NBER WORKING PAPER SERIES

THE ENDOWMENT MODEL AND MODERN PORTFOLIO THEORY

Stephen G. Dimmock
Neng Wang
Jinqiang Yang

Working Paper 25559
http://www.nber.org/papers/w25559

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
February 2019, Revised November 2022

For helpful comments, we thank an anonymous Associate Editor, two anonymous referees, Patrick Bolton, Bruno Biais (Editor), Winston Dou, Thomas Gilbert, Harrison Hong, Steve Kaplan, Monika Piazzesi, Jim Poterba, Tom Sargent, Mark Schroder, and Luis Viceira; seminar participants at Columbia University; and participants at the European Finance Association and NBER New Developments in Long-Term Asset Management conferences. We thank Matt Hamill and Ken Redd of NACUBO and John Griswold and Bill Jarvis of Commonfund for assistance with data. Dimmock gratefully acknowledges financial support from the Singapore Ministry of Education research grant R-315-000-133-133. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

The Endowment Model and Modern Portfolio Theory
Stephen G. Dimmock, Neng Wang, and Jinqiang Yang
NBER Working Paper No. 25559
February 2019, Revised November 2022
JEL No. G11,G23

ABSTRACT

We develop a dynamic portfolio-choice model with illiquid alternative assets to analyze the "endowment model," widely adopted by institutional investors such as pension funds, university endowments, and sovereign wealth funds. In the model, the alternative asset has a lock-up, but can be liquidated at any time by paying a proportional cost. We model how investors can engage in liquidity diversification by investing in multiple illiquid alternative assets with staggered lock-up expirations, and show that doing so increases alternatives allocations and investor welfare. We show how illiquidity from lock-ups interacts with illiquidity from secondary market transaction costs resulting in endogenous and time-varying rebalancing boundaries. We extend the model to allow crisis states and show that increased illiquidity during crises causes holdings to deviate significantly from target allocations.

Stephen G. Dimmock
Division of Finance and Banking
Nanyang Technological University
Singapore
dimmock@nus.edu.sg

Neng Wang
Columbia Business School
3022 Broadway, Uris Hall 812
New York, NY 10027
and NBER
nw2128@columbia.edu

Jinqiang Yang
Shanghai University of Finance
and Economics
Guoding Rd. 777
Shanghai, 200433
China
yang.jinqiang@mail.sufe.edu.cn

1. Introduction

Sophisticated long-term investors hold substantial fractions of their portfolios in illiquid alternative assets. For example, public pension funds allocate 27% of their portfolios to illiquid alternatives and both university endowments and family offices of high-net-worth individuals allocate more than half of their aggregate portfolios to illiquid alternatives.[1] And as of 2020, more than $14.2 trillion was allocated to alternative assets.[2] In recent decades, an investment strategy of high allocations to illiquid assets has been adopted to varying degrees by almost all types of institutional investors. This strategy, called the "endowment model," as it was initially championed by university endowments,[3] advocates that long-term investors should hold high allocations of alternative assets so as to earn illiquidity premiums and exploit the inefficiencies found in illiquid markets.

Despite its perceived success and growing popularity among institutional and high-net-worth investors, the endowment model has significant limitations and lacks theoretical foundations. Standard references of the endowment model are based on static mean-variance analysis adjusted with ad hoc rules of thumb (Swensen, 2000; Takahashi and Alexander, 2002). The endowment model lacks a framework that formalizes the trade-off between the benefits of alternative assets and the costs of their illiquidity. It also lacks a framework for evaluating how investor heterogeneity affects this trade-off. For example, pension funds and wealthy individuals differ greatly in their flexibility to adjust spending across periods – yet the endowment model does not provide guidance on how such differences should affect asset allocation and spending.

In this paper, we develop a normative, tractable, and dynamic asset allocation model based on modern portfolio theory (MPT) to formally assess the heuristic risk-return based endowment model championed by Swensen (2000) and widely adopted by practitioners, e.g., university endowments, family offices, and other institutional investors. To do this,

[1] For public pension plans, see the American Investment Council 2021 Public Pension Study at `https://www.investmentcouncil.org/pensions/`. For university endowments see the 2020 NACUBO Endowment Study `https://www.nacubo.org/Research/2020/Public-NTSE-Tables`. See also Brown et al. (2010), Dimmock (2012), and Brown et al. (2014). For family offices, see the UBS/Campden survey `http://www.globalfamilyofficereport.com/investments/`.

[2] See `https://www.bcg.com/publications/2020/global-asset-management-protect-adapt-innovate`.

[3] David Swensen of Yale University endowment is generally credited with originating the endowment model. See Swensen (2000), Takahashi and Alexander (2002), and Lerner et al. (2008).

we incorporate illiquid investment opportunities and important institutional features of alternative assets, e.g., private equity and hedge funds, into a generalized MPT framework (Merton, 1971). We show that the illiquidity of alternative assets and incomplete markets have first-order effects on the investor's dynamic spending and asset allocations. Further, we show how the features of alternative assets interact with the investor's characteristics.

Our model captures the key features of alternative assets in a manner that is both realistic and analytically tractable. First, the alternative asset's risk is not fully spanned by public equity. Second, the model includes a secondary market for the alternative asset, where the investor can voluntarily transact at any time by paying a proportional transaction cost. This captures a form of illiquidity costs. Third, we allow the alternative asset to provide some "natural" liquidity. This can be in the form of dividends, such as rental cash flows from private real estate. Or it can be in the form of liquidity events, when the alternative asset (or a fraction of it) becomes fully liquid, such as when a hedge fund lock-up expires or a private equity fund makes a distribution to investors. This is an important feature of our model, as advocates of the endowment model argue that natural liquidity, such as private equity cash distributions, offsets much of the apparent illiquidity of alternative assets (see Swensen, 2000; Takahashi and Alexander, 2002) and therefore should influence an investor's asset allocation decisions.

We model how this natural liquidity can provide "liquidity diversification" using a reduced-form approach.[4] In our model, the investor can stagger investments into the alternative asset over time, resulting in distinct positions that have staggered natural liquidity events. Although we do not endogenize the investor's liquidity diversification decisions, we show how staggering investments in the alternative asset affects the investor's welfare, portfolio allocations, and spending. Incorporating both secondary markets and natural liquidity events for alternative assets allows us to closely match the relevant features of alternative assets, and differentiates our model from prior work.

We provide an analytical characterization for the investor's certainty equivalent wealth under optimality, $P(W_t, K_t, t)$, which is the time-t total wealth that makes the investor indifferent between: permanently forgoing the opportunity to invest in the illiquid asset and keeping the status quo with liquid wealth W_t and illiquid wealth K_t with the op-

[4]For an empirical examination of liquidity diversification, see Robinson and Sensoy (2016).

portunity to invest in the illiquid asset. We exploit the model's tractability to provide a quantitative yet intuitive analysis of a long-term investor's optimal portfolio choice, spending rule, and welfare measured by $P(W_t, K_t, t)$.

Our qualitative and quantitative results significantly differ from the standard predictions of MPT. In contrast to the classic MPT prediction, we show that the allocation to the illiquid alternative asset follows a double barrier policy where the allocation can rise or fall until it reaches the endogenous rebalancing boundaries.[5] This result is in sharp contrast to the classic MPT prediction that the allocation ratio between any two assets is constant over time. Our model is among the first to show how double barrier policy arises in a setting with incomplete markets, unspanned risks, and illiquidity.

We examine the endogenous rebalancing boundaries over time, and show that the two types of illiquidity – arising from lock-ups and from transaction costs – interact over time. As an automatic liquidity event approaches, the investor becomes less willing to liquidate alternative assets. The rebalancing policies are also strongly affected by liquidity diversification; investors who stagger the maturities of their alternative asset investments over time can maintain more stable portfolio allocations, which results in higher ex ante allocations to alternatives.

We calibrate our model and evaluate the effects of liquidity diversification – by exogenously varying the number of alternative asset investments and then comparing the investor's optimal portfolio allocations. We show that the investor's ideal allocation to alternative assets is higher as the number of distinct alternative investments increases, and the investor's welfare is higher. The results show that the benefits of liquidity diversification are reached rapidly and only a small number of distinct investments are required to realize most of the benefits.

The calibrated results show that investors' preferences for smooth inter-temporal spending have first-order effects on their allocations to illiquid assets. We use Epstein and Zin (1989) preferences, which separate risk aversion from the elasticity of intertemporal substitution (EIS). This separation is economically important, as by varying the EIS we conveniently capture the heterogeneity in spending flexibility. For example, defined

[5] The double-barrier policy is a standard feature in models with transaction costs. See Davis and Norman (1990) as an early example in the portfolio-choice literature.

benefit pension plans have little spending flexibility and so have a low EIS. In contrast, family offices have high spending flexibility.

In contrast to the full-spanning case in which the investor's portfolio allocation is independent of the EIS, the EIS significantly affects portfolio allocations when the alternative asset is illiquid and its risk is unspanned by publicly traded assets. This is the realistic scenario. An investor with a high EIS can accept higher allocations to the illiquid asset, as they are more willing to substitute spending across states and over time. This flexibility of deferring spending with little utility loss boosts the investor's ability to make long-term illiquid investments.

We show that allocation results crucially depend on the compensation the alternative asset provides for liquidity risk and skill. This is important, as there is significant variation in allocations to alternative assets even *within* investor types (e.g., endowments or pension funds). And it is consistent with empirical findings of large and persistent heterogeneity in investors' realized excess returns on alternative investments.[6]

Our quantitative results show that asset allocations are sensitive to the unspanned volatility of the alternative asset. We further show that, controlling for the level of the alternative asset's total risk, the spanned and unspanned risks have quantitatively very different effects on asset allocation. While the investor can offset the alternative asset's spanned risk by adjusting allocations to public equity, unspanned volatility is specific to the alternative and cannot be hedged. Alternative asset performance metrics such as internal rates of returns (IRRs) and public market equivalent (PMEs), while useful, do not directly guide investors' asset allocation as these metrics ignore the distinction between spanned and unspanned volatilities.

We extend the model to include new capital contributions (donations) into the portfolio (Section 6) and a minimum spending-rate constraint (Section 7). The results for contributions show that contributions increase allocations to alternative assets and, by providing an inflow of liquidity, decrease the variation in allocations over time. Contributions also result in higher and more stable spending rates. The results for a spending constraint show large effects on allocations to alternative assets. Even if the spending

[6]See Lerner, Schoar, and Wongsunwai (2007), Lerner, Schoar, and Wang (2008), and Brown, Garlappi, and Tiu (2010).

constraint rarely binds, the states in which it will bind are those with high allocations to alternative assets, resulting in the investor choosing a significantly lower allocation to illiquid securities.

We also extend the model to include the possibility of crisis states. During crisis states, Robinson and Sensoy (2016) and Brown et al. (2021) document that capital calls are significantly higher in crisis and distributions to investors are much lower, and Ramadorai (2012) and Nadauld et al. (2019) document that secondary market transactions costs are much higher. To capture these important institutional features, we extend our model to include stochastic arrivals of crisis states, e.g., in Barro (2006) and Wachter (2013), during which alternative assets becomes even more illiquid than usual. We find that investors' holdings of alternative assets often significantly deviate from the optimal target allocations and hence the utility loss from being unable to hedge stochastic call and distributions can be large in the crisis state.

Our paper contributes to the literature on portfolio choice with illiquid assets. The prior literature can be broadly divided into two branches. One branch models illiquidity from trading restrictions in which the asset is freely tradable at certain points in time but cannot be traded at other times, e.g., Longstaff (2001), Kahl, Liu, and Longstaff (2003), Gârleanu (2009), Longstaff (2009), Dai et al. (2015), and Ang et al. (2016).[7] The other branch of the literature models illiquidity arising from transaction costs, e.g., Davis and Norman (1990), Grossman and Laroque (1990), Vayanos (1998), Lo et al. (2004), Collin-Dufresne et al. (2012), and Gârleanu and Pedersen (2013, 2016).[8]

Motivated by the structures of private equity and hedge funds, as well as the secondary markets for these illiquid alternatives, we combine the features of both types of models discussed above. In our model, the alternative asset becomes fully liquid at maturity (e.g., when a private equity fund is dissolved). But the alternative asset can also be sold prior to maturity by paying a proportional transaction cost, such as by selling a private equity fund at a discount in the secondary market (see for example Nadaul et al., 2019; Ramadorai, 2012). We show that these two types of illiquidity interact, and that

[7]Benzoni et al. (2007) show how non-tradable human capital affects portfolio choice.

[8]Gallmeyer et al. (2006) model how transaction frictions from the taxation of realized capital gains affect portfolio choice.

this interaction varies over the life-cycle of the alternative asset. We further show how *liquidity diversification* – holding multiple distinct investments in the alternative asset with staggered lock-up expirations – affects portfolio choice. Finally, we generalize our model to include crisis states featuring increases in both types of illiquidity, and show that investors' holdings deviate significantly from target allocations.

Our paper also contributes to the literature on private equity and other alternative asset funds. The theory papers in this literature largely focus on a single investment in isolation (e.g., Sorensen et al., 2014) and do not consider the possibility of staggering alternative investments over time, nor do these papers realistically address alternative assets as a component of a larger portfolio.

Finally, our paper contributes to the literature by providing a rigorous foundation for analyzing the endowment model. Although the endowment model is highly influential to practice and is used to allocate trillions of dollars, it is based on ad hoc rules of thumb and practitioner's lore. Our model formalizes the endowment model by developing a generalized dynamic portfolio theory with the key features of illiquid private equity.

2. Model

We analyze a long-term investor's dynamic spending (or equivalently consumption) and asset allocation decisions by incorporating an illiquid investment opportunity into the classic modern portfolio theory developed by Merton (1969, 1971) and Samuelson (1969). We interpret the illiquid investment opportunity in our model as the representative portfolio of *alternative assets* including private equity, hedge funds, private real estate, etc. For technical convenience, we develop our model in continuous time.

Liquid Investment Opportunities: Bonds and Public Equity. The risk-free bond pays interest at a constant (annualized) risk-free rate r. Public equity can be interpreted as the market portfolio of publicly traded securities, and its cum-dividend market value, S_t, follows a geometric Brownian motion (GBM):

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S d\mathbb{B}_t^S \,, \tag{1}$$

where $\mathbb{B}_t^S$ is a standard Brownian motion, and μ_S and σ_S are the constant drift and volatility parameters. The Sharpe ratio for public equity is $\eta_S = (\mu_S - r)/\sigma_S$. The liquid investment opportunity in our model is the same as in Merton (1971). Next, we introduce the alternative asset, which is the investor's third investment opportunity and the key building block in our model.

2.1. The Alternative Asset

Adding the alternative asset expands the investment opportunity set and thus makes the investor better off. Additionally, provided the alternative asset is not perfectly correlated with public equity, it provides diversification benefits. Unlike public equity, however, alternative assets are generally illiquid and involve some form of lock-up. For example, private equity funds typically have 10 year life spans, hedge funds often have lock-up periods and gate provisions, and private real estate is often has limited liquidity for its secondary market.

A key feature of alternative assets is that their illiquidity is not constant over time. For example, private equity funds are highly illiquid for much of their lives but eventually mature and return liquid capital to their investors. We model these liquidity events as follows. Let $\{A_t; t \geq 0\}$ denote the alternative asset's fundamental value process with a given initial stock A_0. The fundamental value refers to the fully realizable value of the asset if it is held to maturity. However, with illiquidity, at any time t prior to maturity the asset's fundamental value differs from its market value. Let $\{K_t; t \geq 0\}$ denote the accounting value of the alternative asset holding process with a given initial stock K_0. To capture the target finite duration of the lock-up and holding period, we assume every mT years, where m is a positive integer, a δ_T fraction of the stock of illiquid alternative asset K_{mT} automatically becomes liquid at no cost. Naturally, the investor's liquid asset value at time mT increases by $\delta_T K_{mT-}$. Therefore, in the absence of any active acquisition or divestment of the illiquid asset at mT, we have $K_{mT} = (1 - \delta_T)K_{mT-}$.

Fundamental Value Process A for the Alternative Asset. We assume that the fundamental value A, in the absence of a scheduled automatic liquidity event (at time

mT) or any interim acquisition or divestment, evolves via the following GBM:

$$\frac{dA_t}{A_{t-}} = \mu_A dt + \sigma_A d\mathbb{B}_t^A - \delta_A dt\,, \tag{2}$$

where $\mathbb{B}_t^A$ is a standard Brownian motion, μ_A is the cum-payout expected return (net of fees), σ_A is the constant volatility of returns, and δ_A is the alternative asset's payout rate. That is, the alternative asset pays dividends at the rate of $\delta_A A_t$ with an implied payout yield of δ_A. Intuitively, δ_A is one way for illiquid alternative assets to provide liquidity to investors. We use ρ to denote the correlation coefficient between the shocks to alternative assets, $\mathbb{B}_t^A$, and the shocks to public equity, $\mathbb{B}_t^S$.

Note that in complete markets, the investor can frictionlessly and dynamically trade the alternative asset without restrictions or costs. Therefore, the alternative asset's market value equals its fundamental value and the Modigliani-Miller theorem holds, meaning that whether we explicitly model the alternative asset's payout yield δ_A is irrelevant. In this ideal case, the alternative asset is conceptually no different than liquid public equity. In contrast, when the alternative asset is illiquid and not fully spanned by public equity, we must separately keep track of the payout yield δ_A and expected capital gains $\mu_A - \delta_A$. That is, the cum-dividend return μ_A is no longer a sufficient measure of the total expected returns for the alternative asset as its (current) payout yield and expected capital gains influence the investor's portfolio optimization problem differently.

