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NETWORKS, BARRIERS, AND TRADE

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ABSTRACT

We study a non-parametric class of neoclassical trade models with international production networks and arbitrary distortions. We characterize their properties in terms of sufficient statistics useful for growth and welfare accounting as well as for counterfactuals. Using these sufficient statistics, we characterize societal losses from increases in tariffs and iceberg trade costs, and highlight the qualitative and quantitative importance of accounting for intermediates. Finally, we establish a formal duality between open and closed economies and use this to analytically quantify the gains from trade. Our results, which can be used to compute local and global counterfactuals, provide an analytical toolbox for studying large-scale trade models. Therefore, this paper helps bridge the gap between computation and theory.

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1 Introduction

Trade economists increasingly recognize the importance of using large-scale computational general equilibrium models for studying trade policy questions. One of the major downsides of relying on purely computational methods is their opacity: computational models can be black boxes, and it may be hard to know which forces in the model drive specific results. On the other hand, simple stylized models, while transparent and parsimonious, can lead to unreliable quantitative predictions when compared to the large-scale models.

This paper attempts to provide a theoretical map of territory usually explored by machines. It studies output and welfare in open economies with disaggregated and interconnected production structures and heterogeneous consumers. We address two types of questions: (i) how to measure and decompose, à la Solow (1957), the sources of output and welfare changes using ex-post sufficient statistics, and (ii) how to predict the responses of output, welfare, as well as disaggregated prices and quantities, to changes in trade costs or tariffs using ex-ante sufficient statistics. Our analysis is non-parametric and fairly general, which helps us to isolate the common forces and sufficient statistics necessary to answer these questions without committing to a specific parametric set up. We show how accounting for the details of the production structure can theoretically and quantitatively change answers to a broad range of questions in open-economy settings.

In analyzing the structure of open-economy general equilibrium models, we emphasize their similarities and differences to the closed-economy models used to study growth and fluctuations. To fix ideas, consider the following fundamental theorem of closed economies. For a perfectly-competitive economy with a representative household and inelastically supplied factors,

$$\frac{\mathrm{d}\log W}{\mathrm{d}\log A_i} = \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_i} = \frac{sales_i}{GDP}, \tag{1}$$

where W is real income or welfare (measured by equivalent variation), Y is real output or GDP, and A_i is a Hicks-neutral shock to some producer i. Equation (1), also known as Hulten's Theorem, shows that the sales share of producer i is a sufficient statistic for the impact of a shock on aggregate welfare, aggregate income, and aggregate output to a first order. Specifically, Hulten's theorem implies that, to a first order, any disaggregated information beyond the sales share (the input-output network, the number of factors, the degrees of returns to scale, and the elasticities of substitution) is macroeconomically irrelevant.

In this paper, we examine the extent to which the logic of (1) can be transported into international economics. We provide the open-economy analogues of equation (1), and show that although versions of Hulten's theorem continue to hold in open-economies, the sales shares are no longer such universal sufficient statistics. Ultimately, there are two main barriers to naively applying Hulten's theorem in an open-economy: first, in an open-economy, output and welfare are no longer the same since welfare depends on terms-of-trade but output does not (see e.g. Burstein and Cravino, 2015); second, much of trade policy concerns the effects of tariffs, which knocks out the foundation of marginal cost pricing and Pareto efficiency that Hulten's Theorem is built on. Our generalizations make clear precisely the conditions under which a naive-application of (1) to an open-economy is valid. Even when not directly applicable, it proves helpful to think in terms of (1), and deviations from it.

Our framework allows for arbitrary distorting wedges (like tariffs or markups) in the initial equilibrium, and we derive comparative statics with respect to both wedges (like tariffs) and technologies (like iceberg costs of trade) in terms of model primitives. When the initial allocation is Pareto-efficient, because of the first welfare theorem, changes in wedges have no first-order effect on real GDP and world welfare. In this case, we can instead provide second-order approximations. We show that welfare losses to the world as a whole, and to the output of each country, from the imposition of tariffs or other distortions is approximately equal to a Domar-weighted sum of Harberger triangles. This result holds even in the absence of implausible compensating transfers, and we provide explicit formulas for what these Harberger triangles are equal to in terms of microeconomic primitives. We explain how to adjust these formulas to obtain welfare losses. We show that the existence of global value chains dramatically increases the costs of protectionism by inflating both the area of each triangle *and* the Domar weight used to aggregate the triangles. Simple (non-input-output) models, regardless of how they are calibrated, get either the area of the triangles or their weight wrong.

Our comparative static results generalize the local *hat-algebra* of Jones (1965) beyond frictionless $2 \times 2 \times 2$ no input-output economies. These local results can also be numerically integrated to arrive at exact global comparative statics. This provides an alternative to *exact* hat-algebra (e.g. Dekle et al., 2008) common in the literature. Whereas exact hat-algebra requires solving a large nonlinear system of equations once, this differential approach requires solving a smaller linear system repeatedly. Computationally, for large and highly nonlinear models, this differential equation approach is significantly faster.[1]

Finally, we analytically characterize the gains from trade by considering how welfare changes as a country moves towards autarky. To do so, we show that under some conditions, there exists a useful isomorphism between open and closed economies. In particular, for any open-economy with nested-CES import demand there exists a companion (dual) closed economy, and the welfare effects of trade shocks in the open-economy are equal to

[1] We also provide flexible Matlab code for performing these loglinearizations and numerically integrating the results.

the output effects of productivity shocks in the closed economy. Hence, we can use results from the closed-economy literature, principally Hulten (1978) and Baqaee and Farhi (2017a), to characterize the effects of trade shocks on welfare. Our formulas provide a generalization of some of the influential insights of Arkolakis et al. (2012) to environments with disaggregated, non-loglinear (non-Cobb-Douglas) input-output connections. Compared to the loglinear (Cobb-Douglas) production networks common in the literature (e.g. Costinot and Rodriguez-Clare, 2014; Caliendo and Parro, 2015), we find that accounting for nonlinear production networks significantly raises the gains from trade. Accounting for nonlinear input-output networks is as, or more important, as accounting for intermediates in the first place. For example, for the US, the gains from trade increase from 4.5% to 9% once we account for intermediates with a loglinear network, but they increase further to 13% once we account for realistic complementarities in production. The numbers are even more dramatic for more open economies, for example, the gains from trade for Mexico go from 11% in the model without intermediates, to 16% in the model with a loglinear network, to 44.5% in the model with a non-loglinear network.

In Section 7, we present a series of worked-out analytical examples. These examples show how our general results can be applied to study a range of different applied questions, like Dutch disease, the incidence of tariffs on different factors, how global value chains can amplify the losses from protectionism, and how the presence of universal intermediate inputs, like foreign energy, amplify the welfare loss of moving towards autarky.

The outline of the paper is as follows. In Section 2, we set up the model and define the objects of interest. In Section 3, we derive some first-order growth-accounting results useful for measurement and decompositions. In Section 4, we derive first-order comparative statics in terms of microeconomic primitives, useful for prediction. In Section 5, we apply the results in Section 4 to approximate societal losses from tariffs and other wedges to the second order. In Section 6, we establish a (global) dual relationship between closed and open economies and use it to study the gains from trade. Section 7 contains analytical examples. Section 8 contains quantitative examples, calibrated using nested-CES functional forms, to show the sorts of questions our results can be used to answer and to check the accuracy of our local approximations. Section 9 concludes.

Related Literature. This paper is related to three literatures: the literature on the gains from trade, the literature on production networks, and the literature on growth accounting. We discuss each literature in turn starting with the one on the gains (or losses) from trade. Our results generalize some of the results in Arkolakis et al. (2012) and Costinot and Rodriguez-Clare (2014) to environments with non-linear input-output connections. Our framework generalizes the input-output models emphasized in Caliendo and Parro (2015), Caliendo et al. (2017), Morrow and Trefler (2017), Fally and Sayre (2018), and Bernard et al. (2019). In contemporaneous work, Huo et al. (2020) provide a framework for decomposing bilateral GDP comovement into shock transmission and shock correlation using a general equilibrium model with input-output linkages. Their analysis, which focuses on business cycle fluctuations, highlights the role of endogenous factor supply, an issue that we abstract from.

Our results about the effects of trade in distorted economies also relates to Berthou et al. (2018) and Bai et al. (2018). Our results also relate to complementary work with non-parametric or semi-parametric models of trade like Adao et al. (2017), Lind and Ramondo (2018), and Allen et al. (2014). Whereas they study reduced-form general equilibrium demand systems, we show how to construct these general equilibrium objects from microeconomic primitives. The cost is that our approach requires more data, but the benefit is that our analysis does not rely on the invertibility or stability of factor demand systems or gravity equations, assumptions that can be easily violated in models with intermediates or wedges. Our characterization of how factor shares and prices respond to shocks is related to a large literature, for example, Trefler and Zhu (2010), Davis and Weinstein (2008), Feenstra and Sasahara (2017), Dix-Carneiro (2014), Galle et al. (2017), among others. Finally, our computational approach, which, instead of solving a nonlinear system of equations, numerically integrates derivatives, is similar to the way computational general equilibrium (CGE) models are solved (for a survey, see Dixon et al., 2013).

The literature on production networks has primarily been concerned with the propagation of shocks in closed economies, typically assuming a representative agent. For instance, Long and Plosser (1983), Acemoglu et al. (2012), Atalay (2017), Carvalho et al. (2016), Baqaee and Farhi (2017a,b), and Baqaee (2018), among others. A recent focus of the literature, particularly in the context of open economies, has been to model the formation of links, for example Chaney (2014), Lim (2017), Tintelnot et al. (2018), and Kikkawa et al. (2018). Our approach, which builds on the results in Baqaee and Farhi (2017a,b), is different: rather than modelling the formation of links as a binary decision, we use a Walrasian environment where the presence and strength of links is determined by cost minimization subject to some production technology.

Finally, our growth accounting results are related to closed-economy results like Solow (1957), Hulten (1978), as well as to the literature extending growth-accounting to open economies, including Kehoe and Ruhl (2008) and Burstein and Cravino (2015). Perhaps closest to us are Diewert and Morrison (1985) and Kohli (2004) who introduce output indices which account for terms-of-trade changes. Our real income and welfare-accounting measures share their goal, though our decomposition into pure productivity changes and reallocation effects is different. In explicitly accounting for the existence of intermediate inputs, our approach also speaks to how one can circumvent the double-counting problem and spill-overs arising from differences in gross and value-added trade, issues studied by Johnson and Noguera (2012) and Koopman et al. (2014). Relative to these other papers, our approach has the added bonus of easily being able to handle inefficiencies and wedges.

Our approach is general, and relies on duality, along the lines of Dixit and Norman (1980). We differ from the classic analysis, however, in that, in extending Hulten's theorem to open economies, we state our comparative static results in terms of observable sufficient statistics: expenditure shares, changes in expenditure shares, the input-output table, and elasticities of substitution. Our approach relies heavily on the notion of the allocation matrix, which helps give a physical interpretation to the theorems, and is convenient for analyzing inefficient economies. In inefficient economies, the abstract approach that relies on macro-level envelope conditions, like Dixit and Norman (1980) and Chipman (2008), runs into problems. However, our results and their interpretation in terms of the allocation matrix readily extend to inefficient economies.

2 Framework

In this section, we set up the model and define the equilibrium and statistics of interest.

2.1 Model Environment

There is a set of countries C, a set of producers N producing different goods, and a set of factors F. Each producer and each factor is assigned to be within the borders of one of the countries in C. The sets of producers and factors inside country c are N_c and F_c. The set F_c of factors physically located in country c may be owned by any household, and not necessarily the households in country c. To streamline the exposition, we assume that there is a representative agent in each country.[2]

Distortions. Since tax-like wedges can implement any feasible allocation of resources in our model, including inefficient allocations, we use wedges to represent distortions in the model. These tax wedges may be explicit, like tariffs, or they may be implicit, like markups or financial frictions. For ease of notation, to represent a wedge on i's purchases of inputs from k, we introduce a fictitious middleman k' that buys from k and sells to i at a "markup"

[2]See Appendix M for a discussion of how to extend the results to models with heterogeneous households within countries.

$\mu_{k'}$. The revenues collected by these markups/wedges are rebated back to the households in a way we specify below.[3]

Factors. Income is earned by primary factors and revenues generated by the wedges. A primary factor is simply a non-produced (endowment) good, and earns Ricardian rents.[4] To model revenues earned by wedges, for each country $c \in C$, we introduce a "fictitious" factor that collects the markup/wedge revenue accruing to residents of country c. We denote the set of true primary factors by F and the set of true and fictitious factors by F^*. The $C \times (N + F)$ matrix Φ is the ownership matrix, where Φ_{ci} is the share of i's value-added (sales minus costs) that goes to households in country c.

Households. The representative household in country c has homothetic preferences[5]

$$W_c = \mathcal{W}_c(\{c_{ci}\}_{i \in N}),$$

and faces a budget constraint given by

$$\sum_{i \in N} p_i c_{ci} = \sum_{f \in F} \Phi_{cf} w_f L_f + \sum_{i \in N} \Phi_{ci} \left(1 - 1/\mu_i\right) p_i y_i + T_c,$$

where c_{ci} is the quantity of the good i consumed by household c, w_f and L_f is the wage and quantity of factor f, p_i is the price and y_i is the quantity of good i, and T_c is an exogenous lump-sum transfer. The right-hand side is country c's income: the first summand is income earned by primary factors, the second summand is income earned from wedges ("fictitious" factors), and the final summand is net transfers.

[3]These fictitious middlemen are convenient for writing compact formulas, but adding them to the model explicitly is computationally inefficient. In the computational appendix, Appendix K, we discuss these issues in more detail.

[4]That is, we assume factors are inelastically supplied. In Appendix L, we discuss how to endogenize factor supply by using a Roy model and discuss the connection of our results with those in Galle et al. (2017).

[5]In mapping our model to data, we interpret domestic "households" as any agent which consumes resources without producing resources to be used by other agents. Specifically, this means that we include domestic investment and government expenditures in our definition of "households".

Producers. Good $i \in N$ belongs to some country $c \in C$ and is produced using a constant-returns-to-scale production function[6,7]

$$y_i = A_i F_i \left(\{x_{ik}\}_{k \in N}, \{l_{if}\}_{f \in F_c} \right),$$

where y_i is the total quantity of good i produced, x_{ik} is intermediate inputs from k, l_{if} is factor inputs from f, and A_i is an exogenous Hicks-neutral productivity shifter. Producer i chooses inputs to minimize costs and sets prices equal to marginal cost times a wedge $p_i = \mu_i \times mc_i$.

Iceberg Trade Costs. We capture changes in iceberg trade costs as Hicks-neutral productivity changes to specialized importers or exporters whose production functions represent the trading technology. The decision of where trading technologies should be located is ambiguous since they generate no income. It is possible to place them in the exporting country or in the importing country, and this would make no difference in terms of the welfare of agents or the allocation of resources.[8]

Equilibrium. Given productivities A_i, wedges μ_i, and a vector of transfers satisfying $\sum_{c \in C} T_c = 0$, a general equilibrium is a set of prices p_i, intermediate input choices x_{ij}, factor input choices l_{if}, outputs y_i, and consumption choices c_{ci}, such that: (i) each producer chooses inputs to minimize costs taking prices as given; (ii) the price of each good is equal to the wedge on that good times its marginal cost; (iii) each household maximizes utility subject to its budget constraint taking prices as given; and, (iv) the markets for all goods and factors clear so that $y_i = \sum_{c \in C} c_{ci} + \sum_{j \in N} x_{ji}$ for all $i \in N$ and $L_f = \sum_{j \in N} l_{jf}$ for all $f \in F$.

2.2 Definitions and Notation

In this subsection, we define the statistics of interest and introduce useful notation.

[6]This is more general than it might appear. First, production has constant returns to scale without loss of generality, because non-constant-returns can be captured via fixed factors. Second, the assumption that each producer produces only one output good is also without loss of generality. A multi-output production function is a single output production function where all but one of the outputs enter as negative inputs. Finally, productivity shifters are Hicks-neutral without loss of generality. To represent input-augmenting technical change for i's use of input k, introduce a fictitious producer buying from k and selling to i, and hit this fictitious producer with a Hicks-neutral shock.

[7]We rule out fixed costs in our analysis. Our results accommodate an extensive margin of product entry-exit, but only if it operates according to a choke-price, rather than a fixed cost. For an analysis of general equilibrium models with fixed costs see Baqaee and Farhi (2020).

[8]We do not need to take a precise stand at this stage, but we note that this will matter for our conclusions regarding country-level real GDP changes (as pointed out by Burstein and Cravino, 2015).

Nominal Output and Expenditure. Nominal output or Gross Domestic Product (GDP) for country c is the total final value of the goods produced in the country. It coincides with the total income earned by the factors located in the country:

$$GDP_c = \sum_{i \in N} p_i q_{ci} = \sum_{f \in F_c} w_f L_f + \sum_{i \in N_c} (1 - 1/\mu_i)\, p_i y_i,$$

where $q_{ci} = y_i 1_{\{i \in N_c\}} - \sum_{j \in N_c} x_{ji}$ is the "final" or net quantity of good $i \in N$ produced by country c. Note that q_{ci} is negative for imported intermediate goods.

Nominal Gross National Expenditure (GNE) for country c, also known as domestic absorption, is the total final expenditures of the residents of the country. In our model, it coincides with nominal Gross National Income (GNI) which is the total income earned by the factors owned by a country's residents adjusted for international transfers:

$$GNE_c = \sum_{i \in N} p_i c_{ci} = \sum_{f \in F} \Phi_{cf} w_f L_f + \sum_{i \in N} \Phi_{ci} (1 - 1/\mu_i)\, p_i y_i + T_c.$$

To denote variables for the world, we drop the country-level subscripts. Nominal GDP and nominal GNE are *not* the same at the country level, but they are the same at the world level:

$$GDP = GNE = \sum_{f \in F} w_f L_f + \sum_{f \in N} (1 - 1/\mu_i) p_i y_i = \sum_{i \in N} p_i q_i = \sum_{i \in N} p_i c_i,$$

where, for the world, final consumption coincides with net output $c_i = q_i$ because $c_i = \sum_{c \in C} c_{ci} = \sum_{c \in C} q_{ci} = q_i$, net transfers are zero $T = 0$ because $T = \sum_{c \in C} T_c$. Let world GDP be the numeraire, so that $GDP = GNE = 1$. All prices and transfers are expressed in units of this numeraire.

Real Output and Expenditure. To convert nominal variables into real variables, as in the data, we use Divisia indices throughout. The change in real GDP of country c and the corresponding GDP deflator are defined to be

$$\mathrm{d} \log Y_c = \sum_{i \in N} \Omega_{Y_c, i}\, \mathrm{d} \log q_{ci}, \quad \mathrm{d} \log P_{Y_c} = \sum_{i \in N} \Omega_{Y_c, i}\, \mathrm{d} \log p_i,$$

where $\Omega_{Y_c, i} = p_i q_{ci} / GDP_c$ is good i's share in final output of country c.[9]

[9] Our definition of real GDP coincides with the double-deflation approach to measuring real GDP, where the change in real GDP is defined to be the sum of changes in real value-added for domestic producers. We also slightly abuse notation since, at the initial equilibrium, $q_{ci} = 0$ for new goods and $q_{ci} < 0$ for imported intermediates. In these cases, we define $\mathrm{d} \log q_{ci} = \mathrm{d}\, q_{ci} / q_{ci}$.

The change in real GNE of country c and the corresponding deflator are

$$\mathrm{d}\log W_c = \sum_{i\in N} \Omega_{W_c,i}\,\mathrm{d}\log c_{ci}, \quad \mathrm{d}\log P_{W_c} = \sum_{i\in N} \Omega_{W_c,i}\,\mathrm{d}\log p_i,$$

where $\Omega_{W_c,i} = p_i c_{ci}/GNE_c$ is good i's share in country c's consumption basket. By Shephard's lemma, changes in real GNE are equal to changes in welfare for every country .

As with the nominal variables, real GDP and real GNE are *not* the same at the country level. However, these differences vanish at the world level so that, for the world, $\mathrm{d}\log Y = \mathrm{d}\log W$ and $\mathrm{d}\log P_Y = \mathrm{d}\log P_W$.[10] Conveniently, changes in country real GDP and real GNE aggregate up to their world counterparts.[11]

Finally, infinitesimal changes in real GDP and real GNE can be integrated or *chained* into discrete changes by updating the corresponding shares along the integration path. We denote the corresponding discrete changes by $\Delta\log Y$, $\Delta\log Y_c$, $\Delta\log W$, and $\Delta\log W_c$. In the case of GDP, this is how these objects are typically measured in the data, and in the case of GNE, this coincides with the nonlinear change in the welfare of each agent c.

Input-Output Matrices. The Heterogenous-Agent Input-Output (HAIO) matrix is the $(C+N+F)\times(C+N+F)$ matrix Ω whose ijth element is equal to i's expenditures on inputs from j as a share of its total revenues/income

$$\Omega_{ij} = \frac{p_j x_{ij}}{p_i y_i}\mathbf{1}_{\{i\in N\}} + \frac{p_j c_{ij}}{GNE_i}\mathbf{1}_{\{i\in C\}}.$$

The HAIO matrix Ω includes the factors of production and the households, where factors consume no resources (zero rows), while households produce no resources (zero columns). The Leontief inverse matrix is

$$\Psi = (I-\Omega)^{-1} = I + \Omega + \Omega^2 + \dots.$$

Whereas the input-output matrix Ω records the *direct* link from one agent or producer to another, the Leontief inverse matrix Ψ records instead the *direct and indirect* exposures through the production network.

Denote the diagonal matrix of wedges by μ (where non-taxed quantities have wedge

[10]Real GDP and real GNE for the world are defined by aggregating across all countries, so $\mathrm{d}\log Y = \sum_{i\in N}(p_i q_i/GDP)\,\mathrm{d}\log q_i$, $\mathrm{d}\log P_Y = \sum_{i\in N}(p_i q_i/GDP)\,\mathrm{d}\log p_i$, $\mathrm{d}\log W = \sum_{i\in N}(p_i c_i/GNE)\,\mathrm{d}\log c_i$, and $\mathrm{d}\log P_W = \sum_{i\in N}(p_i c_i/GNE)\,\mathrm{d}\log p_i$,.

[11]Namely, $\mathrm{d}\log Y = \sum_{c\in C}(GDP_c/GDP)\,\mathrm{d}\log Y_c$ and $\mathrm{d}\log W = \sum_{c\in C}(GNE_c/GNE)\,\mathrm{d}\log W_c$.

$\mu_i = 1$) and define the *cost-based* HAIO matrix and Leontief inverse to be

$$\tilde{\Omega} = \mu\Omega, \qquad \tilde{\Psi} = (I - \tilde{\Omega})^{-1}.$$

It will sometimes be convenient to treat goods and factors together and index them by $k \in N + F$ where the plus symbol denotes the union of sets. To this effect, we slightly extend our definitions. We interchangeably write y_k and p_k for the quantity L_k and wage w_k of factor $k \in F$.

Exposures. Each $i \in C + N + F$ is *exposed* to each $j \in C + N + F$ through revenues Ψ_{ij} and through costs $\tilde{\Psi}_{ij}$. Intuitively, Ψ_{ij} measures how expenditures on i affect the sales of j (due to backward linkages), whereas $\tilde{\Psi}_{ij}$ measures how the price of j affects the marginal cost of i (due to forward linkages). In the absence of wedges, $\mu_i = 1$ for every i, these two objects coincide.

When i is a household, we use special notation to denote *backward* and *forward* exposure. In particular, let

$$\lambda_k^{W_c} = \Psi_{c,k} = \sum_{i \in N} \Omega_{c,i}\Psi_{ik}, \quad \tilde{\lambda}_k^{W_c} = \tilde{\Psi}_{c,k} = \sum_{i \in N} \tilde{\Omega}_{c,i}\tilde{\Psi}_{ik}.$$

In words, c's exposure to k is the expenditure share weighted average of the exposure of c's suppliers to k. By analogy, the forward and backward exposure of country c's GDP (as opposed to welfare) is defined as

$$\lambda_k^{Y_c} = \sum_{i \in N} \Omega_{Y_c,i}\Psi_{ik}, \quad \tilde{\lambda}_k^{Y_c} = \sum_{i \in N} \Omega_{Y_c,i}\tilde{\Psi}_{ik}, \tag{2}$$

where recall that $\Omega_{Y_c,i} = p_i q_{ci}/GDP_c$ is the share of a good i in GDP. As usual, the world-level backward and forward exposure to k are denoted by suppressing the country subscript: that is, λ_k^Y and $\tilde{\lambda}_k^Y$ respectively.

We sometimes denote exposure to factors with capital Λ or $\tilde{\Lambda}$ to distinguish them from non-factor producers λ or $\tilde{\lambda}$. In other words, when $f \in F^*$, we write $\Lambda_f^{Y_c} = \lambda_f^{Y_c}, \Lambda_f^{W_c} = \lambda_f^{W_c}, \tilde{\Lambda}_f^{W_c} = \tilde{\lambda}_f^{W_c}, \tilde{\Lambda}_f^{W_c} = \tilde{\lambda}_f^{W_c}$ to emphasize that f is a factor.

Sales and Income. Exposures of GDP to a good or factor k at the country and world levels have a direct connection to the sales of k:

$$\lambda_k^{Y_c} = 1_{\{k \in N_c + F_c\}} \frac{p_k y_k}{GDP_c}, \quad \lambda_k = \frac{p_k y_k}{GDP}. \tag{3}$$

Hence, the exposure of world GDP λ_k^Y to k is just the sales share (or *Domar weight*) of k in world output $\lambda_k = p_k y_k / GDP$. Similarly, the exposure of country c's GDP to k is the *local Domar weight* of k in country c, that is $\lambda_k^{Y_c} = 1_{\{k \in N_c + F_c\}} (GDP/GDP_c) \lambda_k$.

We also define *factor income shares*: the share of a factor f in the income of country c and of the world are denoted

$$\Lambda_f^c = \frac{\Phi_{cf} w_f L_f}{GNE_c}, \quad \Lambda_f = \frac{w_f L_f}{GNE}.$$

Since world GNE is equal to world GDP, it follows from (2) and (3) that $\Lambda_f = \Lambda_f^Y = \sum_{i \in N} \Omega_{Y,i} \Psi_{if}$.

3 Comparative Statics: Ex-Post Sufficient Statistics

In this section, we characterize the response of real GDP and welfare to shocks. We state our results in terms of changes in endogenous, but observable, sufficient statistics. In the next section, we solve for changes in these endogenous variables in terms of microeconomic primitives.

Allocation Matrix. To better understand the intuition for the results, we introduce the allocation matrix, which helps give a physical interpretation to the theorems. Following Baqaee and Farhi (2017b), define the $(C + N + F) \times (C + N + F)$ *allocation matrix* $\mathcal{X}$ as follows: $\mathcal{X}_{ij} = x_{ij}/y_j$ is the share of good j used by i, where i and j index households, factors, and producers. Every feasible allocation is defined by a feasible allocation matrix $\mathcal{X}$, a vector of productivities A, and a vector of factor supplies L. In particular, the equilibrium allocation gives rise to an allocation matrix $\mathcal{X}(A, L, \mu, T)$ which, together with A, and L, completely describes the equilibrium.[12]

We decompose changes in any quantity X into changes due to the technological environment, for a given allocation matrix, and changes in the allocation matrix, for given technology. In vector notation:

$$\mathrm{d}\log X = \underbrace{\frac{\partial \log X}{\partial \log A} \mathrm{d}\log A + \frac{\partial \log X}{\partial \log L} \mathrm{d}\log L}_{\Delta \text{ technology}} + \underbrace{\frac{\partial \log X}{\partial \mathcal{X}} \mathrm{d}\,\mathcal{X}}_{\Delta \text{ reallocation}}.$$

Real GDP. The response of real GDP to shocks, stated in terms of country c variables, is given by the following.

[12]Since there may be multiplicity of equilibria, technically, $\mathcal{X}(A, L, \mu, T)$ is a correspondence. In this case, we restrict attention to perturbations of isolated equilibria. As shown by Debreu (1970), we can generically expect equilibria to be locally isolated.

Theorem 1 (Real GDP)**.** *The change in real GDP of country c in response to productivity shocks, factor supply shocks, transfer shocks, and shocks to wedges is:*

$$\mathrm{d}\log Y_c = \underbrace{\sum_{i\in N_c} \tilde{\lambda}_i^{Y_c}\,\mathrm{d}\log A_i + \sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c}\,\mathrm{d}\log L_f + \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\mathrm{d}\log(q_{ci})}_{\Delta\ technology}$$

$$\underbrace{- \sum_{i\in N_c} \tilde{\lambda}_i^{Y_c}\,\mathrm{d}\log \mu_i - \sum_{f\in F_c}^{F} \tilde{\Lambda}_f^{Y_c}\,\mathrm{d}\log \Lambda_f^{Y_c} + \sum_{i\in N-N_c}\left(\Lambda_i^{Y_c} - \tilde{\Lambda}_i^{Y_c}\right)\mathrm{d}\log \Lambda_i^{Y_c}}_{\Delta\ reallocation}, \quad (4)$$

where, for imported intermediates $i \in N - N_c$, the term $\Lambda_i^{Y_c} = \sum_{i\in N_c} \Omega_{Y_c,i}\Psi_{ik} = -p_i q_{ci}/GDP_c$ is expenditure on imported intermediate i as a share of GDP and $\tilde{\Lambda}_i^{Y_c} = \sum_{i\in N_c} \Omega_{Y_c,i}\tilde{\Psi}_{ik}$. The change in world real GDP $\mathrm{d}\log Y$ *can be obtained by simply suppressing the country index c. That is,*

$$\mathrm{d}\log Y = \underbrace{\sum_{i\in N} \tilde{\lambda}_i^{Y}\,\mathrm{d}\log A_i + \sum_{f\in F} \tilde{\Lambda}_f^{Y}\,\mathrm{d}\log L_f}_{\Delta\ technology} \underbrace{- \sum_{i\in N} \tilde{\lambda}_i^{Y}\,\mathrm{d}\log \mu_i - \sum_{f\in F}^{F} \tilde{\Lambda}_f^{Y}\,\mathrm{d}\log \Lambda_f^{Y}}_{\Delta\ reallocation}.$$

To understand equation (4), first consider the case where there are no wedges in the initial equilibrium. Then forward and backward exposures are the same $\tilde{\Lambda}_i^{Y_c} = \Lambda_i^{Y_c}$. Furthermore, since revenues generated by wedges exactly offset the reduction in primary factor income shares $\sum_{i\in N_c} \tilde{\lambda}_i^{Y_c}\,\mathrm{d}\log \mu_i = -\sum_{f\in F_c}^{F} \Lambda_f^{Y_c}\,\mathrm{d}\log \Lambda_f^{Y_c} = -\sum_{f\in F_c}^{F} \tilde{\Lambda}_f^{Y_c}\,\mathrm{d}\log \Lambda_f^{Y_c}$, Theorem 1 simplifies to

$$\mathrm{d}\log Y_c = \sum_{i\in N_c} \lambda_i^{Y_c}\,\mathrm{d}\log A_i + \sum_{f\in F_c} \Lambda_f^{Y_c}\,\mathrm{d}\log L_f.$$

That is, when there are no initial (domestic) wedges, country c's real GDP is equal to a Domar-weighted sum of *domestic* productivity shocks and *domestic* factor endowments. In this case, changes in the allocation matrix have no effect on real GDP.[13] Intuitively, when there are no domestic wedges, there is an envelope theorem for real GDP (the competitive equilibrium maximizes the joint profits of all domestic firms for given prices). Hence, without wedges, reallocations cannot affect real GDP to a first-order. Furthermore, in the absence of wedges, foreign shocks, like shocks to iceberg costs outside c's borders, have no effect on real GDP.

Now, suppose there are pre-existing wedges. There are two major changes. First, on the first line of equation (4), there are "mechanical" technology effects (holding fixed the

[13] Theorem 1 generalizes Hulten (1978), Burstein and Cravino (2015), and Baqaee and Farhi (2017b).

distribution of resources). As in the efficient benchmark, shocks to domestic productivity $\mathrm{d}\log A_i$ and domestic factor-endowments $\mathrm{d}\log L_f$ move real GDP. However, when there are pre-existing wedges, changes in the quantity of imported intermediate inputs $\mathrm{d}\log q_{ci}$ also change real GDP. This happens because expenditure on imported intermediates $\Lambda_i^{Y_c}$ is not equal to the shadow value of imported intermediates $\tilde{\Lambda}_i^{Y_c}$. Imported intermediates are netted out of GDP using expenditures $\Lambda_i^{Y_c}$ and not their shadow-values, so, when there are initial wedges, changes in the quantity of imported intermediates changes real GDP.[14]

The second line of (4) reflects changes in the allocation of resources. When there are pre-existing wedges, reallocation can have first-order effects on real GDP even holding fixed microeconomic productivities, factor endowments, and the quantity of imports. These are genuine changes in efficiency, and they occur because resources are not being used efficiently at the initial equilibrium. The intuition for the second line of (4) is similar to that described in Baqaee and Farhi (2017b) and Baqaee and Farhi (2019) for closed economies.

