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THE LIMITS OF *ONETARY ECONOMICS*: ON MONEY AS A LATENT MEDIUM OF EXCHANGE

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ABSTRACT

We formulate a generalization of the traditional medium-of-exchange function of money in contexts where there is imperfect competition in the intermediation of credit, settlement, or payment services used to conduct transactions. We find that the option to settle transactions directly with money strengthens the stance of sellers of goods and services vis-á-vis intermediaries. We show this mechanism is operative even for sellers who never exercise the option to sell for cash, and that these "latent money demand" considerations imply monetary policy remains effective through medium-of-exchange channels even if the share of monetary transactions is arbitrarily small.

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1. Introduction

We formulate a generalization of the traditional medium-of-exchange function of money in contexts where credit, settlement, or payment services involve financial intermediaries with some degree of market power (e.g., banks, broker-dealers, credit card companies). The option to settle transactions directly with money strengthens the stance of sellers of goods, services, or assets, vis-à-vis intermediaries. This mechanism is operative even for sellers who never exercise the option to trade in cash: the mere threat of monetary exchange restrains the market power of the financial intermediaries they face. From an aggregate perspective, the mechanism operates even if the volume of transactions settled with cash is very small (e.g., because most traders opt for the credit-based settlement alternative to money). Thus, in these contexts, money functions as a *latent* medium of exchange. This latency, or off-equilibrium role of money, is distinct from the traditional medium-of-exchange function that money performs when it is actively exchanged to overcome trading frictions, such as double-coincidence-of-wants problems, lack of commitment, and lack of enforcement.

The role of money as a discipline device for imperfectly competitive financial intermediaries opens a novel conduit for the monetary transmission mechanism that operates through the effect that changes in the opportunity cost of holding money have on money demand, and ultimately on prices and allocations. We show that, unlike the traditional medium-of-exchange role that emphasizes buyer-side incentives to carry money, the seller-side benefits of the option of monetary exchange as safeguard against intermediary market power remains relevant even in cashless limiting economies where credit and settlement are so developed that the transaction velocity of money is arbitrarily large. The changes in incentives to hold money that monetary policy imposes on a negligible-sized population of cash-only sellers can have nonnegligible macroeconomic impact. The logic is that it is the money-demand behavior of the few cash-only sellers that the larger population of sellers who use credit-based settlement threaten to mimic in order to keep the financial intermediaries' market power in check. Because money functions as a *latent* medium of exchange, changes in the value of money influence the terms of trade of everyone with a credible option to trade for money—even those who end up choosing to settle transactions through the intermediary rather than use money.

A large body of work in macroeconomics rests on the premise that artificial economies without money are well suited to study monetary policy. In fact, most of the work in modern monetary economics that caters to policymakers, abstracts from the usefulness of money altogether: there is typically no money in the models, or if there is money, it is merely held as a redundant asset (see, e.g., the textbook treatments of the New Keynesian model in Woodford (2003) or Galí (2008)). What underlies this moneyless approach to monetary economics is the received wisdom that the medium-of-exchange role of money is essentially irrelevant for the transmission of monetary policy in the context of advanced economies whose credit-based settlement mechanisms have developed sufficiently to make the inverse velocity of some monetary aggregates very small. Our theoretical results suggest that, in general, any attempt to assess the macroeconomic effects of monetary policy without medium-of-exchange money-demand considerations is at best incomplete.

The rest of the paper is organized as follows. Section 2 describes the economic environment, presents the solution to the social planner's problem, formulates the individual optimization and bilateral bargaining problems, and defines equilibrium. Section 3 characterizes the equilibrium of the nonmonetary economy. Section 4 characterizes monetary equilibria: stationary (Section 4.1), dynamic (Section 4.2), and sunspots (Section 4.3). For each type of equilibrium, Section 5 derives prices and allocations in the cashless pure-credit limit. Section 6 concludes. The appendix contains all proofs.

2. Model

2.1. Environment

Time is represented by a sequence of periods indexed by $t \in \mathbb{T} \equiv \{0, 1, ...\}$. Each period is divided into two subperiods where different activities take place. There are three types of infinitely lived agents: *bankers, consumers*, and *producers*, denoted B, C, and P, respectively. An agent of type $i \in \{B, C, P\}$ is identified with a point in the set $\mathcal{I}_i = [0, N_i]$, with $N_i \in \mathbb{R}_{++}$. There are two consumption goods in each period: *good 1* and *good 2*.

In every subperiod, each producer has a time endowment that can be used as labor input. Each consumer also has a time endowment that can be used as labor input, but only in the second subperiod. In the first subperiod, producers have access to a linear technology to transform labor into good 1, which is only consumed by consumers. Production of good 1 takes place at the beginning of the first subperiod, before agents engage in any trading activity. In the second subperiod, consumers and producers have access to a linear production technology to transform labor into good 2, which is consumed by all agents. Good 1 and good 2 cannot be stored across periods, but there is within-period storage: producers can transform every unit of unsold inventory of beginning-of-period good 1 into $\underline{\varphi} \in \mathbb{R}_+$ units of end-of-period good 2.

A monetary authority issues a financial security called *money*, which is durable and intrinsically useless (i.e., it is not an argument of any utility or production function, and it is not a formal claim to goods or services). The quantity of money outstanding at the beginning of period t is denoted M_t, with $M_0 \in \mathbb{R}_{++}$ given, and distributed uniformly among consumers. In the second subperiod of every period, the monetary authority injects or withdraws money via lump-sum transfers or taxes to consumers, so that the money supply evolves according to $M_{t+1} = \mu M_t$, with $\mu \in \mathbb{R}_{++}$.

In order to preserve a meaningful role for money as a medium of exchange, we assume that from the standpoint of producers, consumers are unable to commit, so producers cannot enforce consumers' promises (neither individually nor via collective punishment schemes). We assume bankers are endowed with the ability to enforce and commit. In particular, a banker can enforce a future payment promised by a consumer, and can commit to make a future payment to a seller. This special ability to trust consumers and be trusted by producers makes bankers well suited to act as financial intermediaries between consumers and producers. Specifically, some consumers will issue bonds through bankers in the first subperiod of t, with each bond representing a claim to one unit of good 2 to be delivered to bond holders through bankers in the second subperiod of t.[1]

In the second subperiod, all agents can trade good 2 and money in a spot Walrasian market. In the first subperiod, consumers and producers may trade good 1, money, and private bonds, while bankers can trade money and private bonds. Trade in the first subperiod is organized as follows. Two Walrasian markets operate contemporaneously: a *goods market* and a *bond (interbank) market*. All consumers and producers have access to the goods market where they can trade good 1 and money competitively. All bankers have access to the bond market where they can trade bonds and money competitively. Consumers and producers access the bond market indirectly, by engaging in bilateral trades with bankers whom they contact at random. Specifically, let $\alpha_i \in [0, 1]$ denote the probability that an agent of type $i \in \{C, P\}$ contacts a

[1]Absent bankers, there would be complete lack of enforcement: consumers would be unable to borrow, and would have no alternative but to fund first-subperiod consumption of good 1 with money. The equations in the following sections also admit an equivalent interpretation. Instead of assuming that bankers have the special power to enforce and commit, one could assume consumers can themselves commit to repay, but that bond trade must be intermediated by bankers for reasons other than limited enforcement of contracts and limited commitment to honor them.

random banker. Once the agent and the banker have met, the pair negotiates the quantities of bonds and money that the banker will buy or sell in the competitive bond market on behalf of the agent, and an intermediation fee for the banker's service. The banker's fee is expressed in terms good 2 and paid in the second subperiod. The terms of this bilateral trade are determined by Nash bargaining, where an agent of type $i \in \{C, P\}$ has bargaining power $\theta_i \in [0, 1]$.

The individual preferences of an agent of type $i \in \{B, C, P\}$ are represented by

$$\mathbb{E}_0^i \sum_{t=0}^{\infty} \beta^t \left[u\left(y_{it}\right) \mathbb{I}_{\{i=C\}} - \kappa y_{it} \mathbb{I}_{\{i=P\}} + v(x_t) - h_t \right],$$

where the expectation operator, $\mathbb{E}_0^i$, is with respect to the probability measure induced by the random trading process in the first subperiod, $\beta \in (0, 1)$ is the discount factor, $u : \mathbb{R}_+ \to \mathbb{R}$ is the consumer's utility function for good 1, $\mathbb{I}_{\{\cdot\}}$ is an indicator function that equals 1 if the condition in the subscript is satisfied, and 0 otherwise, $\kappa \in \mathbb{R}_{++}$ is the producer's marginal (disutility) cost of producing good 1, y_{it} is the agent's consumption (if $i = C$) or production (if $i = P$) of good 1 in period t, x_t is the agent's consumption of good 2 in period t, and h_t is the agent's disutility of supplying labor input h_t in the second subperiod of period t. We assume $u'' < u(0) = 0 < u'$, $v'' \leq v(0) = 0 < v'$, $\underline{\varphi} < \kappa$, and that there exist $x^*, y^* \in \mathbb{R}_{++}$ such that $v'\left(x^*\right) = 1$ and $u'\left(y^*\right) = \kappa$. For any $\varphi \in \mathbb{R}_+$, let $\mathrm{D}(\varphi) \equiv u'^{-1}\left(\varphi\right)$.

The following result characterizes the efficient allocation that solves the problem of a social planner who maximizes the equally weighted sum of all agents' expected discounted utilities. Let the planner's solution be denoted $\left\{y_{Ct}^*, y_{Pt}^*, (x_{it}^*, h_{it}^*)_{i \in \{B,C,P\}}\right\}_{t=0}^{\infty}$, where y_{Ct}^* is the individual consumption of good 1 in period t, y_{Pt}^* is the individual production of good 1 in period t, x_{it}^* is the individual consumption of good 2 of an agent of type i in period t, and h_{it}^* is the individual production of good 2 of an agent of type i in period t.

Proposition 1. *The efficient allocation is* $y_{Ct}^* = y^*$, $y_{Pt}^* = \frac{N_C}{N_P} y^*$, *and* $x_{it}^* = h_{it}^* = x^*$ *for all* $i \in \{B, C, P\}$ *and all* t.

2.2. Individual optimization, bargaining, and definition of equilibrium

We begin by describing the individual optimization problems in the second subperiod of a typical period. Let $W_t^i(a_t^m, a_t^g)$ denote the maximum expected discounted payoff, at the beginning of the second subperiod of period t, of an agent of type $i \in \{B, C, P\}$ who has $a_t^m \in \mathbb{R}_+$ units of money and a claim to $a_t^g \in \mathbb{R}$ units of good 2. Let $V_t^i\left(a_t^m\right)$ denote the maximum expected discounted payoff of an agent of type $i \in \{B, C, P\}$ with money holding a_t^m at the beginning of the first subperiod of period t. Then

$$W_t^i(a_t^m, a_t^g) = \max_{(x_t, h_t, a_{t+1}^m) \in \mathbb{R}_+^3} \left[v(x_t) - h_t + \beta V_{t+1}^i(a_{t+1}^m) \right], \tag{1}$$

$$\text{s.t. } x_t + \phi_t a_{t+1}^m \leq h_t + \phi_t a_t^m + a_t^g + \phi_t T_t^m \mathbb{I}_{\{i=C\}},$$

where ϕ_t is the real price of unit of money in terms of good 2, and $T_t^m \in \mathbb{R}$ is the time t lump-sum monetary injection to an individual consumer. Next, consider the three individual optimization problems that each agent type faces in the first subperiod of a typical period t.

First, consider the portfolio problem of a banker at the end of the first subperiod of period t, i.e., after the round of bilateral bond-market trades with consumers and producers. Let $\hat{W}_t^B(a_t^m, a_t^g)$ denote the maximum expected discounted payoff of a banker who has money holding a_t^m and a claim to a_t^g units of good 2, as he reallocates his portfolio of money and bonds in the bond market at the end of the first subperiod of period t (i.e., possibly after having executed a trade on behalf of a client).[2] Then

$$\hat{W}_t^B(a_t^m, a_t^g) = \max_{\bar{\boldsymbol{a}}_t \in \mathbb{R}_+ \times \mathbb{R}} W_t^B(\bar{a}_t^m, \bar{a}_t^g) \tag{2}$$

$$\text{s.t. } \bar{a}_t^m + q_t \bar{a}_t^b \leq a_t^m,$$

where $\bar{\boldsymbol{a}}_t = (\bar{a}_t^m, \bar{a}_t^b)$, $\bar{a}_t^g = a_t^g + \bar{a}_t^b$, and q_t is the nominal price of a bond in the bond market of time t. Let $\bar{\boldsymbol{a}}_{Bt}(a_t^m) = (\bar{a}_{Bt}^m(a_t^m), \bar{a}_{Bt}^b(a_t^m))$ denote the solution to the maximization in (2).

Second, consider a consumer who enters period t with money holding a_t^m and is unable to trade bonds through a banker. This could happen either because the agent is unable to contact a banker (an event that happens with probability $1 - \alpha_C$), or contacts a banker (with probability α_C) but the negotiation breaks down. This agent's individual decision problem in the first subperiod of t is to choose the quantity of good 1 to buy in the goods market, $\tilde{y}_{Ct}(a_t^m)$, and the post-trade money holding, $\tilde{a}_{Ct}^m(a_t^m)$, that satisfy

$$(\tilde{y}_{Ct}(a_t^m), \tilde{a}_{Ct}^m(a_t^m)) = \arg \max_{(\tilde{y}_t, \tilde{a}_t^m) \in \mathbb{R}_+^2} u(\tilde{y}_t) + W_t^C(\tilde{a}_t^m, 0), \tag{3}$$

$$\text{s.t. } \tilde{a}_t^m + p_t \tilde{y}_t \leq a_t^m,$$

[2] In principle, the banker may be holding a nonzero bond position when reallocating his own portfolio at the end of the first subperiod. However, as will become clear when we formulate the relevant bargaining problem, it is without loss of generality to assume that the banker's portfolio after having provided intermediation services is the same as the banker's beginning-of-period portfolio, which has zero bonds.

where p_t is the nominal price of good 1 in the goods market of period t.

Third, consider a producer who entered period t with money holding a_t^m, produced y_t at the beginning of the period, and is then unable to trade bonds through a banker in the first subperiod. This could happen either because the producer is unable to contact a banker (an event that happens with probability $1-\alpha_P$) or contacts a banker (with probability α_P) but the negotiation breaks down. This producer's individual decision problem in the first subperiod of t is to choose the quantity of the inventory of good 1 to sell in the goods market, $\tilde{y}_{Pt}(y_t, a_t^m)$, and post-trade money holding, $\tilde{a}_{Pt}^m(y_t, a_t^m)$, that satisfy

$$(\tilde{y}_{Pt}(y_t, a_t^m), \tilde{a}_{Pt}^m(y_t, a_t^m)) = \arg \max_{(\tilde{y}_t, \tilde{a}_t^m) \in \mathbb{R}_+^2} W_t^P(\tilde{a}_t^m, \tilde{a}_t^g), \tag{4}$$

$$\begin{aligned} \text{s.t. } \tilde{a}_t^m &\leq a_t^m + p_t \tilde{y}_t \\ \tilde{y}_t &\leq y_t, \end{aligned}$$

where $\tilde{a}_t^g = (y_t - \tilde{y}_t)\underline{\varphi}$. The first constraint is the budget constraint the producer faces in the first subperiod when only able to trade in the goods market. The second constraint says the producer can at most sell the inventory of good 1 produced at the beginning of the period.

Two bargaining situations arise in the first subperiod of a typical period t: the first, when a consumer contacts a banker, and the second, when a producer contacts a banker. Consider a consumer with beginning-of-period money holding a_t^m. With probability α_C, the consumer contacts a banker and simultaneously chooses the quantity of consumption to buy from the goods market, $\bar{y}_{Ct}(a_t^m)$, and bargains over the post-trade portfolio of bonds and money, $\bar{\boldsymbol{a}}_{Ct}(a_t^m) = (\bar{a}_{Ct}^m(a_t^m), \bar{a}_{Ct}^b(a_t^m))$, as well as the banker's fee, $k_{Ct}(a_t^m)$. The bargaining outcome, $(\bar{y}_{Ct}(a_t^m), \bar{\boldsymbol{a}}_{Ct}(a_t^m), k_{Ct}(a_t^m))$, is the solution to

$$\max_{(\bar{y}_t, \bar{\boldsymbol{a}}_t, k_t) \in \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{R}_+} \left[u(\bar{y}_t) + W_t^C(\bar{a}_t^m, \bar{a}_t^g) - u(\tilde{y}_t) - W_t^C(\tilde{a}_t^m, 0) \right]^{\theta_C} k_t^{1-\theta_C} \tag{5}$$

$$\begin{aligned} \text{s.t. } \bar{a}_t^m + p_t \bar{y}_t + q_t \bar{a}_t^b &\leq a_t^m \\ u(\tilde{y}_t) + W_t^C(\tilde{a}_t^m, 0) &\leq u(\bar{y}_t) + W_t^C(\bar{a}_t^m, \bar{a}_t^g), \end{aligned}$$

where $\bar{\boldsymbol{a}}_t = (\bar{a}_t^m, \bar{a}_t^b)$, $\bar{a}_t^g = \bar{a}_t^b - k_t$, $\tilde{y}_t = \tilde{y}_{Ct}(a_t^m)$, and $\tilde{a}_t^m = \tilde{a}_{Ct}^m(a_t^m)$. The first constraint is the budget constraint the consumer faces in the first subperiod when able to trade simultaneously in the goods market and the bond market. The second constraint ensures the trade is incentive compatible for the consumer (the restriction $k_t \in \mathbb{R}_+$ ensures the trade is also incentive compatible for the banker). If the consumer and the banker were unable to reach an agreement, the consumer can still trade in the goods market. Hence, the outcome (3) acts as the consumer's outside option in his bargaining problem with the banker.

Consider a producer with beginning-of-period money holding a_t^m and good 1 inventory y_t. With probability α_P, the producer contacts a banker and simultaneously chooses the quantity of good 1 to sell in the goods market, $\bar{y}_{Pt}(y_t, a_t^m)$, and bargains over the post-trade portfolio of bonds and money, $\bar{\boldsymbol{a}}_{Pt}(y_t, a_t^m) = (\bar{a}_{Pt}^m(y_t, a_t^m), \bar{a}_{Pt}^b(y_t, a_t^m))$, as well as the banker's fee, $k_{Pt}(y_t, a_t^m)$. The outcome, $(\bar{y}_{Pt}(y_t, a_t^m), \bar{\boldsymbol{a}}_{Pt}(y_t, a_t^m), k_{Pt}(y_t, a_t^m))$, is the solution to

$$\max_{(\bar{y}_t, \bar{\boldsymbol{a}}_t, k_t) \in \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{R}_+} \left[W_t^P(\bar{a}_t^m, \bar{a}_t^g) - W_t^P(\tilde{a}_t^m, \tilde{a}_t^g)\right]^{\theta_P} k_t^{1-\theta_P} \tag{6}$$

$$\begin{aligned} \text{s.t. } \bar{a}_t^m + q_t \bar{a}_t^b &\leq a_t^m + p_t \bar{y}_t \\ \bar{y}_t &\leq y_t \\ W_t^P(\tilde{a}_t^m, \tilde{a}_t^g) &\leq W_t^P(\bar{a}_t^m, \bar{a}_t^g), \end{aligned}$$

where $\bar{\boldsymbol{a}}_t = (\bar{a}_t^m, \bar{a}_t^b)$, $\bar{a}_t^g = \bar{a}_t^b - k_t + (y_t - \bar{y}_t)\underline{\varphi}$, $\tilde{a}_t^m = \tilde{a}_{Pt}^m(y_t, a_t^m)$, and $\tilde{a}_t^g = (y_t - \tilde{y}_{Pt}(y_t, a_t^m))\underline{\varphi}$. The first constraint is the budget constraint the producer faces in the first subperiod when able to trade simultaneously in the goods market and the bond market. The second constraint states that the producer can at most sell the inventory of good 1 produced at the beginning of the period. The third constraint ensures the trade is incentive compatible for the producer (the restriction $k_t \in \mathbb{R}_+$ ensures the trade is also incentive compatible for the banker). Notice that if the producer and the banker were unable to reach an agreement, the producer can still trade in the goods market. Hence, the outcome (4), which determines the gain from selling of a cash-only producer, acts as the cash-and-credit producer's outside option in his bargaining problem with the banker.