Interim Acquisition and Liquidation of the Alternative Asset Holding. At any time, the investor can choose to change the alternative asset holdings through acquisitions or liquidations. Let dL_t denote the amount of the alternative asset that the investor liquidates at any time $t > 0$, and let dX_t denote the amount of the alternative asset that the investor purchases at time t. Then, we can incorporate the investor's acquisition and liquidation options into the alternative asset's fundamental value process as follows:

$$dK_t = (\mu_A - \delta_A)K_{t-}dt + \sigma_A K_{t-} d\mathbb{B}_t^A - dL_t + dX_t - \delta_T K_{t-}\mathbb{I}_{\{t=mT\}}\,. \tag{3}$$

Here, $\mathbb{I}_{\{t=mT\}}$ is the indicator function, which is equal to one if and only if t is an integer multiple, m, of T. The first two terms correspond to the standard drift and volatility terms, the third and fourth terms give the liquidation and acquisition amounts, and the last term captures the lumpy payout to the investor at the scheduled liquidity event dates $t = mT$ where $m = 0, 1, \ldots$

Although the acquisition and liquidation costs for the alternative asset do not appear in (3), they will appear in the liquid wealth accumulation process. We assume that the cost of voluntary liquidation is proportional. That is, by liquidating an amount $dL_t > 0$, the investor realizes only $(1-\theta_L)dL_t$ in net, where the remaining amount $\theta_L dL_t$ is the liquidation cost. Similarly, if the investor acquires an amount $dX_t > 0$, the transaction cost $\theta_X dX_t$ is paid out of the liquid asset holding. Naturally, $0 \leq \theta_L \leq 1$ and $\theta_X \geq 0$. Higher values of θ_L or θ_X indicate that the alternative asset is less liquid.

Intuitively, θ_L can be interpreted as the illiquidity discount on secondary market sales of alternative assets (e.g., see Kleymenova et al., 2012; Albuquerque et al., 2018; Nadauld et al., 2019). Such discounts can arise to compensate buyers for search costs, asymmetric information risks, or due to market power when there are few buyers. The parameter θ_X can be interpreted as the transaction costs of purchasing alternative assets, such as search costs, legal fees, placement agent fees, consultant fees, and etc. The costs of interim liquidation (θ_L) and of purchases (θ_X) can be asymmetric as voluntary liquidation is generally more costly, particularly when there are few buyers and many sellers such as during the recent financial crisis.

Alpha, Beta, and Epsilon (Unspanned Volatility). Suppose that the instantaneous return for the alternative asset, dA_t/A_{t-}, is perfectly measurable. We can then regress dA_t/A_{t-} on dS_t/S_t, and obtain the alternative asset's beta with respect to public equity.

$$\beta_A = \frac{\rho\sigma_A}{\sigma_S}\,. \tag{4}$$

However, in reality, because investors cannot dynamically rebalance their holdings in the illiquid asset without incurring transaction costs, investors will demand compensation in addition to the risk premium implied by the covariance with public equity.

We decompose the total volatility of the alternative asset, σ_A, into two orthogonal components: the part spanned by the public equity, $\rho\sigma_A$, and the remaining unspanned volatility, ε, given by:

$$\varepsilon = \sqrt{\sigma_A^2 - \rho^2\sigma_A^2} = \sqrt{\sigma_A^2 - \beta_A^2\sigma_S^2}\,. \tag{5}$$

This volatility, ε, introduces an additional risk into the investor's portfolio, as markets are incomplete and adjusting the alternative asset holding is costly. We will show that the spanned and unspanned volatilities play distinct roles in the investor's dynamic asset allocation.[9]

Anticipating our subsequent risk-return trade-off analysis in the context of dynamic portfolio construction, we next introduce the α implied by a single-index model using public equity. That is, we define α as follows:

$$\alpha = \mu_A - (r + \beta_A(\mu_S - r)) \,, \tag{6}$$

where β_A is the alternative asset's beta given by (4).

In frictionless capital markets where investors can continuously rebalance their portfolio without incurring any transaction costs, α measures the risk-adjusted excess return after benchmarking against public equity. However, importantly, in our framework with illiquid assets, α also includes compensation for bearing any systematic risk that is unspanned by public equity, which for simplicity, we refer to as illiquidity premium.

2.2. Optimization Problem

Liquid Wealth and Net Worth. We use W to denote the investor's liquid wealth and Π to denote the amount allocated to public equity. The remaining liquid wealth, $W - \Pi$, is allocated to the risk-free bond. Thus, liquid wealth evolves according to:

$$\begin{aligned} dW_t &= (rW_{t-} + \delta_A K_{t-} - C_{t-})\, dt + \Pi_{t-}\left((\mu_S - r)dt + \sigma_S d\mathbb{B}_t^S\right) \\ &\quad + (1-\theta_L)dL_t - (1+\theta_X)dX_t + \delta_T K_{t-}\mathbb{I}_{\{t=mT\}} \,, \end{aligned} \tag{7}$$

where the first two terms in (7) are the standard ones in Merton's consumption/portfolio-choice problem. The third and fourth terms describe the effect on liquid wealth W due to the investor's interim liquidation and purchase of the alternative asset, where θ_L and

[9] Our approach follows the common industry practice of defining β relative to the portfolio of publicly traded equity. Although ε is unspanned by public equity, this does not imply it is purely idiosyncratic volatility. For example, private equity constitutes a substantial fraction of total wealth in the world and is not perfectly correlated with public equity. Theoretically, Cochrane, Longstaff, and Santa Clara (2008) and Eberly and Wang (2009) show that in segmented markets both segments command risk premia. Empirically, Aragon (2007), Sadka (2010), and Franzoni et al. (2012) show that alternative assets earn significant liquidity premia.

θ_X capture the proportional cost of interim liquidations and purchases of the alternative asset, respectively. Finally, the last term captures the lumpy payout to the investor at the automatic liquidity event dates $t = mT$.

Recursive Preferences and Value Functions. The investor's preferences allow for separation of risk aversion and the elasticity of intertemporal substitution (EIS). Epstein and Zin (1989) and Weil (1990) develop this utility in discrete time by building on Kreps and Porteus (1978). We use the continuous-time formulation of this non-expected utility, introduced by Duffie and Epstein (1992). That is, the investor has a recursive preference defined as follows:

$$V_t = \mathbb{E}_t \left[\int_t^\infty f(C_s, V_s) ds \right], \tag{8}$$

where $f(C, V)$ is known as the normalized aggregator for consumption C and the investor's utility V. Duffie and Epstein (1992) show that $f(C, V)$ for Epstein-Zin non-expected homothetic recursive utility is given by:

$$f(C, V) = \frac{\zeta}{1 - \psi^{-1}} \frac{C^{1-\psi^{-1}} - ((1-\gamma)V)^\chi}{((1-\gamma)V)^{\chi-1}}, \tag{9}$$

where

$$\chi = \frac{1 - \psi^{-1}}{1 - \gamma}. \tag{10}$$

The parameter $\psi > 0$ measures the EIS, $\gamma > 0$ is the coefficient of relative risk aversion, and $\zeta > 0$ is the investor's subjective discount rate.

This recursive, non-expected utility formulation allows us to separate the coefficient of relative risk aversion (γ) from the EIS (ψ), which is important for our quantitative analysis. For example, a key source of preferences heterogeneity among investors is the elasticity and flexibility of their spending. The expected CRRA utility is a special case of recursive utility where the coefficient of relative risk aversion, γ, equals the inverse of the EIS, $\gamma = \psi^{-1}$, implying $\chi = 1$.[10]

There are three state variables for the optimization problem: liquid wealth W_t, the alternative asset's value K_t, and calendar time t. Let $V(W_t, K_t, t)$ denote the corresponding value function. The investor chooses consumption C, public equity investment

[10]For the special case of CRRA, $f(C, V) = U(C) - \zeta V$, where $U(C) = \zeta C^{1-\gamma}/(1-\gamma)$. By integrating Eq. (8), we obtain $V_t = \mathbb{E}_t \left[\int_t^\infty e^{-\zeta(s-t)} U(C_s) ds \right]$.

Π, and the alternative asset's cumulative (undiscounted) liquidation L and cumulative (undiscounted) acquisition X to maximize (8).

Naturally, at each automatic liquidity event date iT, if $W_{iT} = W_{(i-1)T} = W$, and $K_{mT} = K_{(m-1)T} = K$, we must have:

$$V(W, K, mT) = V(W, K, (m-1)T) \,. \tag{11}$$

Hence, it is sufficient for us to characterize our model over $(0, T]$, as the solution is stationary every T years.

3. Model Solution

We solve the model as follows. First, we analyze the investor's problem in the region where there is no voluntary adjustment of the alternative asset in the absence of automatic liquidity event (i.e., when $t \neq mT$.) Second, we characterize the investor's voluntary liquidation and acquisition decisions for the alternative asset when $t \neq mT$. Finally, we integrate the periodic liquidity event that occurs at $t = mT$ to complete our analysis.

Dynamic Programming and First-Order Conditions (FOCs). Fix time t within the time interval $((m-1)T, mT)$, where m is a positive integer. Using the standard dynamic programming approach, we have the following standard Hamilton-Jacobi-Bellman (HJB) equation for the investor's value function $V(W_t, K_t, t)$ in the interior region:

$$\begin{aligned} 0 \;\; = \;\; & \max_{C, \Pi} \; f(C, V) + (rW + \delta_A K + (\mu_S - r)\Pi - C)V_W + \frac{(\Pi\sigma_S)^2}{2} V_{WW} \\ & + V_t + (\mu_A - \delta_A) K V_K + \frac{\sigma_A^2 K^2}{2} V_{KK} + \rho \Pi K \sigma_S \sigma_A V_{WK} \,. \end{aligned} \tag{12}$$

The first three terms on the right side of (12) capture the standard effects of consumption and asset allocation (both drift and volatility effects) on the investor's value function, $V(W_t, K_t, t)$ as in Merton (1971). The investor's opportunity to invest in the illiquid alternative asset generates three additional effects on asset allocation: 1) the effect of target holding horizon T captured by V_t; 2) the risk-return and volatility effects of changes in the value of the alternative asset K; and 3) the additional diversification/hedging benefits due to the correlation between public equity and the alternative asset. By optimally choosing C and Π, the investor equates the right side of (12) to zero in the interior region where there is no interim liquidation nor acquisition.

The optimal consumption C is characterized by the following standard FOC:

$$f_C(C, V) = V_W(W, K, t), \tag{13}$$

which equates the marginal benefit of consumption with the marginal value of savings V_W. The optimal investment in public equity is given by:

$$\Pi = -\frac{\eta_S}{\sigma_S}\frac{V_W}{V_{WW}} - \frac{\rho\sigma_A}{\sigma_S}\frac{KV_{WK}}{V_{WW}}. \tag{14}$$

The first term gives the classical Merton's mean-variance demand and the second term captures the investor's hedging demand with respect to the illiquid alternative asset. Note that the hedging demand depends on the cross partial V_{WK}, and is proportional to $\rho\sigma_A/\sigma_S$ (which is equal to β_A as shown in (4)). Both results are intuitive and follow from the standard hedging arguments in Merton (1971); the investor chooses the public equity allocation to fulfill two objectives: to obtain the desired mean-variance exposure and to hedge the fraction of the alternative asset's risk spanned by public equity.

Certainty Equivalent Wealth $P(W, K, t)$. We express the investor's value function $V(W, K, t)$ during the time period $t \in ((m-1)T, mT)$ as:

$$V(W, K, t) = \frac{(b_1 P(W, K, t))^{1-\gamma}}{1-\gamma}, \tag{15}$$

where b_1 is a constant given by:

$$b_1 = \zeta^{\frac{\psi}{\psi-1}}\phi_1^{\frac{1}{1-\psi}}, \tag{16}$$

and ϕ_1 is the constant given by:

$$\phi_1 = \zeta + (1-\psi)\left(r - \zeta + \frac{\eta_S^2}{2\gamma}\right). \tag{17}$$

Guided by MPT, we can interpret $P(W, K, t)$ as the investor's certainty equivalent wealth, which is the minimal amount of total wealth required for the investor to permanently give up the opportunity to invest in the alternative asset. Thus, imagine that at any time t, the investor has two options: either (1) adhere to the optimal portfolio and spending plan prescribed by the model; or (2) surrender both the liquid asset W and illiquid asset holdings K in exchange for immediately and permanently giving up the opportunity to invest in the alternative asset but with a liquid wealth level of Ω, from which the investor can continuously spend and rebalance between public equity and bonds. Liquid wealth $\Omega = P(W, K, t)$ makes the investor indifferent between these two options. Mathematically, in the interim period where $(m-1)T < t < mT$, the following equation holds:

$$V(W, K, t) = J(P(W, K, t)) . \tag{18}$$

Here, $J(\,\cdot\,)$ is the value function for an investor who can invest only in liquid public equity and risk-free bonds. We show that $J(\,\cdot\,)$ is given by

$$J(W) = \frac{(b_1 W)^{1-\gamma}}{1-\gamma} , \tag{19}$$

where b_1 given in (16) is the same constant appearing in the value function for the classic Merton's problem. We emphasize that certainty-equivalent wealth $P(W, K, t)$ is more natural and intuitive than the investor's value function $V(W, K, t)$ to measure welfare. This is because the unit for $P(W, K, t)$ is dollars while the unit for $V(W, K, t)$ is utils. Appendix B contains a proof for the characterization of the certainty-equivalent wealth $P(W, K, t)$.

Homogeneity Property. In our model, the certainty equivalent wealth $P(W, K, t)$ has the homogeneity property in W and K, and hence it is convenient to work with the liquidity ratio $w_t = W_t/K_t$ and the scaled certainty equivalent wealth function $p(w_t, t)$ defined as follows:

$$P(W_t, K_t, t) = p(w_t, t) \cdot K_t . \tag{20}$$

This homogeneity property is due to the Duffie-Epstein-Zin utility and the value processes for public equity and the alternative asset. Importantly, this homogeneity property allows us to conveniently interpret the optimal portfolio rule and target asset allocation.

Endogenous Effective Risk Aversion $\boldsymbol{\gamma_i}$. To better interpret our solution it is helpful to introduce the following measure of endogenous relative risk aversion for the investor, denoted by $\gamma_i(w,t)$ and defined as follows:

$$\gamma_i(w,t) \equiv -\frac{V_{WW}}{V_W} \times P(W,K,t) = \gamma p_w(w,t) - \frac{p(w,t)p_{ww}(w,t)}{p_w(w,t)}\,. \tag{21}$$

In (21) the first identity sign gives the definition of γ_i and the second equality follows from the homogeneity property.

What economic insights does $\gamma_i(w,t)$ capture and what is the motivation for introducing it? First, recall the standard definition of the investor's coefficient of absolute risk aversion is $-V_{WW}/V_W$. To convert this to a measure of relative risk aversion, we need to multiply absolute risk aversion $-V_{WW}/V_W$ with an appropriate economic measure for the investor's total wealth. Under incomplete markets, although there is no market-based measure of the investor's economic well being, the investor's certainty equivalent wealth $P(W,K,t)$ is a natural measure of the investor's welfare. This motivates our definition of γ_i in (21).[11] We will show that the illiquidity of alternative assets causes the investor to be effectively more risk averse, meaning $p_w(w,t) > 1$ and $p_{ww}(w,t) < 0$, so that $\gamma_i(w,t) > \gamma$. In contrast, if the alternative asset is publicly traded (and markets are complete), $\gamma_i(w,t) = \gamma$ as $p_w(w,t) = 1$ and $p_{ww}(w,t) = 0$.

Optimal Policy Rules. Again, by using the homogeneity property, we may express the scaled consumption rule $c(w_t,t) = C(W_t,K_t,t)/K_t$ as follows:

$$c(w,t) = \phi_1\, p(w,t)\, p_w(w,t)^{-\psi}\,. \tag{22}$$

Because illiquidity makes markets incomplete, the investor's optimal consumption policy is no longer linear and depends on both the certainty equivalent wealth $p(w,t)$ and also the marginal certainty equivalent value of liquid wealth $p_w(w,t)$.

The allocation to public equity is $\Pi_t = \pi(w_t,t)K_t$ where $\pi(w,t)$ is given by:

$$\pi(w,t) \;=\; \frac{\eta_S}{\sigma_S}\frac{p(w,t)}{\gamma_i(w,t)} - \frac{\rho\sigma_A}{\sigma_S}\left(\frac{\gamma p(w,t)}{\gamma_i(w,t)} - w\right)\,, \tag{23}$$

where $\gamma_i(\,\cdot\,)$ is the investor's effective risk aversion given by (21). Intuitively, the first term in (23) reflects the mean-variance demand for the market portfolio, which differs from

[11]See Wang et al. (2012) and Bolton et al. (2019) for similar definitions involving endogenous risk aversion but for very different economic applications.

the standard Merton model in two ways: 1) risk aversion γ is replaced by the effective risk aversion $\gamma_i(w,t)$ and 2) net worth is replaced by certainty equivalent wealth $p(w,t)$. The second term in (23) captures the dynamic hedging demand, which also depends on $\gamma_i(w,t)$ and $p(w,t)$.

