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THE WELFARE EFFECTS OF TRANSPORTATION INFRASTRUCTURE IMPROVEMENTS

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The Welfare Effects of Transportation Infrastructure Improvements  
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**ABSTRACT**

We develop a general equilibrium geographic framework to characterize the welfare effect of transportation infrastructure investments. We tackle three distinct but conflating challenges: First, we offer an analytical characterization of the routing problem and, in particular, how infrastructure investment between any two connected locations decreases the total trade costs between all pairs of locations. Second, we characterize how this cost reduction affects welfare within a standard general equilibrium geography setup where market inefficiencies arise due to agglomeration and dispersion spillovers. Finally, we show how our framework admits analytical characterizations of traffic congestion, which creates a critical – albeit tractable – feedback loop between trade costs and the general equilibrium economic system. We apply these results to calculate the welfare effects of improving each of the thousands of segments of the U.S. national highway network. We find large but heterogeneous welfare effects, with the largest gains concentrated in metropolitan areas and along important trading corridors.

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# 1 Introduction

More than a trillion dollars is spent on investment in transportation infrastructure across the world each year (Lefevre, Leipziger, and Raifman, 2014). Is this money being well spent? This is a difficult question to answer, as evaluating the welfare effect of infrastructure improvements faces three paramount challenges: first, there is a “routing problem”: because agents endogenously choose how to travel along through the infrastructure network, an improvement along one segment can change trade costs across the entire network. Second, there is an “economic problem”: a change in trade costs impacts the distribution of economic activity (and hence welfare) in all locations through complex general equilibrium interactions, made all the more complicated by market failures created by the presence of agglomeration and dispersion spillovers. Finally, traffic congestion creates a critical feedback loop as infrastructure improvements change traffic patterns, affecting costs throughout the network, thereby affecting trade flows, which in turn impacts traffic patterns. This myriad of complex interactions makes evaluating the welfare impact of a particular infrastructure investment difficult, let alone choosing where to best target infrastructure improvements.

In this paper, we develop and implement a new framework that overcomes each of the aforementioned challenges, providing a tractable way of assessing the welfare impact of infrastructure improvements. In particular, this paper makes four contributions. First, we solve the “routing problem” by deriving an analytical expression on how an infrastructure investment between any two connected locations in an arbitrary infrastructure network decreases the cost of travel between all bilateral pairs. Second, we solve the “economic problem” by analytically characterizing how equilibrium welfare changes in response to a reduction in trade costs in the competitive equilibrium of a standard general equilibrium “gravity” spatial economic framework with agglomeration and dispersion spillovers. Third, we combine the routing framework with the general equilibrium spatial framework in the presence of traffic congestion – which creates a crucial feedback between the routing problem and the economic problem – to assess the welfare impact of infrastructure improvements. Finally, we apply our framework to the U.S. highway network, estimating several crucial model parameters using readily available data on observed traffic flows and calculating the welfare impact of improving each of the thousands of existing highway segments.

To tackle the routing problem, we assume the economy is composed of many locations arranged on a weighted graph. For each origin-destination pair, a continuum of heterogeneous traders each chooses their own optimal cost-minimizing route. Combining convenient results of extreme-value distributions popularized by Eaton and Kortum (2002) with results from graph theory, we derive an analytical expression for the expected (iceberg) trade cost between

all pairs of locations as a function of the underlying infrastructure network. In the special case when trader heterogeneity goes to zero, this expression is simply equal to the cost incurred along the least-cost route (which can be calculated using the algorithm presented in [Dijkstra \(1959\)](#), albeit without an analytical solution).

Our framework also allows us to derive two additional analytical expressions that exploit the analytical characterization of the routing problem but are essential for implementing our framework. First, we derive how any small change in the infrastructure network changes the expected trade cost between all pairs of locations. The expression is intuitive – the more “out of the way” a segment of the network, the smaller the impact on of its improvement on the iceberg trade cost of a bilateral pair – and provides a straightforward link between the routing problem and the economic framework to evaluate the welfare impact of an infrastructure network change. Second, for any number of traders moving between location pairs, we characterize the amount of traffic along all segments of the network. This result will prove helpful in exploiting traffic data in two ways: first, it allows us to introduce a traffic congestion into the framework (where the cost of traveling along an infrastructure segment depends on the amount of traffic); second, it will prove helpful when estimating the model using observed traffic flows.

To tackle the “economic problem”, we derive the elasticity of equilibrium welfare with respect to a small change in trade costs in a standard general equilibrium spatial framework with labor mobility as in [Allen and Arkolakis \(2014\)](#). In the absence of agglomeration and dispersion spillovers, the competitive equilibrium is efficient and the elasticity of equilibrium welfare to a change in bilateral trade costs is simply equal to the fraction of world trade between those locations.<sup>1</sup> However, in the presence of agglomeration and/or dispersion spillovers, the competitive equilibrium is not efficient, and we show that the response of welfare to trade costs carries a distortion term. The distortion term has a very interesting interpretation in itself: as the positive agglomeration spillovers become stronger the distortion term is increasingly determined by higher-order relationships in the network of locations. In particular, indirect connections obtain a higher weight in determining its value. Overall, in the presence of inefficiencies the entire set of trading connections between locations is necessary for the characterization of the effects of reducing transportation costs.

We then combine the routing and economic problems in the presence of traffic congestion (modeled as proposed by [Vickrey \(1967\)](#)). As more traffic travels along a segment of the network, the cost of traveling along that segment increases, inducing some traders to

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<sup>1</sup>This argument can be shown to be extended in other spatial frameworks without labor mobility where efficiency holds as in [Anderson \(1979\)](#); [Anderson and Van Wincoop \(2003\)](#); [Allen, Arkolakis, and Takahashi \(2018\)](#).

opt to take alternative routes, impacting the cost of traveling along other segments in the network. The extent to which costs increase along each segment depend on the number of traders traveling from each origin to each destination (i.e. trade flows) which are in turn impacted by changes in travel costs along the network, creating a feedback loop between the “economic problem” and the “routing problem.” Despite these complex interactions, we are able to provide analytical characterizations of how welfare changes in response to a small improvement of any segment of the infrastructure network described in form of sufficient statistics observed in the data and model parameters.

Finally, we apply our framework to analyze the welfare impact of improving each of the nearly 7,000 segments of the U.S. National Highway System connecting nearly 900 cities. Using the observed traffic flows and travel times along each segment, we estimate traders are not very heterogeneous (i.e. most take a route that is close to optimal), but traffic congestion spillovers are substantial. Reassuringly, model-predicted traffic flows are strongly correlated with observed traffic flows and, as an out of sample test, the model implied bilateral trade flows between U.S. cities, when aggregated, correlate well with observed state to state trade flows.

Using our estimated parameters we find large and heterogeneous welfare impacts of improving the U.S. highway network. While the welfare gains of adding 10 additional lane-miles range from \$10 to \$20 million for three quarters of the highway segments, we estimate substantially larger gains for segments within metropolitan areas and along important travel corridors, with the returns exceeding \$500 million for two highway segments in the New York City metropolitan area. For every highway segment, these benefits exceed the costs of construction and maintenance (estimated based on local topography), with the greatest returns on investment for highways within the New York City and Los Angeles metropolitan areas and in central Indiana (aptly nicknamed the “Crossroads of America”). Interestingly, the presence of market failures due to agglomeration, dispersion spillovers and traffic congestion have important implications for which segment improvements have the greatest welfare impact.

This paper is connected to different strands of the literature related to transportation economics and infrastructure evaluation. First, there is long literature dating back to [Fogel \(1962, 1964\)](#) that evaluates the economic impact of infrastructure improvements. Indeed, we show that, in the absence of externalities, our finding that the welfare elasticity to a change in bilateral trade costs is equal to the value of trade between those locations, the general equilibrium equivalent of the celebrated “Social Savings” formula Fogel heuristically derives.<sup>2</sup> More recently, there is a burgeoning field evaluating the general equilibrium impacts

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<sup>2</sup>[Crafts \(2004\)](#) in a partial equilibrium setup where incomes are constant, formally shows that the Fogel

of infrastructure investment, e.g. [Donaldson \(2012\)](#), [Allen and Arkolakis \(2014\)](#), [Donaldson and Hornbeck \(2016\)](#) in an inter-city context and the work of [Ahlfeldt, Redding, Sturm, and Wolf \(2015\)](#) in an intra-city context; see [Redding and Turner \(2014\)](#) and [Redding and Rossi-Hansberg \(2017\)](#) for excellent reviews. Relative to this literature, we offer a “routing” framework that allows us to analytically solve for the impact of an improvement to the transportation network in a gravity framework without having to rely on computational methods (such as Dijkstra’s algorithm or the “Fast Marching Method” pioneered by [Osher and Sethian \(1988\)](#)), which allows us to incorporate traffic congestion in a tractable manner.

Second, there is also a long theoretical literature in transportation economics examining agent’s optimal routing decisions; these tools are comprehensively summarized in [Galichon \(2016\)](#). However, the bulk of the existing empirical transportation economics literature is focused on combining agent based (utility maximizing or rule-based) econometric approaches that describe the demand for transportation needs with rule-based descriptions of the supply of transportation and the associated traffic.<sup>3</sup> While the details of these models vary, the endogenous routing choice oftentimes limits their applicability to a few stylized examples. In addition, the only interaction of agents decisions is through transportation demand and supply; other prices are explicitly taken as given so that the literature altogether avoids the general equilibrium modeling. Relative to this literature, we embed a routing problem into a spatial general equilibrium framework.

Third, there is a small (but growing) literature studying spatial policy in general equilibrium.<sup>4</sup> [Ossa \(2014\)](#) and [Ossa \(2015\)](#) follow a computational approach to characterize optimal tariff and tax policies in spatial frameworks, while [Allen, Arkolakis, and Takahashi \(2018\)](#) and [Allen, Arkolakis, and Li \(2015\)](#) derive first order necessary conditions for the characterizations of the optimal investment in infrastructure. [Fajgelbaum and Gaubert \(2018\)](#) characterize the optimal transfers across locations in a general equilibrium framework in the presence of market failures due to agglomeration and congestion forces with many types of labor. Relative this literature, we derive the analytical relationship of how equilibrium welfare responds to changes in bilateral trade frictions in the presence of market failures due to agglomeration and dispersion forces and traffic congestion and show these relationships can be expressed as functions of observables (e.g. trade flows).

Most closely related to this paper are two recent working papers, each of which evalu-

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formula provides an upper bound of consumer gains from price improvement.

<sup>3</sup>See for example Chapter 10 on [De Palma, Lindsey, Quinet, and Vickerman \(2011\)](#) for a review and [De Palma, Kilani, and Lindsey \(2005\)](#) and [Eluru, Pinjari, Guo, Sener, Srinivasan, Copperman, and Bhat \(2008\)](#) for realistic large-scale micro-econometric models of urban systems with traffic.

<sup>4</sup>There is also an expanding literature evaluating infrastructure investment using credible instrumental variables variation, see for example [Baum-Snow \(2007\)](#), [Michaels \(2008\)](#), and [Duranton and Turner \(2011, 2012\)](#).

ates optimal infrastructure policy in general equilibrium with rich geographies. Alder et al. (2014) use a general equilibrium spatial model similar to the one we employ and applies a heuristic algorithm to derive an approximation to the global optimal transportation network. Fajgelbaum and Schaal (2016) consider a spatial equilibrium model with externalities whereby they assume traffic congestion with associated optimal Pigouvian taxes that turn the equilibrium to an efficient one and the optimization problem to a convex optimization problem. We make two contributions relative to these papers: first, we derive an analytical relationship between the infrastructure network and the resulting iceberg trade costs from the optimal routing problem, resulting in a tractable method incorporating traffic congestion into a quantitative general equilibrium gravity framework; second, we derive analytical expressions for how welfare responds to (small) infrastructure improvements in the competitive equilibrium in the presence of market failures. However, unlike both these papers, we consider small improvements to existing infrastructure networks rather than solving for the globally optimal transportation network.

The remainder of the paper proceeds as follows. The next section presents the routing framework for the endogenous transportation costs and derives a number of key properties. Section 3 embeds this routing framework into a spatial economic model and derives the elasticity of aggregate welfare in the presence of agglomeration, dispersion, and congestion externalities. Section 4 uses the framework to evaluate the welfare impact of improving each segment of the U.S. highway network. Section 5 concludes.

## 2 Endogenous transportation costs

In this section, we describe how to calculate the transportation costs between any two locations, accounting for the fact that agents endogenously choose the least cost route between locations.

### 2.1 Setup

Consider a world composed of a finite number of locations  $i \in \{1, \dots, N\}$ . These locations are organized on a weighted graph with an associated *infrastructure matrix*  $\mathbf{T} = [t_{ij} \geq 1]$ , where  $t_{ij}$  indicates the iceberg trade cost incurred from moving *directly* from  $i$  to  $j$  (if  $i$  and  $j$  are not directly connected in the graph, then  $t_{ij} = \infty$ ). The top left panel of Figure 1 provides an example of such a network, where  $N = 25$  and locations are arrayed in a two-dimensional grid, with locations that are directly connected (i.e.  $t_{ij}$  is finite) if they are adjacent and not connected (i.e.  $t_{ij}$  is infinite) otherwise.

Trade between  $i$  and  $j$  is undertaken by a continuum of traders  $\nu \in [0, 1]$  who travel along (endogenously chosen) paths to get from  $i$  to  $j$ . A *path*  $p$  between  $i$  and  $j$  is a sequence of locations beginning with location  $i$  and ending with location  $j$   $\{i = p_0, p_1, \dots, p_K = j\}$ , where  $K$  is the *length* of path  $p$ .<sup>5</sup> The aggregate trade cost from  $i$  to  $j$  on a path  $p$  of length  $K$ ,  $\tilde{\tau}_{ij}(p)$ , is the product of the instantaneous trade costs along the path:

$$\tilde{\tau}_{ij}(p) = \prod_{k=1}^K t_{p_{k-1}, p_k} \quad (1)$$

Each trader also incurs a path-specific idiosyncratic trade cost shock  $\varepsilon_{ij}(p, \nu)$ , so the total cost to trader  $\nu$  of traveling along path  $p$  between  $i$  and  $j$  is  $\tilde{\tau}_{ij}(p) \varepsilon_{ij}(p, \nu)$ . Let  $\tau_{ij}(\nu)$  indicate the cost trader  $\nu$  incurs optimally choosing the path between  $i$  and  $j$  to minimize the iceberg trade costs incurred:

$$\tau_{ij}(\nu) = \min_{p \in P_K, K \geq 0} \tilde{\tau}_{ij}(p) \varepsilon_{ij}(p, \nu).$$

The trade cost between locations  $i$  and  $j$  is *expected trade cost*  $\tau_{ij}$  from  $i$  to  $j$  across all traders as:

$$\tau_{ij} \equiv E_\nu [\tau_{ij}(\nu)]. \quad (2)$$

Notice that in the limit case of no heterogeneity all traders choose the route with the minimum deterministic trade cost, a case that is typically solved in the routing choice literature with the use of the Dijkstra algorithm (see e.g. [Donaldson \(2012\)](#)). The introduction of trader heterogeneity allows us to “convexify” the problem, which we will see – under appropriate assumptions on the idiosyncratic term – results in a tractable analytical solution.

Our baseline heterogeneous traders formulation instead departs from the tradition of the deterministic routing choice algorithms and bears resemblance to stochastic path-assignment methods used in transportation and computer science literature. As pointed out by [Prato \(2009\)](#) a common characteristic of those stochastic methods is that all solutions are heuristic. Even with heterogeneity most of those methods require the implementation of a minimum cost algorithm in advance (a la Dijkstra) in order to a-priori exclude a number of possible paths (e.g. by setting a cutoff of distance or maximum links away from the minimum cost one).