Welfare. We now turn our attention to changes in welfare (real GNE).[15]

Theorem 2 (Welfare)**.** *The change in welfare of country c in response to productivity shocks, factor supply shocks, and transfer shocks can be written as:*

$$\mathrm{d}\log W_c = \underbrace{\sum_{f\in F}\tilde{\Lambda}_f^{W_c}\,\mathrm{d}\log L_f + \sum_{i\in N}\tilde{\lambda}_i^{W_c}\,\mathrm{d}\log A_i}_{\Delta\ technology} \underbrace{- \sum_{i\in N}\tilde{\lambda}_i^{W_c}\,\mathrm{d}\log\mu_i + \sum_{f\in F^*}\left(\Lambda_f^c - \tilde{\Lambda}_f^{W_c}\right)\mathrm{d}\log\Lambda_f + (GNE/GNE_c)\,\mathrm{d}\,T_c}_{\Delta\ reallocation},$$

where $\mathrm{d}\,T_c$ *is the change in net transfers, and* $\tilde{\Lambda}_f^{W_c} = 0$ *whenever f is a fictitious factor. The change* $\mathrm{d}\log W$ *of world real GNE is obtained by suppressing the country index c. That is,*

$$\mathrm{d}\log W = \underbrace{\sum_{f\in F}\tilde{\Lambda}_f^{W}\,\mathrm{d}\log L_f + \sum_{i\in N}\tilde{\lambda}_i^{W}\,\mathrm{d}\log A_i}_{\Delta\ technology} \underbrace{- \sum_{i\in N}\tilde{\lambda}_i^{W}\,\mathrm{d}\log\mu_i - \sum_{f\in F^*}\tilde{\Lambda}_f^{W}\,\mathrm{d}\log\Lambda_f}_{\Delta\ reallocation}.$$

[14] For example, if there is a tariff on an imported intermediate $i \in N - N_c$, then expenditure on the import is less than its shadow-value $\Lambda_i^{Y_c} < \tilde{\Lambda}_i^{Y_c}$. In this case, ceteris paribus, an increase in intermediate input usage $d\log q_{ci} = dq_{ci}/q_{ci} > 0$ will boost real GDP and TFP. Therefore, even in an economy where the allocation of resources across producers is efficient, trade shocks can alter aggregate TFP by changing the quantity of imported intermediates. For an example, see Gopinath and Neiman (2014).

[15] Throughout the paper, if the wedges associated with a fictitious factor f are 1, then we have $\Lambda_f = 0$ and $\mathrm{d}\log\Lambda_f/\,\mathrm{d}\log\mu_i$ is not defined. In this case, elasticities can be replaced with semi-elasticities in a straightforward way, but we omit the details for brevity.

As with real GDP, changes in welfare can be broken into technological effects (holding fixed the distribution of resources) and reallocation effects (holding fixed technology). However, unlike real GDP, reallocation effects are first-order even when there are no wedges. This is because unlike real GDP, even in the absence of wedges, there is no envelope theorem for the welfare of a given country. We discuss the intuition for the technology and reallocation effects in turn.

The direct technology effect of a shock depends on each household's exposures to the technology shock. Since households consume foreign goods, either directly or indirectly through supply chains, this means that technology shocks outside of a country's borders affect the household in that country holding fixed the allocation matrix.

The second line in Theorem 2 captures reallocation effects. The first term is the direct effect of wedges on consumer prices. To see the intuition for the second term on the second line, recall that $\tilde{\Lambda}_f^{W_c} = \tilde{\Psi}_{cf}$ is the cost-based forward exposure of household c to factor f. This captures the total reliance of household c on f, taking into account direct and indirect exposures through supply chains. Intuitively, the reallocation effects consider, for each factor f, how the income earned by the factor changes $\mathrm{d}\log\Lambda_f$, and whether household c is a net seller $\Lambda_f^c - \tilde{\Lambda}_f^{W_c} > 0$ or a net buyer $\Lambda_f^c - \tilde{\Lambda}_f^{W_c} < 0$ of factor f. The final term on the second line is the change in net transfers.

Once we aggregate to the level of the world, if there are no pre-existing wedges, the reallocation effects are zero. In other words, for efficient models, reallocation effects are zero-sum distributive changes. On the other hand, when there are pre-existing wedges, reallocation effects are no longer zero-sum, since they can make everyone better or worse off by changing the efficiency of resource allocation. Appendix H contains a detailed and formal discussion of the reallocation effects. Appendix H emphasizes that these reallocation effects are not the same as changes in the terms-of-trade.

Simple Example. To see the difference between Theorems 1 and 2, consider a productivity shock $\mathrm{d}\log A_i$ to a foreign producer $i \notin N_c$. Suppose there are no wedges and all production and utility functions are Cobb-Douglas. Since there are no wedges, Theorem 1 implies that domestic real GDP does not respond to the foreign productivity shock $\mathrm{d}\log Y_c = 0$.

Now, consider the change in welfare in Theorem 2. The Cobb-Douglas assumption implies that factor income shares do not respond to productivity shocks $\mathrm{d}\log\Lambda_f = 0$. Hence, there are no reallocation effects for welfare either. Nevertheless, domestic welfare does respond to the foreign productivity shock $\mathrm{d}\log W_c = \lambda_i^{W_c}\,\mathrm{d}\log A_i$. Intuitively, even though there are no reallocation effects, an increase in foreign productivity increases the overall amount of goods the world economy can produce and this increases the welfare of country c to the extent that the consumption basket of country c relies on i (directly and indirectly through global supply chains).

Uses of Theorems 1 and 2. Since Theorems 1 and 2 depend on endogenous movements in factor income shares, they cannot be used directly to make predictions. However, despite this fact, they are useful for three reasons: (i) they provide intuition about why and how Hulten's theorem fails to describe welfare and real GDP in open economies, (ii) they can be used to measure and decompose changes into different sources *conditional* on observing the changes in factor shares (extending growth-accounting to open and distorted economies), and (iii) they can be combined with the results in Section 4 to perform counterfactuals.

Outline of the Rest of the Paper. In Section 4, we provide a full characterization of how disaggregated sales shares, prices, and quantities change in terms of microeconomic primitives (ex-ante sufficient statistics) to a first-order. In Section 5, we use these first-order results to approximate the losses to society from the imposition of tariffs and other distortions to a second-order. In Section 6, we use these results to study the effect on welfare of large external shocks, for instance, the cost of moving the economy to autarky. We end the paper with analytical and quantitative examples in Sections 7 and 8.

4 Comparative Statics: Ex-Ante Sufficient Statistics

Section 3 shows that the response of welfare and real GDP to shocks depend on changes in ex-post sufficient statistics (like changes in factor shares). In this section we characterize these ex-post sufficient statistics in terms of microeconomic primitives: the HAIO matrix and elasticities of substitution in production and in consumption (ex-ante sufficient statistics). The results of this section can then be combined with Theorems 1 and 2 to answer counterfactual questions about welfare and real GDP. We focus on two types of shocks: productivity shocks, which nest shocks to factor supply and iceberg costs, and wedge shocks, which nest shocks to tariffs and markups.

4.1 Set Up

To clarify exposition, we specialize production and consumption functions to be nested-CES aggregators, with an arbitrary number of nests and elasticities. This is for clarity not tractability. Appendix A shows that it is very straightforward to generalize the rest of the results in the paper to non-nested-CES economies.

Nested CES economies can be written in many different equivalent ways. We adopt a standardized representation, which we call the *standard-form* representation. We treat every CES aggregator as a separate producer and rewrite the input-output matrix accordingly, so that each producer has a single elasticity of substitution associated with it; the representative household in each country c consumes a single specialized good which, with some abuse of notation, we also denote by c. Importantly, note that this procedure changes the set of producers, which, with some abuse of notation we still denote by N.[16] In other words, every $k \in C + N$ has an associated cost function

$$p_k = \frac{\mu_k}{A_k} \left(\sum_{j \in N+F_c} \tilde{\Omega}_{kj} p_j^{1-\theta_k} \right)^{\frac{1}{1-\theta_k}},$$

where θ_k is the elasticity of substitution.[17]

For nested-CES economies, the input-output covariance turns out to be a central object.

Input-Output Covariance. We use the following matrix notation throughout. For a matrix X, we define $X^{(i)}$ to be its ith row and $X_{(j)}$ to be its jth column. We define the *input-output covariance operator* to be

$$Cov_{\tilde{\Omega}^{(k)}}(\Psi_{(i)}, \Psi_{(j)}) = \sum_{l \in N+F} \tilde{\Omega}_{kl} \Psi_{li} \Psi_{lj} - \left(\sum_{l \in N+F} \tilde{\Omega}_{kl} \Psi_{li} \right) \left(\sum_{l \in N+F} \tilde{\Omega}_{kl} \Psi_{lj} \right).$$

This is the covariance between the ith and jth columns of the Leontief inverse using the kth row of $\tilde{\Omega}$ as the probability distribution. We make extensive use of the input-output covariance operator throughout the rest of the paper.

4.2 Comparative Statics

Sales Shares and Prices. The following characterizes how prices and sales shares, including factor income shares, respond to perturbations in an open-economy.[18]

Theorem 3 (Prices and Sales Shares)**.** *For a vector of perturbations to productivity* $\mathrm{d}\log A$ *and wedges* $\mathrm{d}\log \mu$*, the change in the price of a good or factor* $i \in N + F$ *is*

$$\mathrm{d}\log p_i = \sum_{k \in N} \tilde{\Psi}_{ik} \left(\mathrm{d}\log \mu_k - \mathrm{d}\log A_k \right) + \sum_{f \in F} \tilde{\Psi}_{if} \, \mathrm{d}\log \Lambda_f. \tag{5}$$

[16]See Baqaee and Farhi (2017a) for a more detailed discussion of the standard-form representation.
[17]See the second example in Section 7 for an example of an economy written in standard-form.
[18]Theorem 3 generalizes Propositions 2 and 3 from Baqaee and Farhi (2017b) to open-economies.

The change in the sales share of a good or factor $i \in N+F$ is

$$\mathrm{d}\log\lambda_i = \sum_{k\in N+F}\left(\mathbf{1}_{\{i=k\}} - \frac{\lambda_k}{\lambda_i}\Psi_{ki}\right)\mathrm{d}\log\mu_k + \sum_{k\in N}\frac{\lambda_k}{\lambda_i}\mu_k^{-1}(1-\theta_k)Cov_{\tilde{\Omega}^{(k)}}(\Psi_{(i)},\mathrm{d}\log p)$$
$$+ \sum_{g\in F^*}\sum_{c\in C}\frac{\lambda_i^{W_c}-\lambda_i}{\lambda_i}\Phi_{cg}\Lambda_g\,\mathrm{d}\log\Lambda_g, \tag{6}$$

where $\mathrm{d}\log p$ *is the* $(N+F)\times 1$ *vector of price changes in* (5). *The change in wedge income accruing to household c (represented by a fictitious factor) is*

$$d\log\Lambda_c = \sum_i \frac{\Phi_{ci}\lambda_i}{\Lambda_c}\left(\mu_i^{-1}d\log\mu_i + (1-\mu_i^{-1})d\log\lambda_i\right). \tag{7}$$

Recall that for every fictitious or real factor $i \in F^*$, we interchangeably use λ_i or Λ_i to denote its Domar weight. This means that (6) pins down the change in primary factor income shares and (7) pins down changes in "fictitious" factor income shares. Therefore, substituting the vector of price changes (5) into (6) results in an $F^* \times F^*$ linear system in factor income shares $\mathrm{d}\log\Lambda$. The solution to this linear system gives the equilibrium changes in factor shares, which can be plugged back into equations (5) and (6) to get the change in the sales shares and prices for every (non-factor) good.

We discuss the intuition in detail below, but at a high level, equation (5) captures *forward propagation* of shocks — shocks to suppliers change the prices of their downstream consumers. On the other hand, equation (6) captures *backward propagation* of shocks — shocks to consumers change the sales of their upstream suppliers. Each term in these equations has a clear interpretation.

To see this intuition, start by considering the forward propagation equations (5): the first set of summands show that a change in the price of k, caused either by wedges $\mathrm{d}\log\mu_k$ or productivity $\mathrm{d}\log A_k$, affect the price of i via its direct and indirect exposures $\tilde{\Psi}_{ik}$ through supply chains. The second set of summands capture how changes in factor prices, which are measured by changes in factor income shares, also propagate through supply chains to affect the price of i. These expressions use the cost-based HAIO matrix $\tilde{\Omega}$, instead of the revenue-based HAIO matrix Ω, because Shephard's lemma implies that the elasticity of the price of i to the price of one of its inputs k is given by $\tilde{\Omega}_{ik}$ and not Ω_{ik}.

For the intuition of backward propagation equations (6), we proceed term by the term. The first term captures how an increase in the wedge $\mathrm{d}\log\mu_k$ reduces expenditures on suppliers i. If μ_k increases, then for each dollar k earns, relatively less of it makes it to i, and this reduces the sales of i.

The second term captures the fact that when relative prices change $\mathrm{d}\log p \neq 0$, then every producer k will substitute across its inputs in response to this change. Suppose that $\theta_k > 1$, so that producer k substitutes (in expenditure shares) *towards* those inputs that have become cheaper. If those inputs that became cheap are also heavily reliant on i, then $Cov_{\tilde{\Omega}^{(k)}}(\Psi_{(i)}, \mathrm{d}\log p) < 0$. Hence, substitution by k towards cheaper inputs will increase demand for i. These substitutions, which happen at the level of each producer k, must be summed across all producers.

The last set of summands, on the second line of (6), capture the fact that changes in factor prices change the distribution of income across households in different countries. This affects the demand for i if the different households are differently exposed, directly and indirectly, to i. The overall effect can be found by summing over countries c the increase in c's share of aggregate income $\sum_{g \in F^*} \Phi_{cg} \Lambda_g \, \mathrm{d}\log \Lambda_g$ multiplied by the relative welfare exposure $(\lambda_i^{W_c} - \lambda_i)/\lambda_i$ to i. If every household has the same consumption basket, the last term disappears.

Quantities. Theorem 3 can be used to characterize the response of quantities to shocks.[19]

Corollary 1. *(Quantities) The changes in the quantity of a good or factor i in response to a productivity shock to i is given by:*

$$\mathrm{d}\log y_i = \mathrm{d}\log \lambda_i - \mathrm{d}\log p_i,$$

where $\mathrm{d}\log \lambda$ *and* $\mathrm{d}\log p$ *are given in Theorem 3.*

These results on the responses of prices and quantities to perturbations generalize classic results of Stolper-Samuelson and Rybczynski.

Real GDP and Welfare. Theorem 3 gives the response of factor shares to shocks as a function of microeconomic primitives. These were left implicit in Theorem 2. Furthermore, Theorem 3 and Corollary 1 also pin down changes in the sales and quantities of imported intermediate inputs, which were left implicit in Theorem 1. Hence, Theorem 3 used in conjunction with Theorems 1 and 2 characterizes the response of real GDP and welfare to shocks as a function of microeconomic primitives, up to the first order.

For an efficient model, without wedges, real GDP in Theorem 1 does not depend on changes in factor shares to a first-order. In that, case, Theorem 3 gives the response of real

[19]Recall that prices are expressed in the numeraire where $GDP = GNE = 1$ at the world level.

GDP to shocks to the second order instead:

$$\frac{d\log Y_c}{d\log A_j} = \lambda_j^{Y_c}, \quad \frac{d^2\log Y_c}{d\log A_j\, d\log A_i} = \frac{d\,\lambda_j^{Y_c}}{d\log A_i} = \lambda_j^{Y_c}\left(\frac{d\log\lambda_j}{d\log A_i} - \sum_{f\in N_c}\Lambda_f^{Y_c}\frac{d\log\Lambda_f}{d\log A_i}\right), \quad (8)$$

where $d\,\lambda_j/\,d\log A_i$ and $d\log\Lambda_f/\,d\log A_i$ are given by Theorem 3.[20] For world real GDP, suppress the c subscript.

Non-infinitesimal Shocks. Theorem 3, which is a generalization of hat-algebra (Jones, 1965), is useful for studying small shocks and gaining intuition. For large shocks, the trade literature instead relies on exact-hat algebra (e.g. Dekle et al., 2008; Costinot and Rodriguez-Clare, 2014), which involves solving the non-linear system of supply and demand relationships. Theorem 3 provides an alternative way to make hat-algebra exact by "chaining" together local effects. This amounts to viewing Theorem 3 as a system of differential equations that can be solved by iterative means (e.g. Euler's method or Runge-Kutta). In our quantitative exercises in Section 8, we find that the differential approach is ten times faster than using state-of-the-art nonlinear solvers to perform exact hat-algebra.[21] Furthermore, the non-parametric generalization of Theorem 3 in Appendix A can be used to feed estimates of the elasticity of substitution directly into the differential equation to compute global comparative statics without specifying a closed-form expression for production or cost functions. See Appendix C for more details about global comparative statics.

Other Uses of Theorem 3. Theorem 3 can also be used to characterize other statistics of interest. Appendix E provides the elasticity of the international factor demand system with respect to factor prices and iceberg shocks as a linear combination of microeconomic elasticities of substitution with weights that depend on the input-output table. Figure 4 in Appendix E quantifies these elasticities using input-output data. This relates to insights from Adao et al. (2017), who show that the factor demand system is sufficient for performing certain counterfactuals. As another application, Appendix F writes trade elasticities at any level of aggregation as a linear combination of underlying microeconomic elasticities

[20]The expression for $d^2\log Y_c/(d\log A_j\, d\log A_i)$ abuses notation and must be handled with care. Technically, the change in real GDP from one allocation to another in general depends on the *path* taken. Hulten's theorem guarantees that changes in real GDP are a path integral of the vector field defined by the local Domar weights along a path of productivity changes. Hence, the expression $d^2\log Y_c/(d\log A_j\, d\log A_i)$ is really the derivative of the vector field defined by the local Domar weights. Conditional on the path taken from one allocation to the next, it can be used to compute the second derivative of the change in GDP at any point along that path.

[21]This type of approach is also used in the CGE literature, for example Dixon et al. (1982), to solve high-dimensional models because exact-hat algebra is computationally impracticable for very large models.

of substitution with weights that depend on the input-output table.[22]

5 Losses from Tariffs and Other Distortions

Section 4 shows how changes in wedges affect output and welfare to a first-order. However, starting at an efficient allocation, the response of real GDP and aggregate welfare to changes in wedges is zero to a first-order (due to the envelope theorem). Losses are not zero to a second-order, and in this section, we characterize these losses. We show that losses are approximately equal to a Domar-weighted sum of deadweight-loss triangles. This connects our results to the large, but mostly closed-economy, literature on misallocation. As usual, we present this result in two ways, using ex-post and ex-ante sufficient statistics.

5.1 Losses: Ex-Post Sufficient Statistics

Starting at the efficient point, consider introducing some tariffs or other distortions as $\exp(\Delta \log \mu_i)$. We provide approximations for small wedges $\Delta \log \mu_i$ around the efficient equilibrium, $\log \mu = 0$, for both real GDP and welfare.

Losses in Real GDP. We start by characterizing changes in real output.

Theorem 4 (Real GDP)**.** *Starting at an efficient equilibrium, up to the second order, in response to the introduction of small tariffs or other distortions, changes in the real GDP of country c are given by*

$$\Delta \log Y_c \approx \frac{1}{2} \sum_{i \in N_c} \lambda_i^{Y_c} \Delta \log y_i \Delta \log \mu_i.$$

Changes in world real GDP (and real GNE) are given by suppressing the country subscript.

Hence, for both the world and for each country, the reduction in real GDP from tariffs and other distortions is given by the sum of all the deadweight-loss triangles $1/2 \Delta \log y_i \Delta \log \mu_i$ weighted by their corresponding local Domar weights.[23,24]

[22] Appendix F.3 shows that the effect of supply chains on the trade elasticity, emphasized by Yi (2003), are formally identical to the issues of reswitching and capital reversing identified in the Cambridge Capital Controversy of the 1950s and 60s.

[23] Theorem 4 holds in general equilibrium, but it has a more familiar partial equilibrium counterpart (Feenstra, 2015). For a small open economy operating in a perfectly competitive world market, import tariffs reduce the welfare by $\Delta W \approx (1/2) \sum_i \lambda_i \Delta \log y_i \Delta \log \mu_i$, where μ_i is the ith gross tariff (no tariff is $\mu_i = 1$), y_i is the quantity of the ith import, and λ_i is the corresponding Domar weight (see Appendix I for details). Theorem 4 shows that this type of intuition can be applied in general equilibrium as well.

[24] Harberger (1964) argues that an equation like the one in Theorem 4 can be used to measure welfare as long as there are compensating transfers to keep the distribution of income across households fixed. Theorem

Theorem 4 shows that we only need to track changes in those quantities which are subject to a wedge — if a good is untaxed, or taxed but not included in real GDP (like a tax on imported consumption), then changes in that quantity are not directly relevant for real GDP.

To give some intuition for Theorem 4, we focus on the country level result for simplicity. Starting at an efficient equilibrium, the introduction of tariffs or other distortions leads to changes $\Delta \log y_i$ in the quantities of goods $i \in N_c$ in country c and to changes in the wedges $\Delta \log \mu_i$ between prices and marginal costs. The price-cost margin $p_i \Delta \log \mu_i$ measures the wedge between the marginal contribution to country real GDP and the marginal cost to real GDP of increasing the quantity of good i by one unit. Hence, $\lambda_i^{Y_c} \Delta \log \mu_i$ is the marginal proportional increase in real GDP from a proportional increase in the output of good i. Integrating from the initial efficient point to the final distorted point, we find that $(1/2)\lambda_i^{Y_c} \Delta \log y_i \Delta \log \mu_i$ is the contribution of good i to the change in real GDP.

This formula helps explain why accounting for global value chains matters a great deal for the quantitative effects of tariffs. Intuitively, this is because the triangles $1/2 \Delta \log y_i \Delta \log \mu_i$ are larger, and they are weighted more heavily λ_i and $\lambda_i^{Y_c}$, when there are input-output linkages.

Tariffs vs. Iceberg Trade Costs. It is instructive to compare the costs of tariffs to the costs of an increase in iceberg costs. At the world level, in response to a change $\Delta \log(1/A_i)$ in iceberg trade costs, following equation (8), the change in real GDP or real GNE is given up to a second-order by the sum of trapezoids rather than triangles:

$$\Delta \log Y = \Delta \log W \approx - \sum_{i \in N} \lambda_i \left(1 + \frac{1}{2} \Delta \log \lambda_i \right) \Delta \log(1/A_i).$$

In contrast to equivalent shocks to tariffs, shocks to iceberg trade costs have nonzero first-order effects. This is a way to see why iceberg shocks are typically much more costly than tariffs.

Losses in Welfare. Theorem 4 shows how real GDP responds to changes in tariffs or other distortions. These results do not apply to welfare. At the country level, changes in tariffs and other distortions typically lead to first-order changes (due to reallocation effects). But even at the world level, where these effects wash out, changes in real expenditure no longer coincide with changes in welfare, since changes in world real expenditures $\mathrm{d} \log W$ cannot

4 shows that, in fact, a similar formula can be used for changes in real GDP, even in the absence of compensating transfers. Proposition 5 shows how Harberger's formula must be altered for aggregate welfare in the absence of compensating transfers.

be integrated to arrive at a well-defined social welfare function.[25]

To measure world welfare, we introduce a homothetic social welfare function

$$W^{BS}(W_1, \ldots, W_C) = \sum_c \overline{\chi}_c^W \log W_c,$$

where $\overline{\chi}_c^W$ is the initial income share of country c at the efficient equilibrium. These welfare weights are chosen so that there is no incentive to redistribute across agents at the initial equilibrium. Starting at an efficient allocation, to a first-order approximation, the response of world welfare to the introduction of wedges is zero because of the envelope theorem. Therefore, we consider the reduction in world welfare from the introduction of wedges to a second-order approximation.

We measure the change in welfare by asking what fraction of consumption would society be prepared to give up to avoid the imposition of the tariffs. Formally, we measure changes in welfare by $\Delta \log \delta$, where δ solves the equation

$$W^{BS}(\delta\overline{W}_1, \ldots, \delta\overline{W}_C) = W^{BS}(W_1, \ldots, W_C),$$

where $\overline{W}_c$ and W_c are the values at the initial and final equilibrium. We use a similar definition for country level welfare δ_c.

Define $\chi_c^W = GNE_c/GNE$ to be country c's share of income. Then changes in country income shares are given up to the first order by

$$\Delta \log \chi_c^W \approx \sum_{g \in F} \Phi_{cg} \Lambda_g \Delta \log \Lambda_g + \sum_{i \in N} \Phi_{ci} \lambda_i \Delta \log \mu_i,$$

where the first set of summands show how c's income changes due to changes in factor prices, and the second set of summands capture revenues earned by the wedges accruing to c. Changes in the consumption price index of country c are given up to the first order by

$$\Delta \log P_{W_c} \approx \sum_{i \in N} \lambda_i^{W_c} \Delta \log \mu_i + \sum_{g \in F} \Lambda_g^{W_c} \Delta \log \Lambda_g,$$

where the first term captures changes in consumer prices due to the wedges and the second term captures changes in consumer prices due to changes in factor prices (the log change in the factor price is the same as the log change in the factor income share).

Proposition 5 (Welfare)**.** *Starting at an efficient equilibrium in response to the introduction of small tariffs or other distortions:*

[25] This has to do with the fact that individual household preferences across all countries are non-aggregable.

(i) *changes in world welfare are given up to the second order by*

$$\Delta \log \delta \approx \Delta \log Y + Cov_{\Omega_{\chi^W}} \left(\Delta \log \chi_c^W, \Delta \log P_{W_c} \right);$$

(ii) *changes in country real expenditure or welfare are given up to the first order by*

$$\Delta \log \delta_c \approx \Delta \log W_c \approx \Delta \log \chi_c^W - \Delta \log P_{W_c}.$$

The change in world welfare is the sum of the change in world real expenditure (output) and a redistributive term. The redistributive term is positive whenever the covariance between the changes in household income shares and the changes in consumption price deflators is positive. It captures a familiar deviation from perfect risk sharing. It would be zero if households could engage in perfect risk sharing before the introduction of the tariffs or other distortions. In our applications, this redistributive effect is quantitatively small and so changes in world welfare are approximately equal to changes in world real GDP.

5.2 Losses: Ex-Ante Sufficient Statistics

Theorem 4 and Proposition 5 express the effects of tariffs and other distortions in terms of endogenous individual output changes. In this subsection, we provide formulas for these individual output changes, and hence for the effects of tariffs and other distortions, in terms of primitives: microeconomic elasticities of substitution and the HAIO matrix. To do this, we combine Theorem 4 with Theorem 3 and Corollary 1.[26]

Theorem 6 (Real GDP)**.** *Around an efficient equilibrium, changes in world real GDP/GNE in response to changes in tariffs or other distortions are given, up to the second order, by*

$$\begin{aligned}\Delta \log Y \approx -\frac{1}{2} \sum_{l \in N} \sum_{k \in N} \Delta \log \mu_k \Delta \log \mu_l \sum_{j \in N} \lambda_j \theta_j Cov_{\Omega^{(j)}} (\Psi_{(k)}, \Psi_{(l)}) \\ - \frac{1}{2} \sum_{l \in N} \sum_{g \in F} \Delta \log \Lambda_g \Delta \log \mu_l \sum_{j \in N} \lambda_j \theta_j Cov_{\Omega^{(j)}} (\Psi_{(g)}, \Psi_{(l)}) \\ + \frac{1}{2} \sum_{l \in N} \sum_{c \in C} \chi_c^W \Delta \log \chi_c^W \Delta \log \mu_l (\lambda_l^{W_c} - \lambda_l).\end{aligned}$$

Changes $\Delta \log Y_c$ in the real GDP of country c are similar and in Appendix N.

[26] Theorem 6 generalizes Proposition 5 from Baqaee and Farhi (2017b) to open-economies.

First, all the terms scale with the square of the tariffs or other distortions $\Delta \log \mu$. There is therefore a sense in which misallocation increases with the tariffs and other distortions. Second, all the terms scale with the elasticities of substitution θ of the different producers. There is therefore a sense in which elasticities of substitution magnify the costs of these tariffs and other distortions. Third, all the terms also scale with the sales shares λ of the different producers and with the square of the Leontief inverse matrix Ψ. There is therefore also a sense in which accounting for intermediate inputs magnifies the costs of tariffs and other distortions. Fourth, all the terms mix the wedges, the elasticities of substitution, and of properties of the network.

For a given producer $l \in N$, there are terms in $\Delta \log \mu_l$ on the three lines. Taken together, these terms sum up to the Harberger triangle $(1/2)\lambda_l \Delta \log \mu_l \Delta \log y_l$ corresponding to good l in terms of microeconomic primitives. The three lines break it down into three components, corresponding to three different effects responsible for the change in the quantity $\Delta \log y_l$ of good l.

The term $-\sum_{k \in N} \Delta \log \mu_k \sum_{j \in N} \lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)})$ on the first line corresponds to the change $\Delta \log y_l$ in the quantity of good l coming from *substitutions* by all producers j in response to changes in all tariffs and other distortions $\Delta \log \mu_k$, holding factor wages constant.

The term $\sum_{g \in F} \Delta \log \Lambda_g \sum_{j \in N} \lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)})$ on the second line corresponds to the change $\Delta \log y_l$ in the quantity of good l coming from *substitutions* by all producers j in response to the endogenous changes in factor wages $\Delta \log w_g = \Delta \log \Lambda_g$ brought about by all the changes in tariffs and other distortions.

The term $\sum_{c \in C} \chi_c^W \Delta \log \chi_c^W (\lambda_l^{W_c} - \lambda_l)$ on the third line corresponds to the change $\Delta \log y_l$ in the quantity of good l coming from *redistribution* across agents with different spending patterns, in response to the endogenous changes in factor wages brought about by all the changes in tariffs and other distortions.

It is straightforward to combine Theorem 6 with Proposition 5 to arrive at ex-ante sufficient statistics for the change in welfare.

Corollary 2 (Welfare)**.** *Starting at an efficient equilibrium, changes in world and country welfare $\Delta \log \delta$ and $\Delta \log \delta_c \approx \Delta \log W_c$ are given via Proposition 5, respectively up to the second order (world) and up to the first order (country).*

6 The Gains from Trade

In this section, we characterize the change in welfare caused by trade shocks, for example, the gains relative to autarky. To reach autarky, we would have to raise iceberg trade costs to infinity, at which point our local approximations in Sections 3, 4, and 5 become unusable. In this section, we study the effect of large trade shocks on domestic welfare by relying on a dual representation of trade shocks. Formally, we show that the effects of foreign shocks on welfare are globally equivalent to the effects of productivity shocks on real GDP in a "dual" closed economy.[27] This allows us study the gains from trade by using characterizations of the linear and nonlinear effects of productivity shocks in closed economies provided respectively in Hulten (1978) and Baqaee and Farhi (2017a).

The approach in this section builds on Feenstra (1994) and Arkolakis et al. (2012). We use changes in domestic shares, and the elasticity of substitution between domestic and foreign varieties, to back out changes in the price of imports. We then show that changes in the price of imports affect welfare in a way that is isomorphic to productivity shocks in a fictitious closed economy.

To facilitate exposition, we restrict attention to nested-CES economies where the country of interest has only one primary factor, which we call labor. We also assume that there are no domestic wedges. We discuss how the results may be extended beyond the CES functional form in Appendix A.[28]

6.1 Duality Mapping

Consider an open nested-CES economy c written in standard form. Each producer $i \in N_c$ in the domestic economy has a unit cost-function

$$p_i = \frac{1}{A_i} \left(\sum_{j \in N+F_c} \Omega_{ij} p_j^{1-\theta_i} \right)^{\frac{1}{1-\theta_i}},$$

where θ_i is the elasticity of substitution for i and A_i is a productivity shifter. Since there are no wedges $\tilde{\Omega}_{ij} = \Omega_{ij}$.

Construct a dual closed economy with the same set of producers $i \in N_c$ with CES production functions with the same set of elasticities θ_i and a HAIO matrix $\check{\Omega}$ given by $\check{\Omega}_{ij} = \Omega_{ij}/\Omega_{ic}$, where $\Omega_{ic} = \sum_{j \in N_c} \Omega_{ij}$ is the *domestic input share* of i.[29] The unit-cost

[27] Our results are related in spirit, but different, to those of Deardorff and Staiger (1988).

