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ABSTRACT

The government budget constraint ties the market value of government debt to the expected present discounted value of fiscal surpluses. Bond investors fail to impose this no-arbitrage restriction in the U.S., resulting in a government debt valuation puzzle. Both cyclical and long-run dynamics of tax revenues and government spending make the surplus claim risky. Under a realistic asset pricing model, this risk in surpluses creates a wedge of 299% of GDP between the value of debt and that the surplus claim, and implies an expected return on the debt portfolio that far exceeds the observed yield on Treasuries.

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1 Introduction

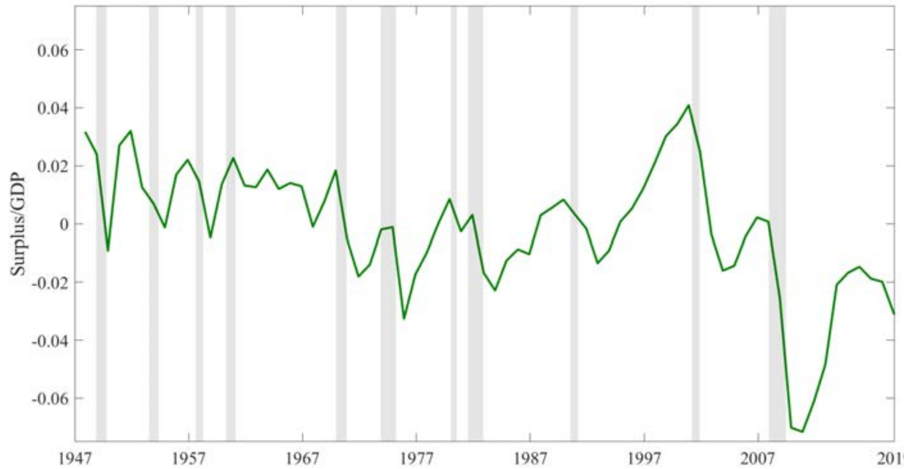
The U.S. Treasury is the largest borrower in the world. At the end of 2019, outstanding federal government debt held by the public was valued at $17 trillion. It doubled after the Great Recession to 78.4% of the U.S. annual GDP. Before the financial crisis, there was widespread concern that the U.S. had embarked on an unsustainable fiscal path (see, e.g., Rubin, Orszag, and Sinai, 2004). Yet, recently, some economists have argued that the U.S. has ample debt capacity to fund additional spending by rolling over its debt because interest rates are below GDP growth rates (Blanchard, 2019). As a case in point, the massive spending increase in response to the covid pandemic is forecast to generate a deficit of 19% of GDP in 2020 ($3.8 trillion) and to increase the debt to 100% of GDP. The debt increase met with little resistance from bond markets so far.

The central idea in this paper is to price the entire portfolio of outstanding Treasury debt, rather than individual bond securities. By the government's dynamic budget constraint and in the absence of bubbles, the market value of outstanding debt must equal the present discounted value of current and future primary surpluses. By the same logic, the expected return on the debt portfolio has to reflect the risk profile of primary surpluses. However, we find that the value of the bond portfolio exceeds the value of the surplus claim, a gap we label *the government debt valuation puzzle*, and that yields on the Treasury bond portfolio are lower than the relevant "interest rate" bond investors ought to be earning, *the government risk premium puzzle*.

To see why, note that the price of a stock is the expected present discount value of future dividends. Risk-free interest rates are below dividend growth rates, yet the price of the stock is finite. Since the stock's dividend growth is pro-cyclical, its cash flows are low when the investor's marginal utility is high. The relevant "interest rate" for the stock contains a risk premium because of the risk exposure of its cash flow. Analogously, a portfolio strategy that buys all new Treasury issues and receives all Treasury coupon and principal payments has as its cash flow the primary surplus of the federal government. Primary surpluses are strongly pro-cyclical just like stock dividends, as shown in Figure 1. Spending by the federal government increases in recessions, while the progressive nature of the tax system produces sharply pro-cyclical revenue. In recessions, when marginal utility is high, surpluses are negative and net bond issuance is high. The Treasury portfolio cash flows have substantial business cycle risk. As explained below, tax revenue and spending also have substantial long-run risk due to cointegration with GDP. Taken together, the relevant "interest rate" for surpluses contains a substantial risk premium reflecting both short- and long-run risk exposures.

The value of the surplus claim is obtained as the difference between the value of a claim to future federal tax revenues, P_t^T, and the value of a claim to future federal spending excluding debt service, P_t^G. The pro-cyclicality of tax revenues makes the tax revenue claim risky; P_t^T is low. The

Figure 1: Government Cash Flows



The figure plots the U.S. federal government primary surplus as a fraction of GDP. The construction of the primary surplus is detailed in Appendix C.1. The data source is NIPA Table 3.2. The sample period is from 1947 to 2019.

counter-cyclicality makes the spending claim safer; P_t^G is high. Quantitatively, we find that the value of the surplus claim, $P_t^S = P_t^T - P_t^G$, has averaged -260.37 percentage of GDP. The market value of outstanding debt has averaged 0.38 times GDP over the same period. The gap is 3.0 times GDP on average over our sample, and has widened dramatically in the last twenty years.

The above argument relies on a realistic model of quantities and prices of risk. When modeling the quantity of risk in fiscal cash flows, adequately capturing the dynamics of government spending and tax revenue is crucial. We model the growth rates of tax revenues-to-GDP and government spending-to-GDP in a VAR alongside macro-economic and financial variables. This structure allows us to capture the cyclical properties of fiscal cash-flows. A second important feature of fiscal cash flows is that tax revenues and spending are co-integrated with GDP, so that revenues, spending, and GDP adjust when revenue-to-GDP or spending-to-GDP are away from their long-run relationship. This imposes a form of long-run automatic stabilization, as discussed by Bohn (1998). With cointegration, GDP innovations permanently alter all future surpluses. A deep recession not only raises current government spending and lowers current tax revenue as a fraction of GDP, it also lowers future spending and raises future revenue as a fraction of future GDP. Both the spending and the revenue claims are exposed to the same long-run risk as GDP.

When modeling the price of risk, we posit a state-of-the-art stochastic discount factor (SDF) model. Rather than committing to a specific utility function, we use a flexible SDF that accurately prices the nominal and real term structure of Treasury bond yields. The model also closely matches stock prices and generates an equity risk premium. The SDF contains a large permanent component (Alvarez and Jermann, 2005). The SDF model's rich implications for the term structure of risk allow it to adequately price short- and long-run risk to spending and tax revenue.

Combining features from both quantities and prices of risk, the long-run discount rates on claims to tax revenues, spending, and GDP must all be equal. A claim to GDP is akin to an unlevered equity claim. In any reasonable asset pricing model with a large permanent component in the SDF, the unlevered equity risk premium exceeds the yield on a long-term government bond (Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and Chernov, 2018). The discount rate for revenues and spending is high. Because of the dynamic government budget constraint, the relevant "interest rate" on the portfolio of government debt must also be high. Treasury investors seem willing to purchase government debt at low yields. The historical return on the U.S. government debt portfolio is only 1.11% in excess of the T-bill rate.

An important consequence is that the risk-free rate cannot be the right discount rate for future surpluses and hence for government debt. While one can roll over a constant dollar amount at the risk-free rate, one cannot roll over a cash flow stream that is pro-cyclical and co-integrated with GDP at the risk-free rate. The latter cash flow stream carries a substantial risk premium. Yet, it is commonplace in the literature to discount government surpluses at the one-period risk-free rate.

Furthermore, if the debt were truly risk-free, then the present value of surpluses would also be risk-free and hence not respond to fiscal shocks. Hansen, Roberds, and Sargent (1991) refer to this as the fiscal measurability condition. This condition imposes that any current increase in spending or decrease in revenue during recessions is fully offset (in present value terms) by future decreases in spending and/or increases in revenue. We find no evidence for such offsets in the data. This should not be surprising. There are no built-in offsets in non-discretionary spending or in the tax system. And politicians have displayed little fiscal discipline on discretionary spending. Instead, we find that the surplus claim responds strongly to economic shocks, much more so than the value of debt. This amounts to a severe violation of the measurability constraints. Put differently, the valuation of the outstanding debt is not responsive enough to news about the fundamentals. U.S. Treasury investors seem largely oblivious to fiscal news, except during the "bond market vigilante" episode of 1993-94. The "excess smoothness" in the Treasury market stands in contrast to the excess volatility in stock markets.

In the last part of the paper, we study several potential resolutions of the government bond valuation and risk premium puzzles. First, the valuation gap can be interpreted as a violation of the transversality condition in the Treasury market, due to a rational bubble. Rational bubbles are unlikely in the presence of long-lived investors unless there are severe limits to arbitrage. Second, the U.S. Treasury may earn a convenience yield on the debt it issues, making Treasury yields lower than the risk-free rate. Convenience yields generate an additional source of revenue which increase the surplus. Furthermore, convenience yields are counter-cyclical and hence reduce the riskiness of the surplus stream. Despite their theoretical appeal, we find that convenience yields do not help much to explain the puzzle. Higher surpluses due to convenience are discounted at a higher rate to result in a similar valuation for the surplus claim. Third, we explore the possibility of a future large fiscal correction that is absent from our sample, but in the minds of investors who value the surplus claim. We back out from the market value of debt what probability investor assign to such an austerity event. The high probability we infer belies the nature of a peso event, and is not consistent with rational expectations. Fourth, missing government assets are too small to resolve the puzzle. Fifth, market segmentation between U.S. bond and equity markets—maybe because of the large Treasury holdings of foreign investors and the Federal Reserve—does not help because the puzzle is as large in a model that only prices bonds. In the absence of arbitrage opportunities, all bond investors must agree on the valuation of bonds.

Related Literature Our paper connects with a long literature which tests the government's inter-temporal budget constraint. Hamilton and Flavin (1986); Trehan and Walsh (1988, 1991); Hansen, Roberds, and Sargent (1991); Bohn (2007) derive general time-series restrictions on the government revenue and spending processes that enforce the government's inter-temporal budget constraint. These authors use the risk-free rate as the discount rate for surpluses. This literature suffers from a joint hypothesis problem. It tests the null hypothesis that the budget constraint holds and that the debt is risk-free so that surpluses can be priced off the risk-free yield curve. Our paper argues that risk premia on the surplus claim and hence on the government bond portfolio are not zero, where risk premia are inferred from no-arbitrage restrictions on bond and stock markets.

There is a parallel literature in asset pricing which tests the present value equation for stocks and other long-lived assets, starting with the seminal work by Shiller (1981); LeRoy and Porter (1981); Campbell and Shiller (1988). The prices of these long-lived assets seem excessively volatile relative to their fundamentals. Government debt is fundamentally different: its valuation does not seem volatile enough relative to the fundamentals.

We contribute to a recent literature at the intersection of asset pricing and public finance. Chernov, Schmid, and Schneider (2016); Pallara and Renne (2019) argue that higher CDS premia for U.S. Treasuries since the financial crisis are related to the underlying fiscal fundamentals. Our puzzle holds even when accounting for default: the market value of defaultable sovereign debt is still be backed by future surpluses. Liu, Schmid, and Yaron (2020) argue that increasing safe asset supply can be risky as more government debt increases corporate default risk premia despite providing more convenience. Croce, Nguyen, Raymond, and Schmid (2019) study cross-sectional differences in firms' exposure to government debt. Corhay, Kind, Kung, and Morales (2018) study how quantitative easing affects inflation by changing the maturity structure of government debt.

The asset pricing model combines a vector auto-regression model for the state variables as in Campbell (1991, 1993, 1996) with a no-arbitrage model for the (SDF) as in Duffie and Kan (1996); Dai and Singleton (2000); Ang and Piazzesi (2003). Lustig, Van Nieuwerburgh, and Verdelhan (2013) study the properties of the price-dividend ratio of a claim to aggregate consumption, the wealth-consumption ratio, and Gupta and Van Nieuwerburgh (2018) evaluate the performance of private equity funds in similar settings. This paper focuses on pricing a claim to government surpluses. Our paper adds novel no-arbitrage restrictions on the aggregate Treasury portfolio, in addition to the no-arbitrage restrictions on individual bonds.

There is a large literature on rational bubbles in asset markets, starting with the seminal work by Samuelson (1958); Diamond (1965); Blanchard and Watson (1982). One interpretation of our results is as a violation of the transversality condition in Treasury markets, consistent with the existence of a rational bubble. We show that a rational patient investor who pursues an investment strategy that buys all corporate equities and shorts the portfolio of all U.S. Treasuries earns a risk premium similar to the equity premium but receives cash flows that hedge the business cycle. This casts doubt on the rational bubble hypothesis, unless there are severe limits to arbitrage (Shleifer and Vishny, 1997). Giglio, Maggiori, and Stroebel (2016) devise a model-free test for bubbles in housing markets. Our test is not model-free, but the results hold in a large class of models in which permanent shocks to the pricing kernel are an important driver of risk premia.

Our work connects to the large literature on the specialness of U.S. government bonds, which finds that U.S. government bonds trade at a premium relative to other risk-free bonds (Longstaff, 2004; Krishnamurthy and Vissing-Jorgensen, 2012; Fleckenstein, Longstaff, and Lustig, 2014; Krishnamurthy and Vissing-Jorgensen, 2015; Nagel, 2016; Bai and Collin-Dufresne, 2019). Greenwood, Hanson, and Stein (2015) study the government debt's optimal maturity in the presence of such premium, and Jiang, Krishnamurthy, and Lustig (2018) study this premium in international finance. We tackle the question of how expensive a portfolio of all Treasuries is relative to the underlying collateral, a claim to surpluses. Using the standard convenience yield estimates of Krishnamurthy and Vissing-Jorgensen (2012), we find that our puzzle remains. This leaves open the possibility that convenience yields are much larger, as suggested by Jiang, Krishnamurthy, and Lustig (2018).

Our approach is to estimate processes for government spending and revenue growth from the data, and to study its implications for the riskiness of the government debt portfolio in a model with realistic asset prices. A large literature, following Barro (1979) and Lucas and Stokey (1983) estimates optimal fiscal policy. Recently, Karantounias (2018) and Bhandari, Evans, Golosov, and

Sargent (2017) bring a richer asset pricing model to this literature and study the optimal maturity structure of government debt.

The rest of the paper is organized as follows. Section 2 presents theoretical results. Section 3 describes the data. Section 4 sets up and solves the quantitative model. Section 5 documents the government risk premium puzzle in that model. Section 6 discusses potential resolutions of the puzzle. Section 7 concludes. The appendix presents proofs of the propositions, and details of model derivation and estimation.

2 Two Equivalence Results

We derive two theoretical results which are general in that they rely on the absence of arbitrage opportunities and two weak assumptions on government cash flows. The first assumption concerns the long run: tax revenues and government spending are cointegrated with GDP; they share a stochastic trend. The second assumption concerns the short-run: spending is counter-cyclical spending and tax revenues are pro-cyclical.

2.1 Value Equivalence

Let G_t denote nominal government spending before interest expenses on the debt, T_t denote nominal government tax revenue, and $S_t = T_t - G_t$ denote the nominal primary surplus. Let $P_t^\$(h)$ denote the price at time t of a nominal zero-coupon bond that pays \$1 at time $t + h$, where h is the maturity. There exists a multi-period stochastic discount factor (SDF) $M_{t,t+h}^\$ = \prod_{k=0}^h M_{t+k}^\$$ is the product of the adjacent one-period SDFs, $M_{t+k}^\$$. By no arbitrage, bond prices satisfy $P_t^\$(h) = \mathbb{E}_t\left[M_{t,t+h}^\$\right] = \mathbb{E}_t\left[M_{t+1}^\$ P_{t+1}^\$(h-1)\right]$. By convention $P_t^\$(0) = M_{t,t}^\$ = M_t^\$ = 1$ and $M_{t,t+1}^\$ = M_{t+1}^\$$. The government bond portfolio is stripped into zero-coupon bond positions $Q_{t,h}^\$$, where $Q_{t,h}^\$$ denotes the outstanding face value at time t of the government bond payments due at time $t + h$. $Q_{t-1,1}^\$$ is the total amount of debt payments that is due today. The outstanding debt reflects all past bond issuance decisions, i.e., all past primary deficits. Let D_t denote the market value of the outstanding government debt portfolio.

Proposition 1 (**Value Equivalence**). In the absence of arbitrage opportunities and subject to a transversality condition, the market value of the outstanding government debt portfolio equals the expected present discounted value of current and future primary surpluses:

$$D_t \equiv \sum_{h=0}^{H} P_t^\$(h) Q_{t-1,h+1}^\$ \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}^\$ (T_{t+j} - G_{t+j})\right] \equiv P_t^T - P_t^G, \tag{1}$$

where the cum-dividend value of the tax claim and value of the spending claim are defined as:

$$P_t^T = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$} T_{t+j} \right], \quad P_t^G = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$} G_{t+j} \right].$$

The proof is given in Appendix A. The proof relies only on the existence of a SDF, i.e., the absence of arbitrage opportunities, not on the uniqueness of the SDF, i.e., complete markets. It imposes a transversality condition (TVC) that rules out a government debt bubble: $E_t \left[M_{t,t+T} D_{t+T} \right] \to 0$ as $T \to \infty$. The market value of debt is the difference between the value of a claim to tax revenue and the value of a claim to government spending. Imposing the TVC rules out rational bubbles. We return to possible violations of the TVC in Section 6.1.

When the government runs a deficit in a future date and state, it will need to issue new bonds to the investing public. If those dates and states are associated with a high value of the SDF for the representative bond investor, that debt issuance occurs at the "wrong" time. The representative investors who buys all debt issues and participates in all redemptions need to be induced by low prices (high yields) to absorb that new debt. To see this, we can rewrite (1) as:

$$D_t = \sum_{j=0}^{\infty} P_t^{\$}(j) \mathbb{E}_t \left[S_{t+j} \right] + \sum_{j=0}^{\infty} \mathrm{Cov}_t \left(M_{t,t+j}^{\$}, T_{t+j} \right) - \sum_{j=0}^{\infty} \mathrm{Cov}_t \left(M_{t,t+j}^{\$}, G_{t+j} \right) \quad (2)$$

The first term on the right-hand side is the present discounted value of all expected future surpluses, using the term structure of risk-free bond prices. It is the PDV for a risk-neutral investor. If the SDF is constant, this is the only term on the right-hand side. Then, the government's capacity to issue debt is constrained by its ability to generate current and future surpluses. The second and third terms encode the riskiness of the government debt portfolio, and arise in the presence of time-varying discount rates. If tax revenues tend to be high when times are good ($M_{t,t+j}$ is low), then the second term is negative. If government spending tends to be high when times are bad ($M_{t,t+j}$ is high), then the third term is positive. If both are true, then the difference between the two covariance terms is negative. Then the covariance terms lower the government's debt capacity. Put differently, the risk-neutral present-value of future surpluses will need to be higher by an amount equal to the absolute value of the covariance terms to support a given, positive amount of government debt D_t. The covariance terms are new to the literature, and this paper quantifies them. Its key finding is that, in a realistic model of risk and return, they have the hypothesized sign and are large in absolute value.

2.2 Risk Premium Equivalence

Define the holding period returns on the bond portfolio, the tax claim, and the spending claim as:

$$R^D_{t+1} = \frac{\sum_{h=1}^{\infty} P^{\$}_{t+1}(h-1)Q^{\$}_{t,h}}{\sum_{h=1}^{\infty} P^{\$}_t(h)Q^{\$}_{t,h}}, \quad R^T_{t+1} = \frac{P^T_{t+1}}{P^T_t - T_t}, \quad R^G_{t+1} = \frac{P^G_{t+1}}{P^G_t - G_t}.$$

The following proposition proves the relationship between the expected returns on these three assets:

Proposition 2 (Risk Premium Equivalence). Under the same assumptions of Proposition 1, we have:

$$\mathbb{E}_t\left[R^D_{t+1}\right] = \frac{P^T_t - T_t}{D_t - S_t}\mathbb{E}_t\left[R^T_{t+1}\right] - \frac{P^G_t - G_t}{D_t - S_t}\mathbb{E}_t\left[R^G_{t+1}\right]. \tag{3}$$

where $D_t - S_t = (P^T_t - T_t) - (P^G_t - G_t)$.

The proof is given in Appendix A. The average discount rate on government liabilities is equal to the average discount rate on government assets, which are a claim to primary surpluses. Since the primary surpluses are tax revenues minus government spending, the discount rate on government debt equals the difference between the discount rates of tax revenues and government spending, appropriately weighted.

By subtracting the risk-free rate on both sides, we can express the relationship in terms of expected excess returns, or risk premia. To develop intuition, we consider a two simple scenarios. First, if the expected returns on tax revenue and spending claims are identical, then the risk premium on government debt is given by:

$$\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right] = \mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] = \mathbb{E}_t\left[R^G_{t+1} - R^f_t\right].$$

Second, if the revenue claim is riskier than the spending claim and earns a higher higher risk premium, then the risk premium on government debt exceeds that on the revenue and the spending claims:

$$\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right] > \mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] > \mathbb{E}_t\left[R^G_{t+1} - R^f_t\right].$$

We show below that the revenue claim is indeed riskier than the spending claim. The risk premium equivalence then implies that the portfolio of government debt ought to carry a positive risk premium. The right discount rate for government debt, given by (3), cannot be the risk-free rate.

To understand the riskiness of the debt claim, we study the short-run and long-run risk properties of the T- and G-claim. To do so, we study spending and revenue strips. A spending strip that pays off G_{t+j} at time $t+j$ and nothing at other times. A revenue strip similarly pays off T_{t+j}. Let $R^{G,j}_{t,t+j}$ and $R^{T,j}_{t,t+j}$ be the holding period returns on these strips.

At the short end of the maturity spectrum (business cycle frequencies j of 1-3 years), the risk premium on the revenue strip exceeds that on the corresponding-maturity spending strip: $\mathbb{E}_t\left[R^{T,j}_{t,t+j} - R^f_t\right] > \mathbb{E}_t\left[R^{G,j}_{t,t+j} - R^f_t\right]$. The reason is that tax revenue is highly pro-cyclical while government spending is counter-cyclical. Since government debt investors have a long position in a riskier claim and a short position in a safer claim, the short end contributes to a positive risk premium on the government debt portfolio.

Next, we turn to long-end of the strip curve. We analyze the limit of the log returns on these strips as $j \to \infty$, denoted by lowercase letters. We distinguish two cases in terms of the time series properties of government spending and tax revenues.

Proposition 3 (Long-run Discount Rates). If the log of government spending G and of tax revenue T is stationary in levels (after removing a deterministic time trend), then the long-run expected log return on spending and revenue strips equals the yield on a long-term government bond as the payoff date approaches maturity.

$$\lim_{j\to\infty} \mathbb{E}_t\left[r^{G,j}_{t,t+j}\right] = y^{\$}_t(\infty), \quad \lim_{j\to\infty} \mathbb{E}_t\left[r^{T,j}_{t,t+j}\right] = y^{\$}_t(\infty),$$

where $y^{\$}_t(\infty)$ is the yield at time t on a nominal government bond of maturity $+\infty$.

The proof is given in Appendix A. The result builds on work by Alvarez and Jermann (2005); Hansen and Scheinkman (2009); Borovička, Hansen, and Scheinkman (2016); Backus, Boyarchenko, and Chernov (2018), among others.

Under this assumption on cash flows, the proposition implies that long-run T- and G-strips can be discounted off the term-structure for zero coupon bonds. In this case, the long-run discount rate on government debt is the yield on a long-term risk-free bond. However, the underlying assumption on cash flows is highly problematic. If there are no permanent shocks to T or G, then it is imperative to assume that GDP and aggregate consumption are not subject to permanent shocks either. But if there are no permanent shocks to marginal utility, then the long bond is the riskiest asset in economy. That clearly seems counterfactual (Alvarez and Jermann, 2005). The gap between the long-run discount rates on strips and the long bond yield is governed by the entropy of the permanent component of the pricing kernel. Explaining the high returns on risky assets such as stocks requires that entropy to be large, not zero (e.g., Borovička, Hansen, and Scheinkman, 2016). Next we consider a more realistic case.

Corollary 1. If the log of government spending/GDP ratio G/GDP (revenue/GDP T/GDP) is stationary in levels, then the long-run expected log excess return on long-dated spending (revenue) strips equals that on GDP strips:

$$\lim_{j\to\infty} \mathbb{E}_t \left[r^{G,j}_{t,t+j} \right] = \lim_{j\to\infty} \mathbb{E}_t \left[r^{T,j}_{t,t+j} \right] = \mathbb{E}_t \left[r^{GDP,\infty}_{t,t+n} \right] \gg y^{\$}_t(\infty).$$

We show below that government spending and tax revenue are cointegrated with GDP in the data; their ratio is stationary in levels. Under this realistic assumption on cash flows, expected returns on long-dated spending and tax revenue strips tend to the expected return on a long-dated GDP strip. A claim to GDP can be thought of as an unlevered equity claim. In the presence of permanent shocks to marginal utility, the long-run discount rate on GDP (unlevered equity) is much higher than the yield on long-term risk-free bonds. This corollary implies that government bond investors have a net long position in a claim that is exposed to the same long-run risk as the GDP claim. It follows immediately from this discount rate argument that the value of the long-run spending minus revenue strips will be smaller than what is predicted by the yields at the long end of the term structure.

Combining the properties of short-run and long-run discount rates, theory predicts that $\mathbb{E}_t \left[R^D_{t+1} - R^f_t \right] > \mathbb{E}_t \left[R^T_{t+1} - R^f_t \right] > \mathbb{E}_t \left[R^G_{t+1} - R^f_t \right]$. To summarize, a model of asset prices will have to confront two forces that push up the equilibrium returns on government debt. First, there is short-run cash flow risk that pushes the expected return on the revenue claim above the expected return on the spending claim. Second, the long-run discount rates are higher than the yield on a long-maturity bond, because of the long-run cash flow risk in the spending and revenue claims equals that of long-run GDP risk. Government debt investors have a net long position in a claim that is exposed to the same long-run cash flow risk as GDP. The excess returns on government debt will tend to be much higher than those on long-maturity bonds. As a result of these two forces, government debt investors earn a larger risk premium on the long end than what they pay on the short end, which increases the fair expected return on the debt claim. Discounting future surpluses using the term structure of risk-free interest rates, as typically done in the literature, is inappropriate. The low observed interest rate, or equivalently the high value, of the government debt portfolio represents a puzzle in light of the fundamental risk of the cash flows backing that debt.

An important implication of (3) is that, if the government wants to reduce the riskiness and hence expected return on government debt, it would need to make the tax claim safer. This would require counter-cyclical tax revenues and hence tax rates. The latter is strongly at odds with the behavior of observed fiscal policy.

2.3 Inflation and Default

Inflation cannot resolve the puzzle. The value and risk premium equivalences are ex-ante relationships. They hold both in nominal and in real terms. Judged by the low break-even inflation rates (below 2%), bond markets do not seem to anticipate that the U.S. government will erode the real value of debt through inflation. Ex-post, the government can erode the real value of outstanding debt by creating surprise inflation. But given the short duration of outstanding debt of around four years in the U.S., that channel has limited potency to reduce debt burdens.[1]

Sovereign default risk cannot not resolve the puzzle. The same inter-temporal budget constraint holds when we allow for sovereign default: the valuation of government debt is still backed by the value of future surpluses. Bond prices adjust to reflect the possibility of default. The proof is given in Appendix A.[2]

2.4 Fiscal Measurability Constraint

The value equivalence in Proposition 1 implies a measurability constraint (Hansen, Roberds, and Sargent, 1991; Aiyagari, Marcet, Sargent, and Seppälä, 2002):

Proposition 4 (Measurability Constraint). Denote a generic state variable by z_t. The value of the surplus claim responds in the same way as the bond portfolio to changes in every state variable:

$$\frac{\partial D_t}{\partial z_t} = \sum_{h=0}^{H} Q^{\$}_{t-1,h+1} \frac{\partial P^{\$}_t(h)}{\partial z_t} = \frac{\partial P^T_t}{\partial z_t} - \frac{\partial P^G_t}{\partial z_t} \tag{4}$$

If a negative economic shock lowers the present value of future surpluses, bond prices must adjust to restore (4). The proof follows readily from that of proposition 1.