Let $V_t^i(a_t^m)$ denote maximum expected discounted payoff of an agent of type $i \in \{B, C, P\}$ who enters the first subperiod of period t with money holding a_t^m. For a banker,

$$\begin{aligned} V_t^B(a_t^m) &= \sum_{i \in \{C,P\}} \alpha_B^i \int W_t^B(\bar{a}_{Bt}^m(a_t^m), \bar{a}_{Bt}^b(a_t^m) + k_{it}(\tilde{a}_t^m)) dH_{it}(\tilde{a}_t^m) \\ &\quad + \left(1 - \alpha_B^C - \alpha_B^P\right) W_t^B(\bar{\boldsymbol{a}}_{Bt}(a_t^m)), \end{aligned} \tag{7}$$

where, for $i \in \{C, P\}$, α_B^i is the probability an individual banker contacts an agent of type i, and H_{it} is the beginning-of-period t cumulative distribution function of money holdings across agents of type i. For a consumer,

$$V_t^C(a_t^m) = \alpha_C[u(\bar{y}_{Ct}(a_t^m)) + W_t^C(\bar{a}_{Ct}^m(a_t^m), \bar{a}_{Ct}^b(a_t^m) - k_{Ct}(a_t^m))] + (1-\alpha_C)\left[u(\tilde{y}_{Ct}(a_t^m)) + W_t^C(\tilde{a}_{Ct}^m(a_t^m), 0)\right]. \quad (8)$$

For a producer,

$$V_t^P(a_t^m) = \max_{y_t \in \mathbb{R}_+}\Big\{ -\kappa y_t + \alpha_P W_t^P(\bar{a}_{Pt}^m(y_t, a_t^m), \bar{a}_{Pt}^g(y_t, a_t^m)) + (1-\alpha_P) W_t^P\left(\tilde{a}_{Pt}^m(y_t, a_t^m), \tilde{a}_{Pt}^g(y_t, a_t^m)\right)\Big\}, \quad (9)$$

where $\bar{a}_{Pt}^g(y_t, a_t^m) = \bar{a}_{Pt}^b(y_t, a_t^m) - k_{Pt}(y_t, a_t^m) + (y_t - \bar{y}_{Pt}(y_t, a_t^m))\underline{\varphi}$, and $\tilde{a}_{Pt}^g(y_t, a_t^m) = (y_t - \tilde{y}_{Pt}(y_t, a_t^m))\underline{\varphi}$. Let $y_{Pt}(a_t^m)$ denote the solution to the maximization in (9).

Let $A_{it}^m = N_i \int a_t^m dF_{it}(a_t^m)$, where F_{it} is the cumulative distribution function over money holdings a_t^m held by agents of type $i \in \{B, C, P\}$ at the beginning of period t. Let $\tilde{A}_{Ct}^m = (1-\alpha_C)N_C \int \tilde{a}_{Ct}^m(a_t^m)\, dF_{Ct}(a_t^m)$ and $\tilde{A}_{Pt}^m = (1-\alpha_P)N_P \int \tilde{a}_{Pt}^m(y_{Pt}(a_t^m), a_t^m)\, dF_{Pt}(a_t^m)$. For asset type $k \in \{m, b\}$, let $\bar{A}_{Bt}^k = N_B \int \bar{a}_{Bt}^k(a_t^m)\, dF_{Bt}(a_t^m)$, $\bar{A}_{Ct}^k = \alpha_C N_C \int \bar{a}_{Ct}^k(a_t^m)\, dF_{Ct}(a_t^m)$, and $\bar{A}_{Pt}^k = \alpha_P N_P \int \bar{a}_{Pt}^k(y_{Pt}(a_t^m), a_t^m)\, dF_{Pt}(a_t^m)$. Also, let $\bar{Y}_{Ct} = \alpha_C N_C \int \bar{y}_{Ct}(a_t^m)\, dF_{Ct}(a_t^m)$, $\tilde{Y}_{Ct} = (1-\alpha_C)N_C \int \tilde{y}_{Ct}(a_t^m)\, dF_{Ct}(a_t^m)$, $\bar{Y}_{Pt} = \alpha_P N_P \int \bar{y}_{Pt}(y_{Pt}(a_t^m), a_t^m)\, dF_{Pt}(a_t^m)$, and $\tilde{Y}_{Pt} = (1-\alpha_P)N_P \int \tilde{y}_{Pt}(y_{Pt}(a_t^m), a_t^m)\, dF_{Pt}(a_t^m)$. We are now ready to define equilibrium.

Definition 1. *An equilibrium is a sequence of prices, $\{p_t, q_t, \phi_t\}_{t=0}^{\infty}$, portfolio allocations and fees in the first subperiod, $\{\bar{\boldsymbol{a}}_{Bt}^k(\cdot), \bar{\boldsymbol{a}}_{it}^k(\cdot), \tilde{\boldsymbol{a}}_{it}^m(\cdot), y_{P_t}(\cdot), \bar{y}_{it}(\cdot), \tilde{y}_{it}(\cdot), k_{it}(\cdot)\}_{i\in\{C,P\}, k\in\{m,b\}, t\in\mathbb{T}}$, and end-of-period money holdings, $\{a_{it+1}^m\}_{i\in\{B,C,P\}, t\in\mathbb{T}}$, such that for all $t \in \mathbb{T}$: (i) taking prices and the bargaining protocol as given, the end-of-period money holdings solve (1) for $i \in \{B, C, P\}$; (ii) the asset holdings and fees in the first subperiod solve (2), (3), (4), (5), (6); (iii) beginning-of-period production $y_{P_t}(\cdot)$ satisfies (9); and (iv) prices are such that all Walrasian markets clear, i.e., $\sum_{i\in\{B,C,P\}} A_{it+1}^m = M_{t+1}$ (the end-of-period t Walrasian market for money clears), $\sum_{i\in\{B,C,P\}} \bar{A}_{it}^b = 0$ (the period t market for bonds clears), $\tilde{Y}_{Ct} + \bar{Y}_{Ct} = \tilde{Y}_{Pt} + \bar{Y}_{Pt}$ (the market for good 1 clears), and $\mathbb{I}_{\{\phi_t>0\}}\left[\sum_{i\in\{C,P\}}(\bar{A}_{it}^m + \tilde{A}_{it}^m) + \bar{A}_{Bt}^m - M_t\right] = 0$ (the first-subperiod money market clears). An equilibrium is "monetary" if $\phi_t > 0$ for all t and "non-monetary" otherwise.*

In what follows, we let $\varphi_t \equiv p_t/q_t$ be the relative price of good 1 in terms of bonds (i.e., claims to good 2) in the first subperiod of t.[3] In an economy where money is valued, it is useful

[3]When considering individual decision problems, we assume $0 < \varphi_t$ (this is without loss of generality, since it will be true in any equilibrium).

to define real money balances as $Z_t \equiv \phi_t M_t$, and to let $\tilde{\varphi}_t \equiv p_t \phi_t$. Intuitively, φ_t will be the relevant relative price of good 1 in terms of good 2 for a producer who sells good 1 in exchange for a debt instrument, as well as for a consumer who issues a debt instrument to pay for good 1 (in an economy with or without money). The price $\tilde{\varphi}_t$ will be the relevant relative price of good 1 in terms of good 2 for a producer who sells good 1 for money, as well as for a consumer who uses money to pay for good 1. In a monetary economy, let

$$\rho_t \equiv \frac{1}{q_t \phi_t} - 1 = \frac{\varphi_t - \tilde{\varphi}_t}{\tilde{\varphi}_t}, \tag{10}$$

which is the equilibrium nominal interest rate implied by the inside bond.[4] Hereafter we specialize the analysis to $0 \leq \rho_t$, since $\rho_t < 0$ entails an arbitrage opportunity inconsistent with equilibrium. For any $z \in \mathbb{R}$, define the correspondences $\varkappa : \mathbb{R} \rightrightarrows \mathbb{R}$ and $\zeta : \mathbb{R} \rightrightarrows [0,1]$ by[5]

$$\varkappa_{(z)} \begin{cases} = \infty & \text{if } z < 0 \\ \in [0, \infty] & \text{if } z = 0 \\ = 0 & \text{if } 0 < z \end{cases} \quad \text{and} \quad \zeta_{(z)} \begin{cases} = 1 & \text{if } 0 < z \\ \in [0, 1] & \text{if } z = 0 \\ = 0 & \text{if } z < 0. \end{cases}$$

Let $q_{t,k}^B$ denote the nominal price in the second subperiod of period t of a T-period risk-free pure discount nominal bond that matures in period $t+k$, for $k = 0, 1, 2, ..., T$ (so k is the number of periods until the bond matures). Imagine the bond is illiquid in the sense that it cannot be traded in the first subperiod. Then in a stationary monetary equilibrium, $q_{t,k}^B = (\beta/\mu)^k$, and

$$\iota = \frac{\mu - \beta}{\beta} \tag{11}$$

is the time-t nominal yield to maturity of the bond with k periods until maturity. Throughout we assume $\beta < \mu$ (but we consider the limiting case $\mu \to \beta$). Since there is a one-to-one mapping between the growth rate of the money supply, μ, and the nominal interest rate ι, we can regard ι as the nominal *policy rate* chosen by the monetary authority.

[4]To see why i_t is the nominal interest rate implicit in the inside bond, notice that with 1 unit of money an investor can buy $\frac{1}{q_t}$ bonds, which in total yield $\frac{1}{q_t}$ general goods in the following subperiod, and this is equivalent to $\frac{1}{q_t \phi_t}$ dollars. Since the bond is repaid within the period, this is also a notion of real rate on these loans, with loan and repayment measured in terms of the general good. To see this, notice that investing $\frac{1}{\phi_t}$ dollars is equivalent to investing 1 unit of the general good. The $\frac{1}{\phi_t}$ dollars allow to buy $\frac{1}{q_t \phi_t}$ bonds, which in total yield $\frac{1}{q_t \phi_t}$ general goods. So the gross real interest in terms of general goods is also $\frac{1}{q_t \phi_t}$.

[5]Below, we use the variants $\bar{\zeta}_{(z)}$ and $\tilde{\zeta}_{(z)}$ to denote correspondences with $\bar{\zeta}_{(z)} = \tilde{\zeta}_{(z)} = \zeta_{(z)}$ for all $z \neq 0$, but possibly $\bar{\zeta}_{(0)} \neq \tilde{\zeta}_{(0)} \neq \zeta_{(0)}$. Similarly, the variants $\{\varkappa_{it(z)}^m\}_{i \in \{B,C,P\}}$ and $\varkappa_{(z)}^p$, denote correspondences with $\varkappa_{it(z)}^m = \varkappa_{(z)}^p = \varkappa_{(z)}$ for all $z \neq 0$ and all $i \in \{B, C, P\}$ and $t \in \mathbb{T}$, but possibly $\varkappa_{it(0)}^m \neq \varkappa_{jt(0)}^m \neq \varkappa_{(0)}^p \neq \varkappa_{(0)}$ for some $t \in \mathbb{T}$ and $i, j \in \{B, C, P\}$ with $i \neq j$.

3. Nonmonetary economy

The following lemma characterizes the first-subperiod outcomes in an economy with no money.

Lemma 1. *Consider the first subperiod of period t of an economy with no money. (i) The solution to the banker's portfolio problem (i.e., (2)) is* $\bar{a}^b_{Bt} = 0$. *(ii) The trade of a consumer who does not contact a banker (i.e., (3)) is* $\tilde{y}_{Ct} = 0$. *The trade of a consumer who contacts a banker (i.e., the solution to (5)) is* $\bar{y}_{Ct} = \mathrm{D}(\varphi_t)$, $\bar{a}^b_{Ct} = -\varphi_t \mathrm{D}(\varphi_t)$, *and* $k_{Ct} = (1-\theta_C)\left[u(\mathrm{D}(\varphi_t)) - \varphi_t \mathrm{D}(\varphi_t)\right]$. *(iii) The post-production trade of a producer who carries inventory* y_t *and does not contact a banker (i.e., (4)) is* $\tilde{y}_{Pt}(y_t) = 0$. *The post-production trade of a producer who carries inventory* y_t *and contacts a banker (i.e., the solution to (6)) is* $\bar{y}_{Pt}(y_t) = \zeta_{(\varphi_t - \underline{\varphi})} y_t$, $\bar{a}^b_{Pt}(y_t) = \varphi_t \bar{y}_{Pt}(y_t)$, *and* $k_{Pt}(y_t) = (1-\theta_P)(\varphi_t - \underline{\varphi})\bar{y}_{Pt}(y_t)$. *(iv) A producer's pre-trade production is* $y_{Pt} = \varkappa_{(\kappa - R^n(\varphi_t))}$, *where*

$$R^n(\varphi_t) \equiv \underline{\varphi} + \alpha_P \theta_P (\varphi_t - \underline{\varphi}) \zeta_{(\varphi_t - \underline{\varphi})}. \tag{12}$$

To find an equilibrium of the nonmonetary economy, it suffices to find the equilibrium path of $\{\varphi_t\}_{t=0}^{\infty}$. Given this path, the rest of the equilibrium is immediate from Lemma 1. The following result offers a characterization of equilibrium based on this relative price.

Proposition 2. *Assume* $\varphi^n < u'(0)$, *where*

$$\varphi^n = \kappa + \left(\frac{1}{\alpha_P \theta_P} - 1\right)(\kappa - \underline{\varphi}). \tag{13}$$

There exists a unique equilibrium, $\{\varphi^n_t\}_{t=0}^{\infty}$, *of the nonmonetary economy, and* $\varphi^n_t = \varphi^n$ *for all* t. *The individual consumption allocation of good 1 (for consumers with access to bankers),* $\bar{y}^n_C = \mathrm{D}(\varphi^n)$, *satisfies*

$$u'(\bar{y}^n_C) = \varphi^n. \tag{14}$$

Consumers without access to bankers do not consume good 1. The individual production allocation of good 1 is $y^n_P = \frac{\alpha_C N_C}{\alpha_P N_P} \bar{y}^n_C$.

Notice that the equilibrium price satisfies $\underline{\varphi} < \kappa \leq \varphi^n$, and $\varphi^n = \kappa$ only if $\alpha_P \theta_P = 1$. Hence, consumption of good 1 is inefficiently low (i.e., $\bar{y}^n_C \leq y^*$) in the nonmonetary economy as long as either not all producers have access to the credit market ($\alpha_P < 1$), or bankers can exert market power over the producers they transact with ($\theta_P < 1$).[6] This inefficiency is due to the fact that a producer must produce good 1 before the moment when he simultaneously sells the good and negotiates with the banker. Given the constant-returns production technology, in an equilibrium with production of good 1, a producer must expect to break even. Specifically, when the producer makes the production decision at the beginning of the period, he anticipates the relative price of good 1 will be some $\varphi \geq \kappa > \underline{\varphi}$, which implies an expected profit equal to $\Pi^n(\varphi) \equiv R^n(\varphi) - \kappa = \alpha_P\theta_P(\varphi - \kappa) + (1 - \alpha_P\theta_P)(\underline{\varphi} - \kappa)$ per unit of good 1 produced. To interpret this beginning-of-period pre-production expected profit, notice that if having produced, the producer contacts a banker and has all the market power, then the per-unit profit is $\varphi - \kappa$, but if the producer either cannot sell in the first subperiod (e.g., because he fails to contact a banker, which happens with probability $1 - \alpha_P$) or if he contacts a banker who has all the market power (i.e., with probability $\alpha_P(1 - \theta_P)$) then the per unit profit is $\underline{\varphi} - \kappa < 0$. Hence, as long as $\alpha_P\theta_P < 1$, the zero-profit equilibrium condition $\Pi^n(\varphi) = 0$ requires $0 < \varphi - \kappa$, which means an inefficient level of consumption and production of good 1. In the general equilibrium, the market power that producers face in the financial market induces them to charge a mark-up for good 1 (i.e., $\theta_P < 1$ implies $\varphi^n - \kappa > 0$, and $\varphi^n - \kappa$ is decreasing in θ_P), even though individual producers have no market power in the market for good 1 (they are competitive price takers in that market).

4. Monetary equilibrium

The following lemma characterizes the first-subperiod outcomes in a monetary economy.

Lemma 2. *Consider the first subperiod of period t of an economy with money. In each case, focus on an agent who enters the period with money holding a_t^m. (i) The solution to the banker's portfolio problem, (i.e., (2)), is $q_t\bar{a}_{Bt}^b(a_t^m) = a_t^m - \bar{a}_{Bt}^m(a_t^m)$ and $\bar{a}_{Bt}^m(a_t^m) = \varkappa_{Bt(\rho_t)}^m$. (ii) The trade of a consumer who does not contact a banker (i.e., (3)) is $p_t\tilde{y}_{Ct}(a_t^m) = \min[p_t\mathrm{D}(\tilde{\varphi}_t), a_t^m]$ with $\tilde{a}_{Ct}^m(a_t^m) = a_t^m - p_t\tilde{y}_{Ct}(a_t^m)$. The trade of a consumer who contacts a banker (i.e., the solution to (5)) is $\bar{y}_{Ct}(a_t^m) = \mathrm{D}(\varphi_t)$, $\bar{a}_{Ct}^m(a_t^m) = \varkappa_{Ct(\rho_t)}^m$, $q_t\bar{a}_{Ct}^b(a_t^m) = a_t^m - [\bar{a}_{Ct}^m(a_t^m) + p_t\bar{y}_{Ct}(a_t^m)]$, and*

$$k_{Ct}(a_t^m) = (1 - \theta_C)\left\{\rho_t\phi_t a_t^m + u(\bar{y}_{Ct}(a_t^m)) - \varphi_t\bar{y}_{Ct}(a_t^m) - [u(\tilde{y}_{Ct}(a_t^m)) - \tilde{\varphi}_t\tilde{y}_{Ct}(a_t^m)]\right\}.$$

[6] The knife-edge case $\kappa = \underline{\varphi}$, which we rule out in our baseline parametrization, is one where producers' gains from trade in the market for good 1 are always equal to zero, so there is never a gain from trade in a meeting between a producer and a banker, which effectively is as if the banker had no market power.