PDE for $p(w,t)$. Substituting the value function (15) and the policy rules for c and π into the HJB equation (12) and using the homogeneity property and the definition of the investor's effective risk aversion, $\gamma_i(w,t)$, given by (21), we obtain the following PDE for $p(w,t)$ at time t, for the liquidity ratio w_t in the interior region, and when $(m-1)T < t < mT$:

$$0 = \left(\frac{\phi_1\left(p_w(w,t)\right)^{1-\psi} - \psi\zeta}{\psi - 1} + \mu_A - \delta_A - \frac{\gamma\sigma_A^2}{2}\right)p(w,t) + p_t(w,t) + \frac{\varepsilon^2 w^2}{2}p_{ww}(w,t) \quad (24)$$

$$+\left(\delta_A - \alpha + \gamma\varepsilon^2\right)wp_w(w,t) - \frac{\gamma\varepsilon^2 w^2}{2}\frac{(p_w(w,t))^2}{p(w,t)} + \frac{(\eta_S - \gamma\rho\sigma_A)^2 p_w(w,t)p(w,t)}{2\gamma_i(w,t)}\,.$$

Because of incomplete spanning (e.g., $\varepsilon \neq 0$), unlike Black-Scholes, (24) is a nonlinear PDE, and moreover, $p_w(w,t) > 1$, as we will show. The numerical solution for $p(w,t)$ involves the standard procedure. Next, we analyze how the investor actively rebalances the allocation to the illiquid alternative asset.

Rebalancing the Illiquid Alternative Asset during the Interim Period. Although under normal circumstances the investor plans to hold the alternative asset until an automatic liquidity event occurs at date mT, under certain circumstances the investor may find it optimal to actively rebalance even at time $t \neq mT$. As acquisition and voluntary liquidation are costly, we have an inaction region at all time including $t = mT$. Let $\underline{W}_t$ and $\overline{W}_t$ denote the lower liquidation boundary and the upper acquisition boundary for liquid wealth W_t at time t, respectively. Next, we sketch out the key steps and relegate technical details to Appendix B.

To help understand how we obtain our key results, we first state the following two key conditions for the unscaled certainty-equivalent wealth $P(W,K,t)$, implied by the standard value-matching and smooth-pasting conditions for the investor's value function $V(W,K,t)$, at the lower liquidation boundary $\underline{W}_t$:

$$P_K(\underline{W_t}, K_t, t) = (1-\theta_L) P_W(\underline{W_t}, K_t, t)\,, \tag{25}$$

$$P_{KW}(\underline{W_t}, K_t, t) = (1-\theta_L) P_{WW}(\underline{W_t}, K_t, t)\,. \tag{26}$$

Equation (25) is the smooth-pasting condition, which states that the marginal certainty equivalent wealth of illiquid wealth, P_K, must equal $(1-\theta_L)P_W$, the marginal certainty equivalent wealth of liquid wealth when the investor sells a unit of the illiquid asset. This corresponds to the investor's indifference condition when liquidating the alternative asset. Since the investor is optimally choosing the liquidation boundary, the derivatives of the two sides of (25) with respect to W must equal. This gives the super-contact condition (26) for the illiquid asset liquidation decision (Dumas, 1991).

Using the homogeneity property to simplify (25) and (26), we obtain the following smooth-pasting and super-contact conditions at the lower liquidation boundary $\underline{w}_t$:[12]

$$p(\underline{w}_t, t) = (1-\theta_L + \underline{w}_t)\, p_w(\underline{w}_t, t)\,, \tag{27}$$

$$p_{ww}(\underline{w}_t, t) = 0\,. \tag{28}$$

Using essentially the same analysis, we obtain the following smooth-pasting and super-contact conditions at the upper acquisition boundary $\overline{w}_t$:

$$p(\overline{w}_t, t) = (1+\theta_X + \overline{w}_t)\, p_w(\overline{w}_t, t)\,, \tag{29}$$

$$p_{ww}(\overline{w}_t, t) = 0\,. \tag{30}$$

In sum it is optimal for the investor to keep the liquidity ratio w_t within the $(\underline{w}_t, \overline{w}_t)$ region by voluntarily liquidating a portion of the alternative asset if w_t is too high and acquiring the alternative asset if w_t is too low.

Next, we analyze the investor's decision at $t = mT$ when the portfolio's liquidity changes discretely due to the automatic liquidity event, i.e., when an alternative investment in the investor's portfolio pays a lumpy liquidating dividend.

[12]As $p \geq 0$ and $p_w \geq 0$, equation (27) implies $\underline{w}_t \geq -(1-\theta_L)$, meaning that the investor can borrow only a fraction of the alternative asset's fundamental value. As a result, the investor can repay the liability with probability one by liquidating the alternative asset. Thus, the investor's debt capacity is endogenously determined by the liquidation value of the alternative asset. Although the investor can borrow, in our numerical exercise, as in reality, borrowing is rare.

Value and Decisions when there is an Automatic Liquidity Event at $t = mT$. At time $t = mT$, a fraction δ_T of the alternative asset automatically becomes fully liquid without any voluntary liquidation. We use $\widehat{W}_{mT}$ and $\widehat{K}_{mT}$ to denote the corresponding levels of liquid wealth and the alternative asset at $t = mT$ if the investor chooses not to do any voluntary rebalancing. It is immediate to see $\widehat{W}_{mT} = \lim_{t \to mT} (W_t + \delta_T K_t)$ and $\widehat{K}_{mT} = \lim_{t \to mT} (K_t - \delta_T K_t)$. Let $\widehat{w}_{mT}$ denote the corresponding liquidity ratio:

$$\widehat{w}_{mT} \equiv \frac{\widehat{W}_{mT}}{\widehat{K}_{mT}} = \lim_{t \to mT} \frac{w_t + \delta_T}{1 - \delta_T} \,. \tag{31}$$

By now, we have outlined the procedures for calculating both $\overline{w}_{mT}$ (ignoring the automatic liquidity event) and $\widehat{w}_{mT}$ (focusing only the automatic liquidity event.) Of course, the investor optimally decides considering both the "marginal analysis" for the liquidity ratio and the automatic liquidity event at $t = mT$. As a result, we have two cases to consider at $t = mT$: Case (i) where $\widehat{w}_{mT} \le \overline{w}_{mT}$ and Case (ii) where $\widehat{w}_{mT} > \overline{w}_{mT}$. As the automatic liquidity event always increases liquid asset holdings, $\widehat{w}_{mT}$ is always larger than $\underline{w}_{mT}$. Hence, we need only consider these two cases.

In Case (i) when $\widehat{w}_{mT} \le \overline{w}_{mT}$, the optimal liquidity ratio at mT is $\widehat{w}_{mT}$ as it is optimal for the investor not to voluntarily rebalance the illiquid alternative asset holding. The intuition is that, even with the automatic increase in liquidity at mT, the liquidity ratio will still lie within the inaction range $\lim_{t \to mT}(\underline{w}_t, \overline{w}_t.)$ Therefore, the continuity of the value function implies $P(W_{mT}, K_{mT}) = P(\widehat{W}_{mT}, \widehat{K}_{mT})$, which can be simplified as:

$$\lim_{t \to mT} p(w, t) = p(\widehat{w}_{mT}, t)(1 - \delta_T) \,, \tag{32}$$

where $\widehat{w}_{mT}$ is given in (31).

In Case (ii) when $\widehat{w}_{mT} > \overline{w}_{mT}$, the optimal liquidity ratio at mT is $\overline{w}_{mT}$ as it is optimal for the investor to voluntarily acquire the illiquid alternative asset. In this case, the automatic liquidity events results in $w_{mT} = \widehat{w}_{mT} > \overline{w}_{mT}$, which means the investor holds too much of the liquid asset. To bring the portfolio liquidity ratio back into the inaction region, the investor must acquire more of the alternative asset, so that $w_{mT} = \overline{w}_{mT}$. In Appendix B, we show

$$\lim_{t \to mT} p(w, t) = p\left(\overline{w}_{mT}, mT\right)\left(1 - \delta_T + \lambda\right) \,, \tag{33}$$

where λ reflects the effect of rebalancing and is given by

$$\lambda = \lim_{t \to mT} \frac{w_t + \delta_T - \overline{w}_{mT}(1 - \delta_T)}{1 + \theta_X + \overline{w}_{mT}}. \tag{34}$$

Finally, the homogeneity property allows us to express the value-matching condition (11) in terms of $p(w,t)$ at $t = mT$:

$$p(w, mT) = p(w, (m-1)T). \tag{35}$$

Next, we summarize the main results of our model.

Proposition 1 *The scaled certainty equivalent wealth $p(w,t)$ in the interim period when $(m-1)T < t \leq mT$ solves the PDE (24) subject to the boundary conditions (27), (30), (28), (30), and (35). Additionally, $p(w_{mT-}, mT-)$ satisfies (32) if $\widehat{w}_{mT} \leq \overline{w}_{mT}$ and (33) if $\widehat{w}_{mT} > \overline{w}_{mT}$, where $\widehat{w}_{mT}$ is given by (31).*

4. Data and Calibration

4.1. Data and Summary Statistics

As a guide to the calibration parameters and as a benchmark for interpreting our findings, we use university endowment fund data from the National Association of College and University Business Officers and Commonfund Endowment Fund Survey (NCES). See Brown et al. (2010), Dimmock (2012), and Brown et al. (2014) for more details. We focus on the cross-section of 774 university endowment funds as of the 2014-2015 academic year end.

Asset Allocation. The NCES provides annual snapshots of endowment funds' portfolio allocations. To link the NCES data to the model, we aggregate endowment allocations in the NCES data into the three asset classes in our model: (1) *the risk-free asset*, which aggregates cash and fixed income, (2) *public equity*, which aggregates public equity and REITs, and (3) *the alternative asset*, which aggregates hedge funds, private equity, venture capital, private real estate, and natural resources. For determining some of the calibration parameters we use the disaggregated sub-asset classes (e.g., venture capital), which are reported in Appendix D.

Table 1: Summary of Endowment Fund Asset Allocation

This table summarizes endowment fund portfolios as of the end of the 2014-2015 academic year for 774 endowments. The first two columns show the equal and value weighted averages, respectively. The columns 0-10% to 90-100% show averages within size-segmented groups of endowment funds. For example, the column "0-10%" shows the value weighted average portfolio allocation for the smallest decile of endowment funds. The table shows summary statistics for endowment fund size (reported in millions of dollars), asset class allocations and spending rates (reported in percentages), the number of alternative asset funds that the endowment holds, and the average target horizon for the alternative assets. *Cash & Fixed Income* includes cash, cash equivalents, and fixed income securities (except for distressed securities). *Public Equity* includes domestic and foreign equity as well as REITs. *Alternatives* includes hedge funds, private equity, venture capital, private real estate, and illiquid natural resources.

	Avg	VW Avg	0-10%	45-55%	90-100%
Endow. Size ($M)	677		17	116	13,409
Cash & Fixed Inc.	21.0%	12.7	33.1	22.4	10.8
Public Equity	50.7%	35.6	60.1	54.7	32.0
Alternatives	28.3%	51.7	6.3	22.9	57.1
Spending Rate	4.2%	4.4	4.5%	3.9	4.5
No. Alt. Funds	16.9	56.0	1.1	7.2	86.5
Alt. Target Horizon	4.2	5.5	3.6	4.0	5.9

Table 1 shows the summary statistics as of the end of the 2014-2015 academic year. The first and second columns show the equal and value weighted averages, respectively. The remaining columns show averages within various size categories of endowment funds (e.g., "0-10%" summarizes the variables for the smallest decile of funds). The average *Endowment Size* is $677 million, but the distribution is highly positively skewed and the median decile size is $116 million. On an equal weighted basis, *Public Equity* has the largest average allocation at 50.7%. On a value weighted bases *Alternative Allocations* has the largest average allocation at 57.1%, compared with 32.0% for public equity and 10.8% for cash and fixed income. The average spending rate is 4.2%.

Portfolio Illiquidity and Target Horizons. Table 1 reports the average number of alternative asset funds held by the endowments, which is an important component of liquidity management. Suppose two endowment funds have the same allocation to alternative assets, but Endowment A holds a single private equity fund with a 10 year lock-up while Endowment B spreads staggers its holdings across 120 different private equity funds such that one lock-up expires every month. Although both endowments have the same allocations, their liquidity exposures are very different. Endowment A can only adjust its exposure through the secondary market, while Endowment B can costlessly adjust its exposure as lock-ups expire each month. Thus, by holding multiple investments with staggered maturities, the endowment can enhance the liquidity of its portfolio, which we refer to as *liquidity diversification*. In our quantitative analysis, we explore the relation between liquidity diversification and investor welfare.

Table 1 shows there is a strong positive relation between endowment size and the number of alternative asset funds. On average, endowments in the largest decile hold 86.5 alternative asset funds; endowments in the smallest decile own only a single fund. Thus, liquidity diversification is more effective for larger endowments, lowering the unspanned risk.

We also estimate the average target holding period for alternative assets based on investors' portfolio allocations and the horizons of each sub-asset class within alternatives.[13] Table 1 reports the average *Alt Target Horizon*, the period when the alternative investment is locked-up, is 4.2 years for the full sample and 5.9 years for the largest decile of funds.

4.2. Parameter Choices and Calibration

Table 2 summarizes the baseline parameter values. Following the literature, we choose the following standard parameter values. The investor's coefficient of relative risk aversion is set to $\gamma = 2$. We set the EIS to be $\psi = 0.5$, so that it corresponds to expected utility with $\gamma = 1/\psi = 2$. We set the annual risk-free rate $r = 4\%$ and we also set the investor's

[13]For hedge funds, we assume a horizon of six months, which approximately equals the sum of the average redemption, advance notice, and lock-up periods reported in Getmansky et al. (2015). For private equity and venture capital, we assume a horizon of 10 years, based on the average commitment period reported in Metrick and Yasuda (2010). For private real estate and illiquid natural resources, we also assume horizons of 10 years, based on the holding periods reported in Collett et al. (2003) and Newell and Eves (2009).

discount rate equal to the risk-free rate, $\zeta = r$. For public equity, we use an annual volatility of $\sigma_S = 20\%$ and an aggregate equity risk premium of $\mu_S - r = 6\%$.

We calibrate the properties of the alternative asset by building up from the university endowments' allocations and the characteristics found in the literature. Appendix D provides the details and additional discussion. We set the alternative asset $\beta_A = 0.6$ and the unspanned volatility of the alternative asset to $\varepsilon = 15\%$. We set the horizon of the representative alternative asset $H = 6$ years.

Table 2: Summary of Key Parameters

This table summarizes the baseline parameter values. For completeness, the table also reports values of implied parameters, i.e., those parameters whose values are determined by other parameters.

Parameter	Symbol	Value
Coefficient of relative risk aversion	γ	2
Elasticity of intertemporal substitution	ψ	0.5
Subjective discount rate	ζ	4%
Risk-free rate	r	4%
Public equity expected return	μ_S	10%
Volatility of market portfolio	σ_S	20%
Beta of the alternative asset	β_A	0.6
Alternative asset alpha	α	2%
Alternative asset expected return	μ_A	9.6%
Volatility of alternative asset	σ_A	19.2%
Alternative asset target holding horizon	H	6
Proportional cost of liquidation	θ_L	0.1
Proportional cost of acquisition	θ_X	0.02
Implied parameters		
Correlation between risky assets	ρ	0.625
Unspanned volatility	ε	15%
Payout rate	δ_A	4.00%

In our model, the alpha of the illiquid alternative investment includes compensation for skill, value-added from governance, liquidity risk, and other risks unspanned by public equities. We set $\alpha = 2\%$, which we view as reasonable given the empirical findings in the literature. For example, Franzoni et al. (2012) find that private equity earns a net-of-fees liquidity risk premium of 3% annually. Aragon (2007) and Sadka (2010) find similar net-of-fees liquidity risk premia for hedge funds. Given this assumed alpha, the expected overall return on the alternative asset is $\mu_A = 0.02 + 0.04 + 0.6 \times (0.10 - 0.04) = 0.096 = 9.6\%$.

For voluntary liquidations, we assume that the proportional transaction cost is $\theta_L = 10\%$. Appendix D provides details showing how we reach this cost by building up from asset allocations and empirical evidence on secondary market discounts (see Kleymenova et al., 2012; Nadauld et al., 2019; Ramadorai, 2012). For acquisitions, we assume that the proportional acquisition cost is $\theta_X = 2\%$, which is equal to the average placement agent fee reported by Rikato and Berk (2015) and Cain et al. (2020).

Calibrating the model also requires a payout parameter, which determines the liquidity generated by automatic liquidity events (e.g., from intertemporally staggered investments in the alternative asset maturing and paying out capital). The payout rate depends on the number of distinct investments into the alternative asset. For example, given the target horizon of $H = 6$ years, an investor with a single alternative asset investment would receive a large payout once every six years. In contrast, an investor with a large number of distinct investments would receive smaller but more frequent payouts. For any given number of investments, denoted by i, Appendix E shows how it is possible to impute the payout rate using the previously described parameter values. For our baseline calibration we use $i \to \infty$, which implies a continuous payout rate of $\delta_A = 4.0\%$. For comparison, we also consider the cases with $i = 1$ and $i = 6$.

5. Quantitative Results

In this section, we analyze the model using the parameter values from Table 2. As a benchmark, we also analyze the case when the alternative asset is fully liquid.[14]

[14] In this case, the alternative asset simply expands the investment opportunity set. Thus, as Appendix B.1 shows, the value function is clearly higher than when the alternative asset is illiquid.

5.1. Certainty Equivalent Wealth and Net Worth

We introduce the widely-used net worth as the accounting value of the investor's portfolio:

$$N_t \equiv W_t + K_t \,. \tag{36}$$

In general, due to illiquidity net worth is not an economic measure of the investor's true welfare.

