THE U.S. PUBLIC DEBT VALUATION PUZZLE

Zhengyang Jiang
Hanno Lustig
Stijn Van Nieuwerburgh
Mindy Z. Xiaolan

The U.S. Public Debt Valuation Puzzle
Zhengyang Jiang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z. Xiaolan
NBER Working Paper No. 26583
December 2019
JEL No. E43,E62,G12,G15

ABSTRACT

The market value of outstanding federal government debt in the U.S. exceeds the expected present discounted value of current and future primary surpluses by a multiple of U.S. GDP. When the pricing kernel fits U.S. equity and Treasury prices and the government surpluses are consistent with U.S. post-war data, a government debt valuation puzzle emerges. Since tax revenues are pro-cyclical while government spending is counter-cyclical, the tax revenue claim has a higher short-run discount rate and a lower value than the spending claim. Since revenue and spending are co-integrated with GDP, the long-run risk discount rates of both claims are much higher than the long Treasury yield. These forces imply a negative present value of U.S. government surpluses. Convenience yields for Treasurys are much larger than previously thought and/or U.S. Treasury markets have failed to enforce the no-bubble condition.

Zhengyang Jiang
Kellogg School of Management
Northwestern University
2211 Campus Drive
Evanston, IL 60610
zhengyang.jiang@kellogg.northwestern.edu

Hanno Lustig
Stanford Graduate School of Business
655 Knight Way
Stanford, CA 94305
and NBER
hlustig@stanford.edu

Stijn Van Nieuwerburgh
Columbia University
Graduate school of Business
Uris Hall, office 809
3022 Broadway
New York, NY 10027
and NBER
svnieuwe@gsb.columbia.edu

Mindy Z. Xiaolan
University of Texas at Austin
2110 Speedway B6600
Austin, TX 78703
mindy.xiaolan@mccombs.utexas.edu

1 Introduction

The U.S. Treasury is the largest borrower in the world. As of December 31 2018, the outstanding federal government debt held by the public was valued at $16.1 trillion. Outstanding debt nearly doubled after the Great Recession to 76.4% of the U.S. annual GDP. Before the financial crisis, there was widespread concern that the U.S. had embarked on an unsustainable fiscal path (see, e.g., Rubin, Orszag, and Sinai, 2004). Yet, recently, some economists have argued that the U.S. has ample debt capacity to fund additional spending by rolling over its debt because T-bill rates are below GDP growth rates (Blanchard, 2019).

We show that, absent a bubble in government debt, the relevant "interest rate" on the portfolio of the entire outstanding debt is higher than Treasury bond rates and higher than GDP growth, reversing the former argument. To see why, note that the price of a stock is the expected present discount value of future dividends. Risk-free interest rates are below dividend growth rates, yet the price of the stock is finite. Why? As the stock's dividend growth is pro-cyclical, cash flows are low when the investor's marginal utility is high. The relevant "interest rate" for the stock contains a risk premium because of the risk exposures.

Analogously, we consider the portfolio strategy that buys all new Treasury issues and receives all Treasury coupon and principal payments. This portfolio's cash flow is the federal government's primary surplus. As shown in Figure 1, the primary surpluses are strongly pro-cyclical just like the dividends. In recessions, when marginal utility is high, surpluses are negative and net bond issuance is high. Non-discretionary spending by the federal government (e.g. transfer payments) automatically increases in recessions, while the progressive nature of the tax system produces sharply pro-cyclical revenue.

Figure 1: Government Cash Flows

The figure plots the U.S. federal government primary surplus as a fraction of GDP. The sample period is from 1947Q1 to 2017Q4.
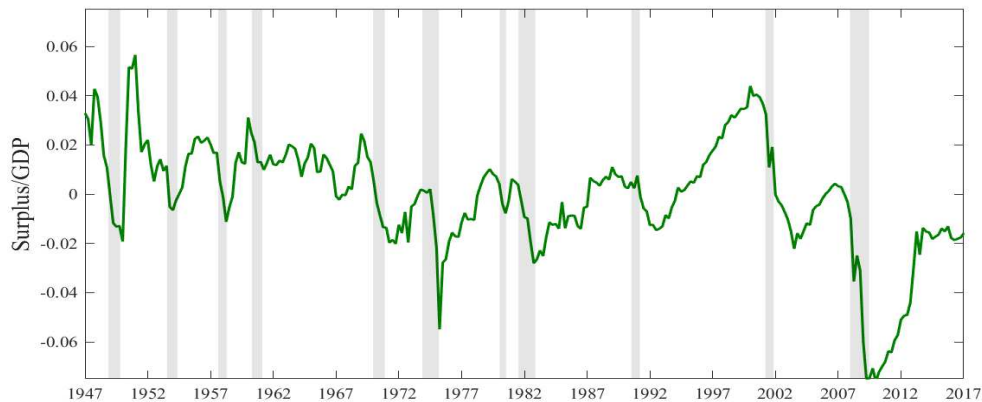
In addition, revenue and spending are cointegrated with GDP and subject, as a result, to the same long run risk as GDP. Therefore, the claim to future government surpluses is risky, and the relevant "interest rate" for government surpluses also contains a risk premium.

More precisely, the value of a claim to current and future government primary surpluses, P_t^S, is the difference between the value of a claim to current and future federal tax revenues, P_t^T, and the value of a claim to current and future federal spending, P_t^G. Since tax revenues are pro-cyclical, the representative investor requires a high risk premium to hold the claim to future tax revenue. Put differently, P_t^T is low. Government spending is counter-cyclical, so that the claim which pays out government spending is a great recession hedge. It commands a lower risk premium. Put differently, P_t^G is high. If the average primary surplus is about zero, as it has been over the past 70 years, the claim to current and future government surpluses, $P_t^S = P_t^T - P_t^G$, should have a negative present discounted value. However, by the government's dynamic budget constraint and in the absence of bubbles, the value of the surplus claim must equal the market value of outstanding debt, which is positive rather than negative. We refer to this difference between the positive valuation of outstanding debt and the negative valuation of the surplus claim as the *government debt valuation puzzle*, which has been 192% of GDP on average since 1947.

In addition, both claims are subject to the same long-run GDP risk. Hence, we expect the long position in the tax claim and short position in the spending claim to earn an additional long-run risk premium, because the long position is larger than the short position and the market value of outstanding debt is positive. This long run risk premium accounts for the bulk of the equity premium (see Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and Chernov, 2018). However, the return on the U.S. government debt portfolio is only 0.93% in excess of three-month Tbill rate. We refer to this as *government debt risk premium puzzle*.

Put in terms of interest rates rather than valuations, the U.S. government's promised payments on its outstanding debt, future surpluses, are a risky cash-flow stream and risk averse investors demand a risk premium to compensate for this risk. Thus, the relevant "interest rate" or discount rate for the government bond portfolio's cash flows is high. Yet, Treasury investors seem willing to purchase government debt at low yields. Government bond yields in the U.S. and other developed bond markets are puzzlingly low.

In addition, we show that the valuation of the outstanding debt is not responsive enough to news about the fundamentals. Instead of excess volatility, as documented in stock markets, we document excess smoothness in public debt markets relative to the fundamentals. U.S. Treasury investors seem largely oblivious to fiscal news, except during isolated episodes such as the 'bond market vigilante' episode between 1993 and 1994.

The above argument relies on a realistic model of risk and asset pricing. First, adequately capturing the dynamics of government spending and tax revenue is crucial. Only when we correctly

specify the risk characteristics of the cash flows to the G-claim and T-claim, will we be able to correctly value the surplus claim. As emphasized above, spending and revenue growth covary with GDP growth. We additionally allow them to covary with inflation, interest rate levels, the slope of the term structure, dividend growth, the price-dividend ratio, and to be predictable by their own lags and the lags of these real and financial variables.

Importantly, we impose that government tax revenues and spending are co-integrated with GDP, and that revenues, spending, and GDP adjust when revenue-to-GDP or spending-to-GDP are away from their long-run relationship. This imposes a form of long-run automatic stabilization. When spending has been higher than usual for a long period, as in the aftermath of the Great Recession, spending growth will be lower than average in the future to return the spending-GDP ratio back to its long-run average. The same mean-reversion is present for tax revenues. Without the assumption of co-integration, all shocks to spending and revenue would permanently affect the levels of spending-to-GDP and revenue-to-GDP. This would render the spending claim even safer and the revenue claim even riskier thereby deepening the puzzle.

Co-integration has important implications for the risk of the T- and G-claims. For example, a deep recession not only raises current government spending and lowers current tax revenue as a fraction of GDP, but also lowers future spending and raises future revenue as a fraction of future GDP. Both the spending and the revenue claims are exposed to the same long-run risk as GDP. As a result, the long-run discount rate for government debt is the same as the rate that investors use when pricing a claim to GDP.

Another way of stating the puzzle is to point out that the surplus claim cannot be risk-free, simply because the government's surpluses trend with GDP; GDP innovations permanently alter all future surpluses. As a result, the risk-free rate cannot be the right discount rate for future surpluses. Put differently, while one can roll over a constant dollar amount at the risk-free rate, one cannot roll over an amount that trends with GDP and is counter-cyclical. Rolling over a claim to a cash flow stream that is co-integrated with GDP at the risk-free rate requires the unlevered equity risk premium to be zero. That requirement flies in the faces of decades of asset pricing research.

Furthermore, if the debt were truly risk-free, then the present value of surpluses would not respond to current fiscal news. Hansen, Roberds, and Sargent (1991) refer to this as the fiscal measurability condition. This condition imposes that any current increase in spending or decrease in revenue during recessions is fully offset (in present value terms) by future decreases in spending and/or increases in revenue. In our data, we find no evidence for future offsets in impulse responses constructed from VARs or local projection methods. This is not surprising. There are no built-in offsets in non-discretionary spending or in the tax system.

Second, to adequately capture attitudes toward risk, we posit a state-of-the-art stochastic discount factor (SDF) model. Rather than committing to a specific utility function, we use a reduced

form SDF that accurately prices the term structure of Treasury bond yields of various maturities in each quarter since 1947. Inflation and GDP growth risk are two key macro-economic sources of risk that affect the price of government bonds. As such, it is by construction consistent with the history of safe interest rates and GDP growth rates. The model matches also the time series of bond risk premia. To obtain a realistic SDF model, we further insist that the model prices a claim to aggregate stock market dividends correctly. Having extracted the market prices of risk associated with the aggregate sources of risk, we have a realistic SDF that can be used to price a claim to future tax revenues and to future government spending. The SDF model's rich implications for the term structure of risk allow it to adequately price not only short-run but also long-run risk to spending and revenue.

One potential resolution of the puzzle is based on the finding that the U.S. government debt earns a convenience yield which lowers the equilibrium returns investors demand for holding government bonds. Convenience yields are an additional source of revenue for the U.S. government which we need to add to the primary surpluses and value properly. Furthermore, convenience yields are counter-cyclical and hence reduce the riskiness of the cash flow stream. Using the measure of Krishnamurthy and Vissing-Jorgensen (2012), adding convenience yield revenue closes about half of the gap between the value of the augmented surplus claim and the value of the debt. To fully close the gap, the additional revenue would need to be more than 12.5% of tax revenue, rather than the observed fraction of 1.74%. Jiang, Krishnamurthy, and Lustig (2018a,b) estimate convenience yields that foreign investors derive from their holdings of dollar safe assets. They find larger estimates of convenience yields, which may help to closing the valuation gap.

We also explore the possibility of a future large fiscal correction that is absent from our sample, but priced into Treasury bonds. We back out from the time series of the market value of government debt that bond investors would have to assign a probability higher than 40% –and even 90% at the end of our sample– to a spending cut equivalent to 7.7% of GDP. Such a high probability seems prima facie implausible and inconsistent with the notion of a peso event. The peso exercise assumes that the large fiscal shock event is not priced. Assuming instead that large fiscal corrections occur in high marginal utility states strikes us as an implausible explanation based on the history of government deficits in the U.S. and other developed economies.

Missing government assets or market segmentation cannot resolve the puzzle either. One final "resolution" to the puzzle is to argue that there is a bubble in U.S. Treasury markets. Indeed, our approach quantifies the bubble as the difference between the value of outstanding government debt and the value of the surplus claim. Over the post-war period, the average size of the bubble is 192% of GDP. Since 2000, the size of the bubble has quadrupled from about 100% to 400% of GDP in 2017 both because the outstanding value of government debt has doubled from about 35% to 75% of GDP and because the value of the surplus claim has fallen dramatically. The Treasury markets do not seem to enforce the transversality condition. The bond market vigilantes seem to

have vanished after the 1990s.[1]

The rest of the paper is organized as follows. We discuss the related literature next. Section 2 introduces the government budget constraint and characterizes the relationship between government surpluses and government debt. Section 3 describes the data and present a high-Sharpe ratio trading strategy exploiting the valuation puzzle. Section 4 sets up and solves the quantitative model. Section 5 documents the government risk premium puzzle in that model. Section 6 revisits the puzzle in a world with convenience yields on Treasury debt. Section 7 discusses other potential resolutions for this puzzle, including a fiscal austerity peso event. Section 8 concludes. The appendix presents the details of model derivation and estimation.

Related Literature   Our paper connects with a long literature which tests the government's inter-temporal budget constraint. Hamilton and Flavin (1986); Trehan and Walsh (1988, 1991); Hansen, Roberds, and Sargent (1991); Bohn (2007) derive general time-series restrictions on the government revenue and spending processes that enforce the government's inter-temporal budget constraint. These authors use the risk-free rate as the discount rate. In the absence of arbitrage opportunities, this approach is valid only if we rule out all risk premia, including the equity premium. Our paper contributes to this literature by enforcing no-arbitrage restrictions across different asset markets.

There is a parallel literature in asset pricing which tests the present value equation for stocks and other long-lived assets, starting with the seminal work by LeRoy and Porter (1981); Campbell and Shiller (1988). The prices of these long-lived assets seem excessively volatile relative to their fundamentals. Government debt is fundamentally different: its valuation does not seem volatile enough relative to the fundamentals.

Our work is the first to formally test for the presence of bubbles in Treasury markets. There is a large literature on rational bubbles in asset markets, starting with the seminal work by Samuelson (1958); Diamond (1965); Blanchard and Watson (1982). We document a violation of the transversality condition in Treasury markets, consistent with the existence of a rational bubble. However, Brock (1982); Tirole (1982); Milgrom and Stokey (1982); Santos and Woodford (1997) argue that rational bubbles are hard to sustain in the presence of long-lived investors, absent other frictions. Indeed, we show that a rational patient investor who pursues an investment strategy that buys all corporate equities and shorts the portfolio of all U.S. Treasuries earns a risk premium higher than the equity premium but receives cash flows that hedge the business cycle. Giglio, Maggiori, and Stroebel (2016) devise a model-free test for bubbles in housing markets. Our test is not model-free, but the results hold in a large class of models in which permanent shocks to the pricing kernel are an important driver of risk premia (Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009;

---

[1]While there are equilibrium models with overlapping generations that feature violations of the transversality condition, the violations typically do not grow faster than GDP, which seems to be the case for U.S. Treasuries. See Giglio, Maggiori, and Stroebel (2016) for an overview of theories of rational bubbles.

Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and Chernov, 2018).

Our asset pricing model builds on Lustig, Van Nieuwerburgh, and Verdelhan (2013), who price a claim to aggregate consumption and study the properties of the price-dividend ratio of this claim, the wealth-consumption ratio. Here we focus on pricing claims to government revenue and spending growth instead. The asset pricing model combines a vector auto-regression model for the state variables as in Campbell (1991, 1993, 1996) with a no-arbitrage model for the (SDF) as in Duffie and Kan (1996); Dai and Singleton (2000); Ang and Piazzesi (2003). Gupta and Van Nieuwerburgh (2018) use a similar framework to evaluate the performance of private equity funds. Our current work focuses on estimating the model with a single regime; Bianchi and Melosi (2014, 2017, 2018) study different regimes of the fiscal policy and their real effects.

Treasuries are expensive. Our work connects to the large literature on the specialness of U.S. government bonds. Longstaff (2004); Krishnamurthy and Vissing-Jorgensen (2012); Fleckenstein, Longstaff, and Lustig (2014); Krishnamurthy and Vissing-Jorgensen (2015); Nagel (2016); Bai and Collin-Dufresne (2019) find that U.S. government bonds are traded at a premium relative to other risk-free bonds. Greenwood, Hanson, and Stein (2015) study the government debt's optimal maturity in the presence of such premium, and Jiang, Krishnamurthy, and Lustig (2018a) study this premium in international finance. We tackle the question of how expensive a portfolio of all Treasuries is relative to the underlying collateral, a claim to surpluses.

Foreign ownership of Treasuries has increased dramatically since the 1990s (Favilukis, Kohn, Ludvigson, and Nieuwerburgh, 2013; Kohn, 2016). There is a growing literature in international economics that emphasizes the special role of the U.S. as the world's safe asset supplier (see Gourinchas and Rey, 2007; Caballero, Farhi, and Gourinchas, 2008; Caballero and Krishnamurthy, 2009; Maggiori, 2017; He, Krishnamurthy, and Milbradt, 2018; Gopinath and Stein, 2018; Krishnamurthy and Lustig, 2019). Foreign official institutions and the Federal Reserve Bank combined have held about two-thirds of U.S. Treasuries over the past twenty years. If this source of demand for U.S. Treasuries is highly inelastic, this could help to resolve the puzzle. However, we do not find that the net payouts to bondholders excluding the Fed and foreign investors are significantly less cyclical. Bolton (2016) and Bolton and Huang (2018) explore the notion that domestic currency debt issued by governments is like equity issued by corporations.

Our work is not about the fiscal theory of the price level, which asserts that the price level and the exchange rate adjust to enforce the government's intertemporal budget constraint (Sargent and Wallace, 1984; Leeper, 1991; Woodford, 1994; Sims, 1994; Cochrane, 2001, 2005, 2019a,b; Jiang, 2019a,b). We do not have to take a stand on whether the fiscal theory holds.

Since the 2008 financial crisis, CDS premia for U.S. Treasuries have been trading in a higher range. Recently, some have related these CDS premia to the underlying fiscal fundamentals (Chernov, Schmid, and Schneider, 2016; Pallara and Renne, 2019). Our puzzle holds even when accounting for default: the market value of debt should still be backed by current and future surpluses.

2 Theoretical Characterizations

We start by deriving some general results that only rely on the absence of arbitrage and two assumptions about cash flows. The first assumption concerns the long-run: a stochastic trend in GDP and co-integration of the size of the government with GDP. The second assumption concerns the short-run: counter-cyclical spending and pro-cyclical tax revenue. These theoretical results underscore that the debt valuation puzzle holds regardless of the specifics of our quantitative asset pricing model.

2.1 Value Equivalence

Let G_t denote nominal government spending before interest expenses, T_t denote nominal government tax revenue, and $S_t = T_t - G_t$ denote the nominal primary government surplus. Let $P_t^\$(h)$ denote the price at time t of a nominal zero-coupon bond that pays \$1 at time $t + h$, where h is the maturity expressed in quarters. Let $Q_{t,h}^\$$ denote the outstanding face value at time t of government bond payments that are due at time $t + h$. Iterating on the one-period government budget constraint, we show two equivalences between the government debt portfolio and primary surpluses.

Proposition 1 (Value Equivalence). Today's market value of the outstanding government debt portfolio equals the expected present discounted value of current and all future primary surpluses:

$$D_t \equiv \sum_{h=0}^{H} P_t^\$(h) Q_{t-1,h+1}^\$ = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^\$ (T_{t+j} - G_{t+j}) \right] \equiv P_t^\tau - P_t^g, \tag{1}$$

where the value of the tax claim and value of the spending claim are defined as:

$$P_t^\tau = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^\$ T_{t+j} \right], \quad P_t^g = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^\$ G_{t+j} \right].$$

The proof is given in Appendix A. The multi-period stochastic discount factor (SDF) $M_{t,t+h}^\$ = \prod_{k=0}^{h} M_{t+k}^\$$ is the product of the adjacent one-period SDFs, $M_{t+k}^\$$. Bond prices satisfy $P_t^\$(h) = \mathbb{E}_t \left[M_{t,t+h}^\$ \right] = \mathbb{E}_t \left[M_{t+1}^\$ P_{t+1}^\$(h-1) \right]$. By convention $P_t^\$(0) = M_{t,t}^\$ = M_t^\$ = 1$ and $M_{t,t+1}^\$ = M_{t+1}^\$$. The government bond portfolio is stripped into zero-coupon bond positions $Q_{t,h}^\$$. $Q_{t-1,1}^\$$ is the total amount of debt payments that is due today. The outstanding debt reflects all past bond issuance decisions, i.e., all past primary deficits. The proof relies only on the existence of a SDF, i.e., the absence of arbitrage opportunities, but not on complete markets. It imposes a transversality condition that rules out a government debt bubble: $E_t \left[M_{t,t+T} D_{t+T} \right] \to 0$ as $T \to \infty$.

Eq. (1) implies that when the government runs a deficit in a future date and state, it will need to issue new bonds to the investing public. If those dates and states are associated with a high value of the SDF for the representative bond investor, that debt issuance occurs at the "wrong"

time. Investors will need to be induced by low prices (high yields) to absorb that new debt. To see this more clearly, we can write the right-hand side of eq. (1) as:

$$D_t = \sum_{j=0}^{\infty} P_t^{\$}(j)\mathbb{E}_t\left[S_{t+j}\right] + \sum_{j=0}^{\infty} \text{Cov}_t\left(M_{t,t+j}^{\$}, T_{t+j}\right) - \sum_{j=0}^{\infty} \text{Cov}_t\left(M_{t,t+j}^{\$}, G_{t+j}\right)$$

The first term on the right-hand side is the present discounted value of all expected future surpluses, using the term structure of risk-free bond prices. It is the PDV for a risk-neutral investor. If the SDF is constant, this is the only term on the right-hand side (Hansen, Roberds, and Sargent, 1991; Sargent, 2012). The government's capacity to issue debt today is constrained by, or collateralized by its ability to generate current and future surpluses. The second and third terms encode the riskiness of the government debt portfolio, and only arise in the presence of time-varying discount rates. Since tax revenues tend to be high when times are good ($M_{t,t+j}$ is low), the second term is expected to be negative. Since government spending tends to be high when times are bad ($M_{t,t+j}$ is high), the third term is expected to be positive. Thus, the difference between the second and the third term is unambiguously negative. The covariance terms lower the government's debt capacity. Put differently, to support a given, positive amount of government debt, D_t, the first term will need to be higher by an amount equal to the absolute value of the covariance terms. This paper quantifies the covariance terms in a realistic model of risk and return, while most macro-economic models imply only small risk premia. The key finding of this paper is that the net covariance term is large in absolute value and negative, on the order of two times GDP.

Five remarks are in order. First, ex ante, equation (1) has to hold both in nominal and real terms. Ex post, the government can erode the real value of outstanding debt by creating surprise inflation. Given the short duration of outstanding debt, that channel has limited potency in the U.S.[2] Furthermore, markets do not seem to anticipate that the U.S. government would erode the real value of debt through inflation. The 10-year break-even inflation rate is currently 1.6% per annum.

Second, the same inter-temporal budget constraint holds when we allow for sovereign default: the valuation of government debt is still backed by future surpluses. The proof is given in Appendix A. In this case, the bond prices satisfy $P_t^{\$}(h) = \mathbb{E}_t\left[M_{t,t+h}^{\$}(1 - \chi_{t,t+h})\right]$, where $\chi_{t,t+h}$ is an indicator variable that is one when the government defaults between t and $t + h$.[3]

Third, we simply assume that there exists an SDF to price these cash flows in fixed income markets. We do not take a stand on whether the marginal investor is foreign or domestic.

---

[2]The duration of the Treasury portfolio is roughly 4 years. A 5 percentage point increase in inflation that lasts as long as the maturity of the longest outstanding debt (30 years) reduces the real value of debt by 20%. See Hall and Sargent (2011); Berndt, Lustig, and Yeltekin (2012) for a decomposition of the forces driving the U.S. debt/GDP ratio including inflation. Cochrane (2019a,b) explores the connection between inflation and the value of government debt.

[3]We assume full default to keep the proof simple, but this is without loss of generality. Since the financial crisis, CDS spreads for U.S. Treasurys have been elevated. Chernov, Schmid, and Schneider (2016); Pallara and Renne (2019) study the response of CDS spreads to news about the fiscal surplus.

Fourth, the gap between the left hand side and the right hand side of (1) can be interpreted as the violation of the transversality condition. Blanchard (2019) describes a class of models in which the government can keep rolling over the debt because the economy's growth rate exceeds the discount rate on the debt.

Fifth, Blanchard (2019) assumes that the right discount rate for government debt is the safe rate. We show that it cannot be, given that the primary surplus is co-integrated with GDP, and the surplus is pro-cyclical.

2.2 Risk Premium Equivalence

Recall that P_t^τ denotes the cum-dividend value of claim to tax revenue, P_t^g denotes the cum-dividend value of a claim to government spending, and D_t denotes the market value of the outstanding government debt portfolio. Define the holding period returns on the bond portfolio, the tax claim, and the spending claim as:

$$r_{t+1}^D = \frac{\sum_{h=1}^\infty P_{t+1}^\$(h-1)Q_{t,h}^\$}{\sum_{h=1}^\infty P_t^\$(h)Q_{t,h}^\$}, \quad r_{t+1}^\tau = \frac{P_{t+1}^\tau}{P_t^\tau - T_t}, \quad r_{t+1}^g = \frac{P_{t+1}^g}{P_t^g - G_t}.$$

We can further prove equivalence between the discount rate of the government surplus claim and that of the government debt portfolio.

Proposition 2 (**Risk Premium Equivalence**). Today's expected holding return $\mathbb{E}_t\left[r_{t+1}^D\right]$ on the government debt portfolio equals the expected holding return $\mathbb{E}_t\left[r_{t+1}^\tau\right]$ on the claim to future tax revenues minus the expected holding return $\mathbb{E}_t\left[r_{t+1}^g\right]$ on the claim to future government spending, weighted appropriately:

$$\mathbb{E}_t[r_{t+1}^D] = \frac{P_t^\tau - T_t}{D_t - S_t}\mathbb{E}_t[r_{t+1}^\tau] - \frac{P_t^g - G_t}{D_t - S_t}\mathbb{E}_t[r_{t+1}^g]. \tag{2}$$

where we have used $D_t - S_t = (P_t^\tau - T_t) - (P_t^g - G_t)$,

This second equivalence can be understood as the Modigliani-Miller theorem in the context of government finance. Absent frictions, the average discount rate on government liabilities is equal to the average discount rate on government assets, which are a claim to primary surpluses. Since the primary surpluses are tax revenues minus government spending, the discount rate on government debt equals the difference between the discount rates of tax revenues and government spending, appropriately weighted.

We can restate this expression in terms of expected excess returns:

$$\mathbb{E}[r_{t+1}^D - r_t^f] = \frac{P_t^\tau - T_t}{D_t - S_t}\mathbb{E}[r_{t+1}^\tau - r_t^f] - \frac{P_t^g - G_t}{D_t - S_t}\mathbb{E}[r_{t+1}^g - r_t^f].$$

To develop intuition, we consider a few simple scenarios. If the expected returns on both claims are identical, $\mathbb{E}[r^\tau_{t+1}] = \mathbb{E}[r^g_{t+1}]$, then the expected return on government debt is given by

$$\mathbb{E}[r^D_{t+1} - r^f] = \mathbb{E}[r^\tau_{t+1} - r^f] = \mathbb{E}[r^g_{t+1} - r^f].$$

However, if the tax revenue claim is riskier than the spending clam and hence earns a higher excess return, $\mathbb{E}[r^\tau_{t+1}] > \mathbb{E}[r^g_{t+1}]$, then the expected return on government debt exceeds the expected excess returns on the revenue and the spending claims:

$$\mathbb{E}[r^D_{t+1} - r^f] > \mathbb{E}[r^\tau_{t+1} - r^f] > \mathbb{E}[r^g_{t+1} - r^f].$$

Long-run Discount Rates   We start by analyzing the long-run discount rates on the tax revenue and government spending claims. We consider a spending (revenue) strip that pays off G_{t+j} (T_{t+j}) at time $t + j$ and nothing at other times. Let $R^g_{t,t+j}$ $(R^{\tau,j}_{t,t+j})$ be the holding period return on such an j-period spending (revenue) strip. We analyze the limit of the log returns on these strips as $j \to \infty$ under two assumptions on the time-series properties of government spending and tax revenues.

Proposition 3 (**Long-run Discount Rates**). If the log of government spending G (tax revenue T) is stationary in levels (after removing a deterministic time trend), then the long-run expected log return on spending (revenue) strips equals the yield on a long-term government bond as the payoff date approaches maturity.

$$\lim_{j \to \infty} \mathbb{E}_t r^{g,j}_{t,t+j} = y^\infty_t, \quad \lim_{j \to \infty} \mathbb{E}_t r^{\tau,j}_{t,t+j} = y^\infty_t,$$

where y^∞_t is the yield at time t on a nominal government bond of maturity $+\infty$. The proof is given in Appendix A. The result builds on work by Alvarez and Jermann (2005); Hansen and Scheinkman (2009); Borovička, Hansen, and Scheinkman (2016); Backus, Boyarchenko, and Chernov (2018), among others.

This result implies that the long-run strips can be discounted off the term-structure for zero coupon bonds. In this case, the long-run discount rate on government debt is the *yield on a long-term risk-free bond*. However, if there are no permanent shocks to T or G, then it is imperative to assume that GDP and aggregate consumption are not subject to permanent shocks either. If there are no permanent shocks to marginal utility, then the long bond is the riskiest asset in economy. That clearly seems counterfactual (Alvarez and Jermann, 2005). Put differently, the gap between the long-run discount rates on strips and the long yields is governed by the entropy of the permanent component of the pricing kernel. Explaining the high returns on risky assets such as stocks requires that entropy to be large, not zero (e.g., Borovička, Hansen, and Scheinkman, 2016). Next we consider a more realistic case.

