THE FISCAL ROOTS OF INFLATION

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ABSTRACT

Unexpected inflation devalues nominal government bonds. It must therefore correspond to a decline in expected future surpluses, or a rise in their discount rates, so that the real value of debt equals the present value of surpluses. I measure each component via a vector autoregression, in response to inflation, recession, surplus and discount rate shocks. Discount rates, rather than deficits, account for most inflation variation. Smooth inflation that slowly devalues outstanding long-term bonds is an important mechanism.

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1   Introduction

This paper measures the fiscal roots of inflation. Higher inflation devalues nominal government debt. It must therefore correspond to lower surplus/GDP ratios, lower growth, or higher discount rates for government debt. I develop a set of linearized identities that express these connections. I measure these components in a simple vector autoregression (VAR).

I find that variation in inflation largely corresponds to variation in discount rates. For example, consider 2009. There was a big recession, and a sharp fall in inflation which raised the real value of nominal debt. Yet deficits exploded. How can this be? Well, perhaps people expected future surplus/GDP ratios to swiftly pay down the accumulated debt and more, making nominal debt more valuable in real terms. Aside from its implausibility, I do not find this pattern in the data. Perhaps people expected the recession to end swiftly, and future growth to be even stronger than before, so that government debt became more valuable. Again, the data confirm one's intuition that this is not the case. But nominal and real interest rates on government debt fell sharply. Perhaps this lower discount rate for government debt increased its value, just as higher expected surpluses or growth would do. I find that this is the case: the fall in expected returns is large and persistent enough quantitatively to account for this kind of episode in the data.

I also find that fiscal shocks correspond to a drawn-out period of inflation, that devalues long-termbonds. Long-term debt and monetary policy accommodation substantially smooth fiscal shocks. Finally, I examine shocks to surpluses and shocks to discount rates. These do not produce much inflation. Shocks to surpluses come largely with lower interest and discount rates, and the two effects offset. Deficits are largely "repaid" by lower returns which bring back the value of debt. Here discount rates answer the fiscal roots of the otherwise-puzzling absence of inflation.

I interpret the results through the lens of the fiscal theory of monetary policy: models with interest rate targets, fiscal theory of the price level, and potentially sticky prices, as described in Cochrane (2019a), Cochrane (2019b). (More literature below.) In this interpretation, changes in expected surpluses and discount rates *cause* unexpected inflation. We study the fiscal roots rather than the fiscal consequences of inflation. This paper establishes a set of facts that will hopefully be useful for constructing such models, as atheoretical VARs guided the construction of conventional monetary models. For example, the fact that discount rates account for much inflation variation is key to making a fiscal-theory analysis reasonable, and to allow a fiscal-theory model to have a hope of accounting for events such as 2009, or the converse rise in

inflation in low-deficit booms. My causal language below refers to this interpretation.

But the identities hold in almost all macroeconomic models used to quantitatively address inflation. (They assume that the present value is finite – loosely that $r > g$. I presume this case without further comment.) Therefore, the results can also be interpreted as measures of the passive-fiscal adjustments to an active-money regime, and the standard new-Keynesian model in particular. A well-specified active-money regime must spell out a realistic passive-fiscal policy. The fact that discount rates do much of the adjusting, rather than the ex-post lump-sum taxes alluded to in many theoretical footnotes, changes the fiscal underpinnings of such models substantially.

Since the analysis is based on identities, I do not test anything, and the empirical results do nothing to establish one or another causal story. But which element in an identity moves is still an interesting measurement.

2   Identities

To develop the identities linking inflation to surpluses, growth, and discount rates, start with a linearized version of the government debt flow identity,

$$\rho v_{t+1} = v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} - s_{t+1}. \tag{1}$$

The log debt to GDP ratio at the end of period $t+1$, v_{t+1}, is equal to its value at the end of period t, v_t, increased by the log nominal return on the portfolio of government bonds r_{t+1}^n less inflation π_{t+1}, less log GDP growth g_{t+1}, and less the real primary surplus to GDP ratio[1] s_t. For brevity, I refer to s_t simply as the "surplus." The parameter ρ is a constant of linearization, $\rho = e^{r-g}$, which I take to be $\rho = 1$ in the numerical results.I derive this identity in the Appendix.

Iterating forward, we have a present value identity,

$$v_t = \sum_{j=1}^{\infty} \rho^{j-1} s_{t+j} + \sum_{j=1}^{\infty} \rho^{j-1} g_{t+j} - \sum_{j=1}^{\infty} \rho^{j-1} \left(r_{t+j}^n - \pi_{t+j} \right). \tag{2}$$

The log value of government debt, divided by GDP, is the present value of future surplus to GDP ratios, discounted at the ex-post real return, and adjusted for growth. One may attach growth

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[1]Precisely, s_t is ρ times the ratio of primary surplus to GDP scaled by the steady-state value to GDP ratio. With $\rho = 1$, s_t can also represent the real primary surplus divided by the previous period's real value of debt. Either definition leads to the same linearization. I impute the surplus from the other terms of (1), so its definition only matters when one wishes to assess the accuracy of approximation, which I do below, or to assess an independent data source on surpluses. With $\rho < 1$ there is also a constant in the linearization, or the variables are deviations from steady state.

g to the surplus, as growth raises total surpluses which pay off a given stock of debt for given surplus to GDP ratio. One can also attach g to the discount rate, producing an effective $r - g$ discount rate.

Taking time $t+1$ innovations $\Delta E_{t+1} \equiv E_{t+1} - E_t$ and rearranging, we have an unexpected inflation identity,

$$\Delta E_{t+1}\pi_{t+1} - \Delta E_{t+1}r_{t+1}^n =$$
$$= -\sum_{j=0}^{\infty}\rho^j \Delta E_{t+1}s_{t+1+j} - \sum_{j=0}^{\infty}\rho^j \Delta E_{t+1}g_{t+1+j} + \sum_{j=1}^{\infty}\rho^j \Delta E_{t+1}\left(r_{t+1+j}^n - \pi_{t+1+j}\right) \quad (3)$$

A decline in the present value of surpluses, coming either from a decline in surplus to GDP ratios, a decline in GDP growth, or a rise in discount rates, must correspond to a lower real value of the debt. This reduction can come about by unexpected inflation, or by a decline in nominal long-term bond prices. I use time $t+1$ to denote unexpected events, and time 1 as the date of a shock in the impulse-response functions.

The second term on the left hand side of (3) is a key point of the analysis. For example, when there is a negative innovation to the present value of surpluses on the right hand side of (3), a decline in nominal long-term bond prices and consequent negative return $\Delta E_{t+1}r_{t+1}^n$ can lower the real value of debt, in place of unexpected inflation $\Delta E_{t+1}\pi_{t+1}$. In this way, long-term debt can buffer fiscal shocks.

What determines the long-term bond return r_{t+1}^n, and whether that mechanism operates? With a geometric maturity structure, in which the face value of maturity j debt declines at rate ω^j, the Appendix develops a second approximate identity,

$$\Delta E_{t+1}r_{t+1}^n = -\sum_{j=1}^{\infty}\omega^j \Delta E_{t+1}r_{t+1+j}^n = -\sum_{j=1}^{\infty}\omega^j \Delta E_{t+1}\left[(r_{t+1+j}^n - \pi_{t+1+j}) + \pi_{t+1+j}\right]. \quad (4)$$

Lower nominal bond prices, and a lower ex-post bond return, mechanically correspond to higher bond expected nominal returns, which in turn are composed of real returns and inflation.

We can eliminate the bond return in (3)-(4) to focus on inflation and fiscal affairs alone,

$$\sum_{j=0}^{\infty}\omega^j \Delta E_{t+1}\pi_{t+1+j} =$$
$$= -\sum_{j=0}^{\infty}\rho^j \Delta E_{t+1}s_{t+1+j} - \sum_{j=0}^{\infty}\rho^j \Delta E_{t+1}g_{t+1+j} + \sum_{j=1}^{\infty}(\rho^j - \omega^j)\Delta E_{t+1}\left(r_{t+1+j}^n - \pi_{t+1+j}\right). \quad (5)$$

I focus on this decomposition.

2.1   Mechanisms

The identities highlight several interesting mechanisms which we can look for in the data.

Consider the simple case with constant expected returns $E_t r_{t+1}^n = E_t \pi_{t+1}$. With one-period debt, $\omega = 0$, there is only one term on the left-hand side, $\Delta E_{t+1} \pi_{t+1}$. Shocks to the present value of surpluses must be soaked up by a price-level jump, which absent default is the only way to devalue outstanding debt. With long-term debt, $\omega > 0$, however, a shock to the present value of surpluses can result in a drawn out period of inflation, which slowly devalues outstanding long-term bonds. In the identity (3), the term r_{t+1}^n marks the future inflation to market, as future inflation in (4) sends that return down.

The latter is a much more realistic vision of the US economy, where we see drawn-out inflation accompanying fiscal problems as in the 1970s. In fact, equation (5) allows the entire effect of the fiscal shock to show up in expected future inflation with no movement in current inflation $\Delta E_{t+1} \pi_{t+1} = 0$. This is how continuous-time models that disallow price-level jumps work (Cochrane (2017)). This case provides an important counterexample to the usual intuition gained from one-period models, in which fiscal shocks only give rise to a one-period inflation surprise. We can productively look for fiscal roots of drawn-out inflation.

In both fiscal and standard new-Keynesian theories, monetary policy, control of the path of nominal interest rates controls the path of expected inflation. For example, in the simplest frictionless model, $i_t = E_t \pi_{t+1}$ so the interest rate target sets expected inflation directly. Thus, fiscal shocks can result in smooth inflation, and a large bond-return term in the one-period accounting (3), if monetary policy accommodates them.

As the maturity structure of government debt lengthens, ω increases, and the discount rate terms vanish. When $\omega = \rho$, almost a perpetuity, the discount rate term drops out. Intuitively, a government that funds itself with near-perpetuities can pay off its current debt while completely ignoring real interest rate variation, just as a household that takes out a fixed rate mortgage is immune from such variation. The discount rate effect in present value terms is the same as the interest cost channel in flow terms – debts can be "repaid" if interest costs decline and debts get worse if interest costs rise, when a government rolls over short-term debt.

If surpluses and growth are constant as well as discount rates in (5), a rise in expected future inflation results in a decline in current inflation. This is an important mechanism for a model with long-term debt to allows monetary policy - a rise in interest rates which eventually raises long-term inflation, without changing fiscal policy – to produce a temporary inflation decline.

With one-period debt, expected inflation may continue, but this fact has no impact on

one-period unexpected inflation or this fiscal accounting. $\Delta E_1 \pi_j$ for $j > 1$ is irrelevant (though interesting). With long-term debt, the weighted sum of changes in expected inflation substitutes for inflation at time 1, but only the ω-weighted sum. Additional persistence in inflation, though interesting for matching data, has no fiscal consequence or consequence for understanding unexpected inflation.

With time-varying expected returns, interesting additional dynamics can emerge. A rise in the expected return of government bonds lowers the present value of surpluses just like a decline in expected surpluses. But with long-term debt, a rise in the discount rate for government bonds also lowers nominal bond prices. The discount rate term in (5) balances these two effects. A higher interest rate, which raises the expected return of nominal bonds, will, if prices are sticky, raise the discount rate. This is an inflationary force overall. If an adverse surplus shock comes with lower interest rates and expected bond returns, and inflation is somewhat sticky, the lower real interest rates raise the value of debt and offset the inflationary effect of the surplus shock. We will see this effect is prominent in the data.

2.2 Literature

The technique in this paper is adapted from asset pricing. The general approach to linearizing the valuation identity follows Campbell and Shiller (1988). The Appendix relates impulse-response calculations to asset price variance decompositions. The summary of this literature in Cochrane (2011b) and the treatment of identities in Cochrane (2007) are obvious precursors. The uniting theme in the former is that asset price and return variation is largely driven by variation in discount rates.

The analysis of government finances, how debt is paid off, grown out of, or inflated away, is a huge literature. Hall and Sargent (1997), Hall and Sargent (2011) are the most important precursors. Hall and Sargent focus on the market value of debt, as I do, not the face value reported by the Treasury, and consequent proper accounting for interest costs.

This paper uses the innovation identities (3) and (5), to focus on inflation, paralleling VAR-based return decompositions from asset pricing such as Campbell and Ammer (1993). A companion paper Cochrane (2019c) decomposes the value of government debt v_t, starting from the value identity (2), paralleling price/dividend ratio variance decompositions.

The fiscal theory of monetary policy is the latest step in a long literature on the fiscal theory of the price level, starting with Leeper (1991), that integrates fiscal theory with sticky-price models and interest rate targets. Sims (2011), Cochrane (2017) and are immediate antecedents. Cochrane (2019a) works out such a model with the S-shaped surplus processes I find here, calcu-

lates inflation decompositions and response functions in the model, and reviews the literature.

Much of the fiscal theory literature has pursued various theoretical controversies. A big point of this paper is to begin productively use fiscal theory to understand US data.

3   Data and VAR estimates

I use data on the market value of government debt held by the public and the nominal rate of return of the government debt portfolio from Hall, Payne, and Sargent (2018). I use standard BEA data for GDP and total consumption. I use the GDP deflator to measure inflation. I use CRSP data for the three-month Treasury rate. I use the 10-year constant maturity government bond yield from 1953 on and the yield on long-term United States bonds before that date to measure a long-term bond yield.

I measure the debt to GDP and surplus to GDP ratios by the ratios of debt and surplus to personal consumption expenditures, times the average consumption to GDP ratio. Debt to GDP ratios are often used to compare countries, but in our time-series application they introduce cyclical variation in GDP. We want only a detrending divisor, and an indicator of the economy's long-run level of tax revenue and spending. Potential GDP has a severe look-ahead bias. Consumption is a decent stochastic trend for GDP.