Figure 1 plots $P(W_t, K_t, t)/N_t$, the ratio of the certainty equivalent wealth to the portfolio's book value (net worth) N_t, as a function of

$$z_t = \frac{K_t}{N_t} = \frac{K_t}{W_t + K_t} = \frac{1}{w_t + 1} \,, \tag{37}$$

the proportion of the portfolio allocated to alternative assets. Recall that we use the liquidity ratio, w_t, as the effective state variable when analyzing the model and its solution in Sections 2 and 3. Here, we use z_t to exposit our quantitative results, as practitioners typically work with portfolio allocations. Also, note that z_t is typically between zero and one making the results easier to interpret.

As the optimal w is a range $(\underline{w}, \overline{w})$ and z decreases with w, the corresponding range for the optimal z is $(\underline{z}, \overline{z})$, where

$$\underline{z} = \frac{1}{\overline{w} + 1} \quad \text{and} \quad \overline{z} = \frac{1}{\underline{w} + 1} \,. \tag{38}$$

Therefore, the lower liquidation boundary $\underline{w}$ maps to the upper liquidation boundary $\overline{z}$ and the upper acquisition boundary $\underline{w}$ maps to the lower acquisition boundary $\underline{z}$. Let z_t^* denote the "desired" target of $z_t = K_t/(W_t + K_t)$. Mathematically, z_t^* is the value of z_t at which the investor attains the highest certainty-equivalent wealth for a given level of N_t, $P(W_t, K_t, t)/N_t$, where

$$\frac{P(W_t, K_t, t)}{N_t} = \frac{p(w_t, t)}{w_t + 1} = z_t p((1 - z_t)/z_t, t) \,. \tag{39}$$

Figure 1 includes the case of $i \to \infty$, and for comparison it also includes the cases of $i = 1$ and $i = 6$. For $i \to \infty$, we see that z lies between $(\underline{z}, \overline{z}) = (27.5\%, 64.9\%)$. That is, if the allocation to alternatives z falls to the endogenous acquisition boundary, $\underline{z} = 27.5\%$, the investor immediately sells just enough units of the liquid assets and

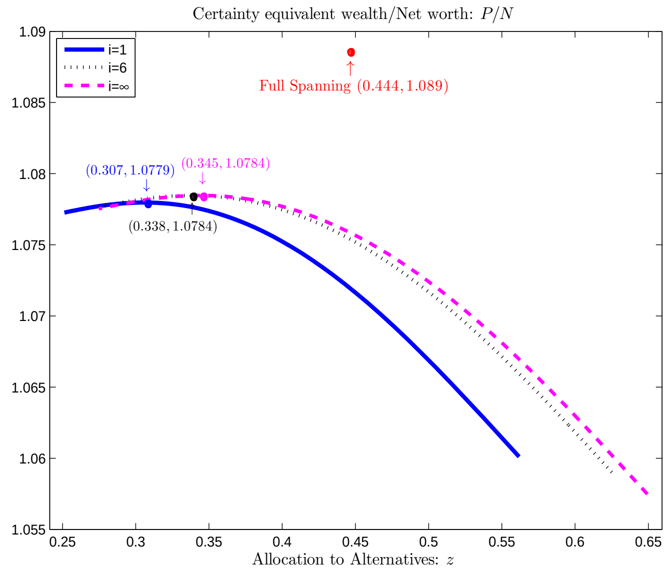

Figure 1: This figure plots $P/N = p(w,t)/(w+1)$, the ratio of the certainty equivalent wealth $P(W,K,t) = p(w,t)K$ and net worth $N = W + K$, as a function of the portfolio's percentage allocation to alternative assets $z = K/N$. For the $i = \infty$ case with $\delta_A = 4\%$, the no-trade region is $(\underline{z}, \overline{z}) = (0.275, 0.649)$ and the desired target is $z^* = 0.345$ which corresponds to the maximal P/N value of 1.078 (the blue circle). Other parameter values are given in Table 2. For the $i = 1$ and $i = 6$ case, the figure shows results at $t = mT$.

invests the proceeds in the illiquid alternative asset to keep $z \geq 27.5\%$. If the allocation to alternatives rises to the endogenous liquidation boundary, $\overline{z} = 64.9\%$, the investor sells just enough units of the illiquid asset so that z falls back to 64.9%.

Hypothetically, if investors could costlessly choose z, they would choose the "desired" target $z^* = 34.5\%$. At this point, certainty equivalent wealth is 7.8% higher than net worth. The curve is noticeably asymmetric and declines more rapidly to the right of the maximum, as the investor approaches the voluntary liquidation boundary, because liquidating alternative assets is more costly than acquiring them (i.e., $\theta_L = 10\% > \theta_X = 2\%$).

In sharp contrast, when the alternative asset is perfectly liquid, as in the case of full spanning, the admissible illiquid alternative asset holding is not a range, but instead is a singleton with the value of $z^* = 44.4\%$.

For the case of $i = 1$, the rebalancing boundaries are further to the left, indicating that the investor holds less of the alternative asset when there is less liquidity diversification from staggering maturities across time. The curve for $i = 6$ is similar to that for $i \to \infty$ indicating that even a moderate number of distinct investments in the alternative asset provides benefits from liquidity diversification.

5.2. Rebalancing Boundaries

Figure 2 shows the rebalancing boundaries over a period $(m-1)T < t < mT$ for any positive integer $m \geq 1$. We plot the optimal liquidation and acquisition boundaries in Panels A and B, respectively. First recall that for the $i = \infty$ case, the rebalancing boundaries are constant over time which correspond to two horizontal lines at $\underline{z} = 27.5\%$ and $\overline{z} = 64.9\%$. Next, we turn to the cases that feature time-varying no-trade regions. The figure shows the cases of $i = 1, 3$, and 6. In our discussion, we focus on the case of $i = 1$ (the investor has made only a single investment into the alternative asset) because the $i = 1$ case provides the greatest contrast with the $i = \infty$ case.

In the case of $i = 1$, there is an automatic liquidity event every six years at which time the alternative asset becomes fully liquid (see Appendix E for details). The differences between the cases of $i \to \infty$ and $i = 1$ highlights one of the unique features of our model – that it can accommodate the *liquidity diversification* from investing in illiquid assets with staggered lock-up expirations.

The initial rebalancing boundaries, at time $t = 0$, are lower for the $i = 1$ case than the $i \to \infty$ case because the effective cost of illiquidity due to trading restrictions is greater, resulting in lower demand for the illiquid asset. The comparison between the cases of $i = 1$ and $i \to \infty$ illustrates the interconnection of illiquidity from transactions costs and from trading restrictions. For the case of $i = 1$, both boundaries increase as $t \to mT$. This means that the investor becomes less willing to liquidate alternative assets and more willing to voluntarily acquire alternative assets as the automatic liquidity event at $t = mT$ approaches. This is intuitive, as the investor's liquid holdings will increase significantly at $t = mT$. Anticipating the natural liquidity event, the investor is more willing to accept large allocations to the illiquid asset. As a result, as $t \to mT$ the liquidation boundary $\overline{z}_t$ becomes exceedingly large and the acquisition boundary, $\underline{z}_t$ also increases. Note that the quantitative effects for the acquisition boundary are smaller than for the liquidation boundary.

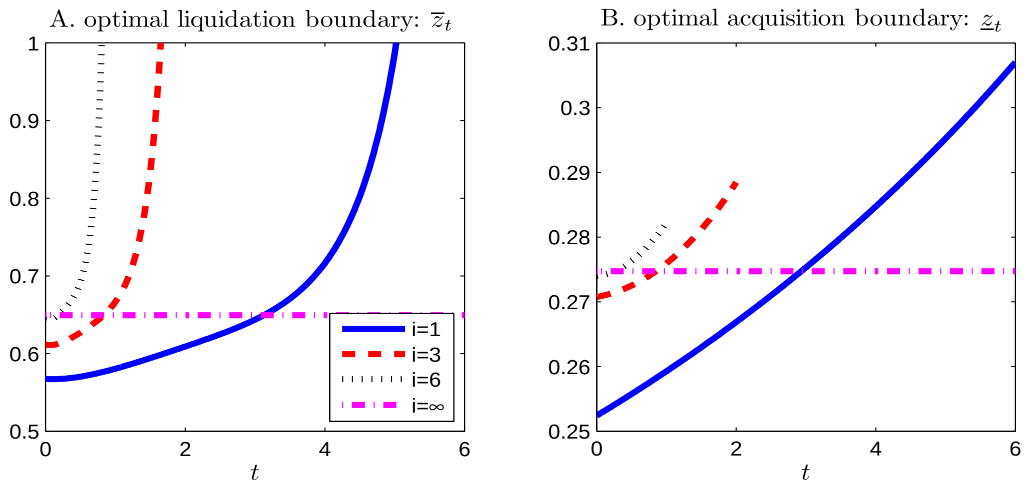

Figure 2: This figure plots the rebalancing boundaries for the portfolio's percentage allocation to alternative assets over time: $z = K/N$, where $N = W + K$. In Panels A and B, we plot the (upper) time-dependent liquidation boundary $\overline{z}_t$ and the (lower) acquisition boundary, $\underline{z}_t$, respectively. All parameter values (other than the number of distinct investments i) are given in Table 2. For the stationary case ($i \to \infty$), the annual payout rate is set at $\delta_A = 4\%$ and the optimal z lies within $(\underline{z}_t, \overline{z}_t) = (0.275, 0.649)$ for all t (see the two dash dotted magenta horizontal lines). For the cases with finite numbers of investments, e.g., $i = 1$, $i = 3$, and $i = 6$, the payout rates are, respectively, $\delta_T = 21.34\%$ every six years ($H = 6$), $\delta_T = 7.69\%$ every two years ($H = 2$), and $\delta_T = 3.92\%$ every year ($H = 1$). The payout rates across the four cases are set so that they have effectively the same total payouts over long time periods.

Figure 2 shows that investors with fewer distinct investments in the alternative assets will experience less stable portfolio allocations and deviate from their desired target to a larger extent. Given that smaller investors are more likely to hold fewer distinct investments, this figure highlights a potential reason for the empirical correlation between investor size and alternative asset allocations – that small investors are less able to engage in liquidity diversification.

5.3. Comparative Statics

In this subsection, we conduct comparative static analysis for the $i \to \infty$ case. As shown earlier, because of transaction costs the model generates an illiquid asset no-trade region

($\underline{z}$, $\overline{z}$). To ease interpretations, we report asset allocations and the spending rate at the desired target, z^*, (at which the investor's value function and certainty equivalent wealth are maximized) defined in the text preceding (39) and shown in Figure 1. The columns "Region" and "Deviation" show, respectively, the rebalancing boundaries for the allocation to the alternative asset and the average deviation from the desired target for the given parameter values. The tables also show the average deviation of the spending rate from the desired target. For each table, the row in bold font shows results for the baseline case where the parameter values are given in Table 2.

5.3.1. EIS ψ

Table 3 shows comparative static effects of changing the EIS ψ. Panel A reports results for four cases ranging from very low to high values of the EIS: $\psi = 0.1, 0.5, 1, 2$. Panel B shows results for the case of full spanning. We see that varying the EIS has very large quantitative effects on the spending rate. An investor who is unwilling to substitute spending over time (e.g., $\psi = 0.1$) has a spending rate of 6.36%, which is relatively high (in light of the permanent-income argument). In contrast, an investor who is willing to substitute consumption over time, (e.g., $\psi = 2$ as in the long-run risk literature following Bansal and Yaron, 2004), has a spending rate of only 1.33%. The intuition is that an investor with a high EIS defers spending to exploit the investment opportunity.

Not only does the EIS have a first-order effect on spending, it also influences asset allocation. As the EIS increases, the investor allocates more to the illiquid alternative asset and less to public equity and bonds. For example, an investor with $\psi = 0.1$ allocates 56.20% to public equity and 30.77% to alternatives, compared to an investor with $\psi = 2$ who allocates 44.39% to public equity and 50.25% to alternatives. The large effect of the EIS on asset allocation in our incomplete markets model is due to the interactive effect between asset allocation and optimal spending policies. An investor with higher EIS is more willing to defer consumption in response to better investment opportunities, rather than engage in costly liquidation of alternative assets. This allows for both a higher target allocation to alternative assets and a wider illiquid asset no-trade region. This contrasts with the case of full spanning, where the EIS has no effect on asset allocation (see equations (C.1) and (C.2)).

Table 3: THE EFFECT OF THE EIS ψ ON ASSET ALLOCATION AND SPENDING

This table reports the comparative static effect of ψ on asset allocation and spending for the $i = \infty$ case. The three columns, Public Equity, Bond, and Alternatives (Alternative Assets), report Π/N, $(W - \Pi)/N$, and K/N, respectively, evaluated at the desired target highlighted in Figure 1. Region (illiquid asset no-trade region) and Dev. (standard deviation) report the range between rebalancing boundaries and the average deviation from the desired target for the alternative asset, respectively. The Spending column reports the corresponding desired target spending rate, C/N, and Dev. reports the average deviation from the spending rate at the desired target. All columns are presented in percent (%), which are omitted for simplicity. Panel A reports results for the case with illiquidity. Panel B reports results for the case of full-spanning. The baseline parameter values are given in Table 2. For the results in this table, we fix risk aversion at $\gamma = 2$.

	Public Equity	Bonds	Alternatives	Region	Dev.	Spending	Dev.
A. Illiquidity case							
$\psi = 0.1$	56.20	13.03	30.77	(24.57, 62.50)	6.9	6.36	0.04
$\boldsymbol{\psi = 0.5}$	**53.93**	**11.59**	**34.48**	**(27.47, 64.94)**	**9.4**	**5.32**	**0.07**
$\psi = 1$	50.47	9.37	40.16	(31.55, 69.44)	10.9	3.95	0.10
$\psi = 2$	44.39	5.36	50.25	(37.04, 82.64)	5.5	1.33	0.01
B. Full-spanning case							
$\psi = 0.5$	48.33	7.22	44.44	(44.44, 44.44)	0	5.35	0

Our model-implied results for the relation between spending flexibility and portfolio liquidity are consistent with empirical facts. Pension plans, which have low spending flexibility, have relatively low allocations to alternative assets as compared to investors with greater flexibility such as endowments and family offices. Over a medium or long horizon, the combined effect of a high EIS – reducing spending and tilting towards investments that earn an illiquidity premium – will have a significant impact on the accumulation of net worth.

5.3.2. Risk Aversion γ

Table 4 shows that the coefficient of relative risk aversion has a very large effect on asset allocation. For a fixed EIS of $\psi = 0.5$, if risk aversion decreases from $\gamma = 2$ to $\gamma = 1$ the investor increases the portfolio allocation to alternative assets from 34.48% to 53.76%.

Table 4: The Effect of γ on Asset Allocation and Spending Rates

This table reports the comparative static effect of γ on asset allocation and spending for the $i = \infty$ case. The three columns, Public Equity, Bond, and Alternatives (Alternative Assets), report Π/N, $(W-\Pi)/N$, and K/N, respectively, evaluated at the desired target highlighted in Figure 1. Region (illiquid asset no-trade region) and Dev. (standard deviation) report the range between rebalancing boundaries and the average deviation from the desired target for the alternative asset, respectively. The Spending column reports the corresponding desired target spending rate, C/N, and Dev. reports the average deviation from the spending rate at the desired target. All columns are presented in percent (%), which are omitted for simplicity. The baseline parameter values are given in Table 2. For the results in this table, we fix the EIS at $\psi = 0.5$.

	Public Equity	Bonds	Alternatives	Region	Dev.	Spending	Dev.
$\gamma = 1$	112.75	-66.51	53.76	(38.91, 135.14)	27.3	6.57	0.21
$\boldsymbol{\gamma = 2}$	**53.93**	**11.59**	**34.48**	**(27.47, 64.94)**	**9.4**	**5.32**	**0.07**
$\gamma = 4$	27.21	55.43	17.36	(13.16, 37.04)	6.2	4.66	0.04

Even more strikingly, the investor changes the portfolio allocation to the risk-free asset from a long position of 11.59% to a short position (borrowing 66.51% of net worth). As a result, the investor increases the portfolio allocation to public equity from 53.93% to a levered position (112.75% of net worth). As risk aversion increases from $\gamma = 2$ to $\gamma = 4$, allocations to bonds significantly increase from 11.59% to 55.43%, allocations to alternative assets decrease by about half from 34.48% to 17.36%, and allocations to public equity decrease from 53.93% to 27.21%.

Table 4 shows that risk aversion has a large effect on the illiquid asset no-trade region and the average deviation of the desired target. Risk aversion affects not only the level of the desired allocation to illiquid assets, but also the tolerance for deviations from the desired allocation. As risk aversion rises, the no-trade region becomes narrower and the average deviation becomes smaller.

Comparing the results for risk aversion in Table 4 with those for the EIS in Table 3, we show that it is important to use Epstein-Zin utility as risk aversion and the reciprocal of the EIS have opposite effects on allocations to public equity and spending. While increasing the coefficient of relative risk aversion causes allocations to public equity to fall, decreasing the EIS causes allocations to public equity to increase. Similarly, while

Table 5: THE EFFECT OF THE PROPORTIONAL LIQUIDATION COST θ_L ON ASSET ALLOCATION AND SPENDING

This table reports the comparative static effect of the proportional liquidation cost θ_L on asset allocation and spending for the $i = \infty$ case. The three columns, Public Equity, Bond, and Alternatives (Alternative Assets), report Π/N, $(W - \Pi)/N$, and K/N, respectively, evaluated at the desired target highlighted in Figure 1. Region (illiquid asset no-trade region) and Dev. (standard deviation) report the range between rebalancing boundaries and the average deviation from the desired target for the alternative asset, respectively. The Spending column reports the corresponding desired target spending rate, C/N, and Dev. reports the average deviation from the spending rate at the desired target. All columns are presented in percent (%), which are omitted for simplicity. The baseline parameter values are given in Table 2.