Corollary 2. If the government only issues one-period risk-free debt ($h = 0$), then the value of the previous period's bond portfolio at the start of the next period cannot depend on any shocks. The measurability conditions become:

$$\frac{\partial P^T_t}{\partial z_t} - \frac{\partial P^G_t}{\partial z_t} = 0 \tag{5}$$

The reason is that the price of one-period debt issues last period is constant: $P^{\$}_t(0) = 1$. Only if condition (5) is satisfied is it appropriate to discount future surpluses at the one-period risk-free

[1]For example, a 5 percentage point increase in inflation that lasts as long as the maturity of the longest outstanding debt reduces the real value of debt by $5\% \times 4 = 20\%$. See Hall and Sargent (2011); Berndt, Lustig, and Yeltekin (2012) for a decomposition of the forces driving the U.S. debt/GDP ratio including inflation. Cochrane (2019a,b) explores the connection between inflation and the value of government debt without imposing no arbitrage restrictions.

[2]Bond prices satisfy $P^{\$}_t(h) = \mathbb{E}_t\left[M^{\$}_{t,t+h}(1-\chi_{t,t+h})\right]$, where $\chi_{t,t+h}$ is an indicator variable that is one when the government defaults between t and $t+h$. We assume full default to keep the proof simple, but this is without loss of generality. Chernov, Schmid, and Schneider (2016) and Pallara and Renne (2019) study the response of CDS spreads to news about the fiscal surplus.

bond rate. We show below that this condition is severely violated in the data.

3 Data

We conduct our analysis at annual frequency, which is a better frequency to study cash flow risk in fiscal revenues and outlays, but all of our results are robust to working at quarterly frequency. We focus on the period from 1947 until 2019.

Nominal federal tax revenue and government spending before interest expense are from the Bureau of Economic Analysis, as is nominal GDP. Constant-maturity Treasury yields are from Fred. Stock price and dividend data are from CRSP; we use the CRSP value-weighted total market to represent the U.S. stock market. Dividends are seasonally adjusted. Details are provided in Appendix C.

As was shown in Figure 1, the surpluses expressed as a fraction of GDP are strongly pro-cyclical. Non-discretionary spending accounts for at least 2/3 of the government's spending. This includes Social Security, Medicare and Medicaid, as well as food stamps and unemployment benefits. Many of these transfer payments rise automatically in recessions. In addition, the government often temporarily increases transfer spending in recessions (e.g., the extension of unemployment benefits in 2009 or 2020). On the tax revenue side, the progressive nature of the tax code generates strongly pro-cyclical variation in revenue as a fraction of GDP.

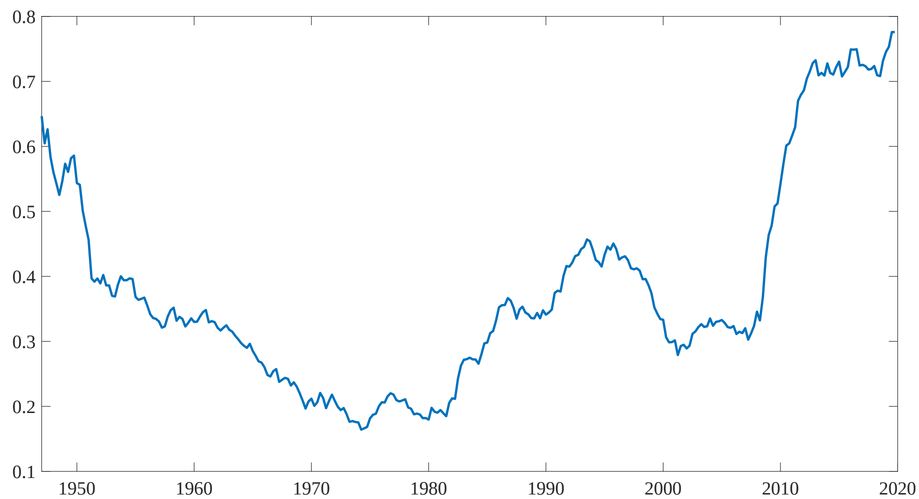
We construct the market value and the total returns of the marketable government bond portfolio using cusip-level data from the CRSP Treasuries Monthly Series. At the end of each period, we multiply the nominal price of each cusip by its total amount outstanding (normalized by the face value), and sum across all issuances (cusips). We exclude non-marketable debt which is mostly held in intra-governmental accounts.[3] Marketable debt includes the Treasury holdings of the Federal Reserve Bank. Hence, we choose not to consolidate the Fed and the Treasury, which would add reserves and subtract the Fed's Treasury holdings on the left hand side of (1). Doing so would mainly tilt the duration of the bond portfolio.

Following Hall and Sargent (2011) and extending their sample, we construct zero coupon bond (strip) positions from all coupon-bearing Treasury bonds (all cusips) issued in the past and outstanding in the current period. This is done separately for nominal and real bonds. Since zero-coupon bond prices are also observable, we can construct the left-hand side of eq. (1) as the market

[3]The largest holders of non-marketable debt are the Social Security Administration (SSA) and the federal government's defined benefit pension plan. Consolidating the SSA and the government DB plans with the Treasury department leads one to include the revenues and spending from the SSA/govt DB plan in the consolidated government revenue and spending numbers, and leads one to net out the SSA holdings of Treasuries since they are an asset of one part of the consolidated government and a liability of the other part. Hence our treatments of debt and cash flows are mutually consistent.

value of outstanding marketable U.S. government debt.[4] Figure 2 plots its evolution over time, scaled by the U.S. GDP. It shows a large and persistent increase in the outstanding debt starting in 2008.

Figure 2: The Market Value of Outstanding Debt to GDP



The figure plots the ratio of the nominal market value of outstanding government debt divided by nominal GDP. GDP Data are from the Bureau of Economic Analysis. The market value of debt is constructed as follows. We multiply the nominal price (bid/ask average) of each cusip by its total amount outstanding (normalized by the face value), and then sum across all issuance (cusip). The series is annual from 1947 until 2019. Data Source: CRSP U.S. Treasury Database, BEA, authors' calculations.

Turning to returns, Table 1 reports summary statistics for the overall Treasury bond portfolio in Panel A and for individual bonds in Panel B. The excess returns on the entire Treasury portfolio realized by an investor who buys all of the new issuances and collects all of the coupon and principal payments is 1.11% per annum, on average. The portfolio has an average duration of 3.62 years. Given the secular decline in interest rates over the past forty years, the observed average return on the bond portfolio is, if anything, an over-estimate of investors' expected return.

4 Quantitative Model

In order to quantify the value of the claims to tax revenue and government spending in (1), we need to (i) take a stance on the time-series properties of revenue and spending, and (ii) a stochastic discount factor $M_{t,t+j}$ to discount these cash flows.

[4]Since the model fits nominal bond prices very well, as shown below, we can equivalently use model-implied bond prices. Similarly, we can use model-implied prices for real zero-coupon bonds.

Table 1: Summary Statistics for Government Bond Portfolio

	Panel A				Panel B			
	R^D	$R^D - R^f$	R^f	Duration	1 Yr	5 Yr	10 Yr	20 Yr
Mean	5.21	1.11	4.10	3.62	4.69	4.72	5.52	5.67
Std.	3.06	2.99	3.14	1.06	1.03	1.70	4.76	6.73
Sharpe Ratio	0.37				0.42	0.30	0.23	0.23

Panel A reports summary statistics for the holding period return on the aggregate government bond portfolio: the mean and the standard deviation of the holding period return, R^D, the excess return, $R^D - R^f$, the three-month Tbill rate, R^f, and the weighted average Macaulay duration. Panel B reports the mean and the standard deviation of the holding period returns of three-month T-bill and T-bonds with time-to-maturity of one year, five years and ten years. All returns are expressed as annual percentage points. Duration is expressed in years. Data source: CRSP Treasuries Monthly Series. The sample period is from 1947 to 2019.

4.1 Cash Flow Dynamics

4.1.1 State Variables

We assume that the $N \times 1$ vector of state variables $\boldsymbol{z}$ follows a Gaussian first-order VAR:

$$\boldsymbol{z}_t = \boldsymbol{\Psi} \boldsymbol{z}_{t-1} + \boldsymbol{u}_t = \Psi \boldsymbol{z}_{t-1} + \boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{\varepsilon}_t, \tag{6}$$

with $N \times N$ companion matrix Ψ and homoscedastic innovations $\boldsymbol{u}_t \sim i.i.d. \mathcal{N}(0, \Sigma)$. The Cholesky decomposition of the covariance matrix, $\boldsymbol{\Sigma} = \Sigma^{\frac{1}{2}} \left(\Sigma^{\frac{1}{2}}\right)'$, has non-zero elements on and below the diagonal. In this way, shocks to each state variable u_t are linear combinations of its own structural shock $\boldsymbol{\varepsilon}_t$, and the structural shocks to the state variables that precede it in the VAR, with $\boldsymbol{\varepsilon}_t \sim i.i.d. \mathcal{N}(0, I)$. Table 2 summarizes the variables we include in the state vector, in order of appearance of the VAR. The vector $\boldsymbol{z}$ contains the state variables demeaned by their respective sample averages.

Table 2: State Variables

Position	Variable	Mean	Description
1	π_t	π_0	Log Inflation
2	x_t	x_0	Log Real GDP Growth
3	$y_t^{\$}(1)$	$y_0^{\$}(1)$	Log 1-Year Nominal Yield
4	$yspr_t^{\$}$	$yspr_0^{\$}$	Log 5-Year Minus 1-Year Nominal Yield Spread
5	pd_t	$\overline{pd}$	Log Stock Price-to-Dividend Ratio
6	Δd_t	μ_d	Log Stock Dividend Growth
7	$\Delta \log \tau_t$	μ_τ	Log Tax Revenue-to-GDP Growth
8	$\Delta \log g_t$	μ_g	Log Spending-to-GDP Growth
9	$\log \tau_t$	$\log \tau_0$	Log Tax Revenue-to-GDP Level
10	$\log g_t$	$\log g_0$	Log Spending-to-GDP Level

4.1.2 Fiscal Policy

Our approach takes spending and tax policy as given, rather than being optimally determined. However, both policies are allowed to depend on a rich set of state variables and are estimated from the data. To capture the government's cash flows, the VAR includes $\Delta \log \tau_t$ and $\Delta \log g_t$, the log change in tax revenue-to-GDP and the log change in government spending-to-GDP in its seventh and eight rows. It also includes the log level of revenue-to-GDP, τ_t, and spending-to-GDP, g_t, in its ninth and tenth rows. This fiscal cash flow structure has two important features.

First, it allows spending and revenue growth to depend not only on its own lag, but also on a rich set of macroeconomic and financial variables. Lagged inflation, GDP growth, interest rates, the slope of the term structure, the stock price-dividend ratio, and dividend growth all predict future revenue and spending growth. And innovations in the fiscal variables are correlated with innovations in these macro-finance variables.

Second, it is crucial to include the level variables τ_t and g_t. When there is a positive shock to spending, spending tends to revert back to its long-run trend with GDP. Similarly, after a negative shock to tax revenue, future revenues tend to increase back to their long-run level relative to GDP. This mean reversion captures the presence of automatic stabilizers and of corrective fiscal action, as pointed out by Bohn (1998). By having spending-to-GDP growth $\Delta \log g_t$ (revenue-to-GDP $\Delta \log \tau_t$) depend on lagged spending g_t (lagged revenue-to-GDP τ_t) with a negative coefficient, the VAR captures this mean reversion. Mean reversion is further amplified when $\Delta \log g_t$ ($\Delta \log \tau_t$) depends on lagged revenue-to-GDP τ_t (g_t) with a positive sign.

Formally, the inclusion of the levels of spending and tax revenue relative to GDP in the VAR is motivated by a cointegration analysis; the system becomes a vector error correction model. Appendix D.2 performs Johansen and Phillips-Ouliaris cointegration tests. The results support two cointegration relationships, one between log tax revenue and log GDP and one between log spending and log GDP. The coefficients estimates of the cointegration relationships tend to vary across sample periods. As a result, we take an a priori stance that the tax-to-GDP ratio $\log \tau$ and the spending-to-GDP ratio $\log g$ are stationary. That is, we assume cointegration coefficients of $(1, -1)$ for both relationships. Put differently, without cointegration all shocks to spending and tax revenues are permanent rather than mean-reverting.

As a technical aside, the in-sample average of $\Delta \log \tau_t$ is $\widehat{\mu}^\tau = -0.7\%$ and the in-sample average of $\Delta \log g_t$ is $\widehat{\mu}^g = 0.2\%$. Because we impose cointegration on the log tax-to-GDP ratio and the log spending-to-GDP ratio, the true unconditional growth rates of the tax-to-GDP ratio and the spending-to-GDP ratio have to be zero ($\mu_0^\tau = \mu_0^g = 0$). In order to be consistent with the cointegration assumption, we remove the in-sample averages of the growth rates, and construct the log

Table 3: VAR Estimates $\boldsymbol{\Psi}$

	π_{t-1}	x_{t-1}	$y^{\$}_{t-1}(1)$	$yspr^{\$}_{t-1}$	pd_{t-1}	Δd_{t-1}	$\Delta \log \tau_{t-1}$	$\Delta \log g_{t-1}$	$\log \tau_{t-1}$	$\log g_{t-1}$
π_t	**0.541**	0.004	**0.214**	*-0.405*	0.008	0.030	**0.044**	0.001	**-0.036**	**0.026**
x_t	*-0.280*	0.162	0.132	0.235	0.000	0.078	-0.015	0.053	-0.050	0.022
$y^{\$}_t(1)$	0.064	0.077	**0.896**	-0.039	0.005	0.042	-0.007	-0.001	-0.030	0.022
$yspr^{\$}_t$	-0.041	**-0.099**	0.007	**0.539**	-0.004	-0.028	0.013	0.009	0.012	**-0.012**
pd_t	**-2.557**	-1.100	0.375	2.302	**0.774**	-0.245	-0.036	0.118	0.240	-0.252
Δd_t	0.186	-0.053	-0.446	-0.675	**0.052**	**0.329**	-0.144	-0.166	**-0.239**	*0.104*
$\Delta \log \tau_t$	**-1.092**	0.100	0.310	**-3.105**	**0.075**	0.152	**0.307**	0.107	**-0.567**	**0.182**
$\Delta \log g_t$	0.517	0.389	**-0.938**	-1.283	**-0.069**	-0.233	0.067	**0.348**	0.099	**-0.266**
$\log \tau_t$	**-1.092**	0.100	0.310	**-3.105**	**0.075**	0.152	**0.307**	0.107	**0.433**	**0.182**
$\log g_t$	0.517	0.389	**-0.938**	-1.283	**-0.069**	-0.233	0.067	**0.348**	0.099	**0.734**

Numbers in bold have t-statistics in excess of 1.96 in absolute value. Numbers in italics have t-statistics in excess of 1.645 but below 1.96.

tax-to-GDP and log spending-to-GDP ratios that enter in the VAR as follows:

$$\log \tau_t = \log \tau_1 + \sum_{k=1}^{t} (\Delta \log \tau_k - \widehat{\mu}^{\tau}), \quad \log g_t = \log g_1 + \sum_{k=1}^{t} (\Delta \log g_k - \widehat{\mu}^{g}),$$

where the initial level $\log g_1$ is the the actual log spending-to-GDP ratio at the start of our sample in 1947, while $\log \tau_1$ is chosen so that the resulting average log surplus-to-GDP ratio is the same as in the unadjusted data. This requires a minor adjustment to the actual 1947 revenue-to-GDP ratio.

4.1.3 VAR Estimates

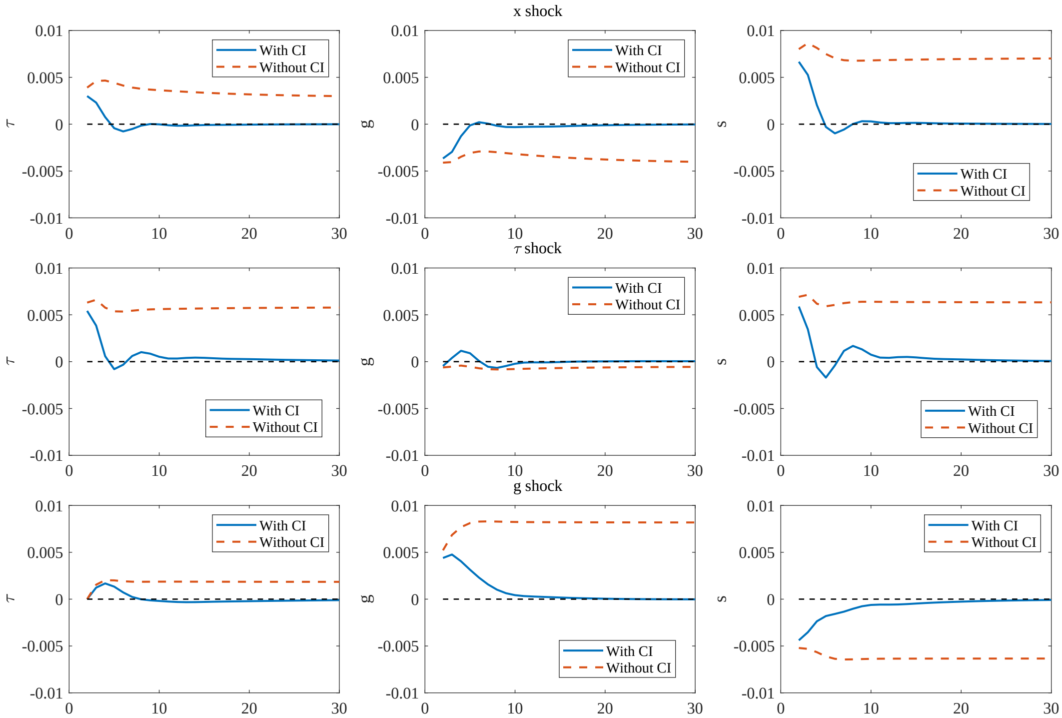
We estimate the first eight equations of (6) using OLS. We do not zero out any of the elements in Ψ even if they are statistically indistinguishable from zero.[5] Since $g_t = \Delta \log g_t + g_{t-1} = e'_{\Delta g}[\Psi + I]z_{t-1} + e'_{\Delta g}\Sigma^{\frac{1}{2}}\varepsilon_t$, where $e_{\Delta g}$ selects the eighth row, and similar for tax revenue-to-GDP, the last two rows of Ψ and $\Sigma^{\frac{1}{2}}$ are implied by the first eight. The last two elements of the VAR do not have independent shocks for the same reason.

The point estimates of $\boldsymbol{\Psi}$ are reported in Table 3. Lagged macro-finance variables affect fiscal variables and vice versa. Consistent with the long-run mean reversion dynamics imposed by cointegration, we find that $\Psi_{[7,9]} = -0.567 < 0$ and $\Psi_{[8,10]} = -0.266 < 0$. Both coefficients are estimated precisely. The cross-terms also have the expected sign: $\Psi_{[7,10]} = 0.182 > 0$ and $\Psi_{[8,9]} = 0.099 > 0$, but only the first one is estimated precisely.

The estimate of $\Sigma^{\frac{1}{2}}$ is reported in Appendix D.1. The innovation in tax revenue-to-GDP growth is positively correlated with the GDP growth rate innovation, while the spending-to-GDP growth shock is negatively correlated with the GDP growth shock. In other words, tax revenues are

[5] None of our main conclusions are sensitive to recursively zeroing out insignificant elements in Ψ. We also find similar results at quarterly frequency.

Figure 3: Fiscal Impulse-Responses



Solid line shows impulse-response functions for the benchmark VAR with cointegration; dashed line is for the VAR without cointegration. The impulse in the top row is a shock to GDP. The x_t shock is defined as the shock that increases x_t by one standard deviation of its VAR residual. The impulse in the middle row is a shock to tax revenues. The impulse in the bottom row is a shock to spending growth.

strongly pro-cyclical and government spending is strongly counter-cyclical.

4.1.4 Spending and Revenue Dynamics

Figure 3 plots the impulse-response functions of the tax revenue-to-GDP ratio (τ_t, left panels), government spending-to-GDP ratio (g_t, middle panels), and surplus-to-GDP (s_t, right panels) to a GDP shock (top row), a revenue shock (middle row), and a spending shock (bottom row). All shocks are one-standard deviation in size. The solid lines, which are for the benchmark VAR system, show mean reversion in spending and revenues in response to the own shock. They also shows the pro-cyclicality of revenues-to-GDP and counter-cyclicality of spending-to-GDP in response to the GDP shock. For comparison, the dashed red lines represent the results under a restricted VAR, in which the first 8 state variables do not load on the cointegration variables $\log \tau_t$ and $\log g_t$. When cointegration is not imposed, the impact of fiscal shocks is permanent.

The impulse-responses show that the VAR system with and without cointegration variables imply very different dynamics in government cash flows. Which one is more consistent with the

Table 4: The Predictability of Government Cash Flow Growth

	Dependent variable: $\Delta \log \tau_{t+k}$				
horizon k (years)	1	2	3	4	5
$\log \tau_t$ – data	-0.33^{***} (0.07)	-0.42^{***} (0.09)	-0.28^{***} (0.05)	-0.11 (0.08)	0.05 (0.12)
$\log \tau_t$ – model	-0.57	-0.42	-0.07	0.11	0.09
$\log g_t$ – data	0.06 (0.06)	0.06 (0.07)	0.05 (0.05)	0.03 (0.04)	-0.03 (0.04)
$\log g_t$ – model	0.18	0.07	-0.07	-0.12	-0.09
	Dependent variable: $\Delta \log g_{t+k}$				
horizon k (years)	1	2	3	4	5
$\log \tau_t$ – data	0.09 (0.06)	0.08 (0.11)	0.003 (0.15)	0.03 (0.09)	0.05 (0.08)
$\log \tau_t$ – model	0.10	0.03	-0.09	-0.11	-0.05
$\log g_t$ – data	-0.10^{*} (0.06)	-0.15^{***} (0.06)	-0.13^{**} (0.05)	-0.09^{**} (0.04)	-0.06 (0.05)
$\log g_t$ – model	-0.27	-0.25	-0.16	-0.09	-0.07

This table reports how the levels of $\log \tau_t$ and $\log g_t$ predict the future tax revenue-to-GDP growth and the future government spending-to-GDP growth. The rows labeled by data report the coefficients from the univariate regression of the annual $\Delta \log \tau_{t+k}$ and $\Delta \log g_{t+k}$ in the following year 1 through 5 on the current $\log \tau_t$ and $\log g_t$. Data are 1947—2019. Standard errors in parentheses are Newey-West with 5 lags. The rows labeled by model report the coefficients implied from the VAR system with cointegration variables.

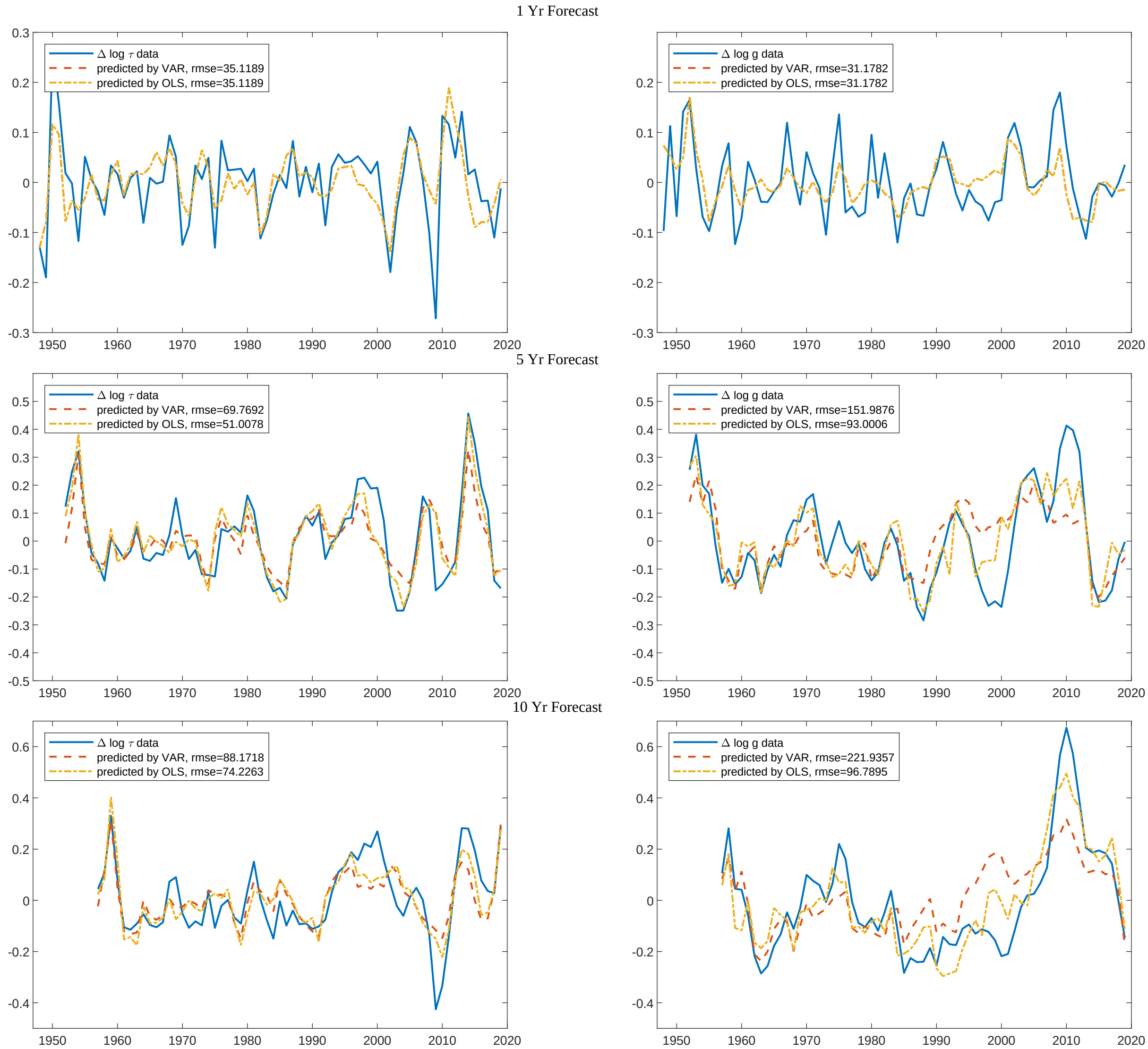
data? Table 4 reports results from predictive regressions of $\Delta \log \tau_{t+k}$ and $\Delta \log g_{t+k}$ in future years $k = 1, \cdots, 5$ on the current-year $\log \tau_t$ and $\log g_t$ levels. In the data, a higher level of $\log \tau_t$ predicts a significantly lower tax revenue-to-GDP growth in the next 1—3 years, and a higher level of $\log g_t$ predicts a lower government spending-to-GDP growth in the next 1—4 years. This mean reversion is the signature of cointegration. Table 4 also reports the model-implied counterparts for the VAR with cointegration. The regression coefficients from the data are quantitatively similar to the conditional expectations implied by the VAR model.