While, of course, we could follow either of these traditions we instead choose to consider all the possible paths and derive precise analytical formulas for the key routing choices,

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<sup>5</sup>Following the literature on graph theory, we assume that  $t_{ii} = \infty$  to exclude paths that stay in the same location; however, we allow traders shipping goods from  $i$  to  $i$  to choose the “null” path where they travel nowhere and incur no trade costs (which is the only admissible path of length 0).



which substantially reduces the computational burden and at the same time allows to convexify the problem.<sup>6</sup> This choice of modeling of the routing problem, constitutes the first computational advantage of our approach and it is twofold: First, it allows us to reduce a problem of large computational complexity (at least polynomial as Dijkstra typically needs to be implemented) to a problem where analytical solutions and immediate implementation is admissible.<sup>7</sup> Second, these analytical solutions can be exploited for the characterization of the traffic problem as we discuss below.

Our first object of interest is the elasticity of the trade cost to changes in infrastructure links in the graph. Given trader’s optimal route choice, an application of the envelope theorem immediately results in the following proposition:

**Proposition 1.** *The elasticity of trade costs between  $i$  and  $j$  to a change in the infrastructure link  $kl$  is:*

$$\frac{\partial \ln \tau_{ij}}{\partial \ln t_{kl}} = \pi_{ij}^{kl}, \quad (3)$$

where  $\pi_{ij}^{kl}$  is the fraction of trade costs incurred from  $i$  to  $j$  on paths through link  $k,l$ .

*Proof.* See Appendix 6.1. □

This expression implies that the elasticity of expected trade costs to investment in infrastructure is given by the share of trade costs spent in link  $k,l$  while traveling from  $i$  to  $j$ . While intuitive, it gives little chance to empirically assess these elasticities as it requires precise traffic flow measurements that document the precise path followed by each trip along with the associated costs incurred along each segment. While access to traffic data is widely available, these data typically reports the volume of traffic along a certain link without information on the origin and destination of each trip. Since our objective is to operationalize traffic flows and other information on transportation infrastructure that will allow us to measure the impact of infrastructure investment on trade costs, we proceed by parameterizing the distribution of path-specific idiosyncratic trade cost shock. We do so so that we

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<sup>6</sup>We should note that the concept of heterogeneous traders formulation has been utilized previously by [Allen and Atkin \(2015\)](#). However, their formulation does not consider the optimal route taken, as the the random trader shocks are realized on the expected cost matrix instead of the infrastructure matrix. The work of [Bell \(1995\)](#) on the other hand is the closer in the routing literature to our approach in that it considers all the paths to compute the minimum routing choice problem. This method is still heuristic, in that distributional assumptions are not made, and thus an analytical characterization is not obtained.

<sup>7</sup>In our case, by taking the logarithm of the above minimization problem, the program can be also expressed as a linear programming problem. This formulation allows the implementation of linear programming path-finding algorithms, such as the widely used A\* algorithm that focuses on likely paths. These algorithms can reduce the number of computations but nevertheless use heuristics to guide the search while still requiring the implementation of the Dijkstra method, resulting in relatively small computational gains (see for example [Zeng and Church \(2009\)](#)).

relate to a voluminous literature in spatial economics and also include the shortest path case analyzed by a Dijkstra as a limit case. For this, we view our formulation as relaxing the previous literature’s focus on the least cost route in order to better understand the impact of changes in the investment matrix on realized trade costs.

## 2.2 Analytical Characterization of Optimal routes

To derive an empirically implementable measure of the elasticity of trade costs to infrastructure investment in what follows, we assume that  $\varepsilon_{ij}(p, \nu)$  is Frechet distributed with shape parameter  $\theta > 0$ . The extent to which traders differ in which path they choose is now determined by the shape parameter  $\theta$ . In a sense, the parameter  $\theta$  can be considered as capturing the possibility of mistakes and randomness in the choice of routes, with higher values indicating greater agreement across traders. In the limit case of no heterogeneity,  $\theta \rightarrow \infty$ , all traders choose the route with the minimum aggregate trade cost. The previous literature in economics has focused exclusively on this limiting case and relied upon computational methods to calculate this least cost route (i.e. using the Dijkstra algorithm as in [Donaldson \(2012\)](#) and [Donaldson and Hornbeck \(2016\)](#) or the Fast Marching Method as in [Allen and Arkolakis \(2014\)](#)).

Let  $P_{ij,K}$  denote the set of all paths of length  $K$  that go from  $i$  to  $j$ . In our formulation we allow traders to choose any possible path to ship a good from  $i$  to  $j$  – including the most meandering of routes.<sup>8</sup> Using the familiar derivations discussed in [Eaton and Kortum \(2002\)](#) we can express the *expected trade cost*  $\tau_{ij}$  from  $i$  to  $j$  across all traders as:

$$\tau_{ij} \equiv E_{\nu} [\tau_{ij}(\nu)] = c \left( \sum_{K=0}^{\infty} \sum_{p \in P_K} \tilde{\tau}_{ij}(p)^{-\theta} \right)^{-\frac{1}{\theta}}, \quad (4)$$

where  $c \equiv \Gamma\left(\frac{\theta-1}{\theta}\right)$ . Substituting equation (1) into equation (4) yields:

$$\tau_{ij}^{-\theta} = c^{-\theta} \sum_{K=0}^{\infty} \sum_{p \in P_K} \prod_{k=1}^K t_{p_{k-1}, p_k}^{-\theta}.$$

In what follows, it is useful to characterize the *weighted adjacency matrix*  $\mathbf{A} = [a_{ij} \equiv t_{ij}^{-\theta}]$ . Note that  $a_{ij} \in [0, 1]$ , where 0 indicates there is no connection between  $i$  and  $j$ ,  $a_{ij} = 1$  indicates a cost-less connection, and  $a_{ij} \in (0, 1)$  indicates a costly connection. By summing over all paths of length  $K$ , we can write the expected trade cost in a more convenient form

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<sup>8</sup>Reassuringly, in the estimation results below, we find that traders are quite homogeneous so that the probability of taking any route that is not very close to optimal is exceedingly small.

by explicitly summing across all locations that are traveled to first, second, etc. as follows:

$$\tau_{ij}^{-\theta} = c^{-\theta} \sum_{K=0}^{\infty} \left( \sum_{k_1=1}^N \sum_{k_2=1}^N \dots \sum_{k_{K-1}=1}^N a_{i,k_1} \times a_{k_1,k_2} \times \dots \times a_{k_{K-2},k_{K-1}} \times a_{k_{K-1},j} \right),$$

where  $k_n$  is the sub-index for the  $n^{\text{th}}$  location arrived at on a particular path. This portion of the expression in the parentheses, however, is equivalent to the  $(i, j)$  element of the weighted adjacency matrix to the power  $K$ , i.e.:

$$\tau_{ij}^{-\theta} = c^{-\theta} \sum_{K=0}^{\infty} A_{ij}^K,$$

where  $\mathbf{A}^K = [A_{ij}^K]$ , i.e.  $A_{ij}^K$  is the  $(i, j)$  element of the matrix  $\mathbf{A}$  to the matrix power  $K$ .<sup>9</sup>

Furthermore, as long as the spectral radius of  $\mathbf{A}$  is less than one, the geometric sum can be expressed as:<sup>10</sup>

$$\sum_{K=0}^{\infty} \mathbf{A}^K = (\mathbf{I} - \mathbf{A})^{-1} \equiv \mathbf{B},$$

where we call  $\mathbf{B} = [b_{ij}]$  the *route cost matrix* (which is simply the Leontief inverse of the weighted adjacency matrix). Finally, the expected trade cost from  $i$  to  $j$  can be written as a simple function of the route cost matrix:

$$\tau_{ij} = cb_{ij}^{-\frac{1}{\theta}}. \quad (5)$$

Equation (5) provides an analytical relationship between any given infrastructure network and the resulting bilateral trade cost between all locations, accounting for traders choosing the least cost route.

## 2.3 Properties of the routing problem solution

In this subsection, we characterize key properties of the endogenous trade cost framework. We first characterize the elasticity of expected trade costs from location  $i$  to location  $j$  to an improvement in link  $t_{kl}$ :

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<sup>9</sup>Note that we could truncate the summation up to some finite  $K$  to restrict consideration to only paths that are not “too” long. It turns out that by allowing for all possible  $K$  admits a more convenient analytical solution. As we will see below, the inclusion of longer routes turns out not to be quantitatively important, as all traders end up taking routes that are optimal or very nearly so.

<sup>10</sup>A sufficient condition for the spectral radius being less than one is if  $\sum_j t_{ij}^{-\theta} < 1$  for all  $i$ . This will necessarily be the case if either trade costs between connected locations are sufficiently large, the adjacency matrix is sufficiently sparse (i.e. many locations are not directly connected), or the heterogeneity across traders is sufficiently small (i.e.  $\theta$  is sufficiently large).

**Proposition 2.** *Assume that trader shocks are Frechet distributed with parameter  $\theta$ . Then the elasticity of expected trade costs to an improvement in link  $kl$  can be expressed as:*

$$\frac{\partial \ln \tau_{ij}}{\partial \ln t_{kl}} = \frac{b_{ik} a_{kl} b_{lj}}{b_{ij}} = \left( c \frac{\tau_{ij}}{\tau_{ik} t_{kl} \tau_{lj}} \right)^\theta. \quad (6)$$

Moreover,  $\frac{\partial \ln \tau_{ij}}{\partial \ln t_{kl}}$  is also the probability a trader going from  $i$  to  $j$  uses link  $kl$ .

*Proof.* See Appendix 6.2. □

Equation (6) is intuitive: the numerator on the right hand side is the expected trade cost on the least cost route from  $i$  to  $j$ , whereas the denominator is the expected trade cost on the least cost route from  $i$  to  $j$  *through the transportation link  $kl$* . Hence, the more costly it is to travel through the link  $kl$  relative to the unconstrained least cost route, the smaller the effect an improvement in  $t_{kl}$  has on the reduction of trade costs between  $i$  and  $j$  and the less likely a trader is to use the link  $kl$ . Of course, because of proposition 1, both the elasticity of expected trade costs to the transportation link  $kl$  and the probability of using link  $kl$  on the way from  $i$  to  $j$  are equal to the fraction of costs incurred along the link, i.e.:

$$\frac{\partial \ln \tau_{ij}}{\partial \ln t_{kl}} = \pi_{ij}^{kl}. \quad (7)$$

Hence, the more “out of the way” the transportation link  $kl$  is from the optimal path between  $i$  and  $j$ , the less frequently that path is used, and the smaller the effect an improvement in  $t_{kl}$  has on the reduction of trade costs between  $i$  and  $j$ .

As an example, consider the network of locations presented in Figure 1. If we assume that all connected locations have the same direct cost  $t_{ij}$ , then the top right panel depicts the probability that an agent going from  $i = 1$  (bottom left) to  $j = 25$  (top left) travels over each connection. As is evident, connections along the direct diagonal route are chosen much more often than other routes, and connections near the diagonal route are chosen more often than those further away. As an additional example, the bottom left panel depicts the probability for agents going from  $i = 1$  (bottom left) to  $j = 15$  (top center). In this case, there are four different paths that share the same least deterministic costs. As is evident, traders are much more likely to use links on these routes than other links. However, there is also heterogeneity in the probability traveled across these different links – for example, the connection from 1 to 7 is traveled more often than the connection from 2 to 8. This occurs because there are many more possible paths going from 1 to 15 that use the 1 to 7 link than

that use the 2 to 8 link, so each trader is more likely to get an idiosyncratic draw that causes her to choose a path using the former link.

### 3 The welfare impact of transportation infrastructure improvements

We now introduce an general equilibrium spatial economic framework with agglomeration and dispersion economies. Using this framework we present a new characterization of how a change in any bilateral trade cost affects the endogenous aggregate welfare. We then combine this result with the optimal routing framework presented in Section 2 in order to characterize the welfare effect of changing the transportation infrastructure. Finally, we show how to extend the framework to allow for traffic congestion, which creates a feedback loop between the optimal routing and economic frameworks.

#### 3.1 A general equilibrium economic geography model

We assume that the world is inhabited by an exogenous measure  $\bar{L}$  of agents and by the locations as described in the previous section. On the consumption side, agents have constant elasticity of substitution (CES) with elasticity of substitution  $\sigma$  preferences over differentiated varieties produced in each location. To purchase goods, workers are randomly matched with traders and hence face trade costs equal to the expected trade costs  $\tau_{ij}$  above. With CES preferences of consumers, the value of goods shipped from location  $i$  to location  $j$  can be written as:

$$X_{ij} = p_{ij}^{1-\sigma} P_j^{\sigma-1} E_j, \quad (8)$$

where  $P_j = \sum_i p_{ij}^{1-\sigma}$  is the ideal Dixit-Stiglitz price index and  $E_j$  is the total expenditure of agents in location  $j$ . The welfare of the agents is given by

$$W_j = C_j \times u_j$$

where  $C_j$  is the CES aggregate consumption bundle in location  $j$  and  $u_j$  is the amenity of the location.

On the production side firms produce differentiated varieties in each location under perfect competition with a constant returns to scale technology that uses labor with productivity  $A_i$ . The workers in location  $i$  are compensated with a wage  $w_i$ . Given these assumptions,

the price for a unit of a good from  $i$  for a consumer in location  $j$  is

$$p_{ij} = \tau_{ij} \frac{w_i}{A_i}.$$

Substituted in expression (8) and using the fact that the only income is labor income, we obtain the standard gravity equation for bilateral flows,

$$X_{ij} = \left( \tau_{ij} \frac{w_i}{A_i} \right)^{1-\sigma} P_j^{\sigma-1} w_j L_j. \quad (9)$$

Finally, we assume, following a large literature in economic geography, that productivities and amenities are subject to spillovers. In particular, following [Allen and Arkolakis \(2014\)](#), we let  $A_i = \bar{A}_i L_i^\alpha$ , and  $u_i = \bar{u}_i L_i^\beta$  whereby  $\bar{A}_i, \bar{u}_i > 0$ . This specification allows for sufficient flexibility in modeling the externalities in production and amenities, while representing a broad range of explanations. Although not a specific requirement we will constraint ourselves to intuitive parametric regions whereby the agglomeration forces are positive and dispersion forces are negative,  $\alpha \geq 0 \geq \beta$  and  $\sigma > 1$ .

To close the model we assume that the following equilibrium conditions hold:

1. Total income  $Y_i$  is equal to total sales:

$$Y_i = \sum_{j=1}^N X_{ij}. \quad (10)$$

2. Total expenditure  $E_i$  is equal to total purchases:

$$E_i = \sum_{j=1}^N X_{ji}. \quad (11)$$

3. Trade balance

$$w_j L_j = Y_i = E_i. \quad (12)$$

Given trade balance, the welfare of a worker is given by  $W_i \equiv \frac{w_i}{P_i} u_i$ . Using this expression to substitute out for the price index and the third equilibrium to substitute for  $E_j$ , we can rewrite equation (8) for the value of bilateral trade as follows:

$$X_{ij} = \left( \frac{\tau_{ij}}{A_i u_j} \right)^{1-\sigma} w_i^{1-\sigma} W_j^{1-\sigma} w_j^\sigma L_j. \quad (13)$$

Substituting the gravity equation (13) and the equilibrium condition (12) into equilibrium

conditions (10) and (11) yields the following system of equilibrium conditions that along with a normalization and the labor adding up constraint,  $\sum_i L_i = \bar{L}$  determines wages and population in all locations along with the aggregate welfare  $W$ .