	Public Equity	Bonds	Alternatives	Region	Dev.	Spending	Dev.
$\theta_L = 0.01$	48.80	8.01	43.20	(32.10, 52.08)	5.7	5.34	0.01
$\theta_L = 0.05$	52.20	10.42	37.38	(29.28, 57.80)	5.8	5.33	0.03
$\boldsymbol{\theta_L = 0.1}$	**53.93**	**11.59**	**34.48**	**(27.47, 64.94)**	**9.4**	**5.32**	**0.07**
$\theta_L = 0.25$	55.51	12.64	31.85	(25.58, 86.21)	11.5	5.31	0.20
$\theta_L = 0.5$	55.63	12.72	31.65	(25.45, 123.46)	11.7	5.31	0.32

increasing the coefficient of relative risk aversion causes spending to fall, decreasing the EIS causes spending to increase.

5.3.3. Proportional Liquidation Cost θ_L

Table 5 reports comparative static effects of changing the proportional liquidation cost θ_L. Changing the liquidation cost θ_L from a low value of 1% to a large cost of 50% results in a moderately large decrease in the allocation to the alternative asset from 43.20% to 31.65%. The investor alters the portfolio allocation to keep the overall portfolio β nearly constant, offsetting the reduced risk exposures of the alternative asset with higher allocations to public equity. We also see that progressively larger increases in θ_L result in progressively smaller changes in asset allocation; as the liquidation cost rises the no-trade region becomes wider, the investor is increasingly unlikely to engage in portfolio rebalancing, and so further increases in rebalancing costs become quantitatively much less important.

5.3.4. Excess Return $\boldsymbol{\alpha}$

Table 6 reports comparative static results for excess return α defined with respect to the single-index model with public equity. For this reason, α includes the risk premium and illiquidity premium of the alternative asset. That is, α is not purely a measure of the alternative asset manager's skills. The results show that asset allocations are quite sensitive to changes in α. For example, increasing α from 2% to 3% increases the alternative asset allocation from 34.48% to 60.24%. As the allocation to the alternative asset increases, the allocation to public equity falls from 53.93% to 37.91% to manage the overall portfolio β and because of the additional liquidity risk.

Table 6: The Effect of α on Asset Allocation and Spending Rates

This table reports the comparative static effect of α on asset allocation and spending for the $i = \infty$ case. The three columns, Public Equity, Bond, and Alternatives (Alternative Assets), report Π/N, $(W - \Pi)/N$, and K/N, respectively, evaluated at the desired target highlighted in Figure 1. Region (illiquid asset no-trade region) and Dev. (standard deviation) report the range between rebalancing boundaries and the average deviation from the desired target for the alternative asset, respectively. The Spending column reports the corresponding desired target spending rate, C/N, and Dev. reports the average deviation from the spending rate at the desired target. All columns are presented in percent (%), which are omitted for simplicity. The baseline parameter values are given in Table 2.

	Public Equity	Bonds	Alternatives	Region	Dev.	Spending	Dev.
$\alpha = 0\%$	75.00	25.00	0.00	(0, 0)	0	5.13	0
$\alpha = 1\%$	67.29	20.02	12.69	(9.09, 38.02)	7.8	5.16	0.05
$\boldsymbol{\alpha = 2\%}$	**53.93**	**11.59**	**34.48**	**(27.47, 64.94)**	**9.4**	**5.32**	**0.07**
$\alpha = 3\%$	37.91	1.87	60.24	(51.02, 88.50)	9.1	5.60	0.10

The sensitivity of the implied portfolio allocations to changes in α is consistent with the large cross-sectional dispersion in endowment funds' allocations to alternative assets. An α of 0% can explain non-participation, while an α of 3% implies allocations that are broadly consistent with those of large endowments such as Yale and Stanford. Thus, with reasonable parameter values, our model is consistent with both the average allocation and also the cross-sectional dispersion of allocations to alternative assets.

The sensitivity of allocations to α is also consistent with the empirically observed strong relation between endowment fund size and allocations to alternative assets. Lerner et al. (2008), Brown et al. (2010), Barber and Wang (2013), and Ang et al. (2018) find that large endowment funds persistently earn significant alphas, which they attribute to superior alternative asset investments, while small endowments do not earn significant alphas. Lerner et al. (2008) and Brown et al. (2011) discuss how large endowments typically have better investment committees, better access to elite managers, and can exploit economies of scale in selecting alternative assets.

5.3.5. Unspanned Volatility ε

Table 7 shows that the unspanned volatility of the alternative asset, ε, has a quantitatively large effect on asset allocation. We use two panels to demonstrate how both the level of the unspanned volatility and the composition of total volatility affect asset allocation.

Table 7: THE EFFECT OF ε ON ASSET ALLOCATION AND SPENDING

This table reports the comparative static effect of ε on asset allocation and spending for the $i = \infty$ case. The three columns, Public Equity, Bond, and Alternatives (Alternative Assets), report Π/N, $(W - \Pi)/N$, and K/N, respectively, evaluated at the desired target highlighted in Figure 1. Region (illiquid asset no-trade region) and Dev. (standard deviation) report the range between rebalancing boundaries and the average deviation from the desired target for the alternative asset, respectively. The Spending column reports the corresponding desired target spending rate, C/N, and Dev. reports the average deviation from the spending rate at the desired target. All columns are presented in percent (%), which are omitted for simplicity. Panel A show the effect of changing ε while β_A is fixed at $\beta_A = 0.6$ and the column "Implied σ_A" shows the total volatility of the alternative asset. Panel B shows the effect of changing ε while the total volatility of the alternative asset is fixed at $\sigma_A = 19.2\%$ and the column "Implied β_A" shows the implied beta of the alternative asset. In both panels, the column "ε^2/σ_A^2" shows the alternative asset's unspanned variance as a percentage of its total variance. The baseline parameter values are given in Table 2.

Panel A. Fixing $\beta_A = 0.6$

	Implied σ_A	ε^2/σ_A^2	Public Equity	Bonds	Alternatives	Region	Dev.	Spending	Dev.
$\varepsilon = 10\%$	15.6%	41.0%	27.79	−4.13	76.34	(51.02, 88.50)	9.1	5.59	0.10
$\boldsymbol{\varepsilon = 15\%}$	**19.2%**	**61.0%**	**53.93**	**11.59**	**34.48**	**(27.47, 64.94)**	**9.4**	**5.32**	**0.07**
$\varepsilon = 17.5\%$	21.2%	68.0%	60.17	15.50	24.33	(18.55, 51.55)	9.6	5.26	0.06
$\varepsilon = 19.2\%$	22.6%	71.9%	63.07	17.33	19.61	(14.53, 44.64)	9.0	5.24	0.06

Panel B. Fixing $\sigma_A = 19.2\%$

	Implied β_A	ε^2/σ_A^2	Public Equity	Bonds	Alternatives	Region	Dev.	Spending	Dev.
$\varepsilon = 10\%$	0.82	27.1%	−2.12	6.88	95.24	(86.21, 135.14)	7.7	5.62	0.10
$\boldsymbol{\varepsilon = 15\%}$	**0.60**	**61.0%**	**53.93**	**11.59**	**34.48**	**(27.47, 64.94)**	**9.4**	**5.32**	**0.07**
$\varepsilon = 17.5\%$	0.40	83.0%	66.41	13.01	20.58	(15.63, 48.54)	9.5	5.25	0.06
$\varepsilon = 19.2\%$	0	100.0%	74.57	13.61	11.82	(8.45, 38.17)	9.0	5.20	0.05

In Panel A of Table 7, we fix $\beta_A = 0.6$ for all rows, which implies that the part of the alternative asset's return volatility spanned by the public market equals $\rho\sigma_A = 0.12$ for all four cases. The total variance, $\sigma_A^2 = (\rho\sigma_A)^2 + \varepsilon^2 = .12^2 + \varepsilon^2$, varies one-to-one with ε^2. We show that changes in the volatility unspanned by public equity have large effects on asset allocation. For example, if we decrease ε from 15% (the baseline case) to 10%, the investor more than doubles the allocation to alternative assets from 34.48% to 76.34%.

In Panel B of Table 7, we fix the total volatility of the alternative asset σ_A at 19.2%. Then, as we increase the unspanned volatility ε, the spanned volatility must decrease to keep the total volatility unchanged. Consider again decreasing ε from 15% (the baseline case) to 10%. The investor reacts to this decrease in unspanned volatility by almost tripling the allocation to alternative assets from 34.48% to 95.24%.

In sum, both the amount of unspanned risk and the composition of total risk have quantitatively large effects on asset allocation. The sensitivity of allocations is striking given the empirical uncertainty associated with these parameter values. Our quantitative results suggest it is worth devoting much more work to improve the empirical estimates of unspanned volatility.

6. Contributions (Inflows) to the Fund

In this section, we extend our baseline model of Section 2 to include contributions (inflows) into the investor's portfolio. Inflows provide new capital, which naturally should influence the fund's portfolio allocation decisions.

We assume that new contributions flow into the portfolio at a rate of $\tau(W_t + K_t)$ where $\tau > 0$. Incorporating $\tau(W_t + K_t)$ into (7) gives the following liquid wealth process:

$$\begin{aligned} dW_t &= \left(rW_{t-} + \delta_A K_{t-} + \tau(W_{t-} + K_{t-}) - C_{t-}\right) dt + \Pi_{t-}\left((\mu_S - r)dt + \sigma_S d\mathbb{B}_t^S\right) \\ &\quad + (1-\theta_L)dL_t - (1+\theta_X)dX_t + \delta_T K_{t-}\mathbb{I}_{\{t=mT\}}\,. \end{aligned} \tag{40}$$

The HJB equation for the value function $V(W_t, K_t, t)$ in the interior region is given by

$$\begin{aligned} 0 &= \max_{C,\Pi}\ f(C,V) + (rW + \delta_A K + \tau(W+K) + (\mu_S - r)\Pi - C)V_W + \frac{(\Pi\sigma_S)^2}{2}V_{WW} \\ &\quad + V_t + (\mu_A - \delta_A)KV_K + \frac{\sigma_A^2 K^2}{2}V_{KK} + \rho\Pi K\sigma_S\sigma_A V_{WK}\,. \end{aligned} \tag{41}$$

By using essentially the same procedure and the homogeneity property as in Section 2, we obtain the following nonlinear PDE for the scaled certainty-equivalent wealth $p(w,t)$ in the interior region and when $(m-1)T < t < mT$:

$$
\begin{aligned}
0 = & \left(\frac{\phi_1 \left(p_w(w,t)\right)^{1-\psi} - \psi\zeta}{\psi - 1} + \mu_A - \delta_A - \frac{\gamma\sigma_A^2}{2}\right) p(w,t) + p_t(w,t) + \frac{\varepsilon^2 w^2}{2} p_{ww}(w,t) \\
& + \left(\left(\delta_A - \alpha + \gamma\varepsilon^2\right) w + \delta_A + \tau(w+1)\right) p_w(w,t) - \frac{\gamma\varepsilon^2 w^2}{2} \frac{(p_w(w,t))^2}{p(w,t)} \\
& + \frac{(\eta_S - \gamma\rho\sigma_A)^2 p_w(w,t) p(w,t)}{2\gamma_(w,t)}
\end{aligned}
\qquad (42)
$$

where ϕ_1 is given by (17).

As in the baseline model of Section 2, the boundary conditions are given in (27), (30), (28), (30), and (35). The scaled certainty equivalent wealth $p(w_{mT-}, mT-)$ is given by (32) and (33) for the $\widehat{w}_{mT} \leq \overline{w}_{mT}$ and $\widehat{w}_{mT} > \overline{w}_{mT}$ cases, respectively, where $\widehat{w}_{mT}$ is given by (31). Finally, the optimal consumption rule is given by (22).

Table 8 shows the effect of the contribution rate, τ, on portfolio allocations and spending. The first row shows the baseline case with $\tau = 0$ and the additional rows report results for $\tau = 1\%$, 2%, and 5%.[15] As the contribution rate τ increases, allocations to the alternative asset increase. Intuitively, current and future capital inflows from contributions effectively make the portfolio more titled towards liquid assets. Anticipating this, the investor reduces allocations to both public equity and bonds. Also consistent with this intuition, the range between the rebalancing boundaries widens, indicating that the investor is willing to tolerate greater deviations from the desired target as τ increases. But deviations from the desired target decrease sharply as τ increases, as the inflows from new contributions allow the investor to maintain the alternative allocation closer to the desired target.

Table 8 also shows that spending increases as the contribute rate τ increases. This is expected as the investor is wealthier and the current spending rate C/N does not take into account future capital inflows. The standard deviation of spending rates also decreases due to the investor's stronger ability to smooth spending over time and across states.

[15] During the 2009-2015 period, the value weighted average and median annual contribution rates to university endowments were 2.8% and 2.0%, respectively.

Table 8: The Effect of Inflow τ on Asset Allocation and Spending Rates

This table reports the comparative static effect of τ, the contribution rate, on asset allocation and spending for the $i = \infty$ case.. The three columns, Public Equity, Bond, and Alternatives (Alternative Assets), report Π/N, $(W - \Pi)/N$, and K/N, respectively, evaluated at the desired target highlighted in Figure 1. Region (illiquid asset no-trade region) and Dev. (standard deviation) report the range between rebalancing boundaries and the average deviation from the desired target for the alternative asset, respectively. The Spending column reports the corresponding desired target spending rate, C/N, and the Dev. column reports the average deviation from the spending rate at the desired target. All columns are presented in percent (%), which are omitted for simplicity. The baseline parameter values are given in Table 2.

	Public Equity	Bonds	Alternatives	Region	Dev.	Spending	Dev.
$\boldsymbol{\tau = 0}$	**53.93**	**11.59**	**34.48**	**(27.47, 64.94)**	**9.4**	**5.32**	**0.07**
$\tau = 1\%$	52.39	10.58	37.04	(29.24, 68.03)	8.0	5.83	0.07
$\tau = 2\%$	50.79	9.53	39.68	(30.77, 71.94)	6.3	6.33	0.06
$\tau = 5\%$	47.22	7.12	45.66	(33.56, 83.33)	2.3	7.84	0.03

In sum, new capital contributions make the investor wealthier in present value and also provide additional flexibility to manage illiquidity risk – resulting in not just higher allocations to alternative assets and higher spending but also lower volatilities for both allocations and spending.

7. Spending Constraint

In this section, we extend our baseline model of Section 2 to include a lower bound on the spending rate. In the United States, most private foundations are required to pay out at least 5% of assets every year to maintain tax exempt status (university endowments are an exception to this rule). A minimum spending requirement reduces flexibility and interacts with illiquidity in potentially important ways.

We assume that the investor's spending C_t as a fraction of total net worth N_t can not fall below $\underline{c}$ at any $t \geq 0$:

$$C_t \geq \underline{c} N_t \,. \tag{43}$$

Table 9: THE EFFECT OF A SPENDING CONSTRAINT

This table reports the comparative static effect of a spending constraint on asset allocation and spending for the $i = \infty$ case. The three columns, Public Equity, Bond, and Alternatives (Alternative Assets), report Π/N, $(W - \Pi)/N$, and K/N, respectively, evaluated at the desired target highlighted in Figure 1. Region (illiquid asset no-trade region) and Dev. (standard deviation) report the range between rebalancing boundaries and the average deviation from the desired target for the alternative asset, respectively. The Spending column reports the corresponding desired target spending rate, C/N, and Dev. reports the average deviation from the spending rate at the desired target. All columns are presented in percent (%), which are omitted for simplicity. The baseline parameter values are given in Table 2.

	Public Equity	Bonds	Alternatives	Region	Dev.	Spend	Dev.
No constraint	**53.93**	**11.59**	**34.48**	**(27.47, 64.94)**	**9.4**	**5.32**	**0.07**
$\underline{c} = 5.2\%$	57.97	14.25	27.78	(22.88, 68.03)	10.7	5.30	0.02

The optimal consumption $c(w_t, t) = C(W_t, K_t, t)/K_t$ is then given by

$$c(w,t) = \max\{\phi_1 \, p(w,t) \, p_w(w,t)^{-\psi}, \; \underline{c} \cdot (w+1)\} \, . \tag{44}$$

The PDE for the scaled certainty equivalent wealth $p(w,t)$ is given by

$$\begin{aligned} 0 = & \left(\frac{\psi c(w,t) \frac{p_w(w,t)}{p(w,t)} - \psi\zeta}{\psi - 1} + \mu_A - \delta_A - \frac{\gamma\sigma_A^2}{2} \right) p(w,t) + p_t(w,t) + \frac{\varepsilon^2 w^2}{2} p_{ww}(w,t) \\ & + \left(\left(\delta_A - \alpha + \gamma\varepsilon^2 \right) w + \delta_A - c(w,t) \right) p_w(w,t) - \frac{\gamma\varepsilon^2 w^2}{2} \frac{(p_w(w,t))^2}{p(w,t)} \\ & + \frac{(\eta_S - \gamma\rho\sigma_A)^2 p_w(w,t) p(w,t)}{2\gamma_i(w,t)} \, , \end{aligned} \tag{45}$$

where $c(w,t)$ is given by (44). All the boundary conditions and other policy functions, e.g., the allocation to public equity, are the same as in our baseline model of Section 2.