Figure 4 adds further credibility to the cash-flow projections by plotting expected cumulative spending and revenue growth over the next one, five, and ten years against realized future spending and revenue growth. To assess predictive accuracy, we compare the prediction of the benchmark annual VAR to that of the best linear forecaster at that horizon. By design, the VAR prediction is the best linear forecaster at the one-year horizon, but not at the five- and ten-year horizons. Prediction accuracy (RMSE) of the VAR is similar to that of the best linear forecaster. The graph shows that the VAR implies reasonable behavior of long-run fiscal cash flows.

4.1.5 Debt in the VAR

Cochrane (2019a,b) includes debt/GDP in the VAR and argues that this affects the dynamics of the surplus in important ways. In particular, a negative shock to GDP or a negative shock to surpluses lowers the surplus on impact. The surplus not only mean reverts in subsequent periods but overshoots. It is this overshooting of the surplus, he argues, that makes government debt

Figure 4: Cash Flow Forecasts



We plot the actual log tax and spending growth rates over 1-year, 5-year and 10-year rolling windows in solid blue lines. The value at each year represents the k-year growth rates that end at that year. We also plot these rates as forecasted by our VAR model in dashed red lines and these rates as forecasted by the OLS model in dash-dotted yellow lines. The value at each year represents the k-year growth rates condition on the information k years ago.

risk-free. Appendix G estimates a VAR that adds the log change in debt-to-GDP and the log level in debt-to-GDP. That is, it adds a third cointegration relationship between debt and GDP. It shows that (i) this specification does not improve the forecast accuracy of spending and revenue growth, (ii) does not generate meaningful overshooting in the surplus, and (iii) results in very similar results for our main exercise that is to follow. For these reasons, we do not include debt in the benchmark VAR.

4.2 Asset Pricing

We take a pragmatic approach and choose a flexible SDF model that only assumes no arbitrage, and prices the term structure of interest rates as well as stocks well. In particular, this approach guarantees that our debt valuation is consistent with observed Treasury bond prices. Motivated by the no-arbitrage term structure literature (Ang and Piazzesi, 2003), we specify an exponentially affine stochastic discount factor (SDF). The nominal SDF $M^{\$}_{t+1} = \exp(m^{\$}_{t+1})$ is conditionally log-normal:

$$m^{\$}_{t+1} = -y^{\$}_t(1) - \frac{1}{2}\Lambda'_t\Lambda_t - \Lambda'_t\varepsilon_{t+1}, \tag{7}$$

The real SDF is $M_{t+1} = \exp(m_{t+1}) = \exp(m^{\$}_{t+1} + \pi_{t+1})$; it is also conditionally Gaussian. The priced sources of risk are the structural innovations in the state vector ε_{t+1} from equation (6). These aggregate shocks are associated with a $N \times 1$ market price of risk vector Λ_t of the affine form:

$$\Lambda_t = \Lambda_0 + \Lambda_1 z_t,$$

The $N \times 1$ vector Λ_0 collects the average prices of risk while the $N \times N$ matrix Λ_1 governs the time variation in risk premia. Asset pricing in this model amounts to estimating the market prices of risk in Λ_0 and Λ_1. All asset pricing results are proven in Appendix B.

4.2.1 Bond Pricing

Nominal bond yields of maturity h are affine in the state vector:

$$y^{\$}_t(h) = -\frac{A^{\$}(h)}{h} - \frac{B^{\$}(h)'}{h} z_t,$$

the scalar $A^{\$}(h)$ and the vector $B^{\$}(h)$ follow ordinary difference equations that depend on the properties of the state vector and of the market prices of risk. There is a similar formula for real bonds. We use this pricing equation to calculate the real interest rate, real bond risk premia, and inflation risk premia on bonds of various maturities.

Since both the nominal short rate ($y^{\$}_t(1)$) and the slope of the term structure ($y^{\$}_t(5) - y^{\$}_t(1)$) are included in the VAR, internal consistency requires the SDF model to price these bonds closely. The nominal short rate is matched automatically; it does not identify any market price of risk parameters. Matching the slope of the yield curve generates $N + 1$ parameter restrictions:

$$-A^{\$}(20)/20 = y^{\$}_0(1) + yspr^{\$}_0 = y^{\$}_0(20) \tag{8}$$

$$-B^{\$}(20)/20 = e_{y1} + e_{yspr} \tag{9}$$

They pin down the fourth element of Λ_0 and the fourth row of Λ_1. We also allow for a non-zero third element of Λ_0 and two non-zero elements in the third row of Λ_1. We pin down these elements by matching bond yields of maturities 2, 10, 20, and 30 years in each year $t \in 1, \cdots, T$. Since they represent $T \times 4$ moments for only 3 parameters, there are $T \times 4 - 3$ over-identifying restrictions. Since the behavior of very long-term interest rates is of great importance for our valuation results—recall the discussion on very long-term bond yields in Section 2,—we impose extra weight on matching the 30-year bond yields.

We also price the yields on real bonds (Treasury inflation-index securities) for maturities 5, 7, 10, 20, and 30 years. They are available over a shorter sample of T_2 years. This adds $T_2 \times 5$ over-identifying restrictions. Again, we overweight the 30-year maturity.

4.2.2 Equity Pricing

The VAR includes both log dividend growth and the log price-dividend ratio. The two time-series imply a time series for returns. We impose that the expected excess return implied by the VAR matches the equity risk premium in the model, which depends on the covariance of the SDF with stock returns. Expressions are provided in the appendix. The equity risk premium conditions pin down the sixth element of Λ_0 and the sixth row of Λ_1.

Let $PD_t^m(h)$ denote the price-dividend ratio of the dividend strip with maturity h (Wachter, 2005; van Binsbergen, Brandt, and Koijen, 2012). Then, the aggregate price-to-dividend ratio can be expressed as

$$PD_t^m = \sum_{h=0}^{\infty} PD_t^m(h). \tag{10}$$

In this SDF model, log price-dividend ratios on dividend strips are affine in the state vector:

$$pd_t^m(h) = \log\left(PD_t^m(h)\right) = A^m(h) + B^{m\prime}(h) z_t.$$

Since the log price-dividend ratio on the stock market in part of the state vector, it is affine in the state vector by assumption; see the left-hand side of (11):

$$\exp\left(\overline{pd} + e_{pd}' z_t\right) = \sum_{h=0}^{\infty} \exp\left(A^m(h) + B^{m\prime}(h) z_t\right), \tag{11}$$

Equation (11) rewrites the present-value relationship (10), and articulates that it implies a restriction on the coefficients $A^m(h)$ and $B^{m\prime}(h)$. We impose this restriction in the estimation; it provides $T \times 1$ additional over-identifying restrictions.

4.2.3 Good Deal Bounds and Regularity Conditions

We impose good deal bounds on the standard deviation of the log SDF in the spirit of Cochrane and Saa-Requejo (2000). Specifically, we impose a penalty for annual Sharpe ratios in excess of 3.

Second, we impose regularity conditions on (unobserved) nominal and real bond yields of maturities of 50 to 1000 years. Specifically, we impose that yields stabilize and that nominal yields remain above real yields by at least long-run expected inflation. This is tantamount to a weak positivity restriction on the inflation risk premium.

Third, we impose that the valuation ratios of the long-run G-claim (T-claims) increase in response to a positive shock to spending (tax revenues). This sign restriction helps identify how spending and tax revenues affect the dynamics of the equity risk premium.

4.3 Estimation Results

Appendix D reports the point estimates for the market price of risk parameters. Appendix E shows that the model matches the time series of nominal bond yields in the data closely. It also shows a reasonable fit for real bond yields. Furthermore, the model closely matches the dynamics of the nominal bond risk premium, and generates reasonable behavior on nominal and real yields at very long horizons. Finally, the model produces reasonable equity risk premium level and dynamics, and provides a close fit to the time-series of the price-dividend ratio. Because it is able to generate an expected equity return that fits the data well, and is large compared to the long-term real rate, the SDF has a large permanent component. Having formulated and estimated a realistic SDF, we now turn to our main exercise.

5 Government Debt Valuation and Risk Premium Puzzles

5.1 Surplus Pricing Model

With the VAR dynamics and the SDF in hand, we can calculate the expected present discounted value of the primary surplus:

$$\mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} S_{t+j}\right] = \sum_{j=0}^{\infty} \mathbb{E}_t\left[M^{\$}_{t,t+j} T_{t+j}\right] - \sum_{j=0}^{\infty} \mathbb{E}_t\left[M^{\$}_{t,t+j} G_{t+j}\right] = P^T_t - P^G_t, \tag{12}$$

where P^T_t is the cum-dividend value of a claim to future nominal tax revenues and P^G_t is the cum-dividend value of a claim to future nominal government spending. The following proposition shows how to price the government cash flows.

Proposition 5 (Pricing Government Cash Flows). (Part a) The price-dividend ratios on the tax claim and the spending claim are the sum of the price-dividend ratios of their strips, whose logs are affine in the state vector z_t:

$$PD_t^T = \frac{P_t^T}{T_t} = \sum_{h=0}^{\infty} \exp(A_\tau(h) + B_\tau'(h) z_t), \tag{13}$$

$$PD_t^G = \frac{P_t^G}{G_t} = \sum_{h=0}^{\infty} \exp(A_g(h) + B_g'(h) z_t). \tag{14}$$

(Part b) The log risk premia on the tax and spending claims are given by:

$$\mathbb{E}_t\left[r_{t+1}^T\right] - y_t^{\$}(1) + Jensen = \left(e_{\Delta\tau} + e_x + e_\pi + \kappa_1^\tau \bar{B}_\tau\right)' \Sigma^{\frac{1}{2}} \left(\Lambda_0 + \Lambda_1 z_t\right), \tag{15}$$

$$\mathbb{E}_t\left[r_{t+1}^G\right] - y_t^{\$}(1) + Jensen = \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)' \Sigma^{\frac{1}{2}} \left(\Lambda_0 + \Lambda_1 z_t\right). \tag{16}$$

The proof is in Appendix B.4. The right-hand side of (15) and (16) denotes the covariance of the claims' returns with the SDF. These covariances are crucially driven by the exposure vectors $\bar{B}_g$ and $\bar{B}_\tau$. The latter capture the risk exposures of all revenue and spending strips to the state variables, captured by the $B_g'(h)$ and $B_\tau'(h)$.

5.2 Main Results

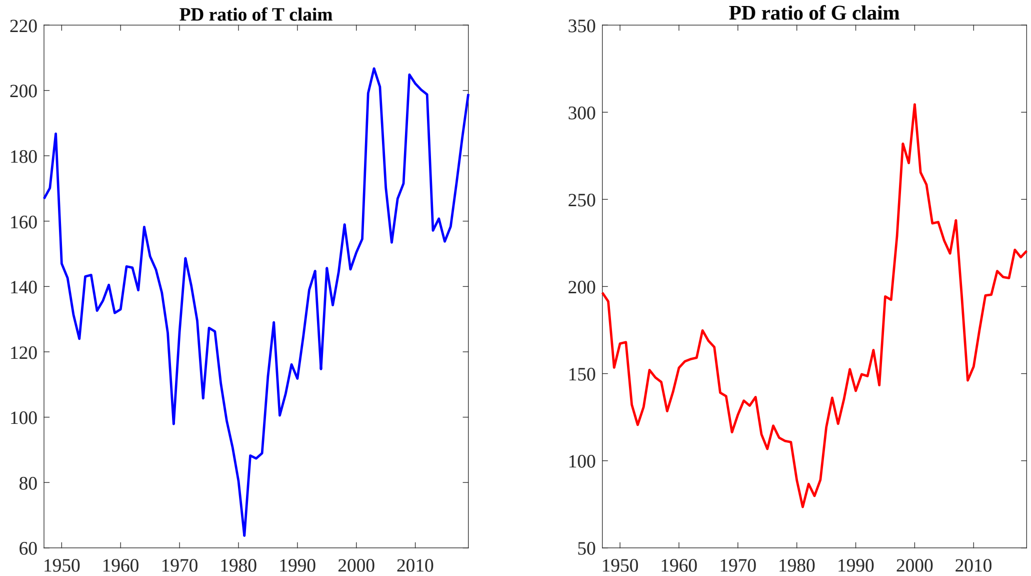
5.2.1 The Valuation Puzzle

The left panel of Figure 5 plots the price-dividend ratio on a claim to future tax revenue, $PD_t^T = P_t^T / T_t$. The time-series average of this ratio is 142.22. In other words, the representative agent would be willing to pay 142.22 times annual tax revenues for the right to receive all current and future tax revenues. In addition, the price-dividend ratio of the tax claim displays substantial time-variation. A pronounced V-shape arises from the inverse V-shape of the long-term real interest rate, which is high in the mid-1970s to mid-1980s and low at the beginning and end of the sample. Intuitively, discounting future tax revenues by a low (high) long-term real rate results in a high (low) valuation ratio.

The time-series average of the price-dividend ratio on a claim to future government spending, $PD_t^G = P_t^G / G_t$ is 164.74. The spending claim is more valuable than the revenue claim, a reflection of its lower riskiness. The price-dividend ratio shows the same inverse V-shape dynamics of the price-dividend ratio on the revenue claim, as shown in the right panel of Figure 5.

Now we are in a position to evaluate the claim to future government surpluses as the tax claim minus the spending claim, the right-hand side of equation (12). Figure 6 plots the present value of government surpluses scaled by GDP as the dashed line. The market value of the US govern-

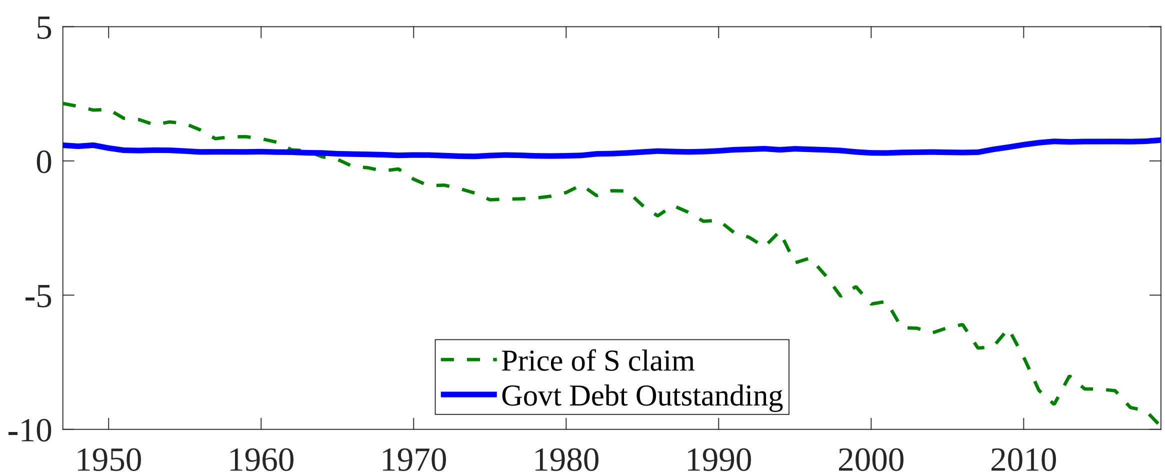
Figure 5: Government Cash Flows and Prices



The left panels plot the (cum-dividend) price-dividend ratio on the claim to current and future tax revenues. The right panel plots the (cum-dividend) price-dividend ratio of a claim to current and future government spending. The sample is 1947 until 2019.

ment debt is plotted as the solid line. We refer to the difference between these two lines as the government debt valuation puzzle. The unconditional average present value of the government surplus is -260.37% of GDP, far below the average market value of outstanding government debt, 0.38 times GDP. The gap is 299% of GDP on average. In the time series, the present value of

Figure 6: Present Value of Government Surpluses and Market Value of Government Debt



The solid line is the market value of government debt. The dashed line is the market value of the surplus claim. Both time series are scaled by the US GDP.

the government surplus does not match the dynamics of government debt value, either. The gap widens dramatically in the last 10 years of the sample, as the level of government debt doubles to 69.4% of the GDP, while the valuation of the surplus claim quadruples in absolute value to about 847% of the GDP. In other words, the U.S. government has been issuing government debt while simultaneously decreasing the expected surpluses to back up the debt. The result has been a widening of the valuation gap to ten times GDP at the end of the sample. The puzzle will further deepen with the large deficits caused by the coronavirus crisis of 2020.

Equation (2) lets us interpret the puzzle further. The first term on the right-hand side, the risk-neutrally discounted present value of surpluses, is just about zero since the average primary surplus is about zero in our sample. Therefore, the entire wedge of 299% of GDP stems from the differential riskiness of the revenue and the spending claims. Put differently, without the covariance terms, the government would need to generate about 75% of GDP in PDV of future surpluses to support 75% in debt relative to GDP. With the covariance terms present, 335% of GDP (75%+260%) in future surpluses are needed to back the same debt.

5.2.2 The Risk Premium Puzzle

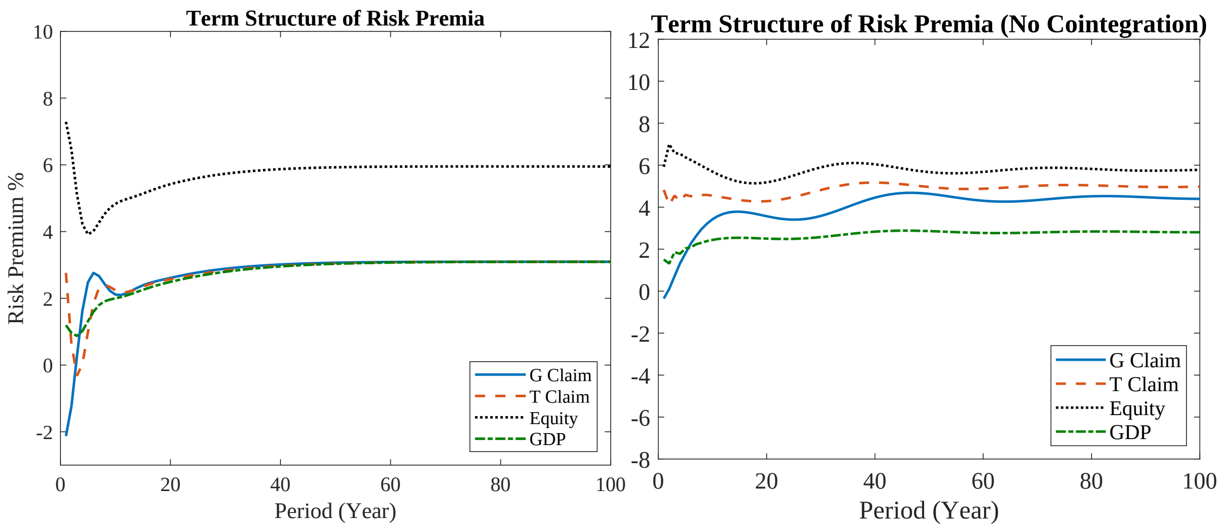
Figure 7 plots the risk premia on revenue and spending strips. For comparison, it also plots the risk premia on GDP strips and stock market dividend strips. The strip maturities run from 1 to 100 years. At the short end of the maturity spectrum (1-5 years), risk premia on spending strips are very low, at -2%. Because spending is counter-cyclical these strips are a hedge. In sharp contrast, short-maturity revenue strips have high risk premia (+2%) because their cash flows are low in high marginal utility times. The average risk premium of tax claim over the five-year horizon is 1.02%, much larger than that of the spending claim, -1.03%.

As we move to long maturities, risk premia on revenue and spending strips converge towards each other. They also converge towards the risk premium on a GDP strip, as noted in Corollary 1. Because of cointegration, revenue and spending claims are as risky as GDP. Claims to GDP are like unlevered equity claims. They have risk premia well in excess of real bond risk premia but below (levered) equity risk premia, as shown in the graph.

In our sample, the average one-year nominal interest rate is $y_0^{\$}(1)$= 4.5% whereas the unconditional average one-year nominal GDP growth rate is $x_0 + \pi_0$= 6.2%. The risk-free interest rate is on average below the growth rate, as emphasized by Blanchard (2019). However, government tax and spending processes are sufficiently risky. Hence, their average nominal discount rates, $r_0^{\tau} = 6.93\%$ and $r_0^{g} = 6.84\%$, are above the average nominal GDP growth rate.[6] We generate these

[6]As derived in Appendix B.4, $r_0^g = x_0 + \pi_0 + \kappa_0^g - \overline{pg}(1 - \kappa_1^g)$, where $\overline{pg}$ is the long-run mean of the log price-dividend ratio on the G-claim, and κ_0^g and κ_1^g are linearization constants, and similar for the T-claim. Note that using

Figure 7: Term Structure of Risk Premia on the T-Claim and the G-Claim



The left panel plots the term structures of risk premia on the G (spending) claim, T (tax claim), equity and GDP claim under our benchmark model. The right panel plots these term structures under an alternative VAR that does not contain the tax-to-GDP ratio and the spending-to-GDP ratio.

discount rates while maintaining an excellent fit for the term structure of Treasury yields. The claim to surpluses reflects the risk of the government's future debt issuance strategy. Future net debt issuances at inopportune (high SDF) times make the overall bond portfolio riskier than individual Treasury bonds. Therefore, even if risk-free interest rates are often below growth rates, the risk premia on government tax and spending processes are large enough to generate a finite valuation for the surplus claim. We recall that when the unconditional expected returns on T- and G-claims are similar, the unconditional expected return on the government debt portfolio is equal to the expected return on the G- and T-claims. That is, the debt portfolio is highly risky.

5.3 Results Without Cointegration

We argued that imposing cointegration between tax revenues and GDP and spending and GDP is imperative to accurately describe fiscal dynamics. To help understand the role cointegration plays, it is useful to contrast the main results with those obtained in a model that does not impose cointegration. Intuitively, the lack of the cointegration dynamics implies that an increase in government spending-to-GDP is not offset by future reductions in spending-to-GDP or future increases in revenue-to-GDP. The increase in the future government spending, which tends to happen during recessions, becomes permanent. This feature makes the spending claim much safer.

this average nominal discount rate in a simple Gordon growth model $PD^G = \frac{1}{r_0^g - (x_0 + \pi_0)}$, delivers an average valuation ratio for the G-claim very close to the one reported in section 5.2.1.

For similar reasons, the lack of the cointegration makes the tax revenue claim much riskier, because a decline in the tax revenue during recessions also becomes permanent. As a result, the long-run discount rates for the revenue claim are much higher than those for the spending claim. This is illustrated in the right panel of Figure 7. The average risk premium on the tax revenue claim is 4.87%, substantially higher than the 4.42% risk premium on the spending claim. A comparison of the left and right panels of Figure 7 also shows that the assumption of cointegration is necessary for the convergence of the long-run risk premium of T- and G-claims to each other and to the risk premium on a GDP claim. In sum, cointegration helps reduce the riskiness of the aggregate debt portfolio.

5.4 Fiscal Measurability Constraint Revisited

The value equivalence in Proposition 1 implies the measurability constraint in (4). If the government can only issues one-period risk-free debt, the condition specializes to (5). Appendix F restates the measurability conditions in our exponentially affine framework. It shows that condition (5) is severely violated in the data; deviations are of the same order of magnitude as GDP. First, surpluses are trending with GDP; recall Figure 1 which shows that the surplus-to-GDP ratio is stationary. Therefore, every innovation to GDP permanently alters the cash flows that accrue to investors in the surplus claim. But with one-period risk-free debt, the value of government debt cannot move with that same GDP growth shock. The long-run GDP risk in the surplus cash flows cannot be replicated with a position in risk-free debt. Second, a positive (negative) innovation to spending (revenues) would need to be offset by future decreases in spending in present value. As discussed in Section 2 and Appendix G, we do not detect any evidence in the data to support this hypothesis. Cointegration imposes mean-reversion but not overshooting of surpluses.

If the yield curve spans all the innovations, as is the case in our affine framework, then there exists a dynamic portfolio in government debt $\widetilde{Q}^{\$}_{t-1,h+1}$ of various maturities that replicates the state-contingency of the surplus claim and satisfies (4). Similar spanning arguments were explored by Angeletos (2002) and Buera and Nicolini (2004). This portfolio looks very different from the government's actual bond portfolio.[7] This is not surprising. We need to construct a Treasury portfolio with long-run risk exposure equivalent to that of a claim to GDP.

[7] An exception is Bhandari, Evans, Golosov, and Sargent (2017) which implies a realistic optimal maturity structure in a Ramsey model with Epstein-Zin preferences. Karantounias (2018) shows such problem behaves very differently from one with standard CRRA preferences.

6 Alternative Explanations

We discuss five alternative explanations for the government valuation and risk premium puzzles but find that, ultimately, all of them fall short.

6.1 Bubbles and Limits to Arbitrage

The valuation gap can be interpreted as violation of the transversality condition (TVC) in Treasury markets, consistent with the presence of a rational bubble in the spirit of Samuelson (1958); Diamond (1965); Blanchard and Watson (1982).

Several pieces of evidence speak against this explanation. First, Brock (1982); Tirole (1982); Milgrom and Stokey (1982); Santos and Woodford (1997) argue that rational bubbles are hard to sustain in the presence of long-lived investors absent other frictions. In Appendix I, we show that a rational patient investor who pursues an investment strategy that buys all corporate equities and shorts the portfolio of all U.S. Treasuries earns a risk premium higher than the equity premium but receives cash flows that hedge the business cycle. While this is not an arbitrage in the strict sense, it is a high-Sharpe ratio strategy with an attractive cash-flow profile. The strategy harvests mostly positive cash flows in anticipation of a correction in Treasury markets. Limits to arbitrage (Shleifer and Vishny, 1997) would be needed to explain why rational investors may choose not to pursue such strategy. Possibly investors would suffer margin calls at inopportune since the market value of the portfolio would be marked down in recessions. Differences in investment horizon of investor and their delegated asset manager may also interfere.

Second, models that violate the TVC typically produce violations of TVCs in all long-lived assets. If the Treasury could run a Ponzi scheme, then why could a AAA-rated corporation not do the same?

Third, as Figure 6 shows, the valuation gap is growing faster than GDP, which is inconsistent with rational bubbles. In rational bubble models, the debt/GDP ratio declines over time.

Fourth, the rise in the sovereign CDS spread after the Great Financial Crisis, documented by Chernov, Schmid, and Schneider (2016); Pallara and Renne (2019), seems inconsistent with a rational bubble in Treasury debt.

Fifth, the TVC is unlikely to be violated because the risk-adjusted discount rate on the portfolio of Treasury debt is higher than the growth rate of GDP.

6.2 Convenience Yield

U.S. government bonds carry a convenience yield which makes Treasury yields lower than the safe rate of interest. Put differently, the convenience yield produces an additional source of revenue, because the U.S. Treasury can sell its bonds for more than their fundamental value. The question is how far this explanations can go towards accounting for the bond valuation puzzle.

The convenience yield, λ_t, is the government bonds' expected returns that investors are willing to forgo under the risk-neutral measure. Assuming a uniform convenience yield across the maturity spectrum, the Euler equation for a Treasury bond with maturity $h+1$ is:

$$e^{-\lambda_t} = \mathbb{E}_t \left[M_{t+1} \frac{P^{\$}_{t+1}(h)}{P^{\$}_t(h+1)} \right].$$

Proposition 6. If the TVC holds, the value of the government debt portfolio equals the value of future surpluses plus the value of future seigniorage revenue:

$$E_t \left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} \left(T_{t+j} - G_{t+j} + (1 - e^{-\lambda_{t+j}}) \sum_{h=1}^{H} Q^{\$}_{t+j,h} P^{\$}_{t+j}(h) \right) \right] = \sum_{h=0}^{H} Q^{\$}_{t-1,h+1} P^{\$}_t(h), \tag{17}$$

where $\sum_{h=0}^{H} Q^{\$}_{t-1,h+1} P^{\$}_t(h)$ on the right-hand side denotes the cum-dividend value of the government's debt portfolio at the start of period t, and $\sum_{h=1}^{H} Q^{\$}_{t+j,h} P^{\$}_{t+j}(h)$ on the left-hand side denotes the ex-dividend value of the government's debt portfolio at the end of period $t+j$.

