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MONETARY AND FISCAL POLICY WHEN PEOPLE HAVE FINITE LIVES

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# MONETARY AND FISCAL POLICY WHEN PEOPLE HAVE FINITE LIVES\*

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## 1. INTRODUCTION

This paper is about the use of the overlapping generations model as a vehicle to understand how fiscal and monetary policy interact with the choices of private agents to determine prices and interest rates. We ask if well established results that hold in the workhorse representative agent model (Leeper and Leith, 2016) also hold in an Overlapping Generations (OLG) model calibrated to an income profile that matches real world data. Our main result is that they do not.

It is well known (Gale, 1973) that overlapping generations models contain at least two steady-state equilibria: one steady state in which the interest rate equals the growth rate – we refer to this as *the golden rule* – and another steady state where there is no trade with future generations – we refer to this steady state as *balanced*. In this paper, we establish a new result. We prove that, for a three-generation overlapping generations model with a hump-shaped endowment, there is a critical value of the intertemporal elasticity of substitution,  $ies_c$ , such that for all values of  $ies < ies_c$  there exist three balanced steady states. We extend our analysis to a 62-generation model with capital,

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and we show that this same result holds when the endowment profile is calibrated to U.S. micro data.

## 2. THE RELATIONSHIP OF OUR WORK TO PREVIOUS LITERATURE

The generic existence of multiple indeterminate steady-state equilibria was established by Kehoe and Levine (1985) in the context of a two-period-lived model with multiple goods and multiple agents.<sup>1</sup> With the exception of their paper, most previous examples of indeterminate equilibria in overlapping generations models have been restricted to two-generation or three-generation examples in which indeterminacy is purely monetary.<sup>2</sup> Our paper is the first to provide a long-lived example of an economy where the endowment profile is matched to U.S. micro data in which there are multiple balanced steady states and where the golden-rule equilibrium is indeterminate when both monetary and fiscal policy are active in the sense of Leeper (1991).

Aiyagari (1988) has demonstrated that the set of equilibria in the OLG model converges to that of the representative agent model as the length of life is increased. He concludes that

... empirically, infinitely lived agents models ... would be good approximations to ... long (but finite) lived overlapping generations models. That is, the “overlapping” structure does not make much difference. Aiyagari (1988, page 104)

This result is sometimes interpreted as an argument in favor of representative agent models. If Aiyagari is correct, the multiplicities and indeterminacies that arise in the OLG model are unlikely to be observed in the real world and researchers can safely restrict attention to the more tractable representative agent framework as a laboratory for the analysis of alternative policy regimes.

We are not persuaded by Aiyagari’s argument. Our reservation stems from work by Reichlin (1992), who has shown that Aiyagari’s convergence result requires the endowments of all agents with positive measure to be bounded away from zero. This is a condition that fails in any OLG model where people die in finite time and it also fails in the perpetual youth model (Blanchard, 1985) where the probability that any agent lives for  $T$  periods approaches zero as  $T \rightarrow \infty$  (Reichlin, 1992).

Kubler and Schmedders (2011) present a second argument to support the case for using representative agent macro models. For the case of zero discounting, they explore the frequency of multiple equilibria in  $T$ -generation overlapping generations economies parameterized by endowments. They show that when endowments are chosen on a random grid in the set  $[0, 1]^T$  that, as  $T$  grows large, economies in which multiple equilibria emerge become increasingly rare. They conclude, as does Aiyagari, that the existence of multiple – possibly indeterminate – equilibria in the OLG model is a phenomenon that is unlikely to be observed in the real world. We disagree. Real-world endowment patterns are not draws from an uncorrelated random grid; they follow a

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<sup>1</sup>It follows from the results of Balasko and Shell (1981) that the two-period assumption is unrestrictive as long as there are multiple agents and multiple goods.

<sup>2</sup>See, for example, Samuelson (1958); Gale (1973); Farmer (1986); Farmer and Woodford (1997) and (Azariadis, 1981) all of which display price level indeterminacy. Kehoe and Levine (1983) present an example of relative price indeterminacy but their parameterization is not calibrated to real-world data. Spear and Young (2023) present a comprehensive, and highly recommended, history of the development of the overlapping generations model.

distinct hump-shaped pattern. In Section 9, we show that multiplicity and indeterminacy of equilibrium *does not* disappear as the length of life increases when the endowment pattern is calibrated to the hump-shape we see in U.S. micro data.