In Table 9, we report the quantitative effect of introducing a lower spending requirement of $\underline{c} = 5.2\%$. Although the desired spending target falls only slightly, from 5.32% to 5.30%, the allocation to the alternative asset decreases from 34.5% to 27.8%, a significant 6.7% fall in levels and 20% fall in percentage terms. Additionally, the illiquid asset no-trade region widens. Panel A of Figure 3 compares the investor's certainty equivalent

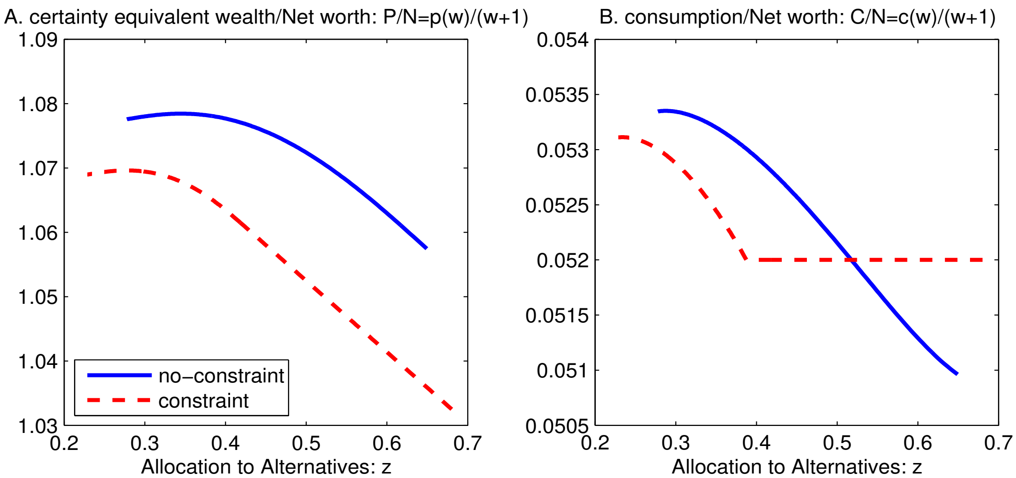

Figure 3: This figure demonstrates the effects of spending constraints on certainty equivalent wealth and consumption for the $i = 1$ case. Panels A and B plot $P/N = p(w,t)/(w+1)$ and $C/N = c(w,t)/(w+1)$ at $t = 0$ as functions of the percentage allocation to alternative assets $z = K/N$, respectively. The solid blue lines and the dashed red lines are for the no-constraint and the case where the spending is constrained to be at least $\underline{c} = 5.2\%$. All other parameter values are given in Table 2.

wealth for the spending constraint case with that of the baseline case. The expected welfare loss from the spending constraint is large: approximately 2% of the investor's certainty equivalent wealth.

Spending constraints reduce investor welfare and significantly alter asset allocation. These results are important given that some politicians, such as Senator Chuck Grassley, advocate such spending requirements.

8. Financial Crisis

In this section, we extend the model to include the possibility of crisis states. This is motivated by empirical findings that alternative asset illiquidity is time-varying and increases in crisis states.[16]

[16]See Franzoni et al. (2012), Kleymenova et al. (2012), Ramadorai (2012), and Nadauld et al. (2019), among others.

8.1. Model and Solution

We assume that there are two states, a normal and a crisis state. The transitions between these two states follow a continuous-time Markov chain. Let s_t denote the state at time t, where $s_t = g$ is the normal state and $s_t = b$ is the crisis state. Over a short time interval, Δ, the state switches from g to b (or from b to g) with a constant probability $\xi_g \Delta$ (or $\xi_b \Delta$). We denote θ_L^g and θ_X^g (θ_L^b and θ_X^b) as the proportional costs of liquidation and acquisition in the normal (crisis) state, respectively. We assume $\theta_L^g < \theta_L^b$, which reflects a higher secondary market liquidation cost (e.g., illiquidity) during the crisis state.

We also assume that in the crisis state the value of the alternative asset is subject to an additional downward jump shock, modeled as in the rare disaster literature, e.g., Barro (2006), Pindyck and Wang (2013), and Wachter (2013). Let $\mathbb{J}$ denote a pure jump process with a constant arrival rate $\lambda > 0$, which is present only in the crisis state. If a jump does not occur at t ($d\mathbb{J}_t = 0$), the alternative asset's fundamental value is continuous: $A_t = A_{t-}$, where $A_{t-} \equiv \lim_{s \uparrow t} A_s$ denotes the left limit of the fundamental value. If a jump occurs at t ($d\mathbb{J}_t = 1$), the alternative asset's fundamental value falls from A_{t-} to $A_t = ZA_{t-}$, reflecting the proportional decline in the value of the alternative asset when the economy transitions into the crisis state. We now write the dynamics of the fundamental value A in the crisis state b as:

$$\frac{dA_t}{A_{t-}} = \mu_A dt + \sigma_A d\mathbb{B}_t^A - \delta_A dt - (1 - Z) d\mathbb{J}_t \,. \tag{46}$$

The dynamics of A in the normal state has no jumps and hence is the same as (2).

Finally, to capture the empirical pattern that capital calls increase and distributions (to investors) decrease during the crisis state,[17] we assume that there is a stochastic call of the alternative asset in the crisis state at the moment the economy is hit by a downward jump shock ($d\mathbb{J}_t = 1$). To be precise, when $d\mathbb{J}_t = 1$, the value of the alternative asset that the investor owns drops from the pre-jump level of K_{t-} to the post-jump level of ZK_{t-} (and for simplicity, we assume that the value of the investor's liquid asset holdings also drops by $(1 - Z)$ fraction).[18] But importantly, at this moment, the investor receives

[17] Robinson and Sensoy (2016) and Nadauld et al. (2019) show that the net cash flows from private equity are countercyclical.

[18] We can relax this assumption at the cost of more involved notations and analysis.

a capital call proportional to ZK_{t-}, where the proportionality constant is $call > 0$. This means that the investor must increase the position in the alternative asset by providing the amount $call \cdot (ZK_{t-})$ to meet this capital call.[19]

To fund the capital call, the investor's liquid wealth decreases from the pre-jump level W_{t-} to $W_t = ZW_{t-} - call \cdot ZK_{t-}$ and the alternative asset position changes from the pre-jump level of K_{t-} to $K_t = Z(1 + call)K_{t-}$, a combination of losses on their original positions and increased allocation (due to the capital call).

Therefore, the alternative asset position in the crisis state b evolves as:

$$dK_t = (\mu_A - \delta_A)K_{t-}dt + \sigma_A K_{t-} d\mathbb{B}_t^A - dL_t + dX_t - \delta_T K_{t-} \mathbb{I}_{\{t=mT\}} - (1-Z)K_{t-}d\mathbb{J}_t + call \cdot ZK_{t-}d\mathbb{J}_t \,. \tag{47}$$

In the crisis state, the investor's liquid wealth evolves as:

$$\begin{aligned} dW_t &= (rW_{t-} + \delta_A K_{t-} - C_{t-})\, dt + \Pi_{t-} \left((\mu_S - r)dt + \sigma_S d\mathbb{B}_t^S \right) + (1 - \theta_L^b)dL_t \\ &\quad -(1 + \theta_X^b)dX_t + \delta_T K_{t-} \mathbb{I}_{\{t=mT\}} - (1 - Z)W_{t-}d\mathbb{J}_t - call \cdot ZK_{t-}d\mathbb{J}_t \,, \end{aligned} \tag{48}$$

where the last two terms capture the value loss from the pre-jump position and the outflow of capital from the liquid asset holdings to meet the stochastic capital call, respectively. For brevity, we do not write down the corresponding equations for the normal state.

Appendix F summarizes the solution. Next, we calibrate this generalized model and analyze the results.

8.2. Quantitative Analysis

We calibrate the cost of liquidation in the crisis state to $\theta_L^b = 0.25$.[20] We set the capital call parameter to $call = 0.2$, based on Robinson and Sensoy (2016) and Brown et al. (2021). The state transition probabilities are set to $\xi_g = 0.1$ and $\xi_b = 0.5$ as in Bolton et al. (2013). To focus on the effect of stochastic capital call in the crisis state, we ignore

[19] In practice, investors prefer not to be called in a crisis precisely for the reason that they do not want their portfolio allocations to be distorted in crisis times. We ignore the negotiation and bargaining between the asset manager and owners.

[20] We obtain this cost by combining the average portfolio weights of endowment funds with the estimated secondary market costs in the financial crisis for hedge funds from Ramadorai (2012) and for private equity from Nadauld et al. (2019).

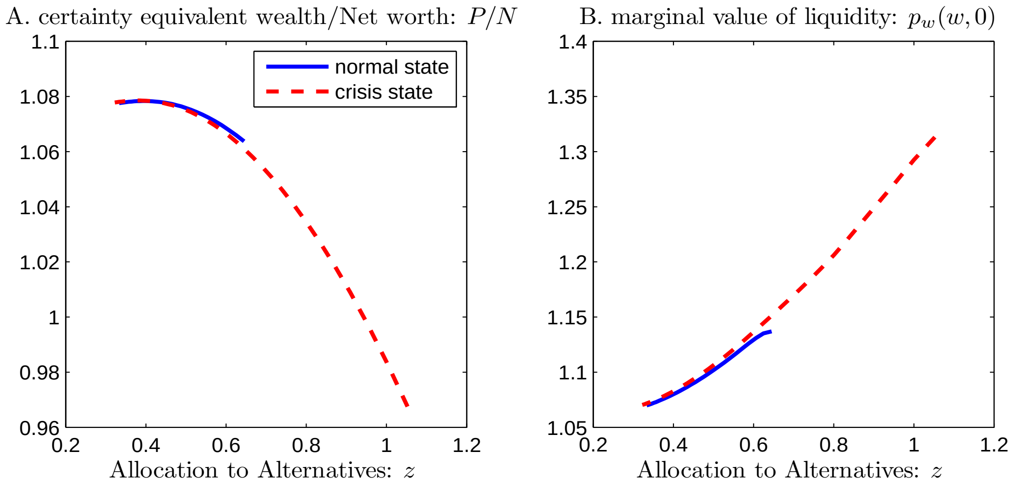

Figure 4: This figure demonstrates the effect of a crisis state on the investor's certainty equivalent wealth and the marginal value of liquidity. We plot $P/N = p(w,t)/(w+1)$ (Panel A) and $p_w(w,t)$ (Panel B) at $t=0$ as functions of the alternative assets $z = K/N$ for the $i=1$ case. The parameter values are: $\theta_X^g = \theta_X^b = 0.02$, $\theta_L^g = 0.1, \theta_L^b = 0.25$, $\xi_g = 0.1, \xi_b = 0.5$, $call = 0.2$, and $\lambda = 0.1$, with an implied payout of $\delta_T = 21.34\%$ every six years ($H = 6$).

the downward jump losses (for both public equity and the alternative asset) in the crisis state (by setting $Z = 1$).

Panel A of Figure 4 plots $P/N = p(w)/(w+1)$, the ratio of the certainty equivalent wealth $P(W,K) = p(w)K$ and net worth $N = W + K$ at time $t = 0$, on the y-axis, as a function of the percentage allocation to alternative assets ($z = K/N$) on the x-axis.

The solid blue curve plots results for the normal state and the red dashed line plots results for the crisis state. The most striking aspect is the extremely wide range of the alternative asset allocation in the crisis state, indicating high reluctance to rebalance in this state. This occurs not only because of the much higher proportional cost, but also because of the option value of waiting for a possible regime switch back to the good state when the transaction cost is lower.

Figure 4 shows that, for a *fixed* allocation to alternative assets (between 0.33 and 0.65), the investor's utility is approximately the same in both states (Panel A) and so is the marginal (certainty equivalent) value of liquidity $p_w(w, t=0)$ (Panel B).[21] However

[21]Note that if the allocation to alternatives exceeds 93%, the ratio P/N falls below one, indicating

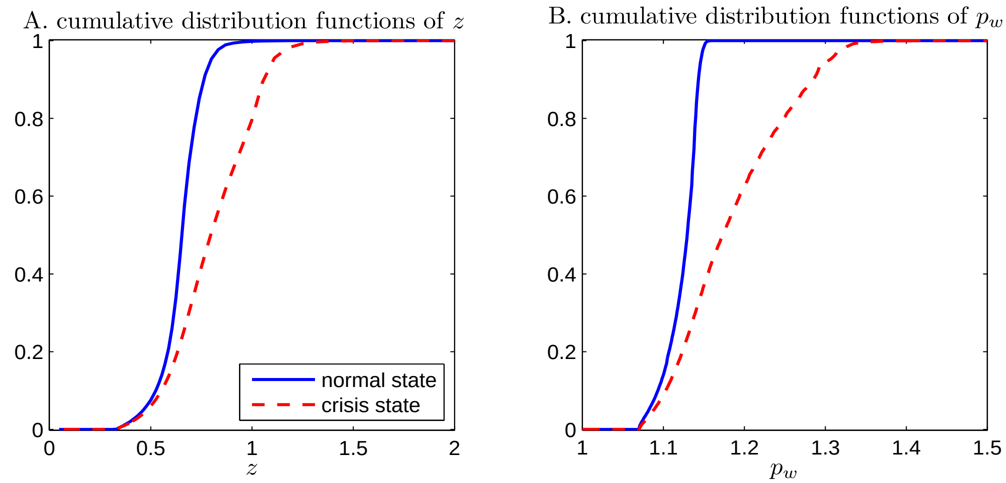

Figure 5: This figure plots the stationary cumulative distributions of the percentage allocation to alternative assets $z = K/(W + K)$ (Panel A) and the marginal value of liquidity $p_w(w,t)$ (Panel B) in both the normal and crisis states for the $i = 1$ case. The parameter values are: $\theta_X^g = \theta_X^b = 0.02$, $\theta_L^g = 0.1, \theta_L^b = 0.25$, $\xi_g = 0.1, \xi_b = 0.5$, $call = 0.2$, and $\lambda = 0.1$ with an implied payout of $\delta_T = 21.34\%$ every six years ($H = 6$).

the distributions of z and marginal value of liquidity are significantly different in the two states, which we discuss next using Figure 5.

In Figure 5, we plot the cumulative distribution functions in the two states for the allocation to the illiquids, z, (Panel A) and the marginal value of liquidity, $p(w_t, t = 0)$ (Panel B). Panel A shows that the distribution of the allocations to alternatives z in the crisis state first-order stochastically dominates the distribution of z in the normal state. That is, the probability to draw a large value of z is higher in the crisis state than in the normal state. Panel B shows that the distribution of the the marginal value $p_w(w,t)$ in the crisis state first-order stochastically dominates that the distribution of $p_w(w,t)$ in the normal state. That is, in the sense of first-order stochastic dominance, the marginal value of liquidity $p_w(w,t)$ is higher in the crisis state than in the normal state, again indicating that the cost of illiquidity is larger in crisis.

Because the distributions of z in the two states are very different, the marginal value of liquidity which measures the cost of investing in alternatives also differs significantly.

that the investor would be willing to *permanently* give up the opportunity to invest in the alternative asset if it were feasible for the investor to costlessly liquidate the alternative asset holdings.

For example, the marginal value of liquidity is $p_w(w, t = 0) = 1.28$ at the 75^{th} percentile of the distribution of z, ($z = 0.95$), in the crisis state, which is economically much larger than $p_w(w, t = 0) = 1.13$ at the 75^{th} percentile of the distribution of z, ($z = 0.69$), in the normal state. This result indicates the high value of liquidity in a crisis.

9. Conclusion

The endowment model, an investment strategy of high allocations to illiquid alternative assets, is widely used by many institutional and high-net-worth investors. We build on the framework of modern portfolio theory to develop a dynamic portfolio-choice model with illiquid alternative assets to analyze conditions under which the endowment model does and does not work. We capture the illiquidity of the alternative asset as follows. First, a fraction of the alternative asset periodically matures and becomes fully liquid, and the investor can benefit from liquidity diversification by holding alternative assets maturing at different dates. Second, the investor can voluntarily buy and sell the illiquid asset at any time by paying a transaction cost. Third, the alternative asset's risk is not fully spanned by publicly traded assets.

We model how investors can engage in liquidity diversification by investing in multiple illiquid alternative assets with staggered lock-up expirations. We show that such liquidity diversification results in higher allocations to alternative assets and higher investor welfare. We also show how illiquidity from lock-ups interacts with transactions costs in the secondary market to create endogenous and time-varying rebalancing boundaries.

Our extended model with crisis states captures stochastic capital calls and much higher secondary market transactions costs. We find that investors' holdings of alternative assets in crisis states often significantly deviate from the optimal target allocations and hence the utility loss from not being able to hedge stochastic calls and distributions can be large.

Appendices

A Public Equity and Bonds with No Alternatives

First, we summarize the solution for the complete-markets special case of our model where an investor with Duffie-Epstein-Zin recursive preferences has the standard investment opportunities defined by the public equity's risky return process given by (1) and a risk-free bond that pays a constant rate of interest r. The investor dynamically adjusts consumption/spending and frictionlessly rebalances the portfolio to maximize the recursive preferences given in (8)-(9). Note that the investor only has liquid wealth W. The following proposition summarizes the solution for this frictionless benchmark.