I infer the primary surplus from the flow identities. This calculation measures how much money the government actually borrows. NIPA surplus data, though broadly similar, does not obey the flow identity.

I infer the surplus for the VAR from the linearized identity (1), at an annual frequency. By doing so, the data obey the identity exactly. Therefore VAR estimates of the decompositions add up exactly with no approximation error. The approximation errors are much smaller than sampling errors, so this choice just produces clearer tables.

To measure the accuracy of the linear approximation, I also infer the monthly real primary surplus from the exact nonlinear flow identity, Appendix equation (11). I then carry the surplus to the end of the year using the government bond return. This procedure produces an annual series for which the nonlinear flow identity (11) continues to hold in annual data.

I approximate around $r = g$ or $\rho = 1$. The variables are all stationary, impulse-responses and expected values converge, so downweighting higher order terms by 0.99^j vs. 1.0 makes little difference to the results. Since the value of the debt v_t is stationary, $\lim_{T \to \infty} E_t v_{t+T} = 0$ without ρ weighting. The parameter ρ is only the arbitrary point about which one takes a Taylor expansion of the one-period flow relation. It is not the long-run value $r - g$ in the economy, and its

choice does not determine whether present values converge, transversality conditions hold, the economy is dynamically efficient, government debt never needs to be repaid, and so forth. One can linearize an economy that has $r > g$ around $r = g$. With $\rho = 1$, the same linearization applies to the surplus to value ratio, which we will shortly see is a bit more accurate. One can also view the unweighted identities as $r \to g$ limits.
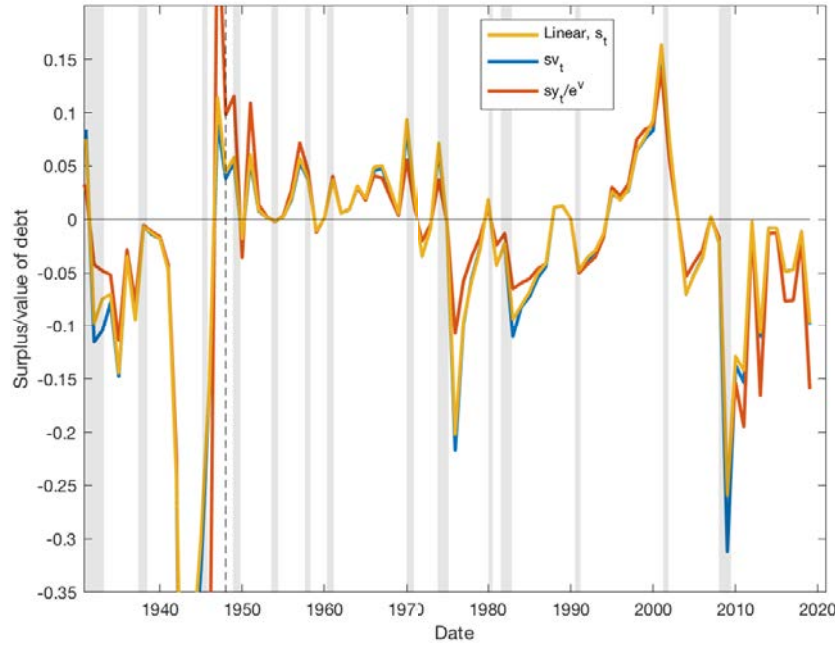

Figure 1: Surplus. "Linear" is inferred from the linearized flow identity, and is the definition used in VAR analysis. "sv" is the exact ratio of the primary surplus to the previous year's market value of the debt. "sy" is the exact ratio of surplus to consumption, scaled by the average consumption to GDP ratio and the average value of debt. Vertical shading denotes NBER recessions.

Figure 1 presents the surplus and compares three measures. The "Linear, s_t" line imputes the surplus from the linearized flow identity (1) directly at the one-year horizon, which is the measure I use in the following analysis. The "sv_t" and "sy_t/e^v" lines both infer the surplus from the exact nonlinear flow identity (11), as above. The "sv_t" line presents the ratio of the exact surplus to the previous year's value of the debt. The "sy_t/e^v" line presents the exact surplus to GDP ratio – actually, the ratio of surplus to consumption, times the average consumption to GDP ratio – scaled by the average value to GDP ratio $e^{E(v_t)}$.

I use a data sample 1947-2018 for the VAR analysis. The immense deficits of WWII would otherwise dominate the analysis, and one may well suspect that financing that war, and expectations and reality of paying it off follows a different pattern than fiscal-monetary policy in the

subsequent decades of largely cyclical deficits. WWII also featured price controls, clouding inflation measurement. The vertical dashed line of Figure 1 indicates the post-1947 sample.

The first piece of news in Figure 1 is that there *are* primary surpluses. One's impression of endless deficits comes from the deficit including interest payments on the debt. Even NIPA measures show regular positive primary surpluses. Steady primary surpluses from 1947 to 1975 helped to pay off WWII debt. The year 1975 started an era of large primary deficits, interrupted by the strong surpluses of the late 1990s. Postwar primary surpluses also have a clear cyclical pattern. The primary surplus correlates very well with the unemployment rate (not shown), a natural result of procyclical tax revenues, automatic (e.g. unemployment insurance) and discretionary countercyclical spending.

The three measures in Figure 1 are close. The graph is a measure of the accuracy of the linearized identity (1). The linearized identity is a slightly closer approximation to the surplus to value ratio sv. The difference is largest when the value of debt is far from its mean, both in WWII and in the 1970s.

3.1   Vector autoregression

Table 1 presents OLS estimates of the VAR coefficients. Each column is a separate regression. I orthogonalize shocks later, so the order of variables has no significance. The VAR includes the central variables for the inflation identity – nominal return on the government bond portfolio r^n, consumption growth rate g, inflation π, surplus s and value v. I include the three-month interest rate i and the 10 year bond yield y as they are important forecasting variables for growth, inflation, and long-term bond returns.

It is important to include the value of debt v_t in the VAR, even if we are calculating terms of the innovation identity (3) that does not reference that value. When we deduce from the present value identity (2) expressions $v_t = E_t(\cdot)$, we must include v_t in the information set that takes the expectation. The surplus typically follows an s-shaped process, in which deficits today are followed by surpluses in the future. The process will not be properly recovered by VARs that do not include the value of debt. (See Cochrane (2019b), Cochrane (2019a) for discussion and examples.)

I use a single lag. Adding the last variable, the long-term rate, already introduces slight wiggles in the impulse-response function indicative of overfitting. More lags are insignificant forecasters, and add additional wiggles.

I compute standard errors from a Monte Carlo. The stars in Table 1 represent one or two standard errors above zero. Since we aren't testing anything, stars are just a visual way to show

	r^n_{t+1}	g_{t+1}	π_{t+1}	s_{t+1}	v_{t+1}	i_{t+1}	y_{t+1}
r^n_t	-0.17**	-0.02	-0.10**	-0.32*	0.28*	-0.08*	0.04*
g_t	-0.27*	0.20*	0.16*	1.37**	-2.00**	0.28**	0.06
π_t	-0.15	-0.14*	0.53**	-0.25	-0.29	0.09	0.04
s_t	0.12**	0.03*	-0.03*	0.35**	-0.24*	-0.04*	-0.04**
v_t	0.01	-0.00	-0.02**	0.04*	0.98**	-0.01	-0.00
i_t	-0.32*	-0.40*	0.29**	0.50	-0.72	0.73**	0.36**
y_t	1.93**	0.54*	-0.17*	-0.04	1.60*	0.11	0.46**
$100 \times std(\varepsilon_{t+1})$	2.18	1.53	1.12	4.75	6.55	1.27	0.82
Corr $\varepsilon, \varepsilon_\pi$	-0.29	-0.24	1.00	-0.14	-0.11	0.21	0.31
R^2	0.71*	0.17*	0.73*	0.48*	0.97*	0.82*	0.90*
$100 \times std(x)$	4.08	1.68	2.16	6.61	37.00	2.96	2.63

Table 1: OLS VAR estimate. Sample 1947-2018. One (two) stars means the estimate is one (two) Monte Carlo standard errors away from zero.

standard errors without another table.

In the first column, the long-term bond yield y_t forecasts the government bond portfolio return r^n_{t+1} (1.93). The negative coefficient on the three-month rate i_t means that the long-short spread also forecasts those returns. Since the y_t and i_t coefficients are not repeated in forecasting inflation and growth, the long rate and long-short spread forecast real, growth-adjusted, and excess returns on government bonds, as we expect from the long literature in which yield spreads forecast bond risk premia (Fama and Bliss (1987), Campbell and Shiller (1991), Cochrane and Piazzesi (2005)). The long rate y_t is thus an important state variable for measuring expected bond returns, the relevant discount rate for our present value computations.

Growth g_t is only very slightly persistent (0.20). The term spread $y_t - i_t$ also predicts economic growth, a common finding, and reinforcing the importance of the interest rates as state variables.

Inflation π_t is moderately persistent (0.53). The interest rate and growth help a bit to predict inflation, but not much else does. We will see inflation responses that mostly look like AR(1) decay.

The surplus is somewhat persistent (0.35). Growth g_t predicts higher surpluses, an important and realistic feedback mechanism. This response does not imply passive fiscal policy, as discussed in more detail below.

The value of the debt is very persistent (0.98). It thus becomes the most important state variable for long-run calculations. A larger surplus s_t results in less market value of debt, v_{t+1}, (-0.24), as one expects. The long-run yield y_t forecasts a rise in the value of debt v_{t+1}, as we expect

given its effect on the expected return r_{t+1}^n.

The short rate i_t is also persistent (0.73). The long yield is also autocorrelated (0.46) and forecast by the persistent interest rate, again reflecting standard yield curve dynamics.

For calculations reported below, I use the standard notation

$$x_{t+1} = Ax_t + \varepsilon_{t+1} \tag{6}$$

to denote this VAR.

4   Responses and decomposition estimates

I start by examining the fiscal roots of a simple inflation shock, an unexpected movement in inflation $\Delta E_1 \pi_1$. I orthogonalize the inflation shock so that all other variables move contemporaneously to the inflation shock. I specify $\varepsilon_1^\pi = 1$. I fill in shocks to the other variables by running regressions of their shocks on the inflation shock. For each variable z, I run

$$\varepsilon_{t+1}^z = b_{z,\pi}\varepsilon_{t+1}^\pi + \eta_{t+1}^z.$$

Then I start the VAR (6) at

$$\varepsilon_1 = - \left[\begin{array}{ccccc} b_{r^n,\pi} & b_{g,\pi} & \varepsilon_1^\pi = 1 & b_{s,\pi} & ... \end{array} \right]'.$$

This procedure is equivalent to the usual orthogonalization of the shock covariance matrix, but it is more transparent and it generalizes more easily later. I denote the VAR innovations as the change in expectations at time 1, i.e. ΔE_1, and thus the response of variable x, j periods in the future is $\Delta E_1 x_j$.

Figure 2 plots responses to this inflation shock. The "inflation" rows of Table 2 presents the terms of the inflation and bond return decompositions for these responses. Figure 2 also presents some of the main terms in the decomposition identities, (3), (4), (5), and includes the formulas for easy reference.

In any interpretation, these responses and calculations answer the question, "if we see an unexpected 1% inflation, how should we revise our forecasts of other variables?" In a fiscal-theoretic interpretation, they answer "what changes in expectations caused the 1% inflation." As shown in the Appendix, the inflation decompositions can also be interpreted as decompositions of the variance of unexpected inflation: They answer the question, "What fraction of the
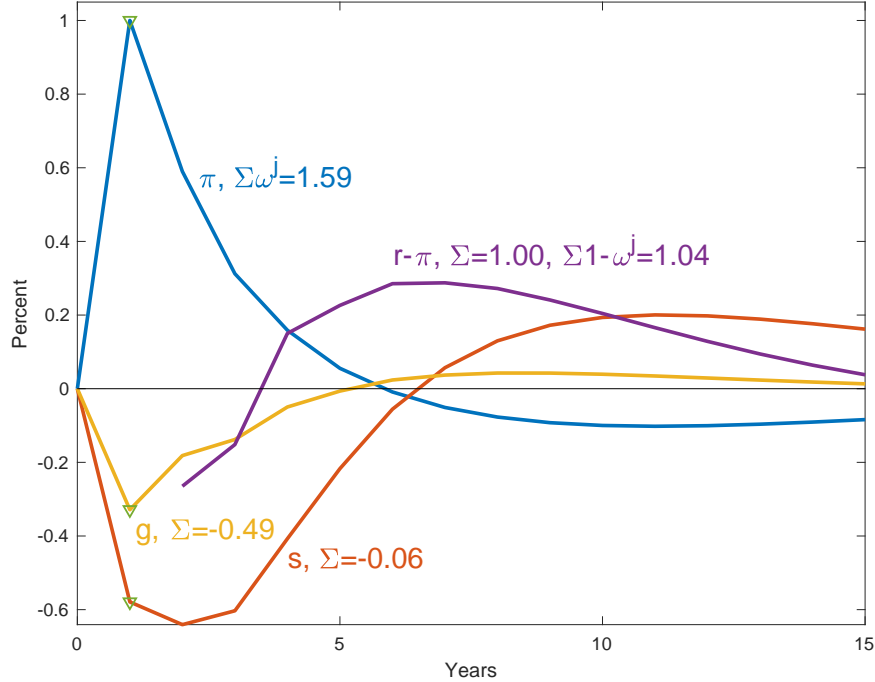
Figure 2: Responses to a 1% inflation shock.

variance of unexpected inflation is due to each component?" Table 3 presents quantiles of the sampling distributions of the terms of the inflation decompositions in Table 2, discussed below.

In Figure 2, the inflation shock is moderately persistent, largely following the AR(1) dynamics we noticed in the VAR coefficients. As result, the weighted sum $\sum_{j=0}^{\infty} \omega^j \Delta E_1 \pi_{1+j} = 1.59\%$, greater than the 1% initial shock.