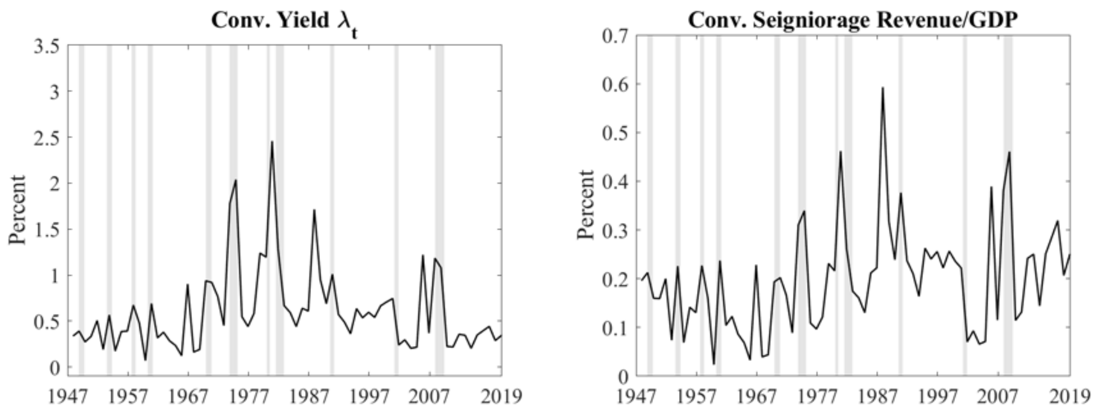
When there is no convenience yield, we end up back in the standard case of Proposition 1. If the quantity of current and future outstanding government debt is positive, then a positive convenience yield will always increase the value of government debt, acting as an additional source of revenue. This additional income is akin to seigniorage revenue and could potentially turn government deficits into surpluses.

As an empirical strategy, we measure the convenience yield following Krishnamurthy and Vissing-Jorgensen (2012). To proxy for λ_t, we use the weighted average of the Aaa-Treasury yield spread and the high-grade commercial papers-bills yield spread where the time series of weights are computed to match the duration of the government bond portfolio period by period. The left panel of Figure 8 shows the time series of the convenience yield. Over the sample period from 1947 to 2019, the average convenience yield is 0.60% per year, which implies average seigniorage revenue of $11.53 billions per year, or 0.20% of U.S. GDP as shown in the right panel of Figure 8. The figure also illustrates the counter-cyclical nature of the convenience yield and seigniorage revenue. Appendix C shows that this convenience yield measure is close to other measures proposed in the literature.

We rewrite equation (17) as:

$$E_t \left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} T_{t+j} K_{t+j} \right] - E_t \left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} G_{t+j} \right] = \sum_{h=0}^{K} Q^{\$}_{t-1,h+1} P^{\$}_t(h),$$

Figure 8: Convenience Yield and Seigniorage Revenue



The left panel reports the annual convenience yield time series time series for λ_t, computed as the weighted average of Aaa-Treasury and high-grade commercial papers-bills yield spreads. The right panel reports time series of the seigniorage revenue from convenience scaled by GDP, $(1 - e^{-\lambda_t})D_t/GDP$. The sample period is from 1947 until 2019.

where:

$$K_{t+j} = 1 + \frac{(1 - e^{-\lambda_{t+j}}) \sum_{h=1}^{H} Q^{\$}_{t+j,h} P^{\$}_{t+j}(h)}{T_{t+j}}.$$

We introduce the log growth rate $\Delta \log K_t$ as an additional state variable in the VAR. The augmented state vector is $\tilde{z}_t = [z_t, \Delta \log K_t]$. The seigniorage term $\log K_t$ follows the process: $\Delta \log K_{t+1} = e_k' \tilde{z}_{t+1}$, with an unconditional mean of zero because $\log K_t$ is stationary.

We use the same method as in Proposition 5 to price the modified tax claim. The new pricing formula for the revenue claim is:

$$E_t \left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} T_{t+j} K_{t+j} \right] = T_t K_t \cdot PD^k_t,$$

where PD^k_t is a function of the state variables $\tilde{z}_t$.

The left panel of Figure 9 reports the present value of the surplus in the red solid line for the model with convenience and in dashed green line for the benchmark model without convenience. The convenience yield increases the present value of the government surplus. On average, the present value of surplus with the seigniorage revenue is -103.72%, compared to the average surplus present value without the seigniorage revenue (-260.37%). Although the effect is sizable, the gap between the market value of public debt and the present value of government surplus remains. This may be a surprising result given the large perceived convenience yield on Treasuries. There are two offsetting effects at work. On the one hand, there is positive seigniorage revenue which increases the surplus and its present value. On the other hand, the higher the convenience yield the higher the true risk-free rate given observed bond yields. Higher safe rates increase the discount rate of future revenues and spending, lowering the present value of surpluses. The positive cash flow effect is offset by the negative discount rate effect, leaving the present value of the surplus nearly unaltered.

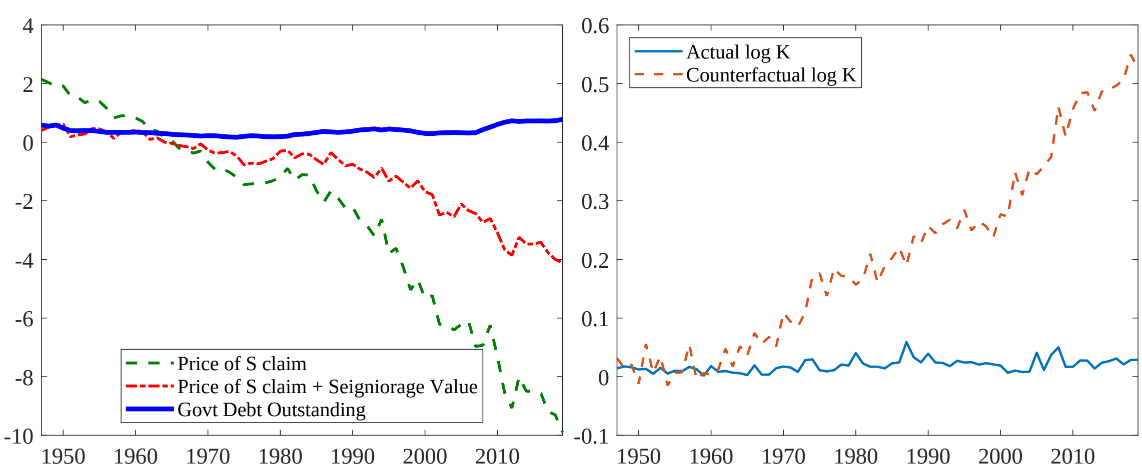
How large does the seigniorage revenue need to be to resolve the puzzle? To answer this question, we fix the VAR and market price of risk parameters and change the seigniorage revenue term from $\log K_t$ to $\log \widetilde{K}_t$ so that:

$$T_t \widetilde{K}_t \widetilde{PD}_t^k - G_t P_t^G = \sum_{h=0}^{H} Q_{t-1,h+1}^{\$} P_t^{\$}(h)$$

Since the last element of $\tilde{z}_t$ is $\Delta \log \widetilde{K}_t$, $\log \widetilde{K}_t$ enters this equation through both $\widetilde{K}_t$ and $\widetilde{PD}_t^k$. We solve for variable $\log \widetilde{K}_t$ in this equation, taking other variables as given. The right panel of Figure 9 reports the resulting $\widetilde{K}_t$ process in the dashed line alongside the actual K_t process in the solid line. Seigniorage revenue would need to be 20.57% of tax revenue on average to match the present value of the government surplus claim to the actual debt value, and more than 41.35% in the last twenty years of the sample. Actual seigniorage revenue only averages 1.90% of tax revenue. In sum, the convenience yield would have to be an order of magnitude larger to bridge the gap.

Some have argued that the convenience yields are larger than implied by the AAA-Treasury

Figure 9: Present Value of Government Surpluses and Seigniorage Revenue



The left panel plots the present value of government surpluses with and without seigniorage revenue, scaled by GDP. The right panel plots the actual and the counterfactual seigniorage revenue process K_t and $\widetilde{K}_t$.

spread. For example, (Jiang, Krishnamurthy, and Lustig, 2018) argue that foreigners earn convenience form dollar assets, including investment-grade corporate bonds. Subtracting Treasury from U.S. AAA corporate yields removes that dollar safety premium. It remains an open question whether the convenience yields needed to close the gap are consistent with the data. As the supply of safe assets increases, convenience yields may decline (Krishnamurthy and Vissing-Jorgensen, 2012) or disappear altogether if the U.S. dollar were to lose its privileged role in the world financial system (Farhi, Gourinchas, and Rey, 2011; Farhi and Maggiori, 2017; He, Krishnamurthy, and Milbradt, 2019).

6.3 Austerity as a Peso Event

Next, we consider a model in which bond investors price in the possibility of a major government spending cut.[8] However, such radical austerity never occurs in our 70-year sample. How large should the spending cut probability be in order to match the market valuation of the government debt to the present value of government surpluses?

We fix the spending cut at 2 times the standard deviation of the log spending-to-GDP shock. When it happens, the spending-to-GDP ratio decreases by $2 \times 3.90\% = 8\%$ of U.S. GDP. Moreover, we assume that the spending cut is permanent. Specifically, we assume that the long-run mean of spending-to-GDP, g_0, falls from its full-sample average of 11.0% of GDP to 3% of GDP. When the peso event happens, log spending g_t shifts down by $\ell = \log(.08/.11)$ as does the long-run mean g_0. The dynamics of the demeaned state variables from that point forward are still given by the benchmark VAR, including the processes of tax and spending. As a result, the price of the G-claim scaled by GDP is simply $\ell g_t PD_t^G$ when the peso event happens. The peso event itself is not priced; we do not change the market prices of risk Λ_t.[9]

Let ϕ_t be the probability of this peso event. We back out ϕ_t by equalizing the market value of the debt to the present value of surpluses:

$$D_t/GDP_t = \tau_t PD_t^T - (1-\phi_t) g_t PD_t^G - \phi_t \ell g_t PD_t^G, \quad \forall t.$$

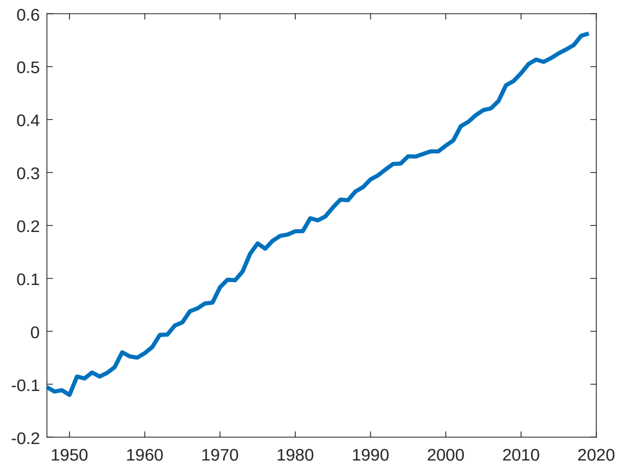
This equation can easily be solved for $\{\phi_t\}$, and the resulting time series is shown in Figure 10. The average gap between the market value of debt and the present value of surpluses under the benchmark model exceeds two hundred percent of GDP and grows in magnitude in the last several decades of the sample. To match such a large gap, the probability of the spending cut has to be large and growing. The spending cut probability is 21.0% on average and rises to 55% at the

[8] Increasing tax revenue would be an alternative way to engineer a fiscal correction; the results would be similar.

[9] If the fiscal correction took place in high marginal utility states, as in a rare disaster model, the implied probability of these fiscal corrections would likely be smaller. But that strikes us as implausible. Governments do not suddenly switch to running large primary surpluses in bad states of the world.

end of the sample. Such a large probability is at odds with the notion of a peso event that never happens in a 70-year sample. We interpret this result as a restatement rather than a resolution of the puzzle.

Figure 10: Probabilities of Spending Cut Implied by Debt-to-GDP Ratio



This figure reports the time series of probabilities of spending cuts implied by the debt to GDP ratio, ϕ_t.

6.4 Other Government Assets and Liabilities

The government owns various assets, including outstanding student loans and other credit transactions, cash balances, and various financial instruments. Based on Congressional Budget Office data, the total value of these government assets is 8.8% of the GDP as of 2018. While these assets bring the net government debt held by the public from 77.8% to 69.1% of GDP, the bulk of the government debt valuation puzzle remains.

Other significant sources of government revenues and outlays are those associated with the Social Security Administration (SSA). Based on the CBO data, net flows from the SSA are close to 0 as of 2018, but will turn into a deficit of 0.7% of GDP per annum from 2020 to 2029. As the SSA turns from a net contributor of primary surpluses into a net contributor to the deficit in 2019 and beyond, the government will need to issue additional debt to the public. Absent new spending cuts or tax increases, this will deepen the puzzle.

6.5 Market Segmentation

One could argue that marginal investors in Treasury bonds do not necessarily overlap with investors in the U.S. equity market. For example, foreign ownership of Treasuries has increased dramatically since the mid 1990s (see Favilukis, Kohn, Ludvigson, and Nieuwerburgh, 2013) to

Table 5: Inelastic Demand: Returns on U.S. Treasury Purchases

	dollar-weighted	time-weighted	gap
Fed	2.58%	4.87%	2.29%
Foreign	3.24%	4.87%	1.63%
Fed+Foreign	3.06%	4.87%	1.81%

Source: Federal Flow of Funds data. Cash flows invested in Bloomberg Treasury Index. The dollar-weighted return is the nominal IRR on all the cash flows invested by foreign investors (the Fed) in the Bloomberg Treasury Index. Flow of Funds Table F106: Monetary authority; other Treasury securities, excluding Treasury bills; asset, and Rest of the world; other Treasury securities, excluding Treasury bills and certificates; asset. The sample is 2000-2019.

about 40% of holdings at the peak in 2008, or as much as holdings of domestic investors excluding the Federal Reserve system (mutual funds, pension funds, banks, and insurance companies). Can market segmentation resolve the government debt risk premium puzzle?

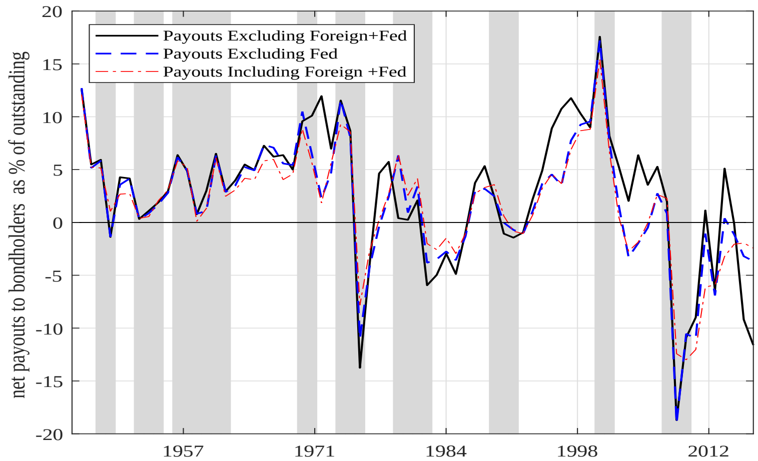
Whatever SDFs foreign investors use, the projections of their SDFs on the state space must agree with those of the domestic investors as far as bond pricing is concerned in the absence of arbitrage. To investigate this possibility, we estimate a model that only focuses on matching bond yields. It is worth noting that zeroing out the stock market risk factors presents an extreme case of segmentation since government bond investors are almost certainly exposed to some U.S. stock market risks. We find that estimates of $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$ remain similar to those in our benchmark estimation. We also find that the debt valuation and risk premium puzzles remain. This type of market segmentation does not resolve the puzzle.

In addition to the rise in foreign holdings, the Fed has substantially increased its holdings of Treasuries in the aftermath of the 2008-09 financial crisis. Table 5 reports the dollar-weighted returns earned by foreign investors and the Fed. The dollar-weighted returns are 181 bps per annum lower than the time-weighted returns (geometric mean return). Foreign investors and the Fed display poor timing skill when investing in U.S. Treasuries. Put differently, they have inelastic demand (Krishnamurthy and Lustig, 2019).

If we take the view that foreign and Fed demand are completely inelastic, it is natural to adjust the net payouts to bond holders by excluding payouts to the Fed and foreign investors. Figure 11 plots the net payouts to bondholders excluding the Fed and foreign investors as a fraction of the face value of the Treasuries outstanding. Especially in the last 2 recessions, the cash flows paid out to bondholders seem just about as pro-cyclical when we exclude the Fed and foreign investors than when we do not. Hence, inelastic demand by the Fed and foreign investors does not mitigate the pro-cyclicality of the cash flows absorbed by U.S. investors.

Finally, if the marginal investor in U.S. Treasuries faces pro-cyclical marginal tax rates, as suggested by Longstaff (2011), then the after-tax cash flows on the entire Treasury portfolio would become less pro-cyclical. This would reduce the riskiness of the Treasury portfolio. Given the

Figure 11: Net Payouts to Bondholders Excluding the Fed and Foreign Investors



The figure plots annual net payouts to bondholders as a fraction of the lagged face value. The red line includes the payouts to the Fed and to foreign investors and includes Fed and foreign holdings in the denominator. The blue excludes the payouts to the Fed and Fed holdings in the denominator. The black line excludes payouts to the Fed and foreign investors. To compute the payouts to bondholders excluding the Fed and foreigners, we start with the Federal government; interest paid (IMA) (FA316130001.A) from Table F106 in the Flow of Funds. The interest paid is scaled down by the fraction of debt held by the Fed (LM713061103.A) and Foreigners (LM713061103.A) from Table L210. To compute net issuance, we take the Federal government; net lending (+) or borrowing (-) (financial account) (FA315000005.A) from Table F106. Then we subtract purchases by the Fed (Monetary authority; Treasury securities; asset; Monetary authority; Treasury securities; asset) and purchases by foreigners (the Rest of the world; Treasury securities; FA263061105.A) from Table F210. Finally, we add the new interest paid series to the new payout series. We divide these payouts by the face value of outstanding bonds excluding Foreign and Fed holdings. Annual data from the Flow of Funds.

large size of foreign, Fed, and tax-exempt domestic institutional holdings of U.S. Treasuries, it is unlikely that this argument has much bite.

7 Conclusion

Because government deficits tend to occur in recessions, times when bond investors face high marginal utility, governments must tap debt markets at inopportune times. This consideration imposes novel no-arbitrage restrictions which affect inference on the riskiness of the overall government debt portfolio. The government debt portfolio is a risky claim whose expected return far exceeds risk-free bond yields. We quantify that the increase in riskiness lowers the government's debt capacity by 299% of GDP. The negative effects of the 2020 covid pandemic on current and future primary surpluses will add to this number. The pricing of U.S. Treasury debt violates the no-arbitrage restrictions implied by the government budget constraint, a violation we call the government debt valuation puzzle. We show that the valuation of debt cannot be reconciled with rational expectations, provided that a no-bubble condition holds. Conventional estimates of convenience yields cannot explain it either. Perhaps investors expect an unprecedented fiscal correction. If so, we show that they have been expecting a correction for a long time, and have been assigning ever-increasing probability to the event, in violation of rational expectations. More work is needed to compare the U.S. to other countries using our approach.

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Online Appendix for The U.S. Public Debt Valuation Puzzle

A Proofs of Propositions

Proposition 1

Proof. All objects in this appendix are in nominal terms but we drop the superscript $^\$$ for ease of notation. The government faces the following one-period budget constraint:

$$G_t - T_t + Q^1_{t-1} = \sum_{h=1}^{H} (Q^h_t - Q^{h+1}_{t-1}) P^h_t,$$

where G_t is total nominal government spending, T_t is total nominal government revenue, Q^h_t is the number of nominal zero-coupon bonds of maturity h outstanding in period t each promising to pay back \$1 at time $t+h$, and P^h_t is today's price for a h-period zero-coupon bond with \$1 face value. A unit of $h+1$-period bonds issued at $t-1$ becomes a unit of h-period bonds in period t. That is, the stock of bonds evolves of each maturity evolves according to $Q^h_t = Q^{h+1}_{t-1} + \Delta Q^h_t$. Note that this notation can easily handle coupon-bearing bonds. For any bond with deterministic cash-flow sequence, we can write the price (present value) of the bond as the sum of the present values of each of its coupons.

The left-hand side of the budget constraint denotes new financing needs in the current period, due to primary deficit $G-T$ and one-period debt from last period that is now maturing. The right hand side shows that the money is raised by issuing new bonds of various maturities. Alternatively, we can write the budget constraint as total expenses equalling total income:

$$G_t + Q^1_{t-1} + \sum_{h=1}^{H} Q^{h+1}_{t-1} P^h_t = T_t + \sum_{h=1}^{H} Q^h_t P^h_t,$$

We can now iterate the budget constraint forward. The period t constraint is given by:

$$\begin{aligned} T_t - G_t \quad &= \quad Q^1_{t-1} - Q^1_t P^1_t + Q^2_{t-1} P^1_t - Q^2_t P^2_t + Q^3_{t-1} P^2_t - Q^3_t P^3_t \\ &\quad + \cdots - Q^H_t P^H_t + Q^{H+1}_{t-1} P^H_t. \end{aligned}$$

Consider the period-$t+1$ constraint,

$$\begin{aligned} T_{t+1} - G_{t+1} \quad &= \quad Q^1_t - Q^1_{t+1} P^1_{t+1} + Q^2_t P^1_{t+1} - Q^2_{t+1} P^2_{t+1} + Q^3_t P^2_{t+1} - Q^3_{t+1} P^3_{t+1} \\ &\quad + \cdots - Q^H_{t+1} P^H_{t+1} + Q^{H+1}_t P^H_{t+1}. \end{aligned}$$

multiply both sides by M_{t+1}, and take expectations conditional on time t:

$$\begin{aligned} \mathbb{E}_t\left[M_{t+1}(T_{t+1} - G_{t+1})\right] \quad &= \quad Q^1_t P^1_t - \mathbb{E}_t[Q^1_{t+1} M_{t+1} P^1_{t+1}] + Q^2_t P^2_t - \mathbb{E}_t[Q^2_{t+1} M_{t+1} P^2_{t+1}] + Q^3_t P^3_t \\ &\quad - \mathbb{E}_t[Q^3_{t+1} M_{t+1} P^3_{t+1}] + \cdots + Q^H_t P^H_t \\ &\quad - \mathbb{E}_t[Q^H_{t+1} M_{t+1} P^H_{t+1}] + Q^{H+1}_t P^{H+1}_t, \end{aligned}$$

where we use the asset pricing equations $\mathbb{E}_t[M_{t+1}] = P^1_t$, $\mathbb{E}_t[M_{t+1} P^1_{t+1}] = P^2_t, \cdots, \mathbb{E}_t[M_{t+1} P^{H-1}_{t+1}] = P^H_t$, and $\mathbb{E}_t[M_{t+1} P^H_{t+1}] = P^{H+1}_t$.

Consider the period $t+2$ constraint, multiplied by $M_{t+1}M_{t+2}$ and take time-t expectations:

$$\begin{aligned} \mathbb{E}_t\left[M_{t+1}M_{t+2}(T_{t+2} - G_{t+2})\right] \quad &= \quad \mathbb{E}_t[Q^1_{t+1} M_{t+1} P^1_{t+1}] - \mathbb{E}_t[Q^1_{t+2} M_{t+1} M_{t+2} P^1_{t+2}] + \mathbb{E}_t[Q^2_{t+1} M_{t+1} P^2_{t+1}] \\ &\quad - \mathbb{E}_t[Q^2_{t+2} M_{t+1} M_{t+2} P^2_{t+2}] + \mathbb{E}_t[Q^3_{t+1} M_{t+1} P^3_{t+1}] - \cdots \\ &\quad + \mathbb{E}_t[Q^H_{t+1} M_{t+1} P^H_{t+1}] - \mathbb{E}_t[Q^H_{t+2} M_{t+1} M_{t+2} P^H_{t+2}] + \mathbb{E}_t[Q^{H+1}_{t+1} M_{t+1} P^{H+1}_{t+1}], \end{aligned}$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M_{t+2}] = P^1_{t+1}$, $\mathbb{E}_{t+1}[M_{t+2} P^1_{t+2}] = P^2_{t+1}$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected discounted surpluses at t, $t+1$, and $t+2$ we get:

$$\begin{aligned} T_t - G_t + \mathbb{E}_t\left[M_{t+1}(T_{t+1} - G_{t+1})\right] + \mathbb{E}_t\left[M_{t+1}M_{t+2}(T_{t+2} - G_{t+2})\right] = \textstyle\sum_{h=0}^{H} Q^{h+1}_{t-1} P^h_t + \\ - \mathbb{E}_t[Q^1_{t+2} M_{t+1} M_{t+2} P^1_{t+2}] - \mathbb{E}_t[Q^2_{t+2} M_{t+1} M_{t+2} P^2_{t+2}] - \cdots - - \mathbb{E}_t[Q^H_{t+2} M_{t+1} M_{t+2} P^H_{t+2}]. \end{aligned}$$

Similarly consider the one-period government budget constraints at times $t+3$, $t+4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t+J$, we get:

$$\sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{J} M_{t,t+j} (T_{t+j} - G_{t+j}) \right] + \mathbb{E}_t \left[M_{t,t+J} \sum_{h=1}^{H} Q_{t+J}^h P_{t+J}^h \right]$$

where we used the cumulate SDF notation $M_{t,t+j} = \prod_{i=0}^{j} M_{t+i}$ and by convention $M_{t,t} = M_t = 1$ and $P_t^0 = 1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next J years plus the present value of the government bond portfolio that will be outstanding at time $t+J$. The latter is the cost the government will face at time $t+J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J \to \infty$:

$$\sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (T_{t+j} - G_{t+j}) \right] + \lim_{J\to\infty} \mathbb{E}_t \left[M_{t,t+J} \sum_{h=1}^{H} Q_{t+J}^h P_{t+J}^h \right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j} - G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the transversality condition:

$$\lim_{J\to\infty} \mathbb{E}_t \left[M_{t,t+J} \sum_{h=1}^{H} Q_{t+J}^h P_{t+J}^h \right] = 0.$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise there is a government debt bubble.

If the transversality condition is satisfied, the outstanding debt today, D_t, reflects the expected present-discounted value of the current and all future primary surpluses:

$$D_t = \sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (T_{t+j} - G_{t+j}) \right].$$

This is equation (1) in the main text.