(iii) The post-production trade of a producer who carries inventory y_t and does not contact a banker (i.e., (4)) is $\tilde{y}_{Pt}(y_t, a_t^m) = \tilde{\zeta}_{(\tilde{\varphi}_t - \underline{\varphi})} y_t$ with $\tilde{a}_{Pt}^m(y_t, a_t^m) = a_t^m + p_t \tilde{y}_{Pt}(y_t, a_t^m)$. The post-production trade of a producer who carries inventory y_t and contacts a banker (i.e., the solution to (6)) is $\bar{y}_{Pt}(y_t, a_t^m) = \bar{\zeta}_{(\varphi_t - \underline{\varphi})} y_t$, $\bar{a}_{Pt}^m(y_t, a_t^m) = \varkappa_{Pt(\rho_t)}^m$, $q_t \bar{a}_{Pt}^b(y_t, a_t^m) = a_t^m + p_t \bar{y}_{Pt}(y_t, a_t^m) - \bar{a}_{Pt}^m(y_t, a_t^m)$, and

$$k_{Pt}(y_t, a_t^m) = (1 - \theta_P) \left\{ \rho_t \phi_t a_t^m + [(\varphi_t - \underline{\varphi}) \mathbb{I}_{\{\underline{\varphi} < \varphi_t\}} - (\tilde{\varphi}_t - \underline{\varphi}) \mathbb{I}_{\{\underline{\varphi} < \tilde{\varphi}_t\}}] y_t \right\}.$$

(iv) A producer's pre-trade production is $y_{Pt}(a_t^m) = \varkappa^p_{(\kappa - R^m(\tilde{\varphi}_t, \varphi_t))}$, where

$$R^m(\tilde{\varphi}_t, \varphi_t) \equiv \underline{\varphi} + \alpha_P \theta_P (\varphi_t - \underline{\varphi}) \mathbb{I}_{\{\underline{\varphi} < \varphi_t\}} + (1 - \alpha_P \theta_P)(\tilde{\varphi}_t - \underline{\varphi}) \mathbb{I}_{\{\underline{\varphi} < \tilde{\varphi}_t\}}. \tag{15}$$

To characterize a monetary equilibrium it suffices to find $\{Z_t, \varphi_t, \tilde{\varphi}_t\}_{t=0}^{\infty}$. Given this path, the nominal prices $1/\phi_t = M_t/Z_t$, $p_t = \tilde{\varphi}_t/\phi_t$, $q_t = p_t/\varphi_t$, $\rho_t = (\varphi_t - \tilde{\varphi}_t)/\tilde{\varphi}_t$, and the rest of the equilibrium are immediate from Lemma 2. The full set of dynamic equilibrium conditions is reported in Lemma 6 in the appendix. The following result characterizes the stationary monetary equilibrium, i.e., a path $\{Z_t, \varphi_t, \tilde{\varphi}_t\}_{t=0}^{\infty}$ such that $Z_t = Z$, $\varphi_t = \varphi$, and $\tilde{\varphi}_t = \tilde{\varphi}$ for all t. Without loss of generality, we focus on economies where good 1 is produced. Also, to simplify and sharpen the exposition, hereafter we specialize the analysis to the case with $\alpha_C \theta_C = 1$ (i.e., all consumers have access to the credit market and can borrow at the competitive rate).[7] We can use part (*ii*) of Lemma 2 to define money velocity as $\mathcal{V}_t \equiv \frac{p_t \mathrm{D}(\varphi_t) N_C}{M_t}$.

4.1. Stationary monetary equilibrium

Proposition 3. *Assume $\varphi^n < u'(0)$, and let*

$$\bar{\iota} \equiv \frac{1}{\alpha_P \theta_P} \frac{\kappa - \underline{\varphi}}{\underline{\varphi}}. \tag{16}$$

There exists a unique stationary monetary equilibrium provided $0 \leq \iota < \bar{\iota}$. In the stationary monetary equilibrium, $Z_t = Z$, $\varphi_t = \varphi$, $\tilde{\varphi}_t = \tilde{\varphi}$, $\rho_t = \rho$ for all t, and $\phi_t = \frac{Z}{M_t}$, $p_t = \frac{\tilde{\varphi}}{Z} M_t$ and $q_t = \frac{\tilde{\varphi}}{\varphi} \frac{M_t}{Z}$. Moreover,

[7] The results for the more general formulation with $\alpha_C \theta_C \leq 1$ are presented in the appendix. Specifically, Proposition 6, Corollary 6, and Corollary 7 are the generalized versions of Proposition 3, Corollary 2, and Corollary 3, respectively. In terms of Proposition 2, notice that θ_C plays no role, and that there is no loss of generality in setting $\alpha_C = 1$.

(i) If $0 < \iota < \bar{\iota}$, then

$$\begin{aligned}
\tilde{\varphi} &= \frac{1}{1+\alpha_P\theta_P\iota}\kappa \\
\varphi &= \frac{1+\iota}{1+\alpha_P\theta_P\iota}\kappa \\
\rho &= \iota \\
\frac{Z}{\tilde{\varphi}} &= (1-\alpha_P)\,\mathrm{D}(\varphi)\,N_C \\
\mathcal{V} &= \frac{1}{1-\alpha_P},
\end{aligned}$$

and the individual consumption allocation of good 1, $\bar{y}_C = \mathrm{D}(\varphi)$, satisfies

$$u'(\bar{y}_C) = \varphi.$$

The individual production allocation of good 1 is $y_P = \frac{N_C}{\alpha_P N_P}\bar{y}_C$.

(ii) As $\iota \to 0$, $\tilde{\varphi} \to \kappa$, $\varphi \to \kappa$, and any $Z \in [(1-\alpha_P)\,\kappa\mathrm{D}(\kappa)\,N_C, \infty)$ is consistent with equilibrium.

(iii) As $\iota \to \bar{\iota}$, $\tilde{\varphi} \to \underline{\varphi}$, and $\varphi \to \varphi^n$.

Notice that the highest policy rate consistent with monetary equilibrium, $\bar{\iota}$, can be written as $\bar{\iota} = \frac{\varphi^n - \underline{\varphi}}{\underline{\varphi}}$, which can be interpreted as the market interest on the inside bond in a nonmonetary economy. In the monetary equilibrium, $\underline{\varphi} < \tilde{\varphi} \leq \kappa \leq \varphi$, where the first "$\leq$" holds with "$=$" if $\alpha_P\theta_P = 1$, and both "$\leq$" hold with "$=$" if $\iota = 0$. Hence, as long as $\alpha_P\theta_P < 1$ and $0 < \iota$, we have $\bar{y}_C < y^*$, i.e., consumption of good 1 is inefficiently low in the monetary equilibrium. This inefficiency is due to the fact that a producer must produce good 1 before the moment when he simultaneously sells it and negotiates with the banker. Given the constant-returns production technology, in an equilibrium with production of good 1, a producer of good 1 must expect to break even. Specifically, when the producer makes the production decision at the beginning of the period, he anticipates the relative price of good 1 (in terms of good 2) will be $\tilde{\varphi} > \underline{\varphi}$ if he sells the good for cash, and $\varphi \geq \kappa \geq \tilde{\varphi}$ if he manages to sell it for bonds, which implies an expected profit equal to

$$\Pi^m(\tilde{\varphi}, \varphi) \equiv R^m(\tilde{\varphi}, \varphi) - \kappa = (1-\alpha_P\theta_P)(\tilde{\varphi} - \kappa) + \alpha_P\theta_P(\varphi - \kappa)$$

per unit of good 1 produced. To interpret this beginning-of-period pre-production expected profit, notice that if having produced, the producer contacts a banker and has all the market power, then the per-unit profit is $\varphi - \kappa$, but if the producer either must sell for cash in the first subperiod (e.g., because he fails to contact a banker, which happens with probability $1 - \alpha_P$), or if he contacts a banker who has all the market power (i.e., with probability $\alpha_P(1 - \theta_P)$) then the per unit profit is $\tilde{\varphi} - \kappa$. As long as $0 < \iota$, we have $\tilde{\varphi} - \kappa < 0$, so the zero-profit equilibrium condition $\Pi^m(\tilde{\varphi}, \varphi) = 0$ requires $0 < \varphi - \kappa$, which means an inefficient level of consumption and production of good 1. In the general equilibrium, the market power that producers face in the financial market induces them to charge a mark-up in the market for good 1, even though individual producers are competitive price takers in that market. In terms of comparative statics, as long as $\alpha_P\theta_P \in (0,1)$, we have $\frac{\partial \tilde{\varphi}}{\partial \iota} < 0 < \frac{\partial \varphi}{\partial \iota}$. The equilibrium approaches the equilibrium of the nonmonetary economy as $\iota \to \bar{\iota}$. The equilibrium consumption allocation converges to the efficient allocation as $\iota \to 0$, or as $\alpha_P\theta_P \to 1$. In the latter case, real balances, Z, and inverse velocity, $1/\mathcal{V}$, approach zero.

4.2. Dynamic monetary equilibrium

In this section we characterize dynamic monetary equilibria for the economy with $\alpha_C\theta_C = 1$ (where $A^m_{Ct} = M_t$ for all t). To streamline the analysis, we consider a version of the model where: (*i*) $N_B = N_C = N_P = 1$ (a normalization); (*ii*) good 1 is produced in equilibrium; and (*iii*) $\underline{\varphi} = 0$ (no storage). These conditions imply $\varkappa^p_{(\kappa - R^m(\tilde{\varphi}_t, \varphi_t))} = \bar{\zeta}_{(\varphi_t - \underline{\varphi})} = \tilde{\zeta}_{(\tilde{\varphi}_t - \underline{\varphi})} = 1$ for all t, and $\varphi^n = \frac{\kappa}{\alpha_P\theta_P}$. It is convenient to define $z_t \equiv \frac{Z_t}{1-\alpha_P}$, i.e., the beginning-of-period t quantity of real money balances outstanding per producer with no access to a banker. The following result offers a characterization of the set of dynamic monetary equilibria.

Proposition 4. *Assume $\varphi^n < u'(0)$, where $\varphi^n = \frac{\kappa}{\alpha_P\theta_P}$. A dynamic monetary equilibrium is a bounded sequence $\{z_t, \varphi_t, \tilde{\varphi}_t, \rho_t\}_{t=0}^{\infty}$ that satisfies*

$$z_t = \begin{cases} \frac{1}{1+\iota} z_{t+1} & \text{if } \kappa \mathrm{D}(\kappa) \leq z_{t+1} \\ \frac{1}{1+\iota} \frac{1-\alpha_P\theta_P}{\alpha_P\theta_P} \frac{f(z_{t+1})}{\varphi^n - f(z_{t+1})} z_{t+1} & \text{if } 0 \leq z_{t+1} < \kappa \mathrm{D}(\kappa) \end{cases} \tag{17}$$

$$\varphi_t = \begin{cases} \kappa & \text{if } \kappa \mathrm{D}(\kappa) \leq z_t \\ f(z_t) & \text{if } 0 \leq z_t < \kappa \mathrm{D}(\kappa) \end{cases} \tag{18}$$

$$\tilde{\varphi}_t = \frac{\alpha_P\theta_P}{1-\alpha_P\theta_P}(\varphi^n - \varphi_t) \tag{19}$$

$$\rho_t = \frac{1}{\alpha_P\theta_P} \frac{\varphi_t - \kappa}{\varphi^n - \varphi_t}, \tag{20}$$

where for any $z \in [0, \kappa \mathrm{D}(\kappa)]$, $f(z)$ *denotes the unique value* $\varphi \in [\kappa, \varphi^n]$ *that satisfies*

$$z = \frac{\alpha_P \theta_P}{1 - \alpha_P \theta_P} (\varphi^n - \varphi) \mathrm{D}(\varphi). \tag{21}$$

Proposition 4 reduces the task of finding dynamic monetary equilibria to finding a bounded solution $\{z_t\}_{t=0}^{\infty}$ to the difference equation (17). The equilibrium prices $\{\varphi_t, \tilde{\varphi}_t\}_{t=0}^{\infty}$ and the interest rate $\{\rho_t\}_{t=0}^{\infty}$ are then obtained from (18), (19), and (20).

Corollary 1. *In any dynamic monetary equilibrium,* $\mathrm{D}(\varphi^n) - \mathrm{D}(\varphi_t) < 0 < \varphi^n - \varphi_t$ *for all* t.

Corollary 1 of Proposition 4 establishes that in any dynamic monetary equilibrium, consumers face a relative price of good 1 (in terms of good 2) that is lower than the relative price they would face in the equilibrium of the same economy without money. Thus, consumption of good 1 (and therefore welfare) is higher in the economy with money than in the one without.

4.3. Sunspot equilibria

In this section we construct equilibria where prices and allocations are time-invariant functions of a *sunspot*, i.e., a random variable on which agents may coordinate actions but that does not directly affect any primitives, including endowments, preferences, and production or trading possibilities. To simplify the exposition, we maintain the assumption $\underline{\varphi} = 0$ and $N_B = N_C = N_P = \alpha_C \theta_C = 1$ (without loss, we focus on equilibria where only consumers hold money between periods). In the appendix (Corollary 9), we provide the equilibrium conditions for a set of sunspot states $\mathbb{S} = \{s_1, ..., s_N\}$, where $s_t \in \mathbb{S}$ follows a Markov chain with $\eta_{ij} = \Pr(s_{t+1} = s_i | s_t = s_j)$. In this context we describe equilibrium with time-invariant functions of the sunspot state, i.e., for any $s_t \in \mathbb{S}$ we use $\tilde{\varphi}(s_t)$, $\varphi(s_t)$, $\rho(s_t)$, and $Z(s_t)$, to denote $\tilde{\varphi}_t$, φ_t, ρ_t, and Z_t, respectively. The following result characterizes a family of sunspot equilibria that contains the nonmonetary equilibrium of Proposition 2 and the monetary equilibrium of Proposition 3.

Proposition 5. *Assume* $\mathbb{S} = \{s_1, s_2\}$, *with* $\eta_{11} \equiv \eta \in [0, 1]$, *and* $\eta_{22} = 1$. *For any arbitrary*

$\eta \in [0,1]$, *there exists a sunspot equilibrium characterized by*

$$\begin{aligned}
\tilde{\varphi}(s_1) &= \frac{\eta}{1+\alpha_P\theta_P\iota-(1-\eta)(1-\alpha_P\theta_P)}\kappa \\
\varphi(s_1) &= \frac{1+\iota}{1+\alpha_P\theta_P\iota-(1-\eta)(1-\alpha_P\theta_P)}\kappa \\
\rho(s_1) &= \frac{\iota+1-\eta}{\eta} \\
Z(s_1) &= (1-\alpha_P)\,\mathrm{D}(\varphi(s_1))\,\tilde{\varphi}(s_1),
\end{aligned}$$

$\tilde{\varphi}(s_2) = Z(s_2) = 0$, *and* $\varphi(s_2) = \rho(s_2)\,\tilde{\varphi}(s_2) = \varphi^n \equiv \frac{1}{\alpha_P\theta_P}\kappa$.

For $\eta = 0$, the equilibrium described in Proposition 5 reduces to the nonmonetary equilibrium of Proposition 2. Conversely, for $\eta = 1$, it reduces to the monetary equilibrium of Proposition 3. By varying η from 0 to 1, we can generate a continuum of proper sunspot equilibria that "convexify" the equilibrium set spanned by the monetary and the nonmonetary equilibrium.

5. Cashless limit

In this section we consider the limit of the equilibrium as $\alpha_P \to 1$, i.e., as the fraction of producers without access to bankers vanishes.

The following corollary of Proposition 2 characterizes the limit of the equilibrium of the nonmonetary economy as $\alpha_P \to 1$.

Corollary 2. *Assume* $\varphi^{n\star} < u'(0)$, *where*

$$\varphi^{n\star} \equiv \lim_{\alpha_P\to 1} \varphi^n = \kappa + \left(\frac{1}{\theta_P} - 1\right)(\kappa - \underline{\varphi}). \tag{22}$$

The individual consumption allocation of good 1, $\bar{y}_C^{n\star} = \mathrm{D}(\varphi^{n\star})$, *satisfies*

$$u'(\bar{y}_C^{n\star}) = \varphi^{n\star}. \tag{23}$$

The individual production allocation of good 1 is $y_P^{n\star} = \frac{N_C}{N_P}\bar{y}_C^{n\star}$.

The following corollary of Proposition 3 characterizes the limit of the stationary monetary equilibrium as $\alpha_P \to 1$. Let $\bar{\iota}^\star \equiv \lim_{\alpha_P\to 1}\bar{\iota} = \frac{1}{\theta_P}\frac{\kappa-\underline{\varphi}}{\underline{\varphi}}$ denote the limit as the cash-and-credit economy converges to the pure-credit economy of the maximum nominal policy rate consistent with existence of a stationary monetary equilibrium.

Corollary 3. *Consider the monetary equilibrium characterized in Proposition 3, with* $0 \leq \iota < \bar{\iota}^{\star}$. *As* $\alpha_P \to 1$,

$$\begin{aligned} \tilde{\varphi} &\to \tilde{\varphi}^{\star} \equiv \frac{\kappa}{1+\theta_P \iota} \\ \varphi &\to \varphi^{\star} \equiv \frac{1+\iota}{1+\theta_P \iota}\kappa \\ \rho &= \iota \\ \frac{Z}{\tilde{\varphi}} &\to 0 \\ \mathcal{V} &\to \infty \end{aligned} \tag{24}$$

$$\bar{y}_C \to \bar{y}_C^{\star}, \text{ where } \bar{y}_C^{\star} \text{ satisfies } u'\left(\bar{y}_C^{\star}\right) = \varphi^{\star} \tag{25}$$

$$y_P \to \frac{N_C}{N_P}\bar{y}_C^{\star}. \tag{26}$$

From (24), (25) and (26), notice that if either $\iota = 0$ or $\theta_P = 1$, then as $\alpha_P \to 1$, the consumption allocation implemented by the monetary equilibrium converges to the Pareto optimal allocation. From (24), we also see that as long as $\theta_P < 1$, the monetary policy ι affects the relative price $\varphi^{\star}$, which according to (25) and (26), in turn affects consumption and output—even as $Z \to 0$ along the cashless limit. In terms of comparative statics, as long as $\theta_P < 1$, we have $\frac{\partial \tilde{\varphi}^{\star}}{\partial \iota} < 0 < \frac{\partial \varphi^{\star}}{\partial \iota}$ in the cashless limit. From (22) and (24),

$$\varphi^{n\star} - \varphi^{\star} = \frac{1-\theta_P}{\theta_P}\left(\frac{1}{1+\theta_P \iota}\kappa - \underline{\varphi}\right).$$

Notice that $\varphi^{n\star} - \varphi^{\star} \geq 0$ for all $\theta_P \in [0,1]$ and all $\iota \in [0, \bar{\iota}^{\star})$, with "=" only if $\theta_P = 1$, so consumption is larger in the cashless limit of the monetary economy than in the nonmonetary economy. In other words, the allocation implemented by the cashless limit of the monetary equilibrium coincides with the allocation implemented by the equilibrium of the nonmonetary economy only if bankers have no market power over producers (i.e., $\theta_P = 1$). A monetary policy that makes money more valuable only makes $\varphi^{n\star} - \varphi^{\star}$ larger, since it improves the producer's outside option of trading good 1 for money, which reduces the banker's effective market power.[8] A key insight to understand this result is that for all $\iota \in [0, \bar{\iota}^{\star})$,

$$\lim_{\alpha_P \to 1} Z/\tilde{\varphi} = 0 < \lim_{\alpha_P \to 1} \frac{Z/\tilde{\varphi}}{1-\alpha_P} = \mathrm{D}\left(\varphi\right) N_C.$$

[8] In contrast, $\lim_{\iota \to \bar{\iota}^{\star}}\left(\varphi^{n\star} - \varphi^{\star}\right) = 0$ even if $\theta_P < 1$. That is, $\bar{y}_C^{\star} - \bar{y}_C^{n\star}$ can be made arbitrarily small by choosing a background monetary policy rate ι high enough to make the value of money sufficiently small. Intuitively, if expected inflation is very high, monetary exchange ceases to be an effective outside option for producers in their negotiations with banks.

That is, aggregate demand for money in the first subperiod converges to zero, but the *individual demand for money* does not, in the sense that any individual producer who belongs to the vanishing population of producers without access to a banker is willing to accept money in exchange for good 1. Hence, when trading with a banker, the producer's threat to sell for cash is credible everywhere along the cashless limit.

The following corollary of Proposition 4 describes the cashless limit (as $\alpha_P \to 1$) of the dynamical system that characterizes any dynamic monetary equilibrium path.