The inflation shock coincides with deficits s, which build with a hump shape. One might think these persistent deficits account for inflation. But surpluses eventually rise to pay back almost all of the incurred debt. The sum of all surplus responses is -0.06%, essentially zero.

The line marked $r^n - \pi$ plots the response of the real discount rate, $\Delta E_1(r^n_{1+j} - \pi_{1+j})$. These points are plotted at the time of the ex-post return, $1 + j$, so they are the expected return one period earlier, at time j. The line starts at time 2, where the terms of the discount-rate sums in the inflation decompositions start, and representing the time-1 expected return. After two periods, this discount rate rises and stays persistently positive. The weighted sum of discount rate terms is 1.04% while the unweighted sum is 1.00% (really 1.004%). The weight ω is 0.69 (chosen to make the identity (4) hold exactly for this response function), so weighting by 1 vs. $1 - \omega^j$ makes little difference in the face of this persistent response.

Weighted or unweighted, the discount rate terms account for 1% inflation. A higher discount rate lowers the value of government debt, an inflationary force.

$$\sum_{j=0}^{\infty} \omega^j \Delta E_1 \pi_{1+j} = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} - \sum_{j=0}^{\infty} \Delta E_1 g_{1+j} + \sum_{j=1}^{\infty}(1-\omega^j)\Delta E_1(r^n_{1+j} - \pi_{1+j})$$

	π	=	s	g	$r^n - \pi$
Inflation	1.59	=	-(-0.06)	-(-0.49)	+(1.04)
Recession	-2.36	=	-(-1.15)	-(-1.46)	+(-4.96)
Surplus	-0.10	=	-(-0.66)	-(-0.34)	+(-1.10)
Disc. Rate	-0.18	=	-(-0.54)	-(-0.28)	+(-1.00)
Surplus, no i	0.38	=	-(-0.52)	-(-0.48)	+(-0.62)

$$\Delta E_1 \pi_1 - \Delta E_1 r^n_1 = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} - \sum_{j=0}^{\infty} \Delta E_1 g_{1+j} + \sum_{j=1}^{\infty} \Delta E_1(r^n_{1+j} - \pi_{1+j})$$

	π	r^n	=	s	g	$r^n - \pi$
Inflation	1.00	-(-0.56)	=	-(-0.06)	-(-0.49)	+(1.00)
Recession	-1.00	-(1.19)	=	-(-1.15)	-(-1.46)	+(-4.79)
Surplus	0.02	-(0.27)	=	-(-0.66)	-(-0.34)	+(-1.25)
Surplus, no i	0.36	-(0.03)	=	-(-0.52)	-(-0.48)	+(-0.67)
Disc. Rate	-0.03	-(0.28)	=	-(-0.54)	-(-0.28)	+(-1.13)
Surplus, no i	0.36	-(0.03)	=	-(-0.52)	-(-0.48)	+(-0.67)

$$\Delta E_1 r^n_1 = -\sum_{j=1}^{\infty} \omega^j \Delta E_1(r^n_{1+j} - \pi_{1+j}) - \sum_{j=1}^{\infty} \omega^j \Delta E_1 \pi_{1+j}$$

	r^n	=	$r^n - \pi$	π
Inflation	-0.56	=	-(-0.03)	-(0.59)
Recession	1.19	=	-(0.17)	-(-1.36)
Surplus	0.27	=	-(-0.15)	-(-0.12)
Disc. Rate	0.28	=	-(-0.13)	-(-0.15)
Surplus, no i	0.03	=	-(-0.05)	-(0.02)

Table 2: Terms of the inflation and bond return identities.

Inflation also is also correlated with a persistent decline in economic growth g. The stagflationary episodes of the 1970s drive this result. The growth decline contributes 0.49% to the inflation decompositions.

Overall, then,

- *A 1% shock to inflation corresponds to a roughly 1.5% decline in the present value of surpluses. A rise in discount rate contributes about 1%, and a decline in growth accounts for about 0.5% of that decline. Changes in the surplus/GDP ratio account for nearly nothing. The additional 0.5% fiscal shock corresponds to a persistent rise in expected inflation, which slowly devalues outstanding long-term bonds, and produces a 1.5% overall rise in inflation weighted by the maturity structure of debt.*

This is an important finding for matching the fiscal theory to data, or for understanding

$$\sum_{j=0}^{\infty} \omega^j \Delta E_1 \pi_{1+j} = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} - \sum_{j=0}^{\infty} \Delta E_1 g_{1+j} + \sum_{j=1}^{\infty}(1-\omega^j)\Delta E_1(r_{1+j}^n - \pi_{1+j})$$

	π	=	s	g	$r^n - \pi$
Inflation 25 %	1.38	=	-(-0.56)	-(-0.77)	+(0.25)
Inflation 75 %	1.67	=	-(0.19)	-(-0.19)	+(1.50)
Recession 25 %	-2.36	=	-(-1.56)	-(-1.86)	+(-5.37)
Recession 75 %	-2.16	=	-(0.02)	-(-0.68)	+(-3.01)
Surplus 25 %	-0.11	=	-(-0.76)	-(-0.42)	+(-1.11)
Surplus 75 %	-0.00	=	-(-0.60)	-(-0.27)	+(-1.00)
Disc. Rate 25 %	-0.29	=	-(-0.61)	-(-0.35)	+(-1.00)
Disc. Rate 75 %	-0.11	=	-(-0.45)	-(-0.22)	+(-1.00)
Surplus, no i 25 %	0.17	=	-(-0.82)	-(-0.49)	+(-0.81)
Surplus, no i 75 %	0.41	=	-(-0.54)	-(-0.21)	+(-0.54)

$$\Delta E_1 \pi_1 - \Delta E_1 r_1^n = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} - \sum_{j=0}^{\infty} \Delta E_1 g_{1+j} + \sum_{j=1}^{\infty} \Delta E_1(r_{1+j}^n - \pi_{1+j})$$

	π	r^n	=	s	g	$r^n - \pi$
Inflation 25%	1.00	-(-0.73)	=	-(-0.56)	-(-0.77)	+(0.19)
Inflation 75%	1.00	-(-0.43)	=	-(0.19)	-(-0.19)	+(1.35)
Recession 25%	-1.00	-(1.01)	=	-(-1.56)	-(-1.86)	+(-5.46)
Recession 75%	-1.00	-(1.37)	=	-(0.02)	-(-0.68)	+(-2.99)
Surplus 25%	-0.00	-(0.20)	=	-(-0.76)	-(-0.42)	+(-1.26)
Surplus 75%	0.06	-(0.29)	=	-(-0.60)	-(-0.27)	+(-1.18)
Disc. Rate 25%	-0.07	-(0.22)	=	-(-0.61)	-(-0.35)	+(-1.19)
Disc. Rate 75%	-0.01	-(0.37)	=	-(-0.45)	-(-0.22)	+(-1.12)
Surplus, no i 25%	0.15	-(-0.04)	=	-(-0.82)	-(-0.49)	+(-0.91)
Surplus, no i 75%	0.32	-(0.08)	=	-(-0.54)	-(-0.21)	+(-0.64)

$$\Delta E_1 r_1^n = -\sum_{j=1}^{\infty} \omega^j \Delta E_1(r_{1+j}^n - \pi_{1+j}) - \sum_{j=1}^{\infty} \omega^j \Delta E_1 \pi_{1+j}$$

	r^n	=	$r^n - \pi$	π
Inflation 25%	-0.73	=	-(-0.18)	-(0.38)
Inflation 75%	-0.43	=	-(0.13)	-(0.67)
Recession 25%	1.01	=	-(-0.17)	-(-1.36)
Recession 75%	1.37	=	-(0.21)	-(-1.16)
Surplus 25%	0.20	=	-(-0.21)	-(-0.12)
Surplus 75%	0.29	=	-(-0.13)	-(-0.07)
Disc. Rate 25%	0.22	=	-(-0.19)	-(-0.21)
Disc. Rate 75%	0.37	=	-(-0.12)	-(-0.10)
Surplus, no i 25%	-0.04	=	-(-0.16)	-(0.01)
Surplus, no i 75%	0.08	=	-(-0.03)	-(0.09)

Table 3: Monte Carlo quantiles of the inflation and bond return identities.

the fiscal side of passive-fiscal models. Thinking in both contexts has focused on the presence or absence of surpluses, or surplus to GDP ratios – lump sum taxes in many discussions – not the discount rate, and not growth effects. Thinking in both contexts has considered one-period unexpected inflation, to devalue one-period bonds, not a rise in expected inflation.

Turn to Table 2 for a more systematic view of the inflation decompositions, and to see the role of one-period bond returns $\Delta E_1 r_1^n$. The top row of the top panel presents the just-discussed overall decomposition (5) of current and expected future inflation in terms of surplus, growth and discount rate shocks. The second and third panels express the inflation decomposition in one-period terms, using the bond return r_1^n. The sum of surpluses and sum of growth rate terms are the same in this second panel as in the top panel, but I repeat them so one can see the terms of each identity more clearly. In the first row of the second panel, the 1% inflation shock corresponds to a roughly 1.56% overall fiscal shock. That shock comes similarly from a tiny 0.06% decline in surpluses, a 1.004% rise in discount rate and 0.49% reduction in growth. Here, the extra 0.56% fiscal shock is absorbed by a 0.56% decline in the value of government debt, r_1^n.

Turning to the last panel, we see that -0.56% return on government debt comes almost entirely from expected inflation (0.59%) not a higher real discount rate (0.03%). Discount rates matter in the inflation decompositions of the top two panels but not in this return decomposition because the former have weights that emphasize long-term movements (1 and $1 - \omega^j$), while the ω^j weights of the bottom panel emphasizes a short-run movement in discount rate. With these weights, the early discount rate declines shown in Figure 2 match nearly exactly the subsequent persistent rise.

Comparing the two analyses, you see how the government bond return essentially marks to market the expected future inflation of the top panel. Here, the roughly 1.5% fiscal shock is absorbed 1% by inflation, and 0.5% by a decline in long-term bond prices. The last panel ties the two decompositions together, showing that the decline in long-term bond prices reflects higher expected future inflation.

In sum,

- *The 1.5% fiscal shock that comes with 1% unexpected inflation is buffered by an 0.5% decline in bond prices, which corresponds to 0.5% additional expected future inflation. The additional expected inflation slowly devalues long-term bonds as they come due, a loss in value marked to market in the fall in bond prices.*

Figure 3 adds detail to the bond pricing responses. The interest rate i, bond yield y, and expected return r^n all move together and persistently. (Again, the graph plots the return r^n, the
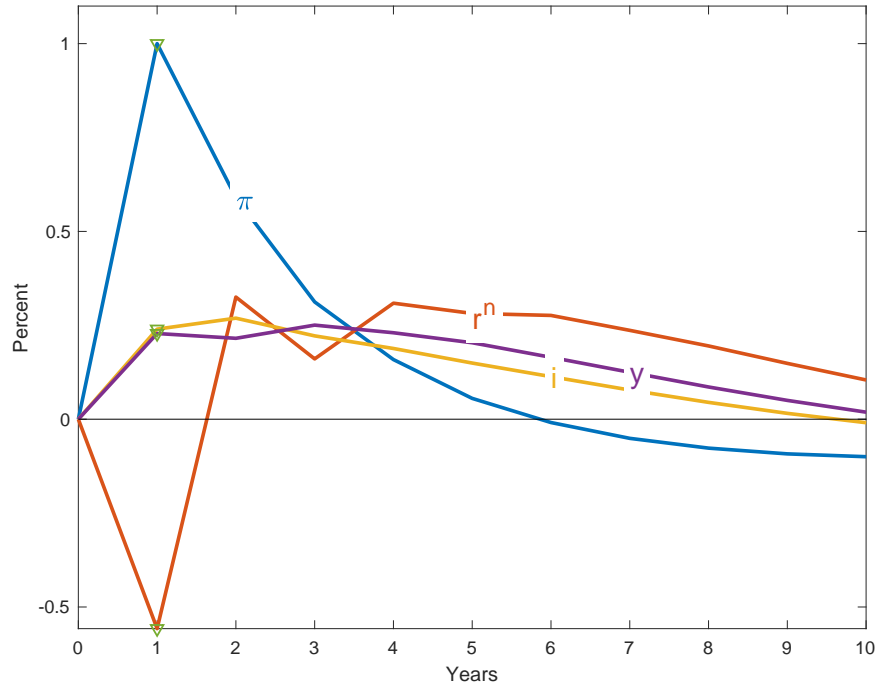
Figure 3: Responses to 1% inflation shock

expected return is one period earlier. The sawtooth pattern in r^n comes from a slightly negative eigenvalue of the VAR, which is far below statistical significance.) The return shock r_1^n moves in the opposite direction as the expected returns, as bond prices decline when yields rise. This is the picture of a "parallel shift" in the yield curve, with no sizeable risk premiums. The rise in real discount rates stems from the much more persistent movement in nominal rates than that of inflation seen on the right hand side of this graph.

Figure 4 plots the response of surplus and value of debt to the unexpected inflation shock. The value of debt declines on impact, as a result of the sharp negative movement in ex-post return r_1^n. The long string of deficits and rise in expected real returns then raises the value of debt. But eventually surpluses rise and pay down the debt.

The s-shaped surplus response is a crucial lesson. It means that early debts are repaid, at least in part, by following surpluses. The surplus does *n*ot follow an AR(1)-like process. Mechanically, this pattern is a result of the VAR coefficient of surplus on lagged debt. Thus, the finding is econometrically robust; it does not rely on a tenuous measurement of high-order surplus autocorrelations.