[28] We also extend duality to the case with multiple domestic factors and tariffs in Appendix J. In Appendix L, we also show that duality can even be extended to Roy models with endogenous factor supply, along the lines of Galle et al. (2017).

[29] This means that the domestic input share of every producer must be greater than zero. If the domestic input share of some producer i is zero, then we treat i as a foreign producer and exclude it from the domestic economy. We can do this because if i's domestic input share is zero, then i generates no value-added for the domestic economy.

function of producer i in the dual closed economy is given by

$$\check{p}_i = \frac{1}{\check{A}_i} \left(\sum_{j \in N_c + F_c} \check{\Omega}_{ij} \check{p}_j^{1-\theta_i} \right)^{\frac{1}{1-\theta_i}}.$$

In words, the closed dual economy has the same set of producers as the open economy with the same elasticities, except the expenditure shares of each producer on foreign goods has been set to zero, and domestic expenditures have been rescaled so they sum to one. Variables with "inverted-hats" are the closed-economy counterparts of the original variable. The shifter $\check{A}_i$ is the productivity shifter in the closed economy, to be defined below.

Denote the set of producers that directly use imports in their production function by $M_c \subseteq N_c$. If i is an importer $i \in M_c$, we sometimes use the notation $\epsilon_i = \theta_i - 1$ since this corresponds to the partial equilibrium trade elasticity for producer i.

6.2 Duality Results

Denote by $\check{W}_c$ the welfare of the dual closed economy. Since the "inverted-hat" economy is closed, welfare is equal to real output $\Delta \log \check{W}_c = \Delta \log \check{Y}_c$.

Theorem 7 (Exact Duality)**.** *The discrete change in welfare $\Delta \log W_c$ of the original open economy in response to discrete shocks to iceberg trade costs or productivities outside of country c is equal to the discrete change in real output $\Delta \log \check{Y}_c$ of the dual closed economy in response to discrete shocks to productivities $\Delta \log \check{A}_i = -(1/\varepsilon_i) \Delta \log \Omega_{ic}$.*

Recall that $\Omega_{ic} = \sum_{j \in N_c} \Omega_{ij}$ is the *domestic input share* of i. In words, when productivity shocks in the closed economy are the negative log change in domestic input shares divided by the trade elasticity, changes in welfare in the closed economy mirror changes in welfare in the open economy. Therefore, we can leverage results from the literature on the real GDP effects of productivity shocks in closed-economies to characterize the welfare effects of trade shocks in open economies.

Corollary 3 (First-Order Duality)**.** *A first-order approximation to the change in welfare of the original open economy is:*

$$\Delta \log W_c = \Delta \log \check{Y}_c \approx \sum_{i \in M_c} \check{\lambda}_i \Delta \log \check{A}_i,$$

where, applying Hulten's theorem, $\check{\lambda}_i$ is the sales share or Domar weight of producer i in the dual closed economy.

Conditional on the size of the associated productivity shocks $\Delta \log \check{A}$, intermediate inputs amplify the gains from trade shocks much in the same way that they amplify productivity shocks in closed economies. This is because sales shares are greater than value-added shares, reflecting an intermediate-input multiplier that magnifies the effect of productivity shocks. This observation is behind the findings of Costinot and Rodriguez-Clare (2014) that allowing for intermediate inputs significantly increases gains from trade.

An easily-missed subtlety is that the sales shares in the closed dual economy $\check{\lambda}$ are *not* the same as the sales shares λ^{Y_c} in the original open economy. To see this, imagine an economy with a representative domestic firm ι. Suppose that the household spends all its income on the domestic firm. Suppose ι is part of a global value chain and its sales are much greater than its value-added, so $\lambda_\iota^{Y_c} > 1$. In this example, the closed economy sales share of this firm is just $\check{\lambda}_\iota = 1 < \lambda_\iota^{Y_c}$, and so the gains from trade in this model are identical to the one-sector model in Arkolakis et al. (2012). This is because input-output linkages outside of a country's borders, although they increase Domar weights, do not amplify trade shocks given changes in observed domestic shares.

We can also use Theorem 7, and the closed-economy results in Baqaee and Farhi (2017a), to provide a second-order approximation of the effect of trade shocks.

Corollary 4 (Second-Order Duality)**.** *The second-derivative of welfare to trade shocks is*

$$\frac{d^2 \log \check{Y}_c}{\mathrm{d} \log \check{A}_j \, \mathrm{d} \log \check{A}_i} = \frac{\mathrm{d}\, \check{\lambda}_i}{\mathrm{d} \log \check{A}_j} = \sum_{k \in N_c} (\theta_k - 1) \check{\lambda}_k Cov_{\check{\Omega}^{(k)}} \left(\check{\Psi}_{(i)}, \check{\Psi}_{(j)} \right).$$

We can re-express the change in welfare in the original open economy as

$$\Delta \log W_c = \Delta \log \check{Y}_c \approx \sum_{i \in M_c} \check{\lambda}_i \Delta \log \check{A}_i + \frac{1}{2} \sum_{k \in N_c} (\theta_k - 1) \check{\lambda}_k Var_{\check{\Omega}^{(k)}} \left(\sum_{i \in M_c} \check{\Psi}_{(i)} \Delta \log \check{A}_i \right).$$

We start by discussing the first equation. It follows from Hulten's theorem that $\mathrm{d} \log \check{Y}_c / \mathrm{d} \log \check{A}_i = \check{\lambda}_i$. This immediately implies that $\mathrm{d}^2 \log \check{Y}_c / (\mathrm{d} \log \check{A}_j \, \mathrm{d} \log \check{A}_i) = \mathrm{d}\, \check{\lambda}_i / \mathrm{d} \log \check{A}_j$. Hence, the nonlinear effect depends on how Domar weights in the closed economy change. The Domar weight of each i changes due to substitution. In response to a shock to j, substitution by k changes the sales of i by $(\theta_k - 1)\check{\lambda}_k Cov_{\check{\Omega}^{(k)}}(\check{\Psi}_{(i)}, \check{\Psi}_{(j)})$. These substitution effects must be weighted by the size $\check{\lambda}_k$ of each k and summed over all k. This equation has a similar intuition, and structure, to the backward propagation equations (6) in Theorem 3.

The second equation in the corollary indicates that the ultimate impact of the shock depends on how heterogeneously exposed each producer k is to the average productivity shock via its different inputs as captured by the term $Var_{\check{\Omega}^{(k)}} \left(\sum_{i \in M_c} \check{\Psi}_{(i)} \Delta \log \check{A}_i \right)$, and on whether these different inputs are complements ($\theta_k < 1$), substitutes ($\theta_k > 1$) or neither ($\theta_k = 1$). It indicates that complementarities lead to negative second-order terms which amplify negative shocks and mitigate positive shocks. Conversely, substitutabilities lead to positive second-order terms which mitigate negative shocks and amplify positive shocks. Of course, there are no second-order terms in the Cobb-Douglas case.

Duality with an Industry Structure. To discuss these results further, we focus on economies with an *industry structure*: producers are grouped into industries and the goods produced in any given industry are aggregated with a CES production function; and all other agents only use aggregated industry goods. In this case, all domestic producers in a given industry are uniformly exposed to any other given domestic producer. This implies that in Corollary 4, only the elasticities of substitution across industries receive non-zero weights. The elasticities of substitution across producers within industries receive a zero weight, and they only matter via their influence on the productivity shocks through the trade elasticities.

In fact, the matrix $\check{\Omega}$ of the dual closed economy can be specified entirely at the industry level where the different producers are the different industries $\iota \in \mathcal{N}_c$. Given the productivity shocks $\Delta \log \check{A}_\iota$ to the importing industries $\iota \in \mathcal{M}_c$, Theorem 7 and Corollaries 3 and 4 can then be applied at the industry level, with this industry level input-output matrix, and with only elasticities of substitution across industries.

Many cases considered in the literature have such an industry structure, and impose the additional assumption that all the elasticities of substitution across industries (and with the factor) in production and in consumption are unitary (but those within industries are above unity). This makes the dual closed economy Cobb-Douglas. Such assumptions are made for example by Arkolakis et al. (2012), Costinot and Rodriguez-Clare (2014), and Caliendo and Parro (2015). In this Cobb-Douglas case, the dual closed economy is exactly log-linear in the dual productivity shocks $\Delta \log \check{A}_\iota$. The effects of shocks to iceberg trade costs or to productivities outside of the country then coincide with the first-order effects of the dual shocks given by Corollary 3. Their second-order effects given by Corollary 4 are zero, and the same goes for their higher-order effects.

Our results therefore generalize some of the insights of Arkolakis et al. (2012) and of Costinot and Rodriguez-Clare (2014) to models with input-output linkages and where elasticities of substitution across industries (and with the factor) are not unitary.[30] In such mod-

[30]Costinot and Rodriguez-Clare (2014) show that the gains from trade are higher in multi-sector economies without input-output linkages when sectors are complements in consumption. Corollary 5 generalizes these results to economies with input-output linkages. This matters quantitatively given that most empirical evidence points to the presence of much more important complementarities in production than in consumption.

els, the dual closed economy is no longer Cobb-Douglas. Deviations from Cobb Douglas generate nonlinearities, which can either mitigate or amplify the effects of the shocks depending on whether there are complementarities or substituabilities, and with an intensity which depends on how heterogeneously exposed the different producers are to the shocks.

Corollary 5 (Exact Duality and Nonlinearities with an Industry Structure)**.** *For country c with an industry structure, we have the following exact characterization of the nonlinearities in welfare changes of the original open economy.*

(i) (Industry Elasticities) Consider two economies with the same initial input-output matrix and industry structure, the same trade elasticities and changes in domestic input shares, but with lower elasticities across industries for one than for the other so that $\theta_\kappa \leq \theta'_\kappa$ for all industries κ. Then $\Delta \log W_c = \Delta \log \check{Y}_c \leq \Delta \log W'_c = \Delta \log \check{Y}'_c$ so that negative (positive) shocks have larger negative (smaller positive) welfare effects in the economy with the lower elasticities.

(ii) (Curvature) Suppose that all the elasticities of substitution θ_κ across industries are less than (greater than) unity, then $\Delta \log W_c = \Delta \log \check{Y}_c$ is concave (convex) in $\Delta \log \check{A}$. So nonlinearities amplify (mitigate) negative shocks and mitigate (amplify) positive shocks compared to a loglinear approximation.

Since elasticities of substitution across industries are likely below one, Corollary 5 suggests that accounting for nonlinearities will amplify the gains relative to autarky, but mitigate the gains from opening up further (for fixed changes in import shares).

7 Analytical Examples

In this section, we consider some simple examples to hone intuition and illustrate the sorts of questions our results can be used to answer. For each example, the section with the relevant propositions is listed in parenthesis.

Two Countries with Arbitrary IO Linkages (Section 4). This example uses the forward and backward propagation equations in Theorem 3 to linearize a model with arbitrary input-output relationships and two single-factor countries.

Consider a two-country economy (home and foreign), with each country owning one primary factor. Hence, $C = F = 2$. Denote foreign variables by an asterisk and let L index

Furthermore, even if the production and consumption elasticities were the same, Corollary 4 shows that given the size of the trade shocks $\Delta \log \check{A}_i$, the nonlinear effects of non-unitary elasticities $\theta_k - 1$ scale with the size of the Leontief inverse and the Domar weights. Therefore, even if the elasticities are identical in consumption and production, input-output linkages amplify the gains from trade.

the home factor and L_* the foreign factor. Assume that there are no wedges, and consider a productivity shock $d\log A_j$ to producer j. Applying Theorem 3, the change in the home factor's share of income is

$$\frac{d\log\Lambda_L}{d\log A_j} = \frac{\sum_k(\theta_k - 1)\lambda_k Cov_{\Omega^{(k)}}\left(\Psi_{(j)}, \frac{\Psi_{(L)}}{\Lambda_L}\right)}{1 + \frac{\Lambda_L}{(1-\Lambda_L)}\sum_k(\theta_k - 1)\lambda_k Var_{\Omega^{(k)}}\left(\frac{\Psi_{(L)}}{\Lambda_L}\right) - \left(\Lambda_L^W - \Lambda_L^{W_*}\right)}. \tag{9}$$

The numerator captures the fact that a shock to j directs demand towards the home factor L if inputs are substitutes $\theta_k > 1$ and exposure to j and L are positively correlated $Cov_{\Omega^{(k)}}(\Psi_{(j)}, \Psi_{(L)}) > 0$ (this is reversed if inputs are complements). In this case, as k substitutes to use inputs most heavily exposed to j, it boosts demand for the home factor L.

The denominator captures the general equilibrium effects of changes in factor prices. An increase in the price of L triggers its own substitution effects and redistributes income between home and foreign. The terms in the denominator reflect these two effects. If inputs are substitutes $\theta_k > 1$ and k is heterogeneously exposed $Var_{\Omega^{(k)}}(\Psi_{(L)}) > 0$ to L, then an increase in the price of L will cause k to substitute away from L and this mitigates the partial equilibrium effect in the numerator. The final term in brackets in the denominator accounts for the fact that an increase in the income share of L raises the income share of the domestic consumer and lowers the income share of the foreign consumer. If the domestic consumer is more heavily exposed to the domestic factor $\Lambda_L^W > \Lambda_L^{W_*}$, then this amplifies the partial equilibrium effect in the numerator.

Since the factor shares must sum to one, we know that $d\Lambda_L = -d\Lambda_{L_*}$. This gives us closed-form equations for changes in both factor shares. Hence, Theorem 3 can be used to obtain closed-form expressions for changes in the sales share and the price of every other producer in the economy.

Dutch Disease in a Cobb-Douglas Model (Section 4). To make the previous example more concrete, we apply Equation (9) to a Cobb-Douglas economy, and use it to characterize conditions under which the home economy experiences Dutch disease. The example below also shows how to map a specific nested-CES model into *standard-form* required by Theorem 3.

Suppose there are n industries at home and foreign. The utility function of home and foreign consumers is

$$W = \prod_{i=1}^{n}(x_{0i})^{\Omega_{0i}}, \qquad W_* = \prod_{i=1}^{n}(x_{0i}^*)^{\Omega_{0i}},$$

where x_{0i} and x_{0i}^* are home and foreign consumption of goods from industry i. The production function of industry i (at home or foreign) is a Cobb-Douglas aggregate of intermediates and the local factor

$$y_i = L_{ij}^{\Omega_{iL}} \prod_{i=1}^{n} x_{ij}^{\Omega_{ij}}.$$

Suppose that the intermediate good x_{ij} is a CES combination of domestic and foreign varieties of j, with initial home share Ω_j and foreign share $\Omega_j^* = 1 - \Omega_j$, and elasticity of substitution $\varepsilon_j + 1$. Since the market share of home and foreign in industry j does not vary by consumer i, this means there is no home-bias.

In standard-form, this economy has $N = 3n$ producers: the first n are industries at home, the second n are industries in foreign, and the last n are CES aggregates of domestic and foreign varieties that every other industry buys. The HAIO matrix for this economy, in standard-form, is $(2 + 3n + 2) \times (2 + 3n + 2)$:

$$\Omega = \left[\begin{array}{c|c|c|c|cc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\, \Omega_{0i} \,\right]_{i=1}^{n} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\, \Omega_{0i} \,\right]_{i=1}^{n} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\, \Omega_{ij} \,\right]_{i,j=1}^{n} & \left[\, \Omega_{iL} \,\right]_{i=1}^{n} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\, \Omega_{ij} \,\right]_{i,j=1}^{n} & \mathbf{0} & \left[\, \Omega_{iL} \,\right]_{i=1}^{n} \\
\hline
\mathbf{0} & \begin{matrix} \Omega_1 & \cdots & 0 \\ & \ddots & \\ 0 & & \Omega_n \end{matrix} & \begin{matrix} \Omega_1^* & \cdots & 0 \\ & \ddots & \\ 0 & & \Omega_n^* \end{matrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right].$$

The first two rows and columns correspond to the households, the next $2n$ rows and columns correspond to home industries and foreign industries respectively. The next n rows and columns correspond to bundles of home and foreign varieties. The last two rows and columns correspond to the home and foreign factor. The vector elasticities of substitution θ for this economy is a vector with $2 + 3n$ elements $\theta = (1, \cdots, 1, \varepsilon_1 + 1, \cdots, \varepsilon_n + 1)$, where ε_i is the trade elasticity in industry i.

Using Equation (9), the change in home's share of income following a productivity shock $d \log A_j$ to some *domestic* producer j is

$$\frac{d \log \Lambda_L}{d \log A_j} = \frac{\lambda_j}{\Lambda_L} \frac{\varepsilon_j \Omega_j^* \Omega_{jL}}{1 + \sum_i \varepsilon_i \frac{\lambda_i \Omega_{iL}}{\Lambda_L} \frac{\Omega_{iL}}{1 - \Lambda_L} \Omega_i^*} \geq 0,$$

which is positive as long as domestic and foreign varieties are substitutes $\varepsilon_j > 0$. The numerator captures the fact that a shock to j will increase demand for the home factor if j

uses the home factor $\Omega_{jL} > 0$. The denominator captures the fact that an increase in the price of the home factor attenuates the increase in demand for the home factor by bidding up the price of home goods.

The positive productivity shock to j will therefore shrink the market share of every other domestic producer, a phenomenon known as Dutch disease. To see this, apply Theorem 3 to some domestic producer $i \neq j$ to get

$$\frac{d\log \lambda_i}{d\log A_j} = -\varepsilon_i \Omega_i^* \frac{\Omega_{iL}}{1-\Lambda_L}\frac{d\log \Lambda_L}{d\log A_j} < 0.$$

In words, the shock to j boosts the price of the home factor, which makes i less competitive in the world market if i relies on the home factor $\Omega_{iL} > 0$. Of course, (9) can easily be used to write down the necessary and sufficient conditions for Dutch disease for the more general model as well.

Incidence of Tariffs with Global Value Chain (Section 4). Theorem 3 can also be used to compute the incidence of tariffs on different factors in the presence of input-output linkages. For example, consider the simple economy depicted in Figure 1. Country 1 is the home country and country 2 represents the rest of the world. The home country has two factors: L_1 representing manufacturing labor and L_3 representing services labor. Manufacturing labor participates in a global value-chain with the rest of the world, whereas services labor sells domestically only.

The rest of the world is kept simple, and foreign factors can either be used as part of the value-chain with home or they can be used directly to supply foreign consumers. To simplify the algebra, we make the stark assumption that the foreign market is perfectly competitive — that is, the elasticity of substitution for foreign consumers $\theta_{H_2} = \infty$.

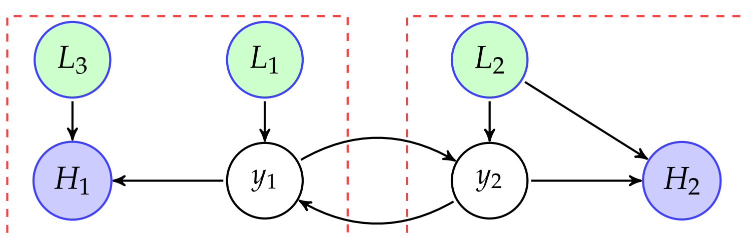

Figure 1: Solid lines show flow of goods. Green, purple, and white nodes are factors, households, and goods. Boundaries of countries are represented by dashed boxes.

Now, suppose that country 1 introduces a tariff $d\log\mu$ on foreign imports in an attempt to shield manufacturing workers from foreign competition. We can use Theorem 3 to calculate the change in the real wages of both types of workers. In this example, the policy backfires, since the real wage of manufacturing workers is

$$\lim_{\theta_{H_2}\to\infty} \frac{d\log \Lambda_{L_1}}{d\log\mu} - \frac{d\log p_{H_1}}{d\log\mu} = -\frac{\Omega}{1-\Omega} < 0,$$

where Ω is the intermediate input share of y_1. The losses increase in the intermediate input share. Intuitively, the tariff raises the marginal cost of y_1. Since the foreign market is perfectly competitive $\theta_{H_2} = \infty$, the price of the y_1 falls by exactly enough to offset the tariff. This comes about via a reduction in manufacturing workers's wages. Service workers are unaffected by the tariff $\lim_{\theta_{H_2}\to\infty} d\log\Lambda_{L_3} - d\log p_{H_1} = 0$. Welfare overall for the home country does not change $d\log W_{H_2} = 0$, because the reduction in the real wages of the manufacturing workers are precisely cancelled out by the real revenue raised by the tariffs.

Trade War with Global Value Chain (Section 5). To see how input-output connections can amplify the losses from protectionism, consider the example depicted in Figure 2. Assume the two countries are symmetric, let Ω be imports as a share of sales, and θ be the elasticity of substitution between intermediates and labor.

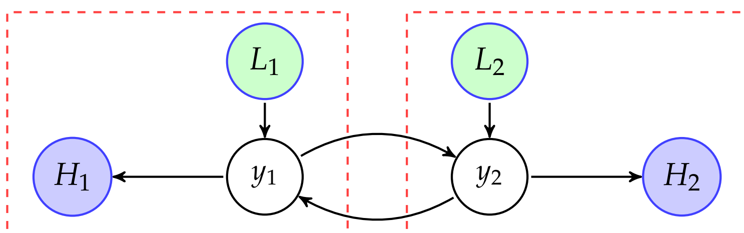

Figure 2: Solid lines show flow of goods. Green, purple, and white nodes are factors, households, and goods. Boundaries of countries are represented by dashed boxes.

Suppose that each country introduces a symmetric tax $\Delta\log\mu$ on its imports from the other country. By symmetry, changes in country real output, country welfare, world real output, and world welfare are all the same. Hence, using Theorem 4 and Theorem 6, up to a second order approximation, the reduction in real GDP and welfare are

$$\Delta\log W = \Delta\log Y \approx -\frac{1}{2}(\lambda_{12}\Delta\log y_{12}\Delta\log\mu + \lambda_{21}\Delta\log y_{21}\Delta\log\mu) \approx \theta\frac{\Omega}{2(1-\Omega)^2}(\Delta\log\mu)^2,$$

where y_{ij} is the quantity of imports from country j by country i, λ_{ij} is the corresponding sales share. By symmetry $y_{12} = y_{21}$ and $\lambda_{12} = \lambda_{21}$.

The losses increase with the elasticity of substitution θ and with the intermediate input share Ω. This is both because the relevant sales shares $\lambda_{12} = \lambda_{21} = \Omega/[2(1-\Omega)]$ and the reductions in the quantities of imports $-\Delta\log y_{12} = -\Delta\log y_{21} = [\theta/(1-\Omega)]\Delta\log\mu$ are increasing in θ and Ω. The latter effect occurs because when Ω is higher, goods effectively cross borders more times, and hence get hit by the tariffs more times, which increases the relative price of imports more and leads to a larger reduction in their quantity.

The Gains from Trade with Critical Inputs (Section 6). The last example uses the duality results in Section 6 to give some intuition for how nonlinearities in the domestic production network affect the gains from trade. In particular, how complementarities in the domestic economy amplify the losses from moving towards autarky and mitigate the gains from further trade liberalization. In this example, we consider how the existence of a universal intermediate input, like foreign energy, can increase the losses of moving to autarky.

Consider country c depicted in Figure 3. The only traded good is energy E.[31] The household consumes domestic goods 1 through to N with some elasticity of substitution θ_0 and equal sales shares $1/N$ at the initial point. Goods 1 through to M are made using labor L and energy E with an elasticity of substitution θ_1, with an initial energy share $(N/M)\check{\lambda}_E$. Energy is a CES aggregate of domestic and foreign energy with elasticity of substitution $\theta_E > 1$. Domestic energy E and consumption goods $M+1$ through to N are made using only domestic labor. Assume that the elasticity of substitution in production $\theta_1 < 1$, and that production has stronger complementarities than consumption $\theta_1 < \theta_0$.

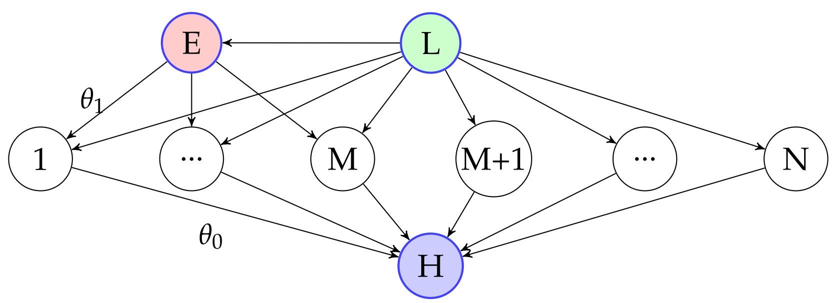

Figure 3: Industries substitutes across labor and energy with elasticity $\theta_1 < 1$. The household substitutes with elasticity of substitution $\theta_0 > \theta_1$. Energy is produced domestically and sourced from foreign with an elasticity of substitution $\theta_E > 1$.

Consider an increase in iceberg trade costs that increase the cost of importing foreign energy. The welfare effect of this trade shock is the same as that of a negative productivity shock to the energy producer of the dual closed economy

$$\Delta \log \check{A}_E = -\frac{1}{\varepsilon_E} \Delta \log \check{\Omega}_{Ec} < 0,$$

where $\varepsilon_E = \theta_E - 1$ is the trade elasticity of the energy composite good E and $\Delta \log \Omega_{Ec}$ is

[31] This example is an open-economy version of an example in Baqaee and Farhi (2017a).

the change of its domestic expenditure share.

Corollary 4 shows that, to a second order, the change in welfare is[32]

$$\Delta \log W_c \approx \check{\lambda}_E \Delta \log \check{A}_E + \frac{1}{2} \sum_{k \in N_c} (\theta_k - 1) \check{\lambda}_k Var_{\check{\Omega}^{(k)}} \left(\check{\Psi}_{(E)} \Delta \log \check{A}_E \right),$$

$$= \check{\lambda}_E \Delta \log \check{A}_E + \frac{1}{2} \check{\lambda}_E \left((\theta_0 - 1) \check{\lambda}_E (\frac{N}{M} - 1) + (\theta_1 - 1)(1 - \frac{N}{M} \check{\lambda}_E) \right) (\Delta \log \check{A}_E)^2.$$

When $M = N$, energy becomes a universal input, and the elasticity of substitution in consumption θ_0 drops out of the expression because $Var_{\check{\Omega}^{(0)}}(\check{\Psi}_E) = 0$. This is because all consumption goods are uniformly exposed to the trade shock, and so substitution by the household is irrelevant. Since $\theta_1 < 1$, nonlinearities captured by the second-order term amplify the negative welfare effects of the trade shock. This is because complementarities between energy and labor imply that the sales share of energy $\check{\lambda}_E$ increases with the shock, thereby amplifying its negative effect.

When $M < N$, the elasticity of substitution in consumption θ_0 matters. Since $\theta_0 > \theta_1$, the nonlinear adverse effect of the trade shock is reduced compared to the case $M = N$ when we keep the initial sales share of energy $\check{\lambda}_E$ constant. This is true generally but the effect is easiest to see when $\theta_0 > 1$ since the household can now substitute away from energy-intensive goods, which mitigates the increase of the sales share of energy $\check{\lambda}_E$, and hence the negative welfare effects of the shock. These effects are stronger, the lower is M, i.e. the more heterogeneous are the exposures of the different goods to energy.

If θ_0 and θ_1 are both less than one, then Corollary 5 implies that domestic welfare is concave in the trade shock $\Delta \log \check{A}_E$. Hence, for the same magnitude change in import shares, complementarities magnify the losses from moving towards autarky and mitigate the benefits of further integration relative to when the domestic input-output network is Cobb-Douglas ($\theta_0 = \theta_1 = 1$).

8 Quantitative Examples

In this section, we use a multi-factor production network model calibrated to match world input-output data. We quantify the way increasing trade costs (tariffs or iceberg) affect output, welfare, and factor rewards, and use our analytical results to give intuition for our findings. We provide flexible Matlab code, detailed in Appendix K, that loglinearizes arbitrary general equilibrium models of the type studied in this paper and computes local

[32] Since the closed dual economy in this example is acyclic, we can actually write the output function in closed-form. See Appendix J for more details.

and global comparative statics.

Calibration. The benchmark model has 40 countries as well as a "rest-of-the-world" composite country, each with four factors of production: high-skilled, medium-skilled, low-skilled labor, and capital. Each country has 30 industries each of which produces a single industry good. The model has a nested-CES structure. Each industry produces output by combining its value-added (consisting of the four domestic factors) with intermediate goods (consisting of the 30 goods). The elasticity of substitution across intermediates is θ_1, between factors and intermediate inputs is θ_2, across different primary factors is θ_3, and the elasticity of substitution of household consumption across industries is θ_0. When a producer or the household in country c purchases inputs from industry j, it consumes a CES aggregate of goods from this industry sourced from various countries with elasticity of substitution $\varepsilon_j + 1$. We use data from the World Input-Output Database (WIOD) (see Timmer et al., 2015) to calibrate the CES share parameters to match expenditure shares in the year 2008.[33]

We use the estimates from Caliendo and Parro (2015) to calibrate the elasticities ε_i between traded and domestic varieties of each industry. We set the elasticity of substitution across industries $\theta_2 = 0.2$, the one between value-added and intermediates $\theta_1 = 0.5$, and the one in consumption $\theta_0 = 0.9$. These elasticities are broadly consistent with the estimates of Atalay (2017), Boehm et al. (2015), Herrendorf et al. (2013), and Oberfield and Raval (2014). Finally, we set the elasticity of substitution among primary factors $\theta_3 = 0.5$. Overall, the evidence suggests that these elasticities are all less than one (sometimes significantly so). Appendix D contains additional details about how the model is mapped to the data.

Effect of Trade Barriers. In Table 1, we report the impact on welfare for a few countries of a universal increase in either the iceberg costs of trade or import tariffs. We compare the nonlinear response of the benchmark economy to the loglinear approximation implied by Theorem 2.

Across the board, and as suggested by the discussion of trapezoids and triangles in Section 5.1, an increase in iceberg trade costs is significantly more costly than an increase in tariffs. For the world, a universal 10% increase in iceberg costs reduces output by 2.26%. A similar increase in tariffs only reduces output by 0.43%. In Appendix B, we show that abstracting from intermediate inputs reduces these estimates by a factor of two or three.

[33]Since most tariffs in 2008 are close to zero, for simplicity, we assume that tariffs are equal to zero at the initial equilibrium. In Appendix G, we show that recomputing the results using initial tariffs does not meaningfully alter the results.

Table 1: Change in welfare, in log points, for a subset of countries in response to a universal 10%, 5%, and 1% change in trade costs, and comparison to a loglinear approximation.

	10% Shock		5% Shock		1% Shock	
Universal Iceberg Shock	Nonlinear	Loglinear	Nonlinear	Loglinear	Nonlinear	Loglinear
China	-1.30	-1.40	-0.69	-0.72	-0.14	-0.15
Great Britain	-2.52	-3.16	-1.42	-1.62	-0.32	-0.33
Luxembourg	-16.86	-19.75	-9.31	-10.11	-2.02	-2.06
Russia	-2.97	-3.00	-1.52	-1.54	-0.31	-0.31
USA	-1.06	-1.32	-0.60	-0.68	-0.13	-0.14
World	-2.26	-2.75	-1.26	-1.41	-0.28	-0.28
Universal Tariff Shock						
China	-0.16	0.93	0.12	0.48	0.08	0.10
Great Britain	-0.65	-0.40	-0.29	-0.20	-0.05	-0.04
Luxembourg	-5.37	-3.05	-2.17	-1.56	-0.33	-0.32
Russia	-1.59	-1.17	-0.74	-0.60	-0.13	-0.12
USA	0.09	0.30	0.07	0.15	0.03	0.03
World	-0.43	-0.85*	-0.15	-0.22*	-0.01	-0.01*

∗ denotes a second order approximation following Theorem 6, because the first-order effect would be zero.

For 1% shocks, the loglinear approximation performs very well for both iceberg shocks and tariff shocks. The approximation performs less well as the shocks get larger. For example, for a 10% universal increase in iceberg costs, a loglinear approximation suggests that world output should fall by 2.75% instead of 2.26%.

To compute the nonlinear effect of the shock, we can either solve the nonlinear system of supply and demand relationships (i.e. exact hat algebra), or we can repeatedly compute first-order approximations and chain the results. For exact hat algebra, we use a state-of-the-art numerical solver (Artelys Knitro), and we provide the solver with analytical derivatives. To compute a new equilibrium, the solver takes around 12 hours on a standard desktop. The linear approximation, on the other hand, takes around four minutes.[34] Therefore, differential exact hat algebra, where we take the derivative 20 times and cumulate the results, is about 10 times faster than exact hat algebra.