Corollary 4. *Assume* $\varphi^{n\star} < u'(0)$*, where* $\varphi^{n\star} = \frac{\kappa}{\theta_P}$*. Let* $\{z_t, \varphi_t, \tilde{\varphi}_t, \rho_t\}_{t=0}^{\infty}$ *be a dynamic monetary equilibrium. (i) As* $\alpha_P \to 1$*,* $\varphi^n \to \varphi^{n\star}$ *and* $\{z_t, \varphi_t, \tilde{\varphi}_t, \rho_t\}_{t=0}^{\infty} \to \{z_t^\star, \varphi_t^\star, \tilde{\varphi}_t^\star, \rho_t^\star\}_{t=0}^{\infty}$*, where*

$$z_t^\star = \begin{cases} \frac{1}{1+\iota} z_{t+1}^\star & \text{if } \kappa \mathrm{D}(\kappa) \leq z_{t+1}^\star \\ \frac{1}{1+\iota} \frac{1-\theta_P}{\theta_P} \frac{g\left(z_{t+1}^\star\right)}{\varphi^{n\star} - g\left(z_{t+1}^\star\right)} z_{t+1}^\star & \text{if } 0 \leq z_{t+1}^\star < \kappa \mathrm{D}(\kappa) \end{cases} \tag{27}$$

$$\varphi_t^\star = \begin{cases} \kappa & \text{if } \kappa \mathrm{D}(\kappa) \leq z_t^\star \\ g(z_t^\star) & \text{if } 0 \leq z_t^\star < \kappa \mathrm{D}(\kappa) \end{cases} \tag{28}$$

$$\tilde{\varphi}_t^\star = \frac{\theta_P}{1-\theta_P} \left(\varphi^{n\star} - \varphi_t^\star\right) \tag{29}$$

$$\rho_t^\star = \frac{1}{\theta_P} \frac{\varphi_t^\star - \kappa}{\varphi^{n\star} - \varphi_t^\star}, \tag{30}$$

where for any $z \in [0, \kappa \mathrm{D}(\kappa)]$*,* $g(z)$ *is the unique* $\varphi \in [\kappa, \varphi^{n\star}]$ *that solves*

$$z = \frac{\theta_P}{1-\theta_P} \left(\varphi^{n\star} - \varphi\right) \mathrm{D}(\varphi). \tag{31}$$

(ii) As long as $\theta_P < 1$*,* $\mathrm{D}(\varphi^{n\star}) - \mathrm{D}(\varphi_t^\star) < 0 < \varphi^{n\star} - \varphi_t^\star$ *for all* t*.*

Part (i) of Corollary 4 describes the set of conditions that characterize the "cashless limiting path" to which the path corresponding to any given dynamic monetary equilibrium converges as $\alpha_P \to 1$. Part (ii) establishes a key result that generalizes the main result in Corollary 3: As long as bankers have market power against producers, i.e., $\theta_P < 1$, in the cashless limit of any dynamic monetary equilibrium, consumers face a relative price of good 1 (in terms of good 2) that is lower than the relative price they would face in the equilibrium of the same economy without money. Thus, consumption of good 1, and therefore welfare, is always strictly higher in in the cashless limit of the monetary equilibrium of economy with money (i.e., as $\alpha_P \to 1$ and aggregate real money balances converge to zero, $Z_t \to 0$) than in the economy without money.

For every $\alpha_P \in [0,1]$, the set of equilibria (indexed by the sunspot probability η) described in Proposition 5 define an equilibrium correspondence that is continuous. The following corollary of Proposition 5 characterizes the limit of the this equilibrium correspondence as $\alpha_P \to 1$.

Corollary 5. *Consider the set of monetary equilibria indexed by $\eta \in [0,1]$ characterized in Proposition 5. For any given $\eta \in [0,1]$, as $\alpha_P \to 1$,*

$$\begin{aligned}
\tilde{\varphi}(s_1) &\to \tilde{\varphi}^{\star}(\eta) \equiv \frac{\eta}{1+\theta_P \iota - (1-\eta)(1-\theta_P)}\kappa \\
\varphi(s_1) &\to \varphi^{\star}(\eta) \equiv \frac{1+\iota}{1+\theta_P \iota - (1-\eta)(1-\theta_P)}\kappa \\
Z(s_1) &\to 0 \\
\varphi(s_2) &\to \varphi^{n\star} \equiv \frac{1}{\theta_P}\kappa.
\end{aligned}$$

This corollary formalizes the intuition that by carefully selecting among the set of (sunspot) equilibria, there is a sense in which one can construct a monetary equilibrium whose cashless limit converges to the nonmonetary equilibrium. The nature of the selection, however, involves decreasing the probability η toward zero as α_P approaches 1, i.e., intuitively, agent's expectations that money will lose its value forever (purely due to self-fulfilling expectations) must converge to 1 along with α_P. More formally, one could focus on the particular joint limit on credit *and beliefs*, $\alpha_P(1-\eta) \to 1$, and in this case, even if $\theta < 1$, one would indeed find $\lim_{\alpha_P(1-\eta)\to 1} \varphi(s_1) = \varphi^{n\star}$. It is our view that this kind of approximation result based on an arbitrary equilibrium selection from a large set of equilibria is too frail to offer a compelling basis for a moneyless approach to monetary economics.

6. Discussion

The basic design of our model builds on Lagos and Wright (2005). The particular market structure is similar to the one we have used in Lagos and Zhang (2015, 2019a,b), which in turn adopts some elements from Duffie et al. (2005). In Lagos and Zhang (2019a) we study the effects of monetary policy in the cashless limit of an economy where investors can settle equity trades using money or margin loans that are intermediated by brokers with market power. That model is calibrated to match the empirical estimates of the asset price responses to monetary policy shocks, and used to obtain quantitative theoretical estimates of these responses in the cashless limit. A key difference with Lagos and Zhang (2019a) is that here, monetary exchange is between buyers and sellers of a consumption good as in canonical monetary models (e.g., Samuelson (1958), Lucas (1980), or Lagos and Wright (2005)). In contrast with canonical monetary models, which emphasize the usefulness of money for buyers with limited access to credit, the baseline formulation of the model we develop here has buyers with unlimited access to credit, and therefore highlights the usefulness of monetary exchange for sellers who need a means to collect payment from buyers they do not trust. In this context, money is essential only to sellers with no access to the intermediated credit-based settlement.

Our main theoretical insight is that if the financial intermediaries who offer the credit-based settlement have market power, then even sellers with access to credit who neither hold, wish to hold, or choose to hold money on the equilibrium path, benefit from having the option to use money to settle sales—even if they never exercise it. The value of this option is reflected in equilibrium prices and allocations even as the measure of sellers with no access to credit vanishes along the trajectory toward a cashless pure-credit economy. As a result, as aggregate real money balances become negligible and the transaction velocity of money becomes arbitrarily large along the cashless limit, the *latent medium-of-exchange channel* of monetary transmission that operates through the opportunity cost of holding monetary assets remains operative, and determines the relative price and the quantities produced of "cash goods" and "credit goods"—even in the cashless limit. It would therefore be incorrect to infer that money and medium-of-exchange considerations cannot matter quantitatively simply based on the observation that monetary transactions account for a small share of total transactions.

For $\theta_P < 1$, our theory provides counterexamples to the claims used to endorse the moneyless approach to monetary economics. For example, Woodford (2003, p. 32) claims that the basic model in his book "abstracts from monetary frictions, in order to focus attention on more essential aspects of the monetary transmission mechanism...". Galí (2008, p. 10) claims that "...there is generally no need to specify a money demand function, unless monetary policy itself is specified in terms of a monetary aggregate, in which case a simple log-linear money demand schedule is postulated." We have shown that unless financial intermediation is perfectly competitive, disregarding medium-of-exchange considerations is not without loss—even in the cashless limit or in near-cashless economies in which liquidity-saving mechanisms have developed sufficiently to make the inverse velocity of some monetary aggregate very small. Any attempt to assess the macroeconomic effects of monetary policy that ignores these considerations is necessarily incomplete.

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A. Planner's problem

Proof of Proposition 1. The planner's problem is to choose a nonnegative sequence

$$\left\{y_{Ct}, y_{Pt}, (x_{it}, h_{it})_{i\in\{B,C,P\}}\right\}_{t=0}^{\infty}$$

that maximizes

$$\sum_{t=0}^{\infty}\beta^t\left\{u\left(y_{Ct}\right)N_C-\kappa y_{Pt}N_P+\sum_{i\in\{B,C,P\}}\left[v(x_{it})-h_{it}\right]N_i\right\}$$
$$\text{s.t. } y_{Ct}N_C=y_{Pt}N_P \text{ and } \sum_{i\in\{B,C,P\}}\left(x_{it}-h_{it}\right)N_i=0.$$

The first-order necessary and sufficient conditions for optimization are $u'\left(y_{Ct}\right)=\kappa$ and $v'\left(x_{it}\right)=1$, so the planner's solution is $y_{Ct}=y^*$, $y^*_{Pt}=\frac{N_C}{N_P}y^*$, and $x_{it}=h_{it}=x^*$ for all $i\in\{B,C,P\}$ and all t. ■

The following remark will be useful in the characterization of equilibrium.

Remark 1. *For $i\in\{B,C,P\}$, the second-subperiod value functions can be written as*

$$W_t^i(a_t^m,a_t^g)=\phi_t a_t^m+a_t^g+\bar{W}_t^i, \tag{32}$$

$$\bar{W}_t^i\equiv\mathbb{I}_{\{i=C\}}\phi_t T_t^m+v\left(x^*\right)-x^*+\max_{a_{t+1}^m\in\mathbb{R}_+}\left[\beta V_{t+1}^i\left(a_{t+1}^m\right)-\phi_t a_{t+1}^m\right] \tag{33}$$

B. Nonmonetary economy

Proof of Lemma 1. Consider a nonmonetary economy, i.e., $M_t=0$ for all t. With a slight abuse, we keep the notation for the value functions of the monetary economy, but simply omit an agent's money holding as an argument in the relevant functions. For example, (32) becomes

$$W_t^i(a_t^g)=a_t^g+\bar{W}_t^i, \tag{34}$$

where $\bar{W}_t^i\equiv v\left(x^*\right)-x^*+\beta V_{t+1}^i$. (i) Problem (2) becomes

$$\hat{W}_t^B(a_t^g)=\max_{\bar{a}_t^b\in\mathbb{R}}W_t^B(a_t^g+\bar{a}_t^b)\text{ s.t. }\frac{\bar{a}_t^b}{\varphi_t}\leq 0.$$

With (34), we have $\bar{a}^b_{Bt} = \arg\max_{\bar{a}^b_t \in \mathbb{R}} \bar{a}^b_t$ s.t. $\bar{a}^b_t/\varphi_t \leq 0$, which given $\varphi_t \in \mathbb{R}_{++}$, implies $\bar{a}^b_{Bt} = 0$. (ii) (a) Condition (3) becomes $\tilde{y}_{Ct} = \arg\max_{\tilde{y}_t \in \mathbb{R}_+} u(\tilde{y}_t) + W^C_t(0)$ s.t. $\varphi_t \tilde{y}_t \leq 0$, which given $\varphi_t \in \mathbb{R}_{++}$, implies $\tilde{y}_{Ct} = 0$. (b) With (34), problem (5) becomes

$$\max_{(\bar{y}_t, k_t, \bar{a}^b_t) \in \mathbb{R}^2_+ \times \mathbb{R}} \left[u(\bar{y}_t) + \bar{a}^b_t - k_t\right]^{\theta_C} k_t^{1-\theta_C}$$
$$\text{s.t. } \varphi_t \bar{y}_t + \bar{a}^b_t \leq 0 \leq u(\bar{y}_t) + \bar{a}^b_t - k_t,$$

and the solution is $\bar{y}_{Ct} = \text{D}(\varphi_t)$, $\bar{a}^b_{Ct} = -\varphi_t \text{D}(\varphi_t)$, and $k_{Ct} = (1-\theta_C)\left[u(\text{D}(\varphi_t)) - \varphi_t \text{D}(\varphi_t)\right]$. So the gain from trade to the consumer is

$$\bar{\Gamma}_{Ct} \equiv u(\bar{y}_{Ct}) + \bar{a}^b_{Ct} - k_{Ct} = \theta_C \left[u(\text{D}(\varphi_t)) - \varphi_t \text{D}(\varphi_t)\right].$$

(iii) (a) Condition (4) becomes $\tilde{y}_{Pt}(y_t) = \arg\max_{\tilde{y}_t \in [0, y_t]} W^P_t((y_t - \tilde{y}_t)\underline{\varphi}) = \arg\max_{\tilde{y}_t \in [0, y_t]} (y_t - \tilde{y}_t)\underline{\varphi}$, so the solution is $\tilde{y}_{Pt}(y_t) = 0$. (b) With (34), problem (5) becomes

$$\max_{(\bar{y}_t, k_t, \bar{a}^b_t) \in \mathbb{R}^2_+ \times \mathbb{R}} (\bar{a}^b_t - k_t - \underline{\varphi}\bar{y}_t)^{\theta_P} k_t^{1-\theta_P}$$

$$\begin{aligned} \text{s.t. } \bar{a}^b_t &\leq \varphi_t \bar{y}_t \\ \bar{y}_t &\leq y_t \\ 0 &\leq \bar{a}^b_t - k_t - \underline{\varphi}\bar{y}_t. \end{aligned}$$

The solution is $\bar{a}^b_{Pt}(y_t) = \varphi_t \bar{y}_{Pt}(y_t)$ and $k_{Pt}(y_t) = (1-\theta_P)(\varphi_t - \underline{\varphi})\bar{y}_{Pt}(y_t)$, with

$$\bar{y}_{Pt}(y_t) \begin{cases} 0 & \text{if } \varphi_t < \underline{\varphi} \\ \in [0, y_t] & \text{if } \varphi_t = \underline{\varphi} \\ y_t & \text{if } \underline{\varphi} < \varphi_t. \end{cases}$$

So the gain from trade to the producer is

$$\begin{aligned} \bar{\Gamma}_{Pt} &\equiv \bar{a}^b_{Pt}(y_t) - k_{Pt}(y_t) - \underline{\varphi}\bar{y}_{Pt}(y_t) \\ &= \theta_P(\varphi_t - \underline{\varphi})\bar{y}_{Pt}(y_t). \end{aligned}$$

(iv) After substituting the bargaining outcomes, (9) becomes

$$V^P_t = \max_{y_t \in \mathbb{R}_+} \left[R^n(\varphi_t) y_t - \kappa y_t + W^P_t(0)\right],$$

where $R^n(\varphi_t)$ as defined in (12). Hence, an individual producer produces

$$y_{Pt} = \arg\max_{y_t \in \mathbb{R}_+} \left[R^n(\varphi_t) - \kappa\right] y_t$$

units of good 1 at the beginning of the first subperiod. ■

Proof of Proposition 2. Part (*iv*) of Lemma 1 implies

$$y_{Pt} = \arg\max_{y_t \in \mathbb{R}_+} \left[R^n\left(\varphi_t\right) - \kappa\right] y_t \equiv \mathrm{Y}\left(\varphi_t\right),$$

so $R^n\left(\varphi_t\right) - \kappa \leq 0$, or equivalently,

$$\varphi_t \leq \varphi^n \tag{35}$$

is a necessary condition for equilibrium, where φ^n is as defined in (13). Hence the solution to the producer's beginning of period production decision is

$$\mathrm{Y}\left(\varphi_t\right) \begin{cases} = 0 & \text{if } \varphi_t < \varphi^n \\ \in [0, \infty) & \text{if } \varphi_t = \varphi^n. \end{cases} \tag{36}$$

Lemma 1 also implies $\tilde{Y}_{Pt} = \tilde{Y}_{Ct} = 0$, $\bar{Y}_{Ct} = \alpha_C N_C \mathrm{D}(\varphi_t)$, and $\bar{Y}_{Pt} = \alpha_P N_P \zeta_{(\varphi_t - \underline{\varphi})} \mathrm{Y}(\varphi_t)$. Given (36), and since $\underline{\varphi} < \varphi^n$, we can write $\bar{Y}_{Pt} = \alpha_P N_P \mathrm{Y}(\varphi_t)$. Thus the market-clearing condition for the goods market can be written as $X_D\left(\varphi_t\right) = 0$, where

$$X_D\left(\varphi_t\right) \equiv \alpha_C N_C \mathrm{D}\left(\varphi_t\right) - \alpha_P N_P \mathrm{Y}\left(\varphi_t\right). \tag{37}$$

For all $\varphi_t \in [0, \varphi^n)$, $0 < X_D\left(\varphi_t\right)$, so equilibrium requires $\varphi^n \leq \varphi_t$, which together with the necessary condition (35), implies $\varphi_t = \varphi^n$ must hold in any equilibrium. From part (*ii*) of Lemma 1, $\bar{y}_{Ct}$ satisfies $u'\left(\bar{y}_{Ct}\right) = \varphi^n$ (the solution is strictly positive since $\varphi^n < u'(0)$), and from the market-clearing condition for good 1, $y_{Pt} = \frac{\alpha_C N_C}{\alpha_P N_P} \bar{y}_{Ct}$. ■

The following corollary of Proposition 2 characterizes the limit of the equilibrium of the nonmonetary economy as $\alpha_P \to 1$.

Corollary 6. *Assume $\kappa + [(\theta_P)^{-1} - 1](\kappa - \underline{\varphi}) < u'(0)$. As $\alpha_P \to 1$, the equilibrium price of good 1 in the nonmonetary economy is*

$$\varphi^{n\star} \equiv \lim_{\alpha_P \to 1} \varphi^n = \kappa + \left(\frac{1}{\theta_P} - 1\right)(\kappa - \underline{\varphi}).$$

The individual consumption allocation of good 1 (for consumers with access to bankers), $\bar{y}_C^{n\star} = \mathrm{D}(\varphi^{n\star})$, satisfies

$$u'\left(\bar{y}_C^{n\star}\right) = \varphi^{n\star}.$$

Consumers without access to bankers do not consume good 1. The individual production allocation of good 1 is $y_P^{n\star} = \frac{\alpha_C N_C}{N_P} \bar{y}_C^{n\star}$.