However, this (0.04) VAR coefficient of surpluses on debt and s-shaped response do not mean that the estimates measure a passive fiscal policy. The active vs. passive fiscal question is how surpluses respond to changes in the value of debt induced by multiple-equilibrium in-
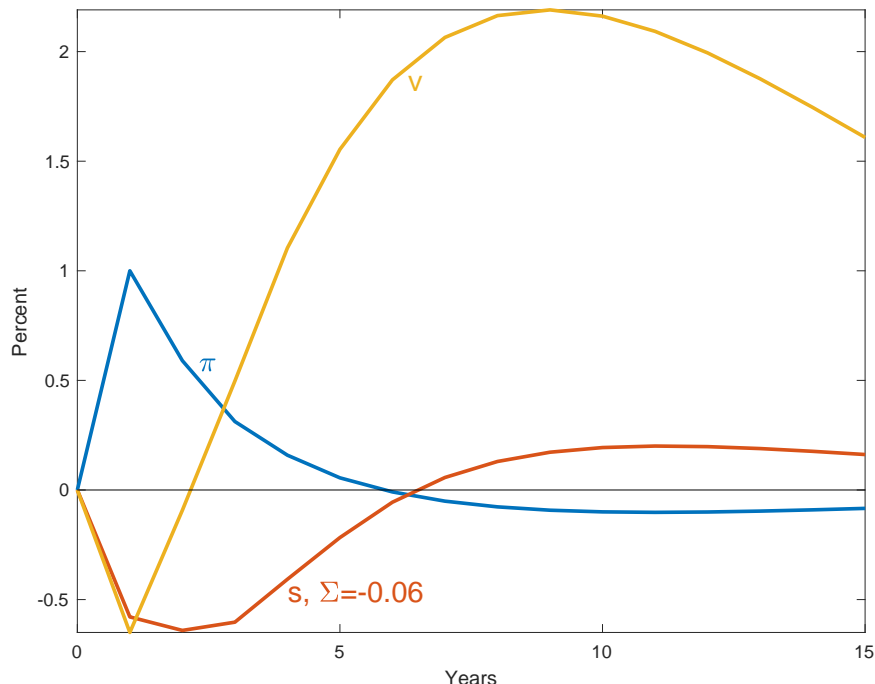
Figure 4: Responses to 1% inflation shock

flation. We cannot measure off-equilibrium responses from data drawn from equilibrium. Suppose, for example, that surpluses are completely exogenous. Suppose that when a government borrows money (negative surplus) it commits to future positive surpluses to repay bondholders, with this s-shaped pattern, but the schedule of those surpluses is fixed. That's active fiscal policy. Yet we observe deficits, which run up debts, and then surpluses which seem to "respond" to those debts. (For more on this point see Cochrane (2019a) and Cochrane (2019b). Leeper and Li (2017) also show that regressions of surplus on debt do not establish passive fiscal policy.)

If we fit an autoregressive process, or any VAR that excludes the value of debt as a forecaster, we will be very unlikely to measure the underlying s-shape of this surplus. If we specify a theoretical model with AR(1) surplus, we miss the crucial fact that governments do promise, and people do expect, subsequent surpluses to pay off debts. Cochrane (2019a) takes up these issues in detail.

I use the words "shock," and "response," which have become conventional in the VAR literature, and compactly describe the calculations. But the calculations do not imply or require a causal structure. In fact, my fiscal theory interpretation offers a reverse causal story: News about future surpluses and discount rates causes inflation to move. That news in turn reflects news about future productivity, fiscal and monetary policy and other truly exogenous or structural disturbances. A "shock" is only an "innovation," a movement in a variable not forecast by

the VAR. A "response" is a change in VAR expectations of a future variable coincident with such a movement. Many VAR exercises do attempt to find an "exogenous" movement in a variable by careful construction of shocks, and "structural" VAR exercises aim to measure causal responses of such shocks. Not here.

Since we start with an identity (1) that holds ex-post, it holds ex-ante using any information set. We do not implicitly assume that agents use only the information in the VAR in order to make these calculations. But "unexpected" here means relative to the VAR information set. People may see a lot more. A decomposition using larger information sets, survey forecasts, or people's full information sets, may be quite different.

4.1 Aggregate demand shocks

We can use the same procedure to understand the fiscal underpinnings of other shocks. For any interesting ε_1, we can compute impulse-response functions, and thereby the terms of the inflation decompositions. I show in the Appendix that we can consider these calculations as a decomposition of the covariance of unexpected inflation with the shock ε_1, rather the decomposition of the variance of unexpected inflation.

I start with a recession shock, which we might also call an aggregate demand shock. The response to the inflation shock in Figure 2 is stagflationary, in that growth falls when inflation rises. Unexpected inflation is, in this sample, negatively correlated with unexpected consumption (and also GDP) growth. The stagflationary episodes in the 1970s outweigh the simple Phillips curve episodes.

However, it is interesting to examine the response to disinflations which come in recessions, and inflations that come in expansions, following a conventional Phillips curve. Such events are common, as in the recession following the 2008 financial crisis. But they pose a fiscal puzzle. In a recession, deficits soar, yet inflation declines. How is this possible? As I outlined in the introduction, future surpluses or lower discount rates could give that deflationary force, needed whether fiscal policy is active or passive. Can we see these effects in the data, and which one is it?

To answer that question, we want to study a shock in which inflation and GDP go in the same direction. I simply specify $\varepsilon_1^{\pi} = -1, \varepsilon_1^g = -1$. The model is linear, so the sign doesn't matter, but the story is clearer for a recession.

Again, we want shocks to other variables to have whatever value they have, on average, conditional on the inflation and output shock. To initialize the other shocks of the VAR, then, I

run a multiple regression

$$\varepsilon_{t+1}^z = b_{z,\pi}\varepsilon_{t+1}^\pi + b_{z,g}\varepsilon_{t+1}^g + \eta_{t+1}^z.$$

I fill in the other shocks at time 1 from their predicted variables given $\varepsilon_1^\pi = -1$ and $\varepsilon_1^g = -1$, i.e. I start the VAR at

$$\varepsilon_1 = -\left[\begin{array}{ccccc} b_{r^n,\pi} + b_{r^n,g} & \varepsilon_1^g = 1 & \varepsilon_1^\pi = 1 & b_{s,\pi} + b_{s,g} & ... \end{array}\right]'.$$

Figure 5 presents responses to this recession shock, and Table 2 collects the inflation decomposition elements in the "Recession" rows.

Both inflation π and growth g responses start at -1%, by construction. Inflation is once again persistent, with a ω-weighted sum of current and expected future inflation equal to 2.36%. Consumption growth g returns rapidly, but does not much overshoot zero, so the level of consumption does not recover much at all. Consumption is roughly a random walk in response to this shock. The nominal interest rate i falls in the recession, and recovers a bit more slowly than inflation. Long-term bond yields y also fall, but not as much as the short-term rate, for about 4 years. We see here the standard upward-sloping yield curve of a recession. The expected bond return follows the long-term yield. The persistent fall in expected return corresponds to a large positive ex-post bond return $\Delta E_1 r_1^n$. The recession includes a large deficit s, which continues for three years. In short, we see a standard picture of an "aggregate demand" recession similar to 2008-2009.

The large deficits in recessions puzzle a simplistic interpretation of the fiscal theory. Why do we not see inflation at such times? Surpluses subsequently turn positive, paying down some of the debt. But the total surplus is still -1.15%. Left to their own devices, surpluses would produce a 1.15% *inflation* during the recession. A potential story that disinflation results from future surpluses more than matching today's deficits is wrong. Growth also adds an inflationary force. The decline in consumption is essentially permanent, so the sum of growth is -1.46%, which would lead on its own to another 1.46% *in*flation.

Discount rates are the central story for deflation in recessions. After one period, expected real returns $r - g$ decline persistently, accounting for 4.96% cumulative deflation.

In terms of the unexpected inflation accounting in the second and third panels of Table 2, again surpluses and growth provide a total $1.15 + 1.46 = 2.61\%$ fiscal loosening. The unweighted sum of future discount rates provides a 4.79% deflationary force, for an overall fiscal shock of 2.19%. Of that, 1% results in unexpected inflation and 1.19% is soaked up by lower long-term bond prices. Again, in the bottom panel, that 1.19% overwhelmingly represents expected infla-
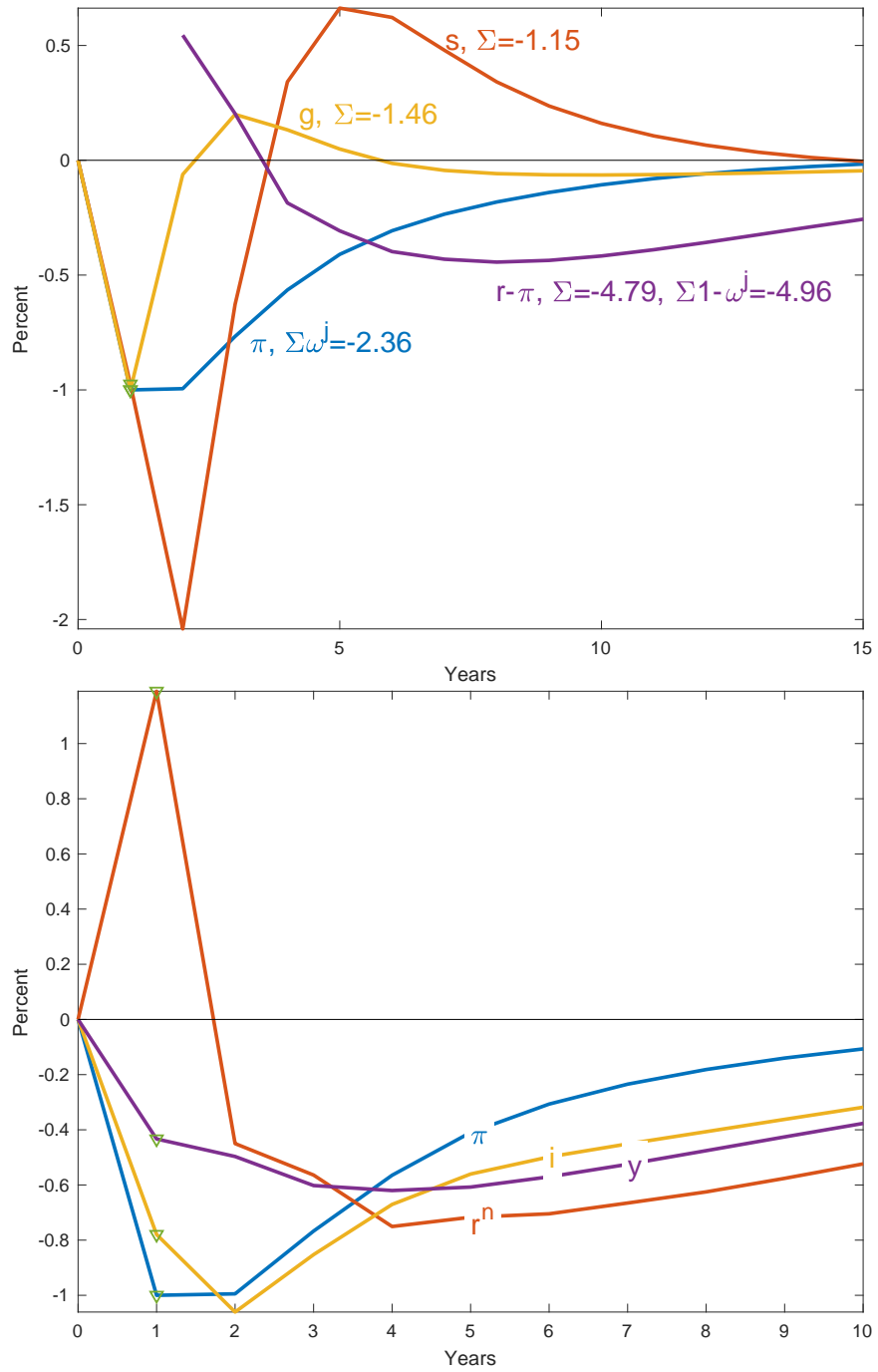
Figure 5: Responses to a recession or aggregate demand shock, $\varepsilon_1^\pi = \varepsilon_1^g = -1$.

tion, essentially marking it to market for a one-period accounting.

In sum, rounding the numbers,

- *Disinflation in a recession, or after an aggregate demand shock that moves output and prices in the same direction, is driven by a lower discount rate, reflected in lower interest rates and bond yields. For each 1% disinflation shock, the expected return on bonds falls so much that the present value of debt rises by nearly 5%. This discount rate shock overcomes a 1.1% inflationary shock coming from persistent deficits, and 1.5% inflationary shock coming from lower growth. The overall fiscal shock is 1.6%, with the extra 0.6% spread to future inflation and soaked up by long-term bond prices.*

The opposite conclusions hold of inflationary shocks in a boom. Discount rate variation gives us a fiscal Phillips curve, accounting for the otherwise puzzling correlation of deficits with disinflation and surpluses with inflation.

The relative magnitudes of the inflation and growth shocks that I used to define a "recession" or "aggregate demand shock" are (obviously) arbitrary. Growth fell about twice as much as inflation in 2008, but inflation fell a bit more than growth in 1982. Other recessions have been stagflationary. To produce a better number one must write a model and find an identification in the data to separate "supply" or "stagflationary" Phillips-curve shift shocks from "demand" or "movement along the Phillips curve" shocks, and one must thereby define precisely just what kind of events we seek to evaluate. Rather than belabor the point with such a calculation, or fill the paper with multiple graphs, I choose a simple and transparent value. The calculations report correctly "How did expectations change if we observe inflation and growth both decline by 1%?" The only quibble is whether that is an interesting question, or whether some other combination of numbers, prefaced by a long identification calculation, might be more interesting.

4.2   Surplus and discount rate shocks

We have studied what happens to surpluses and to discount rates given that we see unexpected inflation. What happens to *inflation* if we see changes in surpluses or discount rates? These are not the same questions. An inflation shock may come, on average, with a discount rate shock, but a discount rate shock may not come on average with inflation.