Table 2 uses the chained-derivatives to decompose welfare changes into technology and reallocation effects following Theorem 2 due to a uniform 10% increase in all iceberg costs. To compute the numbers in Table 2, we compute the reallocation and technology effect locally following Theorem 2 and cumulate the results.[35]

Recall that the technology effect is the direct effect of the iceberg shock on households, holding fixed the distribution of resources. The reallocation effect measures the change in

[34]Solving the linear system described in Theorem 3 takes seconds; the four minutes are almost entirely spent constructing the relevant matrix representation that needs to be inverted.

[35]As with all nonlinear decompositions, the order in which we decompose effects matters. For this nonlinear decomposition, we simultaneously increase all iceberg costs at the same time.

Table 2: Decomposition of welfare changes following Theorem 2 for a 10% universal iceberg shock.

	Δ log Welfare	Δ Technology	Δ Reallocation
China	-1.30	-1.88	0.58
Great Britain	-2.52	-2.51	-0.01
Luxembourg	-16.86	-4.00	-12.86
Russia	-2.97	-2.41	-0.57
USA	-1.06	-1.43	0.37
World	-2.26	-2.25	-0.01

welfare that results from the (endogenous) redistribution of resources. These reallocation effects are not the same as changes in the terms of trade. Instead, they are related to the *factoral terms of trade* introduced by Viner (1937) (see Appendix H for more discussion).

Naturally, small and very open economies, like Luxembourg, Belgium, Ireland, and Taiwan are worst affected by such a shock. Partly, this reflects the fact that their domestic consumers are more exposed to foreign goods, and this effect is captured by the pure technology effect. However, for small open economies, there are also large negative reallocation effects, whereas for large economies, like China or the USA, reallocation effects are positive. Intuitively, as trade becomes more restricted, expenditures shift away from imports and towards domestic factors. For small open economies, this means that the share of income claimed by their domestic factors falls. For very small countries, these negative reallocation effects are as or more important than the direct effect of the technology shock from exposure to traded goods. Naturally, for the world as a whole, there are no reallocation effects.

Gains from Trade: Intermediate Inputs and Nonlinearities. Finally, we use the duality in Theorem 7, to calculate the welfare losses from moving different countries to autarky. For this exercise, we aggregate the factors in each country into a single representative factor. The "dual" productivity shocks corresponding to autarky are $\Delta \log \check{A}_i = -(1/\varepsilon_i) \log \Omega_{ic}$, since all domestic input shares must go to one in autarky.

The gains from trade are in Table 3 for different values of the elasticities of substitution $(\theta_0, \theta_1, \theta_2)$. The first column replicates the results of a multi-sector model without intermediate inputs and with the Cobb-Douglas assumption $(\theta_0, \theta_1, \theta_2) = (1, 1, 1)$, reported in Costinot and Rodriguez-Clare (2014).The second column replicates the results of an a model which allows for intermediate inputs but maintains the Cobb-Douglas assumption, also reported in Costinot and Rodriguez-Clare (2014). As expected, allowing for intermediate inputs increases gains from trade. This is because of the first-order or log-linear

Table 3: Gains from trade for a selection of countries.

$(\theta_0, \theta_1, \theta_2)$	VA $(1,1,1)$	$(1,1,1)$	$(1, 0.5, 0.6)$	$(0.9, 0.5, 0.2)$
France	9.8%	18.5%	24.7%	30.2%
Japan	2.4%	5.2%	5.5%	5.7%
Mexico	11.5%	16.2%	21.3%	44.5%
USA	4.5%	9.1%	10.3%	13.0%

The first column is a multi-sector economy with no intermediates and Cobb-Douglas production/consumption. The second column has intermediates but maintains Cobb-Douglas. The third column has intermediates and complementarities. The final column is our benchmark calibration. The micro trade elasticities are kept constant, so the size of the shock to each industry is the same across all columns.

effect captured by Corollary 3: it reflects the fact that abstracting away from intermediate inputs reduces the volume of imports relative to GDP. The other columns continue to allow for intermediate inputs, but deviate from the Cobb-Douglas assumption, giving rise to nonlinearities. Moving across columns towards more complementarities increases the gains from trade. This is because of the nonlinear effect captured by Corollary 4: more complementarities magnify gains from trade by increasing nonlinearities. Our benchmark calibration is the one on the far right, but the second to last column shows that even with milder complementarities, which are probably more relevant for longer-run applications, the nonlinearities remain sizeable.

The magnitudes of these different effects are different across countries. The importance of accounting for intermediate inputs is largely independent of the degree of openness of the country. By contrast, the importance of accounting for nonlinearities does depend on the degree of openness: the more open the country, the larger are the dual productivity shocks, and hence, the more nonlinearities matter. Overall, it seems that nonlinearities are as important as intermediate goods to the study of gains from trade.

9 Conclusion

This paper establishes a unified framework for studying output and welfare in open and potentially distorted economies. We provide ex-post sufficient statistics for measurement and ex-ante sufficient statistics for conducting local and global counterfactuals. Our formulas bring together results from the open and closed-economy literatures, and provide new characterizations of the gains from trade and the losses from trade protectionism. As discussed in the appendix, these results also have implications for the aggregation of trade elasticities, and the distributional consequences of trade policy.

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Online Appendix to *Networks, Barriers, and Trade*

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A Beyond CES

In this appendix, we show how to generalize the results in the paper beyond nested-CES functional forms. We begin by discussing how Sections 4 and 5 can be generalized beyond CES. We then show how Section 6 can be generalized to non-CES functional forms.

A.1 Generalizing Sections 4 and 5 and Appendix C

In a similar vein to Baqaee and Farhi (2017a), we can extend the results in Sections 4 and 5 to arbitrary neoclassical production functions simply by replacing the input-output covariance operator with the *input-output substitution operator* instead.

For a producer k with cost function $\mathbf{C}_k$, the Allen-Uzawa elasticity of substitution between inputs x and y is

$$\theta_k(x,y) = \frac{\mathbf{C}_k d^2\mathbf{C}_k/(dp_x dp_y)}{(d\mathbf{C}_k/dp_x)(d\mathbf{C}_k/dp_y)} = \frac{\epsilon_k(x,y)}{\Omega_{ky}},$$

where $\epsilon_k(x,y)$ is the elasticity of the demand by producer k for input x with respect to the price p_y of input y, and $\tilde{\Omega}_{ky}$ is the expenditure share in cost of input y. We also use this definition for final demand aggregators.

The *input-output substitution operator* for producer k is defined as

$$\begin{aligned}\Phi_k(\Psi_{(i)},\Psi_{(j)}) &= -\sum_{x,y\in N+F} \tilde{\Omega}_{kx}[\delta_{xy} + \tilde{\Omega}_{ky}(\theta_k(x,y)-1)]\Psi_{xi}\Psi_{yj}, \\ &= \frac{1}{2}E_{\Omega^{(k)}}\left((\theta_k(x,y)-1)(\Psi_i(x)-\Psi_i(y))(\Psi_j(x)-\Psi_j(y))\right), \qquad (10)\end{aligned}$$

where δ_{xy} is the Kronecker delta, $\Psi_i(x) = \Psi_{xi}$ and $\Psi_j(x) = \Psi_{xj}$, and the expectation on the second line is over x and y.

In the CES case with elasticity θ_k, all the cross Allen-Uzawa elasticities are identical with $\theta_k(x,y) = \theta_k$ if $x \neq y$, and the own Allen-Uzawa elasticities are given by $\theta_k(x,x) = -\theta_k(1-\tilde{\Omega}_{kx})/\tilde{\Omega}_{kx}$. It is easy to verify that when $\mathbf{C}_k$ has a CES form we recover the input-output covariance operator:

$$\Phi_k(\Psi_{(i)},\Psi_{(j)}) = (\theta_k - 1)Cov_{\tilde{Omega}^{(k)}}(\Psi_{(i)},\Psi_{(j)}).$$

Even outside the CES case, the input-output substitution operator shares many properties with the input-output covariance operator. For example, it is immediate to verify, that: $\Phi_k(\Psi_{(i)},\Psi_{(j)})$ is bilinear in $\Psi_{(i)}$ and $\Psi_{(j)}$; $\Phi_k(\Psi_{(i)},\Psi_{(j)})$ is symmetric in $\Psi_{(i)}$ and $\Psi_{(j)}$; and

$\Phi_k(\Psi_{(i)}, \Psi_{(j)}) = 0$ whenever $\Psi_{(i)}$ or $\Psi_{(j)}$ is a constant.

All the structural results in the paper can be extended to general non-CES economies by simply replacing terms of the form $(\theta_k - 1)Cov_{\tilde{\Omega}^{(k)}}(\Psi_{(i)}, \Psi_{(j)})$ by $\Phi_k(\Psi_{(i)}, \Psi_{(j)})$.

For example, when generalized beyond nested CES functional forms, Theorem 3 becomes the following.

Theorem 8. For a vector of perturbations to productivity $d\log A$ and wedges $d\log \mu$, the change in the price of a good or factor $i \in N + F$ is the same as (5). The change in the sales share of a good or factor $i \in N + F$ is

$$\begin{aligned} d\log \lambda_i = &\sum_{k\in N+F} \left(\mathbf{1}_{\{i=k\}} - \frac{\lambda_k}{\lambda_i} \Psi_{ki} \right) d\log \mu_k + \sum_{k\in N} \frac{\lambda_k}{\lambda_i} \mu_k^{-1} \Phi_k(\Psi_{(i)}, d\log p) \\ &+ \sum_{g\in F^*} \sum_{c\in C} \frac{\lambda_i^{W_c} - \lambda_i}{\lambda_i} \Phi_{cg} \Lambda_g \, d\log \Lambda_g, \end{aligned}$$

where $d\log p$ is the $(N + F) \times 1$ vector of price changes in (5). The change in wedge income accruing to household c (represented by a fictitious factor) is the same as (7).

A.2 Generalizing Section 6

To generalize the results in Section 6 to non-CES economies, assume that the production function of any importing producer $i \in M_c$ is separable in imported inputs and domestic inputs and let θ_i be the elasticity of substitution between domestic and imported inputs (where θ_i is not necessarily a constant). Then define the dual productivity shock to i via the differential equation

$$d\log \check{A}_i = -\frac{d\log \Omega_{ic}}{\theta_i - 1}.$$

If dual productivity shocks are defined by the equation above, then Theorem 7 holds even when production functions are non-CES. Therefore, Corollary 3 generalizes to non-CES economies without change. Corollary 4 generalizes to non-CES economies if we replace terms of the form $(\theta_k - 1)Cov_{\check{\Omega}^{(k)}}\left(\check{\Psi}_{(i)}, \check{\Psi}_{(j)}\right)$ by $\Phi_k(\check{\Psi}_{(i)}, \check{\Psi}_{(j)})$, where Φ_k is defined by (10).

B Comparison to Models without Intermediates

In this appendix, we discuss how the quantitative results in Table 1 change if we abstract from intermediate inputs. We find that abstracting from intermediates significantly reduces the losses from protectionism, and we discuss the intuition for this using our analytical results.

Value-Added Calibrations

The benchmark model features input-output linkages. To emphasize the importance of taking input-output linkages into account and to connect with our analytical results, we compare the benchmark model with input-output linkages to two alternative value-added calibrations which are common in the trade literature. These alternative calibrations both assume that all production takes place with value-added production functions (no intermediates) but trivialize the input-output connections in two different ways. We call these two calibrations the low-trade value-added (LVA) and the high-trade value-added (HVA) economies. As we shall see, these two value-added calibrations are problematic, because they are not exact representations of the benchmark economy.[1]

Low-trade value-added (LVA): the value-added produced by each producer matches the one in the data. It is then assumed that the fraction of the value added of i which is sold to each country is equal to the corresponding fraction of the sales of i in the data. This calibration matches trade a share of total sales, and therefore, it lowers the volume of trade relative to GDP. We call this the low-trade value-added economy for this reason. This is the procedure used by Costinot and Rodriguez-Clare (2014) in the handbook chapter for their value-added calibration, and it is also how Arkolakis et al. (2012) mapped their model to the data.

High-trade value-added (HVA): the value-added produced by each producer matches the one in the data. However, it is then assumed that the value of i which is sold to each foreign country is equal to the corresponding sales of i in the data. The residual value-added is sold in the domestic country of i. This calibration preserves trade as a share of GDP, so we call this the high-trade value-added economy.

Results

In Table 4, we report the impact on the welfare of a few countries, as well as the effect on world welfare, of a universal 10% increase in either the iceberg costs of trade or a 10% increase in import tariffs. We compare the response of our benchmark economy to those of the LVA and HVA economies (which do not have intermediate goods).

Across the board, and as suggested by the discussion of trapezoids versus triangles in

[1]The only correct way of representing this economy with intermediate inputs as a value-added economy is to follow Adao et al. (2017). For some welfare counterfactuals, this representation can be put to use by directly specifying and estimating a parsimoniously-parameterized factor demand system. This parsimony advantage must of course be traded of against the cost of misspecification, and, as our formulas in Section 4 make clear, the "true" functional form of the factor demand system is likely to be complex for realistic economies. Furthermore, using this approach is more difficult in the presence of intermediate goods, trade costs, and tariffs. In Appendix E, we use Theorem 3 to explicitly characterize the factor-demand system.

Table 4: Percentage change in real income for a subset of the countries in response to a universal 10% change in iceberg trade costs or import tariffs.

	AUS	BEL	BGR	CAN	CHN	DEU	IRL	LUX	USA	World
Benchmark Iceberg	-2.0%	-6.7%	-5.9%	-3.3%	-1.3%	-3.0%	-8.3%	-16.9%	-1.1%	-2.26%
HVA Iceberg	-2.1%	-5.2%	-5.5%	-3.9%	-1.5%	-3.4%	-7.1%	-10.8%	-1.0%	-2.34%
LVA Iceberg	-0.8%	-2.3%	-1.9%	-1.4%	-0.5%	-1.2%	-3.6%	-5.0%	-0.3%	-0.86%
Benchmark Tariff	-0.7%	-2.3%	-0.7%	-1.2%	-0.2%	-0.9%	-3.7%	-5.4%	0.1%	-0.43%
HVA Tariff	-0.5%	-0.6%	-0.1%	-1.5%	-0.1%	-0.7%	-2.1%	-5.3%	0.3%	-0.23%
LVA Tariff	-0.4%	-0.3%	0.3%	-0.7%	0.1%	-0.5%	-1.3%	-0.7%	0.1%	-0.17%

Section 5.1, an increase in iceberg trade costs (or other non-tariff barriers to trade) is significantly more costly than an equivalent increase in tariffs. For example, US welfare actually increases by 0.1% in response to increases in tariffs, but decreases by 1.1% in response to increases in trade costs. World welfare decreases by 0.4% in response to increases in tariffs, but decreases by 2.3% in response to increases in trade costs. Hence, drawing inferences about increases in tariffs by studying increases in iceberg trade costs can be misleading.

In the benchmark economy, the effects of a universal tariff or universal iceberg shock are amplified by global value chains, as pointed out by Yi (2003). Tariffs are compounded each time unfinished goods cross borders, as in the round-about example of Section 5.2, potentially magnifying the impact of the tariff many times. To quantify this double-marginalization effect (where tariffs are paid multiple times) consider taxing traded goods based only on the domestic content of their exports.[2] For the benchmark economy, taxing only the domestic content of exported goods reduces global output by -0.31% instead of the benchmark -0.43%, suggesting that there is a significant degree of re-exporting in world trade.

Next, we compare the benchmark model to the value-added economies. The reduction in world welfare from increases in iceberg trade costs is 2.3% for the benchmark economy, it is also 2.3% for the HVA economy, but it is only 0.9% for the LVA economy. The HVA economy does a better job than the LVA economy because it preserves the volume of trade, and hence, by Theorems 1 and 2, the response of world welfare in that model is, at the first order, identical to that of the benchmark model.[3] The response of country welfare is different at the first order, but for the shock that we consider, these differences seem

[2]Formally, for each traded good i produced in country c, define $\Omega^c_{ij} = \Omega_{ij}\mathbf{1}(i \in c)$ and $\Psi^c = (I - \Omega^c)^{-1}$. Let $\delta_i = \sum_{j \notin c} \Psi^c_{ij}$ and impose the tax $\mu_i = (1 + 0.10^{1-\delta_i})$ on each i. If the traded good i does not rely on foreign inputs in its supply chain, then $\delta_i = 0$, if the traded good i contains no domestic value-added (directly or indirectly) then $\delta_i = 1$.

[3]That means, as long as the shocks are sufficiently small (ruling out nonlinearities), we should expect the benchmark and HVA economies to deliver similar welfare results for the world as a whole.

to be relatively small for most (but not all) countries. The LVA economy, the much more common calibration, does much worse. Since LVA reduces the volume of trade to GDP, it greatly understates, at the first order, the welfare effects of shocks to iceberg trade costs.

The reduction in world welfare from an increase in tariffs is 0.43% for the benchmark economy, but it is only 0.23% for the HVA economy, and it is even less at 0.17% for LVA economy. In this case, neither the LVA nor the HVA economy does a good job of replicating the benchmark model.

Theorem 4 helps explain why: the losses from tariffs are given by $(1/2)\sum_i \lambda_i \Delta \log y_i \times \log \mu_i$, where λ_i is the sales share, $\log y_i$ is the quantity, and $\log \mu_i$ is the (gross) tax for good i. Since the HVA economy preserves the volume of trade, λ_i are the same for the benchmark and the HVA economy. Nevertheless, the response of the HVA economy is half that of the benchmark. This is because in the HVA economy, the reduction in export quantities $\Delta \log y_i$ in response to tariffs is significantly lower. The LVA economy is still hopeless, since it gets both the output elasticity $\Delta \log y_i$ wrong and the trade volumes λ_i wrong.

One reason why the quantities respond less to taxes in the HVA economy than the benchmark is because in the HVA economy imported goods are a larger share of each agent's basket. To match the overall volume of trade relative to value-added, the HVA economy must increase the amount of traded goods consumers buy as a share of their overall consumption basket. Intuitively, the higher is the share of imports in the consumer's consumption basket, the lower is the elasticity of that consumer's demand with respect to the tax. The reason is that increases in the price of imports increase the overall price index by more, and hence reduce substitution away from imports.

Intuition from a Round-About Economy. To gain more intuition, we formally work through the round-about economy depicted in Figure 2. In Section 5, we showed that the welfare and output losses from a universal increase in tariffs in that economy were given by

$$-\frac{1}{2}\sum_{i \neq j} \lambda_{ji} \Delta \log y_{ji} \times \Delta \log \mu_i = \lambda_{ji} \underbrace{\frac{\theta}{1-\Omega} \Delta \log \mu}_{-\Delta \log y_{ji}} \underbrace{\Delta \log \mu}_{\Delta \log \mu_i},$$

with the sales share of each traded good given by $\lambda_{ji} = \Omega/[2(1-\Omega)]$, and where i and j index the origin and destination of the traded good, Ω is the share of traded goods in sales, and θ is the elasticity of substitution between traded and non-traded goods.

The expression above follows from the fact that

$$-\Delta \log y_{ji} = \theta \Delta \log p_{ji} = \theta \frac{1}{1-\Omega} \Delta \log \mu.$$

The term $1/(1-\Omega)$ captures the fact that global value chains amplify the effect of the tariff on the price — each time the good crosses the border, the tariff must be paid again.

Now, imagine that we take data from this economy and calibrate it using the HVA and LVA structures, keeping the elasticity of substitution between traded and non-traded goods (the trade elasticity) constant and equal to θ throughout.[4] The HVA economy has the same value $\lambda_{ji} = \Omega/[2(1-\Omega)]$ for the sales share of traded goods as the round-about economy, but the LVA economy has a lower value for $\lambda_{ji} = \Omega/2$.[5]

Both for the HVA and LVA economies, the reduction in the quantity of traded goods in response to tariffs is given by

$$-\Delta \log y_{ji} = \theta(\Delta \log p_{ji} - \Delta \log P_c) = \theta(1-\lambda_{ji})\Delta \log \mu,$$

where P_c is the consumer price index in both countries. Combining these two facts, the welfare and output losses from tariffs are given by

$$-\frac{1}{2}\sum_{i \neq j} \lambda_{ji} \Delta \log y_{ji} \Delta \log \mu_i = \lambda_{ji} \underbrace{\theta(1-\lambda_{ji})\Delta \log \mu}_{-\Delta \log y_{ji}} \underbrace{\Delta \log \mu}_{\Delta \log \mu_i}, \tag{11}$$

where $\lambda_{ji} = \Omega/[2(1-\Omega)]$ for the HVA economy and $\lambda_{ji} = \Omega/2$ for the LVA economy.

Since for a given tariff $\Delta \log \mu$, the loss is proportional to the product of λ_{ji} and $-\Delta \log y_{ji}$, the HVA and LVA economies will give the wrong answer to the extent that they fail to match the values of these two variables in the round-about economy.

By construction, the HVA economy matches λ_{ji}, whereas the LVA has a lower λ_{ji}. This leaves the change in the quantity $\Delta \log y_{ji}$ to consider. As mentioned before, the HVA economy underpredicts the reduction in imports since it inflates the household's purchases of imported goods to make up for the fact that there are no imported intermediates. This means that the household's price index responds more to a change in the price of imported goods, thereby reducing the extent of substitution.

In Table 4, we saw that both the LVA and HVA economies undershoot the true effect of the tariffs by about the same amount. Equation (11) gives some intuition for why this happens: there, the losses (11) are proportional to $\lambda_{ji}(1-\lambda_{ji})$ — when the volume of trade to GDP is close to its value in the data $\lambda_{ji} \approx 0.5$, both the HVA and LVA give broadly similar results, and both undershoot the round-about economy.

[4] In this context, we define the trade elasticity to be the elasticity of expenditures on foreign goods relative to domestic goods to an iceberg trade shock, holding factor and import prices (before the iceberg cost) constant. This is the notion used by Caliendo and Parro (2015) in the context of economies with IO linkages.

[5] This implies that to a first order, the welfare and output losses from increases in iceberg trade costs in the HVA economy are the same as for the round-about economy, but they are lower in the LVA economy.

C Non-Infinitesimal Shocks

We can conduct global comparative statics by viewing Theorem 3 as a system of differential equations that can be solved by iterative means (e.g. Euler's method or Runge-Kutta). The endogenous terms in Equations (5) and (6) depend only on HAIO and Leontief matrices $(\tilde{\Omega}, \Omega, \tilde{\Psi}, \Psi)$. However, a similar logic to (6) can be used to derive changes in these matrices. In particular, the change in the HAIO matrix $\tilde{\Omega}$ is

$$d\tilde{\Omega}_{ij} = (1-\theta_i) Cov_{\tilde{\Omega}^{(i)}}\left(d\log p, I_{(j)}\right),$$

where $I_{(j)}$ is the jth column of the identity matrix. The change in the Leontief inverse is

$$d\tilde{\Psi}_{ij} = \sum_{k\in N} \tilde{\Psi}_{ik}(1-\theta_i) Cov_{\tilde{\Omega}^{(k)}}\left(d\log p, \tilde{\Psi}_{(j)}\right).$$

Similarly, changes in Ω are

$$d\Omega_{ij} = \mu_i^{-1} d\tilde{\Omega}_{ij} - d\log\mu_i$$

and changes in Ψ are

$$d\Psi_{ij} = \sum_{k\in N} \Psi_{ik}\mu_k^{-1}(1-\theta_k) Cov_{\tilde{\Omega}^{(k)}}(d\log p, \Psi_{(j)}) - \sum_{k} \Psi_{ik}(\Psi_{kj} - \mathbf{1}_{\{k=j\}}) d\log\mu_k.$$

As explained in Appendix K, this means that we can conduct global comparative statics by repeatedly solving a $(C+F)\times(C+F)$ linear system and cumulating the results, instead of solving a system of $(C+N+F)\times(C+N+F)$ nonlinear equations. This type of approach is also used in the CGE literature, for example Dixon et al. (1982), to solve high-dimensional models because exact-hat algebra is computationally impracticable for large models.[6] For the quantitative model in Section 8, the differential approach is more than ten times faster than using state-of-the-art nonlinear solvers to perform exact hat-algebra.

There are two scenarios where the differential equations approach is especially useful. The first is for large models with strong nonlinearities (e.g. low elasticities of substitution). In these cases, repeatedly solving the smaller linear system may be more computationally feasible than solving the larger highly nonlinear system.

Secondly, the differential approach is also useful outside of the nested-CES case where closed-form expressions for the demand system are not available, but estimates of the elasticity of substitution are available at different points of the cost function. In this case, the non-parametric version of Theorem 3 (Theorem 8 in Appendix A) can be used to feed es-

[6] In the CGE literature, supply and demand relationships are log-linearized and then integrated numerically by Euler's method.

timates of the elasticity of substitution directly into the differential equation to compute global comparative statics without specifying a closed-form expression for production or cost functions.[7]

D Data Appendix

To conduct the counterfactual exercises in Section 8, we use the World Input-Output Database (Timmer et al., 2015). We use the 2013 release of the data for the final year which has no-missing data — that is 2008. We use the 2013 release because it has more detailed information on the factor usage by industry. We aggregate the 35 industries in the database to get 30 industries to eliminate missing values, and zero domestic production shares, from the data. In Table 5, we list our aggregation scheme, as well as the elasticity of substitution, based on Caliendo and Parro (2015) and taken from Costinot and Rodriguez-Clare (2014) associated with each industry. We calibrate the model to match the input-output tables and the socio-economic accounts tables in terms of expenditure shares in steady-state (before the shock).

For the growth accounting exercise in Section H.3, we use both the 2013 and the 2016 release of the WIOD data. When we combine this data, we are able to cover a larger number of years. We compute our growth accounting decompositions for each release of the data separately, and then paste the resulting decompositions together starting with the year of overlap. To construct the consumer price index and the GDP deflator for each country, we use the final consumption weights or GDP weights of each country in each year to sum up the log price changes of each good. To arrive at the price of each good, we use the gross output prices from the socio-economic accounts tables which are reported at the (country of origin, industry) level into US dollars using the contemporaneous exchange rate, and then take log differences. This means that we assume that the log-change in the price of each good at the (origin, destination, industry of supply, industry of use) level is the same as (origin, industry of supply) level. If there are differential (changing) transportation costs over time, then this assumption is violated.

To arrive at the contemporaneous exchange rate, we use the measures of nominal GDP in the socioeconomic accounts for each year (reported in local currency) to nominal GDP in the world input-output database (reported in US dollars).

[7] An additional reason why the differential equations approach can be useful is because some statistics, like real GDP, are defined in terms of path integrals. Hence, the differential equation approach *must* be used because the change in real GDP, in general, will depend on the path of integration. See Hulten (1973) for more information.

	WIOD Sector	Aggregated sector	Trade Elasticity
1	Agriculture, Hunting, Forestry and Fishing	1	8.11
2	Mining and Quarrying	2	15.72
3	Food, Beverages and Tobacco	3	2.55
4	Textiles and Textile Products	4	5.56
5	Leather, Leather and Footwear	4	5.56
6	Wood and Products of Wood and Cork	5	10.83
7	Pulp, Paper, Paper , Printing and Publishing	6	9.07
8	Coke, Refined Petroleum and Nuclear Fuel	7	51.08
9	Chemicals and Chemical Products	8	4.75
10	Rubber and Plastics	8	4.75
11	Other Non-Metallic Mineral	9	2.76
12	Basic Metals and Fabricated Metal	10	7.99
13	Machinery, Enc	11	1.52
14	Electrical and Optical Equipment	12	10.6
15	Transport Equipment	13	0.37
16	Manufacturing, Enc; Recycling	14	5
17	Electricity, Gas and Water Supply	15	5
18	Construction	16	5
19	Sale, Maintenance and Repair of Motor Vehicles...	17	5
20	Wholesale Trade and Commission Trade, ...	17	5
21	Retail Trade, Except of Motor Vehicles and...	18	5
22	Hotels and Restaurants	19	5
23	Inland Transport	20	5
24	Water Transport	21	5
25	Air Transport	22	5
26	Other Supporting and Auxiliary Transport....	23	5
27	Post and Telecommunications	24	5
28	Financial Intermediation	25	5
29	Real Estate Activities	26	5
30	Renting of M&Req and Other Business Activities	27	5
31	Public Admin/Defence; Compulsory Social Security	28	5
32	Education	29	5
33	Health and Social Work	30	5
34	Other Community, Social and Personal Services	30	5
35	Private Households with Employed Persons	30	5

Table 5: The sectors in the 2013 release of the WIOD data, and the aggregated sectors in our data.

E Factor Demand System

Adao et al. (2017) show that some trading economies can be represented as if only factors are traded within and across borders, and households have preferences over factors. Theorem 3 can be used to flesh out this representation by locally characterizing its associated reduced-form Marshallian demand for factors in terms of sufficient-statistic microeconomic primitives. For example, in the absence of wedges, the expenditure share of household c on factor f under the "trade-in-factors" representation is given by Ψ_{cf}; the elasticities $\partial \log \Psi_{cf} / \partial \log A_i$ holding factor prices constant then characterize its Marshallian price elasticities as well as its Marshallian elasticities with respect to iceberg trade shocks:

$$\frac{\partial \log \Psi_{cf}}{\partial \log A_i} = \sum_{k \in N} \frac{\Psi_{ck}}{\Psi_{cf}} (\theta_k - 1) Cov_{\Omega^{(k)}}(\Psi_{(f)}, \Psi_{(i)}).$$

The reduced-form factor demand system is locally stable with respect to a single shock $\mathrm{d} \log A_i$ if, and only, if $\partial \log \Psi_{cf} / \partial \log A_i = 0$, with a similar conditions for a combination of such shocks. Similarly, by Theorem 3, we know that the elasticity of the factor income share of some factor j with respect to the price of another factor i, holding fixed all other factor prices, is given by

$$\frac{\partial \log \Lambda_j}{\partial \log w_i} = \sum_{k \in N} (1 - \theta_k) \frac{\lambda_k}{\Lambda_j} Cov_{\Omega^{(k)}}(\Psi_{(i)}, \Psi_{(j)}) + \sum_{c \in C} (\lambda_j^{W_c} / \Lambda_j - 1) \Phi_{ci} \Lambda_i, \tag{12}$$

recalling that for factors $f \in F$, we interchangeably write Λ_f or λ_f to refer to their Domar weight. Figure 4 illustrates these elasticities of the factor demand system for a selection of the countries using the benchmark calibration. The ijth element gives the elasticity of j's world income share with respect to the price of i (holding fixed all other factor prices). Each country has four factors: capital, low, medium, and high skilled labor. Some interesting patterns emerge:

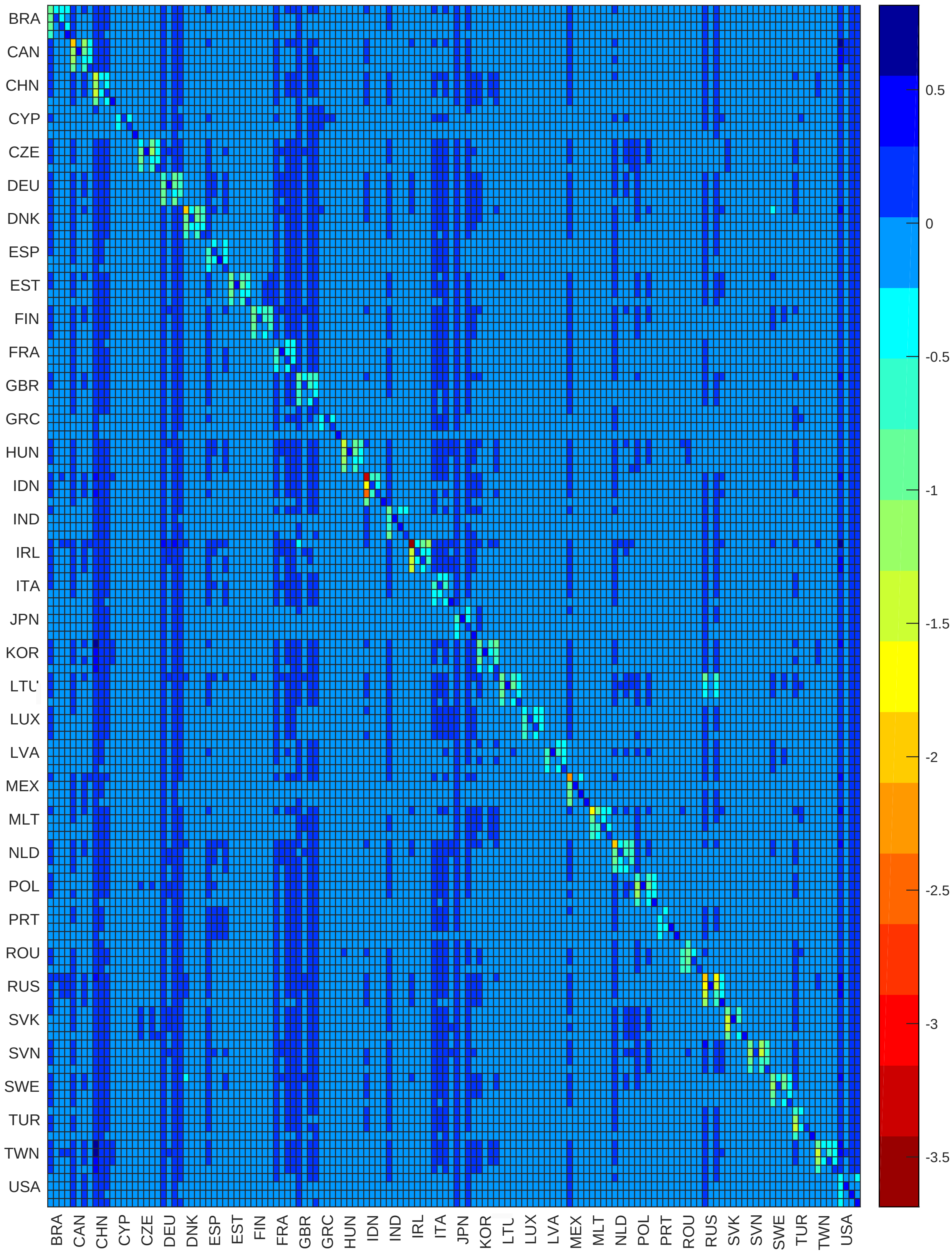
1. There dark blue columns corresponding to factors in major countries like China, Germany, Britain, Japan, and the USA. For these countries, an increase in the price of their countries' factors (except low-skilled labor) strongly raises the share of world income going to foreign countries.