11

Corollary 1. If the log of government spending/GDP ratio G/GDP (revenue/GDP T/GDP) is stationary in levels, then the long-run expected log excess return on long-dated spending (revenue) strips equals that on GDP strips:

$$\lim_{j \to \infty} \mathbb{E}_t r_{t,t+j}^{g,j} = \mathbb{E}_t r_{t,t+n}^{gdp,\infty} >> y_t^\infty, \quad \lim_{j \to \infty} \mathbb{E}_t r_{t,t+j}^{\tau,j} = \mathbb{E}_t r_{t,t+n}^{gdp,\infty} >> y_t^\infty.$$

This corollary implies that government bond investors have a net long position in a claim that is exposed to the same long-run risk as the GDP claim. It follows that the value of the long-run spending minus revenue strips will be smaller than what is predicted by the yields at the long end of the term structure:

$$\lim_{n \to \infty} \left(T_t \widehat{p}_t^n[T] - G_t \widehat{p}_t^n[G] \right) = \lim_{n \to \infty} (T_t - G_t) \, \widehat{p}_t^n[GDP] << \lim_{n \to \infty} (T_t - G_t) \exp(-n y_t^n)$$

where $\widehat{p}_t^n[\cdot]$ is the price of an $n-$period strip. If spending and revenues are cointegrated with GDP, then the long-run discount rate is the *long-run discount rate on* gdp, which we can think of as the expected return on *unlevered equity*. This return is much higher than the yield on long-term risk-free bonds because of permanent shocks to marginal utility.

Short-run Discount Rates   Next, we turn our attention to cyclical risk which drives the expected returns on short maturity strips. Even though revenue and spending claims have the same long-run discount rates, short-run discount rates will likely be higher for the revenue claim because tax revenue is highly pro-cyclical while government spending is counter-cyclical. This property of short-run discount rates deepens the risk premium puzzle, because government debt investors have a net long position in a riskier claim than the short position.

Combining the properties of short-run and long-run discount rates together, theory predicts that $\mathbb{E}[r_{t+1}^D - r^f] > \mathbb{E}[r_{t+1}^\tau - r^f] > \mathbb{E}[r_{t+1}^g - r^f]$. To summarize, a model of asset prices will have to confront two forces that push up the equilibrium returns on government debt. First, the long-run discount rates are higher than the yield on a long-maturity bond, because of the long-run cash flow risk in the spending and revenue claims equals that of long-run GDP risk. Government debt investors have a net long position in a claim that is exposed to the same long-run cash flow risk as GDP. The excess returns on government debt will tend to be much higher than those on long-maturity bonds. Second, there is short-run cash flow risk that pushes the expected return on the revenue claim above the expected return on the spending claim. As a result, government debt investors earn a much larger risk premium on the long end than what they pay on the short end. This further increases the fair expected return on the debt claim. Discounting future surpluses using the (term structure of) risk-free interest rate(s) is inappropriate. Section 5.5 below derives

conditions under which future surpluses can be discounted at the risk-free interest rate and shows that these conditions are severely violated in the data.

3   Empirical Results

3.1   Data

Federal tax revenue, federal government spending before interest expense are from NIPA table 3.2; the nominal GDP and GDP price indexes are from NIPA table 1.1.5 and table 1.1.4 respectively. Constant maturity Treasury yields are from Fred. Stock price and dividend data are from CRSP; we use the CRSP value-weighted total market to represent the U.S. stock market. Dividends are seasonally adjusted. For more data construction details, please see Appendix H.

As shown in Figure 1, the surpluses expressed as a fraction of GDP are strongly pro-cyclical. Non-discretionary spending accounts for at least 2/3 of the government's spending. This includes Social Security, Medicare and Medicaid, as well as food stamps and unemployment benefits. Some of these transfer payments rise automatically in recessions. In addition, the federal government may temporarily increase transfer payments in recessions (e.g., the extension of unemployment benefits in 2009). On the revenue side, the progressive nature of the tax system builds in pro-cyclical variation in revenue as a fraction of GDP.

We construct the market value and the total returns of the marketable government bond port-folio using CRSP Treasuries Monthly Series from 1947.Q1 to 2017.Q4. For each end-of-quarter month, we multiply the nominal price of each cusip by its total amount outstanding (normalized by the face value), and sum across all issuances (cusips).

We exclude non-marketable debt which is mostly held in intra-governmental accounts.[4] The largest holder of non-marketable debt is the Social Security Administration. We exclude debt held in intra-governmental accounts because we consolidate the SSA and federal government's defined benefit plans with the Treasury. We choose not to consolidate the Fed and the Treasury, which would add money and subtract the Fed's Treasury holdings on the left hand side of eq. (1). Doing so would mainly tilt the duration of the bond portfolio.

Following Hall and Sargent (2011) and extending their sample, we construct zero coupon bond (strip) positions from all coupon-bearing Treasury bonds (all cusips) issued in the past and out-standing in the current quarter. This is done separately for nominal and real bonds. Since zero-coupon bond prices are also observable, we can construct the left-hand side of eq. (1) as the market

---

[4]The total amount outstanding includes the amount held in U.S. government accounts and Federal Reserve Banks. The amount held by the public is the total amount outstanding minus the amount held in U.S. government accounts; it includes the amount held by the Federal Reserve system. Note that CRSP Treasuries database only contain the marketable debt, so the nonmarketable debt issued by the Treasury is not included in the market value of outstanding government debt in our analysis. The nonmarketable debt can be held either by the public or in other U.S. government accounts. But, most of the nonmarketable debt belongs to intra-governmental holdings.

value of outstanding marketable U.S. government debt.[5] Figure 2 plots its evolution over time, scaled by the U.S. GDP.

Figure 2: The Market Value of Outstanding Debt to GDP

The figure plots the ratio of the nominal market value of outstanding government debt divided by nominal GDP. GDP Data are from the Bureau of Economic Analysis. The market value of debt is constructed as follows. We multiply the nominal price (bid/ask average) of each cusip by its total amount outstanding (normalized by the face value), and then sum across all issuance (cusip). The series is quarterly from 1947.Q1 until 2017.Q4. Data Source: CRSP U.S. Treasury Database and BEA.
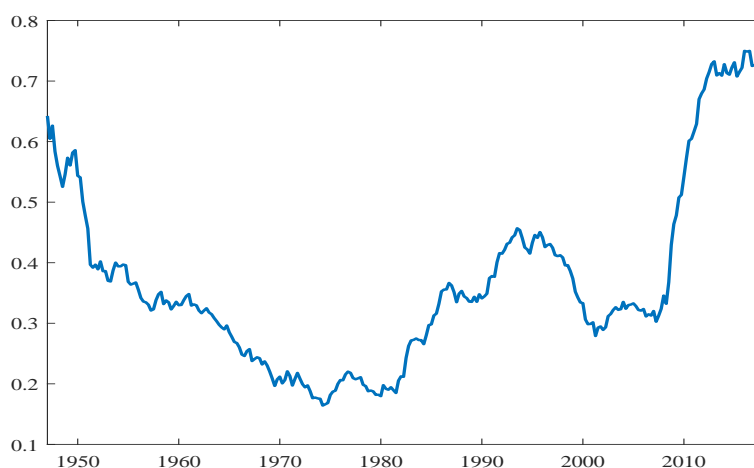
Table 1 reports summary statistics for the overall Treasury bond portfolio in Panel A and for individual bonds in Panel B. The excess returns on the entire Treasury portfolio realized by an investor who buys all of the new issuances and collects all of the coupon and principal payments is 0.93% per annum, on average. The portfolio has an average duration of 4.24 years.

Figure 3 plots the total return on the overall goverment bond portfolio (full line) and breaks it down into the net payout rate (dashed line) and the change in the market value (dotted line). Net payouts to bondholders equal all principal and coupon payments paid by the Treasury minus new issuance. The new issuance equals the primary deficit of the federal government. While the change in the market value is strongly counter-cyclical, the total return is pro-cyclical because the net payout yield is strongly pro-cyclical. The latter occurs because new bond issuance increases in recessions. For an investor who holds the entire outstanding portfolio, Treasuries are not a hedge.

3.2 Betting Against The Treasury

We argue that Treasuries are overpriced relative to other asset classes, such as equities, once you take into account the riskiness of the cash flows. To illustrate this point in a model-free way, we explore a simple investment strategy which shorts the portfolio of all outstanding Treasuries and goes long in all equities.

---

[5]Since the model fits nominal bond prices very well, as shown below, we can equivalently use model-implied bond prices. Similarly, we can use model-implied prices for real zero-coupon bonds.

Table 1: Summary Statistics for Government Bond Portfolio

Panel A reports summary statistics for the holding period return on the aggregate government bond portfolio: the mean and the standard deviation of the holding period return, r^D, the excess return, $r^D - r_f$, the three-month Tbill rate, r_f, and the weighted average Macaulay duration. Panel B reports the mean and the standard deviation of the holding period returns of three-month T-bill and T-bonds with time-to-maturity of one year, five years and ten years. All returns are expressed as annual percentage points. Duration is expressed in years. Data source: CRSP Treasuries Monthly Series. The sample period is from 1947.Q1 to 2017.Q4.

	Panel A			
	r^D	$r^D - r_f$	r_f	Duration
Mean	5.08	0.93	4.16	4.24
Std.	3.92	3.87	3.16	0.70
Sharpe Ratio	0.24			
	Panel B			
	1 Yr	5 Yr	10 Yr	20 Yr
Mean	4.74	4.79	5.56	5.75
Std.	1.04	1.72	4.80	6.80
Sharpe Ratio	0.42	0.29	0.23	0.23

We use the market value of non-financial corporate equity (reported in Table L103 of the Financial Accounts of the U.S.) and the value of all outstanding Treasuries (FL313161105.Q in Table L106).[6] We use the dividend payments (FA106121075.Q) and issuance of equity (FA103164103.Q) by the non-financial corporate sector (from Table F.103). Flow series are seasonally adjusted. We use the CRSP Treasury data to compute the market value of all marketable Treasuries held by the public computed from the zero coupon yield curve. We also use the coupon, principal payment and issuance data from CRSP.

In each year, we implement a zero-cost strategy. We short $1 of the entire Treasury portfolio at the start of each year, and we invest $1 in the entire non-financial corporate sector. Each year, we buy all the newly issued equities net of repurchases and collect dividends. In addition, we make the Treasuries' coupon payments and issue new Treasuries. The net cash flow equals dividends minus net equity issuance per dollar invested minus coupon payments plus net Treasury debt issuance per dollar invested.

The Treasury cash flows on the short leg are strongly pro-cyclical and hence hedge the equity cash flows of the long leg. Figure 4 plots the annual cash flows. Remarkably, despite of the counter-cyclical nature of the cash flows, the annualized Sharpe ratio for this investment strategy is 0.58 and the average excess return is 8.85% per annum. Both are higher than for equities, even though the strategy is a recession hedge, unlike equities.

The pricing of pay-outs to shareholders of non-financial corporations cannot be reconciled with the pricing of the pay-outs to Treasury bondholders, at least not ex post over the past seven

---

[6]Since 2015, the Financial Accounts of the U.S. include some non-marketable debt: the holdings of the federal government employees defined benefit plans.

Figure 3: Excess Returns on the Treasury Portfolio

The figure plots the annual change in the market value in excess of the risk-free rate and the net payouts to bondholders as a fraction of the lagged market value. The total annual return is the sum of the change in the market value and the net payouts to bondholders. The net payouts to bondholders are the the principal plus coupon payments minus new issuance. Annual data is from the CRSP Treasury file.
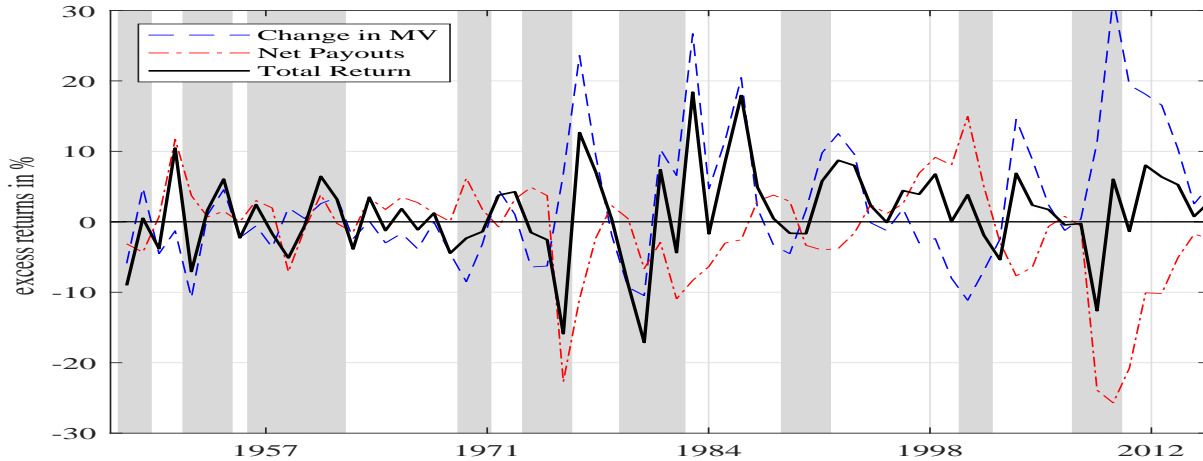

Figure 4: Net Cash Flows from shorting Marketable Treasuries and buying Equities

The figure plots the net annual cash flows per $1 invested (full line) from a zero-cost strategy that shorts $1 of all marketable Treasuries to purchase $1 of the non-financial corporate sector at the start of each year. The cash flows consist of dividends minus net issuance for equities (dashed line), and net lending plus interest for Treasuries.

16
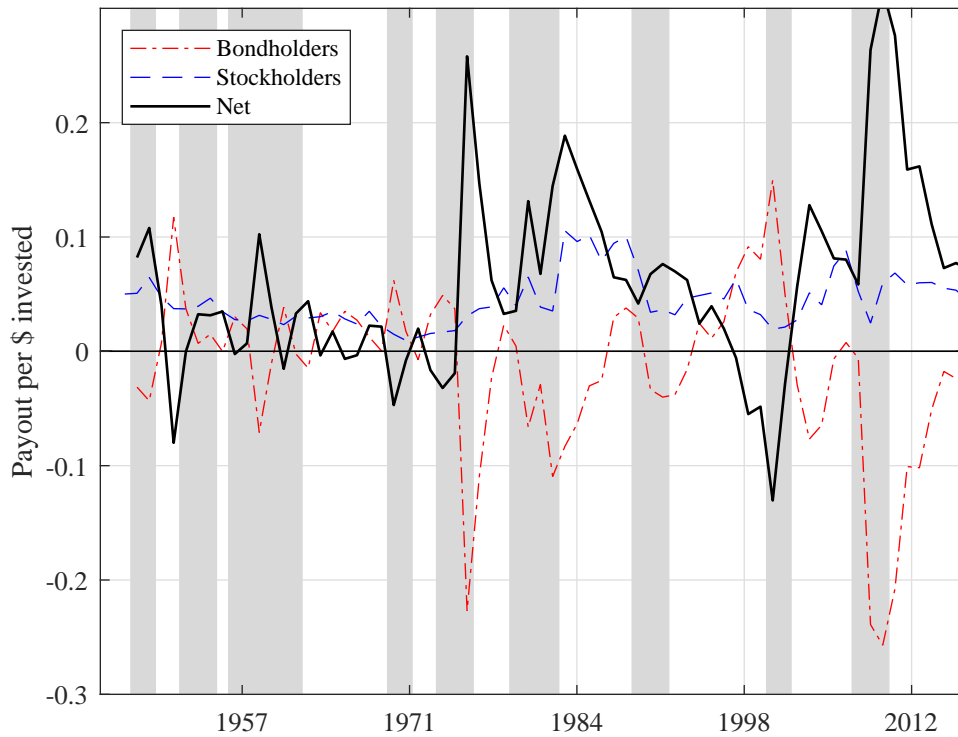
decades. Interest rates on government debt are too low, or alternatively, the government debt portfolio is too expensive. This puzzle is related to the standard equity premium puzzle. In the absence of an equity premium, there would be no debt valuation puzzle. But it is obviously distinct from the equity premium puzzle given the nature of the cash flows.

Limits to arbitrage could explain why rational investors, especially those managing assets on behalf of other investors, may choose not to short the Treasury market in this way (Shleifer and Vishny, 1997). The strategy is a long-term strategy that harvests all of these cash flows in anticipation of a correction in Treasury markets. Furthermore, the market value of the portfolio would be marked down in recessions.

4   Quantitative Asset Pricing Model

Next, we propose a simple no-arbitrage model for stocks, bonds, and government cash flows. We take a pragmatic approach and choose a flexible SDF model that prices the term structure of interest rates precisely.

4.1   State Variables

We take a stance on the key sources of aggregate risk in the economy, and postulate that their dynamics follow a VAR. The goal is to estimate the market prices of these macro-economic risks, such that the model matches observed government bond yields and equity prices. With these market prices of risk in hand, we compute the expected present discounted value of future surpluses, the right-hand side of (1). Our approach takes spending and tax policy as given, rather than being optimally determined. However, both policies are allowed to depend on a rich set of state variables and are estimated from the data.

We assume that the $N \times 1$ vector of state variables follows a Gaussian first-order VAR:

$$z_t = \mathbf{\Psi} z_{t-1} + \mathbf{\Sigma}^{\frac{1}{2}} \varepsilon_t, \tag{3}$$

with shocks $\varepsilon_t \sim i.i.d. \mathcal{N}(0, I)$ whose variance is the identity matrix. The companion matrix $\mathbf{\Psi}$ is a $N \times N$ matrix. The vector z is demeaned. The covariance matrix of the innovations to the state variables is Σ; the model is homoskedastic. We use a Cholesky decomposition of the covariance matrix, $\Sigma = \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}\prime}$, which has non-zero elements only on and below the diagonal. In this way, we interpret the shock to each state variable as a linear combination of its own structural shock ε_t, and the structural shocks to the state variables that precede it in the VAR.

First, we include state variables that govern the term structure of interest rates. We follow the empirical term structure literature and specify a term structure model that contains two key macro-economic sources of risk, inflation (π_t) and real GDP growth (x_t), as well as two interest

rates, the nominal short rate ($y_t^\$(1)$) and the yield spread ($yspr_t^\$$) defined as the difference between the 5-year and 1-quarter nominal bond yields: $yspr_t^\$ = y_t^\$(20) - y_t^\$(1)$. This is akin to a model with two observable macro-economic time series and two latent factors. It is well understood that two latent factors are needed to describe the term structure of interest rates since interest rates are not fully spanned by macro-economic time series (Joslin, Priebsch, and Singleton, 2014).

Second, we include the log price-dividend ratio and the log real dividend growth on the aggregate stock market in the VAR system. Together they encode sufficient information for the time series of stock returns. including stocks is helpful to extract information about the permanent component of the SDF.

Third, to capture the government's cash flows, we include $\Delta \log \tau_t$ and $\Delta \log g_t$, the log change in government revenue to GDP and the log change in government spending to GDP. We denote the ratio of government spending to GDP by g_t, the ratio of tax revenues to GDP by τ_t, and the ratio of the primary surplus to GDP by s_t.

Due to the presence of corrective fiscal actions (Bohn, 1998), the levels of tax revenue and government spending cannot deviate too much from the level of GDP. Appendix D performs the Johansen and Phillips-Ouliaris cointegration tests. The results support two cointegration relationships between log tax revenue and log GDP and between log spending and log GDP. The coefficients estimates of the cointegration relationships, however, tend to vary across sample periods. As a result, we take an a priori stance that the tax-to-GDP ratio $\log \tau$ and the spending-to-GDP ratio $\log g$ are stationary. That is, we assume cointegration coefficients of $(1, -1)$ for both relationships. Tax revenue and government spending may temporarily deviate from the GDP trend, but must eventually mean revert. Imposing cointegration requires us to include the levels of $\log \tau$ and $\log g$ in our vector of state variables. In summary, the VAR variables are:

$$
\begin{aligned}
z_t = & [\pi_t - \pi_0, x_t - x_0, y_t^\$(1) - y_0^\$(1), yspr_t^\$ - yspr_0^\$(1), pd_t - \overline{pd}, \Delta d_t - \mu_d, \\
& \Delta \log \tau_t - \mu_0^\tau, \Delta \log g_t - \mu_0^g, \log \tau_t - \log \tau_0, \log g_t - \log g_0]'.
\end{aligned}
$$

Since we impose cointegration on the level of GDP, tax and spending, the unconditional growth rates of the tax-to-GDP ratio and the spending-to-GDP ratio (μ_0^τ and μ_0^g) have to be zero. The unconditional growth rate of the GDP x_0 is measured from the sample.

We use selector vectors to pick out particular elements of the state vector. For example, we use e_π to denote the vector $[1, 0, \ldots, 0]'$, which picks out the row of the VAR corresponding to π_t: $\pi_t = \pi_0 + e_\pi' z_t$. Similarly, the one-quarter nominal bond yield is $y_t^\$(1) = y_0^\$(1) + e_{yn}' z_t$, where $y_0^\$(1)$ is the unconditional average yield and e_{yn} is a vector that selects the element of the state vector corresponding to the one-month yield. e_x picks out the row of the VAR corresponding to real GDP growth, x_t, $e_{\Delta\tau}$ picks out the row of the VAR corresponding to $\Delta \log \tau_{t+1}$, $e_{\Delta g}$ picks out the row of the VAR corresponding to $\Delta \log g_{t+1}$, and so on.

18

Table 2: VAR Estimates $\mathbf{\Psi}$

	π_{t-1}	x_{t-1}	$y^{\$}_{t-1}(1)$	$yspr^{\$}_{t-1}$	pd_{t-1}	Δd_{t-1}	$\Delta \log \tau_{t-1}$	$\Delta \log g_{t-1}$	$\log \tau_{t-1}$	$\log g_{t-1}$
π_t	0.536	0.065	0.241	0	0	0.028	0	0	0	0.005
x_t	0	0.346	0	0	0	0	0.013	-0.002	-0.010	0.002
$y^{\$}_t(1)$	0.043	0.043	0.958	0	0	0	0	0	0	0
$yspr^{\$}_t$	0	-0.018	0	0.863	0	0	0	0	0	0
pd_t	0	0	0	0	0.978	0	0	0	0	0
Δd_t	0	0	0	0	0	0.453	0.071	-0.092	0	0
$\Delta \log \tau_t$	0	1.199	0	-4.871	0.024	0	-0.141	-0.114	-0.142	0.053
$\Delta \log g_t$	0	-1.275	0	0	-0.018	0	-0.098	-0.060	0.019	-0.064
$\log \tau_t$	0	1.199	0	-4.871	0.024	0	-0.141	-0.114	0.858	0.053
$\log g_t$	0	-1.275	0	0	-0.018	0	-0.098	-0.060	0.019	0.936

4.2 Cash Flow Dynamics

We estimate the VAR system $z_t = \mathbf{\Psi} z_{t-1} + \mathbf{\Sigma}^{\frac{1}{2}} \varepsilon_t$ using OLS with two additional features. First, we iteratively restrict the statistically insignificant elements in $\mathbf{\Psi}$ to 0. The exceptions are the loadings of x, $\Delta \log \tau_t$ and $\Delta \log g_t$ on $\Delta \log \tau_t$, $\Delta \log g_t$, $\log \tau_t$, and $\log g_t$ (i.e. $\mathbf{\Psi}_{[2,7,8],[7,8,9,10]}$), which we do not restrict to 0 even if they are statistically insignificant. We take the a priori stance that the government fiscal position predicts future GDP growth, tax revenue and government spending.

Second, the in-sample average of $\Delta \log \tau_t$ is $\mu_0^\tau = -14$ basis points and the in-sample average of $\Delta \log g_t$ is $\mu_0^g = 2.3$ basis points. Because we assume that the log tax-to-GDP ratio and the log spending-to-GDP ratio are stationary, the unconditional averages of their growth rates $\Delta \log \tau_t$ and $\Delta \log g$ should be 0. In order to obtain a reliable VAR estimate, we remove the in-sample averages of the growth rates. We construct the log tax-to-GDP and log spending-to-GDP ratios that enter in the VAR as follows:

$$\log \hat{\tau}_t = \log \hat{\tau}_0 + \sum_{k=1}^{t} (\Delta \log \tau_k - \mu_0^\tau),$$

$$\log \hat{g}_t = \log \hat{g}_0 + \sum_{k=1}^{t} (\Delta \log g_k - \mu_0^g),$$

where the initial level $\log \hat{g}_0$ is the same as the actual spending $\log g_0$ while $\log \hat{\tau}_0$ is chosen so that the resulting average log surplus-to-GDP ratio is the same as in the unadjusted data.

The estimates of $\mathbf{\Psi}$ are reported in Table 2. In this matrix, the first 4 rows govern the dynamics of bond market variables. It shows substantial diagonal elements (persistence) as well as several non-zero off-diagonal elements. For example, the lagged GDP growth and the lagged short rate predict the inflation rate, and the lagged inflation and the lagged GDP growth also predict the short rate. Lagged revenue-to-GDP growth has little effect on the dynamics of inflation, the short-term interest rate, or the slope of the term structure. The same is true for lagged government spending-to-GDP growth $\Delta \log g_{t-1}$.

The 5th and 6th rows govern the dynamics of stock market variables. The pd ratio is highly persistent, but does not load on other lagged variables. The dividend growth has a quarterly persistence of 0.453, exceeding that of GDP growth of 0.346. Dividend growth is predicted by the tax revenue-to-GDP growth and the spending-to-GDP growth.

The last four rows govern the dynamics of government cash flows. Consistent with the cointegration relationships, the tax revenue-to-GDP growth loads negatively on the lagged level $\log \tau_{t-1}$ ($\Psi_{[7,9]} = -0.142$), and the government spending-to-GDP growth $\Delta \log g_t$ loads negatively on the lagged level $\log g_{t-1}$ ($\Psi_{[8,10]} = -0.064$). These loadings imply long-run mean reversion of the tax-to-GDP ratio and the spending-to-GDP ratio. Moveover, a higher lagged GDP growth predicts a higher tax revenue-to-GDP growth ($\Psi_{[8,2]} > 0$) and a lower spending-to-GDP growth ($\Psi_{[9,2]} < 0$).

The estimates of $\Sigma^{\frac{1}{2}}$ are reported in Appendix C.1. The innovation in tax revenue-to-GDP growth is positively correlated with the GDP growth rate innovation, while the spending-to-GDP growth shock is negatively correlated with the GDP growth shock. In other words, tax revenues are pro-cyclical and government spending is counter-cyclical. The government spending-to-GDP growth shock is also negatively correlated with the dividend growth shock.

4.2.1 Cointegration and Long-run Predictability of Tax Revenue and Spending

Figure 5 plots the impulse responses of the tax revenue-to-GDP ratio ($\log \tau_t$) and the government spending-to-GDP ratio ($\log g_t$) to a x shock, a $\Delta \log \tau_t$ shock, and a $\Delta \log g$ shock. The $\Delta \log \tau_t$ shock is defined as the shock that increases $\Delta \log \tau_t$ by one standard deviation of its VAR residual. By definition, it also raises the level $\log \tau_t$ by the same amount, but it does not affect the GDP growth rate x_t. Conversely, the x_t shock is defined as the shock that increases x_t by one standard deviation of its VAR residual. As it does not affect the the level of government tax and spending, it lowers the tax-to-GDP ratio $\log \tau_t$ and the spending-to-GDP ratio $\log g_t$ by the same amount. The blue curves represent the results under the benchmark VAR system. For example, the $\Delta \log \tau_t$ shock raises the tax-to-GDP ratio $\log \tau_t$ on impact. Then, as the tax-to-GDP ratio is above the long-run average, its growth rate $\Delta \log \tau_t$ adjusts downward, leading to a reversion in the level.

For comparison, the dashed red lines represent the results under a restricted VAR, in which the first 8 state variables do not load on the cointegration variables $\log \tau_t$ and $\log g_t$. When cointegration is not imposed, the impact of the $\Delta \log \tau_t$ shock and the $\Delta \log g_t$ shock is permanent. For example, a positive $\Delta \log \tau_t$ shock raises the tax-to-GDP ratio $\log \tau_t$ permanently.