Proposition 2 *The investor allocates a constant fraction, denoted by π, of wealth W_t to public equity, i.e., the total investment amount in public equity is $\Pi = \pi W$ where*

$$\pi = \frac{\eta_S}{\gamma \sigma_S} = \frac{\mu_S - r}{\gamma \sigma_S^2}\,. \tag{A.1}$$

Note that the optimal asset allocation rule is the same as that in Merton (1969, 1971). Specifically, the EIS has no effect on π in this frictionless benchmark. The optimal spending C_t is proportional to wealth W_t: $C_t = \phi_1 W_t$ where ϕ_1 is given in (17). Note that the optimal spending rule depends on both risk aversion γ and the EIS ψ, which is different from Merton (1969, 1971). The investor's value function $J(W)$ is given by:

$$J(W) = \frac{(b_1 W)^{1-\gamma}}{1-\gamma}\,, \tag{A.2}$$

where b_1 is a constant given by:

$$b_1 = \zeta^{\frac{\psi}{\psi-1}} \phi_1^{\frac{1}{1-\psi}}\,. \tag{A.3}$$

Next, we analyze the general case where the investor can also invest in illiquid alternative assets in addition to public equity and bonds.

B Proof for Proposition 1

Optimal Policy Functions and PDE for $p(w,t)$. We conjecture that the value function $V(W,K,t)$ takes the following form:

$$V(W,K,t) = \frac{(b_1 P(W,K,t))^{1-\gamma}}{1-\gamma} = \frac{(b_1 p(w,t) K)^{1-\gamma}}{1-\gamma}\,, \tag{B.1}$$

where b_1 is given in (A.3). Substituting (B.1) into the consumption FOC given in (13) and the asset-allocation FOC given in (14), we obtain (22) for the scaled consumption rule $c(w,t)$ and (23) for the scaled asset allocation in public equity $\pi(w,t)$, respectively. Finally, substituting the conjectured value function given in (B.1) and the consumption and asset-allocation policy rules, given in (22) and (23), into the HJB equation (12), we obtain the PDE (24) for the certainty equivalent wealth $p(w,t)$.

Lower Liquidation Boundary $\underline{W_t}$ and Upper Acquisition Boundary $\overline{W_t}$. Let (W_t, K_t) denote the investor's time-t holdings in public equity and the alternative asset, respectively. We use Δ to denote the amount of the illiquid alternative asset that the investor is considering to liquidate. The investor's post-liquidation holdings in public equity and the alternative asset, are equal to $K_t - \Delta$ and $W_t + (1-\theta_L)\Delta$, respectively. Because the investor's value function is continuous before and after liquidation, we have

$$V(W_t + (1-\theta_L)\Delta, K_t - \Delta, t) - V(W_t, K_t, t) = 0\,. \tag{B.2}$$

Dividing (B.2) by Δ and letting $\Delta \to 0$, we obtain under differentiability:

$$\begin{aligned} 0 &= \lim_{\Delta\to 0}\frac{1}{\Delta}\left[V(W_t + (1-\theta_L)\Delta, K_t - \Delta, t) - V(W_t + (1-\theta_L)\Delta, K_t, t)\right] \\ &\quad + \lim_{\Delta\to 0}\frac{1-\theta_L}{\Delta(1-\theta_L)}\left[V(W_t + (1-\theta_L)\Delta, K_t, t) - V(W_t, K_t, t)\right] \\ &= -V_K(W_t, K_t, t) + (1-\theta_L)V_W(W_t, K_t, t)\,. \end{aligned} \tag{B.3}$$

The preceding equation implicitly defines the boundary $\underline{W_t}$, in that

$$V_K(\underline{W_t}, K_t, t) = (1-\theta_L)V_W(\underline{W_t}, K_t, t)\,. \tag{B.4}$$

The optimality of $\underline{W_t}$ implies that the derivatives on both sides of (25) are equal. Therefore,

$$V_{KW}(\underline{W_t}, K_t, t) = (1-\theta_L)V_{WW}(\underline{W_t}, K_t, t)\,. \tag{B.5}$$

Substituting the value function given by (B.1) into (B.4), we obtain (25). Similarly, Substituting the value function given by (B.1) into (B.5), we obtain (26).

By using the homogeneity property, we obtain the following: $P_W(W_t, K_t, t) = p_w(w_t, t)$, $P_{WW}(W_t, K_t, t) = p_{ww}(w_t, t)/K_t$, $P_K(W_t, K_t, t) = p(w_t, t) - p_w(w_t, t)w_t$, and $P_{WK}(W_t, K_t, t) = -W_t p_{ww}(w_t, t)/K_t^2$. Substituting these expressions into (25) and (26), we obtain

$$p(\underline{w}_t, t) - p_w(\underline{w}_t, t)\underline{w}_t \;=\; (1-\theta_L)p_w(\underline{w}_t, t) \tag{B.6}$$

$$-p_{ww}(\underline{w}_t, t)\underline{w}_t/K_t \;=\; (1-\theta_L)p_{ww}(\underline{w}_t, t)/K_t\,. \tag{B.7}$$

Simplifying these two equations, we obtain (27) and (28). We can derive the boundary conditions for $\overline{W_t}$ and $\overline{w}_t$ by using essentially the same procedure as the above.

The preceding proof is applicable to the upper and lower barriers for all t such that $t \neq mT$. To complete our analysis for $t = mT$. we need to incorporate the automatic liquidity event that takes place $t = mT$.

Value and Decisions at $t = mT$. When there is an automatic liquidity event at $t = mT$, it is possible that without active rebalancing, the automatic liquidity can cause the portfolio to be overly exposed to liquid assets. In this case, i.e., when $\widehat{w}_{mT} > \overline{w}_{mT}$, the investor may choose to reduce the liquid asset holding.

Suppose that the investor optimally purchases Λ units of the alternative asset such that $\widehat{W}_{mT} - (1+\theta_X)\Lambda = \overline{W}_{mT}$ and the liquidity ratio is then equal to

$$\overline{w}_{mT} = \frac{\widehat{W}_{mT} - (1+\theta_X)\Lambda}{\widehat{K}_{mT} + \Lambda} = \lim_{t\to mT} \frac{W_t + \delta_T K_t - (1+\theta_X)\Lambda}{K_t - \delta_T K_t + \Lambda}, \tag{B.8}$$

Solving the above equation gives the following expression yields the number of units for the alternative asset, $\Lambda = \lambda K_{mT-}$, that the investor plans to purchase at $t = mT$, where λ is given by

$$\lambda = \lim_{t\to mT} \frac{w_t + \delta_T - \overline{w}_{mT}(1-\delta_T)}{1+\theta_X + \overline{w}_{mT}}. \tag{B.9}$$

C Full-Spanning with Liquid Alternative Asset

In this appendix, we summarize the full-spanning case where the alternative asset is fully liquid. An investor with Duffie-Epstein-Zin recursive preferences has three investment opportunities: (*a.*) the public equity whose return process is given by (1), (*b.*) a risk-free bond that pays a constant rate of interest r, and (*c.*) the risky liquid alternative asset. The investor dynamically adjusts consumption/spending and frictionlessly rebalances the portfolio to maximize the recursive preferences given in (8)-(9). Note that the investor's wealth is fully liquid. The following proposition summarizes the solution for this frictionless benchmark.

Proposition 3 *The investor continuously rebalances the portfolio so the investment in public equity,* Π, *and in the alternative asset,* K, *are proportional to net worth* N, *i.e.*

$$\Pi = \frac{\eta_S - \rho\eta_A}{\sigma_S\gamma(1-\rho^2)} N, \tag{C.1}$$

$$K = \frac{\alpha}{\gamma\varepsilon^2} N. \tag{C.2}$$

The remaining wealth, $N - (\Pi + K)$, *is allocated to the risk-free bond. The optimal consumption* C *is proportional to the net worth,* N: $C = \phi_2 N$ *where*

$$\phi_2 = \zeta + (1-\psi)\left(r - \zeta + \frac{\eta_S^2 - 2\rho\eta_S\eta_A + \eta_A^2}{2\gamma(1-\rho^2)}\right). \tag{C.3}$$

The investor's value function $\overline{V}(N)$ *is given by:*

$$\overline{V}(N) = \frac{(b_2 N)^{1-\gamma}}{1-\gamma} = J((b_2/b_1)N), \tag{C.4}$$

where b_2 *is a constant given by*

$$b_2 = \zeta^{\frac{\psi}{\psi-1}} \phi_2^{\frac{1}{1-\psi}}, \tag{C.5}$$

and $J(\cdot)$ *is the value function given in (A.2) for an investor who only has access to public equity and bonds.*

By comparing ϕ_2 given in (C.3) and ϕ_1 given in (17), we see that diversification ($|\rho| < 1$) and an additional risk premium $\eta_A > 0$ both make the investor better. By introducing a new risky (alternative) asset into the investment opportunity set, the investor is better off because $b_2 > b_1$. The second equality in (C.4) implies that $b_2/b_1 - 1$ is the fraction of wealth that the investor would need as compensation to permanently give up the opportunity to invest in the liquid alternative asset and instead invest only in public equity and the risk-free asset.

Proof for the Case of Full-Spanning with the Liquid Alternative Asset. Using the standard dynamic programming method, we have:

$$\begin{aligned} 0 &= \max_{C,\Pi,K} f(C,\overline{V}) + [rN + (\mu_S - r)\Pi + (\mu_A - r)K - C]\overline{V}_N \\ &\quad + \frac{(\Pi\sigma_S)^2 + 2\rho\Pi\sigma_S K\sigma_A + (K\sigma_A)^2}{2}\overline{V}_{NN}\,, \end{aligned} \tag{C.6}$$

and using the FOCs for Π, K and C, we have:

$$f_C(C,\overline{V}) = V_N\,, \tag{C.7}$$

$$\Pi = -\frac{\eta_S}{\sigma_S}\frac{V_N}{V_{NN}} - \frac{\rho\sigma_A}{\sigma_S}K\,, \tag{C.8}$$

$$K = -\frac{\eta_A}{\sigma_A}\frac{V_N}{V_{NN}} - \frac{\rho\sigma_S}{\sigma_A}\Pi\,. \tag{C.9}$$

We conjecture and verify that the value function takes the following form:

$$\overline{V}(N) = \frac{(b_2 N)^{1-\gamma}}{1-\gamma}\,. \tag{C.10}$$

Substituting (C.10) into the FOCs, we obtain $C = \zeta^{\psi} b_2^{1-\psi} N$, (C.1), and (C.2). Finally, substituting them into the HJB equation (C.6) and simplifying the expression, we obtain (C.3).

D Additional Details of Data and Calibration Inputs

This appendix provide details on the inputs and calculations for some of the calibration parameters used in the paper.

Sub-Asset Classes Calibrating the model requires the standard deviation, beta, and unspanned volatility of the representative alternative asset. To obtain these parameters, we build up from the standard deviations and correlations of the sub-asset classes comprising the representative alternative asset. For each sub-asset class a, we combine its β_a and R_a^2 with the standard deviation of the market $\sigma_S = 20\%$ to obtain the implied standard deviation for the asset class: $\sigma_a = \sqrt{\beta_a^2 \sigma_S^2 / R_a^2}$.

Table D1 shows the summary statistics for the more detailed sub-asset categories. Within *Alternative Allocations*, hedge funds has the largest allocation with an equal weighted average allocation of 16.7%. For all of the sub-asset classes the allocations increase with endowment size, particularly for the least liquid categories: private equity, venture capital, private real estate, and illiquid natural resources.

Table D1: Summary of Endowment Fund Asset Allocation Sub-Categories

This table summarizes endowment fund portfolios as of the end of the 2014-2015 academic year for 774 endowments. The first two columns show the equal and value weighted average, respectively. The columns 0-10% to 90-100% show averages within size-segmented groups of endowment funds. For example, the column "0-10%" shows the value weighted average portfolio allocation for the smallest decile of endowment funds. *Hedge Funds* includes managed futures. *Natural Resources* includes illiquid natural resources, such as timberland and oil & gas partnerships.

	Avg	VW Avg	0-10%	20-30%	45-55%	70-80%	90-100%
Cash & Equivalents	5.1%	4.0	7.2	3.5	5.6	3.8	3.5
Fixed Income	15.9%	8.7	25.8	18.2	16.8	11.4	7.4
Public Equity	50.7%	35.6	60.1	57.9	54.7	45.9	32.0
Hedge Funds	16.7%	23.4	4.6	13.0	14.3	22.3	23.8
Private Equity	4.6%	10.9	0.2	3.1	3.1	7.1	12.3
Venture Capital	1.7%	5.7	0.3	0.5	0.4	2.2	6.9
Private Real Estate	2.7%	6.1	0.4	1.8	2.7	3.1	7.0
Natural Resources	2.7%	5.9	0.8	1.9	2.0	4.2	6.7

Panel A of Table D2 shows the β_a, R_a^2, and σ_a for each of the alternative sub-asset classes. For hedge funds, the β and R^2 are taken from Getmansky et al. (2004) and account for return smoothing. For private equity and venture capital, the β and R^2 are taken from Ewens et al. (2013). For private real estate and illiquid natural resources, the variables are based on Pedersen et al. (2014) and account for return smoothing. Panel B of Appendix Table D2 shows the pairwise correlations between the asset classes, which are calculated using index returns over the period 1994-2015.[22] We combine the asset allocations from Appendix Table D1 with the data from Appendix Table D2 to impute portfolio β, σ, and unspanned volatility (ε). Panel C of Table D2 shows the imputed variables for the cross-section of endowment funds.

[22]The indexes are: Bloomberg/Barclays US Aggregate Bond Index, CRSP value weighted index, Credit Suisse/Tremont Aggregate Hedge Fund Index, Cambridge Associates U.S. Private Equity Index, Cambridge Associates U.S. Venture Capital Index, NCREIF Property Index (unsmoothed), and the S&P Global Timber and Forestry Index. For private equity, venture capital, private real estate, and illiquid natural resources the returns are quarterly; the other returns are monthly.

Table D2: Summary of Asset Class Risk and Correlations

Panel A shows β_a, R_a^2, and σ_a for each alternative asset class a. Panel B shows the pairwise correlations between these sub-asset classes. Panel C shows the implied parameters of the representative alternative asset: β_A is the beta, σ_A is the standard deviation, and ε is the unspanned volatility. The first two columns show results for the equal-weighted and value-weighted average portfolios. The remaining columns show allocations for size-segmented groups of endowments. e.g., the column "0-10%" shows the value-weighted statistics for the smallest decile of endowment funds.

Panel A

	β_a	R_a^2	σ_a
Hedge Funds (HF)	0.54	0.32	19.1%
Private Equity (PrivEqu)	0.72	0.32	25.4%
Venture Capital (VC)	1.23	0.30	45.1%
Private Real Estate (PrivRE)	0.50	0.49	16.0%
Natural Resources (NatRes)	0.20	0.07	17.0%

Panel B

	FixedInc	*PubEqu*	*HF*	*PrivEqu*	*VC*	*PrivRE*	*NatRes*
FixedInc	1						
PubEqu	0.02	1					
HF	0.16	0.64	1				
PrivEqu	-0.23	0.78	0.73	1			
VC	-0.18	0.46	0.52	0.66	1		
PrivRE	-0.13	0.35	0.31	0.51	0.17	1	
NatRes	0.04	0.87	0.67	0.70	0.46	0.44	1

Panel C

	Avg.	VW	0-10%	20-30%	45-55%	70-80%	90-100%
β_A	0.58	0.61	0.53	0.55	0.54	0.57	0.62
σ_A	18.1%	18.7	17.7%	17.7	17.3	18.2	18.9
ε	13.9%	14.2	14.1	13.9	13.5	14.2	14.3

Secondary Market Costs For voluntary liquidations, we assume that the proportional transaction cost is $\theta_L = 10\%$, based on empirical findings and the following back-of-the-envelope calculation: For secondary market liquidations of private equity, Kleymenova et al. (2012) and Nadauld et al. (2019) find average discounts of 25.2% and 13.8%, respectively. For secondary market liquidations of hedge funds, Ramadorai (2012) finds an average discount of 0.9%, which rises to 7.8% during the financial crisis. Therefore, we combine the aggregate endowment fund portfolio weights with liquidation costs of 20% for PE and VC, 1% for hedge funds, and 10% for private real estate and timberland, to obtain a proportional liquidation cost of 9.3% for the representative alternative asset. For acquisitions, we assume that the proportional acquisition cost is $\theta_X = 2\%$, which is equal to the average placement agent fee reported by Rikato and Berk (2015) and Cain et al. (2020).

E Calibrating the Payout Rates: δ_A and δ_T

We focus on the steady state in which the investor always has i distinct investments in the alternative asset at any time t. This is feasible provided the investor immediately replaces each investment that exits.

To simplify the exposition, assume that each investment's payoff structure involves only one contribution at its inception and one distribution upon its exit, and the horizon (or equivalently the lock-up period) of each investment is H. At the steady state, i/H investments mature each year, which means that there is one liquidity event every $T = H/i$ years. For example, if the lock-up period for each investment is $H = 6$ and there are three investment in the steady state ($i = 3$), then every two years ($T = 6/3 = 2$) an automatic liquidity event occurs. To ensure that the investor has three investment in the steady state, the investor immediately replaces the exited investment by making a new investment with a 6-year lock-up.

To ensure growth stationarity, we assume that both the growth rate of each investment, g_A, and the growth rate of the inception size for each investment (vintage), g_I, are constant. Consider a vintage-t investment, which refers to the investment that enters the portfolio at time t. Let IS_t denote the investment's initial size (IS) at inception. Its size at $(t + jT)$ is then $e^{g_A jT} IS_t$ where $j = 1, 2, \cdots i$ and hence the investment's size when exiting at time $t + H$ is $e^{g_A H} IS_t$.