2. There is a block-diagonal structure where an increase in domestic capital prices lowers domestic capital's share of world income, but also lowers domestic labor income shares. On the other hand, an increase in labor prices often raises domestic labor income and lowers domestic capital's share of world income. This pattern is interesting

because it emerges due to the HAIO matrix. At the micro level, the elasticity of substitution among domestic factors for every user is symmetric and equal to 0.5. For example, an in increase in US capital prices lowers the income claimed by all types of US labor. However, an increase in US labor prices lowers capital income and raises labor income.

3. For the USA, an increase in domestic factor prices raises the income share of every other country (the biggest beneficiary is Canada). Low-skilled labor has the mildest effects.

Figure 4: The international factor demand system for a selection of countries



The *ij*th element is the elasticity of factor *j* with respect to the price of factor *i*, holding fixed other factor prices, given by equation (12).

F Aggregation and Stability of the Trade Elasticity

In this section, we characterize trade elasticities at different levels of aggregation in terms of microeconomic primitives. We also prove necessary and sufficient conditions for ensuring that the trade elasticity is constant and stable. We also relate the instability of the trade elasticity to the Cambridge Capital controversy — a mathematically similar issue that arose in capital theory in the middle of the 20th century.

F.1 Aggregating and Disaggregating Trade Elasticities

We start by defining a class of aggregate elasticities. Consider two sets of producers I and J. Let $\lambda_I = \sum_{i \in I} \lambda_i$ and $\lambda_J = \sum_{j \in J}$ be the aggregate sales shares of producers in I and J, and let $\chi_i^I = \lambda_i / \lambda_I$ and $\chi_j^J = \lambda_j / \lambda_J$. Let k be another producer. We then define the following aggregate elasticities capturing the bias towards I vs. J of a productivity shock to m as:

$$\varepsilon_{IJ,m} = \frac{\partial(\lambda_I / \lambda_J)}{\partial \log A_m},$$

where the partial derivative indicates that we allow for this elasticity to be computed holding some things constant.

To shed light on trade elasticities, we proceed as follows. Consider a set of producers $S \subseteq N_c$ in a country c. Let J be denote a set of domestic producers that sell to producers in S, and I denote a set of foreign producers that sell to producers in S. Without loss of generality, using the flexibility of network relabeling, we assume that producers in I and J are specialized in selling to producers in S so that they do not sell to producers outside of S.

Consider an iceberg trade cost modeled as a negative productivity shock $\mathrm{d}\log(1/A_m)$ to some producer m. We then define the trade elasticity as $\varepsilon_{IJ,k} = \partial(\lambda_J / \lambda_I) / \partial \log(1/A_m) = \partial(\lambda_I / \lambda_J) / \partial \log A_m$. As already mentioned, the partial derivative indicates that we allow for this elasticity to be computed holding some things constant. There are therefore different trade elasticities, depending on exactly what is held constant. Different versions of trade elasticities would be picked up by different versions of gravity equations regressions with different sorts of fixed effects and at different levels of aggregation.

There are several possibilities for what to hold constant, ranging from the most partial equilibrium to the most general equilibrium. At one extreme, we can hold constant the prices of all inputs for all the producers in I and J and the relative sales shares of all the producers in S:

$$\varepsilon_{IJ,m} = \sum_{s \in S} \sum_{i \in I} \chi_i^I (\theta_s - 1) \frac{\lambda_s}{\lambda_i} Cov_{\Omega^{(s)}}(I_{(i)}, \Omega_{(m)}) - \sum_{s \in S} \sum_{j \in J} \chi_j^J (\theta_s - 1) \frac{\lambda_s}{\lambda_j} Cov_{\Omega^{(s)}}(I_{(j)}, \Omega_{(m)}), \quad (13)$$

where $I_{(i)}$ and $I_{(j)}$ are the ith and jth columns of the identity matrix. An intermediate possibility is to hold constant the wages of all the factors in all countries:

$$\varepsilon_{IJ,k} = \sum_{i \in I} \chi_i^I \Gamma_{ik} - \sum_{j \in J} \chi_j^J \Gamma_{jk}.$$

And at the other extreme, we can compute the full general equilibrium:

$$\varepsilon_{IJ,m} = \sum_{i \in I} \chi_i^I \left(\Gamma_{im} - \sum_{g \in F} \Gamma_{ig} \frac{\mathrm{d} \log \Lambda_g}{\mathrm{d} \log A_m} + \sum_{g \in F} \Xi_{ig} \frac{\mathrm{d} \log \Lambda_g}{\mathrm{d} \log A_m} \right) - \sum_{j \in J} \chi_j^J \left(\Gamma_{jm} - \sum_{g \in F} \Gamma_{jg} \frac{\mathrm{d} \log \Lambda_g}{\mathrm{d} \log A_m} + \sum_{g \in F} \Xi_{jg} \frac{\mathrm{d} \log \Lambda_g}{\mathrm{d} \log A_m} \right),$$

$\mathrm{d} \log \Lambda_f / \mathrm{d} \log A_m$ is given in Theorem 3.

The trade elasticity is a linear combination of microeconomic elasticities of substitution, where the weights depend on the input-output structure. Except at the most microeconomic level where there is a single producer s in S and in the most partial-equilibrium setting where we recover $\epsilon_s - 1$, this means that the aggregate trade elasticity is typically an endogenous object, since the input-output structure is itself endogenous.[8] Furthermore, in the presence of input-output linkages, it is typically nonzero even for trade shocks that are not directly affecting the sales of I to J, except in the most partial-equilibrium setting.

Example: Trade Elasticity in a Round-About World Economy

In many trade models, the trade elasticity, defined holding factor wages constant, is an invariant structural parameter. As pointed out by Yi (2003), in models with intermediate inputs, the trade elasticity can easily become an endogenous object.Consider the two-country, two-good economy depicted in Figure 2. The representative household in each country only consumes the domestic good, which is produced using domestic labor and imports with a CES production function with elasticity of substitution θ. We consider the imposition of a trade cost hitting imports by country 1 from country 2. For the sake of illustration, we assume that the trade cost does not apply to the exports of country 1 to

[8] In Appendix F.3, we provide necessary and sufficient conditions for the trade elasticity to be constant in the way.

country 2.

The trade elasticity holding factor wages and foreign input prices constant is a constant structural parameter, and given simply by

$$\theta - 1.$$

However, echoing our discussion above, the trade elasticity holding factor wages constant is different, and is given by

$$\frac{\theta - 1}{1 - \Omega_{21}\Omega_{12}},$$

where Ω_{ij} is the expenditure share of i on j, e.g. its intermediate input import share. As the intermediate input shares increase, the trade elasticity becomes larger. Simple trade models without intermediate goods are incapable of generating these kinds of patterns.

Of course, since the intermediate input shares Ω_{ij} are themselves endogenous (depending on the iceberg shock), this means that the trade elasticity varies with the iceberg shocks. In particular, if $\theta > 1$, then the trade elasticity increases (nonlinearly) as iceberg costs on imports fall in all countries since intermediate input shares rise. [9]

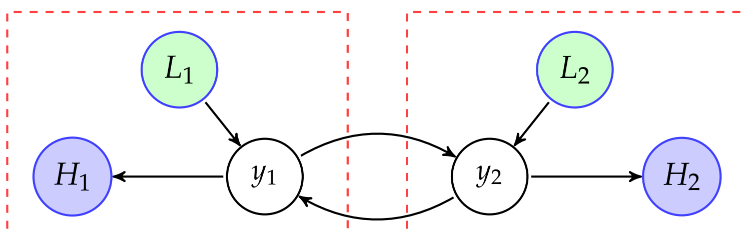


Figure 5: The solid lines show the flow of goods. Green nodes are factors, purple nodes are households, and white nodes are goods. The boundaries of each country are denoted by dashed box.

F.2 Necessary and Sufficient Conditions for Constant Trade Elasticity

In this section, we study conditions under which the trade elasticity (*holding fixed factor prices*) is constant. This trade elasticity between i and j with respect to shocks to k is defined as

$$\varepsilon_{ij,k} = \frac{\partial(\lambda_i/\lambda_j)}{\partial \log A_k},$$

[9]In Appendix F.3, we show that there it is possible to generate "trade re-switching" examples where the trade elasticity is non-monotonic with the trade cost (or even has the "wrong" sign) in otherwise perfectly respectable economies. These examples are analogous to the "capital re-switching" examples at the center the Cambridge Cambridge Capital controversy.

holding fixed factor prices. We say that a good m is *relevant* for $\varepsilon_{ij,k}$ if

$$\lambda_m Cov_{\Omega^{(m)}}(\Psi_{(k)}, \Psi_{(i)}/\lambda_i - \Psi_{(j)}/\lambda_j) \neq 0.$$

If m is not relevant, we say that it is irrelevant. For instance, if some producer m is exposed symmetrically to i and j through its inputs

$$\Omega_{ml}(\Psi_{li} - \Psi_{lj}) = 0 \quad (l \in N),$$

then $\varepsilon_{ij,k}$ is not a function of θ_m and m is irrelevant. Another example is if some producer $m \neq j$ is not exposed to k through its inputs

$$\Psi_{mk} = 0,$$

then $\varepsilon_{ij,k}$ is not a function of θ_m and m is irrelevant.

Corollary 6 (Constant Trade Elasticity)**.** *Consider two distinct goods i and j that are imported to some country c. Then consider the following conditions:*

(i) Both i and j are unconnected to one another in the production network: $\Psi_{ij} = \Psi_{ji} = 0$*, and i is not exposed to itself* $\Psi_{ii} = 1$*.*

(ii) The representative "world" household is irrelevant

$$Cov_{\chi}\left(\Psi_{(i)}, \frac{\Psi_{(i)}}{\lambda_i} - \frac{\Psi_{(j)}}{\lambda_j}\right) = 0,$$

which holds if both i and j are only used domestically, so that only household c is exposed to i and j. That is, $\lambda_i^{W_h} = \lambda_j^{W_h} = 0$ *for all* $h \neq c$*. This assumption holds automatically if i and j are imports and domestic goods and there are no input-output linkages.*

(iii) For every relevant producer l, the elasticity of substitution $\theta_l = \theta$*.*

The trade elasticity of i relative to j with respect to iceberg shocks to i is constant, and equal to

$$\varepsilon_{ij,i} = (\theta - 1).$$

if, and only if, (i)-(iii) hold.

The conditions set out in the example above, while seemingly stringent, actually represent a generalization of the conditions that hold in gravity models with constant trade elasticities. Those models oftentimes either assume away the production network, or assume that traded goods always enter via the same CES aggregator.

A noteworthy special case is when i and j are made directly from factors, without any intermediate inputs. Then, we have the following

Corollary 7. *(Network Irrelevance) If some good i and j are only made from domestic factors, then*

$$\sum_{m \in C,N} \lambda_m Cov_{\Omega^{(m)}}(\Psi_{(i)}, \Psi_{(j)}/\lambda_i - \Psi_{(i)}/\lambda_i) = 1.$$

Hence, if all microeconomic elasticities of substitution θ_m are equal to the same value $\theta_m = \theta$ then $\varepsilon_{ij,j} = \theta$.

Suppose that i is domestic goods and j are foreign imports, both of which are made only from factors (no intermediate inputs are permitted). Then a shock to j is equivalent to an iceberg shock to transportation costs. In this case, the trade elasticity of imports j into the country producing i with respect to iceberg trade costs is a convex combination of the underlying microelasticities. Of course, whenever all micro-elasticities of substitution are the same, the weights (which have to add up to one) become irrelevant, and this is the situation in most benchmark trade models with constant trade elasticities. Specifically, this highlights the fact that having common elasticities is not enough to deliver a constant trade elasticity (holding fixed factor prices) in the presence of input-output linkages as shown in the round-about example in the previous section.

F.3 Trade Reswitching

Yi (2003) shows that the trade elasticity can be nonlinear due to vertical specialization, where the trade elasticity can increase as trade barriers are lowered. Building on this insight, we can also show that, at least in principle, the trade elasticity can even have the "wrong sign" due to these nonlinearities. This relates to a parallel set of paradoxes in capital theory.

To see how this can happen, imagine there are two ways of producing a given good: the first technique uses a domestic supply chain and the other technique uses a global value chain. Whenever the good is domestically produced, the iceberg costs of transporting the good are, at most, incurred once — when the finished good is shipped to the destination. However, when the good is made via a global value chain, the iceberg costs are incurred as many times as the good is shipped across borders. As a function of the iceberg cost parameter τ, the difference in the price of these two goods (holding factor prices fixed) is a polynomial of the form

$$B_n \tau^n - B_1 \tau, \tag{14}$$

where B_n and B_1 are some coefficients and n is the number of times the border is crossed. The nonlinearity in τ, whereby the iceberg cost's effects are compounded by crossing the border, drives the sensitivity of trade volume to trade barriers in Yi (2003). The benefits from using a global value chain are compounded if the good has to cross the border many times.

However, this discussion indicates the behavior of the trade elasticity can, in principle, be much more complicated. In fact, an interesting connection can be made between the behavior of the trade elasticity and the (closed-economy) reswitching debates of the 1950s and 60s. Specifically, equation (14) is just one special case. In general, the cost difference between producing goods using supply chains of different lengths is a polynomial in τ – and this polynomial can, in principle, have more than one root. This means that the trade elasticity can be non-monotonic as a function of the trade costs, in fact, it can even have the "wrong" sign, where the volume of trade decreases as the iceberg costs fall. This mirrors the apparent paradoxes in capital theory where the relationship between the capital stock and the return on capital can be non-monotonic, and an increase in the interest rate can cause the capital stock to increase.

To see this in the trade context, imagine two perfectly substitutable goods, one of which is produced by using 10 units of foreign labor, the other is produced by shipping 1 unit of foreign labor to the home country, back to the foreign country, and then back to the home country and combining it with 10 units of domestic labor. If we normalize both foreign and domestic wages to be unity, then the costs of producing the first good is $10(1 + \tau)$, whereas the cost of producing the second good is $(1 + \tau)^3 + 10$, where τ is the iceberg trade cost. When $\tau = 0$, the first good dominates and goods are only shipped once across borders. When τ is sufficiently high, the cost of crossing the border is high enough that the first good again dominates. However, when τ has an intermediate value, then it can become worthwhile to produce the second good, which causes goods to be shipped across borders many times, thereby inflating the volume of trade.

Such examples are extreme, but they illustrate the point that in the presence of input-output networks, the trade elasticity even in partial equilibrium (holding factor prices constant) can behave quite unlike any microeconomic demand elasticity, sloping upwards when, at the microeconomic level, every demand curve slopes downwards.

Non-Symmetry and Non-Triviality of Trade Elasticities

Another interesting subtlety of Equation (13) is that the aggregate trade elasticities are non-symmetric. That is, in general $\varepsilon_{ij,l} \neq \varepsilon_{ji,l}$. Furthermore, unlike the standard gravity equation, Equation (13) shows that the cross-trade elasticities are, in general, nonzero. Hence, changes in trade costs between k and l can affect the volume of trade between i and j *holding fixed* relative factor prices and incomes. This is due to the presence of global value chains, which transmit shocks in one part of the economy to another independently of the usual general equilibrium effects (which work through the price of factors).

G Results with Initial Tariffs

Table 6 recomputes the exercise in Table 1 but allows for tariffs in the initial equilibrium. Since initial tariffs in 2008 are relatively small, these numbers are broadly similar to those reported in Table 1.

Table 6: Change in welfare, in log points, for a subset of countries in response to a universal 10%, 5%, and 1% change in trade costs, and comparison to a loglinear approximation.

	10% Shock		5% Shock		1% Shock	
Universal Iceberg Shock	Nonlinear	Loglinear	Nonlinear	Loglinear	Nonlinear	Loglinear
China	-1.69	-2.40	-0.96	-1.23	-0.22	-0.25
Great Britain	-2.54	-3.22	-1.43	-1.65	-0.33	-0.34
Luxembourg	-16.81	-19.62	-9.26	-10.04	-2.00	-2.05
Russia	-3.27	-3.87	-1.70	-1.98	-0.36	-0.40
USA	-1.15	-1.57	-0.67	-0.80	-0.15	-0.16
World	-2.37	-3.02	-1.33	-1.55	-0.30	-0.32
Universal Tariff Shock						
China	-0.60	-0.09	-0.19	-0.05	-0.01	-0.01
Great Britain	-0.69	-0.46	-0.31	-0.24	-0.05	-0.05
Luxembourg	-5.28	-2.93	-2.11	-1.50	-0.32	-0.31
Russia	-2.05	-2.07	-1.01	-1.06	-0.21	-0.22
USA	0.00	0.06	0.00	0.03	0.01	0.01
World	-0.56	-0.27	-0.24	-0.14	-0.03	-0.03

Below, we discuss how we map the data to initial tariffs.

Initial Tariff Data. The list of countries we include coincides with the countries in the World Input-Output Database (WIOD) in 2013 Release. We collect bilateral trade and tariffs data for 2016 from the United Nations Statistical Division (UNSD) Commodity Trade (COMTRADE) dataset and the Trade Analysis and Information System (TRAINS). Both trade and tariffs data defined on sectors that use the Harmonized Commodity Description and Coding System (HS) 1988/1992 concorded at the 2-digit ISIC Rev. 3 industry levels. We further aggregate these tradable sectors to match the WIOD data.

The trade value is defined as the import value from partner country to reporter country and is reported in thousand US dollars at current prices. The effective applied tariffs rates (AHS) are reported in both simple average and weighted average rates by the reporter countries. We collect only the weighted average AHS. We also have both the trade and tariffs data available for *World* as a partner and for *European Union (EU)* as a reporter.

We then compute the "aggregate weighted average AHS " (weighted by the import value at the 2-digit ISIC Rev. 3 industries) at the *aggregate* sector level. The following algorithm is used to input the missing values of "aggregate weighted average AHS" (hereafter AAHS) rates of certain country-sectors of our dataset.

1. For reporter countries in the EU, fill in the missings by the AAHS rates reported by the EU in the same year. We do *not* replace the rates by the EU-reported AAHS rates *before* a country joined EU. The timeline of the countries in our dataset who join EU is reported in Table 7.

2. For ROW as a partner, we use data for partner = *World* to fill in the missing values of the corresponding reporter-industry-year. For ROW as a reporter, we use the weighted average of AAHS rates of all other 40 countries in the dataset.

3. If a AAHS rate is still missing after the previous step, use AHHS rate of ROW imputed above to fill in the corresponding industry-year.

4. Finally, when the AAHS rate is not available in a particular year, we input this value using the closest value available in the *past* years. If none of the past-years data are available, we use the closest available one in the proceeding years.

We use the WIOD data on pre-tariff trade flows and WITS to calibrate the model. To do so, we match the dollar values of pre-tariff trade flows and pre-tariff sales from WIOD. Given the pre-tariff IO data, we compute total sales and total trade flows (including tariffs) using the tariff rates from WITS. We readjust WIOD value-added for all sectors in order to match WIOD data on pre-tariff output and total producer costs, where total costs include VA, pre-tariff costs of intermediate goods and tariffs on intermediate goods.

H Reallocation Effects

This appendix discusses the reallocation effects in Theorem 2 and compares them to terms-of-trade effects.

by 1988	by 1995	by 2004	by 2007
Belgium	Austria	Cyprus	Bulgaria
Denmark	Finland	Czech Republic	Romania
France	Sweden	Estonia	
Germany		Hungary	
Greece		Latvia	
Italy		Lithuania	
Ireland		Malta	
Luxembourg		Poland	
Netherlands		Slovakia	
Portugal		Slovenia	
Spain			
United Kingdom			

Table 7: (New) Countries in WIOD that join EU by year

To better understand reallocation effects, we define the change in the *factoral terms of trade* to be $\sum_{f\in F}(\Lambda^c_f - \tilde{\Lambda}^{W_c}_f)\,\mathrm{d}\log w_f$. With fixed factor supplies and in the absence of transfers, the reallocation effect is given by the change in the factoral terms of trade.[10] Intuitively, the factoral terms of trade weighs the change in each factor's price by the households "net" position to the price of that factor, since each factor may contribute to the household's earnings Λ^c_f as well as that household's expenditures $\tilde{\Lambda}^{W_c}_f$. For instance, if household c owns factor f and is the only consumer of services produced by factor f, $\Lambda^c_f = \tilde{\Lambda}^{W_c}_f$ and changes in the price of factor f are irrelevant for welfare.

Once we aggregate to the level of the world, as long as there are no wedges, there are no reallocation effects. This follows from Theorem 2 since $\Lambda^Y_f = \Lambda^W_f = \Lambda_f$ for all factors f and since $\mathrm{d}\,T = 0$. Hence, at the world level, there are only "pure" technology effects but no reallocation effects. Furthermore, the "pure" technology effect and the reallocation effect at the country level aggregate up to their world counterparts. This implies that, if there are no wedges in the initial equilibrium, then country reallocations sum up to zero:

$$\sum_{c\in C}\frac{GNE_c}{GNE}(\mathrm{d}\log\mathcal{W}_c/\,\mathrm{d}\,\mathcal{X})\,\mathrm{d}\,\mathcal{X} = (\mathrm{d}\log\mathcal{W}/\,\mathrm{d}\,\mathcal{X})\,\mathrm{d}\,\mathcal{X} = 0.$$

Reallocation effects can therefore be interpreted as zero-sum distributive changes. When there are pre-existing wedges, reallocation effects are no longer zero-sum, since they can make everyone better or worse off by changing the efficiency of resource allocation.

We now define the terms-of-trade decomposition, and proceed to compare, theoretically

[10] Our definition can be seen as a formalization and a generalization of the "double factoral terms of trade" (in changes) discussed in Viner (1937).

and empirically, the differences between reallocation effects, as defined by Theorem 2 and Terms of Trade effects.

H.1 Terms-of-Trade Decomposition

We characterize an alternative decomposition of welfare in terms of output and terms-of-trade effects. We then contrast this decomposition with the reallocation decomposition, and provide some notable special cases under which the terms-of-trade decomposition or the reallocation decomposition take especially simple forms.

We start by defining each country's share of global GDP and GNE.

$$\chi_c^Y = \frac{GDP_c}{GDP}, \qquad \chi_c^W = \frac{GNE_c}{GNE}.$$

Then we have the following.

Proposition 9 (Welfare-Accounting, Terms of Trade). *The change in welfare of country c in response to productivity shocks, factor supply shocks, and transfer shocks can be decomposed into:*[11]

$$\mathrm{d}\log W_c = \underbrace{\frac{\chi_c^Y}{\chi_c^W}\,\mathrm{d}\log Y_c}_{\Delta Output} + \underbrace{\frac{\chi_c^Y}{\chi_c^W}\,\mathrm{d}\log P_{Y_c} - \mathrm{d}\log P_{W_c}}_{\Delta Terms\ of\ Trade} + \underbrace{\frac{1}{\chi_c^W}\,\mathrm{d}\,T_c + \sum_{f\in F}(\Lambda_f^c - \frac{\chi_c^Y}{\chi_c^W}\Lambda_f^{Y_c})\,\mathrm{d}\log\Lambda_f}_{\Delta Transfers\ and\ Net\ Factor\ Payments},$$

where the change in terms of trade $(\chi_c^Y/\chi_c^W)\,\mathrm{d}\log P_{Y_c} - \mathrm{d}\log P_{W_c}$ *is*

$$\sum_{i\in N}(\tilde{\lambda}_i^{W_c} - \frac{\chi_c^Y}{\chi_c^W}\tilde{\lambda}_i^{Y_c})\,\mathrm{d}\log A_i + \sum_{f\in F}(\tilde{\Lambda}_f^{W_c} - \frac{\chi_c^Y}{\chi_c^W}\tilde{\Lambda}_f^{Y_c})\left(-\,\mathrm{d}\log\Lambda_f + \mathrm{d}\log L_f\right),$$

with $\chi_c^Y/\chi_c^W = GDP_c/GNE_c$*. The change* $\mathrm{d}\log W$ *of world real expenditure can be obtained by simply suppressing the country index c.*

To understand this result, consider for example a unit change in the productivity of producer i on the terms of trade. Intuitively, for given factor wages, the productivity shock affects the terms of trade of country c according to the difference between the country's exposures to producer i in real expenditure and in real output $\tilde{\lambda}_i^{W_c} - (\chi_c^Y/\chi_c^W)\tilde{\lambda}_i^{Y_c}$. The productivity also leads to endogenous changes in the wages of the different factors $\mathrm{d}\log w_f$,

[11] When all factors inside a country are owned by the residents of that country, $\Lambda_f^c = \Lambda_f^{Y_c}$, and so net factor payments are zero. If in addition, there are no transfers so that $T_c = 0$, then $\chi_c^Y = \chi_c^W$ and our decomposition is invariant to changes in the numeraire. Outside of this case, the choice of numeraire influences the breakdown into changes in terms of trade and changes in transfers and net factor payments, but not the sum of the two.

which given that factor supplies are fixed, coincide with the changes in their factor income shares $d\log\Lambda_f$.[12] These changes in factor wages in turn affect the country's terms of trade according to the difference between the country's exposures to producer f in real expenditure and in real output $\tilde{\Lambda}_f^{W_c} - (\chi_c^Y/\chi_c^W)\tilde{\Lambda}_f^{Y_c}$.

At the world level, there are no terms-of-trade effects (and no transfers or net factor payments). Furthermore, changes in real output and real expenditure or welfare and their corresponding deflators for each country aggregate up to their world counterparts. This implies that changes in the country terms of trade sum up to zero:

$$\sum_{c\in C}\chi_c^W[(\chi_c^Y/\chi_c^W)\,d\log P_{Y_c} - d\log P_{W_c}] = d\log P_Y - d\log P_W = 0,$$

where $\chi_c^W = GNE_c/GNE$ and $\chi_c^Y = GDP_c/GDP$. Terms-of-trade effects can therefore be interpreted as zero-sum distributive effects. The same goes for transfers and net factor payments.

H.2 Terms of Trade vs. Reallocation

Proposition 9 and Theorem 2 provide two decompositions of changes in real expenditure or welfare with different economic interpretations: the terms-of-trade and reallocation decompositions. As long as there are no distortions, both the reallocation effects and the terms-of-trade effects (and the net-factor-payments and transfer effects) can be interpreted as zero-sum distributive effects, and both of them can be written in terms of changes in factor shares. The goal of this section is to compare the two decompositions. Throughout this discussion, for simplicity, we assume there are no wedges in the initial equilibrium.

Two Hulten-Like Results

To frame our discussion, it is useful to start by stating two different Hulten-like results for welfare in open economies. We call these "Hulten-like" results because they predict changes in welfare as a function of initial expenditure shares only *without* requiring information on changes in (endogenous) factor shares.

Corollary 8 (Welfare, Two Hulten-like Results)**.** *In the following two special cases, Hulten-like results give changes in the welfare of a country c as exposure-weighted sums of productivity and factor supply shocks (and do not feature changes in factor shares).*

[12] The formula actually still applies with endogenous factor supplies.

(i) Assume that country c receives no transfers from the rest of the world (balanced trade), there are no cross-border factor holdings, and international prices are exogenous and fixed (small-open economy). Then there are only real output effects, and no terms-of-trade, transfer effects, or net factor payment effects, so that the change in welfare is given by

$$\mathrm{d}\log W_c = \mathrm{d}\log Y_c = \sum_{f\in F_c} \Lambda_f^{Y_c} \,\mathrm{d}\log L_f + \sum_{i\in N_c} \lambda_i^{Y_c} \,\mathrm{d}\log A_i.$$

(ii) Assume either that the world economy is Cobb-Douglas or, if it is not, that we keep the allocation of resources (the allocation matrix) constant. Then there are only "pure" technology effects and no reallocation effects, so that the change welfare is given by:

$$\mathrm{d}\log W_c = \sum_{f\in F} \Lambda_f^{W_c} \,\mathrm{d}\log L_f + \sum_{i\in N} \lambda_i^{W_c} \,\mathrm{d}\log A_i.$$

Corollary 8 follows from Theorems 9 and 2. It shows that in some special cases, we can continue to use exposures to predict the effects of productivity and factor supply shocks on welfare in open economies.

The two Hulten-like results are very different. Focusing on productivity shocks, the elasticities $\mathrm{d}\log W_c \,/\, \mathrm{d}\log A_i$ of real expenditure to productivity shocks are given by exposures in real output $\lambda_i^{Y_c}$ in case (i) and by exposures in welfare $\lambda_i^{W_c}$ in case (ii).

The intuitions underlying the two Hulten-like results are also very different. The original Hulten theorem applies in a closed economy (e.g. the world) where there are neither terms-of-trade effects nor reallocation effects. In case (i), there are no terms-of-trade effects but there are reallocation effects. In case (ii), there are no reallocation effects, but there are terms-of-trade effects.

More generally, we can interpret the real output effects in Theorem 9 and the "pure" technology effects in Theorem 2 as Hulten-like terms, and the terms-of-trade effects (together with transfers and net factor payments) and reallocation effects as adjustment terms. As we saw earlier, these adjustment terms are zero-sum and depend on changes in factor shares.

Comparing the Terms-of-Trade and Reallocation Decompositions

Both decompositions can be applied at the level of a country and the world. Both decompositions isolate a distributive zero-sum term, which aggregates up to zero at the level of the world economy. These different distributive terms are responsible for departures from two different versions of Hulten's theorem. The main difference between the two decomposition is their economic interpretations.

Beyond their differences in interpretation, the two decompositions have different robustness and aggregation properties, and different data requirements. In these regards, the reallocation decomposition has several advantages.

First the reallocation decomposition is based on general equilibrium counterfactual: "pure" changes in technology coincide with the change in real expenditure that would arise under the feasible counterfactual allocation which keeps the allocation of resources constant. This is not the case for the terms-of-trade decomposition: changes in real output are *not* the changes in real expenditure that would arise under a specified feasible counterfactual allocation.

Second, as discussed in Section 4, this particular general equilibrium counterfactual is extremely useful conceptually and intuitively in order to unpack our counterfactual results. This is because reallocation effects (but not "pure" technology effects) depend *only* on expenditure substitution by the different producers and households in the economy. By contrast, terms-of-trade effects also include technology effects.

Third, the reallocation decomposition is not sensitive to irrelevant changes in the environment, because it does not use changes in real output. This is not the case for the terms-of-trade decomposition: for example, assuming that changes in iceberg trade costs apply to the importers of a good or to its exporter simply produces different representations of the same underlying changes in the economy and is immaterial for changes in welfare, but it does modify the changes in terms of trade of the importers and of the exporter.

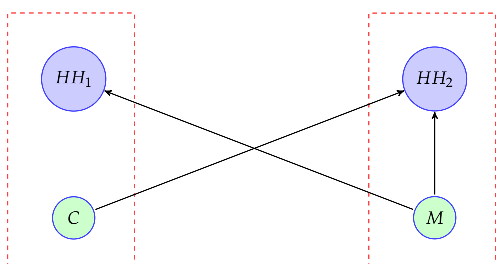


Figure 6: An illustration of the two welfare decompositions in an economy with two countries, two factors, and two goods. Country 1 has an endowment of a commodity good (C), and country 2 has an endowment of the manufacturing good (M). The representative household in country 1 consumes only the manufacturing good, and the representative household in country 2 consumes a CES aggregate of the two goods with an elasticity of substitution θ.