In the economic geography framework, the imposition of welfare equalization results in the following system of equilibrium equations:

$$w_i L_i = W^{1-\sigma} \sum_j \left( \frac{\tau_{ij}}{A_i u_j} \right)^{1-\sigma} w_i^{1-\sigma} w_j^\sigma L_j \quad (14)$$

$$w_i L_i = W^{1-\sigma} \sum_{j=1}^N \left( \frac{\tau_{ji}}{A_j u_i} \right)^{1-\sigma} w_j^{1-\sigma} w_i^\sigma L_i. \quad (15)$$

which can be solved for the equilibrium distribution of wages and population, along with the equilibrium welfare  $W$ . We normalize world income,  $Y^W = \sum_j w_j L_j = 1$ .

In what follows, we use the system of equilibrium conditions above to derive how changes in transportation infrastructure affects the equilibrium welfare of agents. As an intermediate step, we first derive the welfare effects of a changes in a particular (endogenous) trade cost.

### 3.2 Elasticity of welfare to changes in trade costs

We now examine how a change in bilateral trade costs affects the aggregate welfare. To do so, we consider a perturbation of the trade costs around the initial equilibrium. To build intuition we start with the baseline case of no externalities,  $\alpha, \beta = 0$ . In the absence of such externalities across markets, the equilibrium is efficient so an application of the envelope theorem to the corresponding planner's problem yields:

$$\frac{\partial \ln W}{\partial \ln \tau_{ij}} = X_{ij}. \quad (16)$$

Rearranging this formula and expressing it in differentials we obtain

$$\frac{\Delta W}{W} = \frac{p_{ij}}{\tau_{ij}} \frac{X_{ij}}{Y^W} \frac{\Delta \tau_{ij}}{p_{ij}} = \frac{\Delta \tau_{ij}}{\tau_{ij}} X_{ij} \implies$$

and thus

$$\underbrace{\frac{\Delta W}{W}}_{\text{social savings}} = \underbrace{p_{ij} \frac{\Delta \tau_{ij}}{\tau_{ij}}}_{\text{change in price}} \times \underbrace{X_{ij}/p_{ij}}_{\text{initial quantity}}$$

In our general equilibrium setup the result is intuitive as in the equilibrium more workers and more economic activity accumulate in routes with more trade and trade costs reductions proportionately apply to all of them.

This result implies that the impact of infrastructure investment can be summarized by the percentage cost savings offered by the investment multiplied by the volumes of initial trade. Notably, the formula for welfare gains is the same as the formula for the calculation of 'Social Savings' introduced by Fogel (1962, 1964) based on heuristic derivations.<sup>11</sup>

Nevertheless, the case of externalities,  $\alpha, \beta \neq 0$ , is much more intricate. This is because the competitive equilibrium is no longer efficient and so there is no corresponding planner's problem upon which the envelope theorem can be applied. Instead, we proceed by perturbing the system (14) and (15).<sup>12</sup> The following theorem characterizes the welfare gains in the general case:

**Theorem 1.** *The welfare gains from an improvement in a link are given by*

$$\frac{d \ln W}{d \ln \tau_{kl}} = X_{kl} (1 + \kappa_k + \nu_l) \quad (17)$$

where  $\kappa_k, \nu_k$  for  $k = 1, \dots, N$  are given by

$$1 + \kappa_k = (\beta + \alpha) \left( \frac{\sigma - 1}{\gamma_2} \right) \sum_i \frac{L_i}{\bar{L}} \left( (DG)_{ik,22}^{-1} + \frac{\sigma}{\sigma - 1} (DG)_{y,ik,21}^{-1} \right) \frac{1}{Y_k} \quad (18)$$

$$\nu_k = (\beta + \alpha) \left( \frac{\sigma - 1}{\gamma_2} \right) \sum_i \frac{L_i}{\bar{L}} \left( (DG)_{ik,12}^{-1} + \frac{\sigma}{\sigma - 1} (DG)_{ik,11}^{-1} \right) \frac{1}{E_k}, \quad (19)$$

we normalize  $\nu_N = 0$ , and  $(DG)_{ik,mn}^{-1}$  are the  $i, k$  elements of the  $m, n$  submatrix of the inverse of

$$DG = \begin{bmatrix} \frac{\sigma(\alpha+\beta)}{\gamma_2} \mathbf{I} & \boldsymbol{\mu} - \frac{1+\alpha}{\gamma_2} \mathbf{I} \\ \boldsymbol{\lambda}' - \frac{1-\beta}{\gamma_2} \mathbf{I} & \frac{(\sigma-1)(\alpha+\beta)}{\gamma_2} \mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \lambda_{1N} & \dots & \lambda_{N-1,N} & \lambda_{NN} + \frac{\beta}{\alpha} & \mathbf{0} \\ & & & & \mathbf{0} \end{bmatrix}$$

where  $\gamma_2 \equiv 1 + \alpha\sigma + \beta(\sigma - 1)$ ,  $\boldsymbol{\lambda} \equiv \left[ \frac{X_{ij}}{E_j} \right]$  is the matrix of import shares and  $\boldsymbol{\mu} \equiv \left[ \frac{X_{ij}}{Y_i} \right]$  is the matrix of export shares. Note that  $\kappa_k, \nu_k \rightarrow 0$  as  $\alpha, \beta \rightarrow 0$ .

<sup>11</sup>This formula can be derived as an upper bound of the equivalent variation for the gains from infrastructure improvement in partial equilibrium (see e.g. Crafts (2004)). As aggregate demand and aggregate supply respond to prices, the full general equilibrium derivation in our context yields the original Fogel formula.

<sup>12</sup>A perturbation of the system with  $\alpha, \beta = 0$  under symmetry,  $\tau_{ij} = \tau_{ji}$ , yields a one-equation linear system. The perturbation of this system yields the formula 16, a familiar result in Matrix Perturbation theory. Since our system is a multi-equation and highly non-linear an analytical derivation for the general case was not previously available.



*Proof.* See Appendix 6.3. □

This explicit characterization of welfare gains as a function of changes in expected trade costs constitutes the second computational advantage of our approach. It allows us to measure the welfare benefits of transportation costs for small changes in trade costs, even in the presence of externalities. With externalities,  $\kappa_k$  and  $\nu_l$  are the economic distortions that impact the welfare elasticity of trade costs. Notice that these distortions can be written solely as a function of model parameters and the trade links across locations. What is the interpretation of these distortions? To obtain intuition, consider the case of symmetric trade costs, where there is a single distortion for each location, i.e.  $\kappa_k \propto \nu_k$  and we can write:

$$1 + \kappa_k = (\rho - 2) \sum_{i=1}^N \left( \frac{L_i}{\bar{L}} \right) (D_y G)_{ik}^{-1},$$

where  $\rho \equiv \frac{2+\alpha-\beta}{1+(\sigma-1)\beta+\alpha\sigma}$ ,  $\gamma_1 \equiv 1 - \alpha(\sigma - 1) - \sigma\beta$ , and  $D_y G = \boldsymbol{\mu} + (1 - \rho)I = \boldsymbol{\mu} - \frac{\gamma_1}{\gamma_2}I = \frac{\gamma_1}{\gamma_2} \left( I - \frac{\gamma_2}{\gamma_1} \boldsymbol{\mu} \right)$  in the case of unique equilibrium  $\alpha + \beta < 0$ . All in all, in we can express:

$$1 + \kappa_k = (\beta + \alpha) \frac{\sigma - 1}{\gamma_1} \sum_{i=1}^N \frac{L_i}{\bar{L}} \left[ 1 + \frac{\gamma_2}{\gamma_1} \mu_{ik} + \left( \frac{\gamma_2}{\gamma_1} \right)^2 \mu_{ik}^{(2)} + \left( \frac{\gamma_2}{\gamma_1} \right)^3 \mu_{ik}^{(3)} + \dots \right],$$

where  $\mu_{ij}^{(n)}$  is the  $i, j$  element of the  $n$ th power of the export matrix, which can be interpreted as the impact of higher order network effects on the welfare distortion. If agglomeration forces are equal to dispersion forces, i.e.  $\alpha + \beta = 0$ ,  $\gamma_2 = \gamma_1$  and the weights of all ordered effects are equalized. As net agglomeration forces (i.e.  $\alpha + \beta$ ) increase, however,  $\gamma_2/\gamma_1$  increases, meaning that the market distortions depend more on higher order effects of spatial propagation.

### 3.3 Elasticity of welfare to transportation infrastructure improvements

In the absence of traffic congestion externalities, a simple application of the product rule allows us to determine the elasticity of aggregate welfare to infrastructure improvements by combining the elasticity of welfare to bilateral trade costs and the elasticity of bilateral trade costs to infrastructure improvements:

$$\frac{d \ln W}{d \ln t_{ij}} = \sum_{k=1}^N \sum_{l=1}^N \frac{d \ln W}{d \ln \tau_{kl}} \times \frac{d \ln \tau_{kl}}{d \ln t_{ij}}. \quad (20)$$

Note that because traders are optimally choosing their routes, any change in the infrastructure link  $ij$  can potentially affect any bilateral trade flows  $kl$ , which is why we sum across all  $kl$  pairs. From Proposition 1 and Theorem 1, this expression becomes:

$$\frac{d \ln W}{d \ln t_{ij}} = \sum_{k=1}^N \sum_{l=1}^N X_{kl} \times \pi_{kl}^{ij} \times (1 + \kappa_k + \nu_l). \quad (21)$$

There are two things to note about of equation (21). First, given model parameters, all elements comprising the right hand side are in principle observable in the data, namely trade flows, bilateral expenditures on each link, and populations. In the absence of spillovers (i.e.  $\alpha, \beta = 0$ ), we have  $\kappa_k = \nu_l = 0$  for all  $k$  and  $l$ , so that  $\frac{d \ln W}{d \ln t_{ij}} = \sum_{k=1}^N \sum_{l=1}^N X_{kl} \times \pi_{kl}^{ij}$ , i.e. the elasticity of aggregate welfare to an infrastructure improvement along link  $ij$  is equal simply to the total value of goods being shipped over that link.

As an example, return to the two dimensional grid network considered in Figure 1 and suppose all locations have identical productivities and amenities and labor is mobile. The bottom right panel of Figure 1 depicts the elasticity of welfare to a change in each direct connection. As is evident, reducing the cost of traveling over links in the center of the grid have larger impacts on welfare than those in the periphery. Intuitively, this is for two reasons: first, the connections in the center are more likely to be traveled than on the periphery (see the bottom left panel of Figure 1) and hence will have larger effects on bilateral trade costs; second, because they are more centrally located, locations in the center will have greater populations and economic output, so that the trade flows flowing through the central links will be larger on average. As a result, improvements to those links will have larger effects on welfare.

### 3.4 Traffic and congestion

Up until this point, the optimal routing problem and the economic geography model could be considered separately, as there is no interaction between the elasticity of trade costs to infrastructure improvements and the elasticity of welfare to trade costs; this is why equation (20) can be derived immediately from the product rule. In the presence of congestion, however, this is no longer the case: if the amount of trade flowing over a link affects the cost of trading along that link, then the impact of infrastructure improvements will depend in part on how trade adjusts to the new infrastructure. For example, the “fundamental law of road congestion” of [Duranton and Turner \(2011\)](#) suggests that infrastructure improvements will result in negligible reductions in the cost of travel, as they will result in additional traffic

(trade) flowing on the network.<sup>13</sup>

It turns out that with several straightforward assumptions, the analytical framework developed above can easily admit traffic congestion. As we will see, the introduction of congestion creates a complex (but tractable) interplay between the optimal routing problem and the general equilibrium economic framework. Moreover, incorporating congestion both substantially improves the model’s ability to match observed traffic flows and has important implications on which infrastructure improvements result in the greatest welfare gains.

To begin, as above we assume that trader shocks are Frechet distributed; under this assumption, the value of goods flowing over each link has a convenient analytical solution. Define the  $N \times N$  traffic matrix  $\Xi$  whose  $ij^{th}$  element is the value of the flow of goods over link  $ij$ , i.e.  $\Xi_{ij} = \sum_{k=1}^N \sum_{l=1}^N X_{kl} \pi_{kl}^{ij}$ . We then have the following Corollary:

**Corollary 1.** *When trader shocks are Frechet distributed, the traffic matrix has the following analytical representation:*

$$\Xi = \mathbf{A} \odot \mathbf{B}' (\mathbf{X} \oslash \mathbf{B}) \mathbf{B}', \quad (22)$$

where  $\mathbf{X}$  is observed matrix of trade flows and the “ $\odot$ ” and “ $\oslash$ ” indicate the Hadamard product and division operators, respectively, i.e. element by element multiplication/division.

Note that this expression holds regardless of whether or not there is traffic congestion. In addition to proving essential for incorporating congestion, it will also allow us to estimate the parameters governing trade costs using readily observed traffic data when city-to-city trade flows (and link specific expenditure shares) are unobserved.

Second, we assume that the trade cost incurred along the direct connection from  $i$  and  $j$  is a function of the time it takes to travel from  $i$  to  $j$ ,  $time_{ij}$ :

$$t_{ij} = \exp(\kappa \times time_{ij}), \quad (23)$$

where  $c_1 > 0$  and  $time_{ij}$  be the time (in hours) it takes to travel along link  $ij$ . The exponential functional form has been used extensively in the economic geography literature, and has a number of attractive properties including 1. traveling costs are always greater or equal to one, i.e.  $t_{ij} \geq 1$ ; 2. Given expression (1), conditional on travel time, the number of locations through which a trader passes does not affect the trade costs that trader incurs; and 3. the effect of a reduction in travel time on the percentage change in welfare is proportional to the welfare elasticity to a change in the infrastructure cost, i.e.  $\frac{\partial \ln W}{\partial time_{ij}} = \kappa \frac{\partial \ln W}{\partial \ln t_{ij}}$ .

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<sup>13</sup> Indeed, in the following framework it is possible to construct examples where an infrastructure improvement along one segment actually reduces aggregate welfare due to congestion externalities, reminiscent of the famous Braess’ paradox, see e.g. Frank (1981).

Third, to incorporate congestion, we follow a popular specification suggested by [Vickrey \(1967\)](#) by assuming that the time per unit distance (i.e. inverse speed) is related to the traffic (per lane-mile) by the following function:

$$\frac{time_{ij}}{dist_{ij}} = \delta_0^{-1} + \delta_1 \left( \frac{\Xi_{ij}}{dist_{ij} \times lanes_{ij}} \right)^{\delta_2}, \quad (24)$$

where  $\delta_0$  is the unimpeded speed of traffic and  $\delta_1$  and  $\delta_2$  are parameters that govern the strength of congestion; if  $\delta_1 > 0$  and  $\delta_2 > 0$ , then it will take a greater amount of time to travel a particular distance the greater the traffic  $\Xi_{ij}$ . As we will see below, this functional form fits the observed relationship between travel time and traffic remarkably well.