I calculate here how the variables in the VAR react to an unexpected change in current and expected future primary surpluses including growth, $\Delta E_1 \sum_{j=0}^{\infty}(s_{t+j} + g_{t+j}) = 1$, and all shocks to the VAR take their average values given this innovation. I call this a "surplus shock." (The results are almost the same with or without the growth term in the shock definition, as growth

declines in response to a pure surplus shock.) Then I calculate how the variables in the VAR react to an unexpected change in discount rates, $\Delta E_1 \sum_{j=1}^{\infty}(1 - \omega^j)(r_{t+1}^n - \pi_{t+1}) = 1$, again letting all other variables take their average values given this innovation. I call this a "discount rate shock."

The response of the sum of future surpluses and growth to a shock ε_1 is

$$\Delta E_1 \sum_{j=0}^{\infty} (s_{1+j} + g_{t+j}) = (a_s + a_g)' (I - A)^{-1} \varepsilon_1.$$

To calculate how VAR shocks respond to a surplus shock, I run for each variable z a regression

$$\varepsilon_{t+1}^z = b_z \left[(a_s + a_g)' (I - A)^{-1} \varepsilon_{t+1} \right] + \eta_{t+1}^z \tag{7}$$

where a_z pulls variable z from the VAR, $a_z' x_t = z_t$. Then, I start the surplus-shock response function at

$$\varepsilon_1 = - \left[\begin{array}{cccc} b_{r^n} & b_g & b_\pi & ... \end{array} \right]'.$$

Similarly, to calculate responses to a discount-rate shock, I run

$$\varepsilon_{t+1}^z = b_z \left\{ (a_{r^n} - a_\pi)' \left[A(I - A)^{-1} - \omega A(I - \omega A)^{-1} \right] \varepsilon_{t+1} \right\} + \eta_{t+1}^z.$$

I start the discount-rate response function with the negative of these regression coefficients as well, capturing the response to a discount rate decline.

Figure 6 presents the responses to the surplus shock, and Figure 7 presents the responses to the discount rate shock. Table 2 collects relevant contributions to the inflation decompositions.

The sum of surplus and growth responses to the surplus shock is -0.66 - 0.34 = -1.00 by construction. Surpluses still have an s-shaped pattern, but the initial deficits are not matched by subsequent surpluses.

This decline in surpluses and growth has essentially no effect on inflation. Starting in year 2, inflation declines – the "wrong" direction – by less then a tenth of a percent, and the overall weighted sum of inflation declines by a tenth of a percent. Why is there no inflation? Because discount rates also decline, with a weighted sum of 1.10%, almost exactly matching the surplus decline. The lower panel of Figure 6 adds insight. We see a sharp and persistent decline in the interest rate, long-term bond yield, and expected bond return, along with deficits and the growth decline.

This figure paints a picture of an average (in this sample) recession – a decline in growth
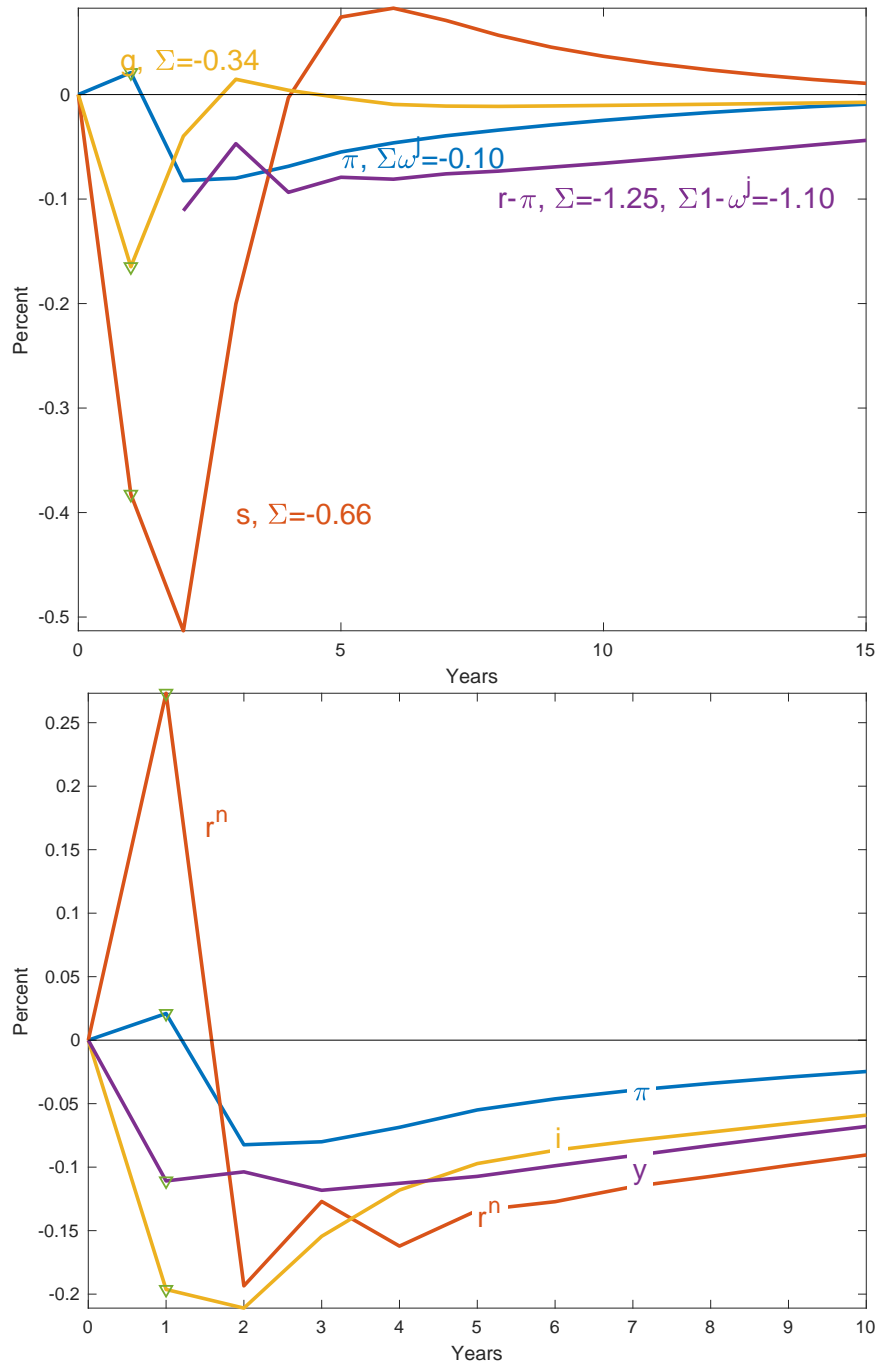
Figure 6: Responses to a surplus and growth shock, $\Delta E_1 \sum_{j=0}^{\infty} \left(s_{1+j} + g_{1+j} \right) = -1$.
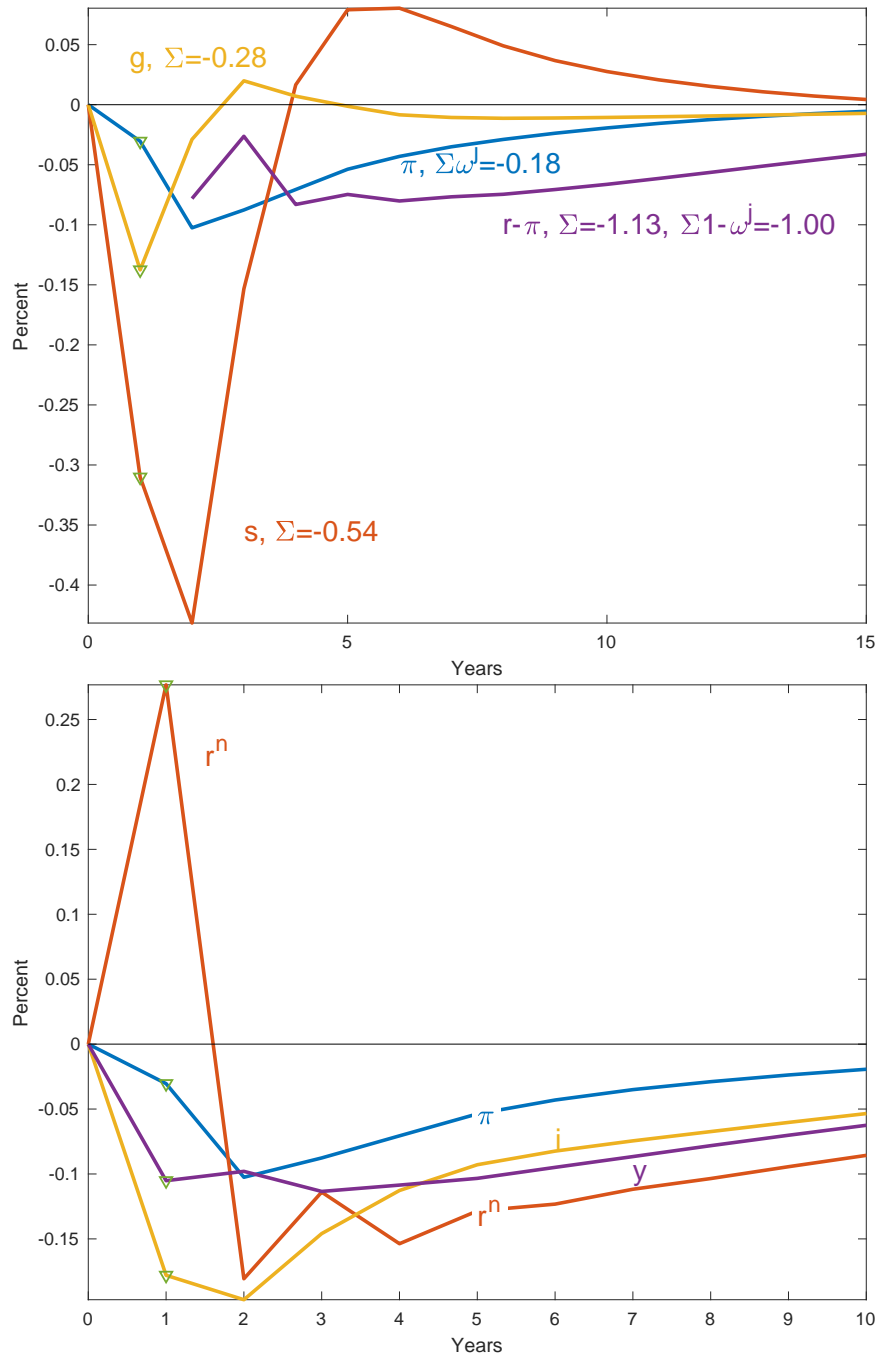
Figure 7: Responses to a discount-rate shock $\Delta E_1 \sum_{j=1}^{\infty} (1 - \omega^j) \left(r_{1+j}^n - \pi_{1+j} \right) = 1$.

and interest rates, accompanied by large deficits. The deficits are on average not directly repaid by subsequent surpluses or growth. Instead, real interest rates decline persistently in the recession and its aftermath. This decline in discount rate essentially pays for the deficits. Ex-post, a low real return brings the value of debt back rather than larger taxes or lower spending. There is, on average, very little inflation or deflation.

The response to the discount rate shock in Figure 7 is, surprisingly, almost exactly the same. The weighted discount rate response ($\sum 1 - \omega^j$) is -1.00 here by construction. This discount rate decline should be deflationary, and it is – but the disinflation peaks at -0.1% and the weighted sum is only -0.18%. A sharp growth and surplus decline accompanies this discount rate decline, with a pattern almost exactly the same as we found from the growth and surplus shock. In the bottom panel, the expected return decline comes with a decline in interest rates and bond yields, as we would expect.

Clearly, the surplus + growth shock and the expected return shock have isolated essentially the same events – recessions, in which growth falls, deficits rise, interest rates fall, and, on average in this sample, inflation doesn't move much, and the converse pattern of expansions. The correlation of the surplus+growth and discount rate shocks is 0.96. The responses to a one-period surplus shock, $\Delta E_1 s_1 = 1$, a pure growth shock $\Delta E_1 g_1 = 1$ and a one-period discount rate shock $\Delta E_1 r_2^n = 1$ are all quite similar as well.

The fiscal roots of the absence of inflation, in the end, characterize most business-cycle movements in the data.

- *Short and long-run surplus, growth, and discount rate shocks all paint the same pictures. Large deficits are mostly not repaid by subsequent growth or surpluses. Instead, they correspond to periods of extended low expected returns. The deficit and discount rate effects largely offset, leaving little inflation on average. Discount rate variation explains why deficits, not repaid by future surpluses, do not result in inflation.*

The fact that interest rates and discount rates move in opposition to the surplus shock is obviously key to this result. What if there is a surplus shock and the Federal Reserve does not accommodate the shock, or its economic correlates with the prominent interest decline seen in Figure 6? To answer this question, I modify the surplus+growth shock so that the short-term interest rate remains constant for two years. I now run

$$\varepsilon_{t+1}^z = b_{z,s}\left[(a_s + a_g)'(I-A)^{-1}\varepsilon_{t+1}\right] + b_{z,i0}\varepsilon_{t+1}^i + b_{z,i1}\left(a_i'A\varepsilon_{t+1}\right) + \eta_{t+1}^z.$$

and I initialize the VAR at

$$\varepsilon_1 = - \left[\begin{array}{cccc} b_{r^n,s} & b_{g,s} & b_{\pi,s} & ... \end{array} \right]'.$$

Figure 8 presents the responses, and Table 2 collects the terms of the identities. Starting in the bottom panel of Figure 8, verify that the interest rate i now stays constant for two years, by construction. This behavior contrasts with the strong interest rate decline in the bottom panel of Figure 6. Except for the one-period expected return decline in year two, the long-term bond yields and expected returns follow the interest rate. All decline eventually.