Fourth, the two decompositions have different economic interpretations. It is useful to provide a simple illustrative example. Consider the economy depicted in Figure 6 with two countries, two factors, and two goods. Country 1 has an endowment of a commodity good (C), and country 2 has an endowment of the manufacturing good (M). The representative household in country 1 consumes only the manufacturing good, and the representative household in country 2 consumes a CES aggregate of the two goods with an elasticity of substitution θ:

$$\left(\bar{\omega}_{2C}(\frac{y_{2C}}{\bar{y}_{2C}})^{\frac{\theta-1}{\theta}}+\bar{\omega}_{2M}(\frac{y_{2M}}{\bar{y}_{2M}})^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}.$$

C and M can either be substitutes ($\theta > 1$) or complements ($\theta < 1$). We denote by λ_2 the sales share of the consumption bundle of producer 2, and by Λ_C and Λ_M the sales shares of C and M (the factor income shares), with $\lambda_2\bar{\omega}_{2C} = \Lambda_C$.

Consider a shock $\mathrm{d}\,\bar{\omega}_{2M} = -\,\mathrm{d}\,\bar{\omega}_{2C} > 0$ which shifts the composition of demand away from C and towards M in country 2. [13] The shock reduces the welfare of country 1 with

$$\mathrm{d}\log W_1 = -\theta\frac{1}{\Lambda_M}\,\mathrm{d}\log\bar{\omega}_{2C} < 0.$$

There are neither real output nor "pure" technology effects, and there are equivalent negative terms-of-trade effects and reallocation effects:

$$\mathrm{d}\log p_C - \mathrm{d}\log p_M = \mathrm{d}\log\Lambda_C - \mathrm{d}\log\Lambda_M = -\theta\frac{1}{\Lambda_M}\,\mathrm{d}\log\bar{\omega}_{2C} < 0.$$

This can be seen as a simple illustration of the Prebisch-Singer hypothesis, whereby demand shifts towards manufacturing as countries develop at the expense of commodity producers.

Consider next a shock $\mathrm{d}\log C > 0$ which increases the endowment of C in country 1. The effect of the shock is different depending on whether C and M are substitutes (complements): it improves (reduces) the welfare of country 1 with

$$\mathrm{d}\log W_1 = (\theta-1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log C;$$

there are positive real output effects $\mathrm{d}\log Y_1 = \Lambda_C\,\mathrm{d}\log C > 0$ and less (more) negative terms-of-trade effects

$$\mathrm{d}\log p_C - \mathrm{d}\log p_M = -\Lambda_C\,\mathrm{d}\log C + (\theta-1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log C;$$

[13] This shock can be modeled as a combination of positive and negative productivity shocks $\mathrm{d}\log A_{2M} = [\theta/(\theta-1)]\,\mathrm{d}\log\mathrm{d}\,\bar{\omega}_{2M}$ and $\mathrm{d}\log A_{2C} = [\theta/(\theta-1)]\,\mathrm{d}\log\mathrm{d}\,\bar{\omega}_{2C}$ for fictitious producers intermediating between C, M, and the representative household of country 2.

there are no "pure" technology effects, and positive (negative) reallocation effects

$$\mathrm{d}\log\Lambda_C - \mathrm{d}\log\Lambda_M = (\theta-1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log C.$$

Finally, consider a shock which increases the endowment of M in country 2. This shock improves the welfare of country 1 as long as goods are not too substitutable

$$\mathrm{d}\log W_1 = \mathrm{d}\log M - (\theta-1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log M;$$

there are no real output effects and positive terms-of-trade effects as long as goods are not too substitutes with

$$\mathrm{d}\log p_C - \mathrm{d}\log p_M = \mathrm{d}\log M - (\theta-1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log M;$$

there are positive "pure" technology effects $\mathrm{d}\log M > 0$ and negative (positive) reallocation effects if C and M are substitutes (complements) with

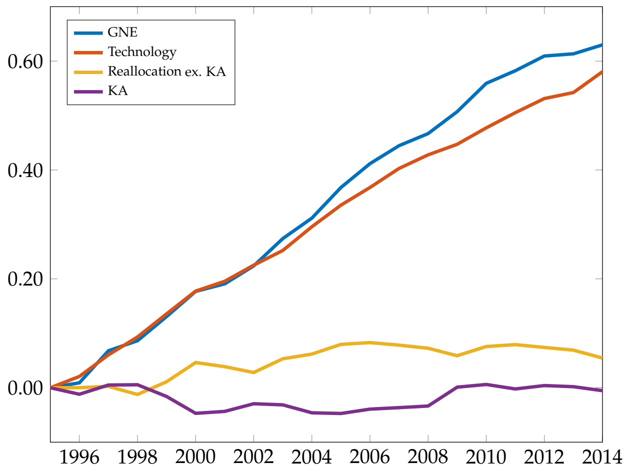
$$\mathrm{d}\log\Lambda_C - \mathrm{d}\log\Lambda_M = -(\theta-1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log M.$$

H.3 Application of Welfare-Accounting Formulas

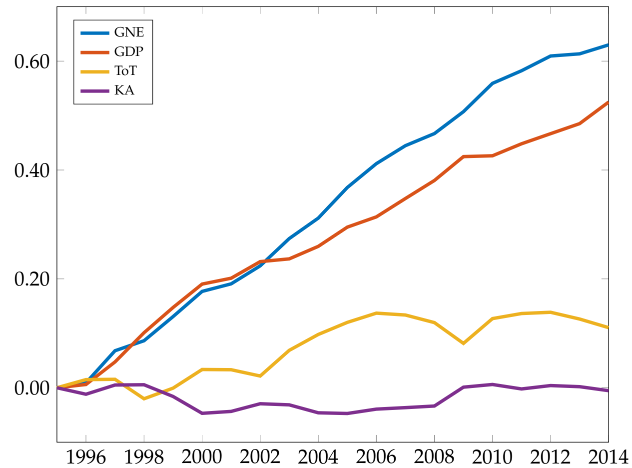
We end this discussion of the welfare-accounting formulas by decomposing the change in real expenditure in different countries over time. We implement our two decompositions: the reallocation decomposition and the terms-of-trade decomposition. We abstract away from distortions. Unlike our previous applications, these decompositions are non-parametric in the sense that they do not require taking a stand on the various elasticities of substitution.

The left column of Figure 7 displays the cumulative change in each component over time of the reallocation decomposition, for a few countries (Canada, China, and Japan). We choose these three countries because they depict a systematic pattern: industrializing countries, like China, and commodities- or services-dependent industrialized countries, like Canada, are experiencing positive reallocation, whereas manufacturing-dependent industrialized countries, like Japan, are experiencing negative reallocation.

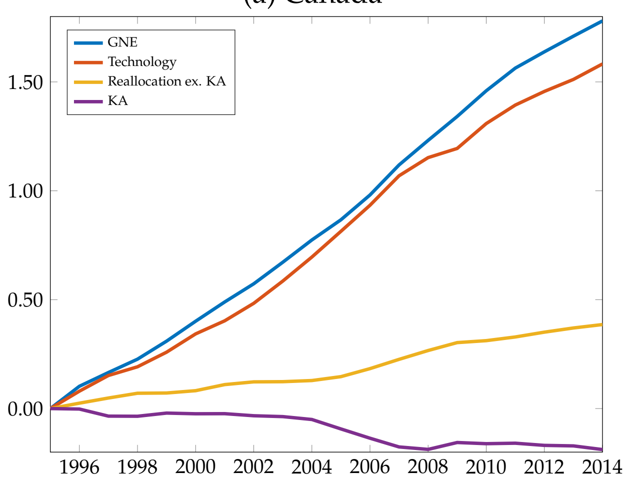
The right column of Figure 7 displays the terms-of-trade decomposition. Commodity producers like Canada experience large movements in terms of trade due to fluctuations in commodity prices. Even for countries for which terms-of-trade effects are small, reallocation effects are typically large, indicating that these countries cannot be taken to be



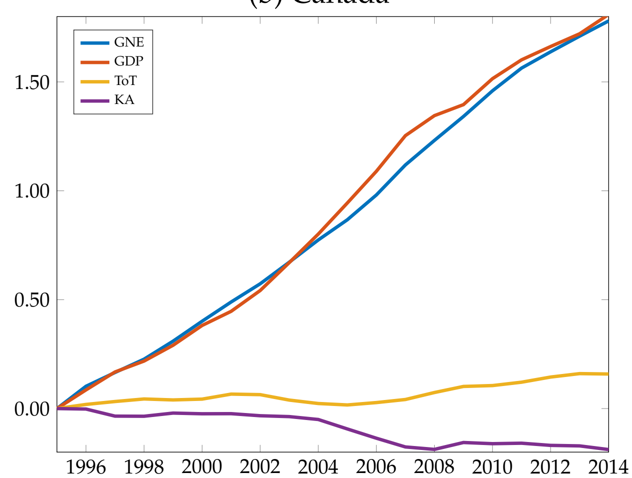
(a) Canada



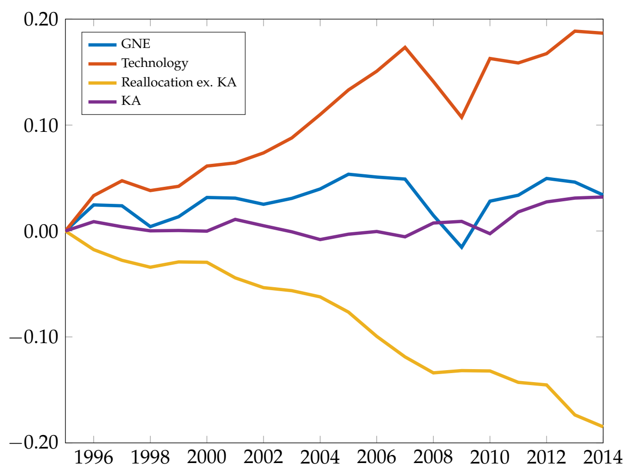
(b) Canada



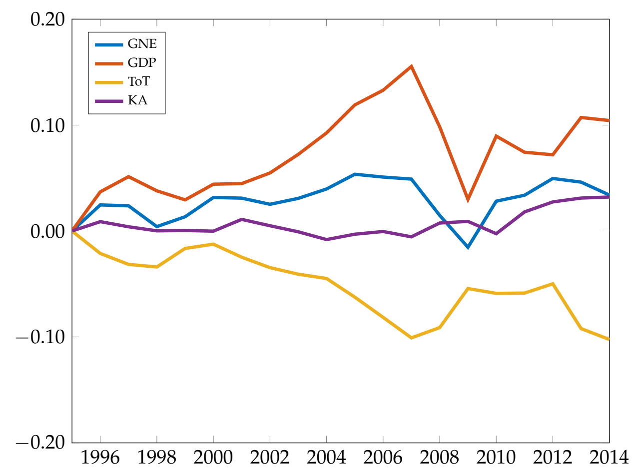
(c) China



(d) China



(e) Japan



(f) Japan

Figure 7: Welfare accounting according to the reallocation decomposition (left column) and according to the terms-of-trade decomposition (right column), for a sample of countries, using the WIOD data. KA are changes due to net transfers.

approximately closed.

Finally, it is interesting to note that the difference between the reallocation effect on the one hand, and the terms of trade effect and the transfer effect on the other hand identifies the following technological residual:[14]

$$\sum_{i\in N}((\chi_c^Y/\chi_c^W)\,\lambda_i^{Y_c}-\lambda_i^{W_c})\,\mathrm{d}\log A_i+\sum_{f\in F}((\chi_c^Y/\chi_c^W)\Lambda_f^{Y_c}-\Lambda_f^{W_c})\,\mathrm{d}\log L_c.$$

This residual is a measure of the difference between country c's technological change and its exposure to world technical change, including the effects of changes in productivities and in factor supplies. For a closed economy, it is always zero. By comparing the two columns of Figure 7, we can see that (and by how much) China and Canada are experiencing faster growth in productivities and factor supplies in their domestic real output than in their consumption baskets, while the pattern is reversed for Japan.

I Partial Equilibrium Counterpart to Theorem 4

Proposition 10. *For a small open economy operating in a perfectly competitive world market, the introduction of import tariffs reduces the welfare of that country's representative household by*

$$\Delta W \approx \frac{1}{2}\sum_i \lambda_i \Delta\log y_i \Delta\log\mu_i,$$

where μ_i is the ith gross tariff (no tariff is $\mu_i = 1$), y_i is the quantity of the ith import, and λ_i is the corresponding Domar weight.

Proof. To prove this, let $e(p)W$ be the expenditure function of the household. We have $e(p)W = p\cdot q + \sum_i(\mu_i - 1)p_i y_i$. Differentiate this once to get $c\cdot \mathrm{d}\,p + e(p)\,\mathrm{d}\,W = q\cdot \mathrm{d}\,p + \mathrm{d}\,q\cdot p + \sum_i \mathrm{d}\,\mu_i p_i y_i + \sum_i(\mu_i-1)\,\mathrm{d}(p_i y_i)$. Theorem 2 implies that this can be simplified to $e(p)\,\mathrm{d}\,W = (q-c)\cdot\mathrm{d}\,p + \sum_i \mathrm{d}\,\mu_i p_i y_i + \sum_i(\mu_i-1)\,\mathrm{d}(p_i y_i) = \sum_i(\mu_i-1)\,\mathrm{d}(p_i y_i)$, where the left-hand side is the equivalent variation. Now differentiate this again, and evaluate at $\mu_i = 1$ to get $\sum_i p_i\,\mathrm{d}\,y_i$. Hence the second-order Taylor approximation, at $\mu = 1$, is $\frac{1}{2}\sum_i \mathrm{d}\,\mu_i p_i\,\mathrm{d}\,y_i = \frac{1}{2}\sum_i \mathrm{d}\log\mu_i p_i y_i\,\mathrm{d}\log y_i$, and our normalization implies $p_i y_i$ is equal to its Domar weight. ■

[14] That is, we compute $(\partial\log\mathcal{W}_h)(\partial\mathcal{X})\,\mathrm{d}\,\mathcal{X} - (1/\chi_c^W)\,\mathrm{d}\,T_c - (\chi_c^Y/\chi_c^W)\,\mathrm{d}\log P_{Y_c} + \mathrm{d}\log P_{W_c}$.

J Duality with Multiple Factors and Tariff Revenues

The duality between trade shocks in an open economy and productivity shocks in a closed economy extends beyond the one-factor case. In the multi-factor case with pre-existing tariffs, external productivity shocks (like iceberg shocks) in the open economy translate into productivity shocks *and* shocks to factor prices in the closed economy. In this section, we establish this duality. As an example application, in Appendix L, we show how the model in Galle et al. (2017), which studies the distributional consequences of trade with a Roy model, can be generalized to economies with production networks.

With multiple factors and tariffs, we must use the change in the dual price deflator $\Delta \log \check{P}_{W_c} = \Delta \log \check{P}_{Y_c} = \Delta \log \check{p}_c$ of the dual economy for given changes in factor prices and not the change in real expenditure or welfare for given factor supplies. This requires the choice of a numeraire in the dual closed economy: we use the nominal GDP, which means that we normalize the nominal GDP of the dual closed economy to one. If there are import tariffs, the input-output table should be written gross of any tariffs (that is, including expenditures on tariffs by importers).

Theorem 11 (Exact Duality)**.** *The discrete change in welfare $\Delta \log W_c$ of the original open economy in response to discrete shocks to iceberg trade costs or productivities outside of country c is equal to (minus) the discrete change in the price deflator $-\Delta \log \check{P}_{Y_c}$ of the dual closed economy in response to discrete shocks to productivities $\Delta \log \check{A}_i = -(1/\varepsilon_i)\Delta \log \Omega_{ic}$ and discrete shocks to the productivities of the factors $\Delta \log \check{A}_f = -\Delta \log \Lambda_f^c$. This duality result is global in that it holds exactly for arbitrarily large shocks.*

In other words, shocks to the open economy are equivalent to productivity and factor price shocks in the closed economy. Note that if there are tariffs, tariff revenues imply reductions in factor income shares in the original open economy, which translates into positive shocks to the productivities of the factors in the dual closed economy.

Corollary 9 (First-Order Duality)**.** *A first-order approximation to the change in welfare of the original open economy is:*

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} \approx \sum_{i \in M_c + F_c} \check{\lambda}_i \Delta \log \check{A}_i,$$

where applying Hulten's theorem, $\check{\lambda}_i$ is the sales share of producer i when $i \in M_c$ and the sales share of factor i in the dual closed economy (which we also sometimes write $\check{\Lambda}_i$).

Corollary 10 (Second-Order Duality). *A second-order approximation to the change in welfare of the original open economy is:*

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} \approx \sum_{i \in M_c + F_c} \check{\lambda}_i \Delta \log \check{A}_i - \frac{1}{2} \sum_{i,j \in M_c + F_c} \frac{d^2 \log \check{P}_{Y_c}}{\mathrm{d} \log \check{A}_j \,\mathrm{d} \log A_i} \Delta \log \check{A}_j \Delta \log \check{A}_i,$$

where applying Baqaee and Farhi (2017a),

$$-\frac{d^2 \log \check{P}_{Y_c}}{\mathrm{d} \log \check{A}_j \,\mathrm{d} \log \check{A}_i} = \frac{\mathrm{d}\, \check{\lambda}_i}{\mathrm{d} \log \check{A}_j} = \sum_{k \in N_c} (\theta_k - 1) \check{\lambda}_k Cov_{\check{\Omega}^{(k)}} \left(\check{\Psi}_{(i)}, \check{\Psi}_{(j)} \right).$$

We can re-express the second-order approximation to the change in welfare of the original open economy as:

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} \approx \sum_{i \in M_c + F_c} \check{\lambda}_i \Delta \log \check{A}_i + \frac{1}{2} \sum_{k \in N_c} (\theta_k - 1) \check{\lambda}_k Var_{\check{\Omega}^{(k)}} \left(\sum_{i \in M_c + F_c} \check{\Psi}_{(i)} \Delta \log \check{A}_i \right).$$

Corollary 11 (Exact Duality and Nonlinearities with an Industry Structure). *For country c with an industry structure, we have the following exact characterization of the nonlinearities in welfare changes of the original open economy.*

- *(i) (Industry Elasticities) Consider two economies with the same initial input-output matrix and industry structure, the same trade elasticities, but with lower elasticities across industries for one than for the other so that $\theta_\kappa \leq \theta'_\kappa$ for all industries κ. Then $\Delta \log W_c = \Delta \log \check{Y}_c \leq \Delta \log W'_c = \Delta \log \check{Y}'_c$ so that negative (positive) shocks have larger negative (smaller positive) welfare effects in the economy with the lower industry elasticities.*
- *(ii) (Cobb-Douglas) Suppose that all the elasticities of substitution across industries (and with the factor) are equal to unity ($\theta_\kappa = 1$), then $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ is linear in $\Delta \log \check{A}$.*
- *(iii) (Complementarities) Suppose that all the elasticities of substitution across industries (and with the factor) are below unity ($\theta_\kappa \leq 1$), then $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ is concave in $\Delta \log \check{A}$, and so nonlinearities amplify negative shocks and mitigate positive shocks.*
- *(iv) (Substituabilities) Suppose that all the elasticities of substitution across industries (and with the factor) are above unity ($\theta_\kappa \geq 1$), then $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ is convex in $\Delta \log \check{A}$, and so nonlinearities mitigate negative shocks and amplify positive shocks.*
- *(v) (Exposure Heterogeneities) Suppose that industry κ is uniformly exposed to the shocks as they unfold, so that $Var_{\check{\Omega}_s^{(\kappa)}} \left(\sum_{\iota \in \mathcal{M}_c + \mathcal{F}_c} \check{\Psi}_{(\iota),s} \Delta \log \check{A}_\iota \right) = 0$ for all s where s indexes the*

dual closed economy with productivity shocks $\Delta \log \check{A}_{\iota,s} = s\Delta \log \check{A}_{\iota}$, *then* $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ *is independent of* θ_κ. *Furthermore*

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} = \sum_{\iota \in \mathcal{M}_c + \mathcal{F}_c} \check{\lambda}_\iota \Delta \log \check{A}_\iota + \int_0^1 \sum_{\kappa \in \mathcal{N}_c} (\theta_\kappa - 1) \check{\lambda}_{\kappa,s} Var_{\check{\Omega}_s^{(\kappa)}} \left(\sum_{\iota \in \mathcal{M}_c + \mathcal{F}_c} \check{\Psi}_{(\iota),s} \Delta \log \check{A}_\iota \right) (1-s) ds.$$

Closed-form Expression for Example in Figure 3

The exact expression for the impact of the trade shock on welfare can be found in closed form by exploiting the recursive structure of the contraction mapping described in the proof of Corollary 5 because this example features no reproducibility:

$$\Delta \log W_c = -\frac{1}{1-\sigma} \log \left(\frac{M}{N} \left(\frac{N}{M} \check{\lambda}_E e^{-(1-\theta)\Delta \log \check{A}_E} + 1 - \frac{N}{M} \check{\lambda}_E \right)^{\frac{1-\sigma}{1-\theta}} + \frac{N-M}{N} \right).$$

K Computational Appendix

This appendix describes our computational procedure, as well as the Matlab code in our replication files. Before running the code, customize your folder directory in the code accordingly.

Writing nested-CES economies in standard-form is useful for intuition, but it is computationally inefficient since it greatly expands the size of the input-output matrix. Therefore, for computational efficiency, we instead use the generalization in Appendix A to directly linearize the nested-CES production functions without first putting them into standard form.

Overview

First, we provide an overview of the different files before providing an in depth description of each.

1. **main_load_data.m**: First part of main code that calculates expenditure shares from data.

2. **main_dlogW.m**: Second part of the main code that loads inputs and calls functions to iterate.

3. **AES_func.m**: Function that calculates Allen-Uzawa elasticities of substitution.

4. **Nested_CES_linear_final.m**: Function that solves the system of linear equations described in Theorem 3.

5. **Nested_CES_linear_result_final.m**: Function that calculates derivatives that are used to derive welfare changes or iterate for large shocks.

While 1. and 3. are specific to our quantitative application, 2., 4. and 5. are general purpose functions that can be used to derive comparative statics and solve any model in the class we study. We now describe each part of the code in some detail.

1. Main code that loads data

Code: main_load_data.m

Data input:

1. Number of countries (C), Number of sectors in each country (N), Number of factors in each country (F)

2. Trade elasticity when a country imports or buys domestic product (`trade_elast`: N by 1 vector)

3. Input-output matrix across country and sectors (`Omega_tilde`: CN by CN matrix, (i, j) element: expenditure share of sector i on sector j)

4. Household expenditure share on heterogenous goods (`beta`: CN by C matrix, (i, c) element: expenditure share of household c on sector i)

5. Value-added share (`alpha`: CN by 1 vector, $(i, 1)$ element: value-added share of sector i), Primary Factor share (`alpha_VA`: CN by F matrix, (i, f) element: expenditure share of sector i on factor f out of factor usage)

6. A ratio of GNE of each country to world GNE (`GNE_weights`: C by 1 vector)

7. (Optional) If economy has initial tariff,

   (a) Tariff matrix when household (column) buys goods (row) – `Tariff_cons_matrix_new`: CN by C matrix ((i, c) element: tariff rate of household c, destination, on sector i, origin)

   (b) Tariff matrix when a sector (row) buys goods (column) –`Tariff_matrix_new`: CN by CN matrix ((i, j) element: tariff rate of sector i, destination, on sector j, origin)

User input:

1. If the economy does not have initial tariff, `initial_tariff_index`= 1. Otherwise, if the economy has initial tariff, =2.

Outputs:

1. `data`, `shock` struct

From the inputs, the code automatically calculates input shares (`beta_s`, `beta_disagg`, `Omega_s`, `Omega_disagg`, `Omega_total_C`, `Omega_total_N`) and the input-output matrix (`Omega_total_ti` `Omega_total`). These variables are used to calculate Allen-Uzawa elasticities of substitution and solve system of linear equations.

2. Main code that loads inputs and calls functions

Code: main_dlogW.m

Data input:

1. `data`, `shock` struct from **main_load_data.m**

User input:

1. Elasticity of substitution parameters for nested CES structure: Elasticity of substitution (1) across Consumption (`sigma`), (2) across Composite Value-added and Intermediates (`theta`), (3) across Primary Factors (`gamma`), and (4) across Intermediate Inputs (`epsilon`)

2. If the economy gets universal iceberg trade cost shock, `shock_index` = 1. Otherwise, if the economy gets universal tariff shock, = 2.

3. When intensity of shock is x%, `intensity` = x.

4. When shock is discretized by x/y% and model cumulates the effect of shocks y times, `ngrid` = y.

5. Ownership structure

   (a) Ownership structure of factor (`Phi_F`: C by CF matrix, (c, f) element: Factor income share of factor f owned by household c)

(b) Ownership structure of tariff revenue (`Phi_T`: C+CN by CN+CF by C 3-D matrix, (i,j,c) element: Tariff revenue share owned by household c when household/sector i buys from sector/factor j)

6. (Optional) Technical details about how to customize iceberg trade cost shock matrix $dlog\tau$ and tariff shock matrix $dlogt$ are described in **Nested_CES_linear_final.m**

Output:

1. `dlogW` (C by `ngrid` matrix) collects change in real income of each country for each iteration of discretized shocks

2. `dlogW_sum` (C by 1 vector) shows change in real income of each country from linearized system by summing up `dlogW`

3. `dlogW_world` (1 by `ngrid` vector) is change in real income of world for each iteration of discretized shocks

4. `dlogR` (C by `ngrid` matrix) collects reallocation terms of each country for each iteration of discretized shocks

5. `dlogR_sum` (C by 1 vector) shows reallocation terms of each country from linearized system by summing up `dlogR`

6. `dlogY_2nd` shows change in world GDP to a 2nd order

3. Allen-Uzawa Elasticity of Substitution (AES)

This code computes Allen-Uzawa elasticities of substitution for each sector. These are then used following Appendix A.

Code: AES_func.m

Inputs:

1. Number of countries (C), Number of sectors in each country (N), Number of factors in each country (F)

2. Elasticity of substitution parameters for nested CES structure: Elasticity of substitution (1) across Consumption (`sigma`), (2) across Composite Value-added and Intermediates (`theta`), (3) across Primary Factors (`gamma`),and (4) across Intermediate Inputs (`epsilon`)

3. Trade elasticity when a country imports or buys domestic product (`trade_elast`: N by 1 vector)

4. Value-added share (`alpha`: CN by 1 vector, $(i, 1)$ element: value-added share of sector i)

5. Input shares:

   (a) b_{ic}(`beta_s` : C by N matrix, (c, i) element: How much household c consumes sector i good)

   (b) ω_j^{ic}(`Omega_s`: CN by N matrix, (ic, j) element: How much sector i in country c uses sector j good)

   (c) $\tilde{\Omega}_{jm}^{0c}$(`Omega_total_C` : C by CN matrix, (c, jm) element: How much household c buys from sector j in country m)

   (d) $\tilde{\Omega}_{jm}^{ic}$(`Omega_total_N` : CN by CN+CF matrix, (ic, jm) element: Hom much sector i in country c buys from good/factor j in country m)

Outputs:

1. $\theta_{0c}(ic', jm)$ (`AES_C_Mat`: CN by CN by C 3-D matrix, (ic', jm, c) element: AES of household in country c that substitutes good i in country c' and good j in country m)

2. $\theta_{kc}(ic', jm)$ (`AES_N_Mat`: CN by CN+CF by CN 3-D matrix, (ic', jm, kc) element: AES of producer of sector k in country c that substitutes good i in country c' and good/factor j in country m)

3. $\theta_{kc}(fc, jm)$ (`AES_F_Mat`: CF by CN+CF by CN 3-D matrix, (fc, jm, kc) element: AES of producer of sector k in country c that substitutes factor f in country c and good j in country m)

To describe how this code functions, we introduce the following notation.

Notation:

Let $p_{ic'}^{kc}$ be the bilateral price when industry or household k in country c buys from industry i in country c'. That is

$$p_{ic'}^{kc} = \tau_{ic'}^{kc} t_{ic'}^{kc} p_{ic'},$$

where $\tau_{ic'}^{kc}$ is an iceberg cost on kc purchasing goods from ic' and $t_{ic'}^{kc}$ is a tariff on kc purchasing goods from ic', and where $p_{ic'}$ is the marginal cost of producer i in country c'.

Define

$$\Omega_{jm}^{ic} = \frac{p_{jm} x_{jm}^{ic}}{p_{ic} y_{ic}}, \quad \tilde{\Omega}_{jm}^{ic} = \frac{t_{jm}^{ic} p_{jm} x_{jm}^{ic}}{p_{ic} y_{ic}},$$

where $p_{jm} x_{jm}^{ic}$ is expenditures of ic on jm not including the import tariff. Notice that every row of $\tilde{\Omega}_{jm}^{ic}$ should always sum up to 1. Also, assume that C is a set of countries, and F_c is the factors owned by Household in country c. Then,

Households: The price of final consumption in country c

$$P_{0c} = \left(\sum_i b_{ic} \left(P_i^{0c} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

where $b_{ic} = \sum_{m \in C} \tilde{\Omega}_{im}^{0c}$. The price of consumption good from industry i in country c

$$P_i^{0c} = \left(\sum_{m \in C} \delta_m^{0c} \left(t_{im}^{0c} \tau_{im}^{0c} p_{im} \right)^{1-\theta_i} \right)^{\frac{1}{1-\theta_i}},$$

where $\delta_m^{0c} = \tilde{\Omega}_{im}^{0c} / \left(\sum_{v \in C} \tilde{\Omega}_{iv}^{0c} \right)$.

Producers: The marginal cost of good i produced by country c

$$p_{ic} = \left(\alpha_{ic} P_{w_{ic}}^{1-\theta} + (1 - \alpha_{ic}) P_{M_{ic}}^{1-\theta} \right)^{\frac{1}{1-\theta}}$$

where $\alpha_{ic} = \sum_{f \in F_c} \tilde{\Omega}_{fc}^{ic}$. The price of value-added bundled used by producer i in country c

$$p_{w_{ic}} = \left(\sum_{f \in F_c} \alpha_f^{ic} w_{fc}^{1-\gamma} \right)^{\frac{1}{1-\gamma}},$$

where $\alpha_f^{ic} = \tilde{\Omega}_{fc}^{ic} / \left(\sum_{d \in F_c} \tilde{\Omega}_{dc}^{ic} \right)$. The price of intermediate bundle used by producer i in country c

$$p_{M_{ic}} = \left(\sum_j \omega_j^{ic} \left(q_j^{ic} \right)^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}},$$

where $\omega_j^{ic} = \left(\sum_{m \in C} \tilde{\Omega}_{jm}^{ic} \right) / (1 - \alpha_{ic})$. The price of intermediate bundle good j used by producer i in country c

$$q_j^{ic} = \left(\sum_{m \in C} \delta_{jm}^{ic} \left(\tau_{jm}^{ic} t_{jm}^{ic} p_{jm} \right)^{1-\theta_i} \right)^{\frac{1}{1-\theta_i}},$$

where $\delta_{jm}^{ic} = \tilde{\Omega}_{jm}^{ic} / \left(\sum_{v \in C} \tilde{\Omega}_{iv}^{ic} \right)$.

Deriving Allen-Uzawa elasticities for nested-CES models is straightforward. To do so, we proceed as follows:

Derivation:

(1) $\theta_{0c}(ic', jm)$ Household demand in country c for good i from c' is

$$x^{0c}_{ic'} = \tilde{\Omega}^{0c}_{ic'} \left(\frac{p^{0c}_{ic'}}{P^{0c}_{i}} \right)^{-\theta_i} \left(\frac{P^{0c}_{i}}{P^{0c}} \right)^{-\sigma} C_c$$

Hence

$$\theta_{0c}(ic', jm) = \frac{1}{\tilde{\Omega}^{0c}_{jm}} \frac{\partial \log x^{0c}_{ic'}}{\partial \log p^{0c}_{jm}} = -\theta_i \frac{\left(\mathbf{1}(jm = ic') - \mathbf{1}(j = i)\delta^{0c}_{jm} \right)}{\tilde{\Omega}^{0c}_{jm}} - \frac{\sigma \left(\mathbf{1}(j = i)\delta^{0c}_{jm} - \tilde{\Omega}^{0c}_{jm} \right)}{\tilde{\Omega}^{0c}_{jm}}.$$

This can be simplified as

$$\theta_{0c}(ic', jm) = \frac{\theta_i}{\sum_{v \in C} \tilde{\Omega}^{0c}_{iv}} + \sigma \left(1 - \frac{1}{\sum_{v \in C} \tilde{\Omega}^{0c}_{iv}} \right) = \frac{\theta_i}{b_{ic}} + \sigma \left(1 - \frac{1}{b_{ic}} \right) \text{ when } i = j \ \&\ ic' \neq jm,$$

$$\theta_{0c}(ic', jm) = -\frac{\theta_i}{\tilde{\Omega}^{0c}_{jm}} + \frac{\theta_i}{b_{ic}} + \sigma \left(1 - \frac{1}{b_{ic}} \right) \text{ when } ic' = jm.$$

Otherwise, $\theta_{0c}(ic', jm) = \sigma$.