Combining these three assumptions with equation (5) yields the following expression for the expected trade costs between any two locations:

$$\tau_{ij}^{1-\sigma} = c \left( \mathbf{I} - \left[ \exp \left( -\frac{c_1}{\delta_0} - c_3 \left( \frac{\Xi_{ij}}{dist_{ij} \times lanes_{ij}} \right)^{\delta_2} \right) \right] \right)^{c_2}, \quad (25)$$

where  $c_1 \equiv \theta \times \kappa$ ,  $c_2 \equiv \frac{\sigma-1}{\theta}$ , and  $c_3 \equiv \theta \times \kappa \times \delta_1$ . Note that equation (25) implicitly defines the trade costs, as traffic  $\Xi_{ij}$  depends in trade costs in all locations (see equation (22)). Intuitively, traffic congestion creates a feedback loop, where greater traffic on one segment of the infrastructure network increases the cost of traveling along that segment. The immediate impact through the optimal routing problem is to divert traffic to other parts of the network, thereby affecting costs elsewhere. However, the extent to which costs elsewhere are impacted depends on the amount of traffic, which in turn depends on trade flows: this is the feedback from the economic model to the routing model. However, by improving a portion of the infrastructure network, the changes in costs will result in changes in equilibrium economic activity and trade flows: this is the feedback from the routing model to the economic model.

Despite this feedback loop, we can still derive the welfare impact of an infrastructure improvement. To do so, we apply the chain rule as follows:

$$\frac{d \ln W}{d \ln lanes_{mn}} = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \frac{d \ln W}{d \ln \tau_{kl}} \times \frac{d \ln \tau_{kl}}{d \ln t_{ij}} \times \frac{d \ln t_{ij}}{d \ln lanes_{mn}}.$$

An increase in the number of lanes on the infrastructure link between  $m$  and  $n$  will potentially impact the congestion between  $i$  and  $j$ , thereby affecting the cost of traveling along the link between  $i$  and  $j$ . However, once we have characterized these (complicated) congestion spillovers, the rest of the calculation is reasonably straightforward: changes in costs along links will impact expected trade costs between all locations, each of which will affect aggregate

welfare.

Applying the inverse function theorem to equation (24) yields the following expression for the congestion externalities:

$$\frac{d \ln t_{ij}}{d \ln lanes_{kl}} = \{A^{-1}\}_{ij,kl} \times \alpha_2 c_2 dist_{kl} \left( \frac{\Xi_{kl}(\mathbf{t})}{dist_{kl} \times lanes_{kl}} \right)^{\alpha_2}$$

where  $\mathbf{A}$  is an  $N^2 \times N^2$  matrix with elements:

$$A_{ij,kl} = \mathbf{1}_{k=i,l=j} - \alpha_2 c_2 dist_{ij} \times \left( \frac{\Xi_{ij}(\mathbf{t})}{dist_{ij} \times lanes_{ij}} \right)^{\alpha_2} \frac{d \ln \Xi_{ij}(\mathbf{t})}{d \ln t_{kl}}$$

and

$$\frac{d \ln \Xi_{ij}(\mathbf{t})}{d \ln t_{kl}} = \sum_{m=1}^N \sum_{n=1}^N \frac{X_{mn} \pi_{mn}^{ij}}{\sum_{k=1}^N \sum_{l=1}^N X_{kl} \pi_{kl}^{ij}} \left( \sum_{k'l'} \frac{d \ln X_{mn}}{d \ln \tau_{k'l'}} \pi_{k'l'}^{kl} + \theta (\pi_{mn}^{kl} - \pi_{mi}^{kl} - \pi_{jn}^{kl}) - \theta \times \mathbf{1}_{i=k,j=l} \right).$$

While complicated, note that the expression depends only on model parameters and observables (e.g. trade flows).<sup>14</sup>

## 4 The welfare effects of improving the U.S. National Highway System

We now use our framework to evaluate the welfare impacts of improving the U.S. National Highway System. The U.S. National Highway System is a network of strategic highways within the United States, including the Interstate Highway System and other roads serving major transport facilities and it constitutes the largest highway system in the world. The main backbone of the National Highway System, the Interstate Highway System, is one of the world's largest infrastructure megaprojects in history (Kaszynski, 2000). It took more than thirty five years to construct at an estimated cost \$560 billion (in 2007 dollars), and total annual maintenance costs are approximately \$130 billion (CBO, 1982; FHA, 2008; NSTIFC, 2009). However, little is known about the relative importance of different segments of the highway system in terms of how each affects the welfare of the U.S. population. Such knowledge is crucial for appropriately targeting future infrastructure investments.

<sup>14</sup>We refer the interested reader to Allen, Arkolakis, and Takahashi (2018) for the analytical expression for  $\frac{d \ln X_{mn}}{d \ln \tau_{k'l'}}$ , which itself depends only on observed trade flows, the trade elasticity, and the strength of agglomeration and dispersion forces.

## 4.1 Data

Data on the length, location, number of lanes, and average annual daily traffic (AADT) of all 1,029,142 segments of the highway system in the continental United States comes from the National Highway Planning Network data-set published by the Federal Highway Association (FHA, 2015), which itself is a compilation of geo-coded datasets from each state’s department of transportation. We combine this data set with the location, population, and median-income of all U.S. cities located within 10 kilometers from the continental U.S. highway network with a population of at least 50,000 people from Edwards (2017); these 893 cities comprise 140 million people, slightly less than half of the total continental U.S. population. The top panel of Figure 2 depicts the highway network and the location of all 893 cities, where the size of the city indicates its population (larger dots indicate greater population) and the color of the highway segment indicates the amount of traffic (blue indicates little traffic, red indicates large amounts of traffic).

To construct the adjacency matrix of the highway network, we use the Network Analyst Toolbox in ArcGIS to calculate the shortest route along the highway network from each U.S. city to each of its 25 nearest neighbors; as long as that route does not pass through the center of another city, we identify those two cities as adjacent.<sup>15</sup> In total, we identify 6,924 links in the adjacency matrix. For each link, we calculate the average AADT, average number of lanes, and total distance across all segments along the shortest route between the cities. Note that this procedure does not impose symmetry and allows for the traffic and number of lanes to differ depending on the direction traveled. The bottom panel of Figure 2 depicts the resulting adjacency matrix between all 893 cities, where color of the links indicate the average amount of traffic over the link (blue indicates little traffic, red indicates large amounts of traffic). For each link, we recover the time of travel from the HERE API using the georoute Stata command by Weber and Péclat (2017).

Finally, to compare the benefits of improving each link of the U.S. highway system to the cost of doing so, we estimate the cost of improving each link by its local topography. In particular, we classify every point in the continental U.S. as belonging to one of four categories: urban, rural-flat (with a slope of less than 5%), rural-rolling (with a slope between 5% and 15%), and rural-mountainous (with a slope greater than 15%). From Exhibit A-1 of (FHA), the construction cost per lane-mile is \$5.598 million in urban areas, \$6.492 million in rural-mountainous areas, \$2.085 million in rural-rolling areas, and \$1.923 million in rural flat-

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<sup>15</sup>The choice of 25 nearest neighbors was made to balance Type I and Type II error; ensuring that nearly all nearby cities were connected without identifying as adjacent city pairs whose shortest route skirt around cities in between. However, the discernible reader will note that the resulting adjacency matrix does not perfectly reflect the actual U.S. highway network; such small errors are unavoidable in any easily replicable process that constructs an adjacency matrix from a road network with many nodes and edges.

areas. For every link, we calculate the average per lane-mile cost by weighting each of these four categories by the percentage of the link traveling over that category. Finally, assuming a (linear) 20 year depreciation (as in Appendix C of [Office of the State Auditor \(2002\)](#)) and a 5% cost of capital, the construction cost is also equal to the annualized construction plus maintenance cost of constructing ten lane-miles along each link. Panel A of Figure 7 depicts this estimated cost, overlaid against the raster identifying the classification of each point of land; as is evident, construction costs are highest in urban segments and on longer stretches near the Rocky Mountains.

## 4.2 Estimation

To estimate the welfare impact of improving the U.S. highway system, we need to have values for the trader shape parameter  $\theta$ , the parameter mapping travel time to trade cost  $\kappa$ , the congestion parameters  $\{\delta_0, \delta_1, \delta_2\}$ , the trade elasticity  $\sigma - 1$ , and the strength of productivity and amenity externalities,  $\alpha$  and  $\beta$ . While there exists a substantial trade and economic geography literature estimating the trade elasticity and the productivity and amenity externalities (upon which we will rely), we will estimate the remaining parameters.

One limitation of using such a detailed representation of the U.S. Highway network between all U.S. cities with more than 50,000 people is that bilateral trade flows between these cities are unobserved. However, traffic flowing along each link of the highway is readily observed. The basic idea of our estimation procedure is to find parameters such that the model predicted value of trade flowing over each segment of the highway network most closely matches the observed traffic on that segment. In order to do so, we assume that the observed traffic over a particular link (measured in vehicles) is proportional to the underlying value of trade on that link:

$$\Xi_{ij} = \lambda \times \text{traffic}_{ij},$$

where  $\lambda > 0$ . This assumption – while strong – is perhaps weaker than it first appears. For example, one might think that this assumption is violated if a greater fraction of traffic within cities is due to commuting rather than the movement of goods. However, we can interpret the act of commuting as a trade from the locations of a worker’s residence to her workplace, as the model is silent about whether a worker trades her labor directly or just the good she produced with her labor.<sup>16</sup> Similarly, the act of an individual driving to a store to go shopping can be interpreted as trade flowing from the location of the store to the worker’s

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<sup>16</sup>[Adao, Costinot, and Donaldson \(2017\)](#) discuss formally an equivalence of an exchange economy to a Ricardian production economy. Our model features externalities and the formal equivalence of our exchange economy with agglomeration and dispersion effects to a Ricardian production economy with externalities is discussed in [Adao, Arkolakis, and Esposito \(2017\)](#).

residence. We verify below the validity of our assumption by comparing the predicted trade flows of our model (estimated using traffic flows) to actual commodity trade flows from the Commodity Flow Survey.

Given this assumption, we first estimate the congestion parameters using observed travel times and traffic flows. Assume that unimpeded traffic can travel seventy miles per hour over each link of the U.S. highway system, i.e.  $\delta_0 = 70$ . Rearranging equation (24) then yields:

$$\log \left( \frac{time_{ij}}{dist_{ij}} - \frac{1}{70} \right) = \ln(\delta_1 \lambda^{\delta_2}) + \delta_2 \log \left( \frac{traffic_{ij}}{dist_{ij} \times lanes_{ij}} \right), \quad (26)$$

where  $traffic_{ij}$  is the observed average annual daily traffic along each link. Figure 3 shows that this log-linear relationship in equation (26) fits the observed relationship between speed of travel and traffic exceedingly well: a local polynomial non-parametric regression is very close to the assumed log-linear relationship, with an estimated  $\delta_2 = 0.4415$  (and a t-statistic of 119).

With an estimated  $\delta_2$ , we now turn to estimating the three (composite) parameters :  $c_1 \equiv \theta \times \kappa$ ,  $c_2 \equiv \frac{\sigma-1}{\theta}$ , and  $c_3 \equiv \theta \times \kappa \times \delta_1$  which, together with the traffic matrix  $\Xi$ , allow us to estimate the bilateral trade costs  $\{\tau_{ij}^{1-\sigma}\}$  using equation (25).

For intuition in how the procedure works, consider first the case where  $c_3 = 0$  so that there is no congestion and equation (25) offers an explicit equation for bilateral trade costs  $\{\tau_{ij}^{1-\sigma}\}$ . Given the observed income  $Y_i$  in each city  $i \in \{1, \dots, N\}$ , and a matrix for  $\{\tau_{ij}^{1-\sigma}\}$ , it is straightforward to show using the results of [Allen, Arkolakis, and Li \(2014\)](#) there exists a unique set of prices,  $p_i = w_i/A_i$ , (to-scale) in each location that ensures markets clear and trade is balanced, i.e. equations (10), (11), and (12) hold, given (9). As a result, there exists a unique set of trade flows  $\{X_{ij}\}$  (normalized so that world income is equal to one) consistent with observed incomes  $\{Y_i\}$ , market clearing conditions, and the trader optimization. Finally, from equation (22), this implies that there exists a unique traffic matrix of goods flows; write this traffic matrix as  $\Xi(\{c_1, c_2\}; \{dist_{kl}\}, \{lanes_{kl}\}, \{Y_k\})$ . Note that  $\Xi$  does not depend on the chosen values of the trade elasticity or the agglomeration forces  $\alpha$  and  $\beta$ , as these parameters do not affect the recovery of the bilateral trade flows given observed incomes and the trade cost matrix  $\{\tau_{ij}^{1-\sigma}\}$ . This discussion proves the following Proposition:

**Proposition 3.** *Assume that  $c_3 = 0$ . Then for any given pair  $\{c_1, c_2\}$  and for given observed data on incomes  $\{Y_i\}$ , distance  $\{dist_{kl}\}$ , and lanes  $\{lanes_{kl}\}$ , there is a unique traffic matrix that rationalizes those observed data.*

When there is congestion – i.e.  $c_3 \neq 0$  – the same argument can be made above given any initial guess for traffic  $\Xi^{(0)}$ , i.e. there exists a unique traffic matrix that rationalizes the



observed data given that initial guess. This allows us to iterate on the procedure to find a fixed point where the equilibrium traffic is consistent with the (implicit) function equation (25), the equilibrium conditions, and observed income, distance, and lanes:

$$\Xi(\{c_1, c_2, c_3\}; \{dist_{kl}\}, \{lanes_{kl}\}, \{Y_k\}) = \lim_{k \rightarrow \infty} \Xi^{(k+1)}(\{c_1, c_2, c_3\}; \{dist_{kl}\}, \{lanes_{kl}\}, \{Y_k\}; \{\Xi^{(k)}\}),$$

where we refer to each element as  $\Xi_{ij}(\{c_1, c_2, c_3\})$  for brevity.<sup>17</sup> We can then estimate  $c_1, c_2$ , and  $c_3$  using our assumption that observed traffic flows are proportional to model predicted value of trade traveling along a link by minimizing the difference between the observed and predicted traffic, where both are logged and de-meanded to difference out the unknown scalar  $\lambda$ :

$$\{c_1^*, c_2^*, c_3^*\} = \arg \min_{c_1, c_2, c_3} \sum_{i,j} \left( \ln \tilde{\Xi}_{ij}(\{c_1, c_2, c_3\}) - \left( \ln \tilde{traffic}_{ij} \right) \right)^2,$$

where the tildes indicate that each has been de-meanded by its sample means.

Figure 4 presents the results of the estimation procedure. In the top panel, we constrain  $c_3 = 0$  (i.e. there is no congestion). Even without congestion, the model predicted traffic flows correlate strongly with observed traffic flows (a correlation of 0.48). Our estimates are  $c_1^* = 16.6$  and  $c_2^* = 0.0394$ . With a trade elasticity of eight (i.e.  $\sigma = 9$ ) as in (Allen and Arkolakis, 2014), this implies  $\theta = 203$  and  $\kappa = 0.08$ , indicating very little heterogeneity across traders (i.e. most traders take the shortest route when traveling). As an illustration, Figure 6 depicts the probability a trader traveling from Seattle, WA to Manhattan borough in New York, NY uses each link; as is evident, all links except for those on the shortest (or nearly shortest) route have virtually zero probability of being used. A  $\kappa = 0.08$  implies that a one hour travel time incurs an ad-valorem equivalent trade cost of 8% (or a 4 hour trip from New York, NY to Washington D.C. an ad valorem equivalent trade cost of 37%).

In the bottom panel of Figure 4, we allow for congestion. While the estimates of  $c_1^*$  and  $c_2^*$  did not change much ( $c_1^* = 11.9$  and  $c_2^* = 0.0455$ , implying  $\theta = 175.8$  and  $\kappa = 0.07$  with a trade elasticity of eight), the data strongly reject no congestion, with  $c_3^* = 103$ . Moreover, the introduction of congestion substantially improves the fit of the model, reducing the distance between observed and predicted trade flows by more than 30% and improving the correlation between observed and predicted trade flows to 0.56. In what follows, we use the estimates with congestion as our preferred estimates.