The impulse responses show that the VAR system with cointegration variables and the VAR system without cointegration variables imply very different dynamics in government cash flows. Which one is more consistent with the data? We regress the annual $\Delta \log \tau_{t+k}$ and $\Delta \log g_{t+k}$ in the following years $k = 1, \cdots, 5$ on the current-year $\log \tau_t$ and $\log g_t$. Table 3 reports the regression results. In the data, a higher level of $\log \tau_t$ predicts a lower tax revenue-to-GDP growth in the

Figure 5: The impulse responses of $\log \tau_t$ and $\log g_t$ to $\Delta \log \tau_t$ shock, $\Delta \log g_t$ shock, and x_t shock. In percentage units.
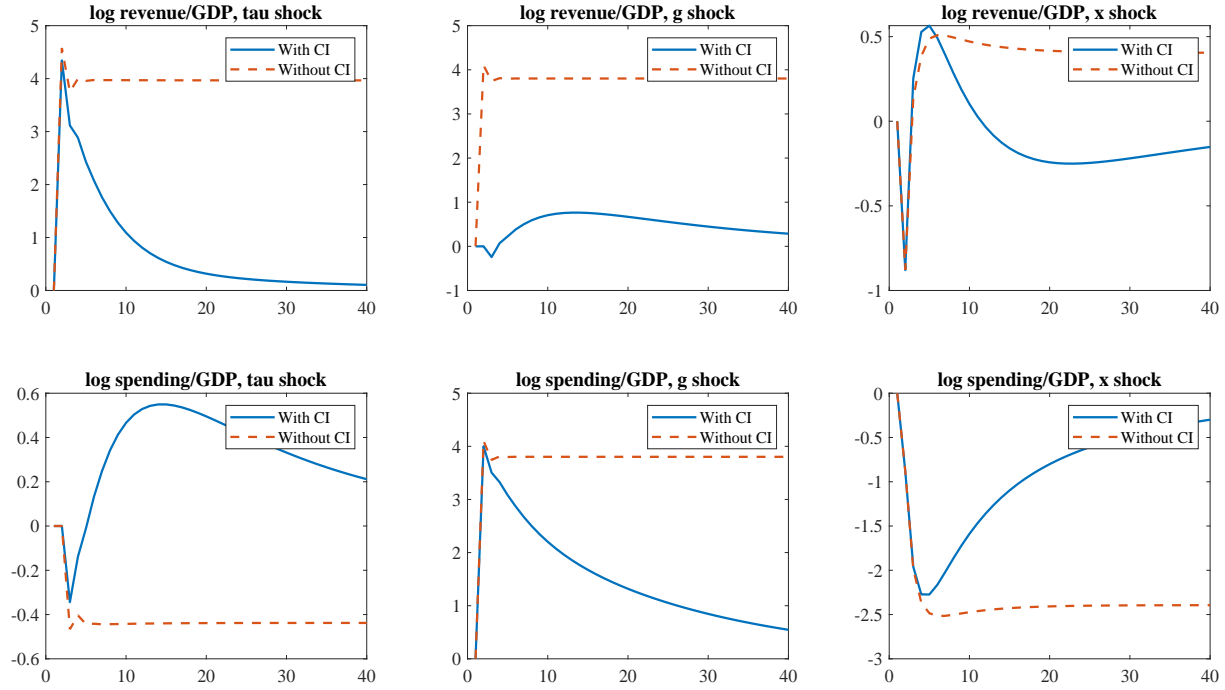

Table 3: The Predictability of Government Cash Flow Growth

This table reports how the levels of $\log \tau_t$ and $\log g_t$ predict the future tax revenue-to-GDP growth and the future government spending-to-GDP growth. The rows labeled by data report the coefficients from the regression of the annual $\Delta \log \tau_{t+k}$ and $\Delta \log g_{t+k}$ in the following year 1 through 5 on the current $\log \tau_t$ and $\log g_t$. Constants are omitted. Standard errors in parentheses are HAC-consistent. The rows labeled by model report the coefficients implied from the VAR system with cointegration variables.

			Dependent variable: $\Delta \log \tau_{t+k}$		
k	1	2	3	4	5
$\log \tau_t$ - data	-0.37 (0.07)	-0.38 (0.08)	-0.21 (0.06)	-0.07 (0.07)	0.06 (0.10)
$\log \tau_t$ - model	-0.46	-0.25	-0.12	-0.06	-0.03
$\log g_t$ - data	0.08 (0.07)	0.05 (0.05)	0.04 (0.05)	0.04 (0.06)	-0.02 (0.06)
$\log g_t$ - model	0.17	0.06	0.00	-0.02	-0.03
			Dependent variable: $\Delta \log g_{t+k}$		
k	1	2	3	4	5
$\log \tau_t$ - data	0.11 (0.10)	0.07 (0.08)	-0.04 (0.13)	0.01 (0.07)	0.03 (0.09)
$\log \tau_t$ - model	0.13	0.07	0.01	-0.02	-0.02
$\log g_t$ - data	-0.15 (0.07)	-0.15 (0.05)	-0.13 (0.05)	-0.10 (0.04)	-0.05 (0.04)
$\log g_t$ - model	-0.24	-0.17	-0.11	-0.08	-0.07

next 3 years, and a higher level of $\log g_t$ predicts a lower government spending-to-GDP growth in the next 4 years. For comparison, Table 3 also reports the model-implied counterparts for the VAR with cointegration. The regression coefficients are quantitatively similar to the conditional expectations implied from the VAR model.

We confirm these results with a local projection approach, detailed in Appendix I, which controls for other covariates when predicting future spending or revenue growth.

4.3 The Asset Pricing Model

Motivated by the no-arbitrage term structure literature (Ang and Piazzesi, 2003), we specify an exponentially affine stochastic discount factor (SDF). The nominal SDF $M_{t+1}^\$ = \exp(m_{t+1}^\$)$ is conditionally log-normal:

$$m_{t+1}^\$ = -y_t^\$(1) - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1}, \tag{4}$$

The real SDF is $M_{t+1} = \exp(m_{t+1}) = \exp(m_{t+1}^\$ + \pi_{t+1})$; it is also conditionally Gaussian. The innovations in the state vector ε_{t+1} from equation (3) are associated with a $N \times 1$ market price of risk vector Λ_t of the affine form:

$$\Lambda_t = \Lambda_0 + \Lambda_1 z_t,$$

The $N \times 1$ vector Λ_0 collects the average prices of risk while the $N \times N$ matrix Λ_1 governs the time variation in risk premia. We specify the restrictions on the market price of risk vector below. Asset pricing in this model amounts to estimating the market prices of risk in Λ_0 and Λ_1.

4.3.1 Bond Pricing

This model offers a simple way to price nominal bonds. Nominal bond yields of maturity h are affine in the state vector:

$$y_t^\$(h) = -\frac{A^\$(h)}{h} - \frac{B^\$(h)'}{h}z_t,$$

the scalar $A^\$(h)$ and the vector $B^\$(h)$ follow ordinary difference equations that depend on the properties of the state vector and of the market prices of risk.

Appendix B presents the proof and also shows a similar formula prices real bonds. We use this affine pricing equation to calculate the real interest rate, real bond risk premia, and inflation risk premia on bonds of various maturities.

Since both the nominal short rate ($y_t^\$(1)$) and the slope of the term structure ($y_t^\$(20) - y_t^\$(1)$) are included in the VAR, the SDF model must price the unconditional mean and the dynamics of the five-year bond yield:

$$-A^\$(20)/20 = y_0^\$(1) + yspr_0^\$ = y_0^\$(20) \tag{5}$$
$$-B^\$(20)/20 = e_{y1} + e_{yspr} \tag{6}$$

4.3.2 Equity Pricing

Let $PD_t^m(h)$ denote the price-dividend ratio of the dividend strip with maturity h (Wachter, 2005; van Binsbergen, Brandt, and Koijen, 2012). Then, the aggregate price-to-dividend ratio can be expressed as

$$PD_t^m = \sum_{h=0}^{\infty} PD_t^m(h). \tag{7}$$

Log price-dividend ratios on dividend strips are affine in the state vector:

$$pd_t^m(h) = \log\left(PD_t^m(h)\right) = A^m(h) + B^{m\prime}(h)z_t.$$

Since we include the log price-dividend ratio on the stock market in the state vector, it is affine in the state vector by assumption; see the left-hand side of (8):

$$\exp\left(\overline{pd} + e_{pd}'z_t\right) = \sum_{h=0}^{\infty} \exp\left(A^m(h) + B^{m\prime}(h)z_t\right), \tag{8}$$

Equation (8) rewrites the present-value relationship (7), and articulates that it implies a restriction on the coefficients $A^m(h)$ and $B^{m\prime}(h)$. We impose this restriction in the estimation.

4.4 Model Estimation

The state vector z_t is observed quarterly from 1947.Q1 until 2017.Q4 (284 observations). Under the VAR system, we estimate the constant market prices of risk Λ_0 and the time-varying market prices of risk Λ_1 that best fit the prices and expected returns on bonds of various maturities and on the aggregate stock market. Appendix C reports the point estimates as well as a detailed discussion of how the market price of risk parameters are identified. We assume that innovations to τ and g that are orthogonal to the innovations of the state variables that precede it are not priced.

4.4.1 Bonds

We use the following moments to estimate the 14 market price of risk parameters that govern the bond block. We include the distance between the observed and model-implied time-series of nominal bond yields for maturities of one quarter, one year, two years, five years, ten years, and thirty years.[7] We also impose the 11 conditions implied by equations (5) and (6). Since it is part of the VAR, we insist on matching the 5-year bond yield precisely. This gives a total of $6T+11$ moments. The unconditional market prices of inflation, GDP growth, the level of interest rates,

---

[7]We use constant-maturity Treasury (CMT) yield data from FRED. For the 1-year, 5-year, and 10-year bonds, we supplement the time series with data from the Federal Reserve Board's FRASER archive for the period 1947.Q1-1953.Q1. The 2-year CMT yields are only available in 1976.Q3 and the 30-year CMT yields are available only for 1977.Q2-2002.Q1 and 2006.Q1-2017.Q3. Since our estimation is quarter by quarter, it can handle missing data points.

and the slope of the yield curve all have the expected sign. The model matches the time series of bond yields in the data closely as shown in Appendix E.

4.4.2   Stocks

We allow for non-zero market prices of risk in the sixth element of Λ_0 and the first six entries of the sixth row of Λ_1; the sixth element is the aggregate dividend growth rate on the U.S. stock market. We use the following moments to identify these parameters. First, we include the distance between the observed and model-implied time-series of the price-dividend ratio on the aggregate stock market in each quarter. The model-implied series is constructed from the dividend strips per (8). Second, we impose that the risk premium in the model matches that in the VAR, both in terms of its unconditional average and its dependence on the state variables. This gives a total of T+11 moments. The model produces reasonable equity risk premia; see Appendix E.

4.4.3   Good deal bounds

Finally, when estimating the market prices of risk, we impose good deal bounds on the standard deviation of the log SDF in the spirit of Cochrane and Saa-Requejo (2000). Specifically, we impose a quadratic penalty for quarterly Sharpe ratios in excess of 1.5.

5   Valuing a Claim to Government Surpluses

5.1   Surplus Pricing Model

With the VAR dynamics and the SDF in hand, we can calculate the expected present discounted value of the primary surplus:

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$} S_{t+j} \right] = \sum_{j=0}^{\infty} \mathbb{E}_t \left[M_{t,t+j}^{\$} T_{t+j} \right] - \sum_{j=0}^{\infty} \mathbb{E}_t \left[M_{t,t+j}^{\$} G_{t+j} \right] = P_t^{\tau} - P_t^{g}, \qquad (9)$$

where P_t^{τ} is the cum-dividend value of a claim to future nominal tax revenues and P_t^{g} is the cum-dividend value of a claim to future nominal government spending.

The following proposition shows us how to price the government cash flows.

Proposition 4 (**Pricing Government Cash Flows**). (a) The price-dividend ratios on the tax claim and the spending claim are the sum of the price-dividend ratios of their strips, whose logs are

affine in the state vector z_t:

$$PD_t^\tau \quad = \quad \frac{P_t^\tau}{T_t} = \sum_{h=0}^{\infty} \exp(A_\tau(h) + B_\tau'(h)z_t), \tag{10}$$

$$PD_t^g \quad = \quad \frac{P_t^g}{G_t} = \sum_{h=0}^{\infty} \exp(A_g(h) + B_g'(h)z_t). \tag{11}$$

(b) After log-linearizing the returns to the tax and spending claims, their risk premia (expected excess returns corrected for a Jensen term) are:

$$\mathbb{E}_t \left[r_{t+1}^\tau \right] - y_t^\$(1) + Jensen \quad = \quad \left(e_{\Delta\tau} + e_x + e_\pi + \kappa_1^\tau \bar{B}_\tau \right)' \Sigma^{\frac{1}{2}} \left(\Lambda_0 + \Lambda_1 z_t \right),$$

$$\mathbb{E}_t \left[r_{t+1}^g \right] - y_t^\$(1) + Jensen \quad = \quad \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g \right)' \Sigma^{\frac{1}{2}} \left(\Lambda_0 + \Lambda_1 z_t \right).$$

The proof is in Appendix B.4. In Part (b), the vectors $\bar{B}_\tau$ and $\bar{B}_g$ describe the exposures of the "price-dividend ratios" of the revenue and spending claims to the state variables. The right-hand side denotes the covariance of the claims' returns with the SDF. These covariances are crucially driven by the exposure vectors $\bar{B}_g$ and $\bar{B}_\tau$.

Our pricing formula are in stark contrast with the case in which the investors treat the government surpluses as safe cash flows. In that case, the government surpluses are discounted at the nominal bond yield:

$$\sum_{j=0}^{\infty} \mathbb{E}_t \left[M_{t,t+j}^\$ \right] \mathbb{E}_t \left[S_{t+j} \right].$$

Since the average nominal GDP growth rate is higher than the average nominal interest rate, and the government surplus is cointegrated with the GDP and therefore growing at the same rate as GDP, this sum is unbounded. This is the case described by Blanchard (2019). The U.S. economy outgrows the risk-free rate and therefore the government has infinite debt capacity.
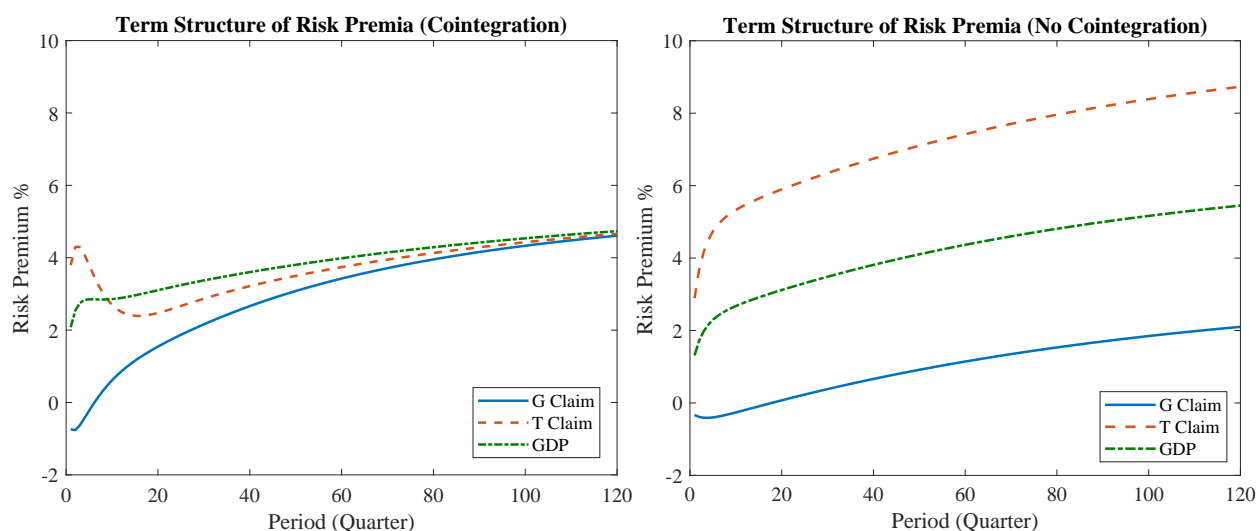
5.2 Results with No Cointegration

We first report the results under the VAR in which the first 8 state variables do not load on the cointegration variables $\log \tau_t$ and $\log g_t$. As shown in Figure 5, tax and spending shocks are permanent under these restricted VAR dynamics.

The time-series average of the price-dividend ratio on a claim to future tax revenue, PD_t^τ in (10), is 26.8 and the average risk premium is 10.1% per year. In other words, the representative agent would be willing to pay 26.8 times annual tax revenues for the right to receive all current and future tax revenues. This low valuation ratio reflects the high risk of tax revenues. Since tax revenues accrue in good times, i.e., low marginal utility times, the tax revenue asset is risky

and therefore has a low valuation ratio. The time-series average of the price-dividend ratio on a claim to future government spending, PD_t^g in (11), is of order 107. This very large valuation ratio translates into an average risk premium of 3.1% per year, much lower that that of the revenue claim. Figure A.4 in the appendix plots the time series for PD_t^τ and PD_t^g.

Intuitively, the lack of the cointegration dynamics implies that an increase in the current $\Delta \log g_t$ is not offset by future reductions in spending to GDP. The increase in the future government spending, which tends to happen during recessions, becomes permanent. This feature makes the spending claim much safer. For similar reasons, the lack of the cointegration makes the tax claim much riskier, because a decline in the tax revenue during recessions also becomes permanent. As a result, the long-run discount rates for the revenue claim are much higher than those for the spending claim. This is illustrated in the right panel of Figure 6 which presents the risk premia of government spending and tax strips over different horizons.

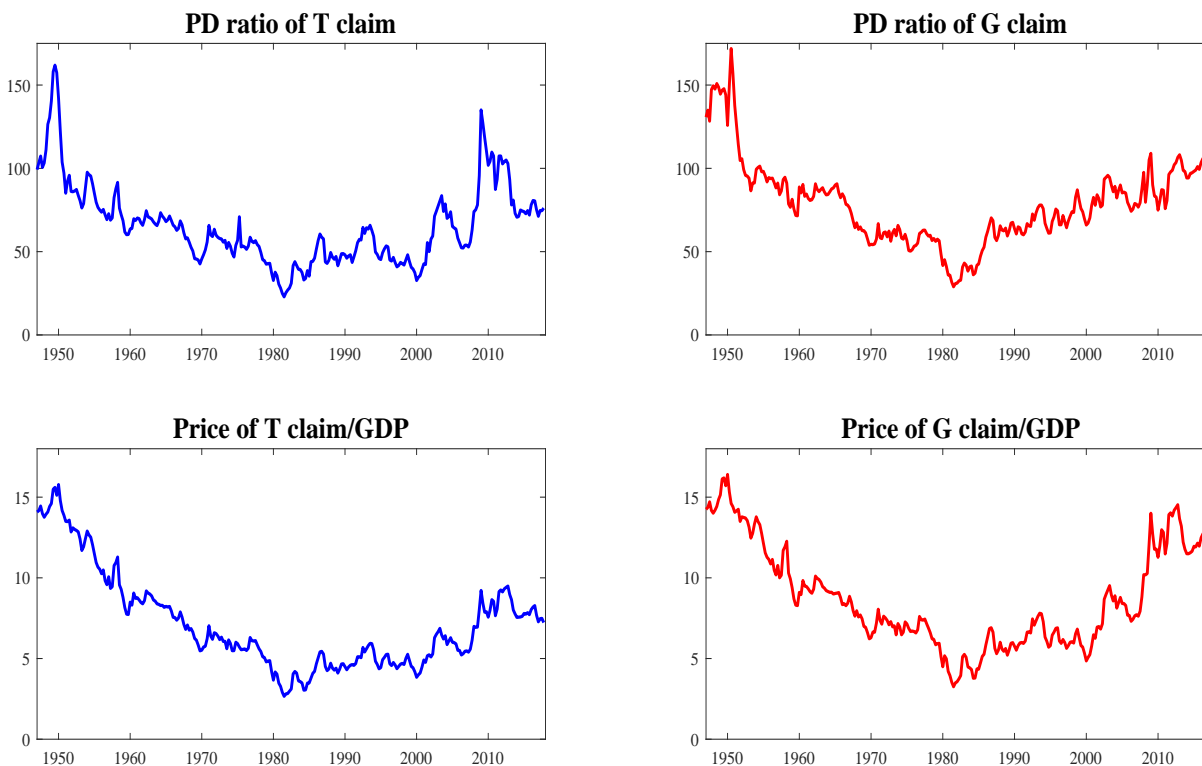Figure 6: Term Structure of Risk Premia on the T-Claim and the G-Claim

5.3 Main Results with Cointegration

Now we report the main pricing results under the benchmark VAR, in which the first 8 state variables can load on the cointegration variables $\log \tau_t$ and $\log g_t$. The top left panel of Figure 7 plots the price-dividend ratio on a claim to future tax revenue, PD_t^τ. The time-series average of this ratio is 65.58. In other words, the representative agent would be willing to pay 65.58 times annual tax revenues for the right to receive all current and future tax revenues. The annual risk premium on the tax claim is 4.71% per year. The risk premium reflects mostly compensation for GDP risk (2.31%), interest rate risk (12.82%), offset by slope risk (-9.73%) and stock market risk

(-0.72%). The high risk premium translates into a low valuation ratio.

Figure 7: Government Cash Flows and Prices

The top panels plot the (cum-dividend) price-dividend ratio on the claim to tax revenues (left) and government spending (middle). Both are annualized (divided by 4). The bottom left panel plots the value of a claim to future tax revenue, scaled by GDP. The middle panel plots the value of a claim to future government spending divided by GDP. The bottom right panel plots the value of future government surpluses scaled by GDP.

In addition, the price-dividend ratio of the tax claim displays substantial time-variation. A pronounced V-shape arises from the inverse V-shape of the long-term real interest rate, which is high in the mid-1970s to mid-1980s and low at the beginning and end of the sample. Intuitively, discounting future tax revenues by a low (high) long-term real rate results in a high (low) valuation ratio.

The time-series average of the price-dividend ratio on a claim to future government spending, PD_t^g is 78.87, and the average risk premium is 4.59% per year. These results are much less extreme than those from the VAR with no cointegration, because a higher government spending today lowers the growth rate of government spending in the future. The spending risk premium reflects mostly compensation for interest rate risk (13.61%) and GDP risk (2.18%), offset by stock market risk (-0.90%) and slope risk (-10.28%). The spending risk premium is lower than the revenue risk premium. Equivalently, the spending claim is more valuable than the revenue claim. The price-dividend ratio shows the same inverse V-shape dynamics of the price-dividend ratio on the revenue claim, as shown in the right panels of Figure 7.

27

Although the gap between the unconditional risk premium on the tax claim and on the spending claim looks small, the term structure of their risk premia behaves very differently, especially over shorter horizons, as shown in the top left panel of Figure 6. The average risk premium of tax claim over the five-year horizon is 3.00%, much larger than that of the spending claim, 0.52%. A claim to short- or medium-horizon government revenues is a safe asset. It pays out high "dividends" in bad economic times, i.e., high marginal utility states of the world. Therefore, agents are willing to pay a higher price/a lower risk premium for such an asset. The unconditional risk premia of two claims are dominated by the risk premia of long term strips, which converge to the long-run risk premium on a GDP strip by virtue of the cointegration restriction. A comparison of the left and right panels of Figure 6 shows that the assumption of cointegration is crucial for the convergence of the long-run risk premium of T- and G-claims.

5.4 The Puzzle

Now we are in a position to evaluate the claim to future government surpluses as the tax claim minus the spending claim, the right-hand side of equation (9). Figure 8 plots the present value of government surpluses scaled by GDP as the dashed line. The value of the surplus claim is not enough to honor the market value of the US government debt, plotted as the solid line. The unconditional average present value of the government surplus is −1.55 times GDP, far below the average market value of outstanding government debt, 0.37 times GDP. The gap is 192% of GDP on average. We refer to this finding as the government debt valuation puzzle.
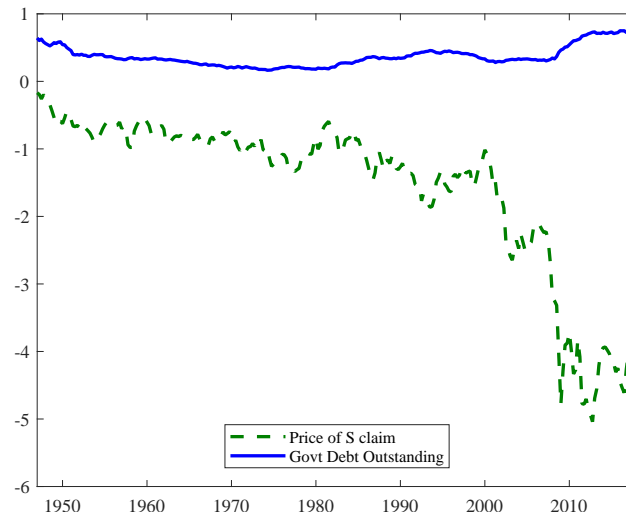
In the time series, the present value of the government surplus does not match the dynamics of government debt value, either. This puzzle deepens in the last 20 years of the sample, as the level of government debt doubles to about 75% of the GDP, while the valuation of the government surplus claim quadruples in absolute value to about −420% of the GDP. In other words, the U.S. government has been issuing government debt while simultaneously reducing the expected government surpluses to back it up.

This gap can be interpreted as a serious violation of the transversality condition in Treasury markets. However, the evidence does not obviously support rational bubbles theories in the spirit of Samuelson (1958); Diamond (1965); Blanchard and Watson (1982) because investors with a long horizon could short the portfolio Treasuries and go long in other long-lived assets that are not subject to a bubble in their valuations, such as equities. As shown in Section 3, they would earn the equity premium yet receive a stream of counter-cyclical cash flows. As Figure 8 shows, the size of the wedge seems to be growing faster than GDP. This is also inconsistent with rational bubbles. Finally, the transversality condition is not violated because the risk-adjusted discount rate on the portfolio of Treasury debt is actually higher than the growth rate of GDP.

This puzzle is deeper than the mere fact that the government does not generate enough surplus

Figure 8: Present Value of Government Surpluses and Market Value of Government Debt

Both time series are scaled by the US GDP.

to cover the debt payments. We can rewrite eq. (1) as

$$
\mathbb{E}\left[\sum_{h=0}^{H} P_t^{\$}(h) Q_{t-1,h+1}^{\$}\right] = \sum_{j=0}^{\infty} \mathbb{E}\left[M_{t,t+j}^{\$}\right] \mathbb{E}\left[T_{t+j} - G_{t+j}\right]
$$
$$
+ \; cov\left(M_{t,t+j}, T_{t+j}\right) - cov\left(M_{t,t+j}, G_{t+j}\right).
$$

On the right-hand side, as the average government surplus has been just about zero in our sample, $\mathbb{E}\left[M_{t,t+j}^{\$}\right] \mathbb{E}\left[T_{t+j} - G_{t+j}\right]$ is approximately 0. Therefore, the entire wedge of 200% of GDP stems from the differential riskiness of the revenue and the spending claims. Put differently, without the covariance terms, the government would need to generate about 75% of GDP in PDV of future surpluses to support 75% in debt relative to GDP. With the covariance terms present, 275% of GDP in future surpluses are needed to back the same debt.

In our estimation sample, the unconditional average 1-quarter log nominal interest rate is $y_0^{\$}(1)$=1.02% whereas the unconditional average 1-quarter log nominal GDP growth rate is $x_0 + \pi_0$= 1.56%. The risk-free interest rate is on average below the growth rate, as highlighted by Blanchard (2019). However, government tax and spending processes are sufficiently risky, so that their average nominal discount rates ($r_0^{\tau} = 1.91\%$ and $r_0^{g} = 1.78\%$) are above the average nominal GDP growth rate. We generate these discount rates while maintaining an excellent fit for the term structure of Treasury yields. The claim to government surpluses reflects the risk of the government's future debt issuance strategy. Future net debt issuances at inopportune (high SDF) times make the overall bond portfolio riskier than an individual Treasury bond. Therefore, even if risk-free interest rates are often below growth rates, the risk premia on government tax and spending processes

are large enough to make these claims to have finite valuation.

5.5 The Fiscal Measurability Constraint

The value equivalence in Proposition 1 implies a measurability constraint that the value of the surplus claim reacts to shocks in the same way as the government bond portfolio (Hansen, Roberds, and Sargent, 1991; Aiyagari, Marcet, Sargent, and Seppälä, 2002).

Proposition 5 (**Measurability Constraint**). The value of the surplus claim responds to all innovations in the same way as the bond portfolio. Exploiting the affine nature of the price-dividend ratio of tax revenue and spending strips and of zero-coupon bond prices, this produces the following system of N equations:

$$\tau_t \sum_{h=0}^{\infty} PD_t^\tau(h) \left(e'_{\Delta\tau} \Sigma^{\frac{1}{2}} + B'_\tau(h) \right) - g_t \sum_{h=0}^{\infty} PD_t^g(h) \left(e'_{\Delta g} \Sigma^{\frac{1}{2}} + B'_g(h) \right) = \sum_{h=0}^{\infty} \frac{Q_{t-1,h+1}^\$}{GDP_t} P_t^\$(h) \cdot B^\$(h)'. \tag{12}$$

Corollary 2. If the government only issues one-period risk-free debt, then the value of the previous period's bond portfolio at the start of the next period cannot depend on any shocks. The measurability conditions become:

$$\tau_t \sum_{h=0}^{\infty} PD_t^\tau(h) \left(e'_{\Delta\tau} \Sigma^{\frac{1}{2}} + B'_\tau(h) \right) - g_t \sum_{h=0}^{\infty} PD_t^g(h) \left(e'_{\Delta g} \Sigma^{\frac{1}{2}} + B'_g(h) \right) = \mathbf{0} \tag{13}$$

Only if condition (13) is satisfied can we discount future surpluses at the one-period risk-free bond rate, as Blanchard (2019) suggests one should do. Hansen, Roberds, and Sargent (1991) deliver a univariate version of this measurability condition.

Condition (13) is severely violated in the data. Figure A.5 in Appendix F plots the left hand side of this equation for our benchmark model estimates as well as the zero line. Deviations from zero are of the same size as GDP.

First, given that the government surpluses are clearly trending with GDP (Figure 1 shows surplus to GDP is stationary), every innovation to GDP permanently alters the cash flows that accrue to investors in the surplus claim. But with one-period risk-free debt, the value of government debt cannot move with that same GDP growth shock. The long-run GDP risk in the surplus cash flows simply cannot be replicated with a position in risk-free debt.

Second, a positive (negative) innovation to spending (revenues) would need to be offset by future decreases in spending in present value. We do not detect any evidence to support this hypothesis in our VAR. Cointegration imposes mean-reversion but not overshooting. The impulse responses from the VAR, shown in Figure 5, do not feature these negative responses in the future. Neither do local projections of future (τ_{t+h}, g_{t+h}) on current (τ_t, g_t) while controlling for

other covariates. Section I in the appendix reports the impulse responses computed using local projections (Jordà, 2005). Policy makers do not seem to enforce the measurability constraints. The federal budget contains 'automatic stabilizers' built into non-discretionary in spending (e.g., the increase in unemployment benefits in recessions) and revenue (e.g., capital gains taxes), but these do not come with automatic offsets for future spending/revenue.