Case with Default

Proof. We consider only full default, without loss of generality. Alternatively, we can write the budget constraint that obtains in case of no default at t:

$$G_t + Q_{t-1}^1 + \sum_{h=1}^{H} Q_{t-1}^{h+1} P_t^h = T_t + \sum_{h=1}^{H} Q_t^h P_t^h,$$

and, in case of default at t, the one-period budget constraint is given by:

$$G_t = T_t + \sum_{h=1}^{H} Q_t^h P_t^h.$$

We can now iterate the budget constraint forward. In case of no default, the period t constraint is given by:

$$\begin{aligned} T_t - G_t \quad &= \quad Q_{t-1}^1 - Q_t^1 P_t^1 + Q_{t-1}^2 P_t^1 - Q_t^2 P_t^2 + Q_{t-1}^3 P_t^2 - Q_t^3 P_t^3 \\ &\quad + \cdots - Q_t^H P_t^H + Q_{t-1}^{H+1} P_t^H. \end{aligned}$$

In case of default, the period t constraint is given by:

$$T_t - G_t \quad = \quad -Q_t^1 P_t^1 - Q_t^2 P_t^2 - Q_t^3 P_t^3 - Q_t^H P_t^H$$

First, consider the period-$t+1$ constraint in case of no default,

$$\begin{aligned} T_{t+1} - G_{t+1} \quad &= \quad Q_t^1 - Q_{t+1}^1 P_{t+1}^1 + Q_t^2 P_{t+1}^1 - Q_{t+1}^2 P_{t+1}^2 + Q_t^3 P_{t+1}^2 - Q_{t+1}^3 P_{t+1}^3 \\ &\quad + \cdots - Q_{t+1}^H P_{t+1}^H + Q_t^{H+1} P_{t+1}^H. \end{aligned}$$

Second, consider the period-$t+1$ constraint in case of default,

$$T_{t+1} - G_{t+1} \quad = \quad -Q_{t+1}^1 P_{t+1}^1 - Q_{t+1}^2 P_{t+1}^2 - Q_{t+1}^3 P_{t+1}^3 - Q_{t+1}^H P_{t+1}^H.$$

We use χ_t as an indicator variable for default. To simplify, we consider only full default with zero recovery. This is without loss of generality. Next, multiply both sides of the no default constraint by $(1-\chi_{t+1})M_{t+1}$, and take expectations conditional on time t:

$$\begin{aligned}\mathbb{E}_t\left[M_{t+1}(1-\chi_{t+1})(T_{t+1}-G_{t+1})\right] &= Q_t^1\mathbb{E}_t\left[M_{t+1}(1-\chi_{t+1})\right]-\mathbb{E}_t[Q_{t+1}^1(1-\chi_{t+1})M_{t+1}P_{t+1}^1]+E_t[(1-\chi_{t+1})M_{t+1}P_{t+1}^1]Q_t^2\\ &-\mathbb{E}_t[Q_{t+1}^2(1-\chi_{t+1})M_{t+1}P_{t+1}^2]+E_t[M_{t+1}(1-\chi_{t+1})P_{t+1}^2]Q_t^3-\mathbb{E}_t[Q_{t+1}^3(1-\chi_{t+1})M_{t+1}P_{t+1}^3]+\\ &\cdots+Q_t^H E_t[M_{t+1}(1-\chi)P_{t+1}^{H-1}]-\mathbb{E}_t[Q_{t+1}^H(1-\chi_{t+1})M_{t+1}P_{t+1}^H]+Q_t^{H+1}E_t[M_{t+1}(1-\chi_{t+1})P_{t+1}^H],\end{aligned}$$

and multiply both sides of the default constraint by $M_{t+1}\chi_{t+1}$

$$\begin{aligned}\mathbb{E}_t\left[M_{t+1}\chi_{t+1}(T_{t+1}-G_{t+1})\right] &= -\mathbb{E}_t[Q_{t+1}^1\chi_{t+1}M_{t+1}P_{t+1}^1]-\mathbb{E}_t[Q_{t+1}^2\chi_{t+1}M_{t+1}P_{t+1}^2]\\ &-\mathbb{E}_t[Q_{t+1}^3\chi_{t+1}M_{t+1}P_{t+1}^3]+\cdots-\mathbb{E}_t[Q_{t+1}^H\chi_{t+1}M_{t+1}P_{t+1}^H].\end{aligned}$$

By adding these 2 constraints, we obtain the following expression:

$$\begin{aligned}\mathbb{E}_t\left[M_{t+1}(T_{t+1}-G_{t+1})\right] &= Q_t^1\mathbb{E}_t\left[M_{t+1}(1-\chi_{t+1})\right]-\mathbb{E}_t[Q_{t+1}^1M_{t+1}P_{t+1}^1]+E_t[(1-\chi_{t+1})M_{t+1}P_{t+1}^1]Q_t^2\\ &-\mathbb{E}_t[Q_{t+1}^2M_{t+1}P_{t+1}^2]+E_t[M_{t+1}(1-\chi_{t+1})P_{t+1}^2]Q_t^3\\ &-\mathbb{E}_t[Q_{t+1}^3M_{t+1}P_{t+1}^3]+\cdots+Q_t^H E_t[M_{t+1}(1-\chi)P_{t+1}^{H-1}]-\mathbb{E}_t[Q_{t+1}^H M_{t+1}P_{t+1}^H]+Q_t^{H+1}E_t[M_{t+1}(1-\chi_{t+1})P_{t+1}^H].\end{aligned}$$

This can be restated as:

$$\begin{aligned}\mathbb{E}_t\left[M_{t+1}(T_{t+1}-G_{t+1})\right] &= Q_t^1P_t^1-\mathbb{E}_t[Q_{t+1}^1M_{t+1}P_{t+1}^1]+Q_t^2P_t^2-\mathbb{E}_t[Q_{t+1}^2M_{t+1}P_{t+1}^2]+Q_t^3P_t^3\\ &-\mathbb{E}_t[Q_{t+1}^3M_{t+1}P_{t+1}^3]+\cdots+Q_t^HP_t^H-\mathbb{E}_t[Q_{t+1}^HM_{t+1}P_{t+1}^H]+Q_t^{H+1}P_t^{H+1},\end{aligned}$$

where we use the asset pricing equations $\mathbb{E}_t\left[M_{t+1}(1-\chi_{t+1})\right]=P_t^1$, $\mathbb{E}_t[M_{t+1}(1-\chi_{t+1})P_{t+1}^1]=P_t^2,\cdots,\mathbb{E}_t[M_{t+1}(1-\chi_{t+1})P_{t+1}^{H-1}]=P_t^H$, and $\mathbb{E}_t[M_{t+1}(1-\chi_{t+1})P_{t+1}^H]=P_t^{H+1}$.

The rest of the proof is essentially unchanged. Consider the period $t+2$ constraint, multiplied by $M_{t+1}M_{t+2}(1-\chi_{t+2})$ in the no-default case, and $M_{t+1}M_{t+2}(\chi_{t+2})$ for the default case, and take time-t expectations (after adding default and no-default states):

$$\begin{aligned}\mathbb{E}_t\left[M_{t+1}M_{t+2}(T_{t+2}-G_{t+2})\right] &= \mathbb{E}_t[Q_{t+1}^1M_{t+1}P_{t+1}^1]-\mathbb{E}_t[Q_{t+2}^1M_{t+1}M_{t+2}P_{t+2}^1]+\mathbb{E}_t[Q_{t+1}^2M_{t+1}P_{t+1}^2]\\ &-\mathbb{E}_t[Q_{t+2}^2M_{t+1}M_{t+2}P_{t+2}^2]+\mathbb{E}_t[Q_{t+1}^3M_{t+1}P_{t+1}^3]-\cdots\\ &+\mathbb{E}_t[Q_{t+1}^HM_{t+1}P_{t+1}^H]-\mathbb{E}_t[Q_{t+2}^HM_{t+1}M_{t+2}P_{t+2}^H]+\mathbb{E}_t[Q_{t+1}^{H+1}M_{t+1}P_{t+1}^{H+1}],\end{aligned}$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M_{t+2}(1-\chi_{t+2})]=P_{t+1}^1$, $\mathbb{E}_{t+1}[M_{t+2}(1-\chi_{t+2})P_{t+2}^1]=P_{t+1}^2$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected discounted surpluses at t, $t+1$, and $t+2$ we get:

$$\begin{aligned}T_t-G_t+\mathbb{E}_t\left[M_{t+1}(T_{t+1}-G_{t+1})\right]+\mathbb{E}_t\left[M_{t+1}M_{t+2}(T_{t+2}-G_{t+2})\right]=\textstyle\sum_{h=0}^H Q_{t-1}^{h+1}P_t^h+\\ -\mathbb{E}_t[Q_{t+2}^1M_{t+1}M_{t+2}P_{t+2}^1]-\mathbb{E}_t[Q_{t+2}^2M_{t+1}M_{t+2}P_{t+2}^2]-\cdots--\mathbb{E}_t[Q_{t+2}^HM_{t+1}M_{t+2}P_{t+2}^H].\end{aligned}$$

Similarly consider the one-period government budget constraints at times $t+3$, $t+4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t+J$, we get:

$$\sum_{h=0}^H Q_{t-1}^{h+1}P_t^h = \mathbb{E}_t\left[\sum_{j=0}^J M_{t,t+j}(T_{t+j}-G_{t+j})\right]+\mathbb{E}_t\left[M_{t,t+J}\sum_{h=1}^H Q_{t+J}^hP_{t+J}^h\right]$$

where we used the cumulate SDF notation $M_{t,t+j}=\prod_{i=0}^j M_{t+i}$ and by convention $M_{t,t}=M_t=1$ and $P_t^0=1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next J years plus the present value of the government bond portfolio that will be outstanding at time $t+J$. The latter is the cost the government will face at time $t+J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J\to\infty$:

$$\sum_{h=0}^H Q_{t-1}^{h+1}P_t^h = \mathbb{E}_t\left[\sum_{j=0}^\infty M_{t,t+j}(T_{t+j}-G_{t+j})\right]+\lim_{J\to\infty}\mathbb{E}_t\left[M_{t,t+J}\sum_{h=1}^H Q_{t+J}^hP_{t+J}^h\right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j}-G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the transversality condition:

$$\lim_{J\to\infty} \mathbb{E}_t \left[M_{t,t+J} \sum_{h=1}^{H} Q^h_{t+J} P^h_{t+J} \right] = 0.$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise there is a government debt bubble.

If the transversality condition is satisfied, the outstanding debt today, D_t, reflects the expected present-discounted value of the current and all future primary surpluses:

$$D_t = \sum_{h=0}^{H} Q^{h+1}_{t-1} P^h_t \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (T_{t+j} - G_{t+j}) \right].$$

This is equation (1) in the main text.

□

Proposition 2

From the time-t budget constraint, we get that the primary surplus

$$-S_t = -Q^1_{t-1} + \sum_{h=1}^{H} (Q^h_t - Q^{h+1}_{t-1}) P^h_t.$$

It follows that

$$D_t - S_t = \sum_{h=0}^{H} Q^{h+1}_{t-1} P^h_t - Q^1_{t-1} + \sum_{h=1}^{H} (Q^h_t - Q^{h+1}_{t-1}) P^h_t = \sum_{h=1}^{H} Q^h_t P^h_t.$$

We obtain equation (3) in the main text.

$$\begin{aligned} r^D_{t+1}(D_t - S_t) &= \sum_{h=0}^{\infty} P^{\$}_{t+1}(h) Q^{\$}_{t,h+1} = D_{t+1} = P^T_{t+1} - P^G_{t+1} \\ &= (P^T_t - T_t) r^\tau_{t+1} - (P^G_{t+1} - G_t) r^g_{t+1}. \end{aligned}$$

□

Proposition 3

Proof. We follow the proof in the working paper version of Backus, Boyarchenko, and Chernov (2018) on page 16 (Example 5). Hansen and Scheinkman (2009) consider the following equation:

$$\mathbb{E}_t[M_{t,t+1} v_{t+1}] = \nu v_t, \tag{A.1}$$

where ν is the dominant eigenvalue and v_t is the eigenfunction. Claims to stationary cash flows earn a return equal to the yield on the long bond. We consider the following decomposition of the pricing kernel:

$$M^1_{t,t+1} = M_{t,t+1} v_{t+1} / \nu v_t, \tag{A.2}$$

$$M^2_{t,t+1} = \nu v_t / v_{t+1}. \tag{A.3}$$

By construction, $\mathbb{E}_t[M^1_{t,t+1}] = 1$. The long yields converge to $-\log \nu$. The long-run bond return converges to $\lim_{n\to\infty} R^n_{t,t+1} = \frac{1}{M^2_{t,t+1}} = v_{t+1}/\nu v_t$. This implies that $\mathbb{E}[\log R^\infty_{t,t+1}] = -\log \nu$.

To value claims to uncertain cash flows with one-period growth rate $g_{t,t+1}$, we define $\widehat{p}^n_t$ to denote the price of a strip that pays off $d_{t,t+n}$, n periods from now.

$$\widehat{p}^n_t = \mathbb{E}_t[M_{t,t+1} g_{t,t+1} \widehat{p}^{n-1}_{t+1}] = \mathbb{E}_t[\widehat{M}_{t,t+1} \widehat{p}^{n-1}_{t+1}],$$

where $\widehat{M}_{t,t+1} = M_{t,t+1} g_{t,t+1}$. Consider the problem of finding the dominant eigenvalue:

$$\mathbb{E}_t[\widehat{M}_{t,t+1} \widehat{v}_{t+1}] = \nu \widehat{v}_t. \tag{A.4}$$

If the cash flows are stationary, then the same ν that solves this equation for $M_{t,t+1}$ in eqn. A.1 solves the one for $\widehat{M}_{t,t+1}$. Hence, if (ν, v_t) solves eqn. A.1, then $(\nu, v_t/d_t)$ solves the hat equation eqn. A.4. □

Proposition 6

Proof. Start from government budget constraint:

$$T_t - G_t = Q_{t-1}^1 + \sum_{k=1}^{K-1} Q_{t-1}^{k+1} P_t^k - \sum_{k=1}^{K} Q_t^k P_t^k$$

We assume these bond prices contain the same convenience yield λ_t:

$$\begin{aligned} E_t[M_{t+1}] &= P_t^1 e^{-\lambda_t}, \\ E_t[M_{t+1} P_{t+1}^1] &= P_t^2 e^{-\lambda_t}, \\ E_t[M_{t+1} P_{t+1}^K] &= P_t^{K+1} e^{-\lambda_t}. \end{aligned}$$

Consider the period-$t+1$ constraint, multiplied by M_{t+1}, and take expectations conditional at time t:

$$\begin{aligned} & E_t\left[M_{t+1}(T_{t+1} - G_{t+1})\right] \\ = \; & E_t[M_{t+1} Q_t^1 + \sum_{k=1}^{K-1} M_{t+1} Q_t^{k+1} P_{t+1}^k - \sum_{k=1}^{K} M_{t+1} Q_{t+1}^k P_{t+1}^k] \\ = \; & P_t^1 e^{-\lambda_t} Q_t^1 + \sum_{k=1}^{K-1} P_t^{k+1} e^{-\lambda_t} Q_t^{k+1} - E_t[M_{t+1} \sum_{k=1}^{K} Q_{t+1}^k P_{t+1}^k]. \end{aligned}$$

Consider the period-$t+2$ constraint, multiplied by $M_{t,t+2}$:

$$\begin{aligned} & E_t\left[M_{t,t+2}(T_{t+2} - G_{t+2})\right] \\ = \; & E_t[M_{t,t+2} Q_{t+1}^1 + \sum_{k=1}^{K-1} M_{t,t+2} Q_{t+1}^{k+1} P_{t+2}^k - \sum_{k=1}^{K} M_{t,t+2} Q_{t+2}^k P_{t+2}^k] \\ = \; & E_t[M_{t+1} P_{t+1}^1 e^{-\lambda_{t+1}} Q_{t+1}^1] + E_t[M_{t+1} \sum_{k=1}^{K-1} P_{t+1}^{k+1} e^{-\lambda_{t+1}} Q_{t+1}^{k+1}] - E_t[M_{t,t+2} \sum_{k=1}^{K} Q_{t+2}^k P_{t+2}^k], \end{aligned}$$

where we have used that

$$E_t[M_{t,t+2} Q_{t+1}^1] = E_t[M_{t,t+1} Q_{t+1}^1 E_{t+1} M_{t+1,t+2}] = E_t[M_{t,t+1} Q_{t+1}^1 e^{-\lambda_{t+1}} P_{t+1}^1],$$

and, similarly, that:

$$E_t[M_{t,t+2} Q_{t+1}^k P_{t+2}^k] = E_t[M_{t,t+1} Q_{t+1}^k E_{t+1} M_{t+1,t+2} P_{t+2}^k] = E_t[M_{t,t+1} Q_{t+1}^k e^{-\lambda_{t+1}} P_{t+1}^{k+1}].$$

By adding up the t, $t+1$ and $t+2$ constraint, we get that $E_t[T_t - G_t + M_{t+1}(T_{t+1} - G_{t+1}) + M_{t,t+2}(T_{t+2} - G_{t+2})]$ equals:

$$\begin{aligned} = \; & Q_{t-1}^1 + \sum_{k=1}^{K-1} Q_{t-1}^{k+1} P_t^k \\ + \; & P_t^1 (e^{-\lambda_t} - 1) Q_t^1 + \sum_{k=1}^{K-1} P_t^{k+1} (e^{-\lambda_t} - 1) Q_t^{k+1} \\ + \; & E_t[M_{t+1} P_{t+1}^1 (e^{-\lambda_{t+1}} - 1) Q_{t+1}^1] + E_t[M_{t+1} \sum_{k=1}^{K-1} P_{t+1}^{k+1} (e^{-\lambda_{t+1}} - 1) Q_{t+1}^{k+1}] - E_t[M_{t,t+2} \sum_{k=1}^{K} Q_{t+2}^k P_{t+2}^k]. \end{aligned}$$

Next, consider the period-$t+3$ constraint, multiplied by $M_{t,t+3}$

$$\begin{aligned} & E_t\left[M_{t,t+3}(T_{t+3} - G_{t+3})\right] \\ = \; & E_t[M_{t,t+3} Q_{t+2}^1 + \sum_{k=1}^{K-1} M_{t,t+3} Q_{t+2}^{k+1} P_{t+2}^k - \sum_{k=1}^{K} M_{t,t+3} Q_{t+2}^k P_{t+2}^k] \\ = \; & E_t[M_{t,t+2} P_{t+2}^1 e^{-\lambda_{t+2}} Q_{t+2}^1] + E_t[M_{t,t+2} \sum_{k=1}^{K-1} P_{t+2}^{k+1} e^{-\lambda_{t+2}} Q_{t+2}^{k+1}] - E_t[M_{t,t+3} \sum_{k=1}^{K} Q_{t+3}^k P_{t+3}^k], \end{aligned}$$

where we use:

$$E_t[M_{t,t+3} Q_{t+2}^1] = E_t[M_{t,t+2} Q_{t+2}^1 E_{t+2} M_{t+2,t+3}] = E_t[M_{t,t+2} Q_{t+2}^1 e^{-\lambda_{t+2}} P_{t+2}^1],$$

and, similarly, that:

$$E_t[M_{t,t+3}Q^k_{t+2}P^k_{t+3}] = E_t[M_{t,t+2}Q^k_{t+2}E_{t+1}M_{t+2,t+3}P^k_{t+3}] = E_t[M_{t,t+2}Q^k_{t+2}e^{-\lambda_{t+2}}P^{k+1}_{t+2}].$$

Iterating forward, and aggregating the discounted surpluses $(T_{t+j} - G_{t+j})$, we obtain:

$$E_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j})\right] + E_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(1 - e^{-\lambda_{t+j}})\sum_{k=1}^{K} Q^k_{t+j}P^k_{t+j}\right] = \sum_{k=0}^{K} Q^{k+1}_{t-1}P^k_t - \lim_{\tau\to\infty} E_t[M_{t,t+\tau}\sum_{k=1}^{K} Q^k_{t+\tau}P^k_{t+\tau}].$$

Let D^{t+j}_t denote the time-t value of the government's debt portfolio at $t+j$. We can restate the previous equation as follows:

$$E_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j})\right] + E_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(1 - e^{-\lambda_{t+j}})D^{t+j}_{t+j}\right] = D^{t-1}_t - \lim_{\tau\to\infty} E_t[M_{t,t+\tau}\sum_{k=1}^{K} Q^k_{t+\tau}P^k_{t+\tau}].$$

If the discounted value of distant future bond portfolio is 0,

$$\lim_{\tau\to\infty} E_t[M_{t,t+\tau}\sum_{k=1}^{K} Q^k_{t+\tau}P^k_{t+\tau}] = 0,$$

then debt value is the present value of future surpluses and future seignorage revenue from issuing bonds that earn convenience yields:

$$E_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j})\right] + E_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(1 - e^{-\lambda_{t+j}})D^{t+j}_{t+j}\right] = D^{t-1}_t.$$

□

B Asset Pricing Model

B.1 Risk-free rate

The real short yield $y_t(1)$, or risk-free rate, satisfies $E_t[\exp\{m_{t+1} + y_t(1)\}] = 1$. Solving out this Euler equation, we get:

$$\begin{aligned} y_t(1) &= y^\$_t(1) - E_t[\pi_{t+1}] - \frac{1}{2}e'_\pi\Sigma e_\pi + e'_\pi\Sigma^{\frac{1}{2}}\Lambda_t \\ &= y_0(1) + \left[e'_{yn} - e'_\pi\Psi + e'_\pi\Sigma^{\frac{1}{2}}\Lambda_1\right]z_t. \quad \text{(A.5)} \\ y_0(1) &\equiv y^\$_0(1) - \pi_0 - \frac{1}{2}e'_\pi\Sigma e_\pi + e'_\pi\Sigma^{\frac{1}{2}}\Lambda_0. \quad \text{(A.6)} \end{aligned}$$

where we used the expression for the real SDF

$$\begin{aligned} m_{t+1} &= m^\$_{t+1} + \pi_{t+1} \\ &= -y^\$_t(1) - \frac{1}{2}\Lambda'_t\Lambda_t - \Lambda'_t\varepsilon_{t+1} + \pi_0 + e'_\pi\Psi z_t + e'_\pi\Sigma^{\frac{1}{2}}\varepsilon_{t+1} \\ &= -y_t(1) - \frac{1}{2}e'_\pi\Sigma e_\pi + e'_\pi\Sigma^{\frac{1}{2}}\Lambda_t - \frac{1}{2}\Lambda'_t\Lambda_t - \left(\Lambda'_t - e'_\pi\Sigma^{\frac{1}{2}}\right)\varepsilon_{t+1} \end{aligned}$$

The real short yield is the nominal short yield minus expected inflation minus a Jensen adjustment minus the inflation risk premium.

B.2 Nominal and real term structure

Proposition 7. Nominal bond yields are affine in the state vector:

$$y^\$_t(h) = -\frac{A^\$(h)}{h} - \frac{B^\$(h)'}{h}z_t,$$

where the coefficients $A^{\$}(h)$ and $B^{\$}(h)$ satisfy the following recursions:

$$A^{\$}(h+1) = -y_0^{\$}(1) + A^{\$}(h) + \frac{1}{2}\left(B^{\$}(h)\right)' \Sigma \left(B^{\$}(h)\right) - \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_0, \tag{A.7}$$

$$\left(B^{\$}(h+1)\right)' = \left(B^{\$}(h)\right)' \Psi - e'_{yn} - \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_1, \tag{A.8}$$

initialized at $A^{\$}(0) = 0$ and $B^{\$}(0) = \mathbf{0}$.

Proof. We conjecture that the $t+1$-price of a τ-period bond is exponentially affine in the state:

$$\log(P^{\$}_{t+1}(h)) = A^{\$}(h) + \left(B^{\$}(h)\right)' z_{t+1}$$

and solve for the coefficients $A^{\$}(h+1)$ and $B^{\$}(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned} P_t^{\$}(h+1) &= E_t[\exp\{m^{\$}_{t+1} + \log\left(P^{\$}_{t+1}(h)\right)\}] \\ &= E_t[\exp\{-y_t^{\$}(1) - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + A^{\$}(h) + \left(B^{\$}(h)\right)' z_{t+1}\}] \\ &= \exp\{-y_0^{\$}(1) - e'_{yn} z_t - \frac{1}{2}\Lambda_t'\Lambda_t + A^{\$}(h) + \left(B^{\$}(h)\right)' \Psi z_t\} \times \\ &\quad E_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\}\right]. \end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned} P_t^{\$}(h+1) &= \exp\left\{-y_0^{\$}(1) - e'_{yn} z_t + A^{\$}(h) + \left(B^{\$}(h)\right)' \Psi z_t + \frac{1}{2}\left(B^{\$}(h)\right)' \Sigma \left(B^{\$}(h)\right)\right. \\ &\quad \left. - \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t)\right\}. \end{aligned}$$

Taking logs and collecting terms, we obtain a linear equation for $\log(p_t(h+1))$:

$$\log\left(P_t^{\$}(h+1)\right) = A^{\$}(h+1) + \left(B^{\$}(h+1)\right)' z_t,$$

where $A^{\$}(h+1)$ satisfies (A.7) and $B^{\$}(h+1)$ satisfies (A.8). The relationship between log bond prices and bond yields is given by $-\log\left(P_t^{\$}(h)\right)/\tau = y_t^{\$}(h)$. □

Define the one-period return on a nominal zero-coupon bond as:

$$r^{b,\$}_{t+1}(h) = \log\left(P^{\$}_{t+1}(h)\right) - \log\left(P^{\$}_t(h+1)\right)$$

The nominal bond risk premium on a bond of maturity τ is the expected excess return corrected for a Jensen term, and equals negative the conditional covariance between that bond return and the nominal SDF:

$$\begin{aligned} E_t\left[r^{b,\$}_{t+1}(h)\right] - y_t^{\$}(1) + \frac{1}{2}V_t\left[r^{b,\$}_{t+1}(h)\right] &= -Cov_t\left[m^{\$}_{t+1}, r^{b,\$}_{t+1}(h)\right] \\ &= \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_t \end{aligned}$$

Real bond yields, $y_t(h)$, denoted without the \$ superscript, are affine as well with coefficients that follow similar recursions:

$$A(h+1) = -y_0(1) + A(h) + \frac{1}{2}(B(h))' \Sigma (B(h)) - (B(h))' \Sigma^{\frac{1}{2}} \left(\Lambda_0 - \Sigma^{\frac{1}{2}\prime} e_\pi\right), \tag{A.9}$$

$$(B(h+1))' = -e'_{yn} + (e_\pi + B(h))' \left(\Psi - \Sigma^{\frac{1}{2}} \Lambda_1\right). \tag{A.10}$$

For $\tau = 1$, we recover the expression for the risk-free rate in (A.5)-(A.6).

B.3 Stocks

B.3.1 Aggregate Stock Market

We define the real return on the aggregate stock market as $R^m_{t+1} = \frac{P^m_{t+1}+D^m_{t+1}}{P^m_t}$, where P^m_t is the ex-dividend price on the equity market. A log-linearization delivers:

$$r^m_{t+1} = \kappa^m_0 + \Delta d^m_{t+1} + \kappa^m_1 pd^m_{t+1} - pd^m_t. \tag{A.11}$$

The unconditional mean log real stock return is $r^m_0 = E[r^m_t]$, the unconditional mean real dividend growth rate is $\mu^m = E[\Delta d^m_{t+1}]$, and $\overline{pd^m} = E[pd^m_t]$ is the unconditional average log price-dividend ratio on equity. The linearization constants κ^m_0 and κ^m_1 are defined as:

$$\kappa^m_1 = \frac{e^{\overline{pd^m}}}{e^{\overline{pd^m}}+1} < 1 \text{ and } \kappa^m_0 = \log\left(e^{\overline{pd^m}}+1\right) - \frac{e^{\overline{pd^m}}}{e^{\overline{pd^m}}+1}\overline{pd^m}. \tag{A.12}$$

Our state vector z contains the (demeaned) log real dividend growth rate on the stock market, $\Delta d^m_{t+1} - \mu^m$, and the (demeaned) log price-dividend ratio $pd^m - \overline{pd^m}$.

$$\begin{aligned} pd^m_t(h) &= \overline{pd^m} + e'_{pd} z_t, \\ \Delta d^m_t &= \mu^m + e'_{divm} z_t, \end{aligned}$$

where e'_{pd} (e_{divm}) is a selector vector that has a one in the fifth (sixth) entry, since the log pd ratio (log dividend growth rate) is the fifth (sixth) element of the VAR.