C. Monetary economy

C.1. Economy with $\alpha_C \theta_C \leq 1$

Proof of Lemma 2. (i) With (32), (2) can be written as

$$\hat{W}_t^B(a_t^m, a_t^g) = \max_{\bar{\boldsymbol{a}}_t \in \mathbb{R}_+ \times \mathbb{R}} (\phi_t \bar{a}_t^m + \bar{a}_t^b + a_t^g + \bar{W}_t^B) \text{ s.t. } \bar{a}_t^m + q_t \bar{a}_t^b \leq a_t^m,$$

and the solution is $q_t \bar{a}_{Bt}^b(a_t^m) = a_t^m - \bar{a}_{Bt}^m(a_t^m)$, with

$$\bar{a}_{Bt}^m(a_t^m) \begin{cases} = \infty & \text{if } \rho_t < 0 \\ \in [0, \infty] & \text{if } \rho_t = 0 \\ = 0 & \text{if } 0 < \rho_t. \end{cases}$$

(ii) (a) With (32), (3) can be written as

$$(\tilde{y}_{Ct}(a_t^m), \tilde{a}_{Ct}^m(a_t^m)) = \arg \max_{(\tilde{y}_t, \tilde{a}_t^m) \in \mathbb{R}_+^2} [u(\tilde{y}_t) + \phi_t \tilde{a}_t^m], \text{ s.t. } \tilde{a}_t^m + p_t \tilde{y}_t \leq a_t^m,$$

so $\tilde{a}_{Ct}^m(a_t^m) = a_t^m - p_t \tilde{y}_{Ct}(a_t^m)$, with

$$p_t \tilde{y}_{Ct}(a_t^m) = \begin{cases} p_t \mathrm{D}(\tilde{\varphi}_t) & \text{if } p_t \mathrm{D}(\tilde{\varphi}_t) \leq a_t^m \\ a_t^m & \text{if } a_t^m < p_t \mathrm{D}(\tilde{\varphi}_t). \end{cases}$$

(ii) (b) With (32), (5) can be written as

$$\max_{(\bar{y}_t, \bar{\boldsymbol{a}}_t, k_t) \in \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{R}_+} \left[u(\bar{y}_t) + \phi_t \bar{a}_t^m + \bar{a}_t^b - k_t - u(\tilde{y}_{Ct}(a_t^m)) - \phi_t \tilde{a}_{Ct}^m(a_t^m) \right]^{\theta_C} k_t^{1-\theta_C}$$

subject $\bar{a}_t^m + p_t \bar{y}_t + q_t \bar{a}_t^b \leq a_t^m$. The solution is $\bar{y}_{Ct}(a_t^m) = \mathrm{D}(\varphi_t)$ and $q_t \bar{a}_{Ct}^b(a_t^m) = a_t^m - [\bar{a}_{Ct}^m(a_t^m) + p_t \mathrm{D}(\varphi_t)]$, with

$$\bar{a}_{Ct}^m(a_t^m) \begin{cases} = \infty & \text{if } \rho_t < 0 \\ \in [0, \infty] & \text{if } \rho_t = 0 \\ = 0 & \text{if } 0 < \rho_t. \end{cases}$$

Hereafter specialize the analysis to $\rho_t \geq 0$, since $\rho_t < 0$ entails an arbitrage opportunity inconsistent with equilibrium. The intermediation fee is

$$\begin{aligned}
\frac{k_{Ct}(a_t^m)}{1-\theta_C} &= u(\bar{y}_{Ct}(a_t^m)) + \phi_t \bar{a}_{Ct}^m(a_t^m) + \bar{a}_{Ct}^b(a_t^m) - [u(\tilde{y}_{Ct}(a_t^m)) + \phi_t \tilde{a}_{Ct}^m(a_t^m)] \\
&= u(\bar{y}_{Ct}(a_t^m)) - \rho_t \phi_t \bar{a}_{Ct}^m(a_t^m) + \frac{1}{q_t}[a_t^m - p_t \bar{y}_{Ct}(a_t^m)] - [u(\tilde{y}_{Ct}(a_t^m)) + \phi_t \tilde{a}_{Ct}^m(a_t^m)] \\
&= u(\bar{y}_{Ct}(a_t^m)) + \frac{1}{q_t}[a_t^m - p_t \bar{y}_{Ct}(a_t^m)] - [u(\tilde{y}_{Ct}(a_t^m)) + \phi_t \tilde{a}_{Ct}^m(a_t^m)] \\
&= u(\bar{y}_{Ct}(a_t^m)) + \frac{1}{q_t}[a_t^m - p_t \bar{y}_{Ct}(a_t^m)] - \{u(\tilde{y}_{Ct}(a_t^m)) + \phi_t [a_t^m - p_t \tilde{y}_{Ct}(a_t^m)]\} \\
&= u(\bar{y}_{Ct}(a_t^m)) - \varphi_t \bar{y}_{Ct}(a_t^m) - [u(\tilde{y}_{Ct}(a_t^m)) - \tilde{\varphi}_t \tilde{y}_{Ct}(a_t^m)] + \rho_t \phi_t a_t^m \\
&= u(\mathrm{D}(\varphi_t)) - \varphi_t \mathrm{D}(\varphi_t) - [u(\min(\mathrm{D}(\tilde{\varphi}_t), a_t^m/p_t)) - \tilde{\varphi}_t \min(\mathrm{D}(\tilde{\varphi}_t), a_t^m/p_t)] \\
&\quad + \rho_t \phi_t a_t^m.
\end{aligned}$$

The gain from trade to the consumer in this case is $\bar{\Gamma}_{Ct}(a_t^m) \equiv \frac{\theta_C}{1-\theta_C} k_{Ct}(a_t^m)$. (iii) (a) With (32), (4) can be written as

$$(\tilde{y}_{Pt}(y_t, a_t^m), \tilde{a}_{Pt}^m(y_t, a_t^m)) = \arg \max_{(\tilde{y}_t, \tilde{a}_t^m) \in \mathbb{R}_+^2} \phi_t \tilde{a}_t^m + (y_t - \tilde{y}_t)\underline{\varphi}$$

subject to $\frac{1}{p_t}(\tilde{a}_t^m - a_t^m) = \tilde{y}_t \leq y_t$, and therefore $\tilde{a}_{Pt}^m(y_t, a_t^m) = a_t^m + p_t \tilde{y}_{Pt}(y_t, a_t^m)$, with

$$\tilde{y}_{Pt}(y_t, a_t^m) \begin{cases} = y_t & \text{if } \underline{\varphi} < \tilde{\varphi}_t \\ \in [0, y_t] & \text{if } \tilde{\varphi}_t = \underline{\varphi} \\ = 0 & \text{if } \tilde{\varphi}_t < \underline{\varphi}. \end{cases}$$

(iii) (b) With (32), (6) can be written as

$$\max_{(\bar{y}_t, \bar{a}_t^m, \bar{a}_t^b, k_t) \in \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{R}_+} \left[\phi_t \bar{a}_t^m + \bar{a}_t^b - k_t + (y_t - \bar{y}_t)\underline{\varphi} - \phi_t \tilde{a}_t^m - (y_t - \tilde{y}_{Pt}(y_t, a_t^m))\underline{\varphi}\right]^{\theta_P} k_t^{1-\theta_P}$$

subject to $\bar{a}_t^m + q_t \bar{a}_t^b \leq a_t^m + p_t \bar{y}_t$ and $\bar{y}_t \leq y_t$. The solution is

$$\bar{a}_{Pt}^b(y_t, a_t^m) = \frac{1}{q_t}\left(a_t^m + p_t \bar{y}_{Pt}(y_t, a_t^m) - \bar{a}_{Pt}^m(y_t, a_t^m)\right),$$

with

$$\bar{y}_{Pt}(y_t, a_t^m) \begin{cases} = y_t & \text{if } \underline{\varphi} < \varphi_t \\ \in [0, y_t] & \text{if } \varphi_t = \underline{\varphi} \\ = 0 & \text{if } \varphi_t < \underline{\varphi} \end{cases}$$

$$\bar{a}_{Pt}^m(y_t, a_t^m) \begin{cases} \infty & \text{if } \rho_t < 0 \\ \in [0, \infty] & \text{if } \rho_t = 0 \\ = 0 & \text{if } 0 < \rho_t. \end{cases}$$

Specialize the analysis to $\rho_t \geq 0$, since $\rho_t < 0$ is inconsistent with equilibrium. The intermediation fee is

$$\begin{aligned}
\frac{k_{Pt}(y_t, a_t^m)}{1-\theta_P} &= \phi_t \bar{a}_{Pt}^m(y_t, a_t^m) + \bar{a}_{Pt}^b(y_t, a_t^m) + (y_t - \bar{y}_{Pt}(y_t, a_t^m))\underline{\varphi} \\
&\quad - \left[\phi_t \tilde{a}_{Pt}^m(y_t, a_t^m) + (y_t - \tilde{y}_{Pt}(y_t, a_t^m))\underline{\varphi}\right] \\
&= \phi_t \bar{a}_{Pt}^m(y_t, a_t^m) + \bar{a}_{Pt}^b(y_t, a_t^m) - \bar{y}_{Pt}(y_t, a_t^m)\underline{\varphi} + \tilde{y}_{Pt}(y_t, a_t^m)\underline{\varphi} - \phi_t \tilde{a}_{Pt}^m(y_t, a_t^m) \\
&= \frac{1}{q_t} a_t^m + (\varphi_t - \underline{\varphi})\bar{y}_{Pt}(y_t, a_t^m) + \tilde{y}_{Pt}(y_t, a_t^m)\underline{\varphi} - \phi_t \tilde{a}_{Pt}^m(y_t, a_t^m) - \rho_t \phi_t \bar{a}_{Pt}^m(y_t, a_t^m) \\
&= \frac{1}{q_t} a_t^m + (\varphi_t - \underline{\varphi})\mathbb{I}_{\{\underline{\varphi} < \varphi_t\}} y_t + \tilde{y}_{Pt}(y_t, a_t^m)\underline{\varphi} - \phi_t \tilde{a}_{Pt}^m(y_t, a_t^m) \\
&= \rho_t \phi_t a_t^m + \left[(\varphi_t - \underline{\varphi})\mathbb{I}_{\{\underline{\varphi} < \varphi_t\}} - (\tilde{\varphi}_t - \underline{\varphi})\mathbb{I}_{\{\underline{\varphi} < \tilde{\varphi}_t\}}\right] y_t.
\end{aligned}$$

The gain from trade to the producer in this case is $\bar{\Gamma}_{Pt}(y_t, a_t^m) \equiv \frac{\theta_P}{1-\theta_P} k_{Pt}(y_t, a_t^m)$. (*iv*) With (32), and substituting the bargaining outcomes from part (*iii*) above, the value function (9) can be written as

$$\begin{aligned}
V_t^P(a_t^m) &= \max_{y_t \in \mathbb{R}_+} \Big\{ -\kappa y_t + \phi_t a_t^m + [\underline{\varphi} + (\tilde{\varphi}_t - \underline{\varphi})\mathbb{I}_{\{\underline{\varphi} < \tilde{\varphi}_t\}}] y_t + \bar{W}_t^P \\
&\quad + \alpha_P \theta_P \{\rho_t \phi_t a_t^m + [(\varphi_t - \underline{\varphi})\mathbb{I}_{\{\underline{\varphi} < \varphi_t\}} - (\tilde{\varphi}_t - \underline{\varphi})\mathbb{I}_{\{\underline{\varphi} < \tilde{\varphi}_t\}}] y_t\} \Big\},
\end{aligned}$$

or equivalently,

$$V_t^P(a_t^m) = \max_{y_t \in \mathbb{R}_+} [R^m(\tilde{\varphi}_t, \varphi_t) - \kappa] y_t + (1 + \alpha_P \theta_P \rho_t)\phi_t a_t^m + \bar{W}_t^P, \qquad (38)$$

with $R^m(\tilde{\varphi}_t, \varphi_t)$ as defined in (15). Hence, an individual producer produces

$$y_{Pt}(a_t^m) = \arg \max_{y_t \in \mathbb{R}_+} [R^m(\tilde{\varphi}_t, \varphi_t) - \kappa] y_t$$

units of good 1 at the beginning of the first subperiod. ■

Lemma 3. *For an agent of type $i \in \{B, C, P\}$, the beginning-of-period value function, $V_t^i(a_t^m)$, can be written as follows. (i) For a producer,*

$$V_t^P(a_t^m) = \max_{y_t \in \mathbb{R}_+} [R^m(\tilde{\varphi}_t, \varphi_t) - \kappa] y_t + (1 + \alpha_P \theta_P \rho_t)\phi_t a_t^m + \bar{W}_t^P.$$

(ii) For a banker,

$$V_t^B(a_t^m) = (1 + \rho_t)\phi_t a_t^m + \bar{W}_t^B + \sum_{i \in \{C,P\}} \alpha_B^i \int k_{it}(\tilde{a}_t^m) dH_{it}(\tilde{a}_t^m).$$

(iii) For a consumer,

$$\begin{aligned} V_t^C(a_t^m) &= u(\min[\mathrm{D}(\tilde{\varphi}_t), a_t^m/p_t]) - \tilde{\varphi}_t \min[\mathrm{D}(\tilde{\varphi}_t), a_t^m/p_t] + \phi_t a_t^m + \bar{W}_t^C \\ &\quad + \alpha_C \theta_C \{\rho_t \phi_t a_t^m + u(\mathrm{D}(\varphi_t)) - \varphi_t \mathrm{D}(\varphi_t) \\ &\quad - [u(\min(\mathrm{D}(\tilde{\varphi}_t), a_t^m/p_t)) - \tilde{\varphi}_t \min(\mathrm{D}(\tilde{\varphi}_t), a_t^m/p_t)]\} \end{aligned}$$

Proof of Lemma 3. (i) The value function $V_t^P(a_t^m)$ is given in (38). (ii) With (32), and part (i) of Lemma 2, (7) can be written as

$$\begin{aligned} V_t^B(a_t^m) &= \phi_t \bar{a}_{Bt}^m(a_t^m) + \bar{a}_{Bt}^b(a_t^m) + \bar{W}_t^B + \sum_{i\in\{C,P\}} \alpha_B^i \int k_{it}(\tilde{a}_t^m) dH_{it}(\tilde{a}_t^m) \\ &= (1+\rho_t)\phi_t a_t^m + \bar{W}_t^B + \sum_{i\in\{C,P\}} \alpha_B^i \int k_{it}(\tilde{a}_t^m) dH_{it}(\tilde{a}_t^m). \end{aligned}$$

(iii) The value function (8) can be written as

$$\begin{aligned} V_t^C(a_t^m) &= u(\tilde{y}_{Ct}(a_t^m)) + W_t^C(\tilde{a}_{Ct}^m(a_t^m), 0) + \alpha_C \bar{\Gamma}_{Ct}(a_t^m) \\ &= u(\tilde{y}_{Ct}(a_t^m)) + \phi_t \tilde{a}_{Ct}^m(a_t^m) + \bar{W}_t^C + \alpha_C \bar{\Gamma}_{Ct}(a_t^m). \end{aligned}$$

where $\bar{\Gamma}_{Ct}(a_t^m) \equiv \frac{\theta_C}{1-\theta_C} k_{Ct}(a_t^m)$ as defined in part (ii) of Lemma 2, and the second line follows from (32). After substituting the trading outcomes in part (ii) of Lemma 2, we arrive at the expression in the statement. ■

Lemma 4. *Consider the money-demand problem at the end of period t (i.e., the maximization on the right side of (33), and let a_{it+1}^m denote the individual money demand of an agent of type $i \in \{B, C, P\}$. Then $\{a_{it+1}^m\}_{i\in\{B,C,P\}}$ must satisfy the following Euler equations:*

$$-\phi_t + \beta \bar{v}_{t+1}^i \phi_{t+1} \leq 0, \text{ with “} = \text{” if } 0 < a_{it+1}^m \text{ for } i \in \{B, P\} \tag{39}$$

and

$$-\phi_t + \beta \bar{v}_{t+1}^C (a_{Ct+1}^m) \phi_{t+1} \leq 0, \text{ with “} = \text{” if } 0 < a_{Ct+1}^m, \tag{40}$$

where $\bar{v}_{t+1}^B \equiv 1 + \rho_{t+1}$, $\bar{v}_{t+1}^P \equiv 1 + \alpha_P \theta_P \rho_{t+1}$, and

$$\bar{v}_{t+1}^C (a_{Ct+1}^m) \equiv 1 + \alpha_C \theta_C \rho_{t+1} + (1 - \alpha_C \theta_C) \left[\frac{u'(\min(\mathrm{D}(\tilde{\varphi}_{t+1}), a_{Ct+1}^m / p_{t+1}))}{\tilde{\varphi}_{t+1}} - 1 \right].$$

Proof of Lemma 4. Take the first-order conditions for the maximization in (33) using the expressions for the value functions reported in Lemma 3. ■

Lemma 5. *Consider a monetary economy. In the first subperiod of period t: (i) The market-clearing condition for good 1 is*

$$\begin{aligned} 0 &= \alpha_C N_C \mathrm{D}(\varphi_t) + (1-\alpha_C)\min(N_C \mathrm{D}(\tilde{\varphi}_t), A^m_{Ct}/p_t) \\ &\quad -[\alpha_P \bar{\zeta}_{(\varphi_t-\underline{\varphi})} + (1-\alpha_P)\tilde{\zeta}_{(\tilde{\varphi}_t-\underline{\varphi})}] N_P \varkappa^p_{(\kappa-R^m(\tilde{\varphi}_t,\varphi_t))}. \end{aligned} \tag{41}$$

(ii) The market-clearing condition for bonds is

$$\begin{aligned} 0 &= A^m_{Bt} - \varkappa^m_{Bt(\rho_t)} N_B \\ &\quad +\alpha_C[A^m_{Ct} - \varkappa^m_{Ct(\rho_t)} N_C - p_t \mathrm{D}(\varphi_t) N_C] \\ &\quad +\alpha_P[A^m_{Pt} - \varkappa^m_{Pt(\rho_t)} N_P + p_t \bar{\zeta}_{(\varphi_t-\underline{\varphi})} \varkappa^p_{(\kappa-R^m(\tilde{\varphi}_t,\varphi_t))} N_P]. \end{aligned} \tag{42}$$

Proof of Lemma 5. (i) From Lemma 2,

$$\begin{aligned} \tilde{Y}_{Ct} &= (1-\alpha_C)N_C \int \tilde{y}_{Ct}(a^m_t)\, dF_{Ct}(a^m_t) = (1-\alpha_C)N_C \int \min(\mathrm{D}(\tilde{\varphi}_t), a^m_t/p_t) dF_{Ct}(a^m_t) \\ \bar{Y}_{Ct} &= \alpha_C N_C \int \bar{y}_{Ct}(a^m_t)\, dF_{Ct}(a^m_t) = \alpha_C \mathrm{D}(\varphi_t) N_C \\ \tilde{Y}_{Pt} &= (1-\alpha_P)N_P \int \tilde{y}_{Pt}(y_{Pt}(a^m_t), a^m_t)\, dF_{Pt}(a^m_t) = (1-\alpha_P)\tilde{\zeta}_{(\tilde{\varphi}_t-\underline{\varphi})} \varkappa^p_{(\kappa-R^m(\tilde{\varphi}_t,\varphi_t))} N_P \\ \bar{Y}_{Pt} &= \alpha_P N_P \int \bar{y}_{Pt}(y_{Pt}(a^m_t), a^m_t)\, dF_{Pt}(a^m_t) = \alpha_P \bar{\zeta}_{(\varphi_t-\underline{\varphi})} \varkappa^p_{(\kappa-R^m(\tilde{\varphi}_t,\varphi_t))} N_P. \end{aligned}$$

So the market-clearing condition for good 1, $\tilde{Y}_{Ct} + \bar{Y}_{Ct} = \tilde{Y}_{Pt} + \bar{Y}_{Pt}$, can be written as in the statement of the lemma. (ii) From Lemma 2,

$$\begin{aligned} \bar{A}^b_{Bt} &= N_B \int \bar{a}^b_{Bt}(a^m_t)\, dF_{Bt}(a^m_t) = \frac{1}{q_t}(A^m_{Bt} - \varkappa^m_{Bt(\rho_t)} N_B) \\ \bar{A}^b_{Ct} &= \alpha_C N_C \int \bar{a}^b_{Ct}(a^m_t)\, dF_{Ct}(a^m_t) = \frac{1}{q_t}\alpha_C[A^m_{Ct} - \varkappa^m_{Ct(\rho_t)} N_C - p_t \mathrm{D}(\varphi_t) N_C] \\ \bar{A}^b_{Pt} &= \alpha_P N_P \int \bar{a}^b_{Pt}(y_{Pt}(a^m_t), a^m_t)\, dF_{Pt}(a^m_t) \\ &= \frac{1}{q_t}\alpha_P[p_t \bar{\zeta}_{(\varphi_t-\underline{\varphi})} \varkappa^p_{(\kappa-R^m(\tilde{\varphi}_t,\varphi_t))} N_P + A^m_{Pt} - \varkappa^m_{Pt(\rho_t)} N_P]. \end{aligned}$$

So the market-clearing condition for the bond, $\sum_{i\in\{B,C,P\}} \bar{A}^b_{it} = 0$, can be written as in the statement of the lemma. ■