If the yield curve spans all the innovations, there exists a dynamic, highly levered long-short portfolio in government debt $Q_{t-1,h+1}^{\$}$ of various maturities that replicates the state-contingency of the surplus claim and satisfies Proposition 5.[8] However, this dynamic portfolio looks very different from the government's actual bond portfolio. The violations of the general measurability condition (12) evaluated at the actual Treasury portfolio are as large as those in shown Figure A.5. This is not surprising. We need to construct a Treasury portfolio with long-run risk exposure to GDP equivalent to that of a claim to GDP.

6 Convenience Yield

The convenience yield λ_t is the government bonds' expected returns that investors are willing to forgo under the risk-neutral measure. Assuming U.S. Treasury bonds carry a uniform convenience yield across the maturity spectrum, the Euler equation for a Treasury bond with maturity $h+1$ is:

$$e^{-\lambda_t} = \mathbb{E}_t \left[M_{t+1} \frac{P_{t+1}^{\$}(h)}{P_t^{\$}(h+1)} \right].$$

Put differently, the convenience yield produces an additional source of revenue, because theU.S. Treasury can sell its bonds for more than their fundamental value. The question is how far this explanations can go towards accounting for the U.S. bond valuation puzzle.

Proposition 6. If the transversality condition holds, the value of the government debt portfolio equals the value of future surpluses plus the value of future seigniorage revenue:

$$E_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$} \left(T_{t+j} - G_{t+j} + (1 - e^{-\lambda_{t+j}}) \sum_{h=1}^{H} Q_{t+j,h}^{\$} P_{t+j}^{\$}(h) \right) \right] = \sum_{h=0}^{H} Q_{t-1,h+1}^{\$} P_t^{\$}(h), \qquad (14)$$
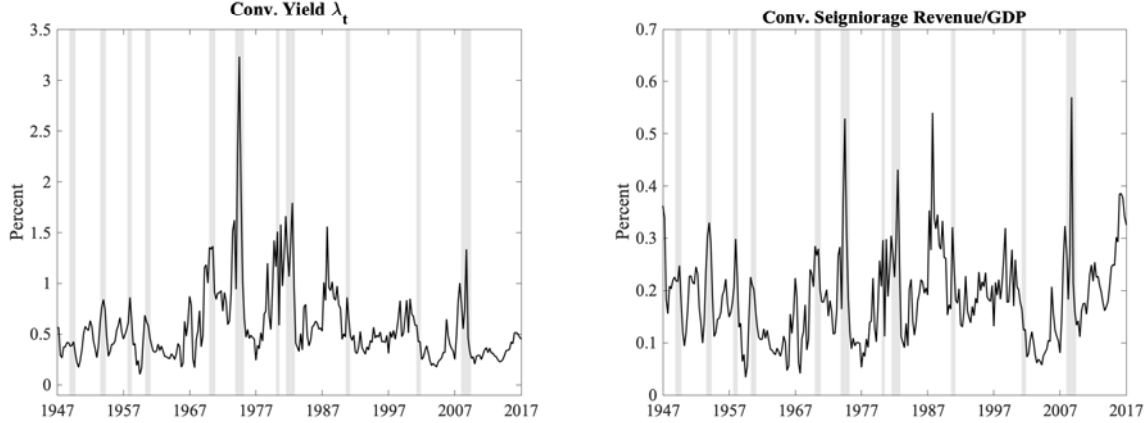
where $\sum_{h=0}^{H} Q_{t-1,h+1}^{\$} P_t^{\$}(h)$ on the right-hand side denotes the cum-dividend value of the government's debt portfolio at the start of period t, and $\sum_{h=1}^{H} Q_{t+j,h}^{\$} P_{t+j}^{\$}(h)$ on the left-hand side denotes the ex-dividend value of the government's debt portfolio at the end of period $t+j$.

When there is no convenience yield, we end up back in the standard case, Proposition 1. When the convenience yield is positive and the quantity of outstanding government debt is positive in

---

[8]Similar spanning arguments were explored by Angeletos (2002) and Buera and Nicolini (2004).

Figure 9: Convenience Yield and Seigniorage Revenue

The left panel reports the time series of Aaa-Treasury yield spreads, high-grade commercial papers-bills yield spreads, and the proxy for λ_t. We measure λ_t using the weighted average of the Aaa-Treasury yield spread and the high-grade commercial papers-bills yield spread. All yields are expressed in percentage per annum. The right panel reports time series of the seigniorage revenue-to-GDP ratio $(1 - \exp(-\lambda_t)D_t/GDP$ and the tax-to-GDP ratio at different scales. The sample period is from 1947-01 to 2017-04.

the future, the convenience yields will always *increase* the value of government debt, acting as an additional source of revenue. We call this additional income seigniorage revenue. It could potentially turn government deficits into surpluses.

As an empirical strategy, we measure the convenience yield following Krishnamurthy and Vissing-Jorgensen (2012). We use the weighted average of the Aaa-Treasury yield spread and the high-grade commercial papers-bills yield spread to proxy λ_t, where the time series of weights are computed to match the duration of the government bond portfolio period by period. The left panel of Figure 9 shows the time series of the convenience yield. Over the sample period from 1947 to 2017, the average convenience yield is 0.57% per year, which implies average seigniorage revenue of $12.28 billions per year, or 0.19% of U.S. GDP as shown in the right panel of Figure 9 .

Then, we rewrite equation (14) as:

$$E_t \left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} T_{t+j} K_{t+j} \right] - E_t \left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} G_{t+j} \right] = \sum_{h=0}^{K} Q^{\$}_{t-1,h+1} P^{\$}_t(h),$$
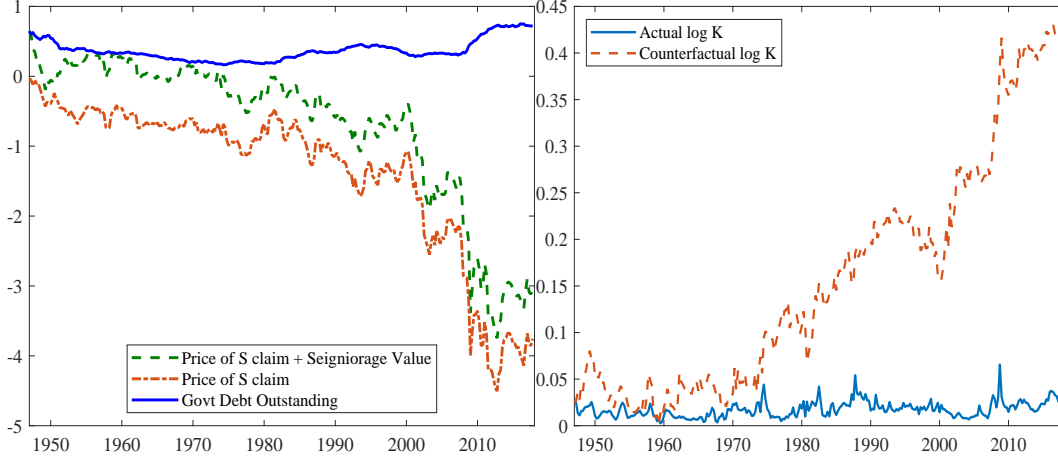
and model:

$$K_{t+j} = 1 + \frac{(1 - e^{-\lambda_{t+j}}) \sum_{h=1}^{H} Q^{\$}_{t+j,h} P^{\$}_{t+j}(h)}{T_{t+j}}$$

as a reduced-form variable. We preserve the structure of the state vectors and the SDF, and introduce the log growth rate $\Delta \log K_t$ as an additional state variable. Specifically, the new state vector

32

Figure 10: Present Value of Government Surpluses and Seigniorage Revenues

The left panel plots the present value of government surpluses with and without seigniorage revenues, scaled by the US GDP. The right panel plots the actual and the counterfactual seigniorage revenue process $\log K_t$.

is $\tilde{z}_t = [z_t, \Delta \log K_t]$. The seigniorage term $\log K_t$ follows the process: $\Delta \log K_{t+1} = e'_k \tilde{z}_{t+1}$, with a mean of zero because $\log K_t$ is stationary.

We use the same method as in Proposition 4 to price the modified tax claim. The new pricing formula for the revenue claim is:

$$E_t \left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} T_{t+j} K_{t+j} \right] = T_t K_t \cdot PD^k_t,$$

where PD^k_t is a function of the state variables $\tilde{z}_t$. The left panel of Figure 10 reports the present value of the government surpluses under the modified model. The convenience yield always increases the present value of the government surpluses. On average, the seigniorage revenue has a present value of 93% of annual U.S. GDP. While substantial, this seigniorage revenue only bridges less than half of the gap between the present value of surpluses and the value of the government debt portfolio. The government debt valuation puzzle remains standing.

How large should the seigniorage revenue be to match the present value of the government surplus claim to the actual debt value? To answer this question, we fix the coefficients of the VAR system for pricing purposes, but change the seigniorage revenue term $\log K_t$ to $\log \tilde{K}_t$ at each point of time t so that

$$T_t \tilde{K}_t \widetilde{PD}^k_t - G_t P^g_t = \sum_{h=0}^{H} Q^{\$}_{t-1,h+1} P^{\$}_t(h)$$

Since the last element of $\tilde{z}_t$ is $\Delta \log \tilde{K}_t$, $\log \tilde{K}_t$ enters this equation through both $\tilde{K}_t$ and $\widetilde{PD}^k_t$, the latter of which is a function of $\tilde{z}_t$. We solve for variable $\log \tilde{K}_t$ in this equation, taking other variables as given. The right panel of Figure 10 reports the resulting $\tilde{K}_t$ process. Seigniorage

revenue would need to be 15.6% of tax revenue on average to match the present value of the government surplus claim to the actual debt value, and more than 40% at the end of the sample. Actual seigniorage revenue only averages 1.74% of tax revenue. In sum, the convenience yield would have to be much larger to bridge the gap.

Recall that the risk premium on the tax (spending) claim is 4.71% per year (4.59% per year) while the return on the Treasury portfolio is 0.93% per year. A risk premium gap of more than 350 bps is too large to close with traditional estimates of convenience yields.

7 Other Potential Resolutions of the Puzzle

7.1 Other Government Assets and Liabilities

The government owns various assets, including outstanding student loans and other credit trans-actions, cash balances, and various financial instruments. Based on Congressional Budget Office data, the total value of these government assets is 8.8% of the GDP as of 2018. While these assets bring the net government debt held by the public from 77.8% to 69.1% of the GDP, the bulk of the government debt valuation puzzle remains.

Other significant sources of government revenues and outlays are those associated with the Social Security Administration (SSA). Based on the CBO data, the net flows from SSA are close to 0 as of 2018, but will turn into a deficit of 0.7% of GDP per annum from 2020 to 2029. As the SSA turns from a net contributor of primary surpluses into a net contributor to the deficit in 2019 and beyond, the government will need to issue additional debt to the public. Absent new spending cuts or tax increases, this will deepen the puzzle.

7.2 Market Segmentation

Can market segmentation resolve the government debt risk premium puzzle? One could argue that marginal investors in Treasury bonds do not necessarily overlap with investors in the U.S. equity market. We conduct the following analysis assuming that investors in the U.S. bond mar-kets are not exposed to stock market risks. We write down the vector of state variables $\hat{z}_t$ without including the log price-dividend ratio and the log real dividend growth on the aggregate stock market:

$$\hat{z}_t = [\pi_t - \pi_0, x_t - x_0, y_t^{\$}(1) - y_0^{\$}(1), yspr_t^{\$} - yspr_0^{\$}(1),$$
$$\Delta \log \tau_t - \mu_0^{\tau}, \Delta \log g_t - \mu_0^{g}, \log \tau_t - \log \tau_0, \log g_t - \log g_0]'.$$

We estimate the VAR system $\hat{z}_t = \hat{\Psi} \hat{z}_{t-1} + \hat{\Sigma}^{\frac{1}{2}} \hat{\epsilon}_t$. Then we estimate the constant market prices of risk $\hat{\Lambda}_0$ and the time-varying market prices of risk $\hat{\Lambda}_1$ to fit the prices and expected returns on

only bonds with different maturities. Both the average and time-varying prices of risk for inflation, real GDP growth, interest rate, and the slope of the term structure are similar to the estimates from our benchmark specification in Section 4.4. It is worth noting that zeroing out the stock market risk factors presents an extreme case of segmentation since government bond investors are almost certainly exposed to some U.S. stock market risks. Our estimates of $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$ remain similar to Λ_0 and Λ_1 in our benchmark estimation (for the non-zero components).

Using the estimated VAR system and market prices of risk, we obtain an average price-dividend ratio for the tax revenue claim of 82.4 and for the spending claim of 94.4. Appendix Figure A.6 shows that the present value of the government surpluses using the SDF that prices Treasury bonds only behaves similarly to the present value of the government surpluses under our benchmark specification. This type of market segmentation does not resolve the puzzle.

Longstaff (2011) finds evidence from municipal bond markets that marginal tax rate increases are associated with a negative risk price, presumably because marginal tax rates are pro-cyclical. If the marginal investor in U.S. Treasuries faces strongly pro-cyclical marginal tax rates, then the after-tax cash flows on the entire Treasury portfolio would become less pro-cyclical. This would reduce the riskiness of the Treasury portfolio. Given the large size of foreign, Fed, and tax-exempt domestic institutional holdings of U.S. Treasuries, it is unclear how much bite this argument has.

7.3 Peso Problem

Lastly, we consider a model in which bond investors price in the possibility of a major government spending cut, but that such a spending cut never realizes in our 70-year sample. How large should the spending cut probability be in order to match the market valuation of the government debt to the present value of government surpluses?

We set the spending cut to be 2 times the standard deviation of the log spending-to-GDP shock. When it happens, the spending-to-GDP ratio decreases by $2 \times 3.85\% = 7.7\%$ of U.S. GDP.

Let $\bar{\phi}$ be the unconditional average of the spending cut probability. We augment the vector of demeaned state variables z_t with the demeaned probability of a spending cut, ϕ_t: $w_t = [z_t; \phi_t]$, and we augment the vector of VAR shocks ε_t with an additional shock to ϕ_t: $u_t = [\varepsilon_t; \varepsilon_t^\phi]$. The new state vector w_t follows:

$$w_t = \tilde{\Psi} w_{t-1} + \tilde{\Sigma}^{\frac{1}{2}} u_t.$$

The time-varying probability ϕ_t can load on w_{t-1} and u_t, with loadings that are to be estimated. We extend Proposition 4, with proof in Appendix B.5, and show that the price-dividend ratios of the tax claim and the spending claim can be expressed in similar form as in the benchmark analysis.

Proposition 7. In the presence of the spending cut, the price-dividend ratios on the tax claim and the spending claim are the sum of the price-dividend ratios of their strips. The log price-dividend

ratios on these strips are affine in the state vector z_t:

$$PD_t^\tau = \sum_{h=0}^{\infty} \exp(\tilde{A}_\tau(h) + \tilde{B}_\tau'(h)w_t),$$

$$PD_t^g = \sum_{h=0}^{\infty} \exp(\tilde{A}_g(h) + \tilde{B}_g'(h)w_t).$$

We estimate the process of the spending cut probability such that the present value of government surpluses is exactly equal to the market valuation of the government debt in every period. The peso event itself is not priced; we do not change the market prices of risk Λ_t. That is, we assume the equity and bond investors still observe the same state variable processes, including the processes of tax and spending. The state variable dynamics are unaffected by the time-varying probability of the spending cut because it never realizes in our sample.

We estimate the unconditional average probability $\bar{\phi}$, the probability's loading on the lagged probability $e_\phi' \Psi e_\phi$, the probability's loading on the contemporaneous GDP shock $e_x u_t$, and the probability's loading on the contemporaneous ϕ shock $e_\phi u_t$. For the benchmark estimation, we set the other loadings of the probability to zero.

Our estimation algorithm proceeds as follows. We start with an initial guess for the aforementioned parameters. Under this set of parameter values, we solve for the ϕ_t process that matches the market value of government debt with the present value of government surpluses. The market value of government debt is observed in the data, and the present value of government surpluses can be calculated following Proposition 7. Then, we calculate the empirical moments of the implied ϕ_t process. The moments are the average value of the probability and the loadings of the residual $\phi_t - e_\phi \tilde{\Psi} w_{t-1}$ on the 9 independent shocks u_t. We search for the parameter values that minimize the L_2 distance between the moments implied from the initial guess and these empirical moments. The resulting parameter estimates are reported in Table 4.

The estimated $\bar{\phi} + \phi_t$ process is shown in Figure 11. The gap between the market value of debt and the present value of surpluses under the benchmark model is nearly two hundred percent of GDP. To match such a large gap, the probability of the potential spending cut has to be large and have large fluctuations. The spending cut probability is around 42% on average and fluctuates strongly between 30% and 90%. It rises sharply after the year 2000. Such a large probability is at

Table 4: Parameter Estimates and Corresponding Moments

The table reports the estimated parameters in the extended model with spending cuts.
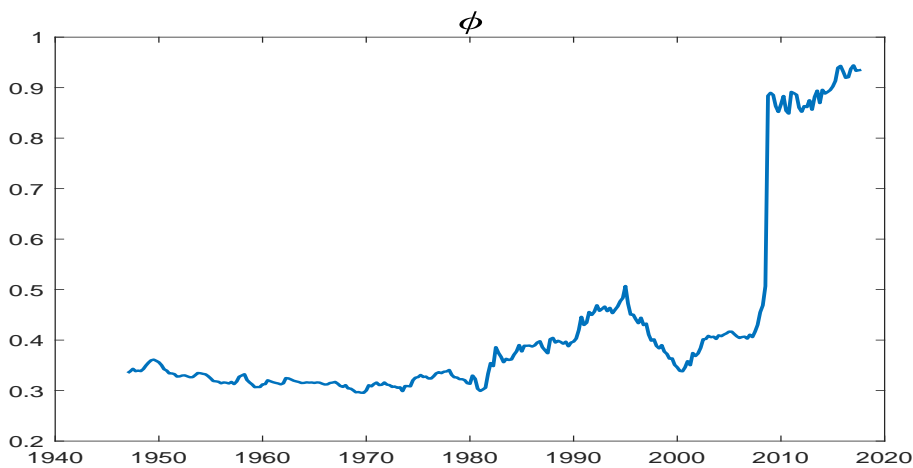
Parameter Estimates			
$\bar{\phi}$	$\tilde{\Psi}_\phi$	$\tilde{\Sigma}^{1/2}(11,2)$	$\tilde{\Sigma}^{1/2}(11,11)$
0.428	0.999	-0.011	0.023

odds with the notion of a peso event that never happens in a 70-year sample. We interpret this result as a restatement of the puzzle.

If the fiscal correction took place in high marginal utility states, as in a rare disaster model, the actual probability of these fiscal corrections could be smaller. But that strikes us as implausible. Governments do not suddenly switch to running large primary surpluses in bad states of the world.

Figure 11: Probabilities of Spending Cut Implied by Debt-to-GDP Ratio

This figure reports the time series of probabilities of spending cuts implied by the debt to GDP ratio, $\overline{\phi} + \phi_t$.
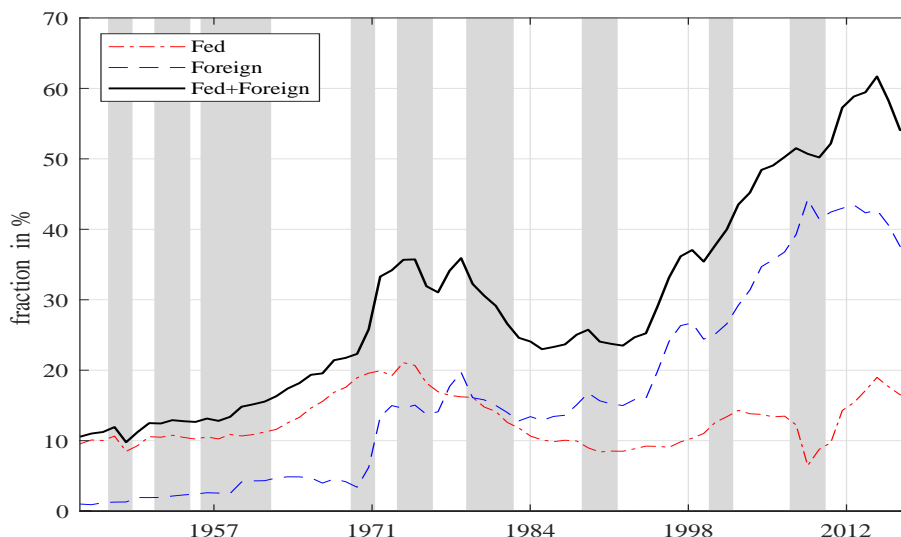

7.4 Inelastic Demand by Foreign Investors and the Fed

Finally, it is important to note that foreign ownership of Treasuries has increased dramatically since the mid 1990s (see Favilukis, Kohn, Ludvigson, and Nieuwerburgh, 2013). Figure 12 plots the holdings of foreign investors and the Fed as a fraction of the total debt. At the end of our sample period, about 40% of U.S. Treasury securities are owned by foreign investors, similar to the holdings of domestic investors excluding the Federal Reserve system (mutual funds, pension funds, banks, and insurance companies). One natural question is whether foreign investors use a different SDF to price Treasury bonds. Whatever SDFs foreign investors use, the projections of their SDFs on the state space z must agree with those of the domestic investors to price bonds. Our benchmark exercise already identifies the SDFs that both foreign and domestic investors consistently use to price government bonds. The exercise in Section 7.2 already showed that estimating the SDF just on bonds did not result in a smaller puzzle.

That said, a growing literature in international economics seeks to understand the role of the U.S. as the world's safe asset supplier and its impact on the global economy (see Gourinchas and Rey, 2007; Caballero, Farhi, and Gourinchas, 2008; Caballero and Krishnamurthy, 2009; Maggiori, 2017; He, Krishnamurthy, and Milbradt, 2018; Gopinath and Stein, 2018; Krishnamurthy and

Lustig, 2019). Jiang, Krishnamurthy, and Lustig (2018a,b) estimate the convenience yields that foreign investors derive from their holdings of U.S. Treasuries. The estimates are larger than the conventional estimates of convenience yields used in Section 6, and could further help resolve the puzzle.

Figure 12: Holdings of Treasurys by Foreign Investors and the Fed

In addition to the rise in foreign holdings, the Fed has substantially increased its holdings of Treasuries in the aftermath of the financial crisis. Table 5 reports the dollar-weighted returns earned by foreign investors and the Fed. The dollar-weighted returns are 181 bps per annum lower than the time-weighted returns (geometric mean return). Foreign investors and the Fed display poor timing skill when investing in U.S. Treasuries. Put differently, they have inelastic demand (see Krishnamurthy and Lustig, 2019, for details).

If we take the view that foreign and Fed demand are completely inelastic, it is natural to adjust the net payouts to bond holders by excluding payouts to the Fed and foreign investors.[9] Figure 13 plots the net payouts to bondholders excluding the Fed and foreign investors as a fraction of the face value of the Treasuries outstanding. Especially in the last 2 recessions, the cash flows paid out to bondholders seem just about as pro-cyclical when we exclude the Fed and foreign investors. Hence, inelastic demand by the Fed and foreign investors does not mitigate the pro-cyclicality of

---

[9]To compute the payouts to bondholders excluding the Fed and foreigners, we start with the Federal government; interest paid (IMA) (FA316130001.A) from Table F106 in the Flow of Funds. The interest paid is scaled down by the fraction of debt held by the Fed (LM713061103.A) and Foreigners (LM713061103.A) from Table L210. To compute net issuance, we take the Federal government; net lending (+) or borrowing (-) (financial account) (FA315000005.A) from Table F106. Then we subtract purchases by the Fed (Monetary authority; Treasury securities; asset; Monetary authority; Treasury securities; asset) and purchases by foreigners (the Rest of the world; Treasury securities; FA263061105.A) from Table F210. Finally, we add the new interest paid series to the new payout series. We divide these payouts by the face value of outstanding bonds excluding Foreign and Fed holdings.

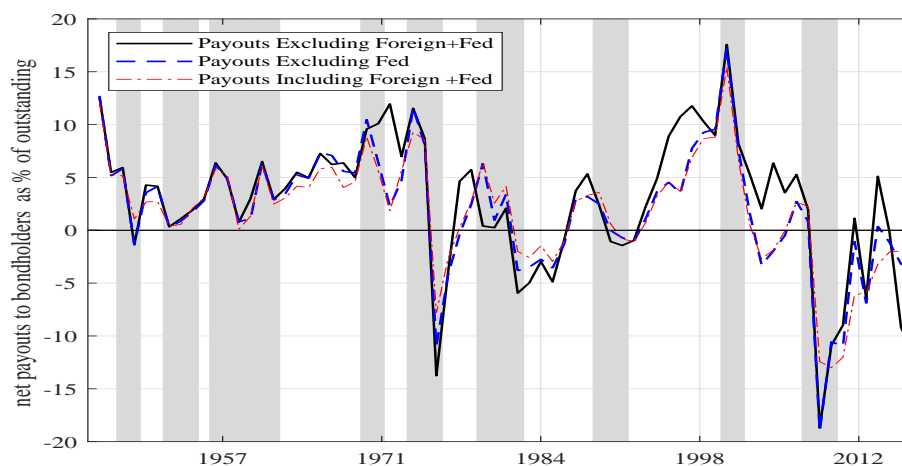Table 5: Inelastic Demand: Returns on U.S. Treasury Purchases

	dollar-weighted	time-weighted	gap
Nominal			
Fed	2.58%	4.87%	2.29%
Foreign	3.24%	4.87%	1.63%
Fed+Foreign	3.06%	4.87%	1.81%
Real			
Fed	0.84%	2.70%	1.87%
Foreign	1.41%	2.70%	1.29%
Fed+Foreign	1.26%	2.70%	1.45%

Source: Federal Flow of Funds data. Cash flows invested in Bloomberg Treasury Index. Sample: 2000-2019. The dollar-weighted return is the IRR on all the cash flows invested by foreign investors (the Fed) in Treasurys. Cash flows invested in Bloomberg Treasury Index. Flow of Funds Table F106: Monetary authority; other Treasury securities, excluding Treasury bills; asset, and Rest of the world; other Treasury securities, excluding Treasury bills and certificates; asset.

the cash flows absorbed by U.S. investors.

Figure 13: Net Payouts to Bondholders Excluding the Fed and Foreign Investors

The figure plots the annual net payouts to bondholders as a fraction of the lagged face value. The red line includes the payouts to the Fed and to foreign investors and includes Fed and foreign holdings from the denominator. The blue excludes the payouts to the Fed and Fed holdings in the denominator. The black line excludes payouts to the Fed and foreign investors. Annual data from the Flow of Funds.

8   Conclusion

Because government deficits tend to occur in recessions, times when bond investors face high marginal utility, governments must tap debt markets at inopportune times. This consideration reduces the government's debt capacity by about 200% of GDP. If tax and spending policies remain on their current course, government debt capacity is negative. Put differently, government debt is a risky claim whose expected return far exceeds risk-free bond yields. We call this violation of the government budget constraint the government debt valuation puzzle. We explore potential resolutions to this puzzle, and we conclude that either convenience yields are much larger than previously thought and/or there is a bubble in U.S. Treasury markets, meaning that Treasury investors have failed to enforce the transversality condition. More work is needed to compare the U.S. to other countries using our approach.

References

Aiyagari, S. R., A. Marcet, T. J. Sargent, and J. Seppälä, 2002, "Optimal Taxation without State-Contingent Debt," *Journal of Political Economy*, 110(6), 1220–1254.

Alvarez, F., and U. Jermann, 2005, "Using Asset Prices to Measure the Measure the Persistence of the Marginal Utility of Wealth.," *Econometrica*, (4), 1977–2016.

Ang, A., and M. Piazzesi, 2003, "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables," *Journal of Monetary Economics*, 50, 745–787.

Angeletos, G.-M., 2002, "Fiscal Policy with Noncontingent Debt and the Optimal Maturity Structure*," *The Quarterly Journal of Economics*, 117(3), 1105–1131.

Backus, D., N. Boyarchenko, and M. Chernov, 2018, "Term structures of asset prices and returns," *J. financ. econ.*, 129(1), 1–23.

Bai, J., and P. Collin-Dufresne, 2019, "The CDS-bond basis," *Financial Management*, 48(2), 417–439.

Berndt, A., H. Lustig, and Ş. Yeltekin, 2012, "How Does the US Government Finance Fiscal Shocks?," *American Economic Journal: Macroeconomics*, 4(1), 69–104.

Bhandari, A., D. Evans, M. Golosov, and T. Sargent, 2017, "The Optimal Maturity of Government Debt," working paper, University of Minnesota working paper.

Bianchi, F., and L. Melosi, 2014, "Dormant shocks and fiscal virtue," *NBER Macroeconomics Annual*, 28(1), 1–46.

——— , 2017, "Escaping the great recession," *American Economic Review*, 107(4), 1030–58.

——— , 2018, "The dire effects of the lack of monetary and fiscal coordination," *Journal of Monetary Economics*.

Blanchard, O., 2019, "Public Debt and Low Interest Rates," *American Economic Review*, 109(4), 1197–1229.

Blanchard, O. J., and M. Watson, 1982, "Bubbles, rational expectations, and financial markets," in *Crises in the Economic and Financial Structure*, ed. by P. Wachtel. Lexington Books.

Bohn, H., 1998, "The behavior of US public debt and deficits," *the Quarterly Journal of economics*, 113(3), 949–963.