In previous versions of this paper we restricted our attention to endowment economies. A number of commentators pointed us to the book by Auerbach and Kotlikoff (1987) and the paper by Ríos-Rull (1996) both of which find, in calibrated versions of production economies, that the balanced steady-state equilibrium is unique.<sup>3</sup> For theoretical results on uniqueness we were referred to Chalk (2000) who analyzes the dynamics of a 2-generation production economy and the set of steady-states of a  $T$ -generation generalization of this model. He proves under certain assumptions that the overlapping generations model with production has, generically, two steady-states.<sup>4</sup> One of Chalk’s assumptions (Chalk, 2000, Assumption B: page 303) is that the aggregate savings function is concave in the steady-state gross real interest rate. This is a very strong assumption. Our examples of economies with multiple balanced steady states violate this assumption and they demonstrate that the coincidence of a hump-shaped endowment profile and a low enough *ies* is sufficient to generate multiple balanced steady-states in an economy with production of the kind studied by Chalk (2000).

### 3. FISCAL AND MONETARY POLICY

In Section 3 we introduce terminology and we explain how the determinacy properties of steady state equilibria are connected with the proposition that either monetary or fiscal policy, but not both, should be active in the sense of Leeper (1991). This literature is known as the Fiscal Theory of the Price Level (FTPL).

We assume that the government purchases  $g_t$  units of a consumption good which it finances with dollar-denominated pure discount bonds and lump-sum taxes,  $\tau_t$ . Let  $B_t$  be the quantity of pure-discount bonds, each of which promises to pay one dollar at date  $t + 1$  and let  $Q_t$  be the date  $t$  dollar price of a discount bond. Further, let  $p_t$  be the date  $t$  dollar price of a consumption good. Using these definitions, government debt accumulation is represented by the following equation,

$$Q_t B_t + p_t \tau_t = B_{t-1} + p_t g_t.$$

Define  $i_t$  to be the net nominal interest rate from period  $t$  to period  $t + 1$ , and let  $\Pi_{t+1}$ , be the gross inflation rate. These variables are given by,

$$i_t \equiv \frac{1}{Q_t} - 1 \quad \text{and,} \quad \Pi_{t+1} \equiv \frac{p_{t+1}}{p_t}.$$

Further, let

$$b_t \equiv \frac{B_{t-1}}{p_t},$$

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<sup>3</sup>Here is Ríos-Rull on the questions of uniqueness:

“In all the cases studied, these functions turned out to be monotone- decreasing, implying a unique steady state.” Ríos-Rull (1996, page 472).

<sup>4</sup>(Chalk, 2000, Proposition 1 page 303).

be the real value of government debt maturing in period  $t$  and define the *real primary deficit* as

$$d_t \equiv g_t - \tau_t,$$

where the negative of  $d_t$  is the *real primary surplus*. Let  $R_{t+1}$  represent the gross real return from  $t$  to  $t + 1$ , which from the Fisher-parity condition equals

$$R_{t+1} \equiv \frac{1 + i_t}{\Pi_{t+1}}. \quad (1)$$

We can combine these definitions to rewrite the government budget equation in purely real terms

$$b_{t+1} = R_{t+1}(b_t + d_t), \quad t = 1, \dots, \infty. \quad (2)$$

We can consolidate these equations into a single government budget constraint by defining  $\mathcal{Q}_t^k$ ,

$$\mathcal{Q}_t^k \equiv \prod_{j=t+1}^k \frac{1}{R_j}, \quad \mathcal{Q}_t^t = 1,$$

to be the relative price at date  $t$  of a commodity for delivery at date  $k$ . Using this definition and iterating Eq. (2) forwards,

$$\frac{B_0}{p_1} = - \sum_{t=1}^{\infty} \mathcal{Q}_1^t d_t + \lim_{T \rightarrow \infty} \mathcal{Q}_1^T b_T. \quad (3)$$

If  $\lim_{T \rightarrow \infty} \mathcal{Q}_1^T b_T$  exists and is equal to zero, Eqn. (3) defines a function relating the initial price level to the real value of all future surpluses.