Turning to the upper panel, the sum of surplus (-0.52) and growth (-0.48) shocks remains -1.00% by construction. Deficits are initially much larger than 0.52%, but much of this immediate deficit is repaid by higher long-term surpluses, so in the end the fiscal shock is split equally between surpluses and growth. The discount rate term is now reduced to 0.62% - 0.67%, however, so the surplus shock now produces 0.36% immediate and 0.38% weighted sum inflation.

In sum, without the interest rate response, which we may interpret as Federal Reserve accommodation of the fiscal shock, or of its economic consequences such as lower growth, the countervailing discount rate effect is much weaker, so the fiscal shock does result in unexpected inflation.

(Holding the interest rate constant for one period produces a similar though slightly weaker result. Interest rates drift down after the one period, so the discount rate effect is slightly stronger and inflation slightly less.)

5   Standard errors

I have delayed a discussion of standard errors because there is nothing important to test. Identities are identities. If $x = y + z$ and x moves, y or z must move, and all we can do is to measure which one. In addition, unlike the case in asset pricing, no important economic hypothesis here rests on whether one of surpluses or discount rates do not move. (Asset pricing finds the hypothesis that expected returns are constant over time interesting to test.) Standard errors only give us a sense of how accurate the measurement is.

To evaluate sampling distributions I run a Monte Carlo. Most of the interesting statistics – variance decompositions, impulse response functions, $(I - A)^{-1}$, etc. – are nonlinear functions of the underlying data, and the near-unit root in value v_t also induces non-normal distributions. For these reasons, I largely characterize the sampling distribution by the interquartile range – the 25% and 75% points of the sampling distribution.

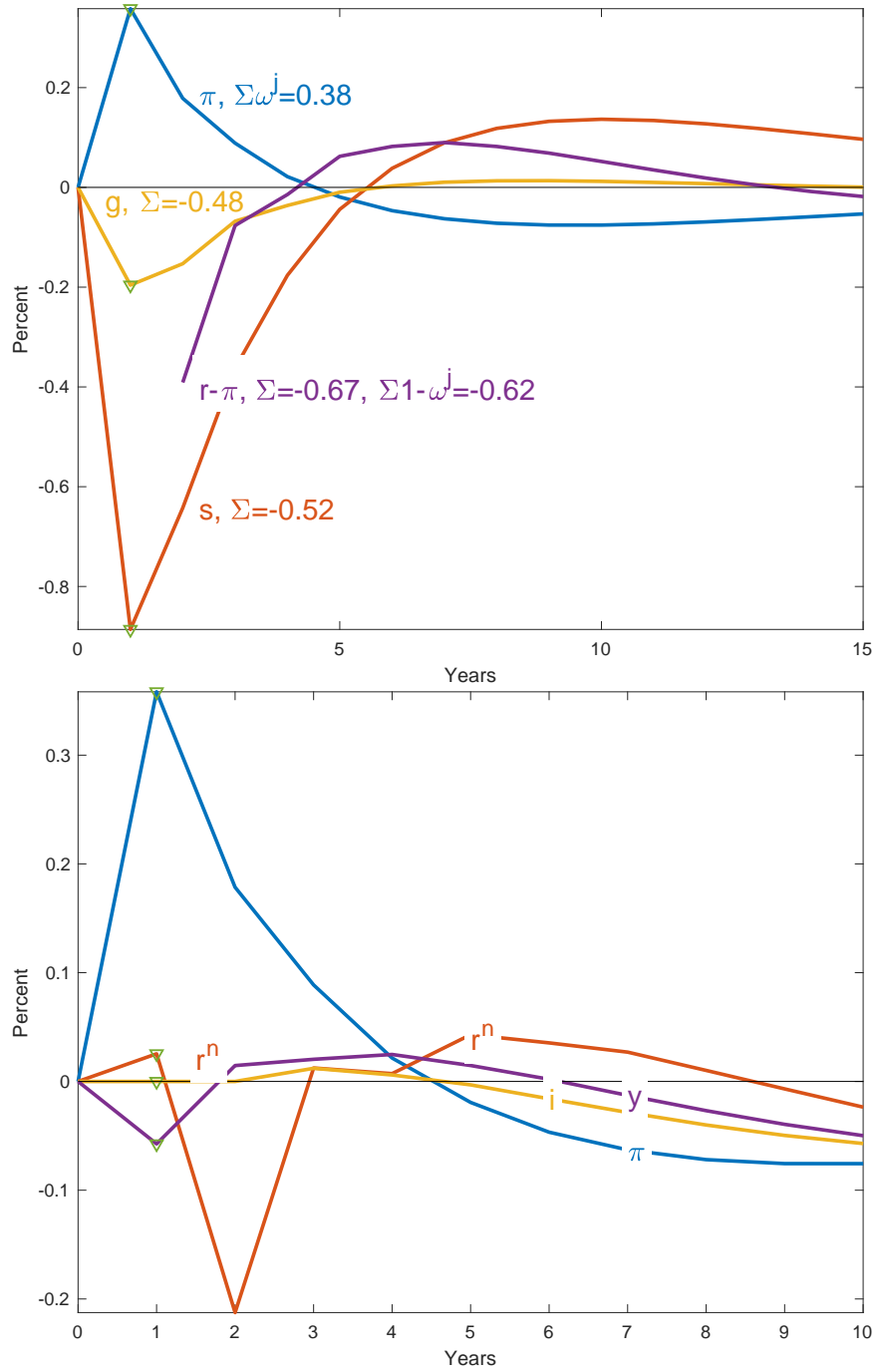Table 3 collects the sampling quantiles for the variance decompositions of Table 2. Fig-

Figure 8: Responses to a surplus and growth shock with no interest rate movement for two years, $\Delta E_1 \sum_{j=0}^{\infty} (s_{1+j} + g_{1+j}) = 1$, $\Delta E_1 i_1 = 0$, $\Delta E_1 i_2 = 0$.
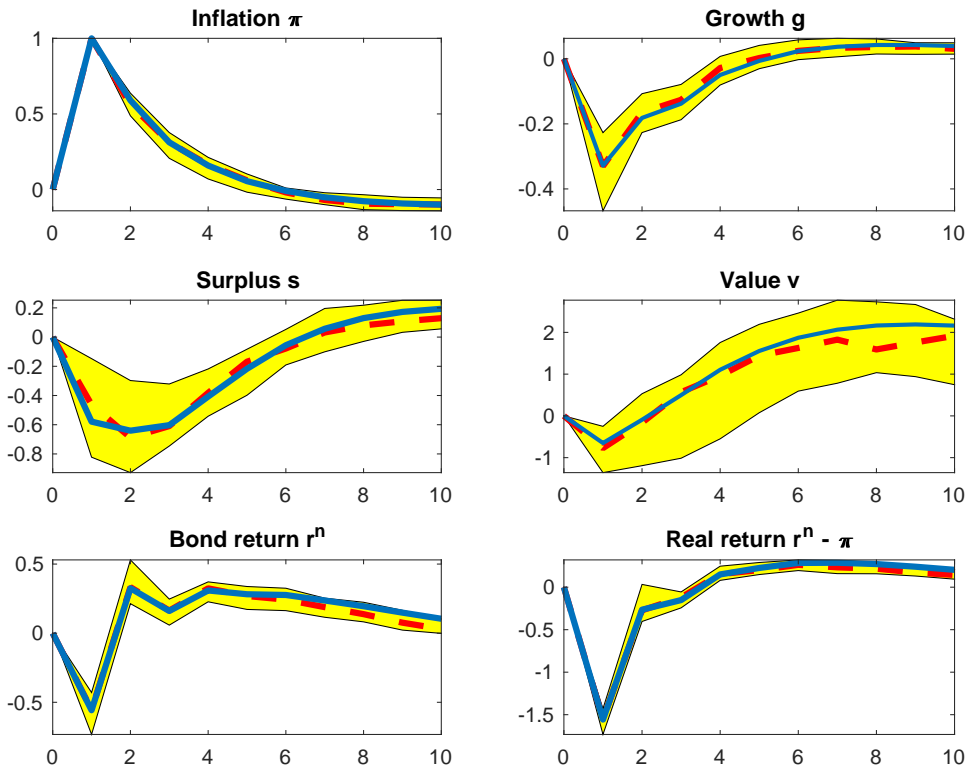
Figure 9: Distribution of the impulse response function, to an inflation shock. The bands are 25% and 75% points of the sampling distribution, the dashed line is the median, and the solid line is the estimate.

ure 9 presents the main components of the impulse-response function relevant to the inflation variance decomposition. The bands are 25% and 75% points of the sampling distribution, the dashed line is the median, and the solid line is the estimate.

Start with the "Inflation" shock in Table 3. In the second panel, inflation quantiles are 1.00 because the shock is defined as a 1% movement in inflation in every sample. The 1.59% weighted sum of inflation has 1.38% to 1.64% quantiles in the top panel. The -0.06% sum of future surpluses has quartiles -0.69% to 0.23%. The -0.49% sum of growth rates has quantiles -0.72% to -0.22% The 1.04% (weighted) and 1.00% (unweighted) discount rate term has quantiles 0.16% to 1.46% and 0.16% to 1.55%. That discount rates matter is a pretty solid conclusion, but negative surpluses may contribute more to unexpected inflation than the point estimate suggests.

There are several sources of this rather large sampling variation. First, the shocks are large. As shown in Table 1, the surplus innovation has a 4.75 percentage point standard deviation, and value 6.55 percentage points, compared to 1.12 percentage points for inflation. Our friend $\sigma/\sqrt{T}$ starts off badly.

Second, the shocks are imperfectly correlated. This matters, because in each case I find movements in other variables contemporaneous with the shock of interest by running a regression of the other shocks on the shock of interest. The sampling uncertainty of this orthogonalization adds to that of the VAR. We see a correspondingly wide band around the initial surplus and growth responses in Figure 9. There is hope in this observation, however. Higher frequency data can better identify shock correlations, at the cost that one must model the strong seasonal in primary surpluses. Moreover, other shock identifications may have better measured correlations.

Third, we measure sums of future surpluses and discount rates. The value of the debt v_t is the main long-run state variable, and uncertainty about its evolution adds to the uncertainty about the sum of surpluses. The coefficient of value v_t on its own lag is 0.98 in Table 1, so small variations in that value lead to large variation in $(I - A)^{-1}$ sums. The Appendix shows that the last two sources of variation contribute about equally.

(It is not always possible to find $\omega \in [0, 1]$ to satisfy the return identity, so many Monte Carlo draws use a best fit value of ω in which the return identity does not hold. This adds to sampling variation, but comparing weighted and unweighted results does not seem to have a large effect overall.)

Table 3 also presents 25% and 75% quantiles for the recession, surplus and discount rate shocks of Table 2. The -1.15% total surplus response to a recession shock has quantiles -1.28% to 0.49%, spanning zero, while the -4.96% and -4.79% discount rate response has quantiles from -4.84% to -2.43% and -4.84% to -2.35%. The conclusion that discount rate variation is a central part of the story for understanding aggregate-demand inflation is fairly solid. The small inflation and offsetting surplus and discount rate responses to surplus and discount rate shocks are similarly measured.

It would be nicer if the elements of the identities were more precisely measured. But there is nothing one can do within the framework of this VAR to improve on them, so it's worth examining point estimates while awaiting more data or other approaches such as model-based estimates that impose prior structure. The rather large sampling variation should, however, discourage one from the inevitable temptation to start eyeballing different regimes and splitting up the sample.

6   Concluding comments

This analysis evidently just scratches the surface. One can apply these decompositions to any VAR, or to the impulse-responses of any theoretical model. Such calculations beckon.

In particular, it is interesting to apply the inflation decompositions model predictions or empirical estimates of monetary and fiscal policy shocks. Suppose, for example, that monetary policy follows $i_t = \phi_{i,\pi}\pi_t + \phi_{i,x}x_t + u_t^i$, and fiscal policy follows $s_t = \phi_{s,\pi}\pi_t + \phi_{s,x}x_t + u_t^s$, with persistent disturbances u_t^i and u_t^s (and in particular an s-shaped moving average of u_t^s, reflecting partial repayment promises). With such a specification, it is interesting to compute responses and inflation decompositions to shocks to u_t^i and u_t^s. The Federal Reserve cannot directly control fiscal policy, so fiscal theory of monetary policy models suggest that it is interesting to shock monetary policy, while holding u_t^s fixed. Yet, it is likely interesting to allow the systematic part of fiscal policy to respond to monetary policy, reflecting the rise in tax revenues with real income and imperfect indexation of the tax code, as well even as predictable stimulus spending in recessions. (That, I think, is the kind of "what happens if we raise rates?" question a Federal Reserve official might have in mind.) A response to u^i, holding $u^s = 0$, but not $s = 0$, answers this question. Cochrane (2019a) presents such calculations from a simple fiscal theory of monetary policy model. It may also be interesting to know what happens without even the systematic fiscal response, which setting $\phi_s = 0$ can answer. Likewise, it is interesting to know what the response to fiscal shocks, changes in u_t^s are, assuming the central bank follows its customary rule represented by ϕ terms and not holding interest rates constant as in my last Figure. New-Keynesian models move both u^i and u^s together by the passive-fiscal assumption. Still, it is interesting to see how important that contemporaneous fiscal shock is to the result.

Alas, making such calculations in data require one to solve the formidable problems of estimating the ϕ coefficients, given that the right hand variables react to the disturbances. The state of the art for identifying monetary policy disturbances and measuring the reaction function goes well beyond the simple recursive and long-run strategies available in the atheoretical annual VAR here, to include highly detailed identification assumptions, high frequency data, narrative approaches, and other devices, and still does does not offer a robustly successful result on which one can build (Ramey (2016), Cochrane (2011a)). And even this voluminous literature has not started to think about how we can identify monetary policy shocks that are independent of fiscal policy shocks. In the data, monetary and fiscal authorities are likely to respond to the same underlying shocks, as we found a strong correlation between interest rate or discount rate shocks and surplus shocks here. Teasing out monetary policy shocks that are orthogonal to fiscal

policy shocks, as well as all the other desired orthogonality, requires some thought. I attempted monetary and fiscal policy shocks by recursive identification in this data, but one-year interest-rate, inflation, and growth shocks are all highly correlated. Assuming all of that correlation flows from interest rates to inflation and growth results in positive effects of interest rates on inflation and growth. Assuming all correlation reflects rule-like responses of interest rates to inflation and growth eliminates the unexpected inflation response we wish to measure. Obviously, reality lies in between.