——— , 2007, "Are stationarity and cointegration restrictions really necessary for the intertemporal budget constraint?," *Journal of Monetary Economics*, 54(7), 1837 – 1847.

Bolton, P., 2016, "Presidential Address: Debt and Money: Financial Constraints and Sovereign Finance," *J. Finance*, 71(4), 1483–1510.

Bolton, P., and H. Huang, 2018, "The Capital Structure of Nations," *Rev Financ*, 22(1), 45–82.

Borovička, J., L. P. Hansen, and J. A. Scheinkman, 2016, "Misspecified recovery," *J. Finance*, 71(6), 2493–2544.

Brock, W. A., 1982, "Asset prices in a production economy," in *The economics of information and uncertainty*. University of Chicago Press, pp. 1–46.

Buera, F., and J. P. Nicolini, 2004, "Optimal maturity of government debt without state contingent bonds," *Journal of Monetary Economics*, 51(3), 531 – 554.

Caballero, R. J., E. Farhi, and P.-O. Gourinchas, 2008, "An Equilibrium Model of "Global Imbalances" and Low Interest Rates," *American Economic Review*, 98(1), 358–93.

Caballero, R. J., and A. Krishnamurthy, 2009, "Global Imbalances and Financial Fragility," *American Economic Review*, 99(2), 584–88.

Campbell, J. Y., 1991, "A Variance Decomposition for Stock Returns," *Economic Journal*, 101, 157–179.

——— , 1993, "Intertemporal Asset Pricing Without Consumption Data," *American Economic Review*, 83(3), 487–511.

——— , 1996, "Understanding Risk and Return," *The Journal of Political Economy*, 104(2), 298–345.

Campbell, J. Y., and R. J. Shiller, 1988, "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors," *Review of Financial Studies*, 1, 195–227.

Chernov, M., L. Schmid, and A. Schneider, 2016, "A Macrofinance View of U.S. Sovereign CDS Premiums," .

Cochrane, J. H., 2001, "Long-Term Debt and Optimal Policy in the Fiscal Theory of the Price Level," *Econometrica*, 69(1), 69–116.

——— , 2005, "Money as Stock," *Journal of Monetary Economics*, 52(3), 501–528.

——— , 2019a, "The Fiscal Roots of Inflation," working paper, National Bureau of Economic Research.

——— , 2019b, "The Value of Government Debt," working paper, National Bureau of Economic Research.

Cochrane, J. H., and J. Saa-Requejo, 2000, "Beyond Arbitrage: 'Good deal' asset price bounds in incomplete markets," *Journal of Political Economy*, 108, 79–119.

Dai, Q., and K. J. Singleton, 2000, "Specification Analysis of Affine Term Structure Models," *Journal of Finance*, 55, 1943–1978.

Diamond, P. A., 1965, "National Debt in a Neoclassical Growth Model," *Am. Econ. Rev.*, 55(5), 1126–1150.

Duffie, D., and R. Kan, 1996, "A Yield Factor Model of Interest Rates," *Mathematical Finance*, 6, 379–406.

Favilukis, J., D. Kohn, S. Ludvigson, and S. V. Nieuwerburgh, 2013, *International Capital Flows and House Prices: Theory and Evidence*NBER Cambridge, MA, chap. 8.

Fleckenstein, M., F. A. Longstaff, and H. Lustig, 2014, "The TIPS-treasury bond puzzle," *J. Finance*, 69(5), 2151–2197.

Giglio, S., M. Maggiori, and J. Stroebel, 2016, "No-Bubble Condition: Model-Free Tests in Housing Markets," *Econometrica*, 84(3), 1047–1091.

Gopinath, G., and J. C. Stein, 2018, "Banking, Trade, and the Making of a Dominant Currency," working paper, Harvard University.

Gourinchas, P.-O., and H. Rey, 2007, "International financial adjustment," *Journal of political economy*, 115(4), 665–703.

Greenwood, R., S. G. Hanson, and J. C. Stein, 2015, "A Comparative-Advantage Approach to Government Debt Maturity," *The Journal of Finance*, 70(4), 1683–1722.

Gupta, A., and S. Van Nieuwerburgh, 2018, "Valuing Private Equity Strip by Strip," Working Paper Columbia Business School and NYU Stern.

Hall, G., and T. Sargent, 2011, "Interest Rate Risk and Other Determinants of Post WW-II U.S. Government Debt/GDP Dynamics," *American Economic Journal: Macroeconomics*, 3, 192–214.

Hamilton, J. D., and M. A. Flavin, 1986, "On the Limitations of Government Borrowing: A Framework for Empirical Testing," *The American Economic Review*, 76(4), 808–819.

Hansen, L. P., W. Roberds, and T. J. Sargent, 1991, *Time Series Implications of Present Value Budget Balance*Westview Press, vol. Rational Expectations Econometrics, chap. 5.

Hansen, L. P., and J. A. Scheinkman, 2009, "Long-term risk: An operator approach," *Econometrica*, 77(1), 177–234.

He, Z., A. Krishnamurthy, and K. Milbradt, 2018, "A Model of Safe Asset Determination," *American Economic Review*.

Jiang, Z., 2019a, "Fiscal Cyclicality and Currency Risk Premia," working paper.

——— , 2019b, "US Fiscal Cycle and the Dollar," working paper.

Jiang, Z., A. Krishnamurthy, and H. Lustig, 2018a, "Foreign Safe Asset Demand and the Dollar Exchange Rate," .

——— , 2018b, "Foreign Safe Asset Demand for US Treasurys and the Dollar," *AEA Papers and Proceedings*, 108, 537–541.

Jordà, Ò., 2005, "Estimation and Inference of Impulse Responses by Local Projections," .

Joslin, S., M. Priebsch, and K. J. Singleton, 2014, "Risk premiums in dynamic term structure models with unspanned macro risks," *The Journal of Finance*, 69(3), 1197–1233.

Kohn, D., 2016, "Addicted to Debt: Foreign Purchases of U.S. Treasuries and the Term Premium," Working Paper IE-PUC, Number 480.

Krishnamurthy, A., and H. N. Lustig, 2019, "Mind the Gap in Sovereign Debt Markets: The U.S. Treasury basis and the Dollar Risk Factor," .

Krishnamurthy, A., and A. Vissing-Jorgensen, 2012, "The aggregate demand for treasury debt," *Journal of Political Economy*, 120(2), 233–267.

——— , 2015, "The impact of Treasury supply on financial sector lending and stability," *Journal of Financial Economics*, 118(3), 571–600.

Leeper, E. M., 1991, "Equilibria under 'Active' and 'Passive' Monetary and Fiscal Policies," *Journal of monetary Economics*, 27(1), 129–147.

LeRoy, S., and R. Porter, 1981, "The Present Value Relation: Tests Based on Variance Bounds," *Econometrica*, 49, 555–557.

Longstaff, F. A., 2004, "The Flight-to-Liquidity Premium in US Treasury Bond Prices," *Journal of Business*, 77(3).

——— , 2011, "Municipal Debt and Marginal Tax Rates: Is There a Tax Premium in Asset Prices?," *J. Finance*, 66(3), 721–751.

Lustig, H., S. Van Nieuwerburgh, and A. Verdelhan, 2013, "The Wealth-Consumption Ratio," *Review of Asset Pricing Studies*, 3(1), 38–94., Review of Asset Pricing Studies.

Maggiori, M., 2017, "Financial Intermediation, International Risk Sharing, and Reserve Currencies," *American Economic Review*, 107(10), 3038–71.

Milgrom, P., and N. Stokey, 1982, "Information, trade and common knowledge," *J. Econ. Theory*, 26(1), 17–27.

Nagel, S., 2016, "The liquidity premium of near-money assets," *The Quarterly Journal of Economics*, 131(4), 1927–1971.

Pallara, K., and J.-P. Renne, 2019, "Fiscal Limits and Sovereign Credit Spreads," .

Rubin, R. E., P. R. Orszag, and A. Sinai, 2004, "Sustained budget deficits: the risk of financial and fiscal disarray," in *AEA-NAEFA Joint Session, Allied Social Science Associations Annual Meetings*. zfacts.com.

Samuelson, P. A., 1958, "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *J. Polit. Econ.*, 66(6), 467–482.

Santos, M. S., and M. Woodford, 1997, "Rational Asset Pricing Bubbles," *Econometrica*, 65(1), 19–57.

Sargent, T. J., 2012, "Nobel Lecture: United States Then, Europe Now," *Journal of Policial Economy*, 120(1).

Sargent, T. J., and N. Wallace, 1984, "Some Unpleasant Monetarist Arithmetic," in *Monetarism in the United Kingdom*. Springer, pp. 15–41.

Shleifer, A., and R. W. Vishny, 1997, "The Limits of Arbitrage," *J. Finance*, 52(1), 35–55.

Sims, C. A., 1994, "A Simple Model for Study of the Determination of the Price Level and the Interaction of Monetary and Fiscal Policy," *Economic Theory*, 4(3), 381–99.

Tirole, J., 1982, "On the Possibility of Speculation under Rational Expectations," *Econometrica*, 50(5), 1163–1181.

Trehan, B., and C. E. Walsh, 1988, "Common trends, the government's budget constraint, and revenue smoothing," *J. Econ. Dyn. Control*, 12(2), 425–444.

——— , 1991, "Testing Intertemporal Budget Constraints: Theory and Applications to U. S. Federal Budget and Current Account Deficits," *J. Money Credit Bank.*, 23(2), 206–223.

van Binsbergen, J., M. Brandt, and R. Koijen, 2012, "On the Timing and Pricing of Dividends," *American Economic Review*, 102(4), 1596–1618.

van Binsbergen, J. H., W. Diamond, and M. Grotteria, 2019, "Risk Free Interest Rates," SSRN Working Paper 3242836.

Wachter, J., 2005, "Solving Models with External Habit," *Finance Research Letters*, 2, 210–226.

Woodford, M., 1994, "Monetary Policy and Price Level Determinacy in a Cash-in-Advance Economy," *Economic Theory*, pp. 345–380.

A Proof

Proposition 1

Proof. All objects in this appendix are in nominal terms but we drop the superscript $\$$ for ease of notation. The government faces the following one-period budget constraint:

$$G_t - T_t + Q_{t-1}^1 = \sum_{h=1}^{H} (Q_t^h - Q_{t-1}^{h+1}) P_t^h,$$

where G_t is total nominal government spending, T_t is total nominal government revenue, Q_t^h is the number of nominal zero-coupon bonds of maturity h outstanding in period t each promising to pay back \$1 at time $t+h$, and P_t^h is today's price for a h-period zero-coupon bond with \$1 face value. A unit of $h+1$-period bonds issued at $t-1$ becomes a unit of h-period bonds in period t. That is, the stock of bonds evolves of each maturity evolves according to $Q_t^h = Q_{t-1}^{h+1} + \Delta Q_t^h$. Note that this notation can easily handle coupon-bearing bonds. For any bond with deterministic cash-flow sequence, we can write the price (present value) of the bond as the sum of the present values of each of its coupons.

The left-hand side of the budget constraint denotes new financing needs in the current period, due to primary deficit $G - T$ and one-period debt from last period that is now maturing. The right hand side shows that the money is raised by issuing new bonds of various maturities. Alternatively, we can write the budget constraint as total expenses equalling total income:

$$G_t + Q_{t-1}^1 + \sum_{h=1}^{H} Q_{t-1}^{h+1} P_t^h = T_t + \sum_{h=1}^{H} Q_t^h P_t^h,$$

We can now iterate the budget constraint forward. The period t constraint is given by:

$$
\begin{aligned}
T_t - G_t \;=\;& Q_{t-1}^1 - Q_t^1 P_t^1 + Q_{t-1}^2 P_t^1 - Q_t^2 P_t^2 + Q_{t-1}^3 P_t^2 - Q_t^3 P_t^3 \\
& + \cdots - Q_t^H P_t^H + Q_{t-1}^{H+1} P_t^H.
\end{aligned}
$$

Consider the period-$t+1$ constraint,

$$
\begin{aligned}
T_{t+1} - G_{t+1} \;=\;& Q_t^1 - Q_{t+1}^1 P_{t+1}^1 + Q_t^2 P_{t+1}^1 - Q_{t+1}^2 P_{t+1}^2 + Q_t^3 P_{t+1}^2 - Q_{t+1}^3 P_{t+1}^3 \\
& + \cdots - Q_{t+1}^H P_{t+1}^H + Q_t^{H+1} P_{t+1}^H.
\end{aligned}
$$

multiply both sides by M_{t+1}, and take expectations conditional on time t:

$$
\begin{aligned}
\mathbb{E}_t\left[M_{t+1}(T_{t+1} - G_{t+1}) \right] \;=\;& Q_t^1 P_t^1 - \mathbb{E}_t[Q_{t+1}^1 M_{t+1} P_{t+1}^1] + Q_t^2 P_t^2 - \mathbb{E}_t[Q_{t+1}^2 M_{t+1} P_{t+1}^2] + Q_t^3 P_t^3 \\
& - \mathbb{E}_t[Q_{t+1}^3 M_{t+1} P_{t+1}^3] + \cdots + Q_t^H P_t^H \\
& - \mathbb{E}_t[Q_{t+1}^H M_{t+1} P_{t+1}^H] + Q_t^{H+1} P_t^{H+1},
\end{aligned}
$$

where we use the asset pricing equations $\mathbb{E}_t[M_{t+1}] = P_t^1$, $\mathbb{E}_t[M_{t+1} P_{t+1}^1] = P_t^2, \cdots, \mathbb{E}_t[M_{t+1} P_{t+1}^{H-1}] = P_t^H$, and $\mathbb{E}_t[M_{t+1} P_{t+1}^H] = P_t^{H+1}$.

Consider the period $t+2$ constraint, multiplied by $M_{t+1} M_{t+2}$ and take time-t expectations:

$$
\begin{aligned}
\mathbb{E}_t\left[M_{t+1} M_{t+2}(T_{t+2} - G_{t+2}) \right] \;=\;& \mathbb{E}_t[Q_{t+1}^1 M_{t+1} P_{t+1}^1] - \mathbb{E}_t[Q_{t+2}^1 M_{t+1} M_{t+2} P_{t+2}^1] + \mathbb{E}_t[Q_{t+1}^2 M_{t+1} P_{t+1}^2] \\
& - \mathbb{E}_t[Q_{t+2}^2 M_{t+1} M_{t+2} P_{t+2}^2] + \mathbb{E}_t[Q_{t+1}^3 M_{t+1} P_{t+1}^3] - \cdots \\
& + \mathbb{E}_t[Q_{t+1}^H M_{t+1} P_{t+1}^H] - \mathbb{E}_t[Q_{t+2}^H M_{t+1} M_{t+2} P_{t+2}^H] + \mathbb{E}_t[Q_{t+1}^{H+1} M_{t+1} P_{t+1}^{H+1}],
\end{aligned}
$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M_{t+2}] = P_{t+1}^1$, $\mathbb{E}_{t+1}[M_{t+2} P_{t+2}^1] = P_{t+1}^2$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected

discounted surpluses at t, $t + 1$, and $t + 2$ we get:

$$T_t - G_t + \mathbb{E}_t \left[M_{t+1}(T_{t+1} - G_{t+1}) \right] + \mathbb{E}_t \left[M_{t+1}M_{t+2}(T_{t+2} - G_{t+2}) \right] = \sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h +$$
$$-\mathbb{E}_t[Q_{t+2}^1 M_{t+1} M_{t+2} P_{t+2}^1] - \mathbb{E}_t[Q_{t+2}^2 M_{t+1} M_{t+2} P_{t+2}^2] - \cdots - \mathbb{E}_t[Q_{t+2}^H M_{t+1} M_{t+2} P_{t+2}^H].$$

Similarly consider the one-period government budget constraints at times $t + 3$, $t + 4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t + J$, we get:

$$\sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{J} M_{t,t+j}(T_{t+j} - G_{t+j}) \right] + \mathbb{E}_t \left[M_{t,t+J} \sum_{h=1}^{H} Q_{t+J}^h P_{t+J}^h \right]$$

where we used the cumulate SDF notation $M_{t,t+j} = \prod_{i=0}^{j} M_{t+i}$ and by convention $M_{t,t} = M_t = 1$ and $P_t^0 = 1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next J years plus the present value of the government bond portfolio that will be outstanding at time $t + J$. The latter is the cost the government will face at time $t + J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J \to \infty$:

$$\sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j}) \right] + \lim_{J \to \infty} \mathbb{E}_t \left[M_{t,t+J} \sum_{h=1}^{H} Q_{t+J}^h P_{t+J}^h \right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j} - G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the transversality condition:

$$\lim_{J \to \infty} \mathbb{E}_t \left[M_{t,t+J} \sum_{h=1}^{H} Q_{t+J}^h P_{t+J}^h \right] = 0.$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise there is a government debt bubble.

If the transversality condition is satisfied, the outstanding debt today, D_t, reflects the expected present-discounted value of the current and all future primary surpluses:

$$D_t = \sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h \quad = \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j}) \right].$$

This is equation (1) in the main text.

Case with Default

Proof. We consider only full default, without loss of generality. Alternatively, we can write the budget constraint that obtains in case of no default at t:

$$G_t + Q_{t-1}^1 + \sum_{h=1}^{H} Q_{t-1}^{h+1} P_t^h = T_t + \sum_{h=1}^{H} Q_t^h P_t^h,$$

and, in case of default at t, the one-period budget constraint is given by:

$$G_t = T_t + \sum_{h=1}^{H} Q_t^h P_t^h.$$

We can now iterate the budget constraint forward. In case of no default, the period t constraint is given by:

$$T_t - G_t \quad = \quad Q_{t-1}^1 - Q_t^1 P_t^1 + Q_{t-1}^2 P_t^1 - Q_t^2 P_t^2 + Q_{t-1}^3 P_t^2 - Q_t^3 P_t^3$$
$$+ \cdots - Q_t^H P_t^H + Q_{t-1}^{H+1} P_t^H.$$

In case of default, the period t constraint is given by:

$$T_t - G_t = -Q_t^1 P_t^1 - Q_t^2 P_t^2 - Q_t^3 P_t^3 - Q_t^H P_t^H$$

First, consider the period-$t+1$ constraint in case of no default,

$$T_{t+1} - G_{t+1} = Q_t^1 - Q_{t+1}^1 P_{t+1}^1 + Q_t^2 P_{t+1}^1 - Q_{t+1}^2 P_{t+1}^2 + Q_t^3 P_{t+1}^2 - Q_{t+1}^3 P_{t+1}^3$$
$$+ \cdots - Q_{t+1}^H P_{t+1}^H + Q_t^{H+1} P_{t+1}^H.$$

Second, consider the period-$t+1$ constraint in case of default,

$$T_{t+1} - G_{t+1} = -Q_{t+1}^1 P_{t+1}^1 - Q_{t+1}^2 P_{t+1}^2 - Q_{t+1}^3 P_{t+1}^3 - Q_{t+1}^H P_{t+1}^H.$$

We use χ_t as an indicator variable for default. To simplify, we consider only full default with zero recovery. This is without loss of generality. Next, multiply both sides of the no default constraint by $(1 - \chi_{t+1})M_{t+1}$, and take expectations conditional on time t:

$$\mathbb{E}_t \left[M_{t+1}(1 - \chi_{t+1})(T_{t+1} - G_{t+1}) \right] = Q_t^1 \mathbb{E}_t \left[M_{t+1}(1 - \chi_{t+1}) \right] - \mathbb{E}_t [Q_{t+1}^1(1 - \chi_{t+1})M_{t+1}P_{t+1}^1] + E_t[(1 - \chi_{t+1})M_{t+1}P_{t+1}^1]Q_t^2$$
$$- \mathbb{E}_t[Q_{t+1}^2(1 - \chi_{t+1})M_{t+1}P_{t+1}^2] + E_t[M_{t+1}(1 - \chi_{t+1})P_{t+1}^2]Q_t^3 - \mathbb{E}_t[Q_{t+1}^3(1 - \chi_{t+1})M_{t+1}P_{t+1}^3] +$$
$$\cdots + Q_t^H E_t[M_{t+1}(1 - \chi)P_{t+1}^{H-1}] - \mathbb{E}_t[Q_{t+1}^H(1 - \chi_{t+1})M_{t+1}P_{t+1}^H] + Q_t^{H+1}E_t[M_{t+1}(1 - \chi_{t+1})P_{t+1}^H],$$

and multiply both sides of the default constraint by $M_{t+1}\chi_{t+1}$

$$\mathbb{E}_t \left[M_{t+1}\chi_{t+1}(T_{t+1} - G_{t+1}) \right] = -\mathbb{E}_t[Q_{t+1}^1\chi_{t+1}M_{t+1}P_{t+1}^1] - \mathbb{E}_t[Q_{t+1}^2\chi_{t+1}M_{t+1}P_{t+1}^2]$$
$$- \mathbb{E}_t[Q_{t+1}^3\chi_{t+1}M_{t+1}P_{t+1}^3] + \cdots - \mathbb{E}_t[Q_{t+1}^H\chi_{t+1}M_{t+1}P_{t+1}^H].$$

By adding these 2 constraints, we obtain the following expression:

$$\mathbb{E}_t \left[M_{t+1}(T_{t+1} - G_{t+1}) \right] = Q_t^1 \mathbb{E}_t \left[M_{t+1}(1 - \chi_{t+1}) \right] - \mathbb{E}_t[Q_{t+1}^1 M_{t+1}P_{t+1}^1] + E_t[(1 - \chi_{t+1})M_{t+1}P_{t+1}^1]Q_t^2$$
$$- \mathbb{E}_t[Q_{t+1}^2 M_{t+1}P_{t+1}^2] + E_t[M_{t+1}(1 - \chi_{t+1})P_{t+1}^2]Q_t^3$$
$$- \mathbb{E}_t[Q_{t+1}^3 M_{t+1}P_{t+1}^3] + \cdots + Q_t^H E_t[M_{t+1}(1 - \chi)P_{t+1}^{H-1}] - \mathbb{E}_t[Q_{t+1}^H M_{t+1}P_{t+1}^H] + Q_t^{H+1}E_t[M_{t+1}(1 - \chi_{t+1})P_{t+1}^H].$$

This can be restated as:

$$\mathbb{E}_t \left[M_{t+1}(T_{t+1} - G_{t+1}) \right] = Q_t^1 P_t^1 - \mathbb{E}_t[Q_{t+1}^1 M_{t+1}P_{t+1}^1] + Q_t^2 P_t^2 - \mathbb{E}_t[Q_{t+1}^2 M_{t+1}P_{t+1}^2] + Q_t^3 P_t^3$$
$$- \mathbb{E}_t[Q_{t+1}^3 M_{t+1}P_{t+1}^3] + \cdots + Q_t^H P_t^H - \mathbb{E}_t[Q_{t+1}^H M_{t+1}P_{t+1}^H] + Q_t^{H+1} P_t^{H+1},$$

where we use the asset pricing equations $\mathbb{E}_t \left[M_{t+1}(1 - \chi_{t+1}) \right] = P_t^1$, $\mathbb{E}_t[M_{t+1}(1 - \chi_{t+1})P_{t+1}^1] = P_t^2, \cdots, \mathbb{E}_t[M_{t+1}(1 - \chi_{t+1})P_{t+1}^{H-1}] = P_t^H$, and $\mathbb{E}_t[M_{t+1}(1 - \chi_{t+1})P_{t+1}^H] = P_t^{H+1}$.

The rest of the proof is essentially unchanged. Consider the period $t+2$ constraint, multiplied by $M_{t+1}M_{t+2}(1 - \chi_{t+2})$ in the no-default case, and $M_{t+1}M_{t+2}(\chi_{t+2})$ for the default case, and take time-t expectations (after adding default and no-default states):

$$\mathbb{E}_t \left[M_{t+1}M_{t+2}(T_{t+2} - G_{t+2}) \right] = \mathbb{E}_t[Q_{t+1}^1 M_{t+1}P_{t+1}^1] - \mathbb{E}_t[Q_{t+2}^1 M_{t+1}M_{t+2}P_{t+2}^1] + \mathbb{E}_t[Q_{t+1}^2 M_{t+1}P_{t+1}^2]$$
$$- \mathbb{E}_t[Q_{t+2}^2 M_{t+1}M_{t+2}P_{t+2}^2] + \mathbb{E}_t[Q_{t+1}^3 M_{t+1}P_{t+1}^3] - \cdots$$
$$+ \mathbb{E}_t[Q_{t+1}^H M_{t+1}P_{t+1}^H] - \mathbb{E}_t[Q_{t+2}^H M_{t+1}M_{t+2}P_{t+2}^H] + \mathbb{E}_t[Q_{t+1}^{H+1} M_{t+1}P_{t+1}^{H+1}],$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M_{t+2}(1 - \chi_{t+2})] = P_{t+1}^1$, $\mathbb{E}_{t+1}[M_{t+2}(1 - \chi_{t+2})P_{t+2}^1] = P_{t+1}^2$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected

49

discounted surpluses at t, $t + 1$, and $t + 2$ we get:

$$T_t - G_t + \mathbb{E}_t\left[M_{t+1}(T_{t+1} - G_{t+1})\right] + \mathbb{E}_t\left[M_{t+1}M_{t+2}(T_{t+2} - G_{t+2})\right] = \sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h +$$
$$-\mathbb{E}_t[Q_{t+2}^1 M_{t+1} M_{t+2} P_{t+2}^1] - \mathbb{E}_t[Q_{t+2}^2 M_{t+1} M_{t+2} P_{t+2}^2] - \cdots - \mathbb{E}_t[Q_{t+2}^H M_{t+1} M_{t+2} P_{t+2}^H].$$

Similarly consider the one-period government budget constraints at times $t + 3$, $t + 4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t + J$, we get:

$$\sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h = \mathbb{E}_t\left[\sum_{j=0}^{J} M_{t,t+j}(T_{t+j} - G_{t+j})\right] + \mathbb{E}_t\left[M_{t,t+J} \sum_{h=1}^{H} Q_{t+J}^h P_{t+J}^h\right]$$

where we used the cumulate SDF notation $M_{t,t+j} = \prod_{i=0}^{j} M_{t+i}$ and by convention $M_{t,t} = M_t = 1$ and $P_t^0 = 1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next J years plus the present value of the government bond portfolio that will be outstanding at time $t + J$. The latter is the cost the government will face at time $t + J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J \to \infty$:

$$\sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j})\right] + \lim_{J \to \infty} \mathbb{E}_t\left[M_{t,t+J} \sum_{h=1}^{H} Q_{t+J}^h P_{t+J}^h\right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j} - G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the transversality condition:

$$\lim_{J \to \infty} \mathbb{E}_t\left[M_{t,t+J} \sum_{h=1}^{H} Q_{t+J}^h P_{t+J}^h\right] = 0.$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise there is a government debt bubble.

If the transversality condition is satisfied, the outstanding debt today, D_t, reflects the expected present-discounted value of the current and all future primary surpluses:

$$D_t = \sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j})\right].$$

This is equation (1) in the main text.

$\square$

Proposition 2 From the time-t budget constraint, we get that the primary surplus

$$-S_t = -Q_{t-1}^1 + \sum_{h=1}^{H} (Q_t^h - Q_{t-1}^{h+1}) P_t^h.$$

It follows that

$$D_t - S_t = \sum_{h=0}^{H} Q_{t-1}^{h+1} P_t^h - Q_{t-1}^1 + \sum_{h=1}^{H} (Q_t^h - Q_{t-1}^{h+1}) P_t^h = \sum_{h=1}^{H} Q_t^h P_t^h.$$

We obtain equation (2) in the main text.

$$r_{t+1}^D (D_t - S_t) = \sum_{h=0}^{\infty} P_{t+1}^\$(h) Q_{t,h+1}^\$ = D_{t+1} = P_{t+1}^\tau - P_{t+1}^g$$
$$= (P_t^\tau - T_t) r_{t+1}^\tau - (P_{t+1}^g - G_t) r_{t+1}^g.$$

50

$\square$

Proposition 3

Proof. We follow the proof in the working paper version of Backus, Boyarchenko, and Chernov (2018) on page 16 (Example 5). Hansen and Scheinkman (2009) consider the following equation:

$$\mathbb{E}_t[M_{t,t+1}v_{t+1}] = \nu v_t, \tag{A.1}$$

where ν is the dominant eigenvalue and v_t is the eigenfunction. Claims to stationary cash flows earn a return equal to the yield on the long bond. We consider the following decomposition of the pricing kernel:

$$
\begin{align}
M^1_{t,t+1} &= M_{t,t+1}v_{t+1}/\nu v_t, \tag{A.2} \\
M^2_{t,t+1} &= \nu v_t/v_{t+1}. \tag{A.3}
\end{align}
$$

By construction, $\mathbb{E}_t[M^1_{t,t+1}] = 1$. The long yields converge to $-\log \nu$. The long-run bond return converges to $\lim_{n\to\infty} R^n_{t,t+1} = \frac{1}{M^2_{t,t+1}} = v_{t+1}/\nu v_t$. This implies that $\mathbb{E}[\log R^\infty_{t,t+1}] = -\log \nu$.