(2) $\theta_{kc}(ic', jm)$ When k is not a household, demand by k in country c for good i from c' is

$$x^{kc}_{ic'} = \tilde{\Omega}^{kc}_{ic'} \left(\frac{p^{kc}_{ic'}}{P^{kc}_{i}} \right)^{-\theta_i} \left(\frac{P^{kc}_{i}}{P^{kc}_{M}} \right)^{-\varepsilon} \left(\frac{P^{kc}_{M}}{p_{kc}} \right)^{-\theta} Y_{kc}.$$

Hence

$$\theta_{kc}(ic', jm) = \frac{1}{\tilde{\Omega}^{kc}_{jm}} \frac{\partial \log x^{kc}_{ic'}}{\partial \log p^{kc}_{jm}} = -\theta_i \frac{\left(\mathbf{1}(jm = ic') - \mathbf{1}(j = i)\delta^{kc}_{jm} \right)}{\tilde{\Omega}^{kc}_{jm}} - \frac{\varepsilon \left(\mathbf{1}(j = i)\delta^{kc}_{jm} - \mathbf{1}(j \notin F)\delta^{kc}_{jm}\omega^{kc}_{j} \right)}{\tilde{\Omega}^{kc}_{jm}}$$
$$- \frac{\theta \left(\mathbf{1}(j \notin F)\delta^{kc}_{jm}\omega^{kc}_{j} - \tilde{\Omega}^{kc}_{jm} \right)}{\tilde{\Omega}^{kc}_{jm}}.$$

This can be simplified as

$$\theta_{kc}(ic',jm)=\frac{\theta_i}{(1-\alpha_{kc})\omega_j^{kc}}+\epsilon\left(\frac{1}{1-\alpha_{kc}}-\frac{1}{(1-\alpha_{kc})\omega_j^{kc}}\right)$$
$$+\theta\left(1-\frac{1}{1-\alpha_{kc}}\right)\text{ when } i=j\in N \;\&\; ic'\neq jm,$$

$$\theta_{kc}(ic',jm)=-\frac{\theta_i}{\tilde{\Omega}_{jm}^{kc}}+\frac{\theta_i}{(1-\alpha_{kc})\omega_j^{kc}}+\epsilon\left(\frac{1}{1-\alpha_{kc}}-\frac{1}{(1-\alpha_{kc})\omega_j^{kc}}\right)+\theta\left(1-\frac{1}{1-\alpha_{kc}}\right)\text{ when } ic'=jm,$$
$$\theta_{kc}(ic',jm)=\frac{\epsilon}{1-\alpha_{kc}}+\theta\left(1-\frac{1}{1-\alpha_{ic}}\right)\text{ when } i\neq j\in N,$$

and when $j\in F$, $\theta_{kc}(ic',jm)=\theta$.

(3) $\theta_{kc}(fc,jm)$ Lastly, when k is not a household, demand by k in country c for factor f is

$$x_{fc}^{kc}=\tilde{\Omega}_{fc}^{kc}\left(\frac{p_{fc}}{p_{w_{kc}}}\right)^{-\gamma}\left(\frac{p_{w_{kc}}}{p^{kc}}\right)^{-\theta}Y_{kc}.$$

Hence,

$$\theta_{kc}(fc,jm)=\frac{1}{\tilde{\Omega}_{jm}^{kc}}\frac{\partial\log x_{fc}^{kc}}{\partial\log p_{jm}^{kc}}=-\gamma\frac{\left(\mathbf{1}(jm=fc)-\mathbf{1}(jm\in F_c)\alpha_j^{ic}\right)}{\tilde{\Omega}_{jm}^{kc}}-\theta\frac{\left(\mathbf{1}(jm\in F_c)\alpha_j^{ic}-\tilde{\Omega}_{jm}^{kc}\right)}{\tilde{\Omega}_{jm}^{kc}}.$$

Notice that $\theta_{kc}(fc,jm)=\theta$ if $j\in N$. Also,

$$\theta_{kc}(fc,jc)=\frac{\gamma}{\sum_{g\in F_c}\tilde{\Omega}_{gc}^{kc}}+\theta\left(1-\frac{1}{\sum_{g\in F_c}\tilde{\Omega}_{gc}^{kc}}\right)=\frac{\gamma}{\alpha_{kc}}+\theta\left(1-\frac{1}{\alpha_{kc}}\right)\text{ when } j\in F \;\&\; m=c,$$

$$\theta_{kc}(fc,jc)=-\frac{\gamma}{\tilde{\Omega}_{fc}^{kc}}+\frac{\gamma}{\alpha_{kc}}+\theta\left(1-\frac{1}{\alpha_{kc}}\right)\text{ when } fc=jm.$$

4. Solving system of linear equations

Code: Nested_CES_linear_final.m

Input:

1. Number of countries (C), Number of sectors in each country (N), Number of factors in each country (F)

2. Allen-Uzawa elasticities of substitution:

   (a) $\theta_{0c}(ic',jm)$ (`AES_C_Mat`: CN by CN by C 3-D matrix)

(b) $\theta_{kc}(ic', jm)$ (`AES_N_Mat`: CN by CN+CF by CN 3-D matrix)

(c) $\theta_{kc}(fc, jm)$ (`AES_F_Mat` CF by CN+CF by CN 3-D matrix)

3. Input-output matrix and Leontief inverse

   (a) $\tilde{\Omega}^{ic}_{jm}$ (`Omega_total_tilde`: C+CN+CF by C+CN+CF matrix) : Standard form of Cost-based IO matrix

   (b) Ω^{ic}_{jm} (`Omega_total`: C+CN+CF by C+CN+CF matrix) : Standard form of Revenue-based IO matrix

   (c) $\tilde{\Psi}^{ic}_{jm}$ (`Psi_total_tilde`) : Leontief inverse of $\tilde{\Omega}^{ic}_{jm}$

   (d) Ψ^{ic}_{jm} (`Psi_total`) : Leontief inverse of Ω^{ic}_{jm}

4. Initial sales share λ_{CN} (`lambda_CN`: C+CN by 1 vector) and factor income Λ_F (`lambda_F`: CF by 1 vector)

5. Ownership structure of factor (`Phi_F`: C by CF matrix) and tariff revenue (`Phi_T`: C+CN by CN by C 3-D matrix) defined in **main_dlogW.m**

6. (Optional) If economy has initial tariff, initial tariff matrix (`init_t`: C+CN by CN matrix) defined in **main_load_data.m**

Current version of code simulates universal iceberg trade cost or tariff shock. If the user wants to specify the shocks, customize

1. universal iceberg trade cost shock matrix (`dlogtau`: C+CN by CN+CF matrix, (i, j) element: log change in iceberg trade cost when household/sector i buys from sector/factor j) or

2. tariff shock matrix (`dlogt`: C+CN by CN+CF matrix, (i, j) element: log change in tariff when household/sector i buys from sector/factor j).

Output:

Let $d\Lambda_F$ be the vector of changes in the sales of primary factors and

$$d\Lambda_{F,c',*} = \sum_{ic}\sum_{jm} \Phi_{c',ic,jm}\Omega^{ic}_{jm}(t^{ic}_{jm} - 1)d\lambda_{ic}$$

be the change in wedge-revenues of household c' due to changes in sales shares, where $\Phi_{c',ic,jm}$ is the share of tax revenues on ic's purchases of jm that go to household c'. The linear system in Theorem 3 can be written as:

$$\begin{bmatrix} d\Lambda_F \\ d\Lambda_{F_*} \end{bmatrix} = A \begin{bmatrix} d\Lambda_F \\ d\Lambda_{F_*} \end{bmatrix} + B$$

This code outputs:

1. `A` (C+CF by C+CF matrix) and `B` (C+CF by 1 vector).

Using these outputs, the code inverts the system and solves for $d\Lambda_F$(`dlambda_F`) and $d\Lambda_{F_*}$ (`dlambda_F_star`), which are used to obtain derivatives calculated by **Nested_CES_linear_result_final.m**. It updates $\tilde{\Omega}$ and other variables which are used in the next iteration.

5. Calculate derivatives

Code: Nested_CES_linear_result_final.m

Input:

All inputs used in **Nested_CES_linear_final.m** are also used in this code. Additionally, it requires

1. `GNE_weights` (C by 1 vector): A ratio of GNE of each country to world GNE
2. $d\Lambda_F$(`dlambda_F`) and $d\Lambda_{F^*}$(`dlambda_F_star`) : Solutions from **Nested_CES_linear_final.m**

Output:

1. $d\lambda$ (`dlambda_result`: C+CN+CF by 1 vector): Change in sales shares;
2. $d\chi$ (`dchi_std`: C+CN+CF by 1 vector): Change in household income shares;
3. $dlogP$ (`dlogP_Vec`: C+CN+CF by 1 vector): Change in either the price index (household), marginal cost (sector), or factor price;
4. $d\tilde{\Omega}^{ic}_{jm}$ (`dOmega_total_tilde`: C+CN+CF by C+CN+CF matrix) : Change in Cost-based IO matrix;
5. $d\Omega^{ic}_{jm}$ (`dOmega_total`: C+CN+CF by C+CN+CF matrix) : Change in Revenue-based IO matrix.

For each iteration, change in real income of country c is

$$d \log W_c = d \log \chi_c - d \log P_c$$

where $d \log P_c$ is change in price index of household c. Meanwhile, outputs are used to update $\lambda, \chi, \Omega, \tilde{\Omega}$, which are used as a simulated data with discretized shock in next iteration.

L Extension to Roy Models

Galle et al. (2017) combine a Roy-model of labor supply with an Eaton-Kortum model of trade to study the effects of trade on different groups of workers in an economy. They prove an extension to the Arkolakis et al. (2012) result that accounts for the distributional consequences of trade shocks. In this section, we show how our framework can be adapted for analyzing such models. We generalize our analysis to encompass Roy-models of the labor market, and show how duality with the closed economy can then be used to study the distributional consequences of trade.

Suppose that H_c denotes the set of households in country c. As in Galle et al. (2017), households consume the same basket of goods, but supply labor in different ways. We assume that each household type has a fixed endowment of labor L_h, which are assigned to work in different industries according to the productivity of workers in that group and the relative wage differences offered in different industries.

As usual, let world GDP be the numeraire. Define Λ_f^h to be type h's share of income derived from earning wages f

$$\Lambda_f^h = \frac{\Phi_{hf} \Lambda_f}{\chi_h},$$

where $\chi_h = \sum_{k \in F} \Phi_{hk} \Lambda_k$. The Roy model of Galle et al. (2017) implies that

$$\frac{\chi_h}{\overline{\chi_h}} = \left(\sum_f \overline{\Lambda}_f^h \left(\frac{w_f}{\overline{w}_f} \right)^{\gamma_h} \right)^{\frac{1}{\gamma_h}} \frac{L^h}{\overline{L}^h},$$

where γ_h is the supply elasticity, variables with overlines are initial values, L^h is the stock of labor h has been endowed with (since we analyze log changes, only shocks to the endowment value are relevant). Galle et al. (2017) show that the above equations can be microfounded via a model where homogenous workers in each group type draw their ability for each job from Frechet distributions, and choose to work in the job that offers them the highest return. The Roy model generalizes the factor market, with $\gamma_h = 1$ representing the case where labor cannot be moved across markets by h. If $\gamma_h > 1$ then h can take advantage of wage differentials to redirect its labor supply and boost its income. When $\gamma \to \infty$, labor mobility implies that all wages in the economy are equalized (and the model collapses to a one-factor model).

Proposition 12 (Exact Duality). *The discrete change in welfare $\Delta \log W_h$ of group $h \in H_c$ of the original open economy in response to discrete shocks to iceberg trade costs or productivities outside of country c is equal to (minus) the discrete change in the price deflator $-\Delta \log \check{P}_{Y_g}$ of the dual closed economy in response to discrete shocks to productivities $\Delta \log \check{A}_i = -(1/\varepsilon_i)\Delta \log \Omega_{ic}$ and discrete shocks to factor wages $\Delta \log \check{A}_f = -\frac{1}{\gamma_h}\Delta \log \Lambda^h_f$. This duality result is global in that it holds exactly for arbitrarily large shocks.*

In the case where $\gamma \to \infty$, we recover the one-factor version of Duality in Theorem 7.

Of course, due to the fact that factor shares Λ^h_f are endogenously respond to factor prices, Theorem 3 can no longer be used to determine how these shares will change in equilibrium. Therefore, we extend those propositions here.

Proposition 13. *The response of the factor prices to a shock* $\mathrm{d}\log A_k$ *is the solution to the following system:*

1. *Product Market Equilibrium:*

$$\Lambda_l \frac{\mathrm{d}\log \Lambda_l}{\mathrm{d}\log A_k} = \sum_{j\in\{H,N\}} \lambda_j(1-\theta_j) Cov_{\Omega^{(j)}}\left(\Psi_{(k)} + \sum_f \Psi_{(f)}\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}, \Psi_{(l)}\right) + \sum_{h\in H}(\lambda_l^{W_h} - \lambda_l)\left(\sum_{f\in F_c}\Phi_{hf}\Lambda_f \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}\right).$$

2. *Factor Market Equilibrium:*

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H} E_{\Phi^{(h)}}\left[\gamma_h\left(E_{\Lambda^{(h)}}\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right)\right) + \left(E_{\Lambda^{(h)}}(\mathrm{d}\log w)\right) + (\mathrm{d}\log L)\right].$$

Given this, the welfare of the hth group is

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log A_k} = \sum_{s\in F}\left(\Lambda^h_s - \Lambda^{W_h}_s\right)\mathrm{d}\log w_s + \lambda_k^{W_h} + \mathrm{d}\log L^h.$$

The product market equilibrium conditions are exactly the same as those in Theorem 3, but now we have some additional equations from the supply-side of the factors (which are no longer endowments). Letting $\gamma_h = 1$ for every $h \in H$ recovers Theorem 3.

M Heterogenous Households Within Countries

To extend the model to allow for a set of heterogenous agents $h \in H_c$ within country $c \in C$, we proceed as follows. We denote by H the set of all households. Each household h in country c maximizes a homogenous-of-degree-one demand aggregator

$$C_h = \mathcal{W}_h(\{c_{hi}\}_{i \in N}),$$

subject to the budget constraint

$$\sum_{i \in N} p_i c_{hi} = \sum_{f \in F} \Phi_{hf} w_f L_f + T_h,$$

where c_{hi} is the quantity of the good produced by producer i and consumed by the household, p_i is the price of good i, Φ_{hf} is the fraction of factor f owned by household, w_f is the wage of factor f, and T_h is an exogenous lump-sum transfer.

We define the following country aggregates: $c_{ci} = \sum_{h \in H_c} c_{hi}$, $\Phi_{cf} = \sum_{h \in H_c} \Phi_{hf}$, and $T_c = \sum_{h \in H_c} T_h$. We also define the HAIO matrix at the household level as a $(H + N + F) \times (H + N + F)$ matrix Ω and the Leontief inverse matrix as $\Psi = (I - \Omega)^{-1}$.

All the definitions in Section 2 remain the same. In addition, we introduce the corresponding household-level definitions for a household h. First, the nominal output and the nominal expenditure of the household are:

$$GDP_h = \sum_{f \in F} \Phi_{hf} w_f L_f, \quad GNE_h = \sum_{i \in N} p_i c_{hi} = \sum_{f \in F} \Phi_{hf} w_f L_f + T_h,$$

where we think of the household as a set producers intermediating the uses by the different producers of the different factor endowments of the household. Second, the changes in real output and real expenditure or welfare of the household are:

$$\mathrm{d}\log Y_h = \sum_{f \in F} \chi_f^{Y_h} \,\mathrm{d}\log L_f, \quad \mathrm{d}\log P_{Y_h} = \sum_{f \in F} \chi_f^{Y_h} \,\mathrm{d}\log w_f,$$

$$\mathrm{d}\log W_h = \sum_{i \in N} \chi_i^{W_h} \,\mathrm{d}\log c_{hi}, \quad \mathrm{d}\log P_{W_h} = \sum_{i \in N} \chi_i^{W_h} \,\mathrm{d}\log p_i,$$

with $\chi_f^{Y_h} = \Phi_{hf} w_f L_f / GDP_h$ and $\chi_i^{W_h} = p_i c_{hi} / GNE_h$. Third, the exposure to a good or factor k of the real expenditure and real output of household h is given by

$$\lambda_k^{W_h} = \sum_{i \in N} \chi_i^{W_h} \Psi_{ik}, \quad \lambda_k^{Y_h} = \sum_{f \in F} \chi_f^{Y_h} \Psi_{fk},$$

where recall that $\chi_i^{W_h} = p_i c_{hi}/GNE_h$ and $\chi_f^{Y_h} = \Phi_{hf} w_f L_f/GDP_h$. The exposure in real output to good or factor k has a direct connection to the sales of the producer:

$$\lambda_k^{Y_h} = 1_{\{k \in F\}} \frac{\Phi_{hk} p_k y_k}{GDP_h},$$

where $\lambda_k^{Y_h} = 1_{\{k \in F\}} \Phi_{hk} (GDP/GDP_h) \lambda_k$ the local Domar weight of k in household h and where $\Phi_{hk} = 0$ for $k \in N$ to capture the fact that the household endowment of the goods are zero. Fourth, the share of factor f in the income or expenditure of the household is given by

$$\Lambda_f^h = \frac{\Phi_{hf} w_f L_f}{GNE_h}.$$

The results in Section 3 go through without modification. Theorems 1 and 2 can be extended to the level of a household h by simply replacing the country index c by the household index h.

The results in Section 4 go through except the term on the second line of (6) must be replaced by

$$\sum_{h \in H} \frac{\lambda_i^{W_h} - \lambda_i}{\lambda_i} \Phi_{hf} \Lambda_f,$$

where we write λ_i and Λ_i interchangeably when $i \in F$ is a factor.

The results in Section 5 go through with the following changes. Theorem 4 goes through without modification, and extends to the household level where $\Delta \log Y_h \approx 0$. Proposition 5 goes through with some minor modifications. The world Bergson-Samuelson welfare function is now $W^{BS} = \sum_h \overline{\chi}_h^W \log W_h$, changes in world welfare are measured as $\Delta \log \delta$, where δ solves the equation $W^{BS}(\overline{W}_1, \ldots, \overline{W}_H) = W^{BS}(W_1/\delta, \ldots, W_H/\delta)$, where $\overline{W}_h$ are the values at the initial efficient equilibrium. We use a similar definition for country level welfare δ_c, and the same notation for household welfare δ_h. Changes in world welfare are given up to the second order by

$$\Delta \log \delta \approx \Delta \log W + Cov_{\chi_h^W} \left(\Delta \log \chi_h^W, \Delta \log P_{W_h} \right),$$

changes in country welfare are given up to the first order by

$$\Delta \log \delta_c \approx \Delta \log W_c \approx \Delta \log \chi_c^W - \Delta \log P_{W_c},$$

and the change in country welfare up to the first order by

$$\Delta \log \delta_h \approx \Delta \log W_h \approx \Delta \log \chi_h^W - \Delta \log P_{W_h}.$$

Theorems 6 goes through with some minor modifications. The final term on the last line must be replaced by

$$\frac{1}{2}\sum_{l\in N}\sum_{c\in H}\chi_c^W \Delta\log\chi_c^W \Delta\log\mu_l(\lambda_l^{W_c}-\lambda_l).$$

N Proofs

Throughout the proofs, let χ_c be the share of total world income accruing to country c.

Proof of Theorem 1. Nominal GDP is equal to

$$P_{Y_c} Y_c = \sum_{i \in N_c} (1 - 1/\mu_i) p_i y_i + \sum_{f \in F_c} w_f L_f$$

Hence

$$\begin{aligned} d\log P_{Y_c} + d\log Y_c &= \sum_{i \in N_c} (1 - 1/\mu_i) \lambda_i^{Y_c} d\log \left((1 - 1/\mu_i) \lambda_i^{Y_c} \right) \\ &\quad + \sum_{f \in F_c} \Lambda_f^{Y_c} \left(d\log w_f + d\log L_f \right) \\ d\log Y_c &= \sum_{i \in N_c} (1 - 1/\mu_i) \lambda_i^{Y_c} d\log \left((1 - 1/\mu_i) \lambda_i^{Y_c} \right) \\ &\quad + \sum_{f \in F_c} \Lambda_f^{Y_c} \left(d\log w_f + d\log L_f \right) - d\log P^{Y_c}. \end{aligned}$$

The price of domestic goods is given by

$$d\log p_i = d\log \mu_i - d\log A_i + \sum_{j \in N_c} \tilde{\Omega}_{ij} d\log p_j + \sum_{j \notin N_c} \tilde{\Omega}_{ij} d\log p_j,$$

which implies that

$$d\log p = (I - \tilde{\Omega}^D)^{-1} \left(d\log \mu_i - d\log A_i + \tilde{\Omega}^F \left(d\log \Lambda - d\log L \right) + \tilde{\Omega}^M d\log p^M \right),$$

where $\tilde{\Omega}^D$ is the cost-based domestic IO table, $\tilde{\Omega}^F$ are cost-based factor shares, and $\tilde{\Omega}^M$ are cost-based intermediate import shares, and $d\log p^M$ represents the change in the price of imported intermediate goods. Use the fact that

$$\begin{aligned} d\log P_{Y_c} &= \sum_{i \in N_c} \Omega_{Y_c, i} d\log p_i - \sum_{i \in N - N_c} \Lambda_i^{Y_c} d\log p_i \\ &= \sum_{i \in N_c} \tilde{\lambda}_i^{Y_c} \left(d\log \mu_i - d\log A_i \right) + \sum_{f \in F_c} \tilde{\Lambda}_f^{Y_c} \left(d\log \Lambda_f - d\log L_f \right) \\ &\quad + \sum_{i \in N - N_c} \tilde{\Lambda}_i^{Y_c} d\log p_i - \sum_{i \in N - N_c} \Lambda_i^{Y_c} d\log p_i. \end{aligned}$$

For an imported intermediate

$$d\log p_i = d\log \Lambda_i^{Y_c} - d\log q_i + d\log GDP$$

Substitute this back to get

$$\begin{aligned}
d\log Y_c &= \sum_{i\in N_c}(1-1/\mu_i)\lambda_i^{Y_c} d\log\left((1-1/\mu_i)\lambda_i^{Y_c}\right) + \sum_{f\in F_c}\Lambda_f^{Y_c}\left(d\log w_f + d\log L_f\right) \\
&- \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c}\left(d\log\mu_i - d\log A_i\right) - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\left(d\log\Lambda_f - d\log L_f\right) \\
&- \sum_{i\in N-N_c}\tilde{\Lambda}_i^{Y_c} d\log p_i + \sum_{i\in N-N_c}\Lambda_i^{Y_c} d\log p_i \\
&= \sum_{f\in F_c^*}\Lambda_f^{Y_c} d\log\Lambda_f - \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c}\left(d\log\mu_i - d\log A_i\right) - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\left(d\log\Lambda_f - d\log L_f\right) \\
&- \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c} - d\log q_i + d\log GDP\right) \\
&= \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log A_i + \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log L_f + \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right) d\log q_i \\
&+ \sum_{f\in F_c^*}\Lambda_f^{Y_c}\left(d\log\Lambda_f^{Y_c} + d\log GDP_c\right) - \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log\mu_i - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\left(d\log\Lambda_f^{Y_c} + d\log GDP_c\right) \\
&- \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c} + d\log GDP\right) \\
&= \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log A_i + \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log L_f + \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right) d\log q_i \\
&- \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log\mu_i - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log\Lambda_f^{Y_c} - \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c}\right) \\
&+ \left[1 - \left(\sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\right) - \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\right] d\log GDP_c \\
&= \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log A_i + \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log L_f + \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right) d\log q_i \\
&- \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log\mu_i - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log\Lambda_f^{Y_c} - \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c}\right).
\end{aligned}$$

The last line follows from the fact that

$$\sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} + \sum_{i\in N-N_c}\tilde{\Lambda}_i^{Y_c} = \left[1 + \sum_{i\in N-N_c}\Lambda_i^{Y_c}\right].$$

■

Proof of Theorem 2. Note that welfare is given by

$$W_c = \frac{\sum_{f \in F} \Phi_{cf} w_f L_f}{P^{W_c}}.$$

Hence, letting world GDP be the numeraire,

$$\mathrm{d}\log W_c = \sum_f \Lambda_f^c \left(\mathrm{d}\log \Lambda_f\right) - \left(\tilde{\Omega}_{(W_c)}\right)' \mathrm{d}\log p.$$

Use the fact that

$$\mathrm{d}\log p_i = \sum_{j \in N} \tilde{\Psi}_{ij} \,\mathrm{d}\log A_j + \sum_{f \in F} \tilde{\Psi}_{if} \left(\mathrm{d}\log \Lambda_f - \mathrm{d}\log L_f\right)$$

to complete the proof. ∎

Proof of Theorem 3. For each good,

$$\lambda_i = \sum_c \Omega_{W_c,i} \chi_c + \sum_i \Omega_{ji} \lambda_j,$$

where χ_c is the share of total income accruing to country c and $\Omega_{W_c,i}$ is the share of income household c spends on good i. This means

$$\lambda_i \,\mathrm{d}\log \lambda_i = \sum_c \chi_c \Omega_{W_c,i} \,\mathrm{d}\log \Omega_{W_c,i} + \sum_j \Omega_{ji} \lambda_j \,\mathrm{d}\log \Omega_{ji} + \sum_j \Omega_{ji} \,\mathrm{d}\,\lambda_j + \sum_c \Omega_{W_c,i} \chi_c \,\mathrm{d}\log \chi_c.$$

Now, note that

$$\mathrm{d}\log \Omega_{W_c,i} = (1-\theta_c)\left(\mathrm{d}\log p_i - \mathrm{d}\log P_{y_c}\right)$$

$$\mathrm{d}\log \Omega_{ji} = (1-\theta_j)\left(\mathrm{d}\log p_i - \mathrm{d}\log P_j + \mathrm{d}\log \mu_j\right) - \mathrm{d}\log \mu_j$$

$$\mathrm{d}\log \chi_c = \sum_{f \in F_c^*} \frac{\Lambda_f}{\chi_c} \,\mathrm{d}\log \Lambda_f + \sum_{i \in c} \frac{\lambda_i}{\mu_i} \,\mathrm{d}\log \mu_i.$$

$$\mathrm{d}\log p_i = \tilde{\Psi}\left(\mathrm{d}\log \mu - \mathrm{d}\log A\right) + \tilde{\Psi}\tilde{\alpha} \,\mathrm{d}\log \Lambda.$$

$$\mathrm{d}\log P_{y_c} = b'\tilde{\Psi}\left(\mathrm{d}\log \mu - \mathrm{d}\log A\right) + b'\tilde{\Psi}\tilde{\alpha} \,\mathrm{d}\log \Lambda.$$

For shock $\mathrm{d}\log \mu_k$, we have

$$\mathrm{d}\log \Omega_{W_c,i} = (1-\theta_c)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \,\mathrm{d}\log \Lambda_f - \sum_j \Omega_{W_c,j}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf} \,\mathrm{d}\log \Lambda_f\right)\right).$$

$$d \log \Omega_{ji} = (1-\theta_j) \left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \, d \log \Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf} \, d \log \Lambda_f \right) - \theta_j \, d \log \mu_j.$$

Putting this altogether gives

$$\begin{aligned} d\lambda_l = & \sum_i \sum_c (1-\theta_c)\chi_c \Omega_{W_c,i} \left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \, d \log \Lambda_f - \sum_j \Omega_{W_c,j} \left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf} \, d \log \Lambda_f \right) \right) \Psi_{il} \\ & + \sum_i \sum_j (1-\theta_j)\lambda_j \mu_j^{-1} \tilde{\Omega}_{ji} \left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \, d \log \Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf} \, d \log \Lambda_f \right) \Psi_{il} \\ & - \theta_k \lambda_k \sum_i \Omega_{ki} \Psi_{il} + \sum_c \chi_c \sum_i \Omega_{W_c,i} \Psi_{il} \, d \log \chi_c. \end{aligned}$$

Simplify this to

$$\begin{aligned} d\lambda_l = & \sum_c (1-\theta_c)\chi_c \left[\sum_i \Omega_{W_c,i} \left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \, d \log \Lambda_f \right) \Psi_{il} \right. \\ & \left. - \left(\sum_i \Omega_{W_c,i} \left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf} \, d \log \Lambda_f \right) \right) \left(\sum_i \Omega_{W_c,i} \Psi_{il} \right) \right] \\ & + \sum_j (1-\theta_j)\lambda_j \mu_j^{-1} \sum_i \tilde{\Omega}_{ji} \left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \, d \log \Lambda_f \right) \Psi_{il} - \left(\sum_i \tilde{\Omega}_{ji} \Psi_{il} \right) \left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf} \, d \log \Lambda_f \right) \\ & - \theta_k \lambda_k \left(\Psi_{kl} - \mathbf{1}(l=k) \right) + \sum_c \chi_c \sum_i \Omega_{W_c,i} \Psi_{il} \, d \log \chi_c. \end{aligned}$$

Simplify this further to get

$$\begin{aligned} d\lambda_l = & \sum_c (1-\theta_c)\chi_c Cov_{b^{(c)}} \left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)} \, d \log \Lambda_f, \Psi_{(l)} \right) \\ & + \sum_j (1-\theta_j)\lambda_j \mu_j^{-1} \sum_i \tilde{\Omega}_{ji} \left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \, d \log \Lambda_f \right) \Psi_{il} \\ & - \left(\sum_i \tilde{\Omega}_{ji} \Psi_{il} \right) \left(\sum_i \tilde{\Omega}_{ji} \tilde{\Psi}_{ik} + \sum_i \tilde{\Omega}_{ji} \sum_f \Psi_{if} \, d \log \Lambda_f \right) \\ & - \theta_k \lambda_k \left(\Psi_{kl} - \mathbf{1}(l=k) \right) + \sum_c \chi_c \sum_i \Omega_{W_c,i} \Psi_{il} \, d \log \chi_c, \end{aligned}$$

Using the input-output covariance notation, write

$$d\lambda_l = \sum_c (1-\theta_c)\chi_c Cov_{\Omega_{(W_c)}} \left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)} \, d \log \Lambda_f, \Psi_{(l)} \right)$$

$$+\sum_j (1-\theta_j)\lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}\left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\, d\log\Lambda_f, \Psi_{(l)}\right)$$

$$-(1-\theta_k)\lambda_k(\Psi_{kl} - \mathbf{1}(l=k)) - \theta_k\lambda_k\left(\Psi_{kl} - \mathbf{1}(l=k)\right) + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\, d\log\chi_c,$$

This then simplifies to give from the fact that $\sum_i \Omega_{W_c,i}\Psi_{il} = \lambda_l^{W_c}$:

$$\lambda_l\, d\log\lambda_l = \sum_{j\in N,C} (1-\theta_j)\lambda_j\mu_j^{-1} Cov(\tilde{\Psi}_{(k)} + \sum_f^F d\log\Lambda_f, \Psi_{(l)})$$
$$-\lambda_k\left(\Psi_{kl} - \mathbf{1}(k=l)\right) + \sum_c \chi_c \lambda_l^{W_c}\, d\log\chi_c.$$

To complete the proof, note that

$$P_{y_c}Y_c = \sum_f w_f L_f + \sum_{i\in N_c}\left(1 - \frac{1}{\mu_i}\right) p_i y_i.$$

Hence,

$$d(P_{y_c}Y_c) = \sum_{f\in c} w_f L_f\, d\log w_f + \sum_{i\in c}\left(1-\frac{1}{\mu_i}\right) p_i y_i\, d\log(p_i y_i) + \sum_{i\in c}\frac{d\left(1-\frac{1}{\mu_i}\right)}{d\log\mu_i} p_i y_i\, d\log\mu_i.$$

In other words, since $P_y Y = 1$, we have

$$d\,\chi_c = \sum_{f\in c}\Lambda_f\, d\log w_f + \sum_{i\in c}\left(1-\frac{1}{\mu_i}\right)\lambda_i\, d\log\lambda_i + \sum_{i\in c}\frac{d\left(1-\frac{1}{\mu_i}\right)}{d\log\mu_i}\lambda_i\, d\log\mu_i.$$

Hence,

$$d\log\chi_c = \sum_{f\in F_c^*}\frac{\Lambda_f}{\chi_c}\, d\log\Lambda_f + \sum_{i\in c}\frac{\lambda_i}{\chi_c}\, d\log\mu_i.$$

■

Proof of Theorem 4. Proof of Part(1):

The expression for $d^2\log Y$ follows from applying part (2) to the whole world. The equality of real GNE and real GDP at the world level completes the proof.