Lemma 6. *For* $i \in \{B, C, P\}$, *let* $\omega_{it} M_t \equiv A^m_{it}$ *and* $\hat{\varkappa}^m_{it(z)} M_t \equiv \varkappa^m_{it(z)}$. *A monetary equilibrium is a bounded sequence*

$$\{Z_t, \varphi_t, \tilde{\varphi}_t, \rho_t, \tilde{\zeta}_{(\tilde{\varphi}_t - \underline{\varphi})}, \bar{\zeta}_{(\varphi_t - \underline{\varphi})}, \varkappa^p_{(\kappa - R^m(\tilde{\varphi}_t, \varphi_t))}, [\omega_{it+1}, \hat{\varkappa}^m_{it(\rho_t)}]_{i \in \{B,C,P\}}\}^\infty_{t=0}$$

that satisfies the market-clearing conditions

$$\begin{aligned}
0 &= \sum\nolimits_{i \in \{B,C,P\}} \omega_{it+1} - 1 \\
0 &= \alpha_C N_C \mathrm{D}(\varphi_t) + (1 - \alpha_C) \min \left[N_C \mathrm{D}(\tilde{\varphi}_t), \frac{\omega_{Ct} Z_t}{\tilde{\varphi}_t} \right] \\
&\quad - [\alpha_P \bar{\zeta}_{(\varphi_t - \underline{\varphi})} + (1 - \alpha_P) \tilde{\zeta}_{(\tilde{\varphi}_t - \underline{\varphi})}] N_P \varkappa^p_{(\kappa - R^m(\tilde{\varphi}_t, \varphi_t))} \\
0 &= (\omega_{Bt} - \hat{\varkappa}^m_{Bt(\rho_t)} N_B) Z_t \\
&\quad + \alpha_C [(\omega_{Ct} - \hat{\varkappa}^m_{Ct(\rho_t)} N_C) Z_t - \tilde{\varphi}_t \mathrm{D}(\varphi_t) N_C] \\
&\quad + \alpha_P [(\omega_{Pt} - \hat{\varkappa}^m_{Pt(\rho_t)} N_P) Z_t + \tilde{\varphi}_t \bar{\zeta}_{(\varphi_t - \underline{\varphi})} \varkappa^p_{(\kappa - R^m(\tilde{\varphi}_t, \varphi_t))} N_P]
\end{aligned}$$

and the optimality conditions

$$\begin{aligned}
0 &= [-\kappa + R^m(\tilde{\varphi}_t, \varphi_t)] \varkappa^p_{(\kappa - R^m(\tilde{\varphi}_t, \varphi_t))} \\
0 &= \left[-Z_t \mu + \beta \bar{v}^i_{t+1} Z_{t+1} \right] \omega_{it+1} \text{ for } i \in \{B, C, P\},
\end{aligned}$$

where

$$\begin{aligned}
\bar{v}^B_{t+1} &\equiv 1 + \rho_{t+1} \\
\bar{v}^P_{t+1} &\equiv 1 + \alpha_P \theta_P \rho_{t+1} \\
\bar{v}^C_{t+1} &\equiv 1 + \alpha_C \theta_C \rho_{t+1} + (1 - \alpha_C \theta_C) \left[\frac{u'\left(\min \left[\mathrm{D}(\tilde{\varphi}_{t+1}), \frac{Z_{t+1}}{\tilde{\varphi}_{t+1}} \frac{\omega_{Ct+1}}{N_C} \right] \right)}{\tilde{\varphi}_{t+1}} - 1 \right] \\
R^m(\tilde{\varphi}_t, \varphi_t) &\equiv \underline{\varphi} + \alpha_P \theta_P (\varphi_t - \underline{\varphi}) \mathbb{I}_{\{\underline{\varphi} < \varphi_t\}} + (1 - \alpha_P \theta_P)(\tilde{\varphi}_t - \underline{\varphi}) \mathbb{I}_{\{\underline{\varphi} < \tilde{\varphi}_t\}} \\
\rho_t &= \frac{\varphi_t - \tilde{\varphi}_t}{\tilde{\varphi}_t}.
\end{aligned}$$

Proof of Lemma 6. By using Definition 1, Lemma 2, Lemma 4, and Lemma 5, we know a monetary equilibrium is a sequence

$$\{Z_t, \varphi_t, \tilde{\varphi}_t, \rho_t, \tilde{\zeta}_{(\tilde{\varphi}_t - \underline{\varphi})}, \bar{\zeta}_{(\varphi_t - \underline{\varphi})}, \varkappa^p_{(\kappa - R^m(\tilde{\varphi}_t, \varphi_t))}, [A^m_{it+1}, \varkappa^m_{it(\rho_t)}]_{i \in \{B,C,P\}}\}^\infty_{t=0}$$

that satisfies the market-clearing conditions

$$
\begin{aligned}
0 &= \sum_{i\in\{B,C,P\}} A^m_{it+1} - M_{t+1} \\
0 &= \alpha_C N_C \mathrm{D}\left(\varphi_t\right) + (1-\alpha_C)\min(N_C \mathrm{D}\left(\tilde{\varphi}_t\right), A^m_{Ct}/p_t) \\
&\quad -[\alpha_P \bar{\zeta}_{(\varphi_t - \underline{\varphi})} + (1-\alpha_P)\tilde{\zeta}_{(\tilde{\varphi}_t - \underline{\varphi})}] N_P \varkappa^p_{(\kappa - R^m(\tilde{\varphi}_t, \varphi_t))} \\
0 &= A^m_{Bt} - \varkappa^m_{Bt(\rho_t)} N_B \\
&\quad +\alpha_C[A^m_{Ct} - \varkappa^m_{Ct(\rho_t)} N_C - p_t \mathrm{D}\left(\varphi_t\right) N_C] \\
&\quad +\alpha_P[A^m_{Pt} - \varkappa^m_{Pt(\rho_t)} N_P + p_t \bar{\zeta}_{(\varphi_t - \underline{\varphi})} \varkappa^p_{(\kappa - R^m(\tilde{\varphi}_t, \varphi_t))} N_P]
\end{aligned}
$$

and the optimality conditions

$$
\begin{aligned}
0 &= \left[-\kappa + R^m\left(\tilde{\varphi}_t, \varphi_t\right)\right] \varkappa^p_{(\kappa - R^m(\tilde{\varphi}_t, \varphi_t))} = 0 \\
0 &\geq -\phi_t + \beta \bar{v}^i_{t+1} \phi_{t+1}, \text{ with " = " if } 0 < a^m_{it+1},
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{v}^B_{t+1} &\equiv 1 + \rho_{t+1} \\
\bar{v}^P_{t+1} &\equiv 1 + \alpha_P \theta_P \rho_{t+1} \\
\bar{v}^C_{t+1} &\equiv 1 + \alpha_C \theta_C \rho_{t+1} + (1-\alpha_C\theta_C)\left[\frac{u'(\min\left(\mathrm{D}\left(\tilde{\varphi}_{t+1}\right), a^m_{Ct+1}/p_{t+1}\right))}{\tilde{\varphi}_{t+1}} - 1\right] \\
R^m\left(\tilde{\varphi}_t, \varphi_t\right) &\equiv \underline{\varphi} + \alpha_P\theta_P(\varphi_t - \underline{\varphi})\mathbb{I}_{\{\underline{\varphi} < \varphi_t\}} + (1-\alpha_P\theta_P)\left(\tilde{\varphi}_t - \underline{\varphi}\right)\mathbb{I}_{\{\underline{\varphi} < \tilde{\varphi}_t\}} \\
\rho_t &= \frac{\varphi_t - \tilde{\varphi}_t}{\tilde{\varphi}_t}.
\end{aligned}
$$

By using the variables ω_{it} and $\hat{\varkappa}^m_{it(z)}$ we arrive at the definition of equilibrium in the statement of the lemma. ■

To characterize a monetary equilibrium it suffices to find $\{Z_t, \varphi_t, \tilde{\varphi}_t\}_{t=0}^{\infty}$, since given this path we know the nominal prices $1/\phi_t = M_t/Z_t$, $p_t = \tilde{\varphi}_t/\phi_t$, $q_t = p_t/\varphi_t$, $\rho_t = (\varphi_t - \tilde{\varphi}_t)/\tilde{\varphi}_t$, and the rest of the equilibrium is immediate from Lemma 2. The following result characterizes the stationary monetary equilibrium, i.e., a path $\{Z_t, \varphi_t, \tilde{\varphi}_t\}_{t=0}^{\infty}$ such that $Z_t = Z$, $\varphi_t = \varphi$, and $\tilde{\varphi}_t = \tilde{\varphi}$ for all t. Without loss of generality, we focus on economies where good 1 is produced. For any $x \in \mathbb{R}_+$, define the function $\varrho : \mathbb{R}_+ \to \mathbb{R}$ as

$$
\varrho\left(x\right) \equiv \frac{\alpha_C\theta_C}{\alpha_P\theta_P}\frac{\kappa - x}{x} + (1-\alpha_C\theta_C)\left[\frac{u'\left(\frac{\alpha_C(1-\alpha_P)}{\alpha_P + \alpha_C(1-\alpha_P)}\mathrm{D}\left(\kappa + \frac{1-\alpha_P\theta_P}{\alpha_P\theta_P}\left(\kappa - x\right)\right)\right)}{x} - 1\right]. \quad (43)
$$

Proposition 6. *Assume $\varphi^n < u'(0)$. There exists a unique stationary monetary equilibrium provided $0 \le \iota < \varrho(\underline{\varphi})$. In any stationary monetary equilibrium, $Z_t = Z$, $\varphi_t = \varphi$, $\tilde{\varphi}_t = \tilde{\varphi}$, $\rho_t = \rho$ for all t, and*

$$\begin{aligned} \phi_t &= \frac{Z}{M_t} \\ p_t &= \frac{\tilde{\varphi}}{Z} M_t \\ q_t &= \frac{\tilde{\varphi}}{\varphi} \frac{M_t}{Z}. \end{aligned}$$

Moreover:

(i) If $\varrho(\kappa) < \iota < \varrho(\underline{\varphi})$, then $\tilde{\varphi} \in (\underline{\varphi}, \kappa)$ is the unique solution to $\varrho(\tilde{\varphi}) = \iota$, and

$$\begin{aligned} \varphi &= \kappa + \frac{1 - \alpha_P \theta_P}{\alpha_P \theta_P} (\kappa - \tilde{\varphi}) \\ \rho &= \frac{1}{\alpha_P \theta_P} \frac{\kappa - \tilde{\varphi}}{\tilde{\varphi}} \\ \frac{Z}{\tilde{\varphi}} &= \frac{\alpha_C (1 - \alpha_P)}{\alpha_P + \alpha_C (1 - \alpha_P)} \mathrm{D}(\varphi) N_C. \end{aligned}$$

The consumption allocation of good 1 for a consumer with access to bankers, $\bar{y}_C$, satisfies

$$u'(\bar{y}_C) = \varphi.$$

The consumption allocation of good 1 for a consumer without access to bankers is

$$\tilde{y}_C = \frac{\alpha_C (1 - \alpha_P)}{\alpha_C (1 - \alpha_P) + \alpha_P} \bar{y}_C.$$

(ii) If $0 < \iota \le \varrho(\kappa)$, then $\tilde{\varphi} = \varphi = \kappa$, $\rho = 0$, and

$$\frac{Z}{\tilde{\varphi}} = \mathrm{D}\left(\frac{1 - \alpha_C \theta_C + \iota}{1 - \alpha_C \theta_C} \kappa \right) N_C.$$

The consumption allocation of good 1 for a consumer with access to bankers, $\bar{y}_C$, satisfies

$$u'(\bar{y}_C) = \kappa.$$

The consumption allocation of good 1 for a consumer without access to bankers, $\tilde{y}_C$, satisfies

$$u'(\tilde{y}_C) = \left(1 + \frac{\iota}{1 - \alpha_C \theta_C} \right) \kappa.$$

The individual production allocation of good 1 is $y_P = [\alpha_C \bar{y}_C + (1 - \alpha_C) \tilde{y}_C] \frac{N_C}{\alpha_P N_P}$.

(iii) As $\iota \to 0$, $\tilde{y}_C \to y^*$, and any $Z \in [\kappa \mathrm{D}(\kappa) N_C, \infty)$ is consistent with equilibrium.

(iv) As $\iota \to \varrho(\underline{\varphi})$, $\tilde{\varphi} \to \underline{\varphi}$, and $\varphi \to \varphi^n$.

Proof of Proposition 6. From Lemma 6, a stationary monetary equilibrium is a vector

$$\left(Z,\varphi,\tilde{\varphi},\tilde{\zeta}_{(\tilde{\varphi}-\underline{\varphi})},\bar{\zeta}_{(\varphi-\underline{\varphi})},\varkappa^{p}_{(\kappa-R^{m}(\tilde{\varphi},\varphi))},[\omega_i,\hat{\varkappa}^{m}_{i(\rho)}]_{i\in\{B,C,P\}}\right)$$

with $Z>0$ that satisfies the market-clearing conditions

$$\begin{aligned}
0 &= \sum\nolimits_{i\in\{B,C,P\}}\omega_i-1 && (44)\\
0 &= \alpha_C\mathrm{D}(\varphi)N_C+(1-\alpha_C)\min\left[\mathrm{D}(\tilde{\varphi})N_C,\frac{\omega_C Z}{\tilde{\varphi}}\right]\\
&\quad-[\alpha_P\bar{\zeta}_{(\varphi-\underline{\varphi})}+(1-\alpha_P)\tilde{\zeta}_{(\tilde{\varphi}-\underline{\varphi})}]N_P\varkappa^{p}_{(\kappa-R^{m}(\tilde{\varphi},\varphi))} && (45)\\
0 &= (\omega_B-\hat{\varkappa}^{m}_{B(\rho)}N_B)Z\\
&\quad+\alpha_C[(\omega_C-\hat{\varkappa}^{m}_{C(\rho)}N_C)Z-\tilde{\varphi}\mathrm{D}(\varphi)N_C]\\
&\quad+\alpha_P[(\omega_P-\hat{\varkappa}^{m}_{P(\rho)}N_P)Z+\tilde{\varphi}\bar{\zeta}_{(\varphi-\underline{\varphi})}\varkappa^{p}_{(\kappa-R^{m}(\tilde{\varphi},\varphi))}N_P] && (46)
\end{aligned}$$

and the optimality conditions

$$\begin{aligned}
0 &= \left[-\kappa+R^{m}(\tilde{\varphi},\varphi)\right]\varkappa^{p}_{(\kappa-R^{m}(\tilde{\varphi},\varphi))} && (47)\\
0 &= \left(-\mu+\beta\bar{v}^{i}\right)\omega_i \text{ for } i\in\{B,C,P\}, && (48)
\end{aligned}$$

where

$$\begin{aligned}
\bar{v}^{B} &\equiv 1+\rho\\
\bar{v}^{P} &\equiv 1+\alpha_P\theta_P\rho\\
\bar{v}^{C} &\equiv 1+\alpha_C\theta_C\rho+(1-\alpha_C\theta_C)\left[\frac{u'\left(\min\left[\mathrm{D}(\tilde{\varphi}),\frac{Z}{\tilde{\varphi}}\frac{\omega_C}{N_C}\right]\right)}{\tilde{\varphi}}-1\right]\\
R^{m}(\tilde{\varphi},\varphi) &\equiv \underline{\varphi}+\alpha_P\theta_P(\varphi-\underline{\varphi})\mathbb{I}_{\{\underline{\varphi}<\varphi\}}+(1-\alpha_P\theta_P)(\tilde{\varphi}-\underline{\varphi})\mathbb{I}_{\{\underline{\varphi}<\tilde{\varphi}\}}\\
\rho &= \frac{\varphi-\tilde{\varphi}}{\tilde{\varphi}}. && (49)
\end{aligned}$$

First, we know that $\tilde{\varphi}\leq\varphi$, since $0\leq\rho$ must hold in any equilibrium. Second, in any equilibrium in which good 1 is produced, we must have: (a) $\kappa=R^{m}(\tilde{\varphi},\varphi)$, or equivalently,

$$\kappa=\underline{\varphi}+\alpha_P\theta_P(\varphi-\underline{\varphi})\mathbb{I}_{\{\underline{\varphi}<\varphi\}}+(1-\alpha_P\theta_P)(\tilde{\varphi}-\underline{\varphi})\mathbb{I}_{\{\underline{\varphi}<\tilde{\varphi}\}}. \quad (50)$$

(b) $\underline{\varphi}<\varphi$, i.e., banked producers never store output. To see why this must be the case, notice that if $\varphi\leq\underline{\varphi}$, then we know that $\tilde{\varphi}\leq\varphi\leq\underline{\varphi}$, and therefore $R^{m}(\tilde{\varphi},\varphi)=\underline{\varphi}<\kappa$

which implies good 1 is never produced. (c) If $\tilde{\varphi} = \varphi$, then (50) implies $\tilde{\varphi} = \varphi = \kappa > \underline{\varphi}$. Third, from the Euler equations for money, we know that $\omega_P = 0$ (since $\bar{v}^P < \bar{v}^B$), and that $\omega_B = \omega_P = 1 - \omega_C = 0$ if $\rho = 0$. In principle, there could be eight types of equilibria, depending on whether $\omega_C = 1 - \omega_B = 0$ or $\omega_C = 1 - \omega_B = 1$, $0 < \rho$ or $\rho = 0$, and $\tilde{\varphi} < \underline{\varphi}$ or $\underline{\varphi} \leq \tilde{\varphi}$. But from the previous observations we know $\rho = 0$ implies $\omega_C = 1$ and $\varphi = \tilde{\varphi} = \kappa$, so there are only five possible equilibrium configurations. Next, we consider each in turn. In every case, let $\tilde{z} \equiv Z/\tilde{\varphi}$.

Configuration 1. $0 < \rho$, $\omega_C = 1$, and $\underline{\varphi} < \tilde{\varphi}$. Under this conjecture, the equilibrium conditions (45)-(49) imply $(\tilde{z}, \varphi, \tilde{\varphi}, \rho)$ must satisfy

$$0 = \alpha_C [\tilde{z} - \mathrm{D}(\varphi) N_C] + \alpha_P \{\alpha_C \mathrm{D}(\varphi) N_C + (1 - \alpha_C) \min[\mathrm{D}(\tilde{\varphi}) N_C, \tilde{z}]\} \tag{51}$$

$$\varphi = \kappa + \frac{1 - \alpha_P \theta_P}{\alpha_P \theta_P} (\kappa - \tilde{\varphi}) \tag{52}$$

$$\iota = \alpha_C \theta_C \rho + (1 - \alpha_C \theta_C) \left[\frac{u'\left(\frac{1}{N_C} \min[\mathrm{D}(\tilde{\varphi}) N_C, \tilde{z}]\right)}{\tilde{\varphi}} - 1 \right] \tag{53}$$

$$\rho = \frac{1}{\alpha_P \theta_P} \frac{\kappa - \tilde{\varphi}}{\tilde{\varphi}}. \tag{54}$$

Condition (51) can be written as $T(\tilde{z}) = 0$ where

$$T(\tilde{z}) \equiv \mathrm{D}(\varphi) N_C - \frac{\alpha_P}{\alpha_C} \{\alpha_C \mathrm{D}(\varphi) N_C + (1 - \alpha_C) \min[\mathrm{D}(\tilde{\varphi}) N_C, \tilde{z}]\} - \tilde{z}.$$

Since T is continuous, with $T' < 0$ and $\lim_{\tilde{z} \to \infty} T(\tilde{z}) < 0 < T(0) = (1 - \alpha_P) \mathrm{D}(\varphi) N_C$, there exists a unique $\tilde{z} \in (0, \infty)$ such that $T(\tilde{z}) = 0$. Moreover, this solution is given by

$$\tilde{z} = \begin{cases} \frac{\alpha_C (1-\alpha_P)}{\alpha_P + \alpha_C (1-\alpha_P)} \mathrm{D}\left(\kappa + \frac{1-\alpha_P\theta_P}{\alpha_P\theta_P}(\kappa - \tilde{\varphi})\right) N_C & \text{if } \tilde{\varphi} \leq \tilde{\varphi}^* \\ \left[(1 - \alpha_P) \mathrm{D}\left(\kappa + \frac{1-\alpha_P\theta_P}{\alpha_P\theta_P}(\kappa - \tilde{\varphi})\right) - \frac{\alpha_P}{\alpha_C}(1 - \alpha_C) \mathrm{D}(\tilde{\varphi})\right] N_C & \text{if } \tilde{\varphi}^* < \tilde{\varphi}, \end{cases}$$

where

$$\tilde{\varphi}^* \in \left(\kappa, \frac{\kappa}{1 - \alpha_P \theta_P}\right) \tag{55}$$

is the unique solution to $\Upsilon(\tilde{\varphi}^*) = 0$, with

$$\Upsilon(\tilde{\varphi}^*) \equiv [\alpha_P + \alpha_C (1 - \alpha_P)] \mathrm{D}(\tilde{\varphi}^*) - \alpha_C (1 - \alpha_P) \mathrm{D}\left(\kappa + \frac{1 - \alpha_P \theta_P}{\alpha_P \theta_P}(\kappa - \tilde{\varphi}^*)\right).$$