To value claims to uncertain cash flows with one-period growth rate $g_{t,t+1}$, we define $\widehat{p}^n_t$ to denote the price of a strip that pays off $d_{t,t+n}$, n periods from now.

$$\widehat{p}^n_t = \mathbb{E}_t[M_{t,t+1}g_{t,t+1}\widehat{p}^{n-1}_{t+1}] = \mathbb{E}_t[\widehat{M}_{t,t+1}\widehat{p}^{n-1}_{t+1}],$$

where $\widehat{M}_{t,t+1} = M_{t,t+1}g_{t,t+1}$. Consider the problem of finding the dominant eigenvalue:

$$\mathbb{E}_t[\widehat{M}_{t,t+1}\widehat{v}_{t+1}] = \nu\widehat{v}_t. \tag{A.4}$$

If the cash flows are stationary, then the same ν that solves this equation for $M_{t,t+1}$ in eqn. A.1 solves the one for $\widehat{M}_{t,t+1}$. Hence, if (ν, v_t) solves eqn. A.1, then $(\nu, v_t/d_t)$ solves the hat equation eqn. A.4. $\square$

B Asset Pricing Model

B.1 Risk-free rate

The real short yield $y_t(1)$, or risk-free rate, satisfies $E_t[\exp\{m_{t+1} + y_t(1)\}] = 1$. Solving out this Euler equation, we get:

$$
\begin{align}
y_t(1) &= y^\$_t(1) - E_t[\pi_{t+1}] - \frac{1}{2}e'_\pi \Sigma e_\pi + e'_\pi \Sigma^{\frac{1}{2}}\Lambda_t \notag \\
&= y_0(1) + \left[e'_{yn} - e'_\pi \Psi + e'_\pi \Sigma^{\frac{1}{2}}\Lambda_1\right]z_t. \tag{A.5} \\
y_0(1) &\equiv y^\$_0(1) - \pi_0 - \frac{1}{2}e'_\pi \Sigma e_\pi + e'_\pi \Sigma^{\frac{1}{2}}\Lambda_0. \tag{A.6}
\end{align}
$$

where we used the expression for the real SDF

$$
\begin{align}
m_{t+1} &= m^\$_{t+1} + \pi_{t+1} \notag \\
&= -y^\$_t(1) - \frac{1}{2}\Lambda'_t\Lambda_t - \Lambda'_t\varepsilon_{t+1} + \pi_0 + e'_\pi \Psi z_t + e'_\pi \Sigma^{\frac{1}{2}}\varepsilon_{t+1} \notag \\
&= -y_t(1) - \frac{1}{2}e'_\pi \Sigma e_\pi + e'_\pi \Sigma^{\frac{1}{2}}\Lambda_t - \frac{1}{2}\Lambda'_t\Lambda_t - \left(\Lambda'_t - e'_\pi \Sigma^{\frac{1}{2}}\right)\varepsilon_{t+1} \notag
\end{align}
$$

The real short yield is the nominal short yield minus expected inflation minus a Jensen adjustment minus the inflation risk premium.

B.2 Nominal and real term structure

Proposition 8. Nominal bond yields are affine in the state vector:

$$y_t^{\$}(h) = -\frac{A^{\$}(h)}{h} - \frac{B^{\$}(h)'}{h} z_t,$$

where the coefficients $A^{\$}(h)$ and $B^{\$}(h)$ satisfy the following recursions:

$$A^{\$}(h+1) = -y_0^{\$}(1) + A^{\$}(h) + \frac{1}{2}\left(B^{\$}(h)\right)' \Sigma \left(B^{\$}(h)\right) - \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_0, \tag{A.7}$$

$$\left(B^{\$}(h+1)\right)' = \left(B^{\$}(h)\right)' \Psi - e'_{yn} - \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_1, \tag{A.8}$$

initialized at $A^{\$}(0) = 0$ and $B^{\$}(0) = \mathbf{0}$.

Proof. We conjecture that the $t+1$-price of a τ-period bond is exponentially affine in the state:

$$\log(P_{t+1}^{\$}(h)) = A^{\$}(h) + \left(B^{\$}(h)\right)' z_{t+1}$$

and solve for the coefficients $A^{\$}(h+1)$ and $B^{\$}(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$
\begin{aligned}
P_t^{\$}(h+1) &= E_t[\exp\{m_{t+1}^{\$} + \log\left(P_{t+1}^{\$}(h)\right)\}] \\
&= E_t[\exp\{-y_t^{\$}(1) - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + A^{\$}(h) + \left(B^{\$}(h)\right)' z_{t+1}\}] \\
&= \exp\{-y_0^{\$}(1) - e'_{yn}z_t - \frac{1}{2}\Lambda_t'\Lambda_t + A^{\$}(h) + \left(B^{\$}(h)\right)' \Psi z_t\} \times \\
&\quad E_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}}\varepsilon_{t+1}\}\right].
\end{aligned}
$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$
\begin{aligned}
P_t^{\$}(h+1) &= \exp\left\{-y_0^{\$}(1) - e'_{yn}z_t + A^{\$}(h) + \left(B^{\$}(h)\right)' \Psi z_t + \frac{1}{2}\left(B^{\$}(h)\right)' \Sigma \left(B^{\$}(h)\right)\right. \\
&\quad \left. - \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}}\left(\Lambda_0 + \Lambda_1 z_t\right)\right\}.
\end{aligned}
$$

Taking logs and collecting terms, we obtain a linear equation for $\log(p_t(h+1))$:

$$\log\left(P_t^{\$}(h+1)\right) = A^{\$}(h+1) + \left(B^{\$}(h+1)\right)' z_t,$$

where $A^{\$}(h+1)$ satisfies (A.7) and $B^{\$}(h+1)$ satisfies (A.8). The relationship between log bond prices and bond yields is given by $-\log\left(P_t^{\$}(h)\right)/\tau = y_t^{\$}(h)$. $\square$

Define the one-period return on a nominal zero-coupon bond as:

$$r_{t+1}^{b,\$}(h) = \log\left(P_{t+1}^{\$}(h)\right) - \log\left(P_t^{\$}(h+1)\right)$$

The nominal bond risk premium on a bond of maturity τ is the expected excess return corrected for a Jensen term, and equals negative the conditional covariance between that bond return and the nominal SDF:

$$
\begin{aligned}
E_t\left[r_{t+1}^{b,\$}(h)\right] - y_t^{\$}(1) + \frac{1}{2}V_t\left[r_{t+1}^{b,\$}(h)\right] &= -Cov_t\left[m_{t+1}^{\$}, r_{t+1}^{b,\$}(h)\right] \\
&= \left(B^{\$}(h)\right)' \Sigma^{\frac{1}{2}}\Lambda_t
\end{aligned}
$$

Real bond yields, $y_t(h)$, denoted without the \$ superscript, are affine as well with coefficients that follow similar recursions:

$$A(h+1) = -y_0(1) + A(h) + \frac{1}{2}(B(h))'\Sigma(B(h)) - (B(h))'\Sigma^{\frac{1}{2}}\left(\Lambda_0 - \Sigma^{\frac{1}{2}'}e_\pi\right), \tag{A.9}$$

$$(B(h+1))' = -e'_{yn} + (e_\pi + B(h))'\left(\Psi - \Sigma^{\frac{1}{2}}\Lambda_1\right). \tag{A.10}$$

For $\tau = 1$, we recover the expression for the risk-free rate in (A.5)-(A.6).

B.3 Stocks

B.3.1 Aggregate Stock Market

We define the real return on the aggregate stock market as $R^m_{t+1} = \frac{P^m_{t+1} + D^m_{t+1}}{P^m_t}$, where P^m_t is the ex-dividend price on the equity market. A log-linearization delivers:

$$r^m_{t+1} = \kappa^m_0 + \Delta d^m_{t+1} + \kappa^m_1 pd^m_{t+1} - pd^m_t. \tag{A.11}$$

The unconditional mean log real stock return is $r^m_0 = E[r^m_t]$, the unconditional mean real dividend growth rate is $\mu^m = E[\Delta d^m_{t+1}]$, and $\overline{pd^m} = E[pd^m_t]$ is the unconditional average log price-dividend ratio on equity. The linearization constants κ^m_0 and κ^m_1 are defined as:

$$\kappa^m_1 = \frac{e^{\overline{pd^m}}}{e^{\overline{pd^m}} + 1} < 1 \text{ and } \kappa^m_0 = \log\left(e^{\overline{pd^m}} + 1\right) - \frac{e^{\overline{pd^m}}}{e^{\overline{pd^m}} + 1}\overline{pd^m}. \tag{A.12}$$

Our state vector z contains the (demeaned) log real dividend growth rate on the stock market, $\Delta d^m_{t+1} - \mu^m$, and the (demeaned) log price-dividend ratio $pd^m - \overline{pd^m}$.

$$pd^m_t(h) = \overline{pd^m} + e'_{pd}z_t,$$
$$\Delta d^m_t = \mu^m + e'_{divm}z_t,$$

where e'_{pd} (e_{divm}) is a selector vector that has a one in the fifth (sixth) entry, since the log pd ratio (log dividend growth rate) is the fifth (sixth) element of the VAR.

We *define* the log return on the stock market so that the log return equation holds exactly, given the time series for $\{\Delta d^m_t, pd^m_t\}$. Rewriting (A.11):

$$r^m_{t+1} - r^m_0 = \left[(e_{divm} + \kappa^m_1 e_{pd})'\Psi - e'_{pd}\right]z_t + (e_{divm} + \kappa^m_1 e_{pd})'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}.$$
$$r^m_0 = \mu^m + \kappa^m_0 - \overline{pd^m}(1 - \kappa^m_1).$$

The equity risk premium is the expected excess return on the stock market corrected for a Jensen term. By the Euler equation, it equals minus the conditional covariance between the log SDF and the log return:

$$\begin{aligned}
1 &= E_t\left[M_{t+1}\frac{P^m_{t+1} + D^m_{t+1}}{P^m_t}\right] = E_t\left[\exp\{m^\$_{t+1} + \pi_{t+1} + r^m_{t+1}\}\right]\\
&= E_t\left[\exp\left\{-y^\$_{t,1} - \frac{1}{2}\Lambda'_t\Lambda_t - \Lambda'_t\varepsilon_{t+1} + \pi_0 + e'_\pi z_{t+1} + r^m_0 + (e_{divm} + \kappa^m_1 e_{pd})'z_{t+1} - e'_{pd}z_t\right\}\right]\\
&= \exp\left\{-y^\$_0(1) - \frac{1}{2}\Lambda'_t\Lambda_t + \pi_0 + r^m_0 + \left[(e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Psi - e'_{pd} - e'_{yn}\right]z_t\right\}\\
&\quad \times E_t\left[\exp\{-\Lambda'_t\varepsilon_{t+1} + (e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right]\\
&= \exp\left\{r^m_0 + \pi_0 - y^\$_0(1) + \left[(e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Psi - e'_{pd} - e'_{yn}\right]z_t\right\}\\
&\quad \times \exp\left\{\frac{1}{2}(e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Sigma(e_{divm} + \kappa^m_1 e_{pd} + e_\pi) - (e_{divm} + \kappa^m_1 e_{pd} + e_\pi)'\Sigma^{\frac{1}{2}}\Lambda_t\right\}
\end{aligned}$$

Taking logs on both sides delivers:

$$r_0^m + \pi_0 - y_0^\$(1) + \left[(e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Psi - e_{pd}' - e_{yn}' \right] z_t \tag{A.13}$$

$$+ \frac{1}{2} (e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Sigma (e_{divm} + \kappa_1^m e_{pd} + e_\pi) = (e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Sigma^{\frac{1}{2}} \Lambda_t$$

$$E_t \left[r_{t+1}^{m,\$} \right] - y_{t,1}^\$ + \frac{1}{2} V_t \left[r_{t+1}^{m,\$} \right] = -Cov_t \left[m_{t+1}^\$, r_{t+1}^{m,\$} \right]$$

The left-hand side is the expected excess return on the stock market, corrected for a Jensen term, while the right-hand side is the negative of the conditional covariance between the (nominal) log stock return and the nominal log SDF. We refer to the left-hand side as the equity risk premium in the data, since it is implied directly by the VAR. We refer to the right-hand side as the equity risk premium in the model, since it requires knowledge of the market prices of risk.

Note that we can obtain the same expression using the log real SDF and log real stock return:

$$E_t \left[r_{t+1}^m \right] - y_{t,1} + \frac{1}{2} V_t \left[r_{t+1}^m \right] = -Cov_t \left[m_{t+1}, r_{t+1}^m \right]$$

$$r_0^m - y_0(1) + \left[(e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Psi - e_{pd}' - e_{yn}' - e_\pi' \Sigma^{1/2} \Lambda_1 \right] z_t$$

$$+ \frac{1}{2} (e_{divm} + \kappa_1^m e_{pd})' \Sigma (e_{divm} + \kappa_1^m e_{pd}) = (e_{divm} + \kappa_1^m e_{pd})' \Sigma^{1/2} (\Lambda_t - \left(\Sigma^{1/2} \right)' e_\pi)$$

We combine the terms in Λ_0 and Λ_1 on the right-hand side and plug in for $y_0(1)$ from (A.6) to get:

$$r_0^m + \pi_0 - y_{0,1}^\$ + \frac{1}{2} e_\pi' \Sigma e_\pi$$

$$+ \frac{1}{2} (e_{divm} + \kappa_1^m e_{pd})' \Sigma (e_{divm} + \kappa_1^m e_{pd}) + e_\pi' \Sigma (e_{divm} + \kappa_1^m e_{pd})$$

$$+ \left[(e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Psi - e_{pd}' - e_{yn}' \right] z_t$$

$$= (e_{divm} + \kappa_1^m e_{pd})' \Sigma^{1/2} \Lambda_t + e_\pi' \Sigma^{\frac{1}{2}} \Lambda_0 + e_\pi' \Sigma^{1/2} \Lambda_1 z_t$$

This recovers equation (A.13).

B.3.2 Dividend Strips

Proposition 9. Log price-dividend ratios on dividend strips are affine in the state vector:

$$p_t^d(h) = \log \left(P_t^d(h) \right) = A^m(h) + B^{m\prime}(h) z_t,$$

where the coefficients $A^m(h)$ and $B^m(h)$ follow recursions:

$$A^m(h+1) = A^m(h) + \mu_m - y_0(1) + \frac{1}{2} (e_{divm} + B^m(h))' \Sigma (e_{divm} + B^m(h))$$

$$- (e_{divm} + B^m(h))' \Sigma^{\frac{1}{2}} \left(\Lambda_0 - \Sigma^{\frac{1}{2}\prime} e_\pi \right), \tag{A.14}$$

$$B^{m\prime}(h+1) = (e_{divm} + e_\pi + B^m(h))' \Psi - e_{yn}' - (e_{divm} + e_\pi + B^m(h))' \Sigma^{\frac{1}{2}} \Lambda_1, \tag{A.15}$$

initialized at $A_0^m = 0$ and $B_0^m = 0$.

Proof. We conjecture the affine structure and solve for the coefficients $A^m(h+1)$ and $B^m(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$P_t^d(h+1) = E_t \left[M_{t+1} P_{t+1}^d(h) \frac{D_{t+1}^m}{D_t^m} \right] = E_t \left[\exp\{ m_{t+1}^\$ + \pi_{t+1} + \Delta d_{t+1}^m + p_{t+1}^d(h) \} \right]$$

$$= E_t \left[\exp\{ -y_{t,1}^\$ - \frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1} + \pi_0 + e_\pi' z_{t+1} + \mu^m + e_{divm}' z_{t+1} + A^m(h) + B(h)^{m\prime} z_{t+1} \} \right]$$

$$
\begin{aligned}
&= \exp\{-y_0^\$(1) - e'_{yn}z_t - \frac{1}{2}\Lambda'_t\Lambda_t + \pi_0 + e'_\pi \Psi z_t + \mu_m + e'_{divm}\Psi z_t + A^m(h) + B(h)^{m'}\Psi z_t\} \\
&\quad \times \mathbb{E}_t\left[\exp\{-\Lambda'_t\varepsilon_{t+1} + (e_{divm} + e_\pi + B^m(h))'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right].
\end{aligned}
$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$
\begin{aligned}
P_t^d(h+1) &= \exp\{-y_0^\$(1) + \pi_0 + \mu_m + A^m(h) + \left[(e_{divm} + e_\pi + B^m(h))'\Psi - e'_{yn}\right]z_t \\
&\quad + \frac{1}{2}(e_{divm} + e_\pi + B^m(h))'\Sigma(e_{divm} + e_\pi + B^m(h)) \\
&\quad - (e_{divm} + e_\pi + B^m(h))'\Sigma^{\frac{1}{2}}(\Lambda_0 + \Lambda_1 z_t)\}
\end{aligned}
$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^d(h+1)$:

$$
p_t^d(h+1) = A^m(h+1) + B^{m'}(h+1)z_t,
$$

where:

$$
\begin{aligned}
A^m(h+1) &= A^m(h) + \mu_m - y_0^\$(1) + \pi_0 + \frac{1}{2}(e_{divm} + e_\pi + B^m(h))'\Sigma(e_{divm} + e_\pi + B^m(h)) \\
&\quad - (e_{divm} + e_\pi + B^m(h))'\Sigma^{\frac{1}{2}}\Lambda_0, \\
B^{m'}(h+1) &= (e_{divm} + e_\pi + B^m(h))'\Psi - e'_{yn} - (e_{divm} + e_\pi + B^m(h))'\Sigma^{\frac{1}{2}}\Lambda_1.
\end{aligned}
$$

We recover the recursions in (A.14) and (A.15) after using equation (A.6). $\qquad\square$

We define the dividend strip risk premium as:

$$
\begin{aligned}
\mathbb{E}_t\left[r_{t+1}^{d,\$}(h)\right] - y_{t,1}^\$ + \frac{1}{2}V_t\left[r_{t+1}^{d,\$}(h)\right] &= -Cov_t\left[m_{t+1}^\$, r_{t+1}^{d,\$}(h)\right] \\
&= (e_{divm} + e_\pi + B^m(h))'\Sigma^{\frac{1}{2}}\Lambda_t
\end{aligned}
$$

B.4 Claim to Future Government Spending and Tax Revenues

This appendix computes P_t^τ, the value of a claim to future tax revenues, and P_t^g, the value of a claim to future government spending.

B.4.1 Spending Claim

Nominal government spending growth equals

$$
\Delta \log G_{t+1} = \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu_0^g + (e_{\Delta g} + e_x + e_\pi)'z_{t+1}. \tag{A.16}
$$

We conjecture the log price-dividend ratios on spending strips are affine in the state vector:

$$
p_t^g(h) = \log(P_t^g(h)) = A^g(h) + B^{g'}(h)z_t.
$$

We solve for the coefficients $A^g(h+1)$ and $B^g(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$
\begin{aligned}
P_t^g(h+1) &= \mathbb{E}_t\left[M_{t+1}P_{t+1}^g(h)\frac{G_{t+1}}{G_t}\right] = \mathbb{E}_t\left[\exp\{m_{t+1}^\$ + \Delta\log g_{t+1} + x_{t+1} + \pi_{t+1} + p_{t+1}^g(h)\}\right] \\
&= \exp\{-y_0^\$(1) - e'_{yn}z_t - \frac{1}{2}\Lambda'_t\Lambda_t + \mu^g + x_0 + \pi_0 + (e_{\Delta g} + e_x + e_\pi + B^g(h))'\Psi z_t + A^g(h)\} \\
&\quad \times \mathbb{E}_t\left[\exp\{-\Lambda'_t\varepsilon_{t+1} + (e_{\Delta g} + e_x + e_\pi + B^g(h))'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right].
\end{aligned}
$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$
\begin{aligned}
P_t^g(h+1) &= \exp\{-y_0^\$(1) + \mu^g + x_0 + \pi_0 + ((e_{\Delta g} + e_x + e_\pi + B^g(h))'\Psi - e_{yn}')z_t + A^g(h) \\
&\quad + \frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right) \\
&\quad - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma^{\frac{1}{2}}(\Lambda_0 + \Lambda_1 z_t)\}
\end{aligned}
$$

Taking logs and collecting terms, we obtain

$$
\begin{aligned}
A^g(h+1) &= -y_0^\$(1) + \mu^g + x_0 + \pi_0 + A^g(h) + \frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right) \\
&\quad - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_0, \\
B^g(h+1)' &= \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Psi - e_{yn}' - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_1,
\end{aligned}
$$

and the price-dividend ratio of the cum-dividend spending claim is

$$
\sum_{h=0}^{\infty} \exp(A^g(h+1) + B^g(h+1)'z_t)
$$

Next, we define the (nominal) return on the claim as $R_{t+1}^g = \frac{P_{t+1}^g}{P_t^g - G_t} = \frac{P_{t+1}^{g,ex} + G_{t+1}}{P_t^{g,ex}}$, where P_t^g is the cum-dividend price on the spending claim and $P_t^{g,ex}$ is the ex-dividend price. We log-linearize the return around $z_t = 0$:

$$
r_{t+1}^g = \kappa_0^g + \Delta \log G_{t+1} + \kappa_1^g pg_{t+1} - pg_t. \tag{A.17}
$$

where $pg_t \equiv \log\left(\frac{P_t^{g,ex}}{G_t}\right) = \log\left(\frac{P_t^g}{G_t} - 1\right)$. The unconditional mean log real stock return is $r_0^g = E[r_t^g]$.

We obtain $\overline{pg}$ from the precise valuation formula Eq. (11) at $z_t = 0$. The linearization constants κ_0^g and κ_1^g are defined as:

$$
\kappa_1^g = \frac{e^{\overline{pg}}}{e^{\overline{pg}} + 1} < 1 \;\; \text{and} \;\; \kappa_0^g = \log\left(e^{\overline{pg}} + 1\right) - \frac{e^{\overline{pg}}}{e^{\overline{pg}} + 1}\overline{pg}. \tag{A.18}
$$

Then, the unconditional expected return is:

$$
r_0^g = x_0 + \pi_0 + \kappa_0^g - \overline{pg}(1 - \kappa_1^g).
$$

We conjecture that the log ex-dividend price-dividend ratio on the spending claim is affine in the state vector and verify the conjecture by solving the Euler equation for the claim.

$$
pg_t = \overline{pg} + \bar{B}_g' z_t \tag{A.19}
$$

This allows us to write the return as:

$$
r_{t+1}^g = r_0^g + \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)' z_{t+1} - \bar{B}_g' z_t. \tag{A.20}
$$

Proof. Starting from the Euler equation:

$$
\begin{aligned}
1 &= E_t\left[\exp\{m_{t+1}^\$ + r_{t+1}^g\}\right] \\
&= \exp\{-y_0^\$(1) - e_{yn}' z_t - \frac{1}{2}\Lambda_t'\Lambda_t + r_0^g + [\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)'\Psi - \bar{B}_g']z_t\} \\
&\quad \times E_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\}\right].
\end{aligned}
$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$
\begin{aligned}
1 \;=\; & \exp\{r_0^g - y_0^\$(1) + [(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g)' \Psi - \bar{B}_g' - e_{yn}']z_t \\
& + \underbrace{\frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)' \Sigma \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)}_{Jensen} \\
& - \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t)\}
\end{aligned}
$$

Taking logs and collecting terms, we obtain the following system of equations:

$$
r_0^g - y_0^\$(1) + Jensen = \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)' \Sigma^{\frac{1}{2}} \Lambda_0 \tag{A.21}
$$

and

$$
\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)' \Psi - \bar{B}_g' - e_{yn}' = \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g\right)' \Sigma^{\frac{1}{2}} \Lambda_1 \tag{A.22}
$$

The left-hand side of this equation is the unconditional expected excess log return with Jensen adjustment. The right hand side is the unconditional covariance of the log SDF with the log return. This equation describes the unconditional risk premium on the claim to government spending. Equation (A.22) describes the time-varying component of the government spending risk premium. Given Λ_1, the system of N equations (A.22) can be solved for the vector $\bar{B}_g$:

$$
\bar{B}_g = \left(I - \kappa_1^g \left(\Psi - \Sigma^{\frac{1}{2}} \Lambda_1\right)'\right)^{-1} \left[\left(\Psi - \Sigma^{\frac{1}{2}} \Lambda_1\right)' \left(e_{\Delta g} + e_x + e_\pi\right) - e_{yn}\right]. \tag{A.23}
$$

$\square$

B.4.2 Revenue Claim

Nominal government revenue growth equals

$$
\Delta \log T_{t+1} = \Delta \log \tau_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu_0^\tau + \left(e_{\Delta \tau} + e_x + e_\pi\right)' z_{t+1}. \tag{A.24}
$$

where $\tau_t = T_t/GDP_t$ is the ratio of government revenue to GDP. Note that this ratio is assumed to have a long-run growth rate of zero. This imposes cointegration between government revenue and GDP. The growth ratio in this ratio can only temporarily deviate from zero.

The remaining proof exactly mirrors the proof for government spending, with

$$
p\tau_t \equiv \log\left(\frac{P_t^{\tau,ex}}{T_t}\right) = \log\left(\frac{P_t^\tau}{T_t} - 1\right) = \overline{p\tau} + B_\tau' z_t \tag{A.25}
$$

$$
r_{t+1}^\tau = r_0^\tau + \left(e_{\Delta \tau} + e_x + e_\pi + \kappa_1^g B_\tau\right)' z_{t+1} - B_\tau' z_t, \tag{A.26}
$$

and

$$
r_0^\tau = x_0 + \pi_0 + \kappa_0^\tau - \overline{p\tau}(1 - \kappa_1^\tau).
$$

$$
r_0^\tau - y_0^\$(1) + Jensen = \left(e_{\Delta \tau} + e_x + e_\pi + \kappa_1^\tau B_\tau\right)' \Sigma^{\frac{1}{2}} \Lambda_0. \tag{A.27}
$$

B.5   Pricing in the Presence of Spending Cut

The original state space is

$$
z_t = \Psi z_{t-1} + \Sigma^{\frac{1}{2}} \varepsilon_t, \tag{A.28}
$$

$$z_t = [\pi_t - \pi_0, x_t - x_0, y_t^\$(1) - y_0^\$(1), yspr_t^\$ - yspr_0^\$(1), pd_t - \overline{pd}, \Delta d_t - \mu_d,$$
$$\Delta \log \tau_t - \mu_0^\tau, \Delta \log g_t - \mu_0^g, \log \tau_t - \log \tau_0, \log g_t - \log g_0]'.$$

We augment it with the probability of a spending cut

$$w_t = [z_t; \phi_t], \tag{A.29}$$

and ϕ_t can load on z_{t-1} and ε_t, and the loadings are to be estimated. ϕ_t describes the demeaned probability of a spending cut in time $t+1$, and $\bar\phi$ is the mean. The spending cut is i.i.d. When it happens, the shock to the growth rate of the tax-GDP ratio increases by c times standard deviation. This increase is separate from the ordinary shocks ε_{t+1}. We denote $\tilde\varepsilon_{t+1} = \varepsilon_{t+1} - ce_{\Delta g}$, and $\tilde w_{t+1} = w_{t+1} - \Sigma^{\frac{1}{2}} ce_{\Delta g}$.