In New-Keynesian models in which the central bank sets an interest rate peg, the initial price level would be indeterminate if the government were constrained to balance its budget for all paths of  $\{\mathcal{Q}_1^t\}_{t=1}^{\infty}$  and all initial price levels (McCallum, 2001). To resolve this apparent indeterminacy of the price level, advocates of the FTPL have argued that the government should be treated differently from other agents in a general equilibrium model.<sup>5</sup> Although Equation (2) is expressed in terms of real variables, the debt instrument issued by the treasury is nominal. It follows that, when Eq. (3) is interpreted as an equilibrium condition, the real value of debt in period 1 is determined by the period 1 price level through the definition

$$b_1 \equiv \frac{B_0}{p_1}.$$

This argument rests critically on the determinacy properties of the steady state and, as we show below, it does not hold in the OLG model when the *ies* is sufficiently low and the endowment profile is hump-shaped.

#### 4. A THREE-GENERATION EXAMPLE

We introduce our main ideas in a 3-generation exchange economy. This example is simple enough that we are able to provide a set of sufficient conditions for the existence of multiple balanced steady-states. These conditions are that the endowment profile has at least one peak and

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<sup>5</sup>See, for example, (Leeper, 1991; Woodford, 1994, 2001). For a recent exposition of the FTPL see Cochrane (2023).

the *ies* is below some critical level that depends on the magnitude of the peak. In Section 8 we show, by means of an example, that this result extends to a 62-generation production economy in which the endowment profile is calibrated to the income distribution, by age, of the U.S. population.

**4.1. Individual Consumption and Savings Functions.** In our 3-generation exchange economy, the preferences of people born in period 2 and later are given by the following utility function;

$$U^t = U(c_t^t, c_{t+1}^t, c_{t+2}^t). \quad (4)$$

We index generations by superscripts and calendar time by subscripts. Thus,  $c_\tau^t$  is the consumption of generation  $t$  in period  $\tau$ . We refer to the people born in period  $T$  and later as *generic generations* and we distinguish them from a set of *non-generic generations*. The non-generic generations are people alive in periods 1 through  $T - 1$  who live for less than  $T$  periods. In the 3-generation model there are 2 non-generic generations; the initial middle-aged and the initial old.

The generic generations maximize utility subject to three budget constraints, one for each period of life,

$$c_t^t + s_{t+1}^t \leq \omega_1, \quad c_{t+1}^t + s_{t+2}^t \leq R_{t+1}s_{t+1}^t + \omega_2, \quad c_{t+2}^t \leq R_{t+2}s_{t+2}^t + \omega_3, \quad (5)$$

where  $\omega \equiv \{\omega_1, \omega_2, \omega_3\}$  is the after-tax endowment profile of a generic generation and  $s_\tau^t$  is the demand for claims to  $\tau + 1$  consumption goods by generation  $t$  in period  $\tau$ . The subscript on the term  $\omega_j$  indexes age and we assume throughout, that  $\omega_j$  does not depend on calendar time. The solution to this problem is fully characterized by a pair of asset demand functions

$$s_{t+1}^t(R_{t+1}, R_{t+2}; \omega), \quad s_{t+2}^t(R_{t+1}, R_{t+2}; \omega),$$

together with the requirement that the three budget constraints characterized in (5) hold with equality.

**4.2. Dynamic Equilibrium Equations.** Let the aggregate demand for assets by all agents alive at date  $t$  be defined by the function

$$S_\omega(R_t, R_{t+1}, R_{t+2}) \equiv s_t^{t-1}(R_t, R_{t+1}; \omega) + s_t^t(R_{t+1}, R_{t+2}; \omega),$$

where the subscript  $\omega$  on the function  $S$  indexes the dependence of the asset demand function on the endowment profile. For this three-generation example,  $S_\omega(\cdot)$  adds up the asset demand of the newborns, this is the term  $s_t^t(\cdot)$ , and the asset demands of the middle-aged, this is the term  $s_t^{t-1}(\cdot)$ . Equilibrium in the asset markets requires that

$$S_\omega(R_t, R_{t+1}, R_{t+2}) = b_t + d_t, \quad (6)$$

where  $b_t + d_t$  is the public sector borrowing requirement in period  $t$  and the dynamics of public borrowing are given by the equation,

$$b_{t+1} = R_{t+1}(b_t + d_t). \quad (7)$$

Beginning with period 2, non-stationary equilibria are characterized by bounded sequences of real interest rates and debt that satisfy equations (6) and (7) and are consistent with a set of initial conditions that arise from the behavior of the initial non-generic generations.