We *define* the log return on the stock market so that the log return equation holds exactly, given the time series for $\{\Delta d^m_t, pd^m_t\}$. Rewriting (A.11):

$$\begin{aligned} r^m_{t+1} - r^m_0 &= \left[(e_{divm} + \kappa^m_1 e_{pd})'\Psi - e'_{pd}\right] z_t + \left(e_{divm} + \kappa^m_1 e_{pd}\right)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}. \\ r^m_0 &= \mu^m + \kappa^m_0 - \overline{pd^m}(1-\kappa^m_1). \end{aligned}$$

The equity risk premium is the expected excess return on the stock market corrected for a Jensen term. By the Euler equation, it equals minus the conditional covariance between the log SDF and the log return:

$$\begin{aligned} 1 &= E_t\left[M_{t+1}\frac{P^m_{t+1}+D^m_{t+1}}{P^m_t}\right] = E_t\left[\exp\{m^\$_{t+1} + \pi_{t+1} + r^m_{t+1}\}\right] \\ &= E_t\left[\exp\left\{-y^\$_{t,1} - \frac{1}{2}\Lambda'_t\Lambda_t - \Lambda'_t\varepsilon_{t+1} + \pi_0 + e'_\pi z_{t+1} + r^m_0 + (e_{divm} + \kappa^m_1 e_{pd})' z_{t+1} - e'_{pd} z_t\right\}\right] \\ &= \exp\left\{-y^\$_0(1) - \frac{1}{2}\Lambda'_t\Lambda_t + \pi_0 + r^m_0 + \left[(e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Psi - e'_{pd} - e'_{yn}\right] z_t\right\} \\ &\quad \times E_t\left[\exp\{-\Lambda'_t\varepsilon_{t+1} + \left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\right] \\ &= \exp\left\{r^m_0 + \pi_0 - y^\$_0(1) + \left[(e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Psi - e'_{pd} - e'_{yn}\right] z_t\right\} \\ &\quad \times \exp\left\{\frac{1}{2}\left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right)' \Sigma \left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right) - \left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right)' \Sigma^{\frac{1}{2}} \Lambda_t\right\} \end{aligned}$$

Taking logs on both sides delivers:

$$\begin{aligned} r^m_0 + \pi_0 - y^\$_0(1) + \left[(e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Psi - e'_{pd} - e'_{yn}\right] z_t & \\ + \frac{1}{2}\left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right)' \Sigma \left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right) &= \left(e_{divm} + \kappa^m_1 e_{pd} + e_\pi\right)' \Sigma^{\frac{1}{2}} \Lambda_t \\ E_t\left[r^{m,\$}_{t+1}\right] - y^\$_{t,1} + \frac{1}{2}V_t\left[r^{m,\$}_{t+1}\right] &= -Cov_t\left[m^\$_{t+1}, r^{m,\$}_{t+1}\right] \end{aligned} \tag{A.13}$$

The left-hand side is the expected excess return on the stock market, corrected for a Jensen term, while the right-hand side is the negative of the conditional covariance between the (nominal) log stock return and the nominal log SDF. We refer to the left-hand side as the equity risk premium in the data, since it is implied directly by the VAR. We refer to the right-hand side as the equity risk premium in the model, since it requires knowledge of the market prices of risk.

Note that we can obtain the same expression using the log real SDF and log real stock return:

$$\begin{aligned} E_t\left[r^m_{t+1}\right] - y_{t,1} + \frac{1}{2}V_t\left[r^m_{t+1}\right] &= -Cov_t\left[m_{t+1}, r^m_{t+1}\right] \\ r^m_0 - y_0(1) &+ \left[(e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Psi - e'_{pd} - e'_{yn} - e'_\pi\Sigma^{1/2}\Lambda_1\right] z_t \end{aligned}$$

$$+\frac{1}{2}(e_{divm}+\kappa_1^m e_{pd})'\Sigma(e_{divm}+\kappa_1^m e_{pd}) \quad = \quad (e_{divm}+\kappa_1^m e_{pd})'\,\Sigma^{1/2}(\Lambda_t - \left(\Sigma^{1/2}\right)' e_\pi)$$

We combine the terms in Λ_0 and Λ_1 on the right-hand side and plug in for $y_0(1)$ from (A.6) to get:

$$\begin{aligned}
& r_0^m + \pi_0 - y_{0,1}^{\$} + \frac{1}{2}e_\pi'\Sigma e_\pi \\
& +\frac{1}{2}(e_{divm}+\kappa_1^m e_{pd})'\Sigma(e_{divm}+\kappa_1^m e_{pd}) + e_\pi'\Sigma\left(e_{divm}+\kappa_1^m e_{pd}\right) \\
& +\left[(e_{divm}+\kappa_1^m e_{pd}+e_\pi)'\Psi - e_{pd}' - e_{yn}'\right] z_t \\
& = \left(e_{divm}+\kappa_1^m e_{pd}\right)'\Sigma^{1/2}\Lambda_t + e_\pi'\Sigma^{\frac{1}{2}}\Lambda_0 + e_\pi'\Sigma^{1/2}\Lambda_1 z_t
\end{aligned}$$

This recovers equation (A.13).

B.3.2 Dividend Strips

Proposition 8. Log price-dividend ratios on dividend strips are affine in the state vector:

$$p_t^d(h) = \log\left(P_t^d(h)\right) = A^m(h) + B^{m\prime}(h)z_t,$$

where the coefficients $A^m(h)$ and $B^m(h)$ follow recursions:

$$\begin{aligned}
A^m(h+1) &= A^m(h) + \mu_m - y_0(1) + \frac{1}{2}\left(e_{divm}+B^m(h)\right)'\Sigma\left(e_{divm}+B^m(h)\right) \\
&\quad - \left(e_{divm}+B^m(h)\right)'\Sigma^{\frac{1}{2}}\left(\Lambda_0 - \Sigma^{\frac{1}{2}\prime}e_\pi\right), \quad (A.14)\\
B^{m\prime}(h+1) &= \left(e_{divm}+e_\pi+B^m(h)\right)'\Psi - e_{yn}' - \left(e_{divm}+e_\pi+B^m(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_1, \quad (A.15)
\end{aligned}$$

initialized at $A_0^m = 0$ and $B_0^m = 0$.

Proof. We conjecture the affine structure and solve for the coefficients $A^m(h+1)$ and $B^m(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}
P_t^d(h+1) &= \mathbb{E}_t\left[M_{t+1}P_{t+1}^d(h)\frac{D_{t+1}^m}{D_t^m}\right] = E_t\left[\exp\{m_{t+1}^{\$}+\pi_{t+1}+\Delta d_{t+1}^m + p_{t+1}^d(h)\}\right] \\
&= \mathbb{E}_t\left[\exp\{-y_{t,1}^{\$} - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + \pi_0 + e_\pi' z_{t+1} + \mu^m + e_{divm}' z_{t+1} + A^m(h) + B(h)^{m\prime}z_{t+1}\}\right] \\
&= \exp\{-y_0^{\$}(1) - e_{yn}'z_t - \frac{1}{2}\Lambda_t'\Lambda_t + \pi_0 + e_\pi'\Psi z_t + \mu_m + e_{divm}'\Psi z_t + A^m(h) + B(h)^{m\prime}\Psi z_t\} \\
&\quad \times\mathbb{E}_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(e_{divm}+e_\pi+B^m(h)\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right].
\end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned}
P_t^d(h+1) &= \exp\{-y_0^{\$}(1) + \pi_0 + \mu_m + A^m(h) + \left[\left(e_{divm}+e_\pi+B^m(h)\right)'\Psi - e_{yn}'\right] z_t \\
&\quad +\frac{1}{2}\left(e_{divm}+e_\pi+B^m(h)\right)'\Sigma\left(e_{divm}+e_\pi+B^m(h)\right) \\
&\quad - \left(e_{divm}+e_\pi+B^m(h)\right)'\Sigma^{\frac{1}{2}}\left(\Lambda_0+\Lambda_1 z_t\right)\}
\end{aligned}$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^d(h+1)$:

$$p_t^d(h+1) \quad = \quad A^m(h+1) + B^{m\prime}(h+1)z_t,$$

where:

$$\begin{aligned}
A^m(h+1) &= A^m(h) + \mu_m - y_0^{\$}(1) + \pi_0 + \frac{1}{2}\left(e_{divm}+e_\pi+B^m(h)\right)'\Sigma\left(e_{divm}+e_\pi+B^m(h)\right) \\
&\quad - \left(e_{divm}+e_\pi+B^m(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_0, \\
B^{m\prime}(h+1) &= \left(e_{divm}+e_\pi+B^m(h)\right)'\Psi - e_{yn}' - \left(e_{divm}+e_\pi+B^m(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_1.
\end{aligned}$$

We recover the recursions in (A.14) and (A.15) after using equation (A.6). □

We define the dividend strip risk premium as:

$$\begin{aligned}
\mathbb{E}_t\left[r^{d,\$}_{t+1}(h)\right] - y^{\$}_{t,1} + \frac{1}{2}V_t\left[r^{d,\$}_{t+1}(h)\right] &= -Cov_t\left[m^{\$}_{t+1}, r^{d,\$}_{t+1}(h)\right] \\
&= \left(e_{divm} + e_\pi + B^m(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_t
\end{aligned}$$

B.4 Government Spending and Tax Revenue Claims

This appendix computes P^T_t, the value of a claim to future tax revenues, and P^G_t, the value of a claim to future government spending. It contains the proof for Proposition 5.

B.4.1 Spending Claim

Nominal government spending growth equals

$$\Delta \log G_{t+1} = \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu^g_0 + \left(e_{\Delta g} + e_x + e_\pi\right)' z_{t+1}. \tag{A.16}$$

We conjecture the log price-dividend ratios on spending strips are affine in the state vector:

$$P^G_t(h) = \log\left(P^G_t(h)\right) = A^g(h) + B^{g\prime}(h) z_t.$$

We solve for the coefficients $A^g(h+1)$ and $B^g(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}
P^G_t(h+1) &= \mathbb{E}_t\left[M_{t+1}P^G_{t+1}(h)\frac{G_{t+1}}{G_t}\right] = \mathbb{E}_t\left[\exp\{m^{\$}_{t+1} + \Delta\log g_{t+1} + x_{t+1} + \pi_{t+1} + P^G_{t+1}(h)\}\right] \\
&= \exp\{-y^{\$}_0(1) - e'_{yn}z_t - \frac{1}{2}\Lambda'_t\Lambda_t + \mu^g + x_0 + \pi_0 + (e_{\Delta g} + e_x + e_\pi + B^g(h))'\Psi z_t + A^g(h)\} \\
&\quad \times \mathbb{E}_t\left[\exp\{-\Lambda'_t\varepsilon_{t+1} + \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right].
\end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned}
P^G_t(h+1) &= \exp\{-y^{\$}_0(1) + \mu^g + x_0 + \pi_0 + ((e_{\Delta g} + e_x + e_\pi + B^g(h))'\Psi - e'_{yn})z_t + A^g(h) \\
&\quad + \frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right) \\
&\quad - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma^{\frac{1}{2}}\left(\Lambda_0 + \Lambda_1 z_t\right)\}
\end{aligned}$$

Taking logs and collecting terms, we obtain

$$\begin{aligned}
A^g(h+1) &= -y^{\$}_0(1) + \mu^g + x_0 + \pi_0 + A^g(h) + \frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right) \\
&\quad - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_0, \\
B^g(h+1)' &= (e_{\Delta g} + e_x + e_\pi + B^g(h))'\Psi - e'_{yn} - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_1,
\end{aligned}$$

and the price-dividend ratio of the cum-dividend spending claim is

$$\sum_{h=0}^{\infty}\exp(A^g(h+1) + B^g(h+1)'z_t)$$

Next, we define the (nominal) return on the claim as $R^g_{t+1} = \frac{P^G_{t+1}}{P^G_t - G_t} = \frac{P^{g,ex}_{t+1} + G_{t+1}}{P^{g,ex}_t}$, where P^g_t is the cum-dividend price on the spending claim and $P^{g,ex}_t$ is the ex-dividend price. We log-linearize the return around $z_t = 0$:

$$r^g_{t+1} = \kappa^g_0 + \Delta\log G_{t+1} + \kappa^g_1 pg_{t+1} - pg_t. \tag{A.17}$$

where $pg_t \equiv \log\left(\frac{P^{g,ex}_t}{G_t}\right) = \log\left(\frac{P^G_t}{G_t} - 1\right)$. The unconditional mean log return of the G claim is $r^g_0 = E[r^g_t]$.

We obtain $\overline{pg}$ from the precise valuation formula (14) at $z_t = 0$. We define linearization constants κ_0^g and κ_1^g as:

$$\kappa_1^g = \frac{e^{\overline{pg}}}{e^{\overline{pg}}+1} < 1 \text{ and } \kappa_0^g = \log\left(e^{\overline{pg}}+1\right) - \frac{e^{\overline{pg}}}{e^{\overline{pg}}+1}\overline{pg}. \tag{A.18}$$

Then, the unconditional expected return is:

$$r_0^g = x_0 + \pi_0 + \kappa_0^g - \overline{pg}(1-\kappa_1^g). \tag{A.19}$$

We conjecture that the log ex-dividend price-dividend ratio on the spending claim is affine in the state vector and verify the conjecture by solving the Euler equation for the claim.

$$pg_t = \overline{pg} + \bar{B}_g' z_t \tag{A.20}$$

This allows us to write the return as:

$$r_{t+1}^g = r_0^g + \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)' z_{t+1} - \bar{B}_g' z_t. \tag{A.21}$$

Proof. Starting from the Euler equation:

$$\begin{aligned} 1 &= E_t\left[\exp\{m_{t+1}^\$ + r_{t+1}^g\}\right] \\ &= \exp\{-y_0^\$(1) - e_{yn}' z_t - \frac{1}{2}\Lambda_t'\Lambda_t + r_0^g + [\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)'\Psi - \bar{B}_g'] z_t\} \\ &\quad \times E_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right]. \end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned} 1 = \; & \exp\{r_0^g - y_0^\$(1) + [\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)'\Psi - \bar{B}_g' - e_{yn}'] z_t \\ & \underbrace{+\frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)'\Sigma\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)}_{Jensen} \\ & -\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)'\Sigma^{\frac{1}{2}}\left(\Lambda_0 + \Lambda_1 z_t\right)\} \end{aligned}$$

Taking logs and collecting terms, we obtain the following system of equations:

$$r_0^g - y_0^\$(1) + Jensen = \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)'\Sigma^{\frac{1}{2}}\Lambda_0 \tag{A.22}$$

and

$$\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)'\Psi - \bar{B}_g' - e_{yn}' = \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)'\Sigma^{\frac{1}{2}}\Lambda_1 \tag{A.23}$$

The left-hand side of this equation is the unconditional expected excess log return with Jensen adjustment. The right hand side is the unconditional covariance of the log SDF with the log return. This equation describes the unconditional risk premium on the claim to government spending. Equation (A.23) describes the time-varying component of the government spending risk premium. Given Λ_1, the system of N equations (A.23) can be solved for the vector $\bar{B}_g$:

$$\bar{B}_g = \left(I - \kappa_1^g\left(\Psi - \Sigma^{\frac{1}{2}}\Lambda_1\right)'\right)^{-1}\left[\left(\Psi - \Sigma^{\frac{1}{2}}\Lambda_1\right)'\left(e_{\Delta g} + e_x + e_\pi\right) - e_{yn}\right]. \tag{A.24}$$

□

B.4.2 Revenue Claim

Nominal government revenue growth equals

$$\Delta \log T_{t+1} = \Delta \log \tau_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu_0^\tau + \left(e_{\Delta\tau} + e_x + e_\pi\right)' z_{t+1}. \tag{A.25}$$

where $\tau_t = T_t/GDP_t$ is the ratio of government revenue to GDP. Note that this ratio is assumed to have a long-run growth rate of zero. This imposes cointegration between government revenue and GDP. The growth ratio in this ratio can only temporarily deviate from zero.

The remaining proof exactly mirrors the proof for government spending, with

$$p\tau_t \equiv \log\left(\frac{P_t^{\tau,ex}}{T_t}\right) = \log\left(\frac{P_t^T}{T_t} - 1\right) = \overline{p\tau} + B_\tau' z_t \tag{A.26}$$

$$r_{t+1}^\tau = r_0^\tau + \left(e_{\Delta\tau} + e_x + e_\pi + \kappa_1^g B_\tau\right)' z_{t+1} - B_\tau' z_t, \tag{A.27}$$

and

$$r_0^\tau = x_0 + \pi_0 + \kappa_0^\tau - \overline{p\tau}(1 - \kappa_1^\tau).$$

$$r_0^\tau - y_0^\$(1) + Jensen = \left(e_{\Delta\tau} + e_x + e_\pi + \kappa_1^\tau B_\tau\right)' \Sigma^{\frac{1}{2}} \Lambda_0. \tag{A.28}$$

C Data Sources

C.1 Primary Surpluses

The primary surpluses are constructed using NIPA Table 3.2 Federal Government Current Receipts and Expenditures from 1947 to 2019. All variables are seasonally adjusted.

The government revenue is the sum of the corporate and personal tax revenue, the net income from the rest of the world, and the federal government dividends income receipts on assets. The personal tax revenue is the total of the current personal tax receipts, the tax revenue from production and imports, the net income from the rest of the world, and surpluses from government-sponsored enterprise net of subsidies. The net income from the rest of the world includes the tax income from the rest of the world, the contributions from government social insurance from the rest of the world, the current transfer receipts from the rest of the world, net of the government transfer payments to the rest of the world and the interest payments to the rest of the world.

The government spending is the domestic net transfer payments before interest payments plus discretionary spending (i.e. consumption expenditures). The domestic net transfer is the domestic current transfer receipts net of the domestic contribution from government social insurances and the domestic current transfer receipts.

The primary surpluses are the government revenue minus the government spending before interest payments.

C.2 State Variables

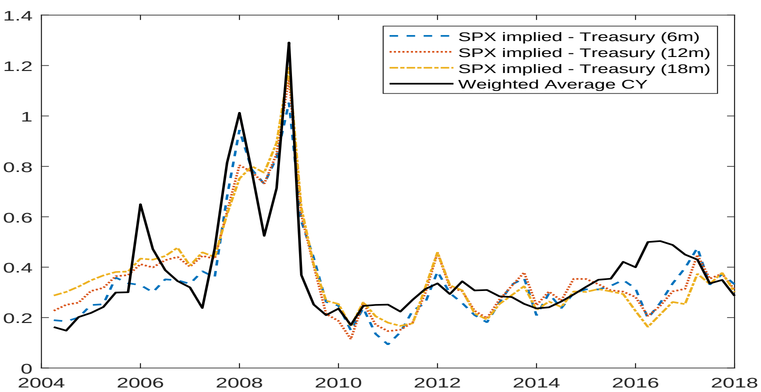
We obtain the time series of GDP from NIPA Table 1.1.5, and inflation is the change in the GDP price index from NIPA Table 1.1.4. The real GDP growth x_t is nominal GDP growth minus inflation. The Treasury yields for all maturities are constant maturity yields from Fred. There are some periods where the 20-year bond was not issued and some periods where the 30 year bond was not issued. The log-price-dividend ratio and the log real dividend growth are computed using CRSP database. Dividends are seasonally adjusted and quarterly. We include the growth of both the government revenue to GDP ratio and the government spending to GDP ratio in the state vector. The government revenue and government spending are defined in Section 1.

C.3 Other Measures of the Convenience Yield

In this section, we compare our measure of the convenience yield with the implied convenience yields from van Binsbergen, Diamond, and Grotteria (2019). Figure A.1 shows the 6-month, 12-month, and 18-month convenience yields from van Binsbergen, Diamond, and Grotteria (2019), which are spreads between the SPX option implied interest rates and government bond rates with corresponding maturities. All measures of the convenience yield exhibit similar time-series patterns over the sample period from 2004-01 to 2017-04.

Figure A.1: Measures of the Convenience Yield

The figure shows the time series of different measures of the convenience yield. The dashed blue line is the spread of 6-month zero coupon interest rates implied from SPX options with 6-month Treasury bill rate. The dotted red line is the spread of 12-month zero coupon interest rates implied from SPX options with 12-month Treasury bill rate. The dashed yellow line is the spread of 18-month zero coupon interest rates implied from SPX options with 18-month Treasury bond rate. The data is from van Binsbergen, Diamond, and Grotteria (2019). The solid black line is the weighted average of the Aaa-Treasury yield spread and the high-grade commercial papers-bills yield spread. All yields are in the quarter frequency, and expressed in percentage per annum. The sample period is from 2014-01 to 2017-04.



D Coefficient Estimates

D.1 The VAR System

The Cholesky decomposition of the residual variance-covariance matrix, $\Sigma^{\frac{1}{2}}$, multiplied by 100 for readability is given by:

$$100 \times \Sigma^{\frac{1}{2}} = \begin{pmatrix} 1.11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.12 & 2.02 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.38 & 0.50 & 1.21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.05 & -0.23 & -0.28 & 0.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.24 & -2.90 & 1.40 & 0.73 & 14.92 & 0 & 0 & 0 & 0 & 0 \\ -1.32 & 1.41 & 0.67 & -0.55 & -1.01 & 4.83 & 0 & 0 & 0 & 0 \\ 0.74 & 2.73 & -0.17 & -0.48 & 0.60 & 1.77 & 4.88 & 0 & 0 & 0 \\ 1.13 & -3.37 & -1.11 & -0.11 & 0.25 & -1.50 & -0.42 & 3.90 & 0 & 0 \\ 0.74 & 2.73 & -0.17 & -0.48 & 0.60 & 1.77 & 4.88 & 0.00 & 0.00 & 0 \\ 1.13 & -3.37 & -1.11 & -0.11 & 0.25 & -1.50 & -0.42 & 3.90 & 0.00 & 0.00 \end{pmatrix}$$

In this matrix, the last two columns are all zero. This is because the dependent variables $\log \tau_t - \log \tau_0$ and $\log g_t - \log g_0$ do not have independent shocks. For example, $\log \tau_t - \log \tau_0$ can be expressed as

$$\begin{aligned} \log \tau_t - \log \tau_0 &= \Delta \log \tau_t + (\log \tau_{t-1} - \log \tau_0) \\ &= (e'_{\Delta\tau}\mathbf{\Psi} + e'_{\tau})z_{t-1} + e'_{\Delta\tau}\mathbf{\Sigma}^{\frac{1}{2}}\boldsymbol{\varepsilon}_t, \end{aligned}$$

which loads on the first eight shocks in the same way as $\Delta \log \tau_t - \mu_0^{\tau}$.

D.2 Cointegration Tests

We perform a Johansen cointegration test by first estimating the vector error correction model :

$$\Delta w_t = A(B'w_{t-1} + c) + D\Delta w_{t-1} + \varepsilon_t, \quad \text{where } w_t = \begin{pmatrix} \log T_t \\ \log G_t \\ \log GDP_t \end{pmatrix}.$$

Both the trace test and the max eigenvalue test do not reject the null of cointegration rank 2 (with p-values of 0.11), but reject the null of cointegration rank 0 and 1 (with p-values lower than 0.01). These results are in favor of two cointegration relationships between variables in w_t.

We also conduct the Phillips-Ouliaris cointegration test on the $\{w_t\}$ matrix with a truncation lag parameter of 2, and reject the null hypothesis that w is not cointegrated with a p-value of 0.03.

D.3 Market Prices of Risk

The constant market price of risk vector is estimated to be:

$$\Lambda_0' = [0, 0.25, -0.43, 0.29, 0, 1.41, 0, 0, 0]$$

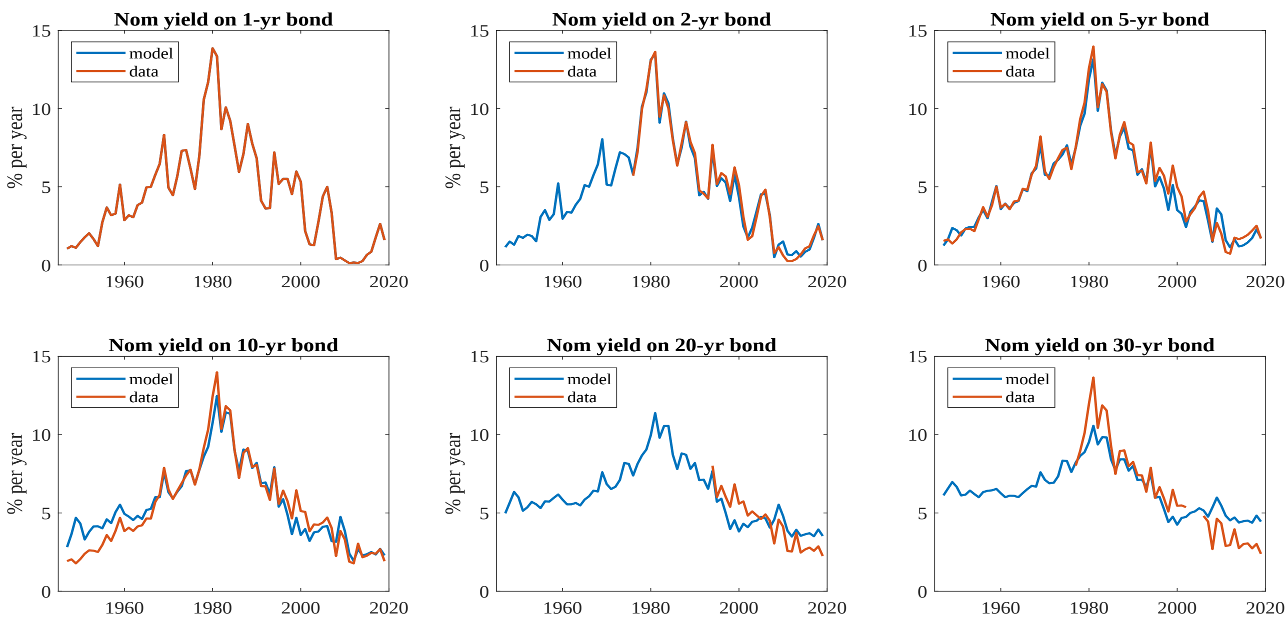
The time-varying market price of risk matrix is estimated at:

$$\Lambda_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8.61 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -9.56 & -79.65 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24.24 & 3.20 & 0.01 & 2.75 & 0.15 & 2.61 & -5.37 & 2.91 & 0.72 & 0.00 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -37.21 & -25.78 & -13.97 & 57.32 & -3.77 & 2.43 & -2.62 & -1.12 & -0.92 & -2.65 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

E Model Fit

Figure A.2 shows that the model matches the nominal term structure in the data closely. The figure plots the observed and model-implied 1-, 2-, 5-, 10-, 20-, and 30-year nominal Treasury bond yields. In the estimation of the market prices of risk, we overweigh matching the 5-year bond yield since it is included in the VAR and the 30-year bond yield since the behavior of long-term bond yields is important for the results.

Figure A.2: Dynamics of the Nominal Term Structure of Interest Rates



The figure plots the observed and model-implied 1-, 2-, 5-, 10-, 20-, and 30-year nominal Treasury bond yields. Yields are measured at the end of the year. Data are from FRED and FRASER. the sample is 1947 until 2019.