We conjecture

$$p_t^g(h) = \log\left(P_t^g(h)\right) = A^g(h) + B^g(h)' w_t,$$

and solve for the coefficients $A^g(h+1)$ and $B^g(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$
\begin{aligned}
P_t^g(h+1) &= \mathbb{E}_t\left[M_{t+1} P_{t+1}^g(h) \frac{G_{t+1}}{G_t}\right] = E_t\left[\exp\{m_{t+1}^\$ + \Delta\log g_{t+1} + x_{t+1} + \pi_{t+1} + p_{t+1}^g(h)\}\right]\\
&= \mathbb{E}_t\left[\exp\{-y_{t,1}^\$ - \tfrac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + \mu^g + x_0 + \pi_0 + A^g(h) + (e_{\Delta g}' + e_x' + e_\pi' + B^g(h)')w_{t+1}\}\right]\\
&= \mathbb{E}_t\Big[(1-\phi_t)\exp\{-y_{t,1}^\$ - \tfrac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + \mu^g + x_0 + \pi_0 + A^g(h) + (e_{\Delta g}' + e_x' + e_\pi' + B^g(h)')w_{t+1}\}\\
&\quad + \phi_t\exp\{-y_{t,1}^\$ - \tfrac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\tilde\varepsilon_{t+1} + \mu^g + x_0 + \pi_0 + A^g(h) + (e_{\Delta g}' + e_x' + e_\pi' + B^g(h)')\tilde w_{t+1}\}\Big]\\
&= \exp\{-y_0^\$(1) - e_{yn}'w_t - \tfrac{1}{2}\Lambda_t'\Lambda_t + \mu^g + x_0 + \pi_0 + (e_{\Delta g} + e_x + e_\pi + B^g(h))'\Psi w_t + A^g(h)\}\\
&\quad \times \mathbb{E}_t\Big[\exp\{(-\Lambda_t' + (e_{\Delta g} + e_x + e_\pi + B^g(h))'\Sigma^{\frac{1}{2}})\varepsilon_{t+1}\}\\
&\quad (1-\phi_t + \phi_t\exp\{((e_{\Delta g} + e_x + e_\pi + B^g(h))'\Sigma^{\frac{1}{2}})(-ce_{\Delta g})\})\Big]
\end{aligned}
$$

For small ϕ_t,

$$
\log(1 - \phi_t + \phi_t\exp\{(e_{\Delta g} + e_x + e_\pi + B^g(h))'\Sigma^{\frac{1}{2}}(-ce_{\Delta g})\}) \approx
$$
$$
-\phi_t\left(1 - \exp\{(e_{\Delta g} + e_x + e_\pi + B^g(h))'\Sigma^{\frac{1}{2}}(-ce_{\Delta g})\}\right)
$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$
\begin{aligned}
p_t^g(h+1) &= -y_0^\$(1) + \mu^g + x_0 + \pi_0 + ((e_{\Delta g} + e_x + e_\pi + B^g(h))'\Psi - e_{yn}')w_t + A^g(h)\\
&\quad + \tfrac{1}{2}(e_{\Delta g} + e_x + e_\pi + B^g(h))'\Sigma(e_{\Delta g} + e_x + e_\pi + B^g(h))\\
&\quad - (e_{\Delta g} + e_x + e_\pi + B^g(h))'\Sigma^{\frac{1}{2}}(\Lambda_0 + \Lambda_1 w_t)\\
&\quad - \left(1 - \exp\{-(e_{\Delta g} + e_x + e_\pi + B^g(h))'\Sigma^{\frac{1}{2}}ce_{\Delta g}\}\right)(\bar\phi + e_\phi'w_t)
\end{aligned}
$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^g(h+1)$:

$$p_t^g(h+1) = A^g(h+1) + B^g(h+1)' w_t,$$

where:

$$
\begin{aligned}
A^g(h+1) &= -y_0^\$(1) + \mu^g + x_0 + \pi_0 + A^g(h) + \tfrac{1}{2}(e_{\Delta g} + e_x + e_\pi + B^g(h))'\Sigma(e_{\Delta g} + e_x + e_\pi + B^g(h))\\
&\quad - (e_{\Delta g} + e_x + e_\pi + B^g(h))'\Sigma^{\frac{1}{2}}\Lambda_0 - \left(1 - \exp\{-(e_{\Delta g} + e_x + e_\pi + B^g(h))'\Sigma^{\frac{1}{2}}ce_{\Delta g}\}\right)\bar\phi,
\end{aligned}
$$

$$B^g(h+1)' = (e_{\Delta g} + e_x + e_\pi + B^g(h))' \Psi - e_{yn}' - (e_{\Delta g} + e_x + e_\pi + B^g(h))' \Sigma^{\frac{1}{2}} \Lambda_1$$
$$- \left(1 - \exp\{-(e_{\Delta g} + e_x + e_\pi + B^g(h))' \Sigma^{\frac{1}{2}} c e_{\Delta g}\}\right) e_\phi'.$$

Then the price of the cum-dividend spending claim is

$$G_t \sum_{h=0}^{\infty} \exp(p_t^g(h))$$

We also obtain the cum-dividend tax claim from our previous formula

$$T_t \sum_{h=0}^{\infty} \exp(p_t^\tau(h)) = T_t \sum_{h=0}^{\infty} \exp(A^\tau(h) + B^\tau(h)' w_t)$$

with

$$A^\tau(h+1) = -y_0^\$(1) + \mu^\tau + x_0 + \pi_0 + A^\tau(h) + \frac{1}{2} (e_{\Delta\tau} + e_x + e_\pi + B^\tau(h))' \Sigma (e_{\Delta\tau} + e_x + e_\pi + B^\tau(h))$$
$$- (e_{\Delta\tau} + e_x + e_\pi + B^\tau(h))' \Sigma^{\frac{1}{2}} \Lambda_0 - \left(1 - \exp\{-(e_{\Delta\tau} + e_x + e_\pi + B^\tau(h))' \Sigma^{\frac{1}{2}} c e_{\Delta g}\}\right) \bar{\phi},$$
$$B^\tau(h+1)' = (e_{\Delta\tau} + e_x + e_\pi + B^\tau(h))' \Psi - e_{yn}' - (e_{\Delta\tau} + e_x + e_\pi + B^\tau(h))' \Sigma^{\frac{1}{2}} \Lambda_1$$
$$- \left(1 - \exp\{-(e_{\Delta\tau} + e_x + e_\pi + B^\tau(h))' \Sigma^{\frac{1}{2}} e_{\Delta g}\}\right) c e_\phi'.$$

Then we can back out ϕ by setting the different in the value of tax and spending claim to the valuation of the government debt. Under this framework, we can run a Kalman filter to find the best-fitting parameters that govern the dynamics of ϕ.

C   Coefficient Estimates

C.1   The VAR System

The Cholesky decomposition of the residual variance-covariance matrix, $\Sigma^{\frac{1}{2}}$, multiplied by 100 for readability is given by:

$$100 \times \Sigma^{\frac{1}{2}} = \begin{pmatrix} 0.36 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.03 & 0.88 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.02 & 0.04 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.00 & -0.01 & -0.07 & 0.09 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.75 & 0.84 & -1.14 & -0.06 & 8.04 & 0 & 0 & 0 & 0 & 0 \\ 0.13 & 0.08 & 0.03 & -0.16 & -0.33 & 1.94 & 0 & 0 & 0 & 0 \\ 0.77 & 1.13 & 0.18 & -0.22 & 0.25 & 0.03 & 4.10 & 0 & 0 & 0 \\ -0.26 & -0.86 & -0.34 & 0.17 & 0.24 & -0.39 & -0.16 & 3.85 & 0 & 0 \\ 0.77 & 1.13 & 0.18 & -0.22 & 0.25 & 0.03 & 4.10 & 0.00 & 0.00 & 0 \\ -0.26 & -0.86 & -0.34 & 0.17 & 0.24 & -0.39 & -0.16 & 3.85 & 0.00 & 0.00 \end{pmatrix}$$

In this matrix, the last two columns are all zero. This is because the dependent variables $\log \tau_t - \log \tau_0$ and $\log g_t - \log g_0$ do not have independent shocks. For example, $\log \tau_t - \log \tau_0$ can be expressed as

$$\begin{aligned} \log \tau_t - \log \tau_0 &= \Delta \log \tau_t + (\log \tau_{t-1} - \log \tau_0) \\ &= (e'_{\Delta \tau} \mathbf{\Psi} + e'_\tau) z_{t-1} + e'_{\Delta \tau} \Sigma^{\frac{1}{2}} \varepsilon_t, \end{aligned}$$

which loads on the first 8 shocks in the same way as $\Delta \log \tau_t - \mu_0^g$.

C.2   Market Prices of Risk

C.2.1   Parameter Estimates

Imposing cointegration requires us to include the levels of $\log \tau$ and $\log g$ in our vector of state variables. In summary, the VAR variables are:

$$\begin{aligned} z_t &= [\pi_t - \pi_0, x_t - x_0, y_t^{\$}(1) - y_0^{\$}(1), yspr_t^{\$} - yspr_0^{\$}(1), pd_t - \overline{pd}, \Delta d_t - \mu_d, \\ &\quad \Delta \log \tau_t - \mu_0^\tau, \Delta \log g_t - \mu_0^g, \log \tau_t - \log \tau_0, \log g_t - \log g_0]'. \end{aligned}$$

The constant market price of risk vector is estimated at:

$$\Lambda_0' = [0.01, 0.33, -0.53, 0.11, 0, 0.60, 0, 0, 0]$$

We impose the restriction that innovations to spending or revenue are not priced. The time-varying market price of risk matrix is estimated at:

$$\Lambda_1 = \begin{bmatrix} 33.47 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -10.53 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -27.45 & -171.15 & 0 & 0 & 0 & 0 & 0 & 0 \\ 37.22 & 34.51 & -15.86 & -65.40 & 0 & 0 & 0 & 0 & -0.11 & 0.13 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 36.14 & 12.03 & -45.44 & -103.91 & -0.97 & 6.77 & 3.61 & 0.28 & -0.17 & 0.33 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C.2.2 Identification

We allow for first four elements of Λ_0 to be non-zero. The market price of the first shock, the inflation shock, will partly capture a shock to expected inflation given the persistence of inflation. Movements in expected inflation are a key determinants of parallel shifts in the term structure of interest rates. i.e., they are the main driver of level of the term structure. Since inflation is usually bad news to the representative agent, we expect a negative price of risk for this shock. Shocks to GDP growth affect the slope of the term structure. They affect long rates more than short rates. We expect a positive price of risk since positive innovations to GDP growth are good news. The third risk price $\Lambda_0(3)$ is the price of risk for a shock to the interest rate that is orthogonal to inflation and GDP growth shocks. As in the classic term structure models of Vasicek and Cox, Ingersoll, and Ross, we expect this risk price to be negative. We expect the shock to the yield spread that is orthogonal to the preceding three shocks to carry a positive risk price $\Lambda_0(4)$, as positive slopes indicate improving economic conditions. This risk price helps the model match the average slope of the term structure. To keep the model tractable, we shut down direct feedback from tax and spending innovations onto the SDF: these innovations are assumed to be non-priced.

We allow for ten non-zero elements in the first four rows (term structure block) of Λ_1, which describes the dynamics in the risk prices. We let the price of inflation risk depend on the level of inflation to capture that periods like the late 1970s, early 1980s may have had elevated inflation risk. We let the price of GDP growth risk depend on the level of GDP growth. We let the price of short rate risk depend on the short rate as well as the term spread. The first dependence is a feature of the Vasicek and Cox, Ingersoll model, for example. The second dependence captures that the slope of the term structure predicts higher future returns on bonds (Capmbell and Shiller). We also need six non-zero elements in the fourth row of Λ_1 in order to allow the model to closely match the dynamics of the slope of the term structure, which is one of the variables included in the VAR. The dynamics of the five-year bond yield must satisfy (6). Given the first three rows of Λ_1, satisfying these conditions requires that the first four elements of the fourth row of Λ_1 all be no-zero.

Lastly, we set all elements in the sixth row of Λ_1 to be non-zero, so that we have enough degrees of freedom to fit (A.13).

D   Cointegration Tests

We run the Johansen cointegration test with the auxiliary specification

$$\Delta w_t \;\; = \;\; A(B'w_{t-1} + c) + D\Delta w_{t-1} + \varepsilon_t, \quad \text{where } w_t = \begin{pmatrix} \log T_t \\ \log G_t \\ \log GDP_t \end{pmatrix}.$$

Both trace and max eigenvalue tests do not reject the null of cointegration rank 1 or 2, but reject the null of cointegration rank 0. In other words, there are at least one cointegration relationship between variables in w_t.

We also conduct the Phillips-Ouliaris cointegration test and reject the null hypothesis that w is not cointegrated with a p value of 0.030 when the truncation lag parameter is 2, or a p value of 0.011 when the truncation lag parameter is 9.

E   Model Fit

Figure A.1 shows that the model matches the time series of bond yields in the data closely.

The top panels of Figure A.2 show the model's implications for the average nominal (left panel) and real (right panel) yield curves at longer maturities. These yields are well behaved, with very long-run nominal (real) yields stabilizing at around 6.60% (3.10%) per year. We impose conditions that ensure that the nominal and real term structure are well behaved at very long maturities, for which we have no data. Specifically, we impose that average nominal (real) yields of bonds with maturities of 600, 800, 1000, 2000, 3000, and 4000 quarters remain above 6.24% (3.05%) per year, which is the long-run nominal (real) GDP growth rate $4x_0 + 4\pi_0$ ($4x_0$) observed in our sample. Second, we impose that nominal yields remain above real yields plus 3.19% expected inflation at those same maturities. This imposes that the inflation risk premium remain positive at very long maturities. Third, we impose that the nominal and real term structures of interest rates flatten out, with an average yield difference between 400 and 200 quarter yields that must not exceed 2% per year and between 1000 and 600 quarters that must not exceed 1% per year. These restrictions are satisfied at the optimum.

The bottom left panel of Figure A.2 shows that the model matches the dynamics of the nominal bond risk premium, defined as the expected excess return on the five-year nominal bond, quite well. Bond risk premia decline in the latter part of the sample,

Figure A.1: Dynamics of the Nominal Term Structure of Interest Rates

The figure plots the observed and model-implied 1-, 4-, 8-, 20-, 40-, and 120-quarter nominal bond yields. Data are from FRED and FRASER.
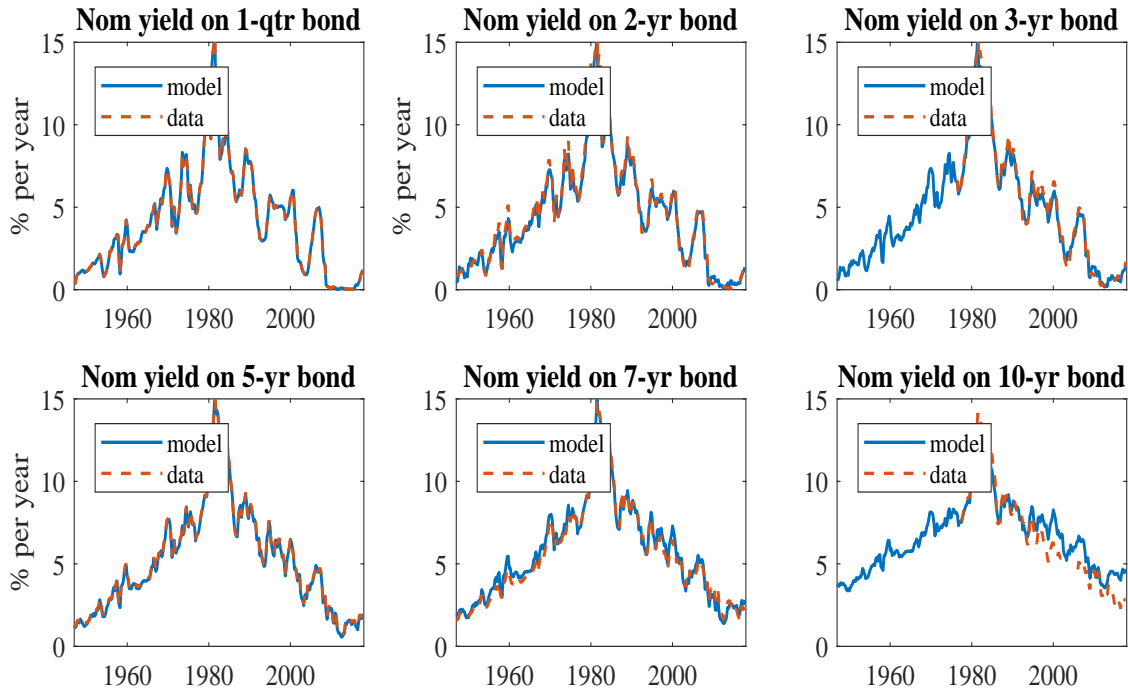

Figure A.2: Long-term Yields and Bond Risk Premia

The top panels plot the average bond yield on nominal (left panel) and real (right panel) bonds for maturities ranging from 1 quarter to 1000 quarters. Yields are annualized. The bottom left panel plots the nominal bond risk premium on the five year bond in model and data. The bottom right panel decomposes the model's five-year nominal bond yield into the five-year real bond yield, the five-year inflation risk premium and the five-year real risk premium.
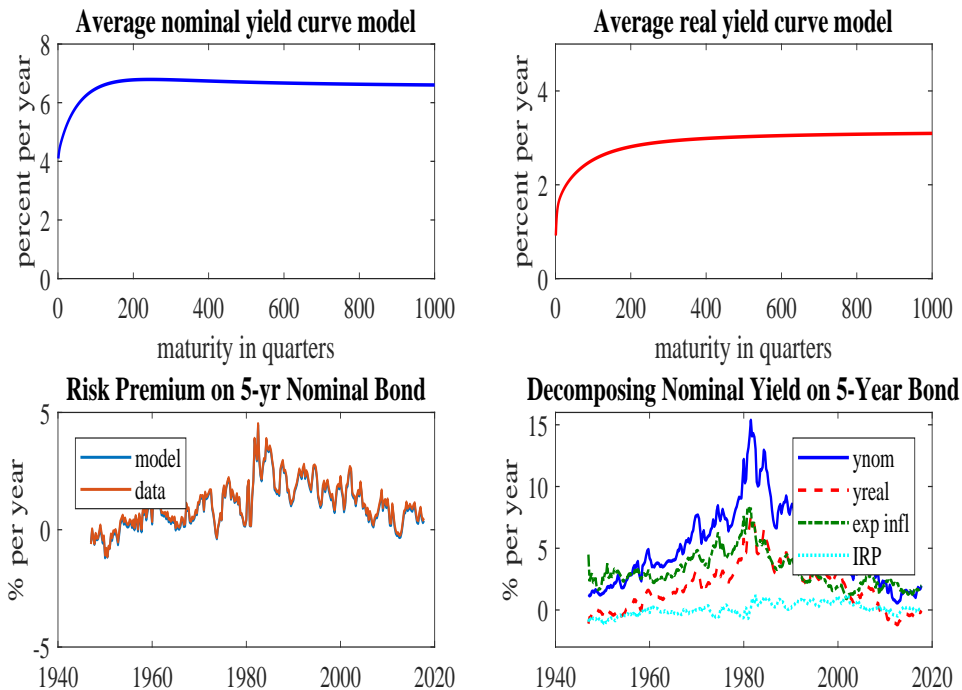
possibly reflecting the arrival of foreign investors who value U.S. Treasuries highly. The bottom right panel shows a decomposition of the nominal bond yield on a five-year bond into the five-year real bond yield, annual expected inflation inflation over the next five years, and the five-year inflation risk premium. On average, the 5.1% nominal bond yield is comprised of a 1.9% real yield, a 3.2% expected inflation rate, and a 0.1% inflation risk premium. The graph shows that the importance of these components fluctuates over time.

Figure A.3 shows the equity risk premium, the expected excess return, in the left panel and the price-dividend ratio in the right panel. The risk premia in the data are the expected equity excess return predicted by the VAR. Their dynamics are sensible, with low risk premia in the dot-com boom of 1999-2000 and very high risk premia in the Great Financial Crisis of 2008-09. Equity risk premia are multiplied by 4 to express them as annual quantities. The VAR-implied quarterly equity risk premium occasionally turns negative. The model-implied one rarely does. We could impose further restrictions on the variables that drive time-variation in expected excess stock returns to limit the in-sample presence of negative equity risk premia. We expect this will make little difference for our results. The figure's right panel shows a tight fit for equity price levels. We conclude that the model captures the observed prices and returns on stocks and bonds well.

Figure A.3: Equity Risk Premium and Price-Dividend Ratio

The figure plots the observed and model-implied equity risk premium on the overall stock market in the left panel and the price-dividend ratio in the right panel. The quarterly equity risk premium in model and data is multiplied by 400 to express it as an annual percentage number. The price-dividend ratio is the price divided by the annualized dividend.
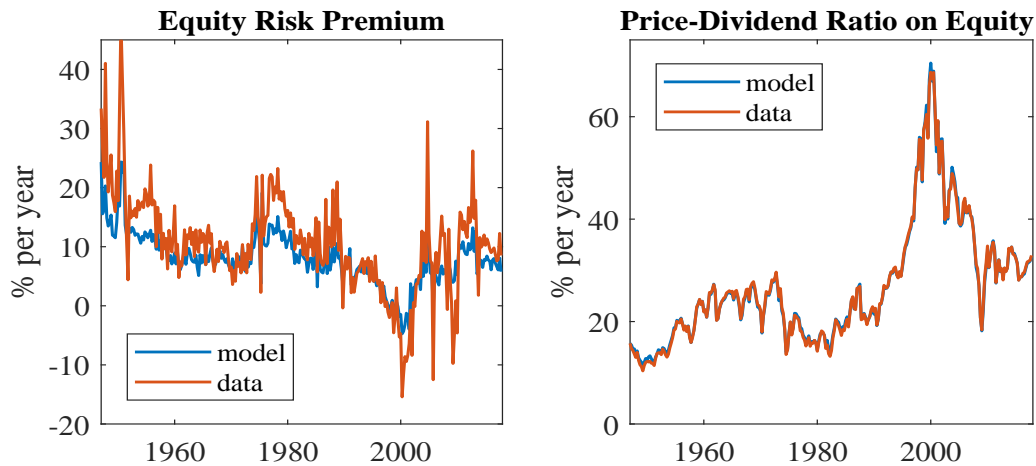
F   Quantitative Results

Figure A.4 and Figure 7 plots the price/dividend ratios of the revenue and spending claims without and with cointegration.

Figure A.4: Government Cash Flows and Prices, No Cointegration

The top panels plot the (cum-dividend) price-dividend ratio on the claim to tax revenues (left) and government spending (middle). Both are annualized (divided by 4). The bottom left panel plots the value of a claim to future tax revenue, scaled by GDP. The middle panel plots the value of a claim to future government spending divided by GDP. The bottom right panel plots the value of future government surpluses scaled by GDP.
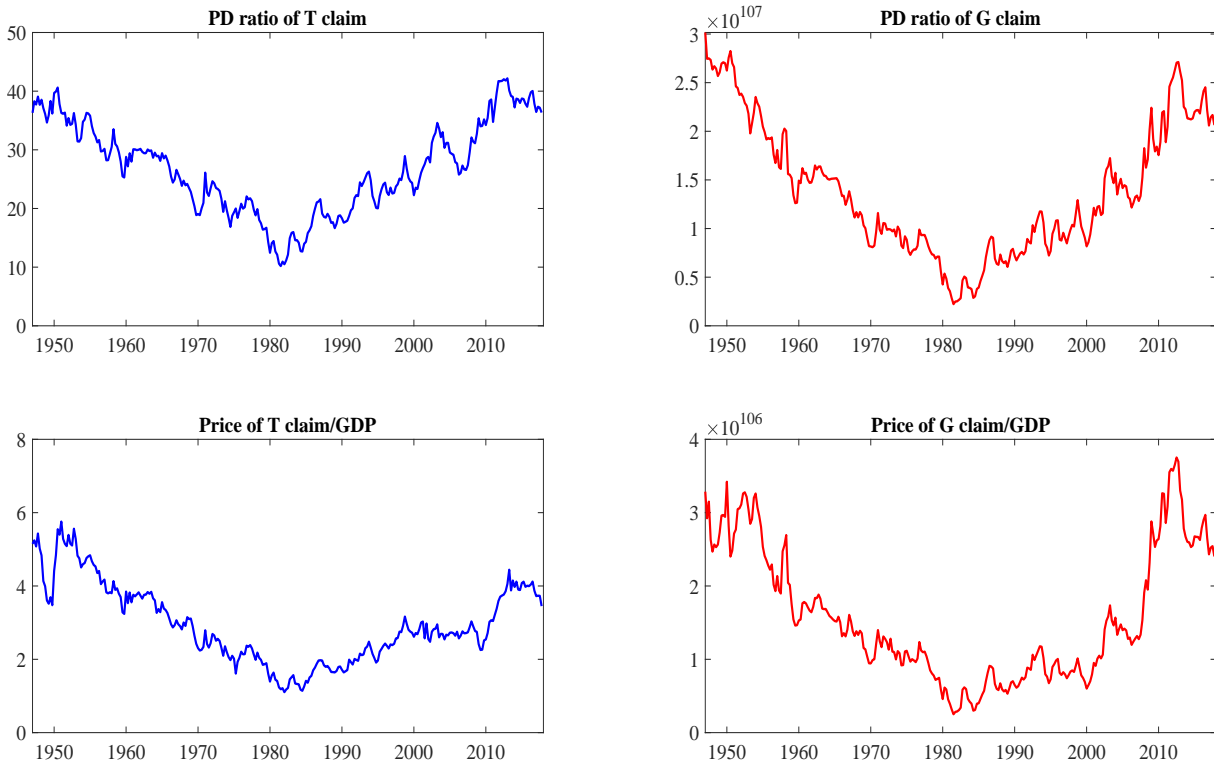
Figure A.5 plots the violations of the measurability constraints.

Finally, A.6 shows the results for a VAR model without a stock market block.

Figure A.5: Violations of the Measurability Constraint With Only One-Period Debt

The figure shows the time series of $\tau_t \sum_{h=0}^{\infty} PD_t^{\tau}(h) \left(e_{\Delta\tau}' \Sigma^{\frac{1}{2}} + B_{\tau}'(h) \right) - g_t \sum_{h=0}^{\infty} PD_t^g(h) \left(e_{\Delta g}' \Sigma^{\frac{1}{2}} + B_g'(h) \right) - 0$, i.e. the violation of the Measurability Constraint, for each shock. They are expressed as a percentage of U.S. annual GDP so that 1 means the surplus claim increases by 1% of the U.S. annual GDP when a certain shock is 1%.
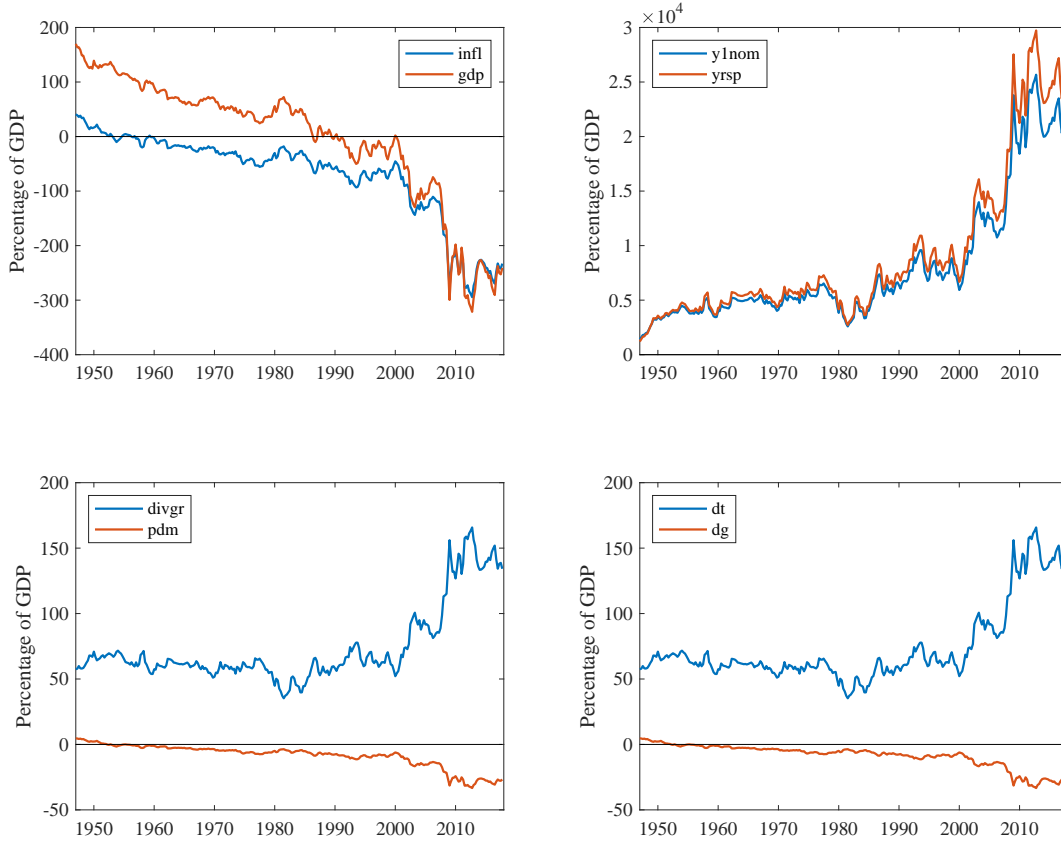
Figure A.6: Present Value of Government Surpluses with the Bond Market SDF
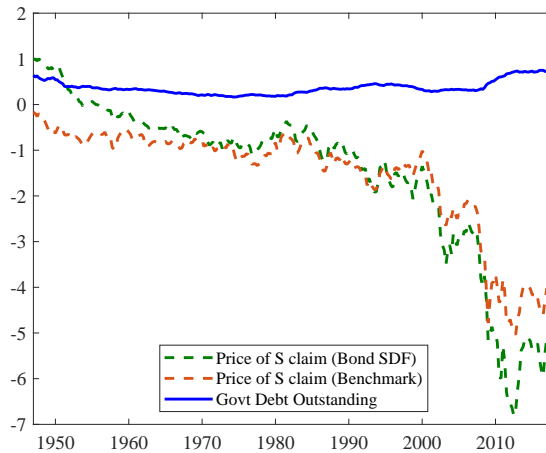
G Government Risk Management

The peso problem formulation has the virtue that it reconciles the market value of outstanding government debt with the present value of future surpluses. With the government budget constraint holding with equality, we can consider the question of optimal government debt portfolio management. As shown by Bhandari, Evans, Golosov, and Sargent (2017), optimal maturity choice in a large class of models amounts to minimizing the variance of government funding needs.

Suppose there is only nominal debt, and let $Q_{t,h}^\$$ denote the outstanding nominal bond quantity at $t+1$ chosen at time t. Then, the funding need at $t+1$ is defined as the time $t+1$ value of its time-t bond portfolio minus the expected discounted value of its future surpluses, holding fixed the government's tax and spending policy rules:

$$Need_{t+1} = \sum_{h=0}^{\infty} P_{t+1}^\$(h) Q_{t,h+1}^\$ - \mathbb{E}_{t+1}\left[\sum_{j=0}^{\infty} M_{t+1,t+1+j}^\$ S_{t+1+j}\right]. \tag{A.30}$$

A positive funding need means that the government faces a funding shortfall and has to issue additional debt at time $t+1$.

Under this formulation, the government's risk management problem amounts to choosing its debt issuance policy along the maturity curve to minimize the variance of its funding needs:

$$\min_{\{Q_{t+1,h}^\$\}} Var_t[Need_{t+1}] \tag{A.31}$$

To develop some intuition, we start by considering a portfolio that is locally immunized against the interest rate shock to $y_{t+1}^\$(1)$. We define this shock as the realization of the shocks ε_{t+1} at time $t+1$ such that $\Sigma^{\frac{1}{2}}\varepsilon_{t+1}$ is zero except the row corresponding to the 3-month interest rate. Recall that the state transition equation is $z_{t+1} = \Psi z_t + \Sigma^{\frac{1}{2}}\varepsilon_{t+1}$. So, if this interest rate shock is 1%, the 1-quarter interest rate becomes:

$$y_{t+1}^\$(1) = y_0^\$(1) + e_{yn}'\Psi z_t + 1\%,$$

while all of the other state variables evolve according to $z_{t+1} = \Psi z_t$.