Hence

$$\begin{aligned} \min[\mathrm{D}(\tilde{\varphi}) N_C, \tilde{z}] &= \begin{cases} \frac{\alpha_C (1-\alpha_P)}{\alpha_P + \alpha_C (1-\alpha_P)} \mathrm{D}\left(\kappa + \frac{1-\alpha_P\theta_P}{\alpha_P\theta_P}(\kappa - \tilde{\varphi})\right) N_C & \text{if } \tilde{\varphi} < \tilde{\varphi}^* \\ \mathrm{D}(\tilde{\varphi}) N_C & \text{if } \tilde{\varphi}^* \leq \tilde{\varphi} \end{cases} \\ &= \left\{ \mathbb{I}_{\{\tilde{\varphi} < \tilde{\varphi}^*\}} \frac{\alpha_C (1 - \alpha_P)}{\alpha_P + \alpha_C (1 - \alpha_P)} \mathrm{D}\left(\kappa + \frac{1 - \alpha_P \theta_P}{\alpha_P \theta_P}(\kappa - \tilde{\varphi})\right) + \mathbb{I}_{\{\tilde{\varphi}^* \leq \tilde{\varphi}\}} \mathrm{D}(\tilde{\varphi}) \right\} N_C. \end{aligned}$$

Substitute this expression into (53), and use (54) to get the following equation in $\tilde{\varphi}$

$$\iota = \frac{\alpha_C \theta_C}{\alpha_P \theta_P} \frac{\kappa - \tilde{\varphi}}{\tilde{\varphi}} + (1 - \alpha_C \theta_C) \left[\frac{u'\left(\frac{\alpha_C(1-\alpha_P)}{\alpha_P + \alpha_C(1-\alpha_P)} \mathrm{D}\left(\kappa + \frac{1-\alpha_P\theta_P}{\alpha_P\theta_P} (\kappa - \tilde{\varphi}) \right) \right)}{\tilde{\varphi}} - 1 \right] \mathbb{I}_{\{\tilde{\varphi} < \tilde{\varphi}^*\}}.$$

This condition can be written as $E(\tilde{\varphi}) = 0$, where

$$\begin{aligned} E(\tilde{\varphi}) \equiv & (1 - \alpha_C\theta_C) \left[u'\left(\frac{\alpha_C (1-\alpha_P)}{\alpha_P + \alpha_C (1-\alpha_P)} \mathrm{D}\left(\kappa + \frac{1-\alpha_P\theta_P}{\alpha_P\theta_P} (\kappa - \tilde{\varphi}) \right) \right) - \tilde{\varphi} \right] \mathbb{I}_{\{\tilde{\varphi} < \tilde{\varphi}^*\}} \\ & + \frac{\alpha_C\theta_C}{\alpha_P\theta_P} (\kappa - \tilde{\varphi}) - \iota \tilde{\varphi}. \end{aligned}$$

Notice that $E(\cdot)$ is continuous and strictly decreasing, with $\lim_{\tilde{\varphi}\to\infty} E(\tilde{\varphi}) < 0 < E(0)$, so there exists a unique $\tilde{\varphi} \in (0, \infty)$ that satisfies $E(\tilde{\varphi}) = 0$. A $\tilde{\varphi}$ that satisfies $E(\tilde{\varphi}) = 0$ is an equilibrium for Configuration 1 only if it also satisfies (a) $\underline{\varphi} < \tilde{\varphi}$; (b) $\tilde{\varphi} < \kappa$ (i.e., $0 < \rho$); and (c) $\bar{v}^B \leq \bar{v}^C$. Condition (a) is equivalent to $0 < E(\underline{\varphi})$, which is equivalent to $\iota < \varrho(\underline{\varphi})$. Condition (b) is equivalent to $E(\kappa) < 0$, which is equivalent to $\varrho(\kappa) < \iota$. Hence, if $\varrho(\kappa) < \iota < \varrho(\underline{\varphi})$, then there exists a unique $\tilde{\varphi} \in (\underline{\varphi}, \kappa)$ that satisfies $E(\tilde{\varphi}) = 0$. For this to be an equilibrium, it only remains to check condition (c), i.e., that $\bar{v}^B < \bar{v}^C$ (which implies $\omega_C = 1$, as conjectured), or equivalently, that

$$\frac{1}{\alpha_P\theta_P} \frac{\kappa - \tilde{\varphi}}{\tilde{\varphi}} < \frac{u'\left(\frac{1}{N_C} \min\left[\mathrm{D}(\tilde{\varphi}) N_C, \tilde{z} \right] \right)}{\tilde{\varphi}} - 1. \tag{56}$$

Under conjectures (a) and (b), we know that the $\tilde{\varphi}$ that satisfies $E(\tilde{\varphi}) = 0$ also satisfies $\tilde{\varphi} \in (\underline{\varphi}, \kappa)$, which given (55), implies $\tilde{\varphi} < \tilde{\varphi}^*$, and therefore (56) is equivalent to

$$\frac{1}{\alpha_P\theta_P} \frac{\kappa - \tilde{\varphi}}{\tilde{\varphi}} < \frac{u'\left(\frac{\alpha_C(1-\alpha_P)}{\alpha_P + \alpha_C(1-\alpha_P)} \mathrm{D}\left(\kappa + \frac{1-\alpha_P\theta_P}{\alpha_P\theta_P} (\kappa - \tilde{\varphi}) \right) \right)}{\tilde{\varphi}} - 1. \tag{57}$$

Since $E(\tilde{\varphi}) = 0$ implies

$$\frac{u'\left(\frac{\alpha_C(1-\alpha_P)}{\alpha_P + \alpha_C(1-\alpha_P)} \mathrm{D}\left(\kappa + \frac{1-\alpha_P\theta_P}{\alpha_P\theta_P} (\kappa - \tilde{\varphi}) \right) \right)}{\tilde{\varphi}} - 1 = \frac{1}{1 - \alpha_C\theta_C} \left(\iota - \frac{\alpha_C\theta_C}{\alpha_P\theta_P} \frac{\kappa - \tilde{\varphi}}{\tilde{\varphi}} \right),$$

condition (57), and therefore condition *(c)*, is equivalent to

$$\frac{\kappa}{1 + \alpha_P\theta_P\iota} < \tilde{\varphi}. \tag{58}$$

Condition (58) is equivalent to $0 < E(\kappa/(1 + \alpha_P\theta_P\iota))$, which is in turn equivalent to

$$0 < \frac{u'\left(\frac{\alpha_C(1-\alpha_P)}{\alpha_P + \alpha_C(1-\alpha_P)} \mathrm{D}\left(\frac{1+\iota}{1+\alpha_P\theta_P\iota} \kappa \right) \right)}{\frac{1+\iota}{1+\alpha_P\theta_P\iota}\kappa} - 1.$$

But this inequality necessarily holds, since

$$0=\frac{u'\left(\mathrm{D}\left(\frac{1+\iota}{1+\alpha_P\theta_P\iota}\kappa\right)\right)}{\frac{1+\iota}{1+\alpha_P\theta_P\iota}\kappa}-1<\frac{u'\left(\frac{\alpha_C(1-\alpha_P)}{\alpha_P+\alpha_C(1-\alpha_P)}\mathrm{D}\left(\frac{1+\iota}{1+\alpha_P\theta_P\iota}\kappa\right)\right)}{\frac{1+\iota}{1+\alpha_P\theta_P\iota}\kappa}-1.$$

To summarize, if $\varrho(\kappa)<\iota<\varrho(\underline{\varphi})$, then there is a unique stationary monetary equilibrium, and it is fully characterized by

$$\begin{aligned}\tilde{z} &= \frac{\alpha_C(1-\alpha_P)}{\alpha_P+\alpha_C(1-\alpha_P)}\mathrm{D}\left(\kappa+\frac{1-\alpha_P\theta_P}{\alpha_P\theta_P}(\kappa-\tilde{\varphi})\right)N_C\\ \varphi &= \kappa+\frac{1-\alpha_P\theta_P}{\alpha_P\theta_P}(\kappa-\tilde{\varphi})\\ \rho &= \frac{1}{\alpha_P\theta_P}\frac{\kappa-\tilde{\varphi}}{\tilde{\varphi}}\end{aligned}$$

where $\tilde{\varphi}\in(\underline{\varphi},\kappa)$ is the unique solution to $E(\tilde{\varphi})=0$. So far, in this construction we have assumed $\underline{\varphi}<\tilde{\varphi}$. Next we consider the case with $\tilde{\varphi}=\underline{\varphi}$, which obtains when $\iota=\varrho(\underline{\varphi})$. In this case, an equilibrium is a vector $(\tilde{z},\varphi,\tilde{\zeta}_{(0)},\rho)$, with $\tilde{\zeta}_{(0)}\in[0,1]$, that satisfies

$$\begin{aligned}0 &= \alpha_C[\alpha_P+(1-\alpha_P)\tilde{\zeta}_{(0)}][\tilde{z}-\mathrm{D}(\varphi)N_C]\\ &\quad+\alpha_P\left\{\alpha_C\mathrm{D}(\varphi)N_C+(1-\alpha_C)\min\left[\mathrm{D}(\underline{\varphi})N_C,\tilde{z}\right]\right\} \qquad (59)\\ \varphi &= \varphi^n \qquad (60)\\ \iota &= \alpha_C\theta_C\rho+(1-\alpha_C\theta_C)\left[\frac{u'\left(\frac{1}{N_C}\min\left[\mathrm{D}(\underline{\varphi})N_C,\tilde{z}\right]\right)}{\underline{\varphi}}-1\right] \qquad (61)\\ \rho &= \frac{1}{\alpha_P\theta_P}\frac{\kappa-\underline{\varphi}}{\underline{\varphi}}. \qquad (62)\end{aligned}$$

Notice that (61) implies

$$\tilde{z}\begin{cases}\in[\mathrm{D}(\underline{\varphi})N_C,\infty) & \text{if } \iota=\frac{\alpha_C\theta_C}{\alpha_P\theta_P}\frac{\kappa-\underline{\varphi}}{\underline{\varphi}}\\ \mathrm{D}\left(\left[1+\frac{\iota-\frac{\alpha_C\theta_C}{\alpha_P\theta_P}\frac{\kappa-\underline{\varphi}}{\underline{\varphi}}}{1-\alpha_C\theta_C}\right]\underline{\varphi}\right)N_C & \text{if } \frac{\alpha_C\theta_C}{\alpha_P\theta_P}\frac{\kappa-\underline{\varphi}}{\underline{\varphi}}<\iota.\end{cases}$$

From the previous analysis we know that $E(\underline{\varphi})=0$ if and only if $\iota=\varrho(\underline{\varphi})$, and $\frac{\alpha_C\theta_C}{\alpha_P\theta_P}\frac{\kappa-\underline{\varphi}}{\underline{\varphi}}<\varrho(\underline{\varphi})$, so $E(\underline{\varphi})=0$ implies

$$\begin{aligned}\tilde{z} &= \mathrm{D}\left(\left[1+\frac{\varrho(\underline{\varphi})-\frac{\alpha_C\theta_C}{\alpha_P\theta_P}\frac{\kappa-\underline{\varphi}}{\underline{\varphi}}}{1-\alpha_C\theta_C}\right]\underline{\varphi}\right)N_C\\ &= \frac{\alpha_C(1-\alpha_P)}{\alpha_P+\alpha_C(1-\alpha_P)}\mathrm{D}(\varphi)N_C,\end{aligned}$$

where φ is given by (60). Hence,

$$\tilde{z} - \mathrm{D}(\varphi) N_C = -\frac{\alpha_P}{\alpha_P + \alpha_C (1 - \alpha_P)} \mathrm{D}(\varphi) N_C$$

and with this, condition (59) implies $\tilde{\zeta}_{(0)} = 1$.

Configuration 2. $0 < \rho$, $\omega_C = 1$, and $\tilde{\varphi} < \underline{\varphi}$. Under this conjecture, the equilibrium conditions (45)-(49) imply $(\tilde{z}, \varphi, \tilde{\varphi}, \rho)$ must satisfy

$$0 = \alpha_C \tilde{z} + (1 - \alpha_C) \min [\mathrm{D}(\tilde{\varphi}) N_C, \tilde{z}] \tag{63}$$

$$\varphi = \varphi^n \tag{64}$$

$$\iota = \alpha_C \theta_C \rho + (1 - \alpha_C \theta_C) \left[\frac{u'\left(\frac{1}{N_C} \min [\mathrm{D}(\tilde{\varphi}) N_C, \tilde{z}]\right)}{\tilde{\varphi}} - 1 \right] \tag{65}$$

$$\rho = \frac{\varphi - \tilde{\varphi}}{\tilde{\varphi}}. \tag{66}$$

Notice that (63) can only hold if $\tilde{z} = 0$, so this configuration is inconsistent with monetary equilibrium.

Configuration 3. $0 < \rho$, $\omega_C = 0$, and $\underline{\varphi} \le \tilde{\varphi}$. Under this conjecture, the equilibrium conditions (45)-(49) imply $(\tilde{z}, \varphi, \tilde{\varphi}, \rho)$ must satisfy

$$\begin{aligned} \rho &= \iota \\ \tilde{z} &= \alpha_C (1 - \alpha_P) \mathrm{D}(\varphi) N_C \\ \varphi &= \frac{1 + \iota}{1 + \alpha_P \theta_P \iota} \kappa \\ \tilde{\varphi} &= \frac{\kappa}{1 + \alpha_P \theta_P \iota}. \end{aligned}$$

For $\omega_C = 0$ to be part of equilibrium, we need two conditions to hold: (a) $\bar{v}^C < \bar{v}^B$; and (b) $\underline{\varphi} \le \tilde{\varphi}$. The former is equivalent to

$$u'(0) - \kappa < \left[\kappa - \alpha_P \theta_P u'(0)\right] \iota \tag{67}$$

and the latter is equivalent to

$$\iota \le \frac{1}{\alpha_P \theta_P} \frac{\kappa - \underline{\varphi}}{\underline{\varphi}}. \tag{68}$$

The maintained assumption $\varphi^n < u'(0)$ is equivalent to

$$\left(\frac{1}{\alpha_P \theta_P} - 1\right)(\kappa - \underline{\varphi}) < u'(0) - \kappa$$

and therefore implies $0 < u'(0) - \kappa$. Hence (67) can only hold if $0 < \kappa - \alpha_P \theta_P u'(0)$, and in this case conditions (67) and (68) can be summarized as

$$\frac{u'(0) - \kappa}{\kappa - u'(0)\,\alpha_P \theta_P} < \iota \leq \frac{1}{\alpha_P \theta_P} \frac{\kappa - \underline{\varphi}}{\underline{\varphi}}.$$

Notice that

$$\frac{u'(0) - \kappa}{\kappa - u'(0)\,\alpha_P \theta_P} < \frac{1}{\alpha_P \theta_P} \frac{\kappa - \underline{\varphi}}{\underline{\varphi}}$$

if and only if $u'(0) < \varphi^n$, which is contradicts our maintained assumption. Thus, this equilibrium configuration cannot be an equilibrium.

Configuration 4. $0 < \rho$, $\omega_C = 0$, and $\tilde{\varphi} < \underline{\varphi}$. Under this conjecture, the equilibrium conditions (45) and (46) imply $\tilde{z} = 0$, so this configuration is inconsistent with monetary equilibrium.

Configuration 5. $\rho = 0$, $\omega_C = 1$, and $\underline{\varphi} < \varphi = \tilde{\varphi} = \kappa$. Under this conjecture, the equilibrium conditions (45)-(49) imply $(\tilde{z}, \varphi, \tilde{\varphi}, \rho, [\hat{\varkappa}^m_{i(0)}]_{i \in \{B,C,P\}})$ must satisfy

$$\begin{aligned}
\iota &= (1 - \alpha_C \theta_C) \left[\frac{u'\left(\frac{1}{N_C} \min\left[\mathrm{D}(\kappa)\, N_C, \tilde{z}\right] \right)}{\kappa} - 1 \right] && (69) \\
0 &= -(1 - \alpha_P)\, \alpha_C \mathrm{D}(\kappa)\, N_C \\
&\quad + \alpha_P (1 - \alpha_C) \min\left[\mathrm{D}(\kappa)\, N_C, \tilde{z}\right] \\
&\quad + \left[\alpha_C - \left(\hat{\varkappa}^m_{B(0)} N_B + \alpha_C \hat{\varkappa}^m_{C(0)} N_C + \alpha_P \hat{\varkappa}^m_{P(0)} N_P \right) \right] \tilde{z} && (70) \\
\varphi &= \tilde{\varphi} = \kappa && (71) \\
\rho &= 0 && (72) \\
\hat{\varkappa}^m_{i(0)} &\in [0, \infty], \text{ for } i \in \{B, C, P\}. && (73)
\end{aligned}$$

Condition (69) implies

$$\tilde{z} \begin{cases} = 0 & \text{if } \iota = (1 - \alpha_C \theta_C) \left[\frac{u'(0)}{\kappa} - 1 \right] \\ = \mathrm{D}\left(\frac{1 - \alpha_C \theta_C + \iota}{1 - \alpha_C \theta_C} \kappa \right) N_C & \text{if } 0 < \iota < (1 - \alpha_C \theta_C) \left[\frac{u'(0)}{\kappa} - 1 \right] \\ \in \left[\mathrm{D}(\kappa)\, N_C, \infty\right) & \text{if } \iota = 0. \end{cases} \qquad (74)$$

Condition (70) can be written as

$$\hat{\varkappa}^m_{B(0)} N_B + \alpha_C \hat{\varkappa}^m_{C(0)} N_C + \alpha_P \hat{\varkappa}^m_{P(0)} N_P = \alpha_C \left[1 - (1 - \alpha_P) \frac{\mathrm{D}(\kappa)\, N_C}{\tilde{z}} \right] + \alpha_P (1 - \alpha_C) \min\left[\frac{\mathrm{D}(\kappa)\, N_C}{\tilde{z}}, 1 \right].$$

From (74), we know that as long as $0 < \iota$, $\tilde{z} = \text{D}\left(\frac{1-\alpha_C\theta_C+\iota}{1-\alpha_C\theta_C}\kappa\right)N_C < \text{D}(\kappa)N_C$, so condition (70) implies

$$\hat{\varkappa}^m_{B(0)}N_B + \alpha_C\hat{\varkappa}^m_{C(0)}N_C + \alpha_P\hat{\varkappa}^m_{P(0)}N_P = \alpha_C\left[1-(1-\alpha_P)\frac{\text{D}(\kappa)N_C}{\tilde{z}}\right] + (1-\alpha_C)\alpha_P$$

From (73) we know that the left side must be a nonnegative number, so for this condition to be satisfied in equilibrium, the right side must be nonnegative, i.e., we must have

$$\frac{\alpha_C(1-\alpha_P)}{\alpha_P+\alpha_C(1-\alpha_P)}\text{D}(\kappa) \le \frac{1}{N_C}\tilde{z}.$$

With (74), this inequality can be written as

$$\frac{\alpha_C(1-\alpha_P)}{\alpha_P+\alpha_C(1-\alpha_P)}\text{D}(\kappa) \le \text{D}\left(\frac{1-\alpha_C\theta_C+\iota}{1-\alpha_C\theta_C}\kappa\right)$$

or equivalently $\iota \le \varrho(\kappa)$. Therefore, this configuration is a monetary equilibrium for any ι that satisfies $0 \le \iota \le \varrho(\kappa)$. ■

The following corollary of Proposition 6 characterizes the limit of the stationary monetary equilibrium as $\alpha_P \to 1$.