At time $(t + H)$ the investor holds a total of i illiquid alternative investments ranging from vintage-t to vintage-$(t + (i - 1)T)$. Note that the value of the vintage-$(t + j)$ investment is $e^{g_A(i-j+1)T} \times (IS_t e^{g_I(j-1)T})$ as its inception size is $IS_t e^{g_I(j-1)T}$ and has grown at the rate of g_A per year for $(i - j + 1)T$ years. Summing across all vintages, we obtain:

$$\sum_{j=1}^{i} e^{g_A(i-j+1)T} \times (IS_t e^{g_I(j-1)T}) = e^{g_I H} IS_t \times \sum_{j=1}^{i} e^{(g_A - g_I)jT} .$$

The net payout at time $(t + H)$ is given by the difference between $e^{g_A H} IS_t$, the size of the exiting vintage-t investment, and $e^{g_I H} IS_t$, the size of the new vintage-$(t + H)$ investment. As the payout occurs once every T years, the annualized net payout rate is then:

$$\frac{1}{T} \frac{e^{g_A H} IS_t - e^{g_I H} IS_t}{e^{g_I H} IS_t \times \sum_{j=1}^{i} e^{(g_A - g_I)jT}} = \frac{1}{T} \frac{e^{(g_A - g_I)H} - 1}{\sum_{j=1}^{i} e^{(g_A - g_I)jT}} = \frac{1}{T} \left(1 - e^{-(g_A - g_I)T}\right) . \tag{E.1}$$

Next, we use this annualized net payout rate to calibrate δ_A and δ_T. Although, for the sake of generality, the model includes both δ_A and δ_T, in any single calibration we use only one of either δ_A or δ_T. Next, we provide three examples.

First, consider the case when $i \to \infty$, and with fixed finite holding period H for each investment, $T \equiv H/i \to 0$. Therefore, the investor continuously receives payout at a constant rate. This maps to the parameter δ_A in our model. By applying L'Hopital's rule to (E.1), we obtain, as one may expect, the following simple expression for the net payout rate,

$$\delta_A = g_A - g_I \,, \tag{E.2}$$

which is simply the difference between the incumbent investment growth rate g_A and the growth of the new investment's initial size g_I. For our calibration, we set $\mu_A = g_A = 9.6\%$ and $g_I = 5.6\%$ (approximately equal to the average endowment fund growth rate over the past 20 years) resulting in $\delta_A = 4\%$.

Second, consider the case when the investor has only one investment outstanding at each point in time. Then, $T = H$, the payout occurs once every H years, and we use δ_T to capture the payout rate for this case. That is, when T is relatively large, δ_T is given by:

$$\delta_T = 1 - e^{-(g_A - g_I)T} \,. \tag{E.3}$$

Note that δ_T as defined in the model is not annualized. Thus, with $g_A = 9.6\%$ and $g_I = 5.6\%$, for a single investment ($i = 1$) in the portfolio and $H = 6$, $\delta_T = 21.34\%$, which is equivalent to an annualized payout of 3.28%.

Third, consider an intermediate case when the investor has six distinct investment at each point in time. Then, we have $T = H/i = 6/6 = 1$, and we could use $\delta_T = \delta_1 = 1 - e^{-0.04} = 3.92\%$ to capture the payout. Alternatively, we could approximate with a continuous constant dividend yield by annualizing δ_T and using this annualized value as δ_A in the calibration. In this case, we would have $\delta_A \approx (1 + \delta_T)^{1/T} - 1 = \delta_T = \delta_1 = 3.92\%$ when $T = 1$. As one can see, the difference between the two approximations is not noticeable.

F Proofs for the Model with Financial Crisis and Stochastic Calls

Let $V^g(W_t, K_t, t)$ and $V^b(W_t, K_t, t)$ denote the value functions in the normal and crisis state, respectively. The HJB equation for the value function in the normal state, $V^g(W_t, K_t, t)$, is:

$$\begin{aligned} 0 \;=\; \max_{C,\Pi} \; & f(C, V^g) + (rW + \delta_A K + (\mu_S - r)\Pi - C)V^g_W + \frac{(\Pi\sigma_S)^2}{2} V^g_{WW} \\ & + V^g_t + (\mu_A - \delta_A) K V^g_K + \frac{\sigma_A^2 K^2}{2} V^g_{KK} + \rho \Pi K \sigma_S \sigma_A V^g_{WK} \\ & + \xi_g (V^b(W, K, t) - V^g(W, K, t)) \,. \end{aligned} \tag{F.1}$$

Similarly, the HJB equation for the value function in the crisis state, $V^b(W_t, K_t, t)$, is given by

$$
\begin{aligned}
0 \;=\; & \max_{C,\Pi}\; f(C, V^b) + (rW + \delta_A K + (\mu_S - r)\Pi - C)V^b_W + \frac{(\Pi\sigma_S)^2}{2}V^b_{WW} + V^b_t \\
& +(\mu_A - \delta_A)KV^b_K + \frac{\sigma_A^2 K^2}{2}V^b_{KK} + \rho\Pi K\sigma_S\sigma_A V^b_{WK} + \xi_b(V^g(W,K,t) - V^b(W,K,t)) \\
& +\lambda(\mathbb{E}(V^b(Z(W - \mathit{call})K), Z(1+\mathit{call})K, t)) - V^b(W,K,t)) \,. \qquad \text{(F.2)}
\end{aligned}
$$

Using the FOCs for Π and C, we obtain that same portfolio choice and consumption rules, given by (13) and (14), respectively, as for our baseline model.

Using the homogeneity property, we obtain the following two inter-connected ODEs for the investor's scaled certainty equivalent wealth:

$$
\begin{aligned}
0 = & \left(\frac{\phi_1\left(p^g_w(w,t)\right)^{1-\psi} - \psi\zeta}{\psi - 1} + \mu_A - \delta_A - \frac{\gamma\sigma_A^2}{2}\right)p^g(w,t) + p^g_t(w,t) + \frac{\varepsilon^2 w^2}{2}p^g_{ww}(w,t) \\
& + \left(\left(\delta_A - \alpha + \gamma\varepsilon^2\right)w + \delta_A\right)p^g_w(w,t) - \frac{\gamma\varepsilon^2 w^2}{2}\frac{(p^g_w(w,t))^2}{p^g(w,t)} + \frac{(\eta_S - \gamma\rho\sigma_A)^2 p^g_w(w,t)p^g(w,t)}{2\gamma^g_i} \\
& + \frac{\xi_g}{1-\gamma}\left(\left(\frac{p^b(w,t)}{p^g(w,t)}\right)^{1-\gamma} - 1\right)p^g(w,t) \qquad \text{(F.3)}
\end{aligned}
$$

and

$$
\begin{aligned}
0 = & \left(\frac{\phi_1\left(p^b_w(w,t)\right)^{1-\psi} - \psi\zeta}{\psi - 1} + \mu_A - \delta_A - \frac{\gamma\sigma_A^2}{2}\right)p^b(w,t) + p^b_t(w,t) + \frac{\varepsilon^2 w^2}{2}p^b_{ww}(w,t) \\
& + \left(\left(\delta_A - \alpha + \gamma\varepsilon^2\right)w + \delta_A\right)p^b_w(w,t) - \frac{\gamma\varepsilon^2 w^2}{2}\frac{(p^b_w(w,t))^2}{p^b(w,t)} + \frac{(\eta_S - \gamma\rho\sigma_A)^2 p^b_w(w,t)p^b(w,t)}{2\gamma^b_i} \\
& + \frac{\xi_b}{1-\gamma}\left(\left(\frac{p^g(w,t)}{p^b(w,t)}\right)^{1-\gamma} - 1\right)p^b(w,t) + \frac{\lambda}{1-\gamma}\left(\mathbb{E}\left(\frac{Z(1+\mathit{call})p^b(\frac{w-\mathit{call}}{1+\mathit{call}}, t)}{p^b(w,t)}\right)^{1-\gamma} - 1\right)p^b(w,t)\,. \qquad \text{(F.4)}
\end{aligned}
$$

Finally, the boundary conditions for (F.3) and (F.4) are the same as those in our baseline case. (We index the two parameters, θ_L and θ_X, with the states g and b, i.e., $\theta^g_L(\theta^b_L)$ and $\theta^g_X(\theta^b_X)$.)

References

Albuquerque, R., Cassel, J., Phalippou, L., Schroth, E., 2018. Liquidity provision in the secondary market for private equity fund stakes. Working paper, Boston College.

Ang, A., Ayala, A., Goetzmann, W.N., 2018. Investment beliefs of endowments. European Financial Management, 24, 3-33.

Ang, A., Papanikolaou, D., Westerfield, M.M., 2016. Portfolio choice with illiquid assets. Management Science, 60, 2737-2761.

Aragon, G.O., 2007. Share restrictions and asset pricing: Evidence from the hedge fund industry. Journal of Financial Economics, 83, 33-58.

Bansal, R., Yaron, A., 2004. Risks for the long run: A potential resolution of asset pricing puzzles. Journal of Finance, 59, 1481-1509.

Barber, B.M., Wang, G., 2013. Do (some) university endowments earn alpha? Financial Analysts Journal, 69, 26-44.

Barro, R.J., 2006. Rare disasters and asset markets in the twentieth century. Quarterly Journal of Economics, 121, 823-866.

Benzoni, L., Collin-Dufresne, P., Goldstein, R.S., 2007. Portfolio choice over the life-cycle when the stock and labor markets are cointegrated. Journal of Finance, 62, 2123-2167.

Bolton, P., Chen, H., Wang, N., 2013. Market timing, investment, and risk management. Journal of Financial Economics, 109, 40-62.

Bolton, P., Wang, N., Yang, J., 2019. Optimal contracting, corporate finance, and valuation with inalienable human capital. Journal of Finance, 74, 1363-1429.

Brown, G., Harris, R., Hu, W., Jenkinson, T., Kaplan, S.N., Robinston, D.T., 2021. Can investors time their exposure to private equity? Journal of Financial Economics, 139, 561-577.

Brown, J.R., Dimmock, S.G., Kang, J.K., Richardson, D., Weisbenner, S., 2011. The governance of university endowments: Insights from a TIAA-CREF institute survey. TIAA-CREF Research Institute Research Dialogue, 102.

Brown, J.R., Dimmock, S.G., Kang, J.K., Weisbenner, S., 2014. How university endowments respond to financial market shocks: Evidence and implications. American Economic Review, 104, 931-962.

Brown, K.C., Garlappi, L., Tiu, C., 2010. Asset allocation and portfolio performance: Evidence from university endowment funds. Journal of Financial Markets, 13, 268-294.

Cain, M.D., McKeon, S.B., Solomon, S.D., 2020. Intermediation in private equity: The role of placement agents. Journal of Financial and Quantitative Analysis, 55, 1095-1116.

Cochrane, H.H., Longstaff, F.A., Santa-Clara, P., 2008. Two trees. Review of Financial Studies, 21, 247-385.

Collett, D., Lizieri, C., Ward, C., 2003. Timing and holding periods in institutional real estate. Real Estate Economics, 31, 205-222.

Collin-Dufresne, P., Daniel, K., Moallemi, C., Saglam, M., 2012. Strategic asset allocation with predictable returns and transaction costs. Working paper, Ecole Polytechnic Federale de Lausanne.

Dai, M., Li, P., Liu, H., Wang, Y., 2015. Portfolio choice with market closure and implications for liquidity premia. Management Science, 62, 368-386.

Davis, M.H.A., Norman, A.R., 1990. Portfolio selection with transaction costs. Mathematics of Operations Research, 15, 676-713.

Dimmock, S.G., 2012. Background risk and university endowment funds. Review of Economics and Statistics, 94, 789-799.

Duffie, D., Epstein, L.G., 1992. Stochastic differential utility. Econometrica, 60, 353-394.

Dumas, B., 1991. Super contact and related optimality conditions. Journal of Economic Dynamics and Controls, 15, 675-685.

Eberly, J., Wang, N., 2009. Capital reallocation and growth. American Economic Review (Papers & Proceedings), 99, 560-566.

Epstein, L.G., Zin, S.E., 1989. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. Econometrica, 57, 937-969.

Ewens, M., Jones, C.M., Rhodes-Kropf, M., 2013. The price of diversifiable risk in venture capital and private equity. Review of Financial Studies, 26, 1853-1889.

Franzoni, F., Nowak, E., Phalippou, L., 2012. Private equity performance and liquidity risk. Journal of Finance, 67, 2341-2373.

Gallmeyer, M.F., Kaniel, R., Tompaidis, S., 2006. Tax management strategies with multiple risky assets. Journal of Financial Economics, 80, 243-291.

Gârleanu, N., 2009. Portfolio choice and pricing in illiquid markets. Journal of Economic Theory, 144, 532-564.

Gârleanu, N., Pedersen, L.H., 2013. Dynamic trading with predictable returns and transaction costs. Journal of Finance, 68, 2309-2340.

Gârleanu, N., Pedersen, L.H., 2016. Dynamic portfolio choice with frictions. Journal of Economic Theory, 165, 487-516.

Getmansky, M., Liang, B., Schwarz, C., Wermers, R., 2015. Share restrictions and investor flows in the hedge fund industry. Working paper, University of Massachusetts.

Getmansky, M., Lo, A.W., Makarov, I., 2004. An econometric model of serial correlation and illiquidity in hedge fund returns. Journal of Financial Economics, 74, 529-609.

Grossman, S.J., Laroque, G., 1990. Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods. Econometrica, 58, 25-51.

Kahl, M., Liu, J., Longstaff, F.A., 2003. Paper millionaires: How valuable is stock to a stockholder who is restricted from selling it? Journal of Financial Economics, 67, 385-410.

Kleymenova, A., Talmor, E., Vasvari, F.P., 2012. Liquidity in the secondaries private equity market. Working paper, London Business School.

Kreps, D. M., Porteus, E.L., 1978. Temporal resolution of uncertainty and dynamic choice theory. Econometrica, 46, 185-200.

Lerner, J., Schoar, A., Wang, J., 2008. Secrets of the academy: The drivers of university endowment success. Journal of Economic Perspectives, 22, 207-222.

Lerner, J., Schoar, A., Wongsunwai, W., 2007. Smart Institutions, foolish choices? The limited partner performance puzzle. Journal of Finance, 62, 731-764.

Lo, A.W., Mamaysky, H., Wang, J., 2004. Asset prices and trading volume under fixed transaction costs. Journal of Political Economy, 112, 1054-1090.

Longstaff, F.A., 2001. Optimal portfolio choice and the valuation of illiquid securities. Review of Financial Studies, 14, 407-431.

Longstaff, F.A., 2009. Portfolio claustrophobia: Asset pricing in markets with illiquid assets. American Economic Review, 99, 1119-1144.

Merton, R.C., 1969. Lifetime portfolio selection under uncertainty: The continuous-time case. Review of Economics and Statistics, 51, 247-257.

Merton, R.C., 1971. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory, 3, 373-413.

Metrick, A., Yasuda, A., 2010. The economics of private equity funds. Review of Financial Studies, 23, 2303-2341.

Nadauld, T.D., Sensoy, B.A., Vorkink, K., Weisbach, M.S., 2019. The liquidity cost of private equity investments: Evidence from secondary market transactions. Journal of Financial Economics, 132, 158-181.

Newell, G., Eves, C., 2009. The role of U.S. timberland in real estate portfolios. Journal of Real Estate Portfolio Management, 15, 95-106.

Pedersen, N., Page, S., He, F., 2014. Asset allocation: Risk models for alternative investments. Financial Analysts Journal, 70, 34-45.

Pindyck, R.S. and Wang, N., 2013. The economic and policy consequences of catastrophes. American Economic Journal: Economic Policy, 5(4), pp.306-39.

Ramadorai, T., 2012. The secondary market for hedge funds and the closed hedge fund premium. Journal of Finance, 67, 479-512.

Rikato, M., Berk, A.S., 2015. Costliness of placement agents. Journal of Financial Services Research, 48, 263-287.

Robinson, D.T., Sensoy, B.A., 2016. Cyclicality, performance measurement, and cash flow liquidity in private equity. Journal of Financial Economics, 122, 521-543.

Sadka, R., 2010. Liquidity risk and the cross-section of hedge-fund returns. Journal of Financial Economics, 98, 54-71.

Samuelson, P.A., 1969. Lifetime portfolio selection by dynamic stochastic programming. Review of Economics and Statistics, 51, 239-246.

Sorensen, M., Wang, N., Yang, J., 2014. Valuing private equity. Review of Financial Studies, 27, 1977-2021.

Swensen, D.F., 2000. Pioneering portfolio management: An unconventional approach to institutional investing. New York, NY, The Free Press.

Takahashi, D., Alexander, D., 2002. Illiquid alternative asset fund modeling. Journal of Portfolio Management, 28, 90-100.

Vayanos, D., 1998. Transaction costs and asset prices: A dynamic equilibrium model. Review of Financial Studies, 11, 1-58.

Wang, C., Wang, N., Yang, J., 2012, A unified model of entrepreneurship dynamics. Journal of Financial Economics, 106, 1-23.

Wachter, J.A., 2013. Can time-varying risk of rare disasters explain aggregate stock market volatility? Journal of Finance, 68, 987-1035.

Weil, P., 1990, Nonexpected utility in macroeconomics. Quarterly Journal of Economics, 105, 29-42.