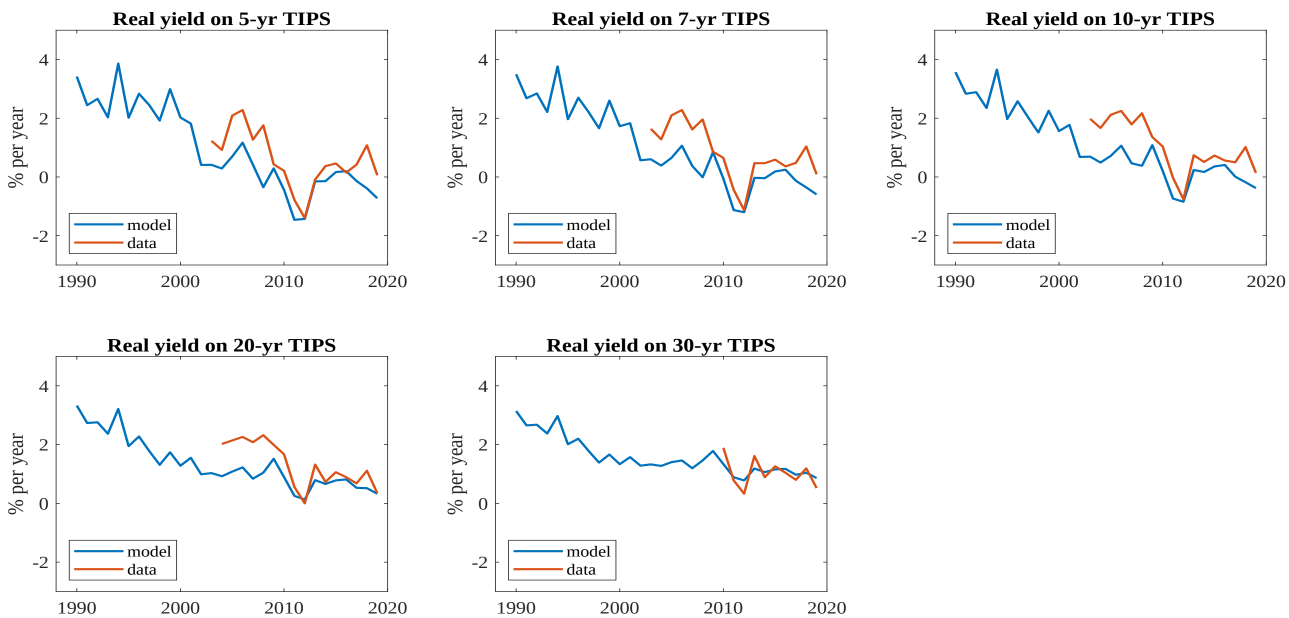
Figure A.3 shows that the model matches the real term structure in the data closely. The figure plots the observed and model-implied 5-, 7-, 10-, 20-, and 30-year real Treasury bond yields (Treasury Inflation Indexed securities). In the estimation of the market prices of risk, we overweigh matching the 30-year bond yield since the behavior of long-term bond yields is important for the results.

The top panels of Figure A.4 show the model's implications for the average nominal (left panel) and real (right panel) yield curves at longer maturities. These yields are well behaved, with very long-run nominal (real) yields stabilizing at around 7.54% (2.84%) per year. We impose conditions that ensure that the nominal and real term structure are well behaved at very long maturities, for which we have no data. Specifically, we impose that average nominal (real) yields of bonds with maturities of 50, 100, 200, 300, and 400 years remain above 6.23% (3.04%) per year, which is the long-run nominal (real) GDP growth rate $x_0 + \pi_0$ (x_0) observed in our sample. Second, we impose that nominal yields remain above real yields plus 3.18% expected inflation at those same maturities. This imposes that the inflation risk premium remain positive at very long maturities. Third, we impose that the nominal and real term structures of interest rates flatten out, with an average yield difference between 100- and 50-year yields that must not exceed 2% per year and between 300- and 100-year maturity that must not exceed 1% per year. These restrictions are satisfied at the optimum.

The bottom left panel of Figure A.4 shows that the model matches the dynamics of the nominal bond risk premium, defined as the expected excess return on the five-year nominal bond, quite well. Bond risk premia decline in the latter part of the sample, possibly reflecting the arrival of foreign investors who value U.S. Treasuries highly. The bottom right panel shows a decomposition of the nominal bond yield on a five-year bond into the five-year real bond yield, annual expected inflation inflation over the next five years, and the five-year inflation risk premium. On average, the 4.9% nominal bond yield is comprised of a 1.5% real yield, a 3.2% expected inflation rate, and a 0.3% inflation risk premium. The graph shows that the importance of these components fluctuates over time.

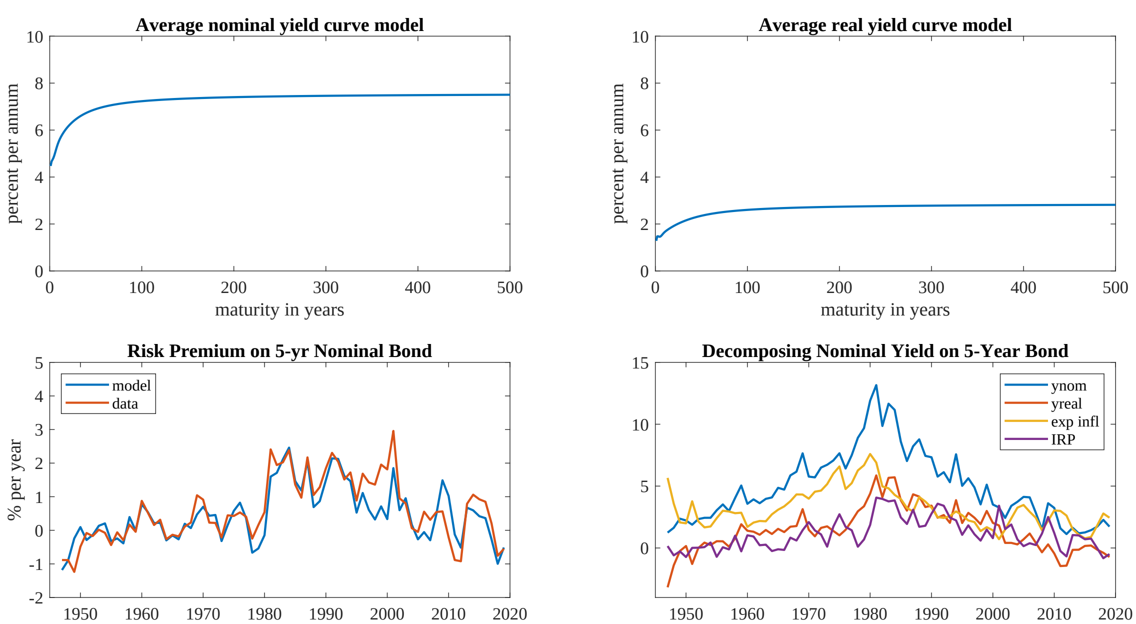
Figure A.5 shows the equity risk premium, the expected excess return, in the left panel and the price-dividend ratio in the right panel. The risk premia in the data are the expected equity excess return predicted by the VAR. Their dynamics are sensible, with low risk premia in the dot-com boom of 1999-2000 and very high risk premia in the Great Financial Crisis of 2008-09. The VAR-implied equity risk premium occasionally turns negative. The figure's right panel shows a tight fit for equity price levels. Hence, the model fits both the behavior of expected returns and stock price levels.

Figure A.3: Dynamics of the Real Term Structure of Interest Rates



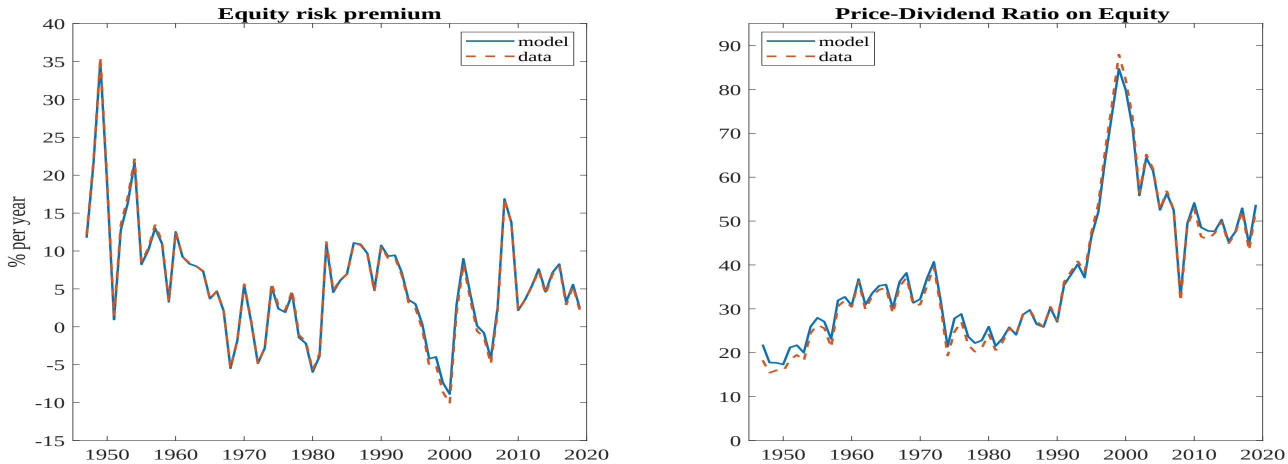
The figure plots the observed and model-implied 5-, 7-, 10-, 20-, and 30-year real bond yields. Data are from FRED and start in 2003. For ease of readability, we start the graph in 1990 but the model of course implies a real yield curve for the entire 1947-2019 period.

Figure A.4: Long-term Yields and Bond Risk Premia



The top panels plot the average bond yield on nominal (left panel) and real (right panel) bonds for maturities ranging from 1 to 500 years. The bottom left panel plots the nominal bond risk premium on the five year bond in model and data. The nominal bond risk premium is measured as the five year bond yield minus the expected one-year bond yield over the next five years. The bottom right panel decomposes the model's five-year nominal bond yield into the five-year real bond yield, the five year expected inflation, and the five-year inflation risk premium.

Figure A.5: Equity Risk Premium and Price-Dividend Ratio



The figure plots the observed and model-implied equity risk premium on the overall stock market in the left panel and the price-dividend ratio in the right panel. The price-dividend ratio is the price divided by the annualized dividend. Data are from 1947-2019. Monthly stock dividends are seasonally adjusted.

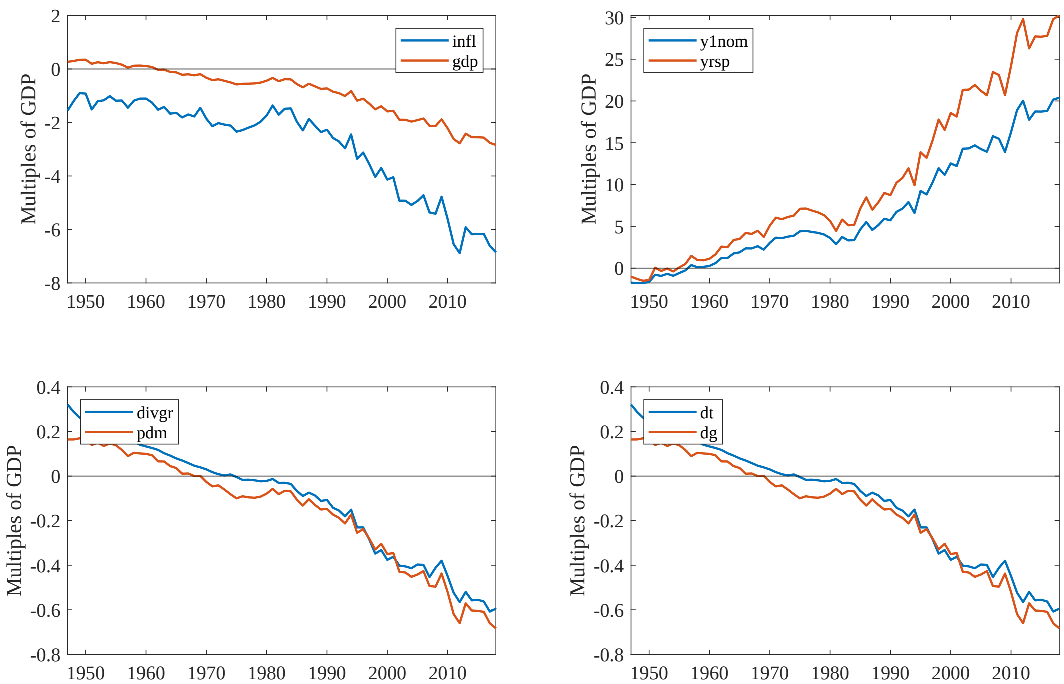
F Measurability Constraints

Exploiting the affine exponentially-nature of bond prices and price-dividend ratios of spending and revenue strips, the generic measurability restriction in (4) becomes:

$$\tau_t \sum_{h=0}^{\infty} PD_t^{\tau}(h) \left(e'_{\Delta\tau}\Sigma^{\frac{1}{2}} + B'_{\tau}(h) \right) - g_t \sum_{h=0}^{\infty} PD_t^{g}(h) \left(e'_{\Delta g}\Sigma^{\frac{1}{2}} + B'_{g}(h) \right) = \sum_{h=0}^{H} \frac{Q^{\$}_{t-1,h+1}}{GDP_t} P_t^{\$}(h) \cdot B^{\$}(h)', \tag{A.29}$$

where we scaled both the left- and the right-hand side by GDP. When there is only one-period risk-free government debt, the right-hand side of (A.29) is equal to zero. This is a special case of equation (5). Figure A.6 plots the violations of this measurability constraint with only one-period risk-free debt. Since the left hand side of (A.29) is a vector of 8 variables, we plot their time series in 4 panels of 2 time series each. Each time series is interpreted as the change in the valuation of debt in response to one of the state variables z_t. All these time series are away from zero, suggesting severe violations of the measurability restriction (4).

Figure A.6: Violations of the Measurability Constraint With Only One-Period Debt



The figure shows the time series of $\tau_t \sum_{h=0}^{\infty} PD_t^{\tau}(h)\left(e'_{\Delta\tau}\Sigma^{\frac{1}{2}} + B'_{\tau}(h)\right) - g_t \sum_{h=0}^{\infty} PD_t^{g}(h)\left(e'_{\Delta g}\Sigma^{\frac{1}{2}} + B'_{g}(h)\right) - 0$, i.e. the violation of the measurability constraint with only one-period risk-free debt. Each panel plots the measurability conditions for two of the state variables. The deviations from zero are expressed as a multiple of U.S. annual GDP so that 1 means the surplus claim increases by 100% of U.S. annual GDP when that state variable changes by 1%.

G Model with Debt in the VAR

In this appendix, we augment the VAR with government debt as suggested by Cochrane (2019a,b). We impose the natural assumption that the market value of debt and GDP are cointegrated. This leads us to add both the log change in the debt-to-GDP ratio and the log level of the debt-to-GDP ratio as two additional elements in the state vector, and two equations in the VAR. The dynamics of the level of debt-to-GDP and its innovations are implied by those in the change of debt-to-GDP, just like we do for tax revenue and government spending. We place no other restrictions on the VAR dynamics. Debt-to-GDP is allowed to predict and be contemporaneously correlated with all other state variables. The estimated Ψ matrix and $\Sigma^{\frac{1}{2}}$ matrix are reported in Table A.1.

One potential issue of including debt-to-GDP in the VAR is that the $\log(debt/GPD)$ series is not stationary. The augmented Dickey-Fuller test can not reject the null hypothesis of the presence of the unit root in the log debt-to-GDP ratio in our sample period. A sign of this non-stationarity is also that the coefficient on lagged debt in the debt equation exceeds 1 ($\Psi_{12,12} = 1.046 > 1$). To deal with the non-stationarity, we estimate a structural break around 2007 for the debt-to-GDP ratio (See Figure A.7). Chow tests for several potential structural breakpoints for the debt-to-GDP ratio from 1947 to 2019 show that we can only reject the null hypothesis of no structural break in 2007 at the 1% level. We demean the debt-to-GDP ratio with two different sample (before and after 2007) average, and then re-estimate the VAR (Lettau and Van Nieuwerburgh, 2008). The Ψ and $\Sigma^{\frac{1}{2}}$ matrices for the VAR with the structural break-adjusted debt series are reported in Table A.2. Note that now $\Psi_{12,12} = 0.96 < 1$.

Cash Flows in VAR The dynamics of the tax revenue, government spending, and surplus are affected little by the presence of debt in the VAR (both with and without the structural break). In particular, we find little evidence for overshooting of the surplus in response to a shock that lowers the surplus on impact. The third column of Figure A.8 shows the response of the surplus to a shock in GDP (row 1), tax revenues (row 2), spending (row 3), and debt (row 4). For example, in response to a negative tax revenue shock (τ shock) or a positive government spending shock (g shock), the surplus declines on impact and then recovers. There is some overshooting in the surplus in the VAR system with debt, but the magnitude of the overshooting is minor. The responses of the surplus are not very different from the benchmark model without debt in the VAR, both in an economic and a statistical sense. The last row shows that an increase in debt results in persistently higher future surpluses, after an initial decline in the surplus, but again the effects are minor. The effects are shorter-lived for the break-adjusted debt series.

The first and the second columns of Figure A.8 show the impulse responses of the tax revenue and spending to shocks. In response to shocks in GDP, tax revenues, and spending, the tax revenue and spending behave similarly across all three VAR models. For example, in response to a negative τ shock or a positive g shock, the tax revenue increases and then declines. Again, the responses

Table A.1: VAR with debt-to-GDP Ratio

$\Psi =$

	$l.\pi$	$l.x$	$l.r$	$l.tp$	$l.dp$	$l.dd$	$l.\Delta\tau$	$l.\tau$	$l.\Delta g$	$l.g$	$l.\Delta debt$	$l.debt$
π	0.450	-0.159	0.134	-0.122	0.006	0.047	0.051	-0.042	-0.006	0.025	-0.049	-0.016
x	-0.325	-0.091	0.272	0.901	0.005	0.106	-0.025	-0.079	0.081	0.043	-0.163	0.002
r	0.006	-0.019	0.838	0.119	0.004	0.052	-0.001	-0.033	-0.007	0.021	-0.025	-0.011
tp	-0.030	-0.032	-0.033	0.359	-0.005	-0.036	0.016	0.019	0.001	-0.018	0.044	-0.001
dp	-2.067	0.220	0.353	-0.563	0.769	-0.386	-0.040	0.334	0.079	-0.304	0.616	0.064
dd	0.379	0.028	0.004	-0.471	0.062	0.323	-0.182	-0.268	-0.101	0.137	-0.112	0.050
$\Delta\tau$	-0.863	-0.122	1.176	-1.898	0.097	0.181	0.236	-0.648	0.243	0.261	-0.389	0.077
τ	-0.863	-0.122	1.176	-1.898	0.097	0.181	0.236	0.352	0.243	0.261	-0.389	0.077
Δg	0.320	0.396	-1.489	-1.760	-0.082	-0.237	0.114	0.141	0.266	-0.312	0.185	-0.056
g	0.320	0.396	-1.489	-1.760	-0.082	-0.237	0.114	0.141	0.266	0.688	0.185	-0.056
$\Delta debt$	0.860	0.011	0.938	1.314	0.009	-0.097	-0.019	0.180	0.126	-0.013	0.433	0.045
$debt$	0.860	0.011	0.938	1.314	0.009	-0.097	-0.019	0.180	0.126	-0.013	0.433	1.045

$100\Sigma^{\frac{1}{2}}$

	$l.\pi$	$l.x$	$l.r$	$l.tp$	$l.dp$	$l.dd$	$l.\Delta\tau$	$l.\tau$	$l.\Delta g$	$l.g$	$l.\Delta debt$	$l.debt$
π	0.981	0	0	0	0	0	0	0	0	0	0	0
x	-0.210	1.784	0	0	0	0	0	0	0	0	0	0
r	0.255	0.518	1.196	0	0	0	0	0	0	0	0	0
tp	0.036	-0.123	-0.303	0.383	0	0	0	0	0	0	0	0
dp	-0.486	-1.463	1.219	-0.163	14.735	0	0	0	0	0	0	0
dd	-1.209	1.159	0.778	-0.397	-1.079	4.762	0	0	0	0	0	0
$\Delta\tau$	0.810	2.278	0.086	-0.157	0.464	1.491	4.596	0	0	0	0	0
τ	0.810	2.278	0.086	-0.157	0.464	1.491	4.596	0	0	0	0	0
Δg	1.097	-3.208	-1.152	-0.067	0.373	-1.344	-0.186	0	3.833	0	0	0
g	1.097	-3.208	-1.152	-0.067	0.373	-1.344	-0.186	0	3.833	0	0	0
$\Delta debt$	-1.045	-4.746	-2.884	-0.428	-0.731	-0.454	-1.511	0	2.088	0	3.311	0
$debt$	-1.045	-4.746	-2.884	-0.428	-0.731	-0.454	-1.511	0	2.088	0	3.311	0

$\Sigma^{\frac{1}{2}}$ is the Cholesky decomposition of the residual variance-covariance matrix. It is multiplied by 100 for readability.

of tax revenues and spending in the VAR with debt (with and without structural break) are similar to those from our benchmark VAR, both qualitatively and quantitatively.

Forecast Errors Adding debt to the VAR does not meaningfully improve the forecast errors of spending and tax revenues in the long-run (5-, 10- and 20-years hence). As shown in Figure A.9 and A.10, the long-run forecast errors of future spending and the tax revenues are similar across all three VAR specifications. The forecasting performance for $\Delta \log \tau$ and $\Delta \log g$ is also slightly improved using the VAR system with debt and structural break (the third columns) compared to the VAR system with debt (the second column).

However, the VAR system with debt and structural break performs substantially better than the VAR system with debt only when it comes to forecasting the long-run evolution of debt. The long-run forecast errors for $\Delta \log(debt/GDP)$ in the VAR with debt and structural break are much lower than those in the VAR with debt (Figure A.11).

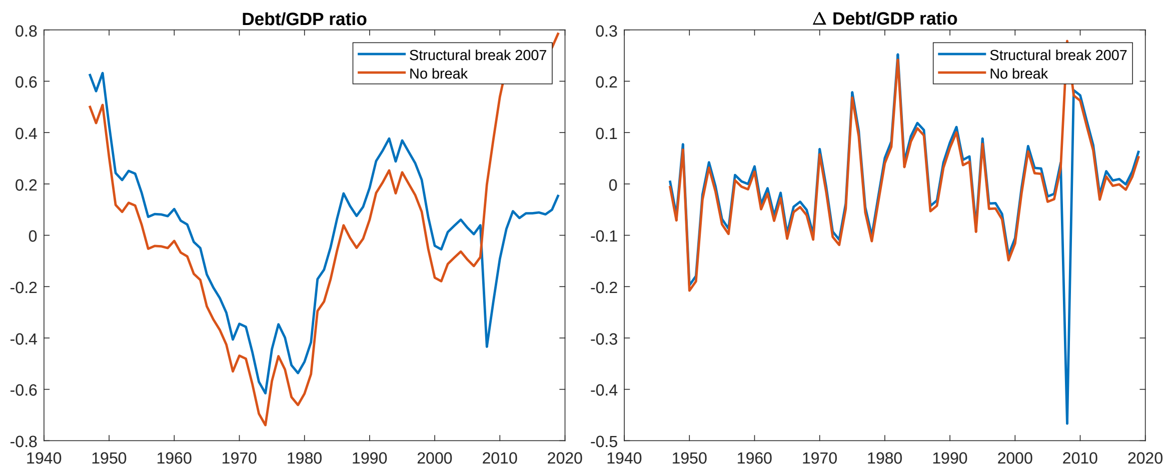
Valuation When we re-estimate the market prices of risk for this model and reprice the claims to tax revenue and spending, we find a broadly similar valuation puzzle. We estimate the market prices of risk for the VAR system with structural break to avoid the non-stationary valuation ratios in the presence of the unit root in the debt-to-GDP ratio. In addition, the VAR system with structural change in the debt-to-GDP ratio performs better in forecasting fiscal cash flows and debt dynamics in the long run.

Figure A.13 shows the value of the surplus claim in the model with debt and the market value of debt in the data. It shows the same pattern of a large and widening gap in the last three decades of the sample. The present value of future supluses is on average -226% of GDP, implying the size of the puzzle (the gap) is -264% of GDP. The right panel of Figure A.13 plots the strip risk premium curve for the model with debt. As in the main model, risk premia on T-, G-, and GDP-strips converge for long-dated strips. The government debt portfolio continues to carry a much higher expected return that the safe rate of interest.

Cross-Equation Restrictions Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2020) shows that in a model where debt/GDP is included in the state vector, a no arbitrage relationship pins down the risk-adjusted discounted sum of surpluses. In other words, when debt is included, surpolus dynamics are redundant and can be backed out from this restriction. In our VAR system with debt described above, the present value of future surpluses at any horizon k is implied by the value of the state vector z_t:

$$\frac{E_t \sum_{j=1}^{k} M_{t,t+j} S_{t+j}}{Y_t} = D_t/Y_t - P_t^D(k) = \exp(e_d' z_t) - \exp(A_0^d(k) + A_1^d(k) z_t). \tag{A.30}$$

Figure A.7: log $Debt/GDP$ ratio with and without Structural Break



The left panel plots the time series of the log $Debt/GDP$ ratio.The red line is the demeaned log $Debt/GDP$ ratio. The blue line is the log $Debt/GDP$ ratio demeaned by two different sample means (before and after 2007). The Chow test show that we can only reject the null hypothesis of no structural break in 2007 at the 1% level (F statistics is 8.81). The right panel plots the demeaned $\Delta \log Debt/GDP$ with and without the structural break.