Corollary 7. *Assume $\varphi^{n\star} < u'(0) < \infty$, and define the function $\varsigma : \mathbb{R}_+ \to \mathbb{R}$ as*

$$\varsigma(x) \equiv \frac{\alpha_C\theta_C}{\theta_P}\frac{\kappa-x}{x} + (1-\alpha_C\theta_C)\left[\frac{u'(0)}{x}-1\right].$$

Consider the monetary equilibrium characterized in Proposition 6. As $\alpha_P \to 1$,

(i) If $\varsigma(\kappa) < \iota < \varsigma(\underline{\varphi})$, then

$$\begin{aligned}
\tilde{\varphi} &\to \tilde{\varphi}^\star \equiv \left[1-\theta_P\frac{\iota-\varsigma(\kappa)}{(1-\theta_P)\alpha_C\theta_C+\theta_P(1+\iota)}\right]\kappa \in (\underline{\varphi},\kappa)\\
\varphi &\to \varphi^\star \equiv \kappa + \frac{1-\theta_P}{\theta_P}(\kappa-\tilde{\varphi}^\star)\\
\rho &\to \frac{1}{\theta_P}\frac{\kappa-\tilde{\varphi}^\star}{\tilde{\varphi}^\star}\\
\frac{Z}{\tilde{\varphi}} &\to 0\\
\bar{y}_C &\to \bar{y}^\star_C, \text{ where } \bar{y}^\star_C \text{ satisfies } u'(\bar{y}^\star_C) = \varphi^\star\\
\tilde{y}_C &\to 0.
\end{aligned}$$

(ii) If $0 < \iota \le \varsigma(\kappa)$, then $\tilde{\varphi}$, φ, ρ, Z, $\bar{y}_C$, and $\tilde{y}_C$ remain as in part (ii) of Proposition 6.

C.2. Economy with $\alpha_C \theta_C = 1$

C.2.1. Stationary monetary equilibrium

Proof of Proposition 3. From Lemma 6, in an economy with $\alpha_C \theta_C = 1$, a stationary monetary equilibrium is a vector

$$(Z, \varphi, \tilde{\varphi}, \tilde{\zeta}_{(\tilde{\varphi}-\underline{\varphi})}, \bar{\zeta}_{(\varphi-\underline{\varphi})}, \varkappa^{p}_{(\kappa-R^{m}(\tilde{\varphi},\varphi))}, [\omega_i, \hat{\varkappa}^{m}_{i(\rho)}]_{i\in\{B,C,P\}})$$

with $Z > 0$ that satisfies the market-clearing conditions

$$0 = \sum\nolimits_{i\in\{B,C,P\}} \omega_i - 1 \tag{75}$$

$$0 = \mathrm{D}(\varphi) N_C - [\alpha_P \bar{\zeta}_{(\varphi-\underline{\varphi})} + (1-\alpha_P)\tilde{\zeta}_{(\tilde{\varphi}-\underline{\varphi})}] N_P \varkappa^{p}_{(\kappa-R^{m}(\tilde{\varphi},\varphi))} \tag{76}$$

$$\begin{aligned} 0 = {} & (\omega_B - \hat{\varkappa}^{m}_{B(\rho)} N_B) Z \\ & + (\omega_C - \hat{\varkappa}^{m}_{C(\rho)} N_C) Z - \tilde{\varphi} \mathrm{D}(\varphi) N_C \\ & + \alpha_P [(\omega_P - \hat{\varkappa}^{m}_{P(\rho)} N_P) Z + \tilde{\varphi} \bar{\zeta}_{(\varphi-\underline{\varphi})} \varkappa^{p}_{(\kappa-R^{m}(\tilde{\varphi},\varphi))} N_P] \end{aligned} \tag{77}$$

and the optimality conditions

$$0 = [-\kappa + R^m(\tilde{\varphi}, \varphi)] \varkappa^{p}_{(\kappa-R^{m}(\tilde{\varphi},\varphi))} \tag{78}$$

$$0 = \left(-\mu + \beta \bar{v}^i\right) \omega_i \text{ for } i \in \{B, C, P\}, \tag{79}$$

where

$$\begin{aligned} \bar{v}^B &\equiv 1 + \rho \\ \bar{v}^P &\equiv 1 + \alpha_P \theta_P \rho \\ \bar{v}^C &\equiv 1 + \rho \\ R^m(\tilde{\varphi}, \varphi) &\equiv \underline{\varphi} + \alpha_P \theta_P (\varphi - \underline{\varphi}) \mathbb{I}_{\{\underline{\varphi}<\varphi\}} + (1 - \alpha_P \theta_P)(\tilde{\varphi} - \underline{\varphi}) \mathbb{I}_{\{\underline{\varphi}<\tilde{\varphi}\}} \\ \rho &= \frac{\varphi - \tilde{\varphi}}{\tilde{\varphi}}. \end{aligned} \tag{80}$$

First, we know that $\tilde{\varphi} \leq \varphi$, since $0 \leq \rho$ must hold in any equilibrium. Second, in any equilibrium in which good 1 is produced, we must have: (a) $\kappa = R^m(\tilde{\varphi}, \varphi)$, or equivalently,

$$\kappa = \underline{\varphi} + \alpha_P \theta_P (\varphi - \underline{\varphi}) \mathbb{I}_{\{\underline{\varphi}<\varphi\}} + (1 - \alpha_P \theta_P)(\tilde{\varphi} - \underline{\varphi}) \mathbb{I}_{\{\underline{\varphi}<\tilde{\varphi}\}}. \tag{81}$$

(b) $\underline{\varphi} < \varphi$, i.e., banked producers never store output. To see why this must be the case, notice that if $\varphi \leq \underline{\varphi}$, then we know that $\tilde{\varphi} \leq \varphi \leq \underline{\varphi}$, and therefore $R^m(\tilde{\varphi}, \varphi) = \underline{\varphi} < \kappa$ which implies good 1 is never produced. (c) If $\tilde{\varphi}=\varphi$, then (50) implies $\tilde{\varphi}=\varphi=\kappa>\underline{\varphi}$. Third, from the Euler equations for money, we know that $\omega_P=0$ (since $\bar{v}^P<\bar{v}^C$), and that any pair $\omega_B,\omega_C\in[0,1]$ with $\omega_B+\omega_P=1$ is consistent with equilibrium (since $\bar{v}^B=\bar{v}^C$). From the previous observations we know that the indeterminacy in the pair (ω_B,ω_C) is immaterial, and also that $\rho=0$ implies $\varphi=\tilde{\varphi}=\kappa$, so there are only three possible relevant equilibrium configurations. Next, we consider each in turn. In every case, let $\tilde{z}\equiv Z/\tilde{\varphi}$.

Configuration 1. $0<\rho$ and $\underline{\varphi}\leq\tilde{\varphi}$. Under this conjecture, the equilibrium conditions (76)-(80) imply $(\tilde{z},\varphi,\tilde{\varphi},\rho)$ must satisfy

$$\begin{aligned}\tilde{z} &= (1-\alpha_P)\,\mathrm{D}(\varphi)\,N_C\\ \varphi &= \kappa+\frac{1-\alpha_P\theta_P}{\alpha_P\theta_P}(\kappa-\tilde{\varphi})\\ \rho &= \iota\\ \tilde{\varphi} &= \frac{\kappa}{1+\alpha_P\theta_P\iota}.\end{aligned}$$

For this to be an equilibrium, it only remains to check that $\underline{\varphi}\leq\tilde{\varphi}$, which is equivalent to $\iota\leq\bar{\iota}$, with $\bar{\iota}$ as defined in (16).

Configuration 2. $0<\rho$ and $\tilde{\varphi}<\underline{\varphi}$. Under this conjecture, the equilibrium conditions (76) and (77) imply $\tilde{z}=0$, so this configuration is inconsistent with monetary equilibrium.

Configuration 3. $\rho=0$ and $\underline{\varphi}<\varphi=\tilde{\varphi}=\kappa$. Under this conjecture, the equilibrium conditions (45)-(49) imply $(\tilde{z},\varphi,\tilde{\varphi},\rho,[\hat{\varkappa}^m_{i(0)}]_{i\in\{B,C,P\}})$ must satisfy

$$\rho = \iota=0 \tag{82}$$

$$\varphi = \tilde{\varphi}=\kappa \tag{83}$$

$$0 = -(1-\alpha_P)\,\mathrm{D}(\kappa)\,N_C + \left[1-\left(\hat{\varkappa}^m_{B(0)}N_B+\hat{\varkappa}^m_{C(0)}N_C+\alpha_P\hat{\varkappa}^m_{P(0)}N_P\right)\right]\tilde{z} \tag{84}$$

$$\hat{\varkappa}^m_{i(0)} \in [0,\infty],\text{ for } i\in\{B,C,P\}. \tag{85}$$

Condition (84) implies

$$\hat{\varkappa}^m_{B(0)}N_B+\hat{\varkappa}^m_{C(0)}N_C+\alpha_P\hat{\varkappa}^m_{P(0)}N_P=1-(1-\alpha_P)\frac{\mathrm{D}(\kappa)\,N_C}{\tilde{z}}.$$

From (85) we know that the left side must be a nonnegative number, so for this condition to be satisfied in equilibrium, the right side must be nonnegative, i.e., we must have

$$(1-\alpha_P)\,\mathrm{D}(\kappa)\,N_C\leq\tilde{z}.$$

Therefore, the monetary equilibrium is given by (82), (83), and (85), and any

$$\tilde{z} \in [(1-\alpha_P)\,\mathrm{D}\,(\kappa)\,N_C, \infty) \cap \mathbb{R}_{++}.$$

■

C.2.2. Dynamic deterministic monetary equilibrium

In this section we characterize deterministic dynamic monetary equilibria for the economy with $\alpha_C\theta_C = 1$ (where $\omega_{Ct} = 1$ for all t). To simplify the analysis, we consider a version of the model where: (*i*) $N_B = N_C = N_P = 1$ (a normalization); (*ii*) good 1 is produced in equilibrium; and (*iii*) $\underline{\varphi} = 0$ (no storage). These conditions imply $\varkappa^p_{(\kappa - R^m(\tilde{\varphi}_t,\varphi_t))} = \bar{\zeta}_{(\varphi_t - \underline{\varphi})} = \tilde{\zeta}_{(\tilde{\varphi}_t - \underline{\varphi})} = 1$ for all t, and $\varphi^n = \frac{\kappa}{\alpha_P\theta_P}$. The following result is a corollary of Lemma 6.

Corollary 8. *A monetary equilibrium can be characterized by a bounded sequence*

$$\{Z_t, \varphi_t, \tilde{\varphi}_t, \rho_t, [\hat{\varkappa}^m_{it(\rho_t)}]_{i\in\{B,C,P\}}\}_{t=0}^{\infty}$$

that satisfies

$$(1-\alpha_P)\,\tilde{\varphi}_t\mathrm{D}\,(\varphi_t) = \left(1 - \hat{\varkappa}^m_{Bt(\rho_t)} - \hat{\varkappa}^m_{Ct(\rho_t)} - \alpha_P\hat{\varkappa}^m_{Pt(\rho_t)}\right) Z_t \tag{86}$$

$$Z_t = \frac{1}{1+\iota}\frac{(1-\alpha_P\theta_P)\,\varphi_{t+1}}{\kappa - \alpha_P\theta_P\varphi_{t+1}} Z_{t+1} \tag{87}$$

$$\rho_t = \frac{1}{\alpha_P\theta_P}\frac{\varphi_t - \kappa}{\varphi^n - \varphi_t} \tag{88}$$

$$\tilde{\varphi}_t = \frac{\kappa - \alpha_P\theta_P\varphi_t}{1-\alpha_P\theta_P} \tag{89}$$

$$\hat{\varkappa}^m_{it(\rho_t)} \begin{cases} \in [0,\infty] & \textit{if } \rho_t = 0 \\ = 0 & \textit{if } 0 < \rho_t. \end{cases} \tag{90}$$

Proof of Proposition 4. The proof builds on Corollary 8. Consider two cases. (*i*) Suppose $\rho_{t+1} = 0$. Then (88) implies $\varphi_{t+1} = \kappa$, and then (87) implies

$$z_t = \frac{1}{1+\iota} z_{t+1}. \tag{91}$$

(*ii*) Suppose $0 < \rho_{t+1}$, then (86), (89), and (90) imply $z_{t+1} = h\left(\varphi_{t+1}\right)$, where

$$h\left(\varphi_{t+1}\right) \equiv \frac{\alpha_P\theta_P}{1-\alpha_P\theta_P}\left(\varphi^n - \varphi_{t+1}\right)\mathrm{D}\left(\varphi_{t+1}\right).$$

Notice that $h' < 0$, and

$$h(\varphi^n) = 0 < h(\kappa) = \kappa \mathrm{D}(\kappa),$$

so for every $z_{t+1} \in [0, \kappa \mathrm{D}(\kappa)]$, there exists a unique $\varphi_{t+1} \in [\kappa, \varphi^n]$ given by $\varphi_{t+1} = f(z_{t+1})$, where $f(z_{t+1}) \equiv h^{-1}(z_{t+1})$. By substituting $\varphi_{t+1} = f(z_{t+1})$ into (87), we obtain

$$z_t = \frac{1}{1+\iota} \frac{1 - \alpha_P \theta_P}{\alpha_P \theta_P} \frac{f(z_{t+1})}{\varphi^n - f(z_{t+1})} z_{t+1}. \tag{92}$$

The condition for case (ii) is $0 < \rho_{t+1}$, i.e.,

$$\rho_{t+1} = \frac{1}{\alpha_P \theta_P} \frac{f(z_{t+1}) - \kappa}{\varphi^n - f(z_{t+1})} > 0.$$

This condition is equivalent to $0 < f(z_{t+1}) - \kappa$, which is in turn equivalent to $z_{t+1} < h(\kappa) = \kappa \mathrm{D}(\kappa)$. Therefore, putting (91) (from case (i)) and (92) (from case (ii)) together, an equilibrium path $\{z_t\}_{t=0}^{\infty}$ must satisfy (17). Once the path $\{z_t\}_{t=0}^{\infty}$ that solves (17) has been found, the equilibrium path for $\{\varphi_t\}_{t=0}^{\infty}$, $\{\rho_t\}_{t=0}^{\infty}$, and $\{\tilde{\varphi}_t\}_{t=0}^{\infty}$, are obtained from $\varphi_t = f(z_t)$, (88), and (89), respectively. ■

Proof of Corollary 1. The key observation is that $\varphi_t = f(z_t) < f(0) = \varphi^n$ for all t. The equalities follow from (21). The inequality follows from $f' < 0$, and the fact that in any monetary equilibrium, $0 < z_t$ for all t. ■

Proof of Corollary 4. The expressions (27)-(31) are immediate from (17)-(21). Next, we show that $z_t^\star > 0$ holds for all t, i.e., that the quantity of real money balances per producer with no access to a banker, remains strictly positive as we take the pure-credit limit (i.e., $\alpha_P \to 1$) of any dynamic monetary equilibrium. For this, it suffices to show that $0 < G(0)$, where for any $z_{t+1}^\star \in \mathbb{R}_+$,

$$G(z_{t+1}^\star) \equiv \frac{1}{1+\iota} \frac{1 - \theta_P}{\theta_P} \frac{g(z_{t+1}^\star)}{\varphi^{n\star} - g(z_{t+1}^\star)} z_{t+1}^\star.$$

From (31),

$$g'(z_{t+1}^\star) = \frac{1 - \theta_P}{\theta_P} \frac{1}{(\varphi^{n\star} - g(z_{t+1}^\star)) \mathrm{D}'(g(z_{t+1}^\star)) - \mathrm{D}(g(z_{t+1}^\star))},$$

so using L'Hôpital's rule

$$\begin{aligned} G(0) &= -\frac{1}{1+\iota}\frac{1-\theta_P}{\theta_P}\frac{\varphi^{n\star}}{\lim_{z^\star_{t+1}\to 0} g'\left(z^\star_{t+1}\right)} \\ &= -\frac{1}{1+\iota}\frac{\varphi^{n\star}}{\lim_{z^\star_{t+1}\to 0}\frac{1}{\left(\varphi^{n\star}-g\left(z^\star_{t+1}\right)\right)\text{D'}\left(g\left(z^\star_{t+1}\right)\right)-\text{D}\left(g\left(z^\star_{t+1}\right)\right)}} \\ &= \frac{\varphi^{n\star}\text{D}\left(\varphi^{n\star}\right)}{1+\iota} > 0. \end{aligned} \tag{93}$$

Suppose $z^\star_t = 0$ for some t. Then for (27) to hold, it is necessary that $G\left(z^\star_{t+1}\right) = 0$, but this is impossible since (27) and (93) imply $0 < G\left(z^\star_{t+1}\right)$ for all $z^\star_{t+1} \geq 0$. Thus $0 < z^\star_t$ for all t. We conclude that $\varphi^\star_t = g\left(z^\star_t\right) < g(0) = \varphi^{n\star}$, where the equalities follow from (31), and the inequality follows from the fact that $g' < 0 < z^\star_t$ for all t. ■

C.2.3. Sunspot equilibria

In this section we construct equilibria where prices and allocations are time-invariant functions of a *sunspot*, i.e., a random variable on which agents may coordinate actions but that does not directly affect any primitives, including endowments, preferences, and production or trading possibilities. Specifically, let $\mathbb{S} = \{s_1, ..., s_N\}$ denote the support of the sunspot, and assume $s_t \in \mathbb{S}$ follows a Markov chain, $\eta_{ij} = \Pr\left(s_{t+1} = s_i | s_t = s_j\right)$. The following corollary of Lemma 6 summarizes the conditions that characterize a recursive monetary sunspot equilibrium. For simplicity, we assume $\underline{\varphi} = 1$ and $N_B = N_C = N_P = \alpha_C\theta_C = 1$ (and without loss, focus on equilibria where only consumers hold money between periods).

Corollary 9. *A recursive monetary sunspot equilibrium is a collection of functions of s,*

$$\left\langle Z(s), \varphi(s), \tilde{\varphi}(s), \rho(s), \varkappa^p_{(\kappa - R^m(\tilde{\varphi}(s),\varphi(s)))}, [\hat{\varkappa}^m_{i(\rho(s))}]_{i\in\{B,C,P\}} \right\rangle,$$

that satisfies the market-clearing conditions

$$\begin{aligned} 0 &= \text{D}\left(\varphi(s)\right) - \varkappa^p_{(\kappa - R^m(\tilde{\varphi}(s),\varphi(s)))} \\ 0 &= \left(1 - \hat{\varkappa}^m_{B(\rho(s))} - \hat{\varkappa}^m_{C(\rho(s))} - \alpha_P\hat{\varkappa}^m_{P(\rho(s))}\right) Z(s) \\ &\quad + \alpha_P\tilde{\varphi}(s)\,\varkappa^p_{(\kappa - R^m(\tilde{\varphi}(s),\varphi(s)))} - \tilde{\varphi}(s)\,\text{D}\left(\varphi(s)\right) \end{aligned}$$

and the optimality conditions

$$
\begin{aligned}
0 &= \left[-\kappa + R^m\left(\tilde{\varphi}(s), \varphi(s)\right)\right] \varkappa^p_{(\kappa - R^m(\tilde{\varphi}(s),\varphi(s)))} \\
Z(s_i) &= \frac{1}{1+\iota}\sum_{j=1}^{N} \eta_{ij}\left(1+\rho(s_j)\right) Z(s_j) \ \text{ for all } s_i \in \mathbb{S},
\end{aligned}
$$

where

$$
\begin{aligned}
R^m\left(\tilde{\varphi}(s), \varphi(s)\right) &\equiv \alpha_P\theta_P\varphi(s) + (1-\alpha_P\theta_P)\tilde{\varphi}(s) \\
\rho(s) &= \frac{\varphi(s) - \tilde{\varphi}(s)}{\tilde{\varphi}(s)}
\end{aligned}
$$

and for all $i \in \{B, C, P\}$,

$$
\hat{\varkappa}^m_{i(\rho(s))} \begin{cases} = \infty & \text{if } \rho(s) < 0 \\ \in [0, \infty] & \text{if } \rho(s) = 0 \\ = 0 & \text{if } 0 < \rho(s). \end{cases}
$$

Proof of Proposition 5. It is easy to check that the proposed equilibrium satisfies all the equilibrium conditions in Corollary 9. ■