As mentioned above, an advantage of our estimation approach is that it relies on readily available traffic data rather than difficult to obtain bilateral trade data between the 893 U.S. cities in our sample. Indeed, a by-product of our estimation procedure is the full

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<sup>17</sup>In practice, the fixed point is found after very few iterations.

matrix of estimated city-to-city level bilateral trade flows.<sup>18</sup> As a final test of our estimation procedure, we aggregate these city-to-city predicted trade flows up to the state-to-state level and compare them to the publicly available 2012 Commodity Flow Survey (CFS) data (CFS, 2012). Note that these bilateral commodity trade flows were not used as inputs or otherwise targeted in our estimation procedure. Figure 5 shows that the model’s predicted trade flows match well with the flows reported in the CFS, with a correlation (in logs) of intra-state trade flows 0.73 and a correlation (in logs) of inter-state trade flows of 0.47 (the correlations in levels are even higher).<sup>19</sup>

### 4.3 The welfare effects of improving the U.S. National Highway System

Finally, we estimate the welfare effects of improving the U.S. National Highway System. To do so, for each link, we calculate the percentage increase in aggregate welfare from adding ten additional lane-miles. We then convert this percentage welfare increase to a dollar amount using a compensating variation approach, i.e. how much would the the annual U.S. real GDP have to increase (in millions of chained 2012 U.S. dollars) holding prices constant to achieve the same welfare increase we estimate.

Note that the direct impact of adding additional lane-miles is to speed up travel by reducing congestion on a particular link (see equation (24)), thereby reducing trade costs on that link (see equation (24)). This reduction in congestion will result in traders endogenously altering which route they choose to get from an origin to a destination; moreover, by changing prices, wages, and where people reside, the amount of trade between each pair of locations will also change. The results that follow incorporates the entirety of this complex feedback process.

The bottom panel of Figure 7 depicts the welfare gains of adding ten additional lane-miles to each highway segment in the U.S., assuming an elasticity of substitution,  $\sigma$ , of nine, a productivity spillover  $\alpha = 0.1$ , and a amenity spillover  $\beta = -0.3$ . As is evident, the gains are both large and heterogeneous. Adding 10 lane-miles to any of the bottom bottom half of highway segments would result in welfare gains equivalent to \$10.2 - \$12.2 million a year. Benefits for the third quartile range from \$12.2 million to \$19 million. However, the best quartile of links, the estimated welfare impacts are much much larger, exceeding \$500 million

<sup>18</sup>This dataset is available for download on [Allen’s website](#).

<sup>19</sup>This strong correlation is not only driven by the predicted trade flows matching the observed city level incomes: the partial  $R^2$  of a regression of observed (log) state-to-state trade flows from the CFS on the model predicted (log) state-to-state trade flows with origin-state and destination-state fixed effects is 0.49, indicating that the model captures roughly half of the residual variation in observed bilateral trade flows even after conditioning on total exports from each origin and total imports from each destination.

annually in two cases in the New York City metropolitan area.<sup>20</sup>

For all highway links, the benefits of adding additional lane-miles substantially exceed the annualized costs of construction, which we estimate are between \$1.9 and \$5.6 million per year, depending on the local topography (see panel A). Where is the biggest bang for the buck? Figure 9 depicts the estimated rate of return for each link, measured as the log ratio of annualized benefit to annualized cost. Several patterns emerge. Given the large estimated welfare impacts for within greater metropolitan areas, these links also have the highest rates of return. For longer routes, however, the benefits are the largest if there are not many alternative routes or if the link connects major economic centers; for example, the gains of adding additional lanes to highway segments in Indiana are in the top decile of rates of returns (therefore justifying its slogan “the cross roads of America”). Consistent with these patterns, Table 1 list the ten highway segments with the greatest returns of investment: seven of the top 10 are in the New York City area, one is in the Los Angeles area, and two are in Indiana.

How does incorporating externalities and traffic congestion affect our estimates of how to best improve existing infrastructure? The top panel of Figure 8 takes our estimated parameters without congestion and compares the welfare elasticities of infrastructure improvements with and without externalities; the extremely high correlation indicates that including externalities without congestion does not substantially impact which highway links have the greatest welfare impact. The bottom panel of Figure 8 compares the welfare impacts with and without congestion (in both cases allowing for externalities). While there is a clear positive relationship in the estimates, there is also substantial disagreement on which links are most important. Hence, incorporating congestion not only improves the model fit as we saw above, it also has important implications on how to best target infrastructure improvements.

## 5 Conclusion

We have presented a tractable approach that combines elements of graph theory with general equilibrium spatial analysis to tackle three main challenges of evaluating infrastructure investment: the characterization of the routing problem of how infrastructure investment between any two locations affects the trade cost between all locations, the derivation of the elasticity of equilibrium welfare with respect to changes in trade costs, and the analytical characterizations of traffic congestion, which creates a critical – albeit tractable – feedback loop between the routing problem and the general equilibrium economic system. Our ap-

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<sup>20</sup>North Hempstead to Queens on 495W (\$719m annually) and White Plains to Greenburgh on 287W (\$510m annually).

proach allows the quantitative analysis of infrastructure investments to scale: we evaluated the welfare implications of changing thousands of links of the US National Highway System connecting hundreds of major US cities. Our approach has used a specific general equilibrium framework and it was oriented towards applied policy but focused on small infrastructure changes. However, the tools we have developed can be used to derive the first order necessary conditions of a global optimum in more generalized frameworks. We see it as an extremely fruitful future research avenue to apply these results to characterize globally optimal transportation networks in the presence of externalities.

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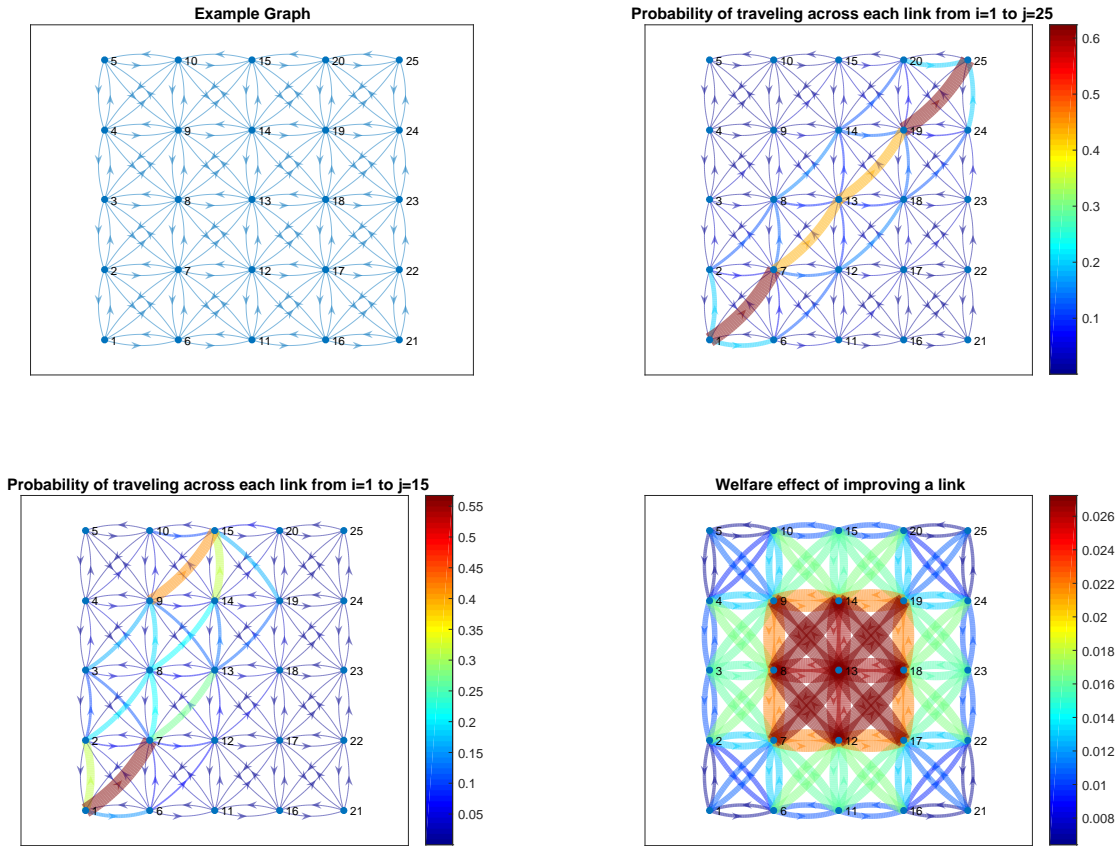
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# Tables and Figures

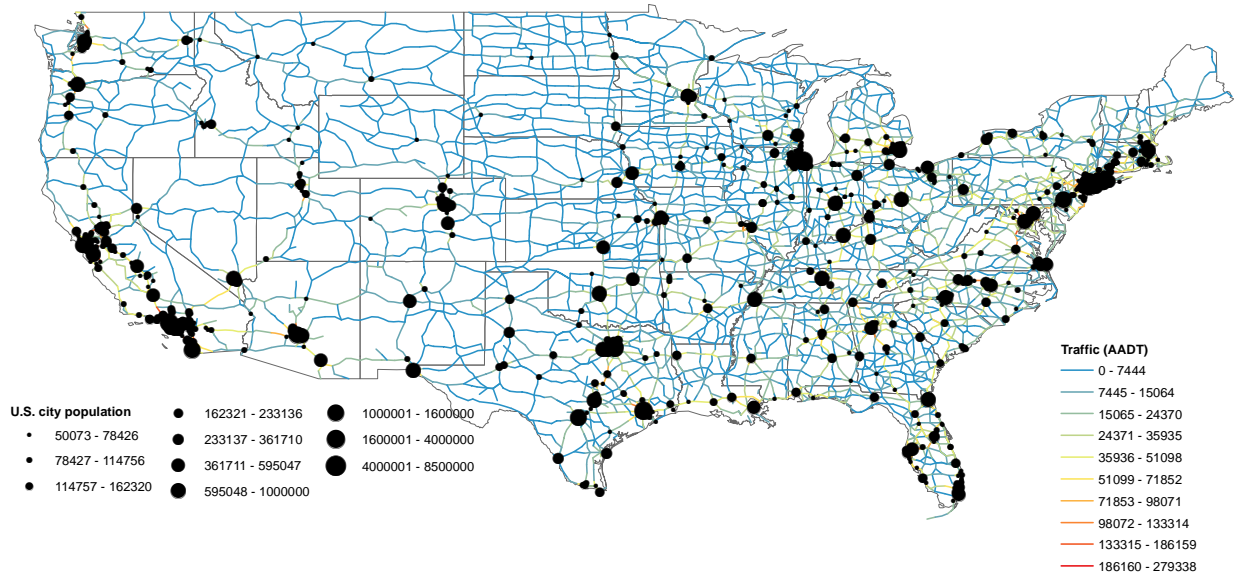
Figure 1: AN EXAMPLE GEOGRAPHY



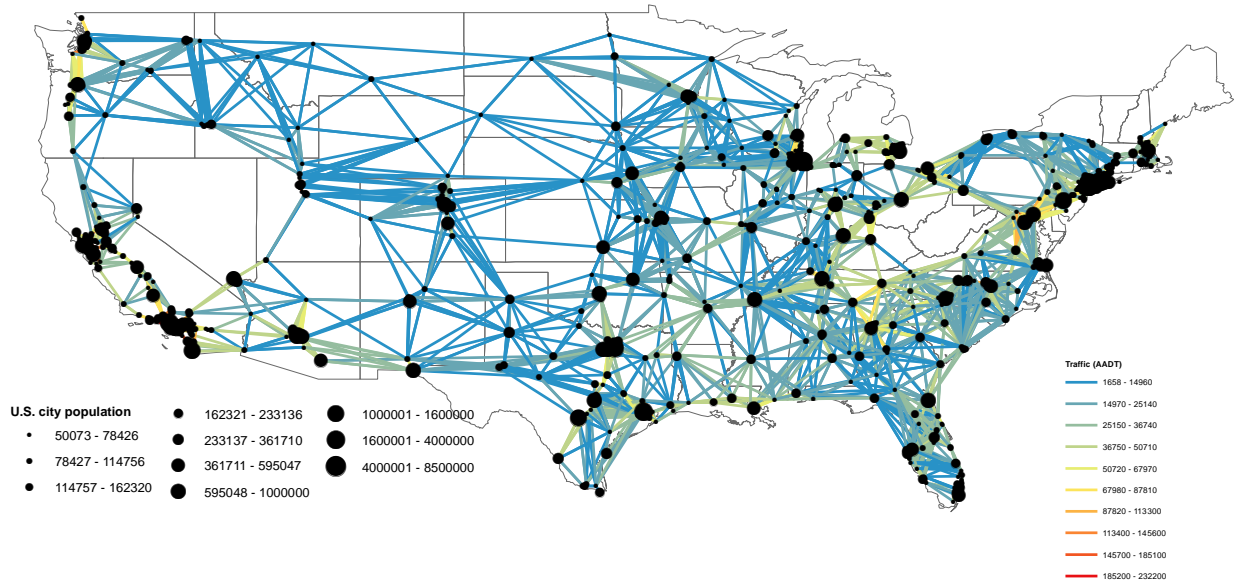
*Notes:* This figure provides an example geography. The top left frame depicts the graph of the connections between each of the 25 locations, where all connections are assumed to incur an equal cost. The top right depicts the probability of traveling along each connection when beginning at location 1 (bottom left) and traveling to location 25 (top right). The bottom left frame also depicts the probability of traveling along each connection when beginning at location 1 (bottom left) but now traveling to location 15 (top center). Finally, the bottom right figure depicts the elasticity of aggregate welfare to an improvement in each of the connections.