Table A.2: VAR with Debt and Structural Break

$\Psi =$

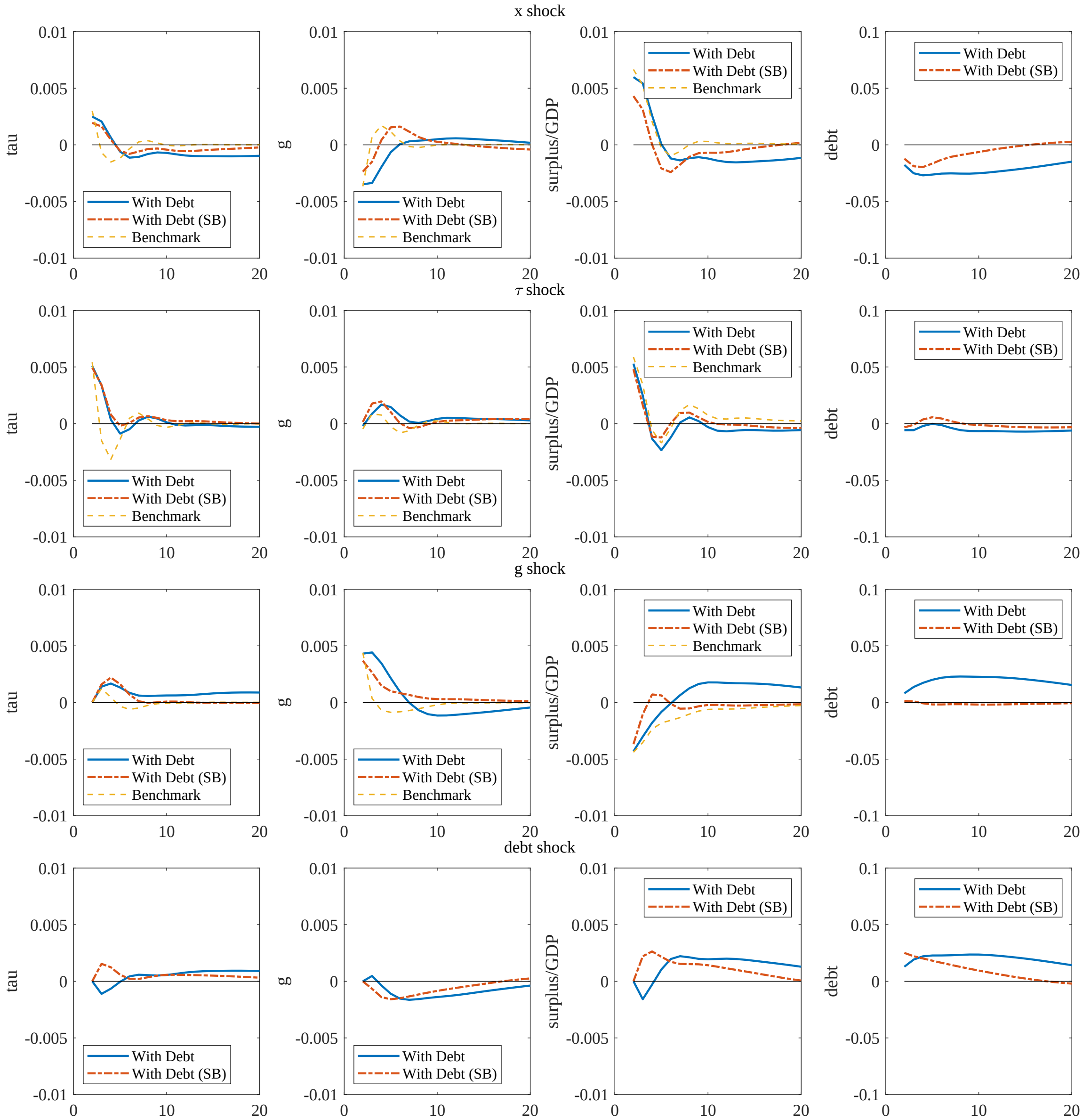
	$l.\pi$	$l.x$	$l.r$	$l.tp$	$l.dp$	$l.dd$	$l.\Delta\tau$	$l.\tau$	$l.\Delta g$	$l.g$	$l.\Delta debt$	$l.debt$
π	0.380	-0.076	0.026	-0.319	-0.008	0.033	0.064	-0.036	-0.015	-0.015	0.002	-0.036
x	-0.090	0.210	0.391	0.272	0.021	0.081	-0.038	-0.059	0.067	0.078	-0.037	0.042
r	0.048	0.043	0.899	0.048	0.005	0.046	-0.005	-0.035	-0.005	0.022	-0.019	-0.004
tp	-0.073	-0.092	-0.050	0.485	-0.008	-0.031	0.017	0.016	0.008	-0.024	0.018	-0.007
dp	-2.697	-1.347	0.354	2.909	0.769	-0.219	-0.020	0.207	0.089	-0.260	-0.129	-0.032
dd	0.477	0.453	-0.395	-1.913	0.063	0.276	-0.178	-0.173	-0.105	0.122	0.263	0.066
$\Delta\tau$	-0.683	0.488	0.642	-3.876	0.106	0.121	0.258	-0.535	0.164	0.258	0.131	0.092
τ	-0.683	0.488	0.642	-3.876	0.106	0.121	0.258	0.465	0.164	0.258	0.131	0.092
Δg	-0.451	0.054	-2.189	-1.198	-0.170	-0.235	0.188	0.129	0.266	-0.536	0.119	-0.215
g	-0.451	0.054	-2.189	-1.198	-0.170	-0.235	0.188	0.129	0.266	0.464	0.119	-0.215
$\Delta debt$	0.154	-0.931	0.773	4.527	-0.025	-0.306	0.109	0.001	-0.042	0.019	-0.072	-0.041
$debt$	0.154	-0.931	0.773	4.527	-0.025	-0.306	0.109	0.001	-0.042	0.019	-0.072	0.959

$100\Sigma^{\frac{1}{2}} =$

	$l.\pi$	$l.x$	$l.r$	$l.tp$	$l.dp$	$l.dd$	$l.\Delta\tau$	$l.\tau$	$l.\Delta g$	$l.g$	$l.\Delta debt$	$l.debt$
π	0.922	0	0	0	0	0	0	0	0	0	0	0
x	0.555	1.835	0	0	0	0	0	0	0	0	0	0
r	0.363	0.494	1.207	0	0	0	0	0	0	0	0	0
tp	-0.098	-0.188	-0.274	0.426	0	0	0	0	0	0	0	0
dp	-3.411	-2.075	1.168	0.784	14.791	0	0	0	0	0	0	0
dd	-0.140	1.022	0.906	-0.798	-0.721	4.228	0	0	0	0	0	0
$\Delta\tau$	2.324	1.774	0.102	-0.487	0.923	0.819	4.473	0	0	0	0	0
τ	2.324	1.774	0.102	-0.487	0.923	0.819	4.473	0	0	0	0	0
Δg	-0.899	-2.168	-1.295	-0.403	0.050	-1.306	0.137	0	3.283	0	0	0
g	-0.899	-2.168	-1.295	-0.403	0.050	-1.306	0.137	0	3.283	0	0	0
$\Delta debt$	-2.495	-3.244	-1.304	-0.320	2.735	2.037	-0.859	0	0.367	0	6.351	0
$debt$	-2.495	-3.244	-1.304	-0.320	2.735	2.037	-0.859	0	0.367	0	6.351	0

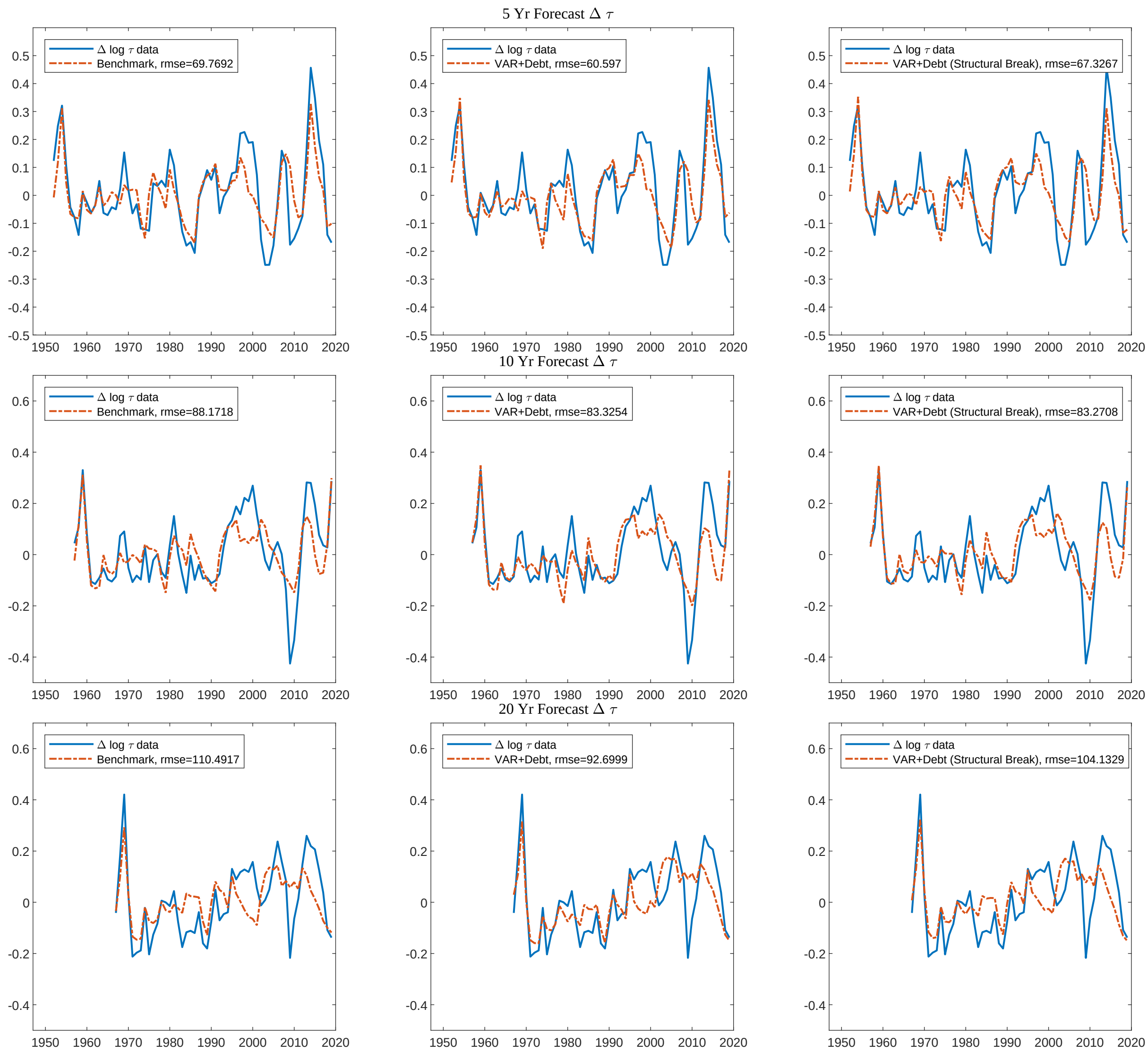
$\Sigma^{\frac{1}{2}}$ is the Cholesky decomposition of the residual variance-covariance matrix. It is multiplied by 100 for readability.

Figure A.8: Fiscal Impulse-Responses with the Debt/GDP ratio in the VAR



Solid line shows impulse-response functions for the VAR with changes in the log debt/GDP ratio and the level of log debt/GDP; dashed red line is for the VAR system with changes in the log debt/GDP ratio and the level of log debt/GDP after applying the structural change in the level of log debt/GDP; dashed yellow line is for the VAR without the debt variables. The impulse in the top row is a shock to GDP. The x_t shock is defined as the shock that increases x_t by one standard deviation of its VAR residual. The impulse in the second row is a shock to tax revenues. The impulse in the third row is a shock to spending growth. The impulse in the last row is a shock to log debt/GDP growth.

Figure A.9: Long Term Forecast for $\Delta\tau$

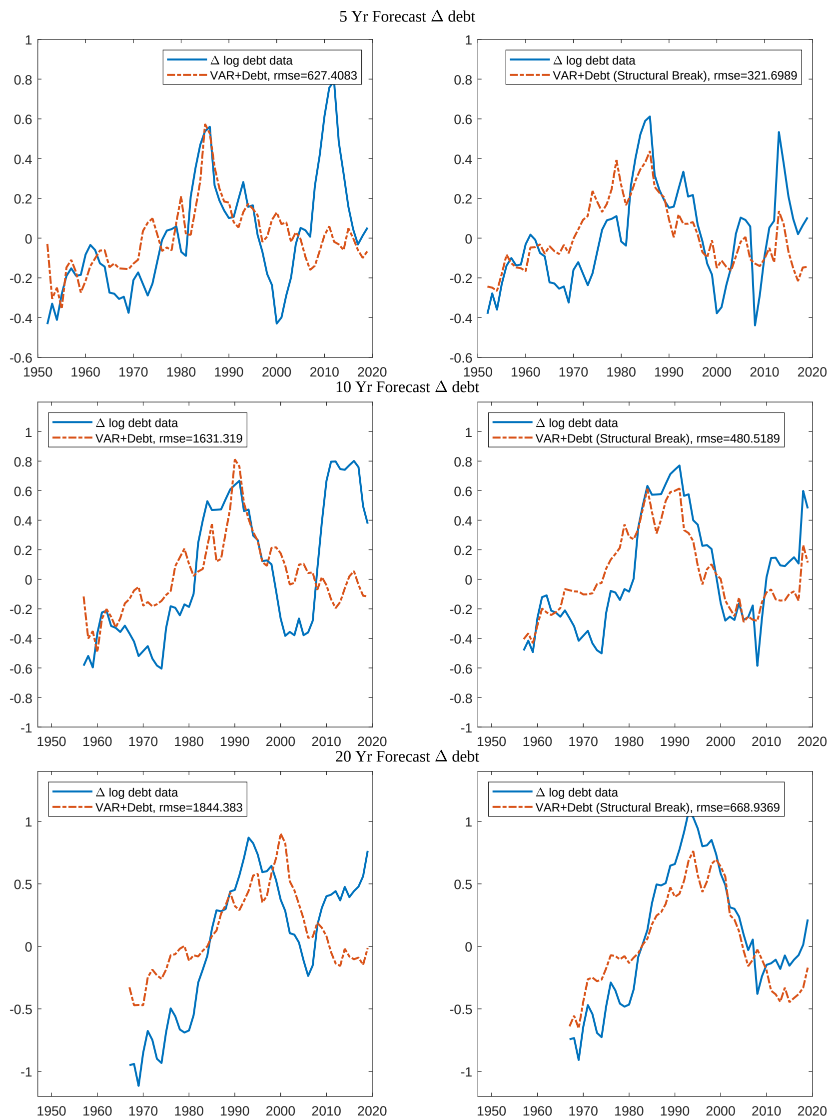


We plot the actual log tax growth rates over 5-year, 10-year and 20-year rolling windows in solid blue lines. The value at each year represents the *k*-year growth rates that end at that year. We also plot these rates as forecasted by the benchmark model (the first column), the rates forecasted by the VAR with debt/gdp ratio (the middle column), and these rates as forecasted by the VAR with debt/gdp ratio and structural change (the last column). The value at each year represents the *k*-year growth rates condition on the information *k* years ago.

Figure A.10: Long Term Forecast for Δg

We plot the actual log spending growth rates over 5-year, 10-year and 20-year rolling windows in solid blue lines. The value at each year represents the k-year growth rates that end at that year. We also plot these rates as forecasted by the benchmark model (the first column), the rates forecasted by the VAR with debt/gdp ratio (the middle column), and these rates as forecasted by the VAR with debt/gdp ratio and structural change (the last column). The value at each year represents the k-year growth rates condition on the information k years ago.

Figure A.11: Long Term Forecast for $\Delta \log(debt/GDP)$



We plot the actual log debt/gdp growth rates over 5-year, 10-year and 20-year rolling windows in solid blue lines. The value at each year represents the k-year growth rates that end at that year. We also plot these rates as forecasted by the benchmark model (the first column), the rates forecasted by the VAR with debt/gdp ratio (the middle column), and these rates as forecasted by the VAR with debt/gdp ratio and structural change (the last column). The value at each year represents the k-year growth rates condition on the information k years ago.

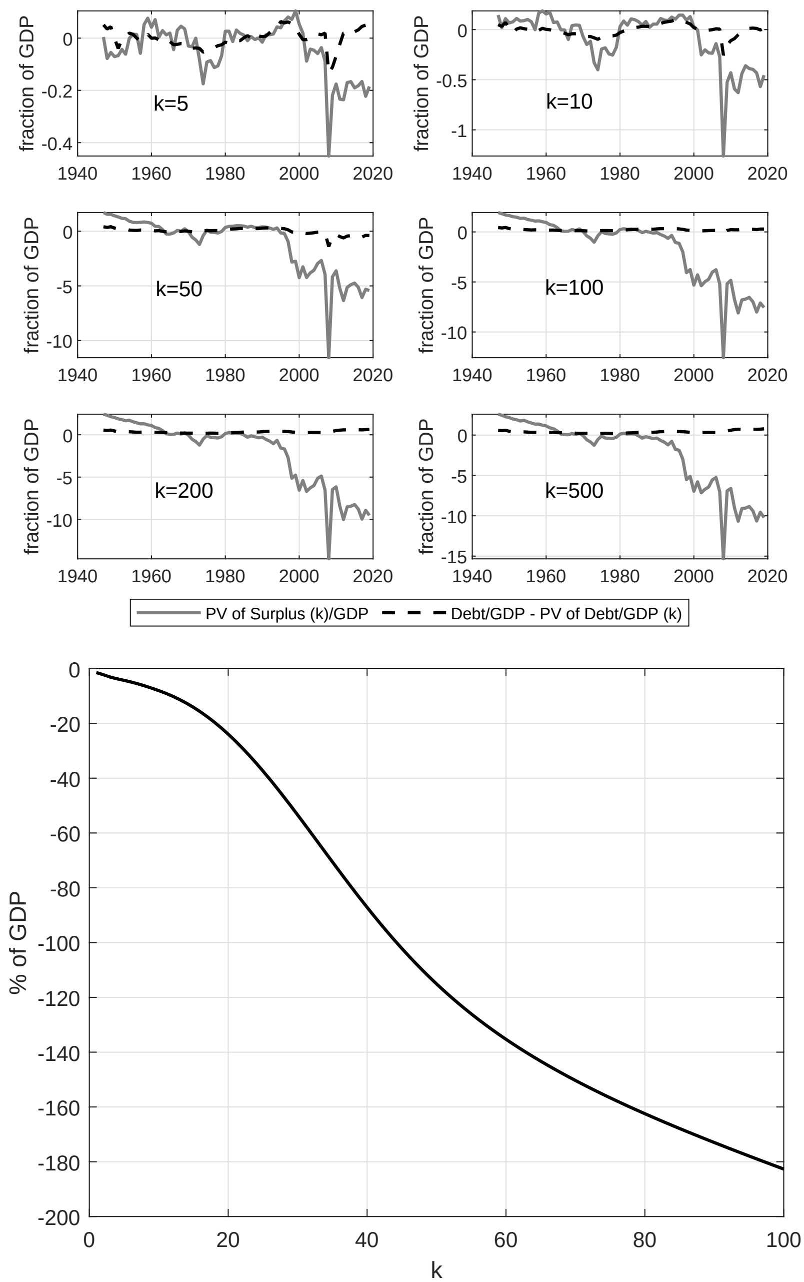
Given that the log price-dividend ratios on spending strips are affine in the state vector, the cross-equation restrictions can be stated as:

$$\begin{aligned} & \tau_t \sum_{h=1}^{k} P_t^{\tau}(h) - g_t \sum_{h=1}^{k} P_t^{G}(h) \\ = \quad & \exp(\log \tau_t) \sum_{h=1}^{k} (\exp(A^{\tau}(h) + B^{\tau\prime}(h) \boldsymbol{z}_t) - \exp(\log g_t) \sum_{h=1}^{k} (\exp(A^{g}(h) + B^{g\prime}(h) \boldsymbol{z}_t) \\ = \quad & \exp(\boldsymbol{e}_d^{\prime} \boldsymbol{z}_t) - \exp(A_0^d(k) + A_1^d(k) \boldsymbol{z}_t), \forall k \end{aligned} \tag{A.31}$$

where $\boldsymbol{e}_d^{\prime}$ selects the log debt/GDP ratio in the VAR, and $A_0^d(k)$ and $A_1^d(k)$ is solved in Appendix of Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2020).

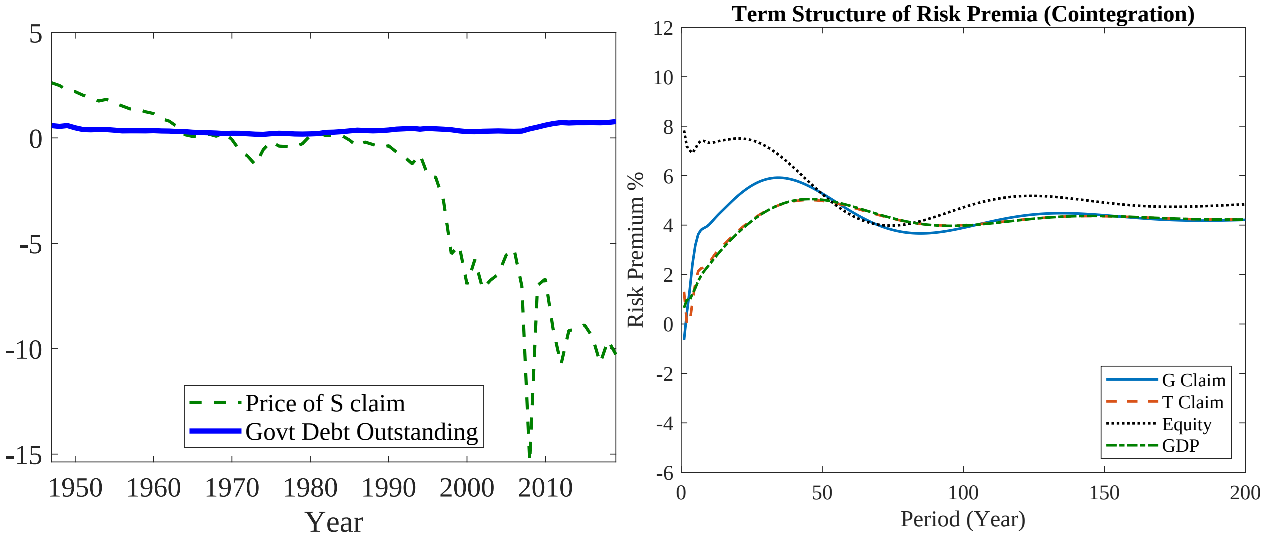
Based on the estimation of the VAR system with debt and structural break, we evaluate the cross-equation no-arbitrage restriction A.31, and Figure A.12 shows that the restrictions are severely violated over 1yr-, 5yr-, 25yr-, 50yr-, 100yr-, 500yr- horizons.

Figure A.12: Testing No-Arbitrage Restrictions



Top panel shows the violation of the cross equation restrictions over 1yr-, 5yr-, 25yr-, 50yr-, 100yr-, 500yr- horizons. The blue solid line is the present value of future surplus over a k-period horizon: $\tau_t \sum_{h=1}^{k} P_t^{\tau}(h) - g_t \sum_{h=1}^{k} P_t^{G}(h)$ where τ_t is the tax revenue-to-GDP ratio and g is the spending-to-GDP ratio from 1947 to 2019. The red dashed line is the debt-to-GDP ratio minus the present value of the debt in k period $D_t/Y_t - P_t^D(k)$. The bottom panel show the terms structure of the violation $\tau_t \sum_{h=1}^{k} P_t^{\tau}(h) - g_t \sum_{h=1}^{k} P_t^{G}(h) - (D_t/Y_t - P_t^D(k))$.

Figure A.13: Valuation and Risk Premium Puzzles in the Model with Debt



The left panel illustrates the government debt valuation puzzle. The solid line is the market value of government debt. The dashed line is the market value of the surplus claim. Both time series are scaled by the US GDP. the right panel shows the risk premia on strips of government spending, tax revenue, equity and GDP.

H Model with Priced Fiscal Shock

Next, we explore the case in which we allow for non-zero market price of risk for shocks to government spending growth. We re-estimate our SDF, fixing the first 6 elements of Λ_0 and the first 6 rows of Λ_1, but freeing up $\Lambda_0(8)$ and the 8th row of Λ_1. Since the stocks and bonds do not load on the government spending shock, their pricing remains the same. On the other hand, $\Lambda_0(8)$ and the 8th row of Λ_1 changes how the government surpluses are priced. We estimate these parameters by minimizing the squared errors between the debt-to-GDP ratio and the model-implied present value of primary surpluses.

The constant market price of risk vector is estimated to be:

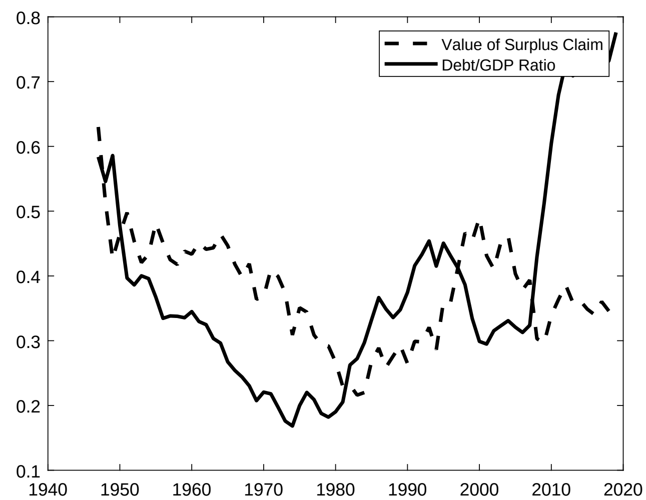
$$\Lambda_0' = [0, 0.25, -0.43, 0.29, 0, 1.41, 119.84, 0, 0]$$

The time-varying market price of risk matrix is estimated at:

$$\Lambda_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8.61 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -9.56 & -79.65 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24.24 & 3.20 & 0.01 & 2.75 & 0.15 & 2.61 & -5.37 & 2.91 & 0.72 & 0.00 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -37.21 & -25.78 & -13.97 & 57.32 & -3.77 & 2.43 & -2.62 & -1.12 & -0.92 & -2.65 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 431.70 & -228.91 & 12.79 & -67.73 & -2.16 & 223.19 & -1.57 & -36.57 & -188.88 & 47.14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Lastly, Figure A.14 plots the implied present value of government surpluses. While we do not have a perfect fit, this modification brings the present value of government surpluses much closer to the market value of government debt. However, this is done at the cost of permitting very large Sharpe ratio: In fact, the government spending has to carry a very high price of risk in order to reduce its valuation. The market price of risk associated with the spending shock is 119.84, compared with 1.41 that is associated with the stock. As a result, the implied maximal Sharpe ratio permitted by the new SDF is 119.84, whereas that of the benchmark SDF is 1.50.

Figure A.14: Government debt-GDP Ratio and Present Value of Government Surpluses, with Priced Spending Shock



I Betting Against The Treasury

The government bond valuation puzzle implies that Treasuries are overpriced relative to other asset classes, such as equities, once the riskiness of the cash flows is taken into consideration. To illustrate this point in a model-free way, this appendix explores a simple investment strategy which shorts the portfolio of all outstanding Treasuries and goes long in all equities.

I.1 Portfolio Construction

The portfolio goes long all outstanding corporate equities, taking into account equity issuances and repurchases, and short all outstanding U.S. Treasuries, taking into account all bond issuances and redemptions.

We use the market value of non-financial corporate equity (reported in Table L103 of the Financial Accounts of the U.S.) and the value of all outstanding Treasuries (FL313161105.Q in Table L106). Since 2015, the Financial Accounts of the U.S. include some non-marketable debt: the holdings of the federal government employees defined benefit plans. We use the dividend payments (FA106121075.Q) and issuance of equity (FA103164103.Q) by the non-financial corporate sector (from Table F.103). Flow series are seasonally adjusted. We use the CRSP Treasury data to compute the market value of all marketable Treasuries held by the public computed from the zero coupon yield curve. We also use the coupon, principal payment and issuance data from CRSP.

In each year, we implement a zero-cost strategy. We short $1 of the entire Treasury portfolio at the start of each year, and we invest $1 in the entire non-financial corporate sector. Each year, we buy all the newly issued equities net of repurchases and collect dividends. In addition, we make the Treasuries' coupon payments and issue new Treasuries. The net cash flow equals dividends minus net equity issuance per dollar invested minus coupon payments plus net Treasury debt issuance per dollar invested.

I.2 Cyclicality of Cash Flows and Returns

The Treasury cash flows on the short leg are strongly pro-cyclical and hence hedge the equity cash flows of the long leg. Figure A.15 plots the annual cash flows. Remarkably, despite of the counter-cyclical nature of the cash flows, the annualized Sharpe ratio for this investment strategy is 0.58 and the average excess return is 8.85% per annum. Both are higher than for equities, even though the strategy is a recession hedge, unlike equities.

The pricing of pay-outs to shareholders of non-financial corporations cannot be reconciled with the pricing of the pay-outs to Treasury bondholders, at least not ex-post over the past seven decades. Interest rates on government debt are too low, or alternatively, the government debt portfolio is too expensive. This puzzle is related to the standard equity premium puzzle. In the absence of an equity premium, there would be no debt valuation puzzle. But it is obviously distinct from the equity premium puzzle given the pro-cyclical nature of equity cash flows and the counter-cyclical nature of the cash flows on this investment strategy.

Figure A.15: Net Cash Flows from shorting Marketable Treasuries and buying Equities

The figure plots the net annual cash flows per $1 invested (full line) from a zero-cost strategy that shorts $1 of all marketable Treasuries to purchase $1 of the non-financial corporate sector at the start of each year. The cash flows consist of dividends minus net issuance for equities (dashed line), and net lending plus interest for Treasuries.