Quarterly or monthly data are attractive, offering potentially better measurement of correlations and shock orthogonalization but requiring us to model the strong seasonality in surpluses. Debt data go back centuries, allowing and requiring us to think what is the same and different across different periods of history. Inflation through wars and under the gold standard may well have different fiscal foundations than in the postwar environment. A narrative counterpart, especially for big episodes such as the 1970s and 1980s, awaits. Different countries under different monetary and exchange rate regimes and different fiscal constraints will behave differently. A parallel investigation of exchange rates beckons, following Jiang (2019a), Jiang (2019b). One could define shocks in many additional interesting ways. The treatment of debt can be refined in many ways. In particular, the maturity structure is not geometric, and varies over time.

I omitted analysis of the remaining shocks in the VAR. A shock to any other variable, orthogonal to the inflation shock, can move all of the other terms of the inflation identities. Such movements must offset: In (5), if a shock does not move the inflation term, but does move the sum of future surpluses, then it must also move the sum of growth rates or discount rates. These additional effects are large. The variation in $\Delta E_1 \sum_{j=0}^{\infty} s_{1+j}$ when other shocks move is large; the corresponding movement in the discount rate term is also large, and the two movements are negatively correlated. We get a hint of that behavior in the surplus+growth and discount rate shock responses. I do not pursue this question because it is much more interesting if one can give some structural or economic interpretation to the shocks to other variables, which requires a model.

Perhaps most of all, linking these theory-free characterizations to explicit fiscal theory of monetary policy models such as Cochrane (2019a), or at least to explicit models of discount rates and long-term debt management, is an obviously important step. However, such models need to be elaborated to the point that they can match data, which requires considerable complication of elements such as the IS and Phillips curve, and to surmount difficult identification and estimation challenges. It's important, but not easy.

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Online Appendix to "The Fiscal Roots of Inflation"

A   Derivation of the linearized identities

In this appendix I derive the linearized identities (1) (2) and (3),

$$v_{t+1} = v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} - s_{t+1} \tag{8}$$

$$v_t = \sum_{j=1}^{T} s_{t+j} - \sum_{j=1}^{T} \left(r_{t+j}^n - \pi_{t+j} - g_{t+j} \right) + v_{t+T}$$

and

$$\Delta E_{t+1} \pi_{t+1} - \Delta E_{t+1} \left(r_{t+1}^n - g_{t+1} \right)$$
$$= -\sum_{j=0}^{\infty} \Delta E_{t+1} s_{t+1+j} + \sum_{j=1}^{\infty} \Delta E_{t+1} \left(r_{t+1+j}^n - \pi_{t+1+j} - g_{t+1+j} \right). \tag{9}$$

I also define the variables more carefully.

The symbols are as follows:

$$V_t = M_t + \sum_{j=0}^{\infty} Q_t^{(t+1+j)} B_t^{(t+1+j)}$$

is the nominal end-of-period market value of debt, where M_t is non-interest-bearing money, $B_t^{(t+j)}$ is zero-coupon nominal debt outstanding at the end of period t and due at the beginning of period $t + j$, and $Q_t^{(t+j)}$ is the time t price of that bond, with $Q_t^{(t)} = 1$. Taking logs,

$$v_t \equiv \log \left(\frac{V_t}{Y_t P_t} \right)$$

is log market value of the debt divided by GDP, where P_t is the price level and Y_t is real GDP or another stationarity-inducing divisor such as consumption, potential GDP, population, etc. I use consumption times the average GDP to consumption ratio in the empirical work, but I will call Y and ratios to Y "GDP" for brevity.

$$R_{t+1}^n \equiv \frac{M_t + \sum_{j=1}^{\infty} Q_{t+1}^{(t+j)} B_t^{(t+j)}}{M_t + \sum_{j=1}^{\infty} Q_t^{(t+j)} B_t^{(t+j)}} \tag{10}$$

is the nominal return on the portfolio of government debt, i.e. how the change in prices overnight

from the end of t to the beginning of $t+1$ affects the value of debt held overnight, and

$$r_{t+1}^n \equiv \log(R_{t+1}^n)$$

is the log nominal return on that portfolio.

$$\pi_t \equiv \log\left(\frac{P_t}{P_{t-1}}\right), \; g_t \equiv \log\left(\frac{Y_t}{Y_{t-1}}\right)$$

are log inflation and GDP growth rate.

We can accommodate explicit default, and yes, the formulas also apply to countries in the Euro. An explicit default is a reduction in the nominal quantity of debt overnight. The $B_t^{(t+j)}$ in the numerator of (10) represents the post-default number of bonds outstanding, i.e. on the morning of time $t+1$, while the $B_t^{(t+j)}$ in the denominator represents the pre-default number of bonds outstanding, i.e. in the evening of time t. A partial default then shows up as a low return. To handle default one would, of course, add notation distinguishing these two values.

Now, I establish the nonlinear flow and present value identities. In period t, we have

$$\sum_{j=0}^{\infty} Q_t^{(t+j)} B_{t-1}^{(t+j)} + M_{t-1} = P_t sp_t + \sum_{j=0}^{\infty} Q_t^{(t+1+j)} B_t^{(t+1+j)} + M_t, \qquad (11)$$

where sp_t denotes the real primary (not including interest payments) surplus or deficit. Money M_t at the end of period t is equal to money brought in from the previous period M_{t-1} plus the effects of bond sales or purchases at price $Q_t^{(t+j)}$, less money soaked up by primary surpluses.

The left hand side of (11) is the beginning-of-period market value of debt, i.e. before debt sales or repurchases $B_t^{(t+j)} - B_{t-1}^{(t+j)}$ have taken place. It turns out to be more convenient here to express equations in terms of the end-of-period market value of debt. To that end, shift the time index forward one period and rearrange to write

$$\sum_{j=0}^{\infty} Q_{t+1}^{(t+1+j)} B_t^{(t+1+j)} + M_t = P_{t+1} sp_{t+1} + \sum_{j=0}^{\infty} Q_{t+1}^{(t+2+j)} B_{t+1}^{(t+2+j)} + M_{t+1},$$

$$\sum_{j=1}^{\infty} Q_{t+1}^{(t+j)} B_t^{(t+j)} + M_t = P_{t+1} sp_{t+1} + \sum_{j=1}^{\infty} Q_{t+1}^{(t+1+j)} B_{t+1}^{(t+1+j)} + M_{t+1},$$

$$\left(M_t + \sum_{j=1}^{\infty} Q_t^{(t+j)} B_t^{(t+j)}\right) R_{t+1}^n = P_{t+1} sp_{t+1} + \left(M_{t+1} + \sum_{j=1}^{\infty} Q_{t+1}^{(t+1+j)} B_{t+1}^{(t+1+j)}\right),$$

$$\frac{M_t + \sum_{j=1}^{\infty} Q_t^{(t+j)} B_t^{(t+j)}}{P_t Y_t} \frac{R_{t+1}^n}{G_{t+1}} \frac{P_t}{P_{t+1}} = \frac{sp_{t+1}}{Y_{t+1}} + \frac{M_{t+1} + \sum_{j=1}^{\infty} Q_{t+1}^{(t+1+j)} B_{t+1}^{(t+1+j)}}{P_{t+1} Y_{t+1}}. \tag{12}$$

We can iterate this flow identity (12) forward to express the nonlinear government debt valuation identity as

$$\frac{M_t + \sum_{j=1}^{\infty} Q_t^{(t+j)} B_t^{(t+j)}}{P_t Y_t} = \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} \frac{1}{R_{t+k}/G_{t+k}} \right) \frac{sp_{t+j}}{Y_{t+j}}. \tag{13}$$

The market value of government debt at the end of period t, as a fraction of GDP, equals the present value of primary surplus to GDP ratios, discounted at the government debt rate of return less the GDP growth rate. (I assume here that the right hand side converges. Otherwise, keep the limiting debt term or iterate a finite number of periods.)

The nonlinear identities (12) and (13) are cumbersome. I linearize the flow equation (12) and then iterate forward to obtain a linearized version of (13). Write (12) as

$$\frac{V_t}{P_t Y_t} R_{t+1}^n \frac{P_t}{P_{t+1}} \frac{Y_t}{Y_{t+1}} = \frac{sp_{t+1}}{Y_{t+1}} + \frac{V_{t+1}}{P_{t+1} Y_{t+1}}.$$

Taking logs,

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = \log\left(\frac{sp_{t+1}}{Y_{t+1}} + \frac{V_{t+1}}{P_{t+1} Y_{t+1}} \right) \tag{14}$$

I linearize in the *level* of the surplus, not its log as one conventionally does in asset pricing, since the surplus is often negative. To linearize in terms of the surplus/GDP ratio, Taylor expand the last term,

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = \log(e^v + sy) + \frac{e^v}{e^v + sy} (v_{t+1} - v) + \frac{1}{e^v + sy} (sy_{t+1} - sy)$$

where

$$sy_{t+1} \equiv \frac{sp_{t+1}}{Y_{t+1}} \tag{15}$$

denotes the surplus to GDP ratio, and variables without subscripts denote a steady state of (14). With $r \equiv r^n - \pi$,

$$r - g = \log \frac{e^v + sy}{e^v}.$$

Then,

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = \left[\log(e^v + sy) - \frac{e^v}{e^v + sy} \left(v + \frac{sy}{e^v} \right) \right] + \frac{e^v}{e^v + sy} v_{t+1} + \frac{e^v}{e^v + sy} \frac{sy_{t+1}}{e^v}$$

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = \left[v + r - g - \frac{e^v}{e^v + sy} \left(v + \frac{e^v + sy}{e^v} - 1 \right) \right] + \rho v_{t+1} + \rho \frac{sy_{t+1}}{e^v}$$

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = [r - g + (1 - \rho)(v - 1)] + \rho v_{t+1} + \rho \frac{sy_{t+1}}{e^v} \tag{16}$$

where

$$\rho \equiv e^{-(r-g)}. \tag{17}$$

Suppressing the small constant, and thus interpreting variables as deviations from means, the linearized flow identity is

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = \rho \frac{sy_{t+1}}{e^v} + \rho v_{t+1}. \tag{18}$$

Iterating forward, the present value identity is

$$v_t = \sum_{j=1}^{T} \rho^{j-1} \left[\rho \frac{sy_{t+j}}{e^v} - \left(r_{t+j}^n - \pi_{t+j} - g_{t+j} \right) \right] + \rho^T v_T. \tag{19}$$

If we linearize around $r - g = 0$, then the constant in (18) is zero ($sy = 0$), and we obtain the linearized flow and present value identities (8) and (9), with the symbol s_t representing sy_t/e^v. There is nothing wrong with expanding about $r = g$. The point of expansion need not be the sample mean.

To approximate in terms of the surplus to value ratio, write (14) as

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = \log \left(\frac{V_t}{P_t Y_t} \frac{\frac{sp_{t+1}}{Y_{t+1}}}{\frac{V_t}{P_t Y_t}} + \frac{V_{t+1}}{P_{t+1} Y_{t+1}} \right)$$

$$r_{t+1}^n - \pi_{t+1} - g_{t+1} = \log \left(\frac{\frac{sp_{t+1}}{Y_{t+1}}}{\frac{V_t}{P_t Y_t}} + \frac{\frac{V_{t+1}}{P_{t+1} Y_{t+1}}}{\frac{V_t}{P_t Y_t}} \right)$$

$$r_{t+1}^n - \pi_{t+1} - g_{t+1} = \log \left(sv_{t+1} + e^{v_{t+1} - v_t} \right).$$

At a steady state

$$r - g = \log (1 + sv). \tag{20}$$

$$e^{r-g} = 1 + sv.$$

Taylor expanding around a steady state,

$$r_{t+1}^n - \pi_{t+1} - g_{t+1} = \log (1 + sv) + \frac{1}{(1 + sv)} (sv_{t+1} - sv + v_{t+1} - v_t)$$

$$v_t + (1 + sv) \left[r^n_{t+1} - \pi_{t+1} - g_{t+1} \right] = \left[(1 + sv) \log (1 + sv) - sv \right] + sv_{t+1} + v_{t+1} \qquad (21)$$

The linearized flow identity (8) follows, with the symbol s_t representing the surplus to value ratio $s_t = sv_t$, if we suppress the constant, using deviations from means in the analysis, or if we use $r = g$ or $sv = 0$, as a point of expansion.

The linearizations in terms of the surplus to value ratio sv_t are more accurate. The units of the flow identities (8), (18) are rates of return. Dividing the surplus by the previous period's value gives a better approximation to the growth in value, when the value of debt is far from the steady state.

With stationary v_t, the term v_{t+T} does not vanish in (9), where the term $\rho^T v_{t+T}$ vanishes in (19). In this paper, the presence of the v_{t+T} term is not a difficulty. I study innovations $\Delta E_{t+1} v_{t+T}$ and $\Delta E_{t+1} v_{t+T} \to 0$. For other purposes, one may wish to use the surplus to GDP linearization and $r > g$ steady state, so that the limiting term vanishes.