Figure 2: THE U.S. NATIONAL HIGHWAY NETWORK

Panel A: *The Observed U.S. National Highway Network*

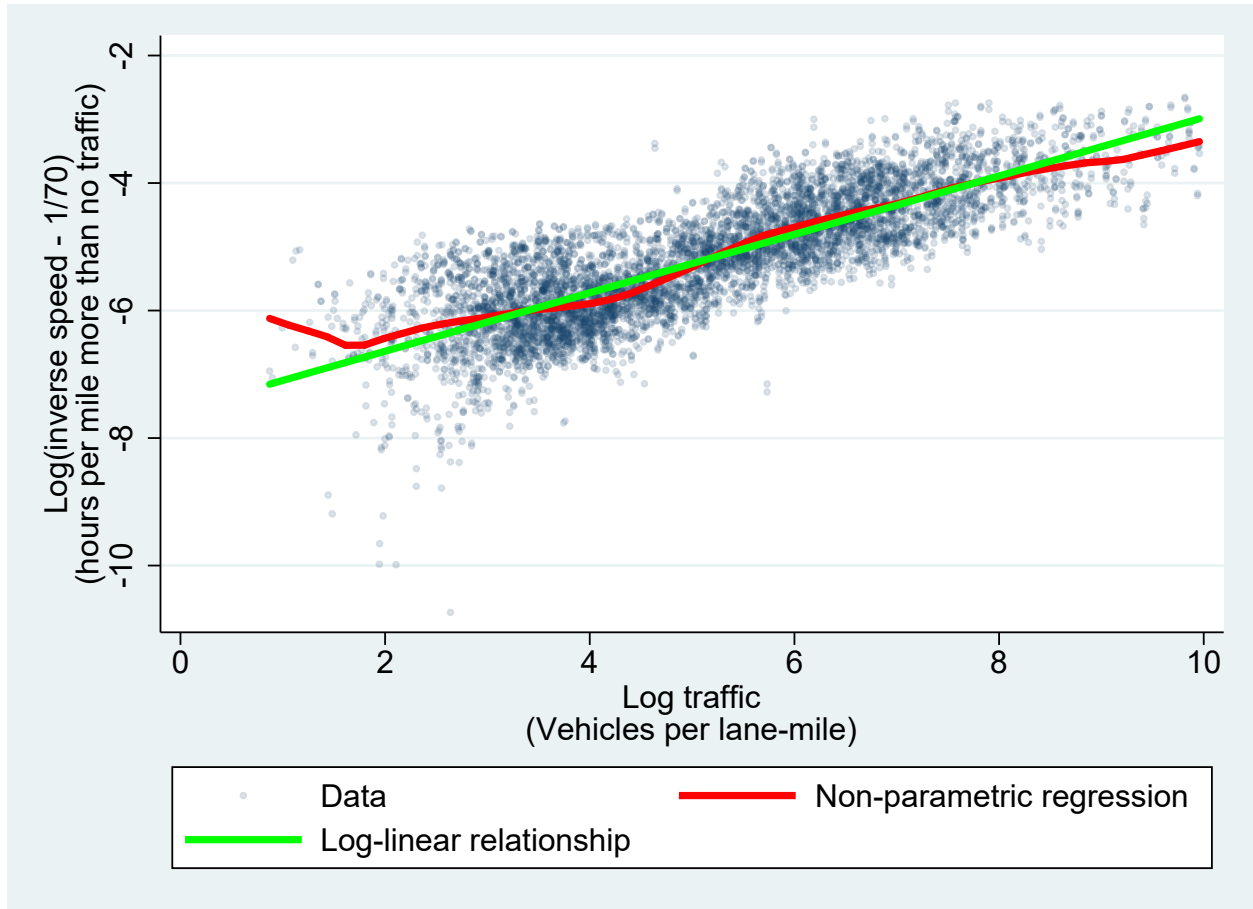


Panel B: *A Graphical Representation of the U.S. National Highway Network*



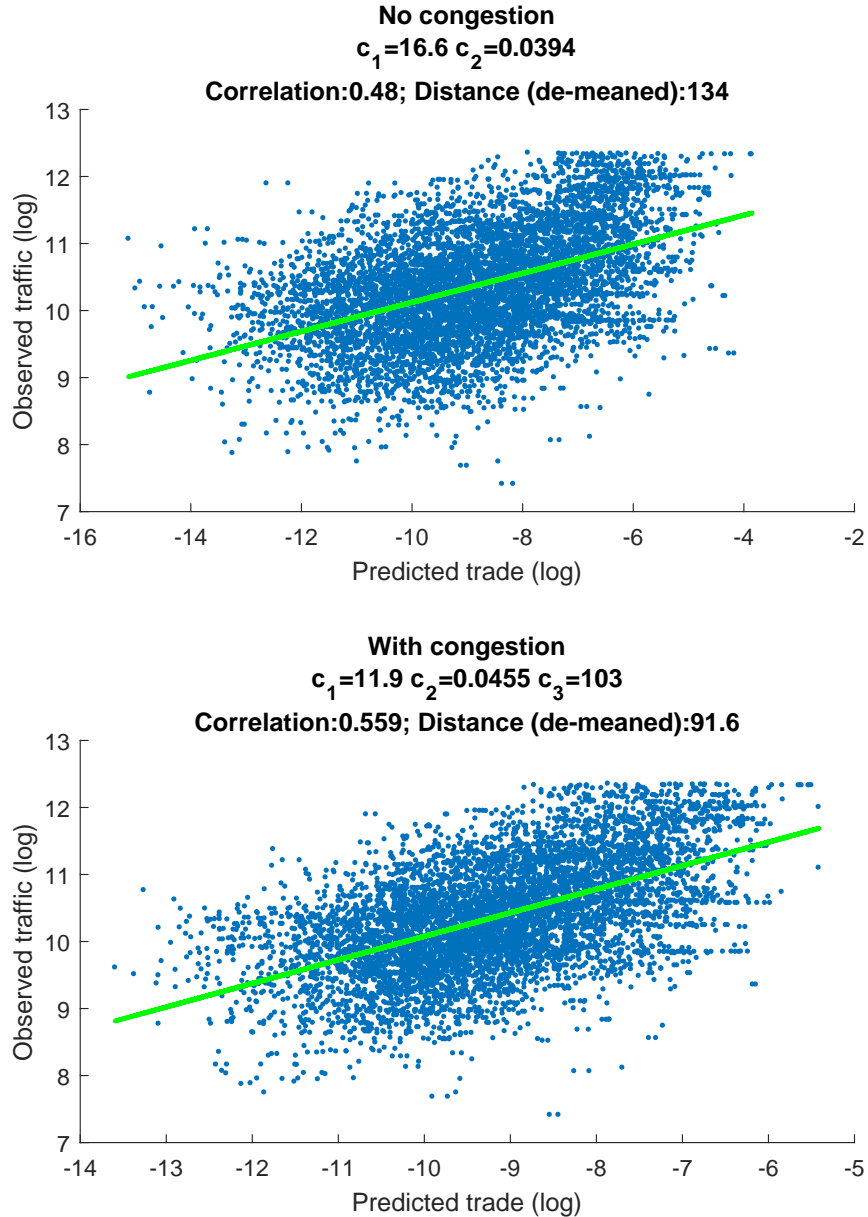
*Notes:* This figure depicts the U.S. highway network. Panel A shows the observed National Highway Network along with all cities with populations greater than 50,000 within 10 kilometers from the highway network. Panel B depicts the graphical representation of the constructed adjacency matrix between cities along the highway system. Two cities are considered connected if the fastest route between the locations along the highway system does not pass through another city. The color of the links indicates the traffic between connected cities, with blue indicating a less traffic and red indicating greater traffic. The size of the city circle indicates its population, with larger sizes indicating greater population.

Figure 3: TRAFFIC AND SPEED



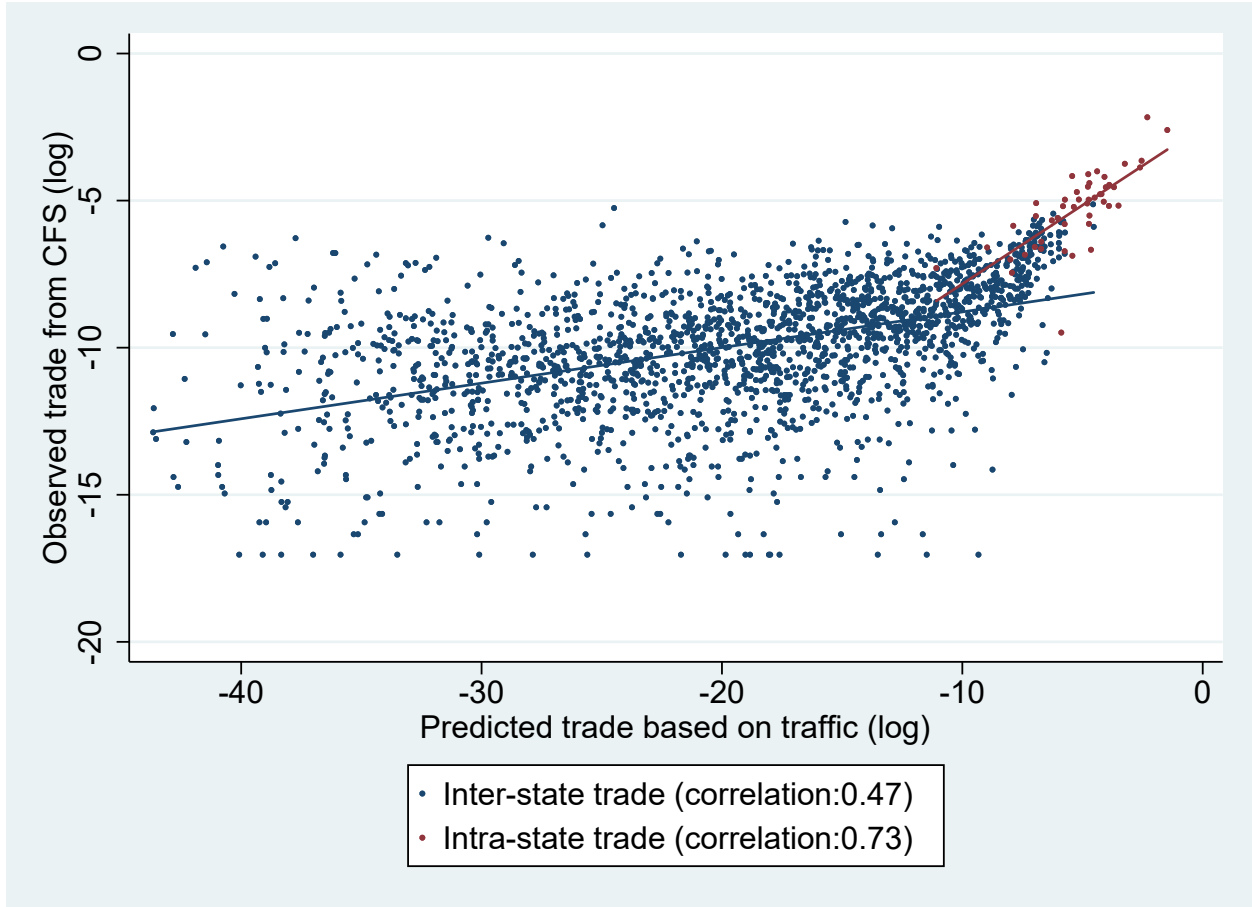
*Notes:* This figure plots the relationship between the (log) average traffic per lane-mile and the observed (log) additional time it takes to travel an additional mile relative to the time it would have taken going seventy miles per hour. The observed time of travel is calculated using API. The traffic data by ( [US](#) ). The green line is a log-linear regression fit over the data and the red line is a local non-parametric regression with an Epanechnikov kernel and a 0.25 bandwidth. The coefficient of the linear regression is 0.4415, with a t-statistic of 119.

Figure 4: MODEL FIT



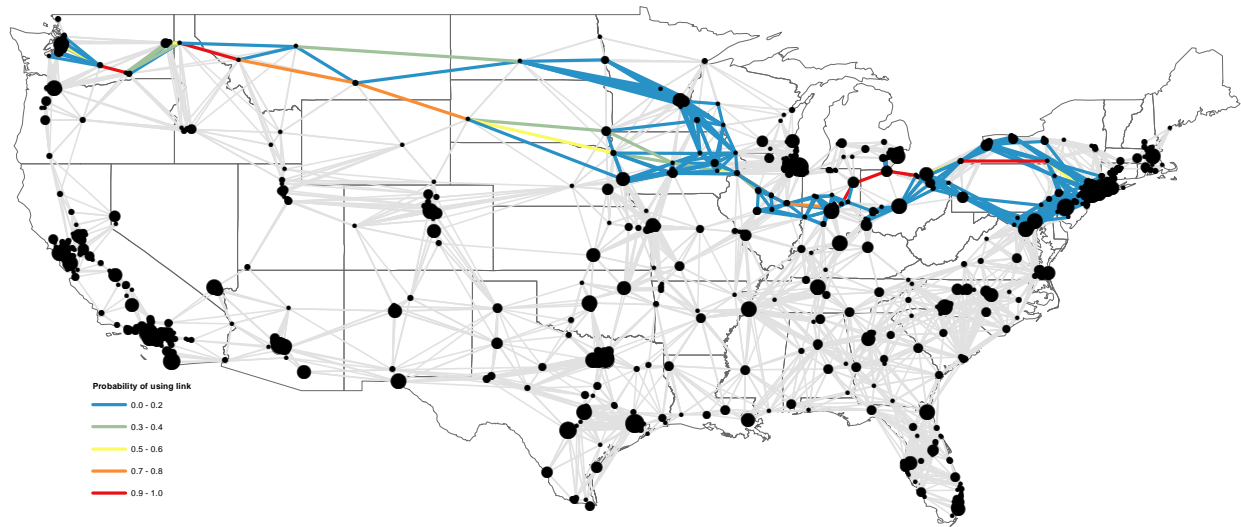
*Notes:* This figure depicts the ability of the model to match the observed traffic data. The top panel reports the relationship between the (log) predicted value of trade flowing on each network link and the (log) observed traffic along each link when there is no traffic congestion. The bottom panel shows how the same figure changes when there is congestion, i.e. the cost of traveling along a link is allowed to endogenously increase in response to increased trade. With congestion, the model fit (measured as the norm between the de-meaned log traffic and log predicted trade flows) improves by more than 30% and the correlation between observed traffic flows and predicted trade flows increases from 0.48 to 0.56.

Figure 5: PREDICTED VERSUS OBSERVED TRADE FLOWS



*Notes:* This figure shows the relationship between the predicted state-to-state bilateral trade flows from the model (calculated by aggregating up from the predicted city-to-city bilateral trade flow data) and the observed state-to-state bilateral trade flows from the 2012 Commodity Flow Survey (CFS). Note that the CFS data was not used in the estimation procedure (the predicted trade flows were estimated by instead matching observed traffic flows.)

Figure 6: PROBABILITY OF USING DIFFERENT HIGHWAYS FROM SEATTLE TO MANHATTAN

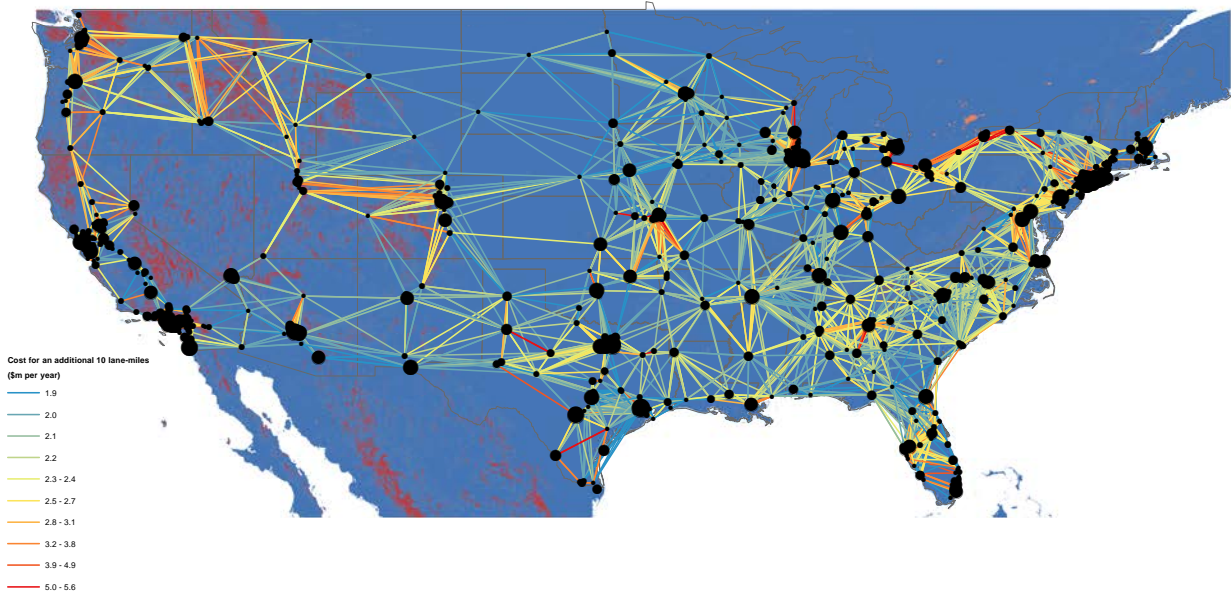


*Notes:* This figure depicts the probability of a trader going from Seattle, WA to Manhattan Borough in New York, NY using each particular connection of the National Highway Network (without congestion). This is equivalent to the elasticity of trade costs from Seattle to Manhattan to a change in the cost of traveling over each highway segment.