Proof of Part (2):

Denote the set of imports into country c by M_c. Then, we can write:

$$\frac{d\log Y_c}{d\log \mu_i} = \sum_{f\in F_c} \Lambda_f^{Y_c} \frac{d\log \Lambda_f}{d\log \mu_i} + \sum_j \frac{d\,\lambda_j}{d\log \mu_i} \frac{\left(1-\frac{1}{\mu_j}\right)}{P_{Y_c} Y_c} + \frac{\lambda_i^{Y_c}}{\mu_i} - \frac{d\log P_{Y_c}}{d\log \mu_i},$$

where

$$\frac{d\log P_{Y_c}}{d\log \mu_i} = \sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c} \frac{d\log \Lambda_f}{d\log \mu_i} + \sum_{m\in M_c} \tilde{\lambda}_m^{Y_c} \frac{d\log p_m}{d\log \mu_i} - \tilde{\lambda}_i^{Y_c} - \sum_{m\in M_c} \Lambda_m^{Y_c} \frac{d\log p_m}{d\log \mu_i},$$

and

$$\tilde{\lambda}_i^{Y_c} = \sum_j \Omega_{Y_c,j} \tilde{\Psi}_{ji}.$$

Combining these expressions, we get

$$\begin{aligned}\frac{d\log Y_c}{d\log \mu_i} &= \sum_{f\in F_c} \left(\Lambda_f^{Y_c} - \tilde{\Lambda}_f^{Y_c}\right) \frac{d\log \Lambda_f}{d\log \mu_i} + \sum_{m\in M_c} \left(\lambda_m^{Y_c} - \tilde{\lambda}_m^{Y_c}\right) \frac{d\log p_m}{d\log \mu_i} \\ &+ \sum_{j\in N_c} \lambda_j^{Y_c} \frac{d\log \lambda_j}{d\log \mu_i} \left(1 - \frac{1}{\mu_j}\right) + \frac{\lambda_i^{Y_c}}{\mu_i} - \tilde{\lambda}_i^{Y_c}.\end{aligned}$$

At the efficient point,

$$\begin{aligned}\frac{d^2\log Y_c}{d\log \mu_i\, d\log \mu_k} &= \sum_{f\in F_c} \left(\frac{d\,\Lambda_f^{Y_c}}{d\log \mu_i} - \frac{d\,\tilde{\Lambda}_f^{Y_c}}{d\log \mu_i}\right) \frac{d\log \Lambda_f}{d\log \mu_k} \\ &+ \sum_{m\in M_c} \left(\frac{d\,\lambda_m^{Y_c}}{d\log \mu_i} - \frac{d\,\tilde{\lambda}_m^{Y_c}}{d\log \mu_i}\right) \frac{d\log p_m}{d\log \mu_k} - \frac{d\,\tilde{\lambda}_k^{Y_c}}{d\log \mu_i} \\ &+ \lambda_k^{Y_c} \left(\frac{d\log \lambda_k^{Y_c}}{d\log \mu_i} - \delta_{ki}\right) + \frac{1}{P_{Y_c} Y_c} \frac{d\,\lambda_i^{Y_c}}{d\log \mu_k},\end{aligned}$$

where δ_{ki} is the a Kronecker delta.

Using Lemma 15,

$$\begin{aligned}\frac{d^2\log Y_c}{d\log \mu_i\, d\log \mu_k} &= -\sum_{f\in F_c} \lambda_i^{Y_c} \Psi_{if} \frac{d\log \Lambda_f}{d\log \mu_k} - \sum_{m\in M_c} \lambda_i^{Y_c} \Psi_{im} \frac{d\log p_m}{d\log \mu_k} - \lambda_i^{Y_c} \left(\Psi_{ik} - \delta_{ik}\right) \\ &- \lambda_k^{Y_c} \delta_{ik} + \frac{d\,\lambda_i}{d\log \mu_k} \frac{1}{P_{Y_c} Y_c}, \\ &= -\sum_{f\in F_c} \lambda_i^{Y_c} \Psi_{if} \frac{d\log \Lambda_f}{d\log \mu_k} - \sum_{m\in M_c} \lambda_i^{Y_c} \Psi_{im} \frac{d\log p_m}{d\log \mu_k} - \lambda_i^{Y_c} \Psi_{ik}\end{aligned}$$

$$+ \lambda_i^{Y_c} \left(\frac{\mathrm{d}\log p_i}{\mathrm{d}\log \mu_k} + \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k} \right),$$
$$= \lambda_i^{Y_c} \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k}.$$

■

Lemma 14. *Let χ_h be the income share of country h at the initial equilibrium. Then*

$$\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log \mu_k} - \sum_h \overline{\chi}_h \frac{\mathrm{d}\log \tilde{\lambda}_j^{W_h}}{\mathrm{d}\log \mu_k} = \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i} \lambda_j^{W_h} - \lambda_i \left(\Psi_{ij} - \delta_{ij} \right).$$

Proof. Let μ be the diagonal matrix of μ_i and I_{μ_k} be a matrix of all zeros except μ_k for its kth diagonal element. Then

$$\overline{\chi}' \frac{\mathrm{d}\,\tilde{\lambda}}{\mathrm{d}\log \mu_k} = \chi' \frac{d\tilde{\Omega}_{(W)}}{\mathrm{d}\log \mu_k} + \chi' \frac{\mathrm{d}\,\tilde{\lambda}}{\mathrm{d}\log \mu_k} \mu\Omega + \chi'\tilde{\lambda} I_{\mu_k}\Omega + \chi'\tilde{\lambda}\mu \frac{\mathrm{d}\,\Omega}{\mathrm{d}\log \mu_k},$$

where $\tilde{\Omega}_{(W)}$ is a matrix whose cith element is household c's expenditure share $\tilde{\Omega}_{W_c,i}$ on good i.

On the other hand,

$$\lambda = \chi'\tilde{\Omega}_{(W)} + \lambda\Omega.$$

Form this, we have

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k} = \frac{\mathrm{d}\,\chi'}{\mathrm{d}\log \mu_k}\tilde{\Omega}_{(W)} + \chi' \frac{d\tilde{\Omega}_{(W)}}{\mathrm{d}\log \mu_k} + \lambda \frac{\mathrm{d}\,\Omega}{\mathrm{d}\log \mu_k} + \frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k}\Omega.$$

Combining these two expressions

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k} - \overline{\chi}' \frac{\mathrm{d}\log \tilde{\lambda}}{\mathrm{d}\log \mu_k} \right) = \left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k} - \overline{\chi}' \frac{\mathrm{d}\log \tilde{\lambda}}{\mathrm{d}\log \mu_k} \right) \Omega + \frac{\mathrm{d}\,\chi}{\mathrm{d}\log \mu_k}\tilde{\Omega}_{(W)} - \chi'\tilde{\lambda}^{(h)} I_{\mu_k}\Omega.$$

Rearrange this to get

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k} - \overline{\chi}' \frac{\mathrm{d}\log \tilde{\lambda}}{\mathrm{d}\log \mu_k} \right) = \frac{\mathrm{d}\,\chi}{\mathrm{d}\log \mu_k}\tilde{\Omega}_{(W)}\Psi - \chi'\tilde{\lambda}^{(h)} I_{\mu_k}(\Psi - I),$$

or

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k} - \overline{\chi}' \frac{\mathrm{d}\log \tilde{\lambda}}{\mathrm{d}\log \mu_k} \right) = \frac{\mathrm{d}\,\chi}{\mathrm{d}\log \mu_k}\tilde{\Omega}_{(W)}\Psi - \lambda I_{\mu_k}(\Psi - I).$$

■

Lemma 15. *At the efficient steady-state*

$$\frac{\mathrm{d}\,\lambda_j^{Y_c}}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_j^{Y_c}}{\mathrm{d}\log\mu_k} = -\lambda_k^{Y_c}\left(\Psi_{kj} - \delta_{kj}\right).$$

Proof. Start from the relations

$$\lambda_j^{Y_c} = \chi_j^{Y_c} + \sum_i \lambda_i^{Y_c}\Omega_{ij},$$

and

$$\tilde{\lambda}_j^{Y_c} = \chi_j^{Y_c} + \sum_i \tilde{\lambda}_i^{Y_c}\mu_i\Omega_{ij}.$$

Differentiate both to get

$$\frac{\mathrm{d}\,\lambda_j^{Y_c}}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_j^{Y_c}}{\mathrm{d}\log\mu_k} = \sum_i \left(\frac{\mathrm{d}\,\lambda_j^{Y_c}}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_j^{Y_c}}{\mathrm{d}\log\mu_k}\right)\Omega_{ij} - \lambda_k^{Y_c}\Omega_{ki}.$$

Rearrange this to get the desired result. ∎

Proof of Corollary 5. Let $\overline{\chi}_h^W$ be the elasticity of social welfare with respect to the consumption of country h (i.e. log Pareto weight). Then

$$\frac{\mathrm{d}\log W^{BS}}{\mathrm{d}\log\mu_k} = \sum_{h\in H}\overline{\chi}_h^W \frac{\mathrm{d}\log W_h}{\mathrm{d}\log\mu_k} = \sum_h \overline{\chi}_h^W\left(\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k}\right).$$

$$\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_k} = \sum_{f\in F_c}\frac{\Lambda_f}{\chi_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \sum_{i\in N_h}\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{(1-\frac{1}{\mu_i})}{\chi_h}.$$

$$\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k} = \sum_{f\in F}\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \tilde{\lambda}_k^{W_h}.$$

Hence, assuming the normalization $P_Y Y = 1$ gives

$$\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i} = \sum_h \overline{\chi}_h^W\left(\sum_f \frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{1}{\chi_h^W} + \sum_f\frac{\Lambda_f}{\chi_h^W}\frac{\mathrm{d}^2\log\Lambda_f}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}\right.$$
$$-\sum_f\frac{\Lambda_f}{\chi_h^W}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i} + \frac{\mathrm{d}\,\lambda_k}{\mathrm{d}\log\mu_i}\frac{1}{\chi_h^W\mu_k} - \frac{\lambda_k}{\chi_h^W\mu_k}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i} - \frac{\lambda_k}{\chi_h^W\mu_k}\delta_{ki}$$
$$\sum_i\frac{\mathrm{d}^2\,\lambda_j}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{1}{\mu_i\chi_h^W} + \sum_j\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h^W}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i}$$

$$-\sum_f \frac{\mathrm{d}\,\tilde{\Lambda}_f^{W_h}}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} - \sum_f \tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}^2\log\Lambda_f}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_k^{W_h}}{\mathrm{d}\log\mu_i}\Bigg).$$

At the efficient point, this simplifies to

$$\begin{aligned}\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i} = \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}&\left(\frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_i} - \sum_h \overline{\chi_h^W}\frac{\mathrm{d}\,\tilde{\Lambda}_f^{W_h}}{\mathrm{d}\log\mu_i}\right)\\ &+ \frac{\mathrm{d}\,\lambda_k}{\mathrm{d}\log\mu_i} - \sum_h \overline{\chi_h^W}\frac{\mathrm{d}\,\tilde{\lambda}_k^{W_h}}{\mathrm{d}\log\mu_i} - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i}\\ &- \lambda_k\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i} - \lambda_k\delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}.\end{aligned}$$

By Lemma 14, at the efficient point,

$$\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_i} - \sum_h \overline{\chi}_h^W\frac{\mathrm{d}\,\tilde{\lambda}_j^{W_h}}{\mathrm{d}\log\mu_i} = \sum_h \frac{\mathrm{d}\,\chi_h^W}{\mathrm{d}\log\mu_i}\tilde{\lambda}_j^{W_h} - \lambda_i\left(\Psi_{ij} - \delta_{ij}\right).$$

Whence, we can further simplify the previous expression to

$$\begin{aligned}\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i} &= \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h^W}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i\Psi_{if}\right)\\ &\quad+ \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i(\Psi_{ik} - \delta_{ik}) - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\\ &\quad- \frac{\lambda_k}{\mathrm{d}\log\chi_h}\mathrm{d}\log\mu_i - \lambda_k\delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k},\\ &= \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i\Psi_{if}\right)\\ &\quad+ \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i\Psi_{ik} - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\\ &\quad- \frac{\lambda_k}{\mathrm{d}\log\chi_h}\mathrm{d}\log\mu_i + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k},\end{aligned}$$

and using $\mathrm{d}\log\lambda_i = \mathrm{d}\log p_i + \mathrm{d}\log y_i$,

$$= \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i\Psi_{if}\right)$$

$$
\begin{aligned}
&+ \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \\
&- \frac{\lambda_k}{\mathrm{d}\log\chi_h}\mathrm{d}\log\mu_i + \lambda_i \frac{\mathrm{d}\log p_i}{\mathrm{d}\log\mu_k} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k}, \\
&= \sum_{f,h} \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} - \lambda_i \sum_f \Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \\
&+ \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \\
&- \lambda_k \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} \\
&+ \lambda_i \left(\sum_f \Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \Psi_{ik} \right), \\
&= \sum_{f,h} \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\chi_h \tilde{\Lambda}_f^{W_h} - \Lambda_f\right) \\
&+ \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_k \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\left(\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \tilde{\lambda}_k^{W_h}\right) \\
&- \sum_{f,h} \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f - \lambda_k \sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k} \\
&- \left(\sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f\right)\left(\sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\right) - \lambda_k \sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + Cov_\chi\left(\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k}\right),
\end{aligned}
$$

since

$$
-\sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f = -\sum_f \frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_k} = \frac{\mathrm{d}\left(1 - \sum_j \lambda_j (1 - \frac{1}{\mu_j})\right)}{\mathrm{d}\log\mu_k} = -\lambda_k
$$

at the efficient point, and

$$
\sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} = 0.
$$

■

Proof of Theorem 6. From Theorem 4, we have

$$\mathcal{L} = -\frac{1}{2}\sum_{l}(d\log\mu_l)\lambda_l d\log y_l.$$

With the maintained normalization $PY = 1$, we also have

$$d\log y_l = d\log\lambda_l - d\log p_l,$$

$$d\log p_l = \sum_f \Psi_{lf} d\log\Lambda_f + \sum_k \Psi_{lk} d\log\mu_k,$$

where, from Theorem 3,

$$d\log\lambda_l = \sum_k(\delta_{lk} - \frac{\lambda_k}{\lambda_l}\Psi_{kl})d\log\mu_k - \sum_j \frac{\lambda_j}{\lambda_l}(\theta_j - 1)Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log\mu_k - \sum_g \Psi_{(g)} d\log\Lambda_g, \Psi_{(l)})$$
$$+ \frac{1}{\lambda_l}\sum_{g\in F^*}\sum_c\left(\lambda_l^{W_c} - \lambda_l\right)\Phi_{cg}\Lambda_g d\log\Lambda_g,$$

and

$$d\log\Lambda_f = -\sum_k \lambda_k \frac{\Psi_{kf}}{\Lambda_f} d\log\mu_k - \sum_j \lambda_j(\theta_j - 1)Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log\mu_k - \sum_g \Psi_{(g)} d\log\Lambda_g, \frac{\Psi_{(f)}}{\Lambda_f})$$
$$+ \frac{1}{\Lambda_f}\sum_{g\in F^*}\sum_c\left(\Lambda_i^{W_c} - \Lambda_f\right)\Phi_{cg}\Lambda_g d\log\Lambda_g.$$

We will now use these expressions to replace in formula for the second-order loss function. We get

$$\mathcal{L} = -\frac{1}{2}\sum_l\sum_k(\frac{\delta_{lk}}{\lambda_k} - \frac{\Psi_{kl}}{\lambda_l} - \frac{\Psi_{lk}}{\lambda_k})\lambda_k\lambda_l d\log\mu_k d\log\mu_l + \frac{1}{2}\sum_l \lambda_l d\log\mu_l \sum_f \Psi_{lf} d\log\Lambda_f$$
$$+ \frac{1}{2}\sum_l\sum_j(d\log\mu_l)\lambda_j(\theta_j - 1)Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log\mu_k - \sum_g \Psi_{(g)} d\log\Lambda_g, \Psi_{(l)})$$
$$- \frac{1}{2}\sum_l d\log\mu_l\left(\sum_g\sum_c\left(\lambda_l^{W_c} - \lambda_l\right)\Phi_{cg}\Lambda_g d\log\Lambda_g\right)$$

$$\mathcal{L} = -\frac{1}{2}\sum_l\sum_k(\frac{\delta_{lk}}{\lambda_k} - \frac{\Psi_{kl}}{\lambda_l} - \frac{\Psi_{lk}}{\lambda_k})\lambda_k\lambda_l d\log\mu_k d\log\mu_l + \frac{1}{2}\sum_l \lambda_l d\log\mu_l \sum_f \Psi_{lf} d\log\Lambda_f$$

$$+\frac{1}{2}\sum_{l}\sum_{j}(d\log\mu_l)\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\sum_k\Psi_{(k)}d\log\mu_k-\sum_g\Psi_{(g)}d\log\Lambda_g,\Psi_{(l)})$$

$$-\frac{1}{2}\sum_l\left(\sum_c\left(\lambda_l^{W_c}-\lambda_l\right)\chi_c d\log\chi_c\right)d\log\mu_l$$

We can rewrite this expression as

$$\mathcal{L}=\mathcal{L}_I+\mathcal{L}_X+\mathcal{L}_H$$

where

$$\mathcal{L}_I=\frac{1}{2}\sum_k\sum_l[\frac{\Psi_{kl}-\delta_{kl}}{\lambda_l}+\frac{\Psi_{lk}-\delta_{lk}}{\lambda_k}+\frac{\delta_{kl}}{\lambda_l}-1]\lambda_k\lambda_l d\log\mu_k d\log\mu_l$$
$$+\frac{1}{2}\sum_k\sum_l\sum_j d\log\mu_k d\log\mu_l\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)}),$$

$$\mathcal{L}_X=\frac{1}{2}\sum_l\sum_f(\frac{\Psi_{lf}}{\Lambda_f}-1)\lambda_l\Lambda_f d\log\mu_l d\log\Lambda_f$$
$$-\frac{1}{2}\sum_l\sum_g d\log\mu_l d\log\Lambda_g\sum_j\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)}),$$

$$\mathcal{L}_H=-\frac{1}{2}\sum_l\left(\sum_c\left(\lambda_l^{W_c}-\lambda_l\right)\chi_c d\log\chi_c\right)d\log\mu_l,$$

where $d\log\Lambda$ is given by the usual expression.[15] Finally, using Lemma 17, we can write

$$\mathcal{L}_{\mathcal{I}}=\frac{1}{2}\sum_l\sum_k(d\log\mu_l)(d\log\mu_k)\sum_j\lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)}).$$

and

$$\mathcal{L}_X=-\frac{1}{2}\sum_l\sum_g d\log\mu_l d\log\Lambda_g\sum_j\lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)}).$$

[15]We have used the intermediate step

$$\mathcal{L}_X=\frac{1}{2}\sum_l\sum_k\lambda_k\lambda_l d\log\mu_k d\log\mu_l+\frac{1}{2}\sum_l\sum_f d\log\mu_l d\log\Lambda_f\lambda_l\Psi_{lf}$$
$$-\frac{1}{2}\sum_l\sum_g d\log\mu_l d\log\Lambda_g\sum_j\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)}).$$

■

Lemma 16. *The following identity holds*

$$\sum_j \lambda_j \left(\tilde{\Psi}_{jk}\Psi_{jl} - \sum_m \Omega_{jm}\tilde{\Psi}_{mk}\Psi_{ml} \right) = \tilde{\lambda}_k \lambda_l.$$

Proof. Write Ω so that it contains all the producers, all the households, and all the factors *as well as* a new row (indexed by 0) where $\Omega_{0i} = \chi_i$ if $i \in C$ and 0 otherwise. then, letting e_0 be the standard basis vector corresponding to the 0th row, we can write

$$\lambda' = e_0' + \lambda'\Omega,$$

or equivalently

$$\lambda'(I - \Omega) = e_0'.$$

Let X^{kl} be the vector where $X^{kl}_m = \tilde{\Psi}_{mk}\Psi_{ml}$. Then

$$\sum_j \lambda_j \left(\tilde{\Psi}_{jk}\Psi_{jl} - \sum_m \Omega_{jm}\tilde{\Psi}_{mk}\Psi_{ml} \right) = \lambda'(I - \Omega)X^{kl},$$

$$= e_0'(I - \Omega)^{-1}(I - \Omega)X^{kl}, \quad = e_0' X^{kl} = \tilde{\Psi}_{0k}\Psi_{0l} = \tilde{\lambda}_k \lambda_l.$$

■

Lemma 17. *The following identity holds*

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \lambda_l \lambda_k [\frac{\tilde{\Psi}_{lk} - \delta_{lk}}{\lambda_k} + \frac{\Psi_{kl} - \delta_{kl}}{\lambda_l} + \frac{\delta_{lk}}{\lambda_k} - \frac{\tilde{\lambda}_k}{\lambda_k}].$$

Proof. We have

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) =$$

$$\sum_j \lambda_j \mu_j^{-1} \left[\sum_m \tilde{\Omega}_{jm}\tilde{\Psi}_{mk}\Psi_{ml} - \left(\sum_m \tilde{\Omega}_{jm}\tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm}\Psi_{ml} \right) \right],$$

or

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) =$$

$$\sum_j \lambda_j \sum_m \Omega_{jm} \tilde{\Psi}_{mk} \Psi_{ml} - \sum_j \lambda_j \mu_j^{-1} \left(\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm} \Psi_{ml} \right),$$

or

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) =$$
$$\sum_j \lambda_j \sum_m \Omega_{jm} \tilde{\Psi}_{mk} \Psi_{ml} - \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl}$$
$$+ \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl} - \sum_j \lambda_j \mu_j^{-1} \left(\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm} \Psi_{ml} \right),$$

or using, Lemma 16

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = -\tilde{\lambda}_k \lambda_l + \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl} - \sum_j \lambda_j \left(\tilde{\Psi}_{jk} - \delta_{jk} \right) \left(\Psi_{jl} - \delta_{jl} \right),$$

and finally

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \lambda_l \lambda_k [\frac{\tilde{\Psi}_{lk} - \delta_{lk}}{\lambda_k} + \frac{\Psi_{kl} - \delta_{kl}}{\lambda_l} + \frac{\delta_{lk}}{\lambda_k} - \frac{\tilde{\lambda}_k}{\lambda_k}].$$

■

Proof of Theorem 7. Here we assume that there is only one factor in the domestic economy and normalize its price to one. Define the "fictitious domestic" IO matrix

$$\check{\Omega}_{ij} \equiv \frac{\Omega_{ij}}{\sum_{k \in N_c} \Omega_{ik}},$$

with associated Leontief-inverse matrix

$$\check{\Psi} \equiv (1 - \check{\Omega})^{-1}.$$

Applying Feenstra (1994), for each producer $i \in N_c$, we have

$$d \log p_i = \sum_{j \in N_c} \check{\Omega}_{ij} d \log p_j + \frac{d \log \lambda_{ic}}{\theta_i - 1},$$

where λ_{ic} is the domestic cost share of producer i. The solution of this system of equations is

$$d\log p_i = \sum_{j\in N_c} \check{\Psi}_{ij} \frac{d\log\lambda_{jc}}{\theta_j - 1}.$$

From this we can get welfare gains

$$d\log Y_c = -\sum_{i\in N_c} \check{b}_i d\log p_i = -\sum_{i\in N_c}\sum_{j\in N_c} \check{b}_i \check{\Psi}_{ij} \frac{d\log\lambda_{jc}}{\theta_j - 1} = -\sum_{j\in N_c} \check{\lambda}_j \frac{d\log\lambda_{jc}}{\theta_j - 1},$$

where

$$\check{\lambda}_i \equiv \sum_{j\in N_c} \check{b}_j \check{\Psi}_{ji}.$$

This can be thought of as hitting the fictitious domestic economy with productivity shocks $-d\log\lambda_{jc}/(\theta_j - 1)$. Since relative domestic prices in the closed and open economy are identical, the relative expenditure shares on domestic goods moves in the same way in both economies. ■

Proof of Corollary 5. Result (ii) follows immediately from Corollary 4 applied at the industry level, which allows us to determine that at every point, the Hessian of the function $\Delta\log\check{Y}_c(\Delta\log\check{A})$ is negative semi-definite when all cross-industry elasticities of substitution are below one and positive semi-definite when they are above one. The same logic can be used to prove a local version of (i) since the Hessians of the two economies at the original point are ordered (using the semi-definite condition partial ordering).

There is an alternative way to prove this corollary via the contraction mapping theorem. To see this, introduce the mapping T, which to every vector of price changes $\Delta\log\check{p}$ for the goods of the dual closed economy, associates a new vector of price changes $\Delta\log\check{p}' = T(\Delta\log\check{p};\Delta\log\check{A})$ given by:

$$\Delta\log\check{p}'_i = -\Delta\log\check{A}_i + \frac{1}{1-\theta_i}\log\left(\sum_{j\in N_c+F_c} \check{\Omega}_{ij} e^{(1-\theta_i)\Delta\log\check{p}_j}\right).$$

It is easy to verify that $T(\cdot;\Delta\log\check{A})$ is a contraction mapping, the fixed-point of which gives the response of prices to productivity shocks in the dual closed economy: $\Delta\log\check{p} = \lim_{n\to\infty} T^n(\Delta\log\check{p}_{\text{init}};\Delta\log\check{A})$, for all $\Delta\log\check{p}_{\text{init}}$. The response of welfare in the original economy is equal to the response of real output in the dual closed economy and is given by $\Delta\log W_c = \Delta\log\check{Y}_c = -\Delta\log\check{p}_c$, where recall that we denote by c the final good consumed by the representative agent of the dual closed economy.

(i) and (ii) can also be derived using $\Delta\log W_c = \Delta\log\check{Y}_c = -\Delta\log\check{p}_c$ with $\Delta\log\check{p} = \lim_{n\to\infty} T^n(\Delta\log\check{p}_{\text{init}};\Delta\log\check{A})$, where the mapping $\Delta\log\check{p}' = T(\Delta\log\check{p};\Delta\log\check{A})$ is applied at the industry level. Indeed, the mapping is always linear in $\Delta \log \check{A}$. If all the elasticities of substitution across industries are below (above) unity, then the mapping is convex (concave) in $\Delta \log p$, and hence the mapping preserves convexity (concavity) in $\Delta \log A$. This immediately implies (ii). The result (i) also follows from the fact that the T mapping for the high-elasticity and low-elasticity economy are monotonically ordered and the ordering is preserved by iterating on T.

■

Proof of Proposition 12. The proof closely follows that of Proposition 11. Notably, using an insight from Galle et al. (2017), which builds on Feenstra (1994), we note that

$$\mathrm{d}\log \chi_g = \mathrm{d}\log w_g + \frac{1}{\gamma_g}\,\mathrm{d}\log \Lambda_s^g$$

for any s, and hence,

$$\mathrm{d}\log \chi_g = \sum_f \check{\Lambda}_f^g \left(\mathrm{d}\log w_f + \frac{1}{\gamma_g}\,\mathrm{d}\log \Lambda_f^g\right),$$

where $\check{\Lambda}^g$ and $\check{\lambda}^g$ are the Domar weights under the closed-economy IO matrix. Then we can combine this with the fact that

$$\mathrm{d}\log P_g^c = \sum_f \check{\Lambda}_f^g\,\mathrm{d}\log w_f - \sum_i \check{\lambda}_i\,\mathrm{d}\log \check{A}_i$$

and choosing household g's nominal income as the numeraire to get

$$\mathrm{d}\log W_g = \sum_f \check{\Lambda}_f^g \frac{\mathrm{d}\log \Lambda_s^g}{\gamma_g} + \sum_i \check{\lambda}_i\,\mathrm{d}\log \check{A}_i.$$

■

Proof of Proposition 13. Loglinearizing this, we get

$$\Lambda_f\,\mathrm{d}\log \Lambda_f = \mathrm{d}\log w_f \sum_{h\in H} \Lambda_f^h \chi_h \gamma_h - \sum_{h\in H} (\gamma_h - 1)\chi_h \Lambda_f^h \sum_{l\in F} \delta_{hl}\,\mathrm{d}\log w_l + \sum_{h\in H} \chi_h \Lambda_f^h\,\mathrm{d}\log L^h.$$

We can put this back into familiar notation

$$\Lambda_f\,\mathrm{d}\log \Lambda_f = \mathrm{d}\log w_f \sum_{h\in H} \Phi_{hf}\Lambda_f\gamma_h - \sum_{h\in H} (\gamma_h - 1)\Phi_{hf}\Lambda_f \sum_{l\in F} \frac{\Phi_{hl}\Lambda_l}{\chi_h}\,\mathrm{d}\log w_l + \sum_{h\in H} \Phi_{hf}\Lambda_f\,\mathrm{d}\log L^h.$$

Simplify this to get

$$\mathrm{d}\log\Lambda_f = \mathrm{d}\log w_f \sum_{h\in H} \Phi_{hf}\gamma_h - \sum_{h\in H}(\gamma_h - 1)\Phi_{hf} \sum_{l\in F} \frac{\Phi_{hl}\Lambda_l}{\chi_h}\,\mathrm{d}\log w_l + \sum_{h\in H}\Phi_{hf}\,\mathrm{d}\log L^h.$$

We can beautify this a bit as

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H}\gamma_h E_{\Phi^{(h)}}\left(E_{\Lambda^{(h)}}\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right)\right) + \sum_{h\in H} E_{\Phi^{(h)}}\left(E_{\Lambda^{(h)}}(\mathrm{d}\log w)\right) + \sum_{h\in H} E_{\Phi^{(h)}}\left(\mathrm{d}\log L\right).$$

or

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H} E_{\Phi^{(h)}}\left[\gamma_h\left(E_{\Lambda^{(h)}}\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right)\right) + \left(E_{\Lambda^{(h)}}(\mathrm{d}\log w)\right) + (\mathrm{d}\log L)\right].$$

or

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H} E_{\Phi^{(h)}}\left[E_{\Lambda^{(h)}}\left(\gamma_h\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right) + (\mathrm{d}\log w)\right) + (\mathrm{d}\log L)\right].$$

The case with immobile labor is given by $\gamma_h = 1$ for every $h \in H$, in which case $\mathrm{d}\log w_f = \mathrm{d}\log\Lambda_f$. Combine this with demand for the factors to finish the characterization

$$\Lambda_l \frac{\mathrm{d}\log\Lambda_l}{\mathrm{d}\log A_k} = \sum_{i\in\{H,N\}} \lambda_j(1-\theta_j) Cov_{\Omega^{(j)}}\left(\Psi_{(k)} + \sum_f \Psi_{(f)}\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}, \Psi_{(l)}\right)$$
$$+ \sum_{h\in H}(\lambda_l^h - \lambda_l)\left(\sum_{f\in F_c}\Phi_{hf}\Lambda_f\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}\right).$$

This means that we can also redo the welfare accounting and write

$$\mathrm{d}\log W_g = \mathrm{d}\log\chi_g - \mathrm{d}\log P_g^c,$$

where χ_g is the (nominal) income of household g. This can be written as

$$\frac{\mathrm{d}\log W_g}{\mathrm{d}\log A_k} = \sum_{s\in F}\left(\Lambda_s^g - \Lambda_s^{W_g}\right)\mathrm{d}\log w_s + \lambda_k^{W_g}\,\mathrm{d}\log A_k + \mathrm{d}\log L^g,$$

or

$$\frac{\mathrm{d}\log W_g}{\mathrm{d}\log A_k} = \sum_{s\in F}\left(\frac{\Phi_{gs}}{\chi_g}\Lambda_s - \Lambda_s^{W_g}\right)\mathrm{d}\log w_s + \lambda_k^{W_g} + \mathrm{d}\log L^g.$$

■

Proposition 18 (Structural Output Loss)**.** *Starting at an efficient equilibrium in response to the* *introduction of small tariffs or other distortions, changes in the real output of country c are, up to the second order, given by*

$$\Delta \log Y_c \approx -\frac{1}{2} \sum_{l \in N_c} \sum_{k \in N} \Delta \log \mu_k \Delta \log \mu_l \sum_{j \in N} \lambda_j^{Y_c} \theta_j Cov_{\Omega^{(j)}}\left(\Psi_{(k)}, \Psi_{(l)}\right)$$
$$- \frac{1}{2} \sum_{l \in N_c} \sum_{g \in F} \Delta \log \Lambda_g \Delta \log \mu_l \sum_{j \in N} \lambda_j^{Y_c} \theta_j Cov_{\Omega^{(j)}}\left(\Psi_{(g)}, \Psi_{(l)}\right)$$
$$+ \frac{1}{2} \sum_{l \in N_c} \sum_{c \in C} \chi_c^W \Delta \log \chi_c^W \Delta \log \mu_l (\lambda_l^{W_c} - \lambda_l) / \chi_c^Y.$$

Proof. The proof follows along the same lines as Theorem 6. ■