A constant ratio of surplus to market value of debt for any price level path leads to a passive fiscal policy. An unexpected deflation raises the real value of debt. If surpluses always rise in response, they validate the lower price level. Thus, although on the equilibrium path one can describe dynamics via either linearization, if one wants to think about how fiscal-theory equilibria are formed, it is better to describe a surplus that does not react to price level changes, so only one value v_t emerges, as is the case in (19). For such purposes, the surplus to GDP definition is appropriate, as well as adopting a linearization point $r > g$ and $\rho < 1$. It's also better to use the nonlinear versions of the identities for determinacy issues. The analysis of this paper is about what happens in equilibrium, and does not require an active-fiscal assumption, so the difference is irrelevant here.

I infer the surplus from the linearized flow identity (8) so which concept the surplus corresponds to makes no difference to the analysis. The difference is only the accuracy of approximation, how close the surplus recovered from the linearized flow identity corresponds to a surplus recovered from the nonlinear exact identity (14).

B A variance decomposition

I use the elements of the impulse response function and their sums to calculate the terms of the unexpected inflation identity (3). We can interpret this calculation as an decomposition of the variance of unexpected inflation. Multiply both sides of (3) by $\Delta E_{t+1} \pi_{t+1}$ and take expectations,

giving

$$var\left(\Delta E_{t+1}\pi_{t+1}\right) - cov\left[\Delta E_{t+1}\pi_{t+1}, \Delta E_{t+1}\left(r_{t+1}^n - g_{t+1}\right)\right] \tag{22}$$

$$= -\sum_{j=0}^{\infty} cov\left[\Delta E_{t+1}\pi_{t+1}, \Delta E_{t+1}s_{t+1+j}\right] + \sum_{j=1}^{\infty} cov\left[\Delta E_{t+1}\pi_{t+1}, \Delta E_{t+1}\left(r_{t+1+j}^n - \pi_{t+1+j} - g_{t+1+j}\right)\right].$$

Unexpected inflation may only vary to the extent that it covaries with current bond returns, or if it forecasts surpluses or real discount rate.

Dividing by $var\left(\Delta E_{t+1}\pi_{t+1}\right)$, we can express each term as a fraction of the variance of unexpected inflation coming from that term. This decomposition adds up to 100%, within the accuracy of approximation, but it is not an orthogonal decomposition, nor are all the elements necessarily positive. Each term is also a regression coefficient of the other terms on unexpected inflation.

The two approaches give exactly the same result – the terms of (22) are exactly the terms of the impulse-response function, to an inflation shock orthogonalized last, i.e. a shock that moves all variables at time 1 including $\Delta E_1 \pi_1$.

To see this fact, write the VAR in standard notation

$$x_{t+1} = Ax_t + \varepsilon_{t+1} \tag{23}$$

so

$$\Delta E_{t+1}\sum_{j=1}^{\infty} x_{t+j} = (I - A)^{-1}\varepsilon_{t+1}.$$

Let a denote vectors which pull out each variable, i.e.

$$\pi_t = a'_\pi x_t, \ s_t = a'_s x_t, \tag{24}$$

etc. Then the present value identity (3) reads and may be calculated as

$$a'_\pi \varepsilon_{t+1} - (a_{r^n} - a_g)'\varepsilon_{t+1} = -a'_s(I - A)^{-1}\varepsilon_{t+1} + a'_{rg}(I - A)^{-1}A\varepsilon_{t+1} \tag{25}$$

where

$$a_{rg} \equiv a_{r^n} - a_\pi - a_g.$$

We can calculate the variance decomposition (22) by

$$a'_\pi \Sigma a_\pi - (a_{r^n} - a_g)'\Sigma a_\pi = -a'_s(I - A)^{-1}\Sigma a_\pi + a'_{rg}(I - A)^{-1}A\Sigma a_\pi$$

where $\Sigma = cov(\varepsilon_{t+1}, \varepsilon'_{t+1})$, and then divide by $a'_\pi \Sigma a_\pi$ to express the result as a fraction,

$$1 - (a_{r^n} - a_g)' \frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi} = -a'_s (I - A)^{-1} \frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi} + a'_{rg} (I - A)^{-1} A \frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi}. \tag{26}$$

To show that this variance decomposition is the same as the elements and sum of elements of the impulse-response function to an inflation shock, orthogonalized last, note that the regression coefficient of any other shock ε^z on the inflation shock is

$$b_{\varepsilon^z, \varepsilon^\pi} = \frac{cov(\varepsilon_{t+1}^z, \varepsilon_{t+1}^\pi)}{var(\varepsilon_{t+1}^\pi)} = \frac{a'_z \Sigma a_\pi}{a'_\pi \Sigma a_\pi},$$

so the VAR shock, consisting of a unit movement in inflation $\varepsilon_1^\pi = 1$ and movements $\varepsilon_1^z = b_{\varepsilon^z, \varepsilon^\pi}$ in each of the other variables is given by

$$\varepsilon_1 = \frac{\Sigma a_\pi}{a'_\pi \Sigma a_\pi}.$$

We recognize in (26) the responses and sums of responses to this shock. Dividing (22) by the variance of unexpected inflation, or examining the terms of (26), we recognize that each term is also the coefficient in a single regression of each quantity on unexpected inflation.

In an analogous way, we can interpret the responses to other shocks as a decomposition of the *co*variance of unexpected inflation with that shock, based on

$$cov\left(\Delta E_{t+1} \pi_{t+1} \varepsilon_{t+1}\right) - cov\left[\varepsilon_{t+1}, \Delta E_{t+1} \left(r_{t+1}^n - g_{t+1}\right)\right]$$
$$= -\sum_{j=0}^{\infty} cov\left[\varepsilon_{t+1}, \Delta E_{t+1} s_{t+1+j}\right] + \sum_{j=1}^{\infty} cov\left[\varepsilon_{t+1}, \Delta E_{t+1} \left(r_{t+1+j}^n - \pi_{t+1+j} - g_{t+1+j}\right)\right].$$

This variance decomposition is similar in style to the decomposition of return variance in Campbell and Ammer (1993). To avoid covariance terms, however, it follows the philosophy of the price/dividend variance decomposition in Cochrane (1992), extended to a multivariate context. With $x = y + z$, I explore $var(x) = cov(x, y) + cov(x, z)$ rather than $var(x) = var(y) + var(z) + 2cov(y, z)$.

C   Formulas for geometric maturity structure

Here I derive the linearized identity

$$r_{t+1}^n \approx \omega q_{t+1} - q_t,$$

which leads to (4),

$$\Delta E_{t+1} r_{t+1}^n = -\sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} \left[(r_{t+1+j}^n - \pi_{t+1+j}) + \pi_{t+1+j} \right].$$

I also derive expectations-hypothesis bond-pricing equations. These are used in the sticky-price model Cochrane (2019a).

$$E_t r_{t+1}^n = i_t$$

$$\omega E_t q_{t+1} - q_t = i_t.$$

Suppose the face value of debt follows a geometric pattern, $B_t^{(t+j)} = B_t \omega^{j-1}$. Then the nominal market value of debt is

$$\sum_{j=1}^{\infty} B_t^{(t+j)} Q_t^{(t+j)} = B_t \sum_{j=1}^{\infty} \omega^{j-1} Q_t^{(t+j)}.$$

Define the price of the government debt portfolio as

$$Q_t = \sum_{j=1}^{\infty} \omega^{j-1} Q_t^{(t+j)}.$$

The return on the government debt portfolio is then

$$R_{t+1}^n = \frac{\sum_{j=1}^{\infty} B_t^{(t+j)} Q_{t+1}^{(t+j)}}{\sum_{j=1}^{\infty} B_t^{(t+j)} Q_t^{(t+j)}} = \frac{\sum_{j=1}^{\infty} \omega^{j-1} Q_{t+1}^{(t+j)}}{\sum_{j=1}^{\infty} \omega^{j-1} Q_t^{(t+j)}} = \frac{1 + \omega \sum_{j=1}^{\infty} \omega^{j-1} Q_{t+1}^{(t+1+j)}}{\sum_{j=1}^{\infty} \omega^{j-1} Q_t^{(t+j)}} = \frac{1 + \omega Q_{t+1}}{Q_t}.$$

I loglinearize as

$$r_{t+1}^n = \log\left(\frac{1 + \omega Q_{t+1}}{Q_t}\right) = \log\left(1 + \omega e^{q_{t+1}}\right) - q_t \approx \log\left(\frac{1 + \omega Q}{Q}\right) + \frac{\omega Q}{1 + \omega Q}\tilde{q}_{t+1} - \tilde{q}_t \qquad (27)$$

where as usual variables without subscripts are steady state values and tildes are deviations from steady state.

In a steady state,

$$Q^{(t+j)} = \frac{1}{(1+i)^j}$$

$$Q = \sum_{j=1}^{\infty} \omega^{j-1} \frac{1}{(1+i)^j} = \left(\frac{1}{1+i}\right)\left(\frac{1}{1 - \frac{\omega}{1+i}}\right) = \frac{1}{1+i-\omega}. \qquad (28)$$

The limits are $\omega = 0$ for one-period bonds, which gives $Q = 1/(1+i)$, and $\omega = 1$ for perpetuities, which gives $Q = 1/i$. The terms of the approximation (27) are then

$$\frac{1 + \omega Q}{Q} = 1 + i$$

$$\frac{\omega Q}{1 + \omega Q} = \frac{\omega}{1 + i}$$

so we can write (27) as

$$r^n_{t+1} \approx i + \frac{\omega}{1+i} \tilde{q}_{t+1} - \tilde{q}_t.$$

since $i < 0.05$ and $\omega \approx 0.7$, I further approximate to

$$r^n_{t+1} \approx i + \omega \tilde{q}_{t+1} - \tilde{q}_t. \tag{29}$$

To derive (4), iterate (29) forward to express the bond price in terms of future returns,

$$\tilde{q}_t = -\sum_{j=1}^{\infty} \omega^j \tilde{r}^n_{t+j}$$

Take innovations, move the first term to the left hand side, and divide by ω,

$$\Delta E_{t+1} \tilde{r}^n_{t+1} = -\sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} \tilde{r}^n_{t+1+j} \tag{30}$$

then add and subtract inflation to get (4),

$$\Delta E_{t+1} \tilde{r}^n_{t+1} = -\sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} \left[(\tilde{r}^n_{t+1+j} - \tilde{\pi}_{t+1+j}) + \tilde{\pi}_{t+1+j} \right]. \tag{31}$$

The expectations hypothesis states that expected returns on bonds of all maturities are the same,

$$E_t r^n_{t+1} = i_t$$

$$i + \omega E_t \tilde{q}_{t+1} - \tilde{q}_t = i_t$$

$$\omega E_t \tilde{q}_{t+1} - \tilde{q}_t = \tilde{\imath}_t$$

In the text, all variables are deviations from steady state, so I drop the tilde notation.

	r^n	g	π	s	v	i	y
	Regression of other shocks on inflation shock						
Coefficient	-0.56	-0.33	1.00	-0.58	-0.65	0.24	0.23
Std. err.	(0.25)	(0.18)	(0.00)	(0.45)	(0.71)	(0.14)	(0.25)
	Correlation matrix of VAR shocks						
r^n	1.00	-0.25	-0.29	-0.27	0.63	-0.74	-0.93
g	-0.25	1.00	-0.24	0.39	-0.56	0.41	0.20
π	-0.29	-0.24	1.00	-0.14	-0.11	0.21	0.31
s	-0.27	0.39	-0.14	1.00	-0.88	0.35	0.26
v	0.63	-0.56	-0.11	-0.88	1.00	-0.63	-0.60
i	-0.74	0.41	0.21	0.35	-0.63	1.00	0.75
y	-0.93	0.20	0.31	0.26	-0.60	0.75	1.00

Table 4: Regression of other shocks on inflation shock, and correlation matrix of VAR shocks

The yield y_t on the government bond portfolio is the i_t that solves (28) for given Q_t,

$$y_t = \frac{1}{Q_t} + \omega - 1$$

To find the yield as deviation from steady state, given the bond portfolio price as deviation from steady state, write

$$q_t = \log \frac{1}{1+i-\omega} + \tilde{q}_t$$
$$y_t = e^{-\log \frac{1}{1+i-\omega} + \tilde{q}_t} + \omega - 1$$
$$\tilde{y}_t = e^{-\log \frac{1}{1+i-\omega} + \tilde{q}_t} - e^{-\log \frac{1}{1+i-\omega}} = \left(e^{\tilde{q}_t} - 1\right)\left(1 + i - \omega\right).$$

D   Sources of sampling variation

Table 4 includes the regression of other shocks on inflation shock that starts off the main inflation decomposition, and thus determines the instantaneous response in Figures 2 and 9. The table also includes the correlation matrix of the shocks.

To measure the relative contribution of the shock correlation and the long-run response function given the shock identification as sources of variation, Table 5 includes two other sampling calculations. The "no b" columns resample data using the original regression of shocks ε_{t+1}^z on inflation shocks ε_{t+1}^π, the top row of Table 4, in each sample. The VAR coefficients still vary across samples, but the identification of the inflation shock does not. The "no A" columns

Component	Fraction Estimate	25%	75%	No b 25%	No b 75%	No A 25%	No A 75%
Inflation π_1	1.00						
Bond return $(r_1^n - g_1)$	-0.23	-0.45	0.00	-0.23	-0.23	-0.45	0.00
Future Σs	-0.06	-0.69	0.23	-0.60	0.14	-0.69	0.23
Future $\Sigma r - g$	1.17	0.42	1.57	0.63	1.37	0.42	1.57

Table 5: Decomposition of unexpected inflation variance – distribution quantiles. No b holds the initial response constant across trials. No A holds the VAR regression coefficients constant across trials

likewise keep constant the VAR regression coefficients, but reestimate the shock regression in each sample. Turning off either source of sampling variation reduces that variation, but not as much as you might think. Sampling variation is still large in either case, and variances add, not standard deviations. Moreover the sampling variation associated with shock orthogonalization – the "no A" exercise – does not go away no matter how small the shocks. Both left and right hand sides of the shock on shock regressions get smaller at the same rate.