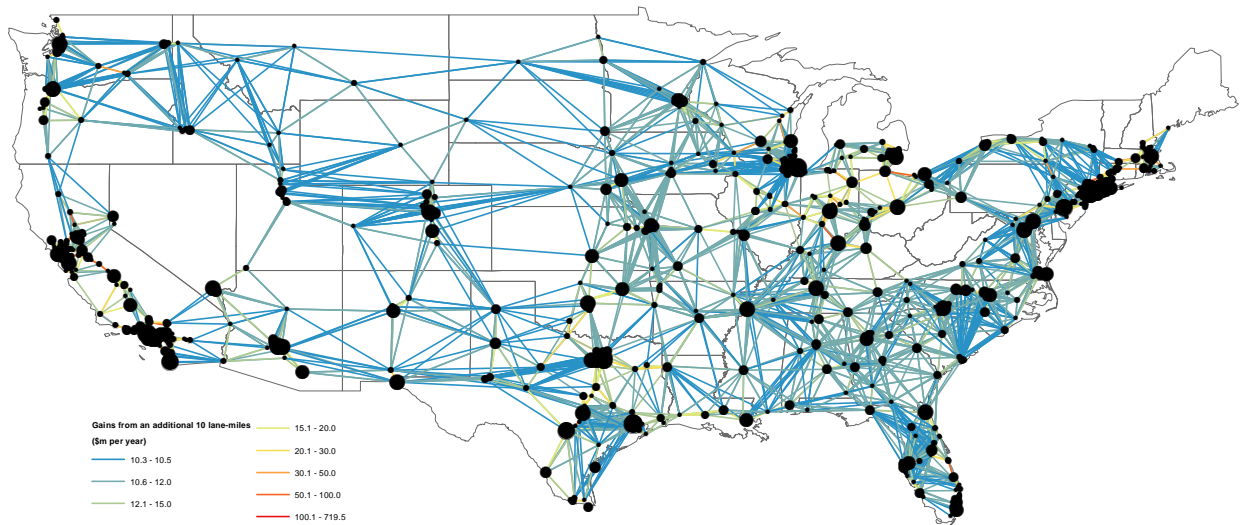


Figure 7: THE COSTS AND BENEFITS OF TRANSPORTATION INFRASTRUCTURE IMPROVEMENTS

(a) Estimated annualized cost of ten additional lane-miles (\$m)

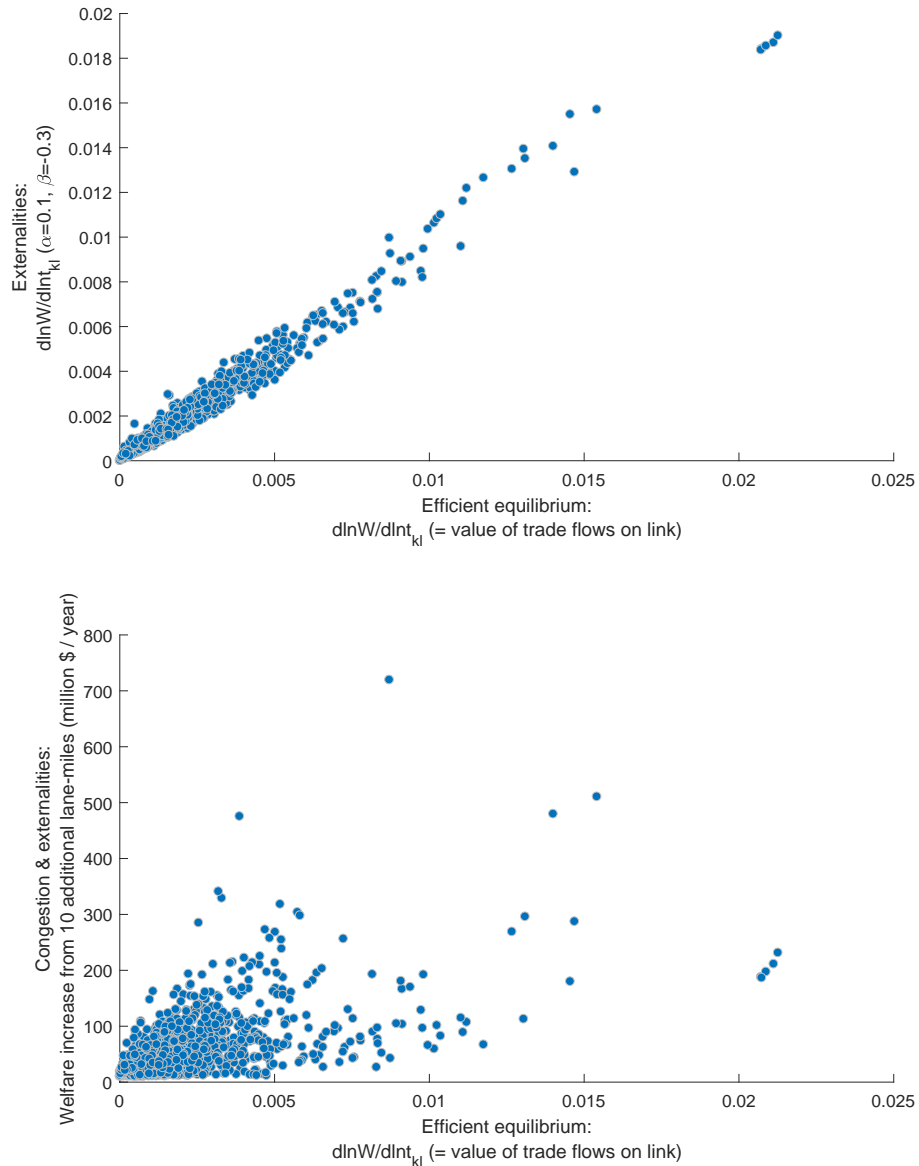


(b) Estimated annualized benefit of ten additional lane-miles (\$m)



*Notes:* Panel A reports the estimated annual cost (in millions of chained 2012 U.S. dollars) of adding ten additional lane-miles to each segment of the highway system; the background color indicates the cost of constructing a highway on each pixel. See the text for details of this calculation). Panel B reports the estimated annual value (in millions of chained 2012 U.S. dollars) of adding ten additional lane-miles to the highway system; this effect incorporates the endogenous response to congestion across all links, the optimal routing of all traders, and the general equilibrium response of wages and population in all cities. The color of a link indicates its cost (top panel) or benefit (bottom panel), with red indicating a higher value and blue indicating a lower value.

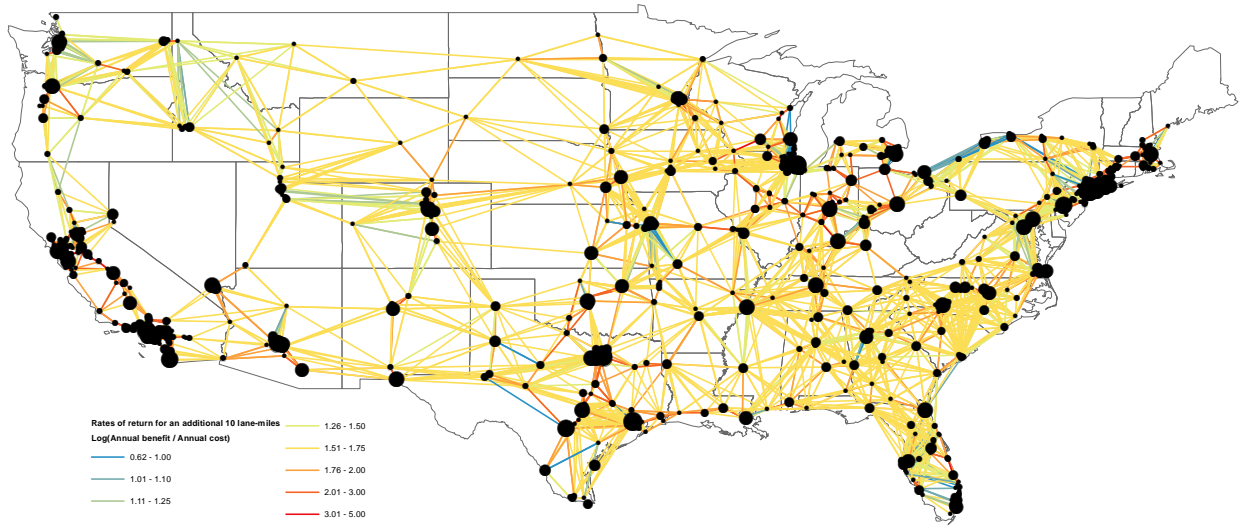
Figure 8: THE EFFECT OF EXTERNALITIES AND CONGESTION ON THE WELFARE IMPACTS OF INFRASTRUCTURE IMPROVEMENTS



*Notes:* This top panel of this figure depicts the relationship between the imputed value of trade traveling over each segment of the U.S. Highway System and the estimated welfare elasticity in the presence of spillovers. Note that in the absence of spillovers, the welfare elasticity should be exactly equal to the value of trade traveling over a segment; hence, the only reason the correlation is not perfect is due to market inefficiencies caused by the presence of spillovers. The bottom panel compares the imputed value of trade to the welfare gains (measured in millions of 2012 U.S. dollars) to adding ten additional lane-miles; this correlation is not perfect due to market inefficiencies arising both from the presence of spillovers and from congestion externalities.



Figure 9: THE RATES OF RETURN ON TRANSPORTATION INFRASTRUCTURE IMPROVEMENTS



*Notes:* This figure reports the rate of return (calculated as the log ratio of benefits to costs) of improving each segment of the highway system. The color of each link indicates its rate of return, with red indicating a higher value and blue indicating a lower value.

Table 1: TOP 10 HIGHWAYS SEGMENTS

Rank	Origin City	Origin State	Destination City	Destination State	Cost (\$m / yr)	Benefit (\$m / yr)	Rate of return (log ratio)
1	White Plains	New York	Greenburgh	New York	3.76	510.50	4.91
2	North Hempstead	New York	Queens	New York	5.35	719.46	4.90
3	Islip	New York	Brookhaven	New York	1.92	257.49	4.90
4	Greenburgh	New York	White Plains	New York	3.76	479.62	4.85
5	Center	Indiana	Indianapolis	Indiana	2.06	206.89	4.61
6	Indianapolis	Indiana	Center	Indiana	2.06	194.91	4.55
7	Bayonne	New Jersey	Staten Island	New York	1.92	179.89	4.54
8	Hempstead (Town)	New York	Queens	New York	5.11	475.31	4.53
9	Hempstead (Town)	New York	Hempstead	New York	3.76	318.10	4.44
10	Costa Mesa	California	Newport Beach	California	3.76	303.79	4.39

*Notes:* This table reports the ten highway links that have greatest benefit to cost ratio in the United States.

## 6 Appendix

### 6.1 Proof of Proposition 1

We have that

$$\tau_{ij}(\nu) = \min_{p \in P_K, K \geq 0} \tilde{\tau}_{ij}(p) \varepsilon_{ij}(p, \nu).$$

where we have defined

$$\tilde{\tau}_{ij}(p) = \prod_{k=1}^K t_{p_{k-1}, p_k}.$$

Replacing in the formula we need to compute

$$\frac{d \ln \tau_{ij}}{d \ln t_{kl}} = \frac{d \ln \left( E \min_{p \in P_K, K \geq 0} \left( \prod_{k=1}^K t_{p_{k-1}, p_k} \right) \varepsilon_{ij}(p, \nu) \right)}{d \ln t_{kl}}$$

Notice that we can split the set  $P_K$  into elements that include paths passing through  $kl$  (indicated as  $P_{K \cap kl}$ ) and ones that do not ( $P_{K \setminus kl}$ ). Thus,

$$\frac{d \ln \tau_{ij}}{d \ln t_{kl}} = \frac{d \left( E \min_{p \in P_{K \cap kl}, K \geq 0} \left( \prod_{k=1}^K t_{p_{k-1}, p_k} \right) \varepsilon_{ij}(p, \nu) + E \min_{p \in P_{K \setminus kl}, K \geq 0} \left( \prod_{k=1}^K t_{p_{k-1}, p_k} \right) \varepsilon_{ij}(p, \nu) \right)}{\tau_{ij}} t_{kl}.$$

By definition the derivative of the second term is zero, and thus,

$$\frac{d \ln \tau_{ij}}{d \ln t_{kl}} = \frac{E_\varepsilon \left( \min_{p \in P_{K \cap kl}, K \geq 0} \left( \prod_{k=1}^K t_{p_{k-1}, p_k} \right) \varepsilon_{ij}(p, \nu) \right)}{\tau_{ij}},$$

which is indeed the cost of the routes from  $i$  to  $j$  passing through  $kl$  versus the expected cost of  $i$  to  $j$ .

### 6.2 Proof of Proposition 2

First note that this elasticity is equal to the elasticity of an element of the route cost matrix to a change in the element of the weighted adjacency matrix  $\mathbf{A}$ :

$$\frac{\partial \ln \tau_{ij}}{\partial \ln t_{kl}} = \frac{\partial \ln \left( c b_{ij}^{-\frac{1}{\theta}} \right)}{\partial \ln \left( a_{kl}^{-\frac{1}{\theta}} \right)} = \frac{\partial \ln b_{ij}}{\partial \ln a_{kl}}.$$

To calculate this elasticity, we parameterize the weighted adjacency matrix as a function of a variable  $t$  follows:

$$\mathbf{A}_{kl}(t) = \begin{cases} a_{ij} & \text{if } k \neq i \text{ or } l \neq j \\ t & \text{if } k = i \text{ and } l = j \end{cases},$$

i.e. changing  $t$  just increases  $a_{kl}$ . By defining  $\mathbf{C}_{kl}(t) \equiv \mathbf{I} - \mathbf{A}_{kl}(t)$ , we have:

$$\frac{\partial \ln b_{ij}}{\partial \ln a_{kl}} = \left[ \frac{d\mathbf{C}_{kl}(t)^{-1}}{dt} \right]_{ij} \times \frac{a_{kl}}{b_{ij}} \quad (27)$$

Using the familiar expression for the derivative of an inverse of a parameterized matrix (see e.g. [Weber and Arfken \(2003\)](#)), we have:

$$\frac{d\mathbf{C}_{kl}(t)^{-1}}{dt} = -\mathbf{C}_{kl}(t)^{-1} \frac{d\mathbf{C}_{kl}(t)}{dt} \mathbf{C}_{kl}(t)^{-1}.$$

Note that  $\frac{d\mathbf{C}_{kl}(t)}{dt} = \begin{cases} 0 & \text{if } k \neq i \text{ or } l \neq j \\ -1 & \text{if } k = i \text{ and } l = j \end{cases}$  and  $\mathbf{C}_{kl}(t)^{-1} = \mathbf{B}$  so the derivative becomes:

$$\frac{d\mathbf{C}_{kl}(t)^{-1}}{dt} = \mathbf{B}\mathbf{E}_{kl}\mathbf{B},$$

where  $\mathbf{E}_{kl}$  is an  $N \times N$  matrix equal to one at  $(k, l)$  and zeros everywhere else. The  $(i, j)$  component of this inverse is hence:

$$\left[ \frac{d\mathbf{C}_{kl}(t)^{-1}}{dt} \right]_{ij} = [\mathbf{B}\mathbf{E}_{kl}\mathbf{B}]_{ij} = \left[ \sum_n \sum_m b_{im} E_{mn}^{kl} b_{nj} \right]_{ij} = b_{ik} b_{lj}, \quad (28)$$

so combining equations (27) and (28) and using the relationships  $a_{kl} = t_{kl}^{-\theta}$  and  $\tau_{ij} = cb_{ij}^{-\frac{1}{\theta}}$  yields expression (7).