Definition 1. A government bond portfolio $\{Q_{t+1,h}^*\}$ is locally hedged against a change in short-term interest rates if the dollar value of the surplus claim adjusts by the same amount as the dollar value of the outstanding bond portfolio in response to an interest rate shock:

$$\sum_{h=0}^{\infty} Q_{t+1,h}^* \frac{-\partial P_{t+1}^\$(h)}{\partial y_{t+1}^\$(1)} = \frac{-\partial \mathbb{E}_{t+1}\left[\sum_{j=0}^{\infty} M_{t+1,t+1+j}^\$ S_{t+1+j}\right]}{\partial y_{t+1}^\$(1)} \tag{A.32}$$

We refer to these derivatives above as dollar durations. Note they are the dollar value changes with respect to the 3-month interest rate, *holding the shocks to other state variables to be zero*. An increase in the nominal yield will decrease the value of the bond portfolio and the surplus claim. The negative signs turn the dollar durations into positive numbers. Rewritten in percentages, they become elasticities:

$$\frac{\sum_{h=0}^{\infty} P_t^\$(h) Q_{t+1,h}^* \frac{-\partial \log P_{t+1}^\$(h)}{\partial y_{t+1}^\$}}{\sum_{h=0}^{\infty} P_t^\$(h) Q_{t+1,h}^*} = \frac{\mathbb{E}_t\left[\sum_{j=1}^{\infty} M_{t,t+j}^\$ S_{t+j}\right] \frac{-\partial \log \mathbb{E}_{t+1}\left[\sum_{j=0}^{\infty} M_{t+1,t+1+j}^\$ S_{t+1+j}\right]}{\partial y_{t+1}^\$}}{\mathbb{E}_t\left[\sum_{j=1}^{\infty} M_{t,t+j}^\$ S_{t+j}\right]}$$

where $-\partial \log P_{t+1}^\$(h)/\partial y_{t+1}^\$$ is the modified duration of the zero coupon bond.

The left panel of Figure A.7 shows the dollar duration of the tax and spending claims, expressed as a percent of annual GDP. The right panel shows the dollar duration of government surpluses, the difference between the two lines in the left panel, and that of the actual government bond portfolio. The dollar duration of the outstanding government debt portfolio grows in recent years in part because the amount of debt outstanding grows. The modified durations in the middle panel take out this size-of-debt effect. The modified duration of the government surplus claim is positive, like that of the debt portfolio, but substantially larger at various times during the sample. For example, in the mid-2000s, the government debt portfolio has a modified duration of 3 years whereas the surplus claim has a modified duration of 8 years. Most of the time, the actual government bond portfolio insufficiently hedges the interest rate risk of the surplus claim.

We can now generalize the notion of duration to other shocks. Minimize the variability of funding needs requires that the government's bond portfolio should also hedge those shocks.

Figure A.7: The Durations of the Government Bond Portfolio and Government Surpluses

The figure plots the duration of the government bond portfolio and that of government surpluses, weighted by annual GDP. The series is quarterly from 1940.Q1 until 2017.Q4. Data Source: CRSP U.S. Treasury Database and BEA.
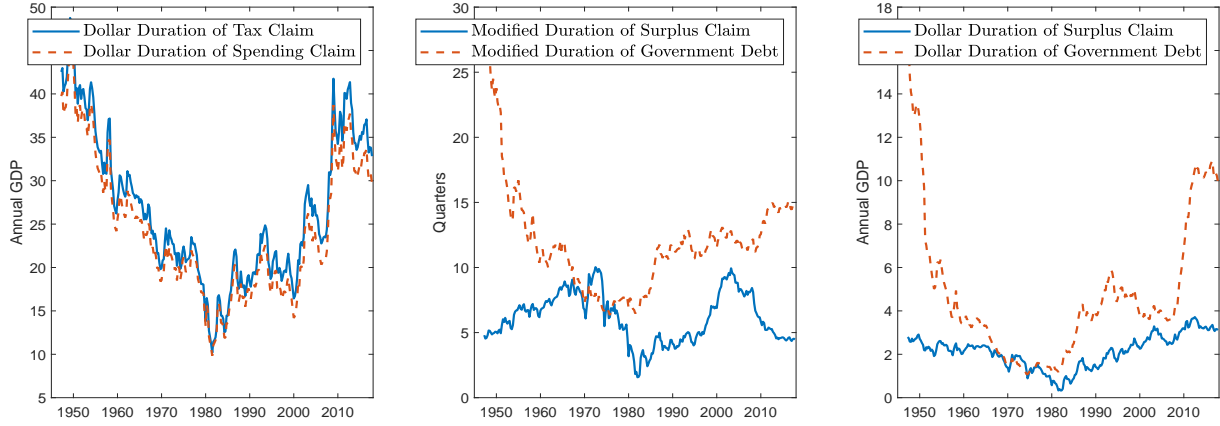
Definition 2. A government bond portfolio $\{Q^*_{t+1,h}\}$ is fully immunized against any shock to the state of the economy provided that the changes in value of the outstanding bond portfolio equal the changes in the value of the surplus claim for each shock:

$$\sum_{h=0}^{\infty} Q^*_{t+1,h} \frac{\partial P^{\$}_{t+1}(h)}{\partial z_{t+1}} = \frac{\partial \mathbb{E}_{t+1}\left[\sum_{j=0}^{\infty} M^{\$}_{t+1,t+1+j} S_{t+1+j}\right]}{\partial z_{t+1}} \tag{A.33}$$

Equation (A.33) describes a system of N equations, where N is the number of state variables (shocks) in the VAR. Appendix J contains the details. If the government can issue N different bond maturities, it is possible to fully immunize its debt portfolio against all economic shocks. The immunization conditions look similar to the measurability conditions discussed in section 5.5. However, note that we are imposing the government budget constraint here through the peso model assumption.

Our objective is to evaluate how far the actual government debt portfolio (maturity structure) deviates from full immunization. To measure the deviation, we express the conditional standard deviation of the funding needs as:

$$\sqrt{Var_t\left[Need_{t+1}\right]} = \left(Var_t\left[\sum_{h=0}^{\infty} P^{\$}_{t+1}(h) Q^{\$}_{t,h} - P^{\tau}_{t+1} + P^{g}_{t+1}\right]\right)^{1/2} = (U_t \Sigma U'_t)^{1/2} \tag{A.34}$$

$$\text{where } U_t = \left(\sum_{h=0}^{\infty} Q^{\$}_{t,h} \frac{\partial P^{\$}_{t+1}(h)}{\partial z_{t+1}} - \frac{\partial \mathbb{E}_{t+1}\left[\sum_{j=0}^{\infty} M^{\$}_{t+1,t+1+j} S_{t+1+j}\right]}{\partial z_{t+1}}\right)$$

The left plot in Figure A.8 reports this standard deviation. In 2017.Q3, the standard deviation of the government's funding need is around 3% of the annual GDP. This is a sizeable deviation from full immunization. The right plot decomposes this standard deviation to the component $e_i U_t \Sigma^{\frac{1}{2}}$ driven by each shock i. By Eq. (A.34), the square root of the sum of their squares is the total variation in the funding need:

$$\sqrt{Var_t\left[Need_{t+1}\right]} = \left(\sum_i (e_i U_t \Sigma^{\frac{1}{2}})^2\right)^{1/2}$$

This calculation suggests that the deviations from full immunization arise mainly from failure to hedge exposure to the spending cut probability and the stock price-to-dividend ratio.

Figure A.8: Conditional Standard Deviation of Funding Needs

The left figure plots the standard deviation of the funding needs, $(Var_t[Need_{t+1}])^{1/2}$, scaled by the annual GDP. The right figure decomposes the standard deviation to the variation due to each shock. Data Source: CRSP U.S. Treasury Database and BEA.
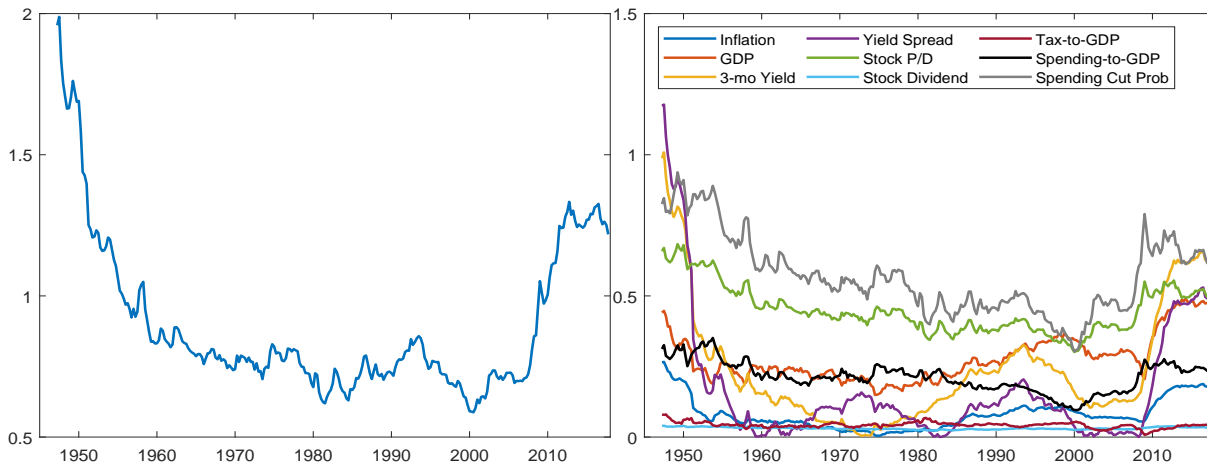
H   Data Appendix

H.1   Primary Surpluses

The primary surpluses are constructed using NIPA Table 3.2 Federal Government Current Receipts and Expenditures from 1947.Q1 to 2017.Q4. All variables are seasonally adjusted.

The government revenue is the sum of the corporate and personal tax revenue, the net income from the rest of the world, and the federal government dividends income receipts on assets. The personal tax revenue is the total of the current personal tax receipts, the tax revenue from production and imports, the net income from the rest of the world, and surpluses from government-sponsored enterprise net of subsidies. The net income from the rest of the world includes the tax income from the rest of the world, the contributions from government social insurance from the rest of the world, the current transfer receipts from the rest of the world, net of the government transfer payments to the rest of the world and the interest payments to the rest of the world.

The government spending is the domestic net transfer payments before interest payments plus discretionary spending (i.e. consumption expenditures). The domestic net transfer is the domestic current transfer receipts net of the domestic contribution from government social insurances and the domestic current transfer receipts.

The primary surpluses are the government revenue minus the government spending before interest payments.

H.2   State Variables

We obtain the time series of GDP from NIPA Table 1.1.5, and GDP price index from NIPA Table 1.1.4. The real GDP growth x_t is the real GDP growth per capita. The Treasury yields for all maturities are constant maturity yields from Fred. The log-price-dividend ratio and the log real dividend growth are computed using CRSP database. Dividends are seasonally adjusted and quarterly. We include the growth of both the government revenue to GDP ratio and the government spending to GDP ratio in the state vector. The government revenue and government spending are defined in Section 1.

H.3   Other Measures of the Convenience Yield

In this section, we compare our measure of the convenience yield with the implied convenience yields from van Binsbergen, Diamond, and Grotteria (2019). Figure A.9 shows the 6-month, 12-month, and 18-month convenience yields from van Binsbergen, Diamond, and Grotteria (2019), which are spreads between the SPX option implied interest rates and government bond rates with corresponding maturities. All measures of the convenience yield exhibit similar time-series patterns over the sample period from 2004-01 to 2017-04.

Figure A.9: Measures of the Convenience Yield

The figure shows the time series of different measures of the convenience yield. The dashed blue line is the spread of 6-month zero coupon interest rates implied from SPX options with 6-month Treasury bill rate. The dotted red line is the spread of 12-month zero coupon interest rates implied from SPX options with 12-month Treasury bill rate. The dashed yellow line is the spread of 18-month zero coupon interest rates implied from SPX options with 18-month Treasury bond rate. The data is from van Binsbergen, Diamond, and Grotteria (2019). The solid black line is the weighted average of the Aaa-Treasury yield spread and the high-grade commercial papers-bills yield spread. All yields are in the quarter frequency, and expressed in percentage per annum. The sample period is from

2014-01 to 2017-04.
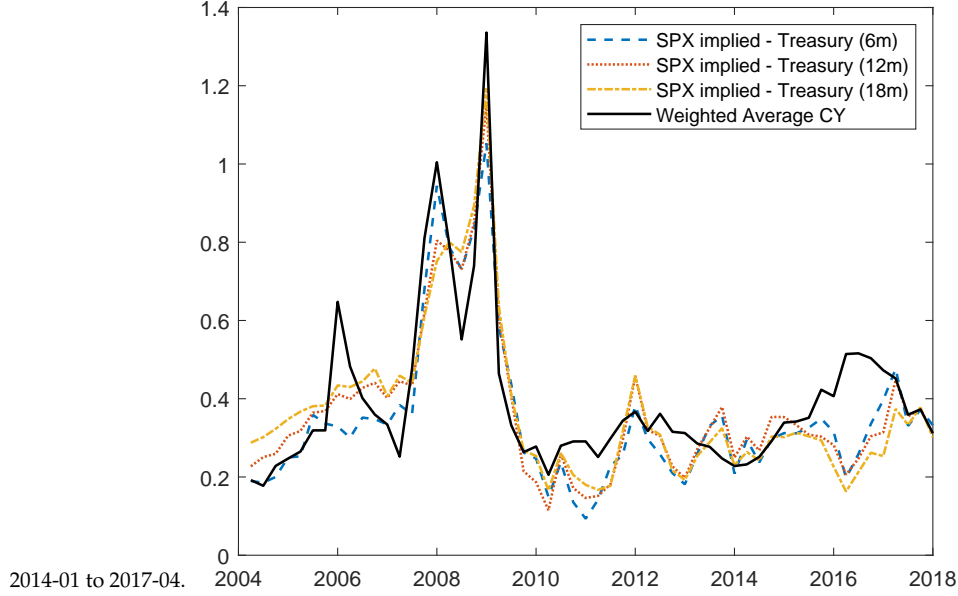
I Local Projection

Figure 5 plots the impulse response of the tax revenue-to-GDP ratio ($\log \tau_t$) to a $\Delta \log \tau_t$ shock and the impulse response of the government spending-to-GDP ratio ($\log g_t$) to a $\Delta \log g$ shock. The $\Delta \log \tau_t$ shock is defined as the shock that increases $\Delta \log \tau_t$ by the standard deviation of its VAR residuals. By definition, it also raises the level $\log \tau_t$ by the same amount, but it does not affect the GDP growth rate x_t.

The blue curves represent the results under the benchmark VAR system. For example, the $\Delta \log \tau_t$ shock raises the tax-to-GDP ratio $\log \tau_t$ on impact. Then, as the tax-to-GDP ratio is above the long-run average, its growth rate $\Delta \log \tau_t$ adjusts downward, leading to a reversion in the level.

For comparison, the red curves represent the results under a restricted VAR. In this case, the first 8 state variables do not load on the cointegration variables $\log \tau_t$ and $\log g_t$. Then, the impact of the $\Delta \log \tau_t$ shock and the $\Delta \log g_t$ shock is permanent. For example, a positive $\Delta \log \tau_t$ shock raises the tax-to-GDP ratio $\log \tau_t$ permanently.

In order to validate our VAR dynamics, we run local projection (Jordà, 2005) of future government tax-to-GDP ratio τ_{t+h} and government spending-to-GDP ratio g_{t+h} on current state variables. The regression equations are

$$\tau_{t+h} \quad = \quad \alpha + \beta_\tau \tau_t + \gamma_\tau g_t + \varepsilon_{t+h}, \tag{A.35}$$

$$g_{t+h} \quad = \quad \alpha + \beta_g \tau_t + \gamma_g g_t + \varepsilon_{t+h}. \tag{A.36}$$

We report β_τ and γ_g as functions of the forecast horizon h in Figure A.10 below. The error bands are from Newey-West standard errors with 8 lags. The coefficient estimates imply that a higher level of government tax-to-GDP ratio predicts a lower growth in the tax-to-GDP ratio in the next 8 quarters, while a higher level of government spending-to-GDP ratio predicts a lower growth in the spending-to-GDP ratio in the next 16 quarters. In other words, the tax and spending processes are mean-reverting, with the long-run responses tending towards the average level.
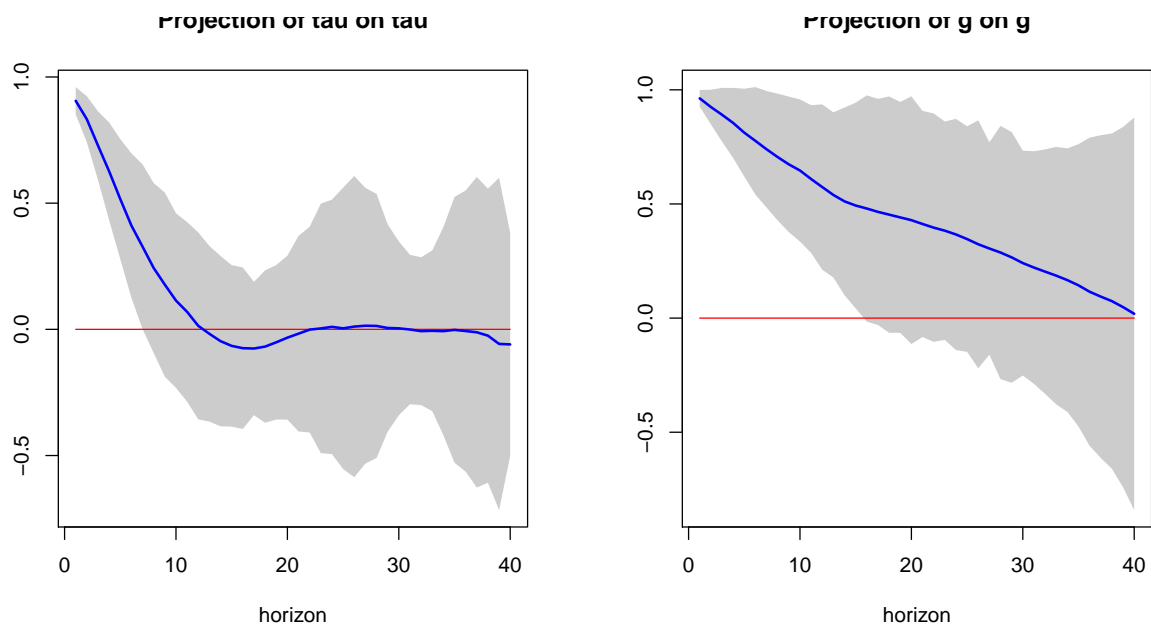
69

Figure A.10: Local Projection of τ_{t+h} and g_{t+h}.

Now we also control for the value of outstanding debt:

$$\tau_{t+h} \quad = \quad \alpha + \beta_\tau \tau_t + \gamma_\tau g_t + \theta_1 b_t + \varepsilon_{t+h} \tag{A.37}$$

$$g_{t+h} \quad = \quad \alpha + \beta_g \tau_t + \gamma_g g_t + \theta_1 b_t + \varepsilon_{t+h} \tag{A.38}$$

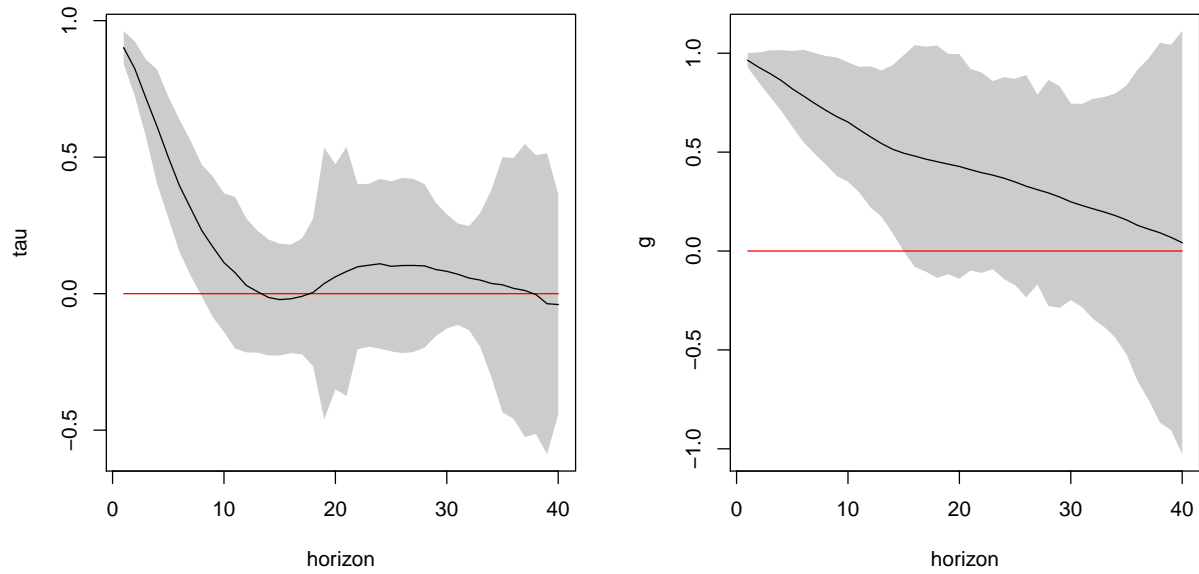We report β_τ and γ_g as functions of the forecast horizon h below:

Figure A.11: Local Projection of τ and g.

Now we also control for other state variables:

$$\tau_{t+h} = \alpha + \beta_\tau \tau_t + \gamma_\tau g_t + \theta_1 b_t + \theta_2 \pi_t + \theta_3 x_t + \theta_4 y_t^{(1)} + \theta_5 y_t^{(20-1)} + \theta_6 \Delta d_t + \theta_7 dp_t + \varepsilon_{t+h} \tag{A.39}$$

$$g_{t+h} = \alpha + \beta_g \tau_t + \gamma_g g_t + \theta_1 b_t + \theta_2 \pi_t + \theta_3 x_t + \theta_4 y_t^{(1)} + \theta_5 y_t^{(20-1)} + \theta_6 \Delta d_t + \theta_7 dp_t + \varepsilon_{t+h} \tag{A.40}$$

We report β_τ and γ_g as functions of the forecast horizon h below:
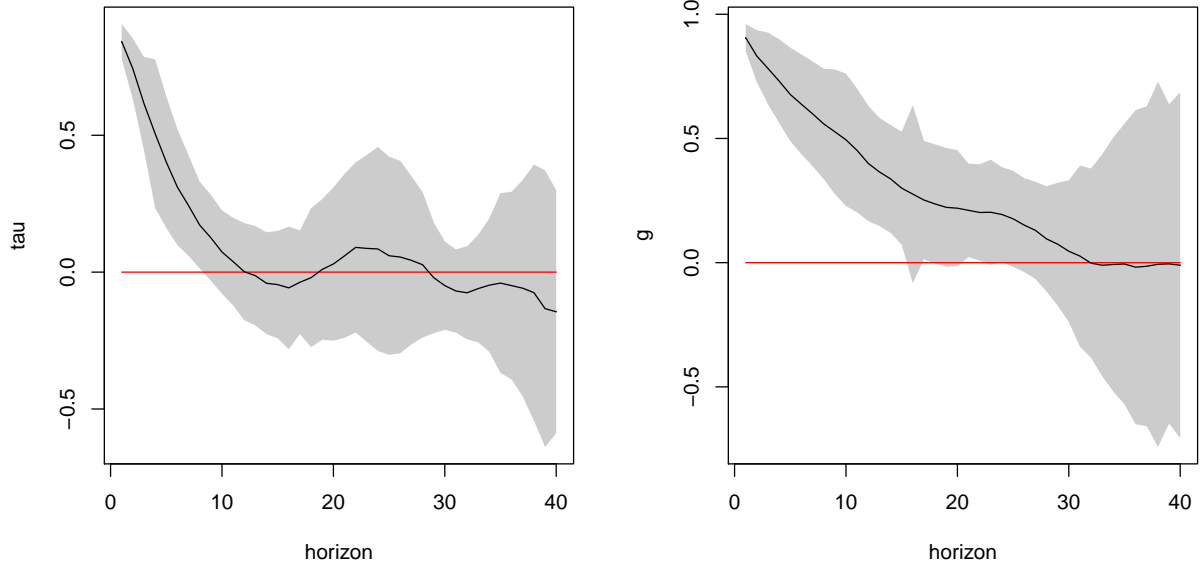

Figure A.12: Local Projection of τ and g.

J Immunization

To immunize against all shocks, we construct a replicating bond portfolio for the surplus claim. For the G-claim, we can approximate the change in the valuation of the spending claim as:

$$
\begin{aligned}
P_{t+1}^g - P_t^g &= G_{t+1} \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'z_{t+1}) - G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'z_t) \\
&= G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'z_{t+1} + \Delta \log G_{t+1}) - G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'z_t) \\
&= G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'(\mathbf{\Psi} z_t + \mathbf{\Sigma}^{\frac{1}{2}} \varepsilon_{t+1}) + x_0 + \pi_0 + \mu^g \\
&\quad + (e_g + e_x + e_\pi)'(\mathbf{\Psi} z_t + \mathbf{\Sigma}^{\frac{1}{2}} \varepsilon_{t+1})) - G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'z_t) \\
&\approx G_t \sum_{h=0}^{\infty} \exp(A_g(h) + x_0 + \pi_0 + \mu^g + (B_g(h) + e_g + e_x + e_\pi)' \mathbf{\Psi} z_t) \\
&\quad \cdot (1 + (B_g(h) + e_g + e_x + e_\pi)' \mathbf{\Sigma}^{\frac{1}{2}} \varepsilon_{t+1}) - G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'z_t) \\
&= G_t \sum_{h=0}^{\infty} \{ \exp(A_g(h) + x_0 + \pi_0 + \mu^g + (B_g(h) + e_g + e_x + e_\pi)' \mathbf{\Psi} z_t) \\
&\quad - \exp(A_g(h) + B_g(h)'z_t) \} + \{ G_t \sum_{h=0}^{\infty} \exp(A_g(h) + x_0 + \pi_0 + \mu^g
\end{aligned}
$$

$$+ \quad \left(B_g(h) + e_g + e_x + e_\pi\right)' \Psi z_t\right) \left(B_g(h) + e_g + e_x + e_\pi\right)' \Sigma^{\frac{1}{2}} \}\varepsilon_{t+1}$$

Similarly, we can approximate the change in the valuation of the T-claim, assuming a constant tax revenue-to-GDP ratio, as:

$$
\begin{aligned}
P_{t+1}^\tau - P_t^\tau \quad &\approx \quad T_t \sum_{h=0}^{\infty} \{\exp(A_\tau(h) + x_0 + \pi_0 + \mu^\tau + (B_\tau(h) + e_\tau + e_x + e_\pi)' \Psi z_t) \\
&- \quad \exp(A_\tau(h) + B_\tau(h)'z_t)\} + \{T_t \sum_{h=0}^{\infty} \exp(A_\tau(h) + x_0 + \pi_0 + \mu^\tau \\
&+ \quad (B_\tau(h) + e_\tau + e_x + e_\pi)' \Psi z_t) \left(B_\tau(h) + e_\tau + e_x + e_\pi\right)' \Sigma^{\frac{1}{2}}\}\varepsilon_{t+1}
\end{aligned}
$$

After collecting terms, we can state the change in the valuation of the surplus claim as:

$$(P_{t+1}^\tau - P_t^\tau) - (P_{t+1}^g - P_t^g) \approx a_t^s + b_t^s \Sigma^{\frac{1}{2}} \varepsilon_{t+1}$$

Next, we compute the sensitivity of the nominal bond price to the state variables, for a generic bond of maturity h quarters:

$$\log P_{t+1}^\$(h) - \log P_t^\$(h+1) = A^\$(h) - A^\$(h+1) + \left[(B^\$(h))'\Psi) - (B^\$(h+1))'\right] z_t + (B^\$(h))'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}.$$

Hence, we can approximate the change in the price of the bond as:

$$
\begin{aligned}
\left(P_{t+1}^\$(h) - P_t^\$(h+1)\right) \quad &\approx \quad P_t^\$(h+1)\left(A^\$(h) - A^\$(h+1)\right) + P_t^\$(h+1)\left[((B^\$(h))'\Psi) - (B^\$(h+1))'\right] z_t \\
&+ P_t^\$(h+1)(B^\$(h))'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}.
\end{aligned}
$$

We can state the latter, collecting terms, as:

$$\left(P_{t+1}^\$(h) - P_t^\$(h+1)\right) = P_t^\$(h+1)a_t^\$(h) + P_t^\$(h+1)B^\$(h)\Sigma^{\frac{1}{2}}\varepsilon_{t+1}$$

where

$$a_t^\$(h) = \left(A^\$(h) - A^\$(h+1)\right) + \left[((B^\$(h))'\Psi) - (B^\$(h+1))'\right] z_t$$

The sensitivity of real bonds takes the exact same expression, except without the dollar superscripts:

$$(P_{t+1}(h) - P_t(h+1)) = P_t(h+1)a_t(h) + P_t(h+1)B(h)\Sigma^{\frac{1}{2}}\varepsilon_{t+1}$$

where

$$a_t(h) = (A(h) - A(h+1)) + \left[((B(h))'\Psi) - (B(h+1))'\right] z_t$$

Let $Q_{t,h}^\$$ $(Q_{t,h})$ denote the observed position in the h-quarter nominal (real) zero coupon bond in the data. If the government is immunizing the risk exposure of its funding shock according to Bhandari, Evans, Golosov, and Sargent (2017), the dollar exposure of the government bond portfolio to each shock should equal the dollar exposure of the surplus claim:

$$
\begin{aligned}
\sum_{h=0}^{H} Q_{t,h+1}^\$ P_t^\$(h+1)a_t^\$(h) + \sum_{h=0}^{H} Q_{t,h+1} P_t(h+1)a_t(h) \quad &= \quad a_t^s \\
\sum_{h=0}^{H} Q_{t,h+1}^\$ P_t^\$(h+1)B^\$(h) + \sum_{h=0}^{H} Q_{t,h+1} P_t(h+1)B(h) \quad &= \quad b_t^s
\end{aligned}
$$

We quantify the differences between the left-hand side and the right-hand side.