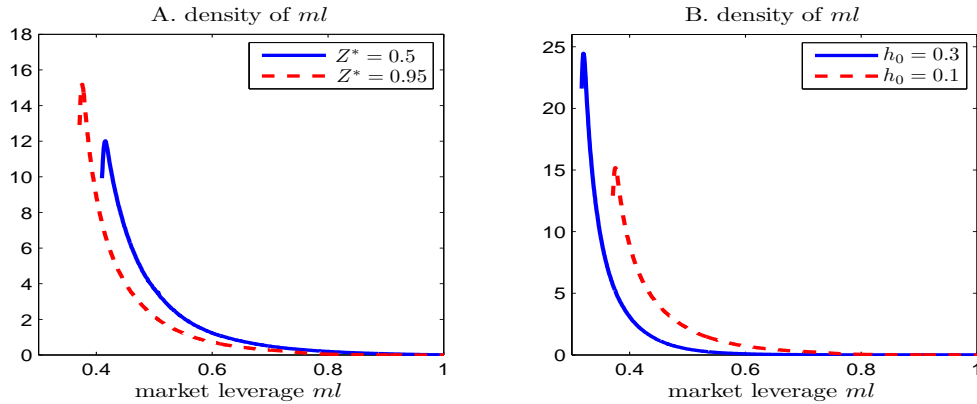


Figure 11: **The density functions of market leverage ml : Comparative statics with respect to Z^* and h_0 .** All parameter values other than Z^* and/or h_0 are given in Table 1. In Panel A, $h_0 = 0.1$. In Panel B, $Z^* = 0.95$. Other parameter values are given in Table 1.

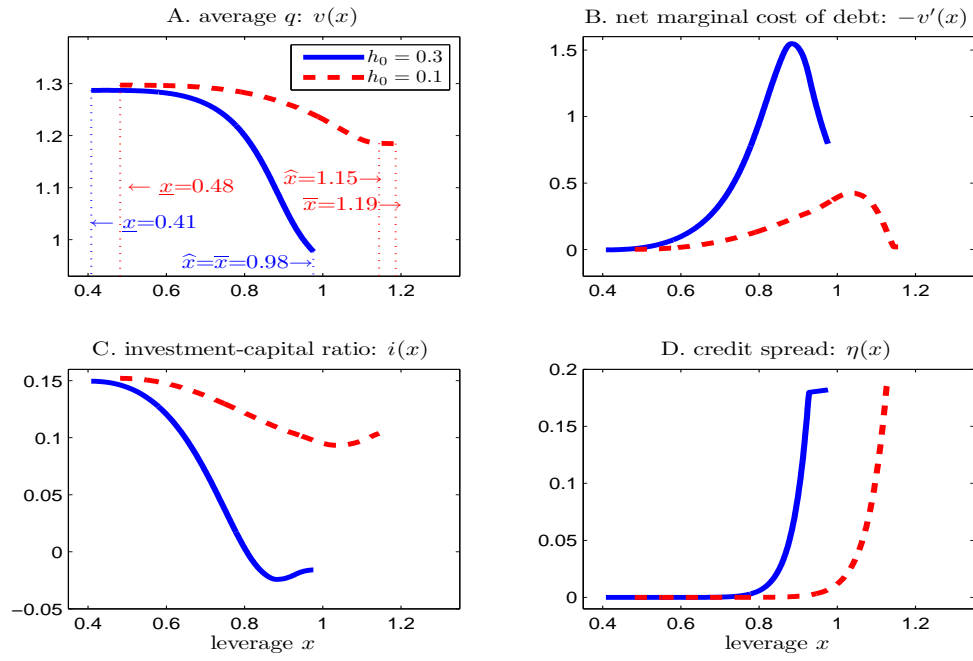


Figure 12: **The effects of external financing costs h_0 .** All parameter values other than h_0 and $Z^* = 0.95$ are given in Table 1.

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