We now derive the probability of using link  $kl$  along the path from  $i$  to  $j$ . Given the extreme value distribution, the probability of taking any particular path  $p$  of length  $K$  can be written as:

$$\pi_{ij}(p) = \frac{\tau_{ij}(p)^{-\theta}}{\sum_{K=0}^{\infty} \sum_{p' \in P_{ij,K}} \tau_{ij}(p')^{-\theta}}.$$

$$\begin{aligned}
\pi_{ij}(p) &= \frac{\tau_{ij}(p)^{-\theta}}{\sum_{K=0}^{\infty} \sum_{p' \in P_{ij,K}} \tau_{ij}(p')^{-\theta}} \iff \\
\pi_{ij}(p) &= \frac{1}{b_{ij}} \prod_{k=1}^K a_{p_{k-1}, p_k},
\end{aligned} \tag{29}$$

where the second line used the definition of  $\tau_{ij}(p)$  from equation (1) and the derivation for equation (5). As a result, we can calculate the probability of using link  $t_{kl}$  when traveling from  $i$  to  $j$  by summing across all paths from  $i$  to  $j$  that use the link  $t_{kl}$ :

$$\pi_{ij}^{kl} = \frac{1}{b_{ij}} \sum_{K=0}^{\infty} \sum_{p \in P_{ij,K}^{kl}} \prod_{k=1}^K a_{p_{k-1}, p_k},$$

where  $P_{ij,K}^{kl}$  is the set of all paths from  $k$  to  $l$  of length  $K$  that use link  $t_{ij}$ .

Note that for any  $p \in P_{ij,K}^{kl}$ , there must exist some length  $B \in [1, 2, \dots, K-1]$  at which the path arrives at link  $t_{kl}$  so that this can be written as:

$$\pi_{ij}^{kl} = \frac{1}{b_{ij}} \sum_{K=0}^{\infty} \sum_{B=0}^{K-1} \left( \sum_{p \in P_{ik,B}} \prod_{k=1}^B a_{p_{k-1}, p_k} \right) \times a_{kl} \times \left( \sum_{q \in P_{kj, K-B-1}} \prod_{k=1}^{K-B-1} a_{q_{k-1}, q_k} \right)$$

As above, we can then explicitly enumerate all possible paths from  $i$  to  $k$  of length  $B$  and all possible paths from  $l$  to  $j$  of length  $K-B-1$ :

$$\pi_{ij}^{kl} = \frac{1}{b_{ij}} \sum_{K=0}^{\infty} \sum_{B=0}^{K-1} \left( \sum_{k_1=1}^N \cdots \sum_{k_{B-1}=1}^N a_{i, k_1} \times \cdots \times a_{k_{B-1}, k} \right) \times a_{kl} \times \left( \sum_{k_1=1}^N \cdots \sum_{k_{K-B-1}=1}^N a_{l, k_1} \times \cdots \times a_{k_{K-B-1}, j} \right),$$

which again can be expressed more simply as elements of matrix powers of  $\mathbf{A}$  :

$$\pi_{ij}^{kl} = \frac{1}{b_{ij}} \sum_{K=0}^{\infty} \sum_{B=0}^{K-1} A_{ik}^B \times a_{kl} \times A_{lj}^{K-B-1}$$

This is where it gets a little more difficult. Recall from matrix calculus that the derivative of the power of a matrix can be written as:

$$DS_K(\mathbf{A}) \mathbf{C} = \sum_{B=0}^{K-1} \mathbf{A}^B \mathbf{C} \mathbf{A}^{K-B-1}, \tag{30}$$

where  $S_K(\mathbf{A}) = \mathbf{A}^K$  and  $\mathbf{C}$  is an arbitrary matrix. Furthermore, recall from above that the geometric series of a power of matrices can be written as:

$$\sum_{K=0}^{\infty} \mathbf{A}^K = (\mathbf{I} - \mathbf{A})^{-1}.$$

Hence, we can right multiply both sides of equation (30) by  $\mathbf{C}$  and differentiate to yield:

$$\sum_{K=0}^{\infty} DS_K(\mathbf{A}) \mathbf{C} = DT(\mathbf{A}) \mathbf{C},$$

where  $T(\mathbf{A}) \equiv (\mathbf{I} - \mathbf{A})^{-1}$ . Recall from matrix calculus (see e.g. [Weber and Arfken \(2003\)](#)) that  $DT(\mathbf{A}) \mathbf{C} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{C} (\mathbf{I} - \mathbf{A})^{-1}$  so that we have:

$$\sum_{K=0}^{\infty} \sum_{B=0}^{K-1} \mathbf{A}^B \mathbf{C} \mathbf{A}^{K-B-1} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{C} (\mathbf{I} - \mathbf{A})^{-1}. \quad (31)$$

Define  $\mathbf{C}$  to be an  $N \times N$  matrix that takes the value of  $a_{kl}$  at row  $k$  and column  $l$  and zeros everywhere else. Using equation (31) we obtain our result:

$$\pi_{ij}^{kl} = \frac{b_{ik} a_{kl} b_{lj}}{b_{ij}}.$$

### 6.3 Proof of Theorem 1

Define the auxiliary variables

$$\begin{aligned} x_i &= \left( W^{\frac{1-\sigma}{\alpha(\sigma-1)+\beta(\sigma-1)}} L_i \right)^{\beta(\sigma-1)+1} w_i^\sigma, \\ y_i &= \left( W^{\frac{1-\sigma}{\alpha(\sigma-1)+\beta(\sigma-1)}} L_i \right)^{\alpha(\sigma-1)} w_i^{1-\sigma}, \end{aligned}$$

so that

$$w_i = x_i^{\frac{\alpha}{\beta(\sigma-1)+1+\alpha\sigma}} y_i^{\frac{-1}{\sigma-1} \frac{\beta(\sigma-1)+1}{\beta(\sigma-1)+1+\alpha\sigma}}.$$

Also, define

$$\rho \equiv \frac{2 + \alpha - \beta}{1 + (\sigma - 1) \beta + \alpha \sigma}$$

for convenience. Given those definitions the two equation system given by equations (10)-(11) can be transformed into

$$x_i^{\frac{\alpha+1}{\beta(\sigma-1)+1+\alpha\sigma}} (y_i)^{\frac{-\beta+1}{\beta(\sigma-1)+1+\alpha\sigma}} = \sum_{j=1}^N K_{ij} y_i x_j,$$

$$x_i^{\frac{\alpha+1}{\beta(\sigma-1)+1+\alpha\sigma}} (y_i)^{\frac{-\beta+1}{\beta(\sigma-1)+1+\alpha\sigma}} = \sum_{j=1}^N K_{ji} x_i y_j.$$

We chose to impose normalization condition  $w_N = 1$ , We now proceed to solve for the comparative statics  $\frac{d \ln x_i}{d \ln \tau_{kl}}, \frac{d \ln y_i}{d \ln \tau_{kl}}$  from the following system ( $\mathbf{D}_{\ln \tau_{kl}} \mathbf{G} = \mathbf{0}$ : 2N equations, 2N unknown) by replacing the last equation with the linearized version of the normalization condition

$$G_1^1 = \ln \sum_{j=1}^N K_{1j} y_1 x_j - \frac{\alpha+1}{\beta(\sigma-1)+1+\alpha\sigma} \ln x_1 - \frac{-\beta+1}{\beta(\sigma-1)+1+\alpha\sigma} \ln y_1$$

.....

$$G_N^1 = \ln \sum_{j=1}^N K_{Nj} y_N x_j - \frac{\alpha+1}{\beta(\sigma-1)+1+\alpha\sigma} \ln x_N - \frac{-\beta+1}{\beta(\sigma-1)+1+\alpha\sigma} \ln y_N$$

$$G_1^2 = \ln \sum_{j=1}^N K_{j1} y_1 x_j - \frac{\alpha+1}{\beta(\sigma-1)+1+\alpha\sigma} \ln x_1 - \frac{-\beta+1}{\beta(\sigma-1)+1+\alpha\sigma} \ln y_1$$

.....

$$G_{N-1}^2 = \ln \sum_{j=1}^N K_{jN-1} y_1 x_j - \frac{\alpha+1}{\beta(\sigma-1)+1+\alpha\sigma} \ln x_{N-1} - \frac{-\beta+1}{\beta(\sigma-1)+1+\alpha\sigma} \ln y_{N-1}$$

$$G_N^2 = \frac{\alpha}{\beta(\sigma-1)+1+\alpha\sigma} \ln x_N - \frac{1}{\sigma-1} \frac{\beta(\sigma-1)+1}{\beta(\sigma-1)+1+\alpha\sigma} \ln y_N$$

The last row of  $DG$  matrix needs to be slightly modified.

$$DG = \begin{bmatrix} \frac{\sigma(\beta+\alpha)}{\beta(\sigma-1)+1+\alpha\sigma} \mathbf{I} & \boldsymbol{\mu} - \frac{\alpha+1}{\beta(\sigma-1)+1+\alpha\sigma} \mathbf{I} \\ \boldsymbol{\lambda}' - \frac{1-\beta}{\beta(\sigma-1)+1+\alpha\sigma} \mathbf{I} & \frac{(\sigma-1)(\beta+\alpha)}{\beta(\sigma-1)+1+\alpha\sigma} \mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \lambda_{1N} & \dots & \lambda_{N-1,N} & \lambda_{NN} + \frac{\beta}{\alpha} & \mathbf{0} \\ & & & & \mathbf{0} \end{bmatrix}.$$

Taking total derivative of this system we can solve for the following

$$\frac{\partial \ln y_i}{\partial \ln \tau_{kl}} = \begin{cases} (\sigma-1) \left[ (DG)_{y,ik,1}^{-1} \mu_{kl} + (DG)_{x,il,1}^{-1} \lambda_{kl} \right] & \text{if } l \neq N \\ (\sigma-1) (DG)_{y,ik,1}^{-1} \mu_{kl} & \text{if } l = N \end{cases}.$$

Similarly,

$$\frac{\partial \ln x_i}{\partial \ln \tau_{kl}} = \begin{cases} (\sigma - 1) \left[ (DG)_{y,ik,2}^{-1} \mu_{kl} + (DG)_{x,il,2}^{-1} \lambda_{kl} \right] & \text{if } l \neq N \\ (\sigma - 1) (DG)_{y,ik,2}^{-1} \mu_{kl} & \text{if } l = N \end{cases}.$$

We can expression the derivative of welfare in terms of these derivatives

$$\frac{d \ln W}{d \ln \tau_{kl}} = - \left( \frac{2 - \rho}{2\sigma - 1} \right) \sum_i \frac{L_i}{\bar{L}} \left[ \frac{d \ln x_i}{d \ln \tau_{kl}} + \frac{\sigma}{\sigma - 1} \frac{d \ln y_i}{d \ln \tau_{kl}} \right],$$

and substituting in:

$$\frac{d \ln W}{d \ln \tau_{kl}} = \begin{cases} - (2 - \rho) \left( \frac{\sigma - 1}{2\sigma - 1} \right) \mu_{kl} \left( \sum_i \frac{L_i}{\bar{L}} \left( (DG)_{y,ik,2}^{-1} + \frac{\sigma}{\sigma - 1} (DG)_{y,ik,1}^{-1} \right) \right) - \\ (2 - \rho) \left( \frac{\sigma - 1}{2\sigma - 1} \right) \lambda_{kl} \left( \sum_i \frac{L_i}{\bar{L}} \left( (DG)_{x,il,2}^{-1} + \frac{\sigma}{\sigma - 1} (DG)_{x,il,1}^{-1} \right) \right) & \text{if } l \neq N \\ - (2 - \rho) \left( \frac{\sigma - 1}{2\sigma - 1} \right) \mu_{kl} \left( \sum_i \frac{L_i}{\bar{L}} \left( (DG)_{y,ik,2}^{-1} + \frac{\sigma}{\sigma - 1} (DG)_{y,ik,1}^{-1} \right) \right). & \text{if } l = N \end{cases}$$

To proceed with further characterizing this expression we make the following definitions:

$$\begin{aligned} - (1 + \kappa_k) &= (2 - \rho) \tilde{\sigma} \left( \sum_i \frac{L_i}{\bar{L}} \left( (DG)_{y,ik,2}^{-1} + \frac{\sigma}{\sigma - 1} (DG)_{y,ik,1}^{-1} \right) \right) \frac{1}{Y_k} \iff \\ \left( \frac{1 - \sigma\beta - \alpha(\sigma - 1)}{1 + (\sigma - 1)\beta + \alpha\sigma} \right) \tilde{\sigma} &\left( \sum_i \frac{L_i}{\bar{L}} \left( (DG)_{y,ik,2}^{-1} + \frac{\sigma}{\sigma - 1} (DG)_{y,ik,1}^{-1} \right) \right) \frac{1}{Y_k} = 1 + \kappa_k, \end{aligned}$$

and

$$\begin{aligned} -\nu_k &= (2 - \rho) \tilde{\sigma} \left( \sum_i \frac{L_i}{\bar{L}} \left( (DG)_{x,ik,2}^{-1} + \frac{\sigma}{\sigma - 1} (DG)_{x,ik,1}^{-1} \right) \right) \frac{1}{E_k} \iff \\ \left( \frac{1 - \sigma\beta - \alpha(\sigma - 1)}{1 + (\sigma - 1)\beta + \alpha\sigma} \right) \tilde{\sigma} &\left( \sum_i \frac{L_i}{\bar{L}} \left( (DG)_{x,ik,2}^{-1} + \frac{\sigma}{\sigma - 1} (DG)_{x,ik,1}^{-1} \right) \right) \frac{1}{E_k} = \nu_k, \end{aligned}$$

which gives us expressions (18) and (19), where

$$DG = \begin{bmatrix} \frac{\sigma(\beta + \alpha)}{\beta(\sigma - 1) + 1 + \alpha\sigma} \mathbf{I} & \boldsymbol{\mu} - \frac{\alpha + 1}{\beta(\sigma - 1) + 1 + \alpha\sigma} \mathbf{I} \\ \boldsymbol{\lambda}' - \frac{1 - \beta}{\beta(\sigma - 1) + 1 + \alpha\sigma} \mathbf{I} & \frac{(\sigma - 1)(\beta + \alpha)}{\beta(\sigma - 1) + 1 + \alpha\sigma} \mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \lambda_{1N} & \dots & \lambda_{N-1,N} & \lambda_{NN} + \frac{\beta}{\alpha} & \mathbf{0} \end{bmatrix}.$$

Thus, we finally arrive to the expression of the Theorem,

$$\frac{d \ln W}{d \ln \tau_{kl}} = \begin{cases} X_{kl} (1 + \kappa_k + \nu_l) & \text{if } l \neq N \\ X_{kl} (1 + \kappa_k) & \text{if } l = N \end{cases},$$

with the corresponding definitions of  $\kappa_k$  and  $\nu_l$ .