**4.3. Steady-State Equilibrium Equations.** Let  $f_\omega(R) \equiv S_\omega(R, R, R)$  be the aggregate *steady-state* demand for assets by the private sector. A steady-state equilibrium is a non-negative real number  $R$  and a real number  $b$  such that

$$f_\omega(R) = b + d, \tag{8}$$

$$b = R(b + d). \tag{9}$$

When  $d = 0$ , inspection of Eqn. (9) shows that there are, generically, at least two steady-state equilibria; one in which  $b = 0$  and one in which  $R = 1$ .<sup>6</sup> We refer to an equilibrium in which  $b = 0$  as a *balanced* steady state and the equilibrium in which  $R = 1$  as the *golden rule* steady state. Let

$$R_{gr} = 1 \quad \text{and} \quad b_{gr},$$

be the gross real interest rate and the real value of government debt at the golden rule steady-state equilibrium and let

$$R_{bal_i} \quad \text{and} \quad b_{bal_i} = 0,$$

be the corresponding values of the equilibrium real interest rate and real government debt at a balanced steady-state equilibrium. Replacing (9) in (8) we see that  $b_{gr} = f_\omega(1)$ , is the aggregate demand for debt at the golden-rule and the set of balanced interest rates  $\{R_{bal_i}\}$ , are solutions to the equation  $f_\omega(R) = 0$ . We seek conditions under which this equation has multiple solutions.

## 5. MULTIPLICITY OF STEADY STATES

We begin by constructing an economy in which we are able to express the equilibrium relative price in the balanced steady-state equilibrium as a function of the parameters of the model. In our example, the endowment profile is given by,  $\omega_\lambda \equiv \{1, \lambda^\eta, \lambda^{2\eta}\}$  where  $0 < \lambda \leq 1$  is a parameter that determines the rate at which the endowment declines with age and  $\eta$  is the intertemporal elasticity of substitution. We chose an income profile in which the endowment process is related to the preference parameter  $\eta$  in order to derive a simple closed form expression for the balanced steady-state interest rate. For this choice of the endowment process we show in Appendix A that there is a unique balanced equilibrium which occurs at  $R = \lambda/\beta$ .

Let

$$h_\omega(R) \equiv \frac{\omega_1}{1} + \frac{\omega_2}{R} + \frac{\omega_3}{R^2},$$

be the steady-state human wealth of a new-born, as a function of  $R$ , for an economy with endowment profile  $\omega$ . An equilibrium balanced gross interest rate is a positive number  $R_{bal}$  that solves the

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<sup>6</sup>The adjective ‘generically’ is required because there exists a set of measure zero in the parameter space for which the two steady states coincide. When  $d \neq 0$ , a continuity argument establishes that there is an open set  $d \in (d_L, d_U)$ , which contains  $d = 0$ , for which the number of steady-state equilibria and the determinacy properties of each steady-state equilibrium is the same as the case where  $d = 0$ . It follows that, as long as the primary budget deficit is not too large, our analysis of the properties of equilibria for the case of  $d = 0$  carries over to the case where the treasury runs a primary deficit or a primary surplus.



equation

$$f_{\omega_\lambda}(R_{bal}) = 0.$$

We show in Appendix A that if  $R_{bal}$  is a balanced equilibrium for an economy with endowment profile  $\omega_\lambda$  then it is also a balanced equilibrium of an economy with an alternative endowment profile  $\tilde{\omega}$  as long as

$$h_{\tilde{\omega}}(R_{bal}) = h_{\omega_\lambda}(R_{bal}) \quad (10)$$

and

$$\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3 = 1 + \lambda^\eta + \lambda^{2\eta}. \quad (11)$$

Equation (10) says that, when the interest rate is equal to  $R_{bal}$ , the new-born human wealth at the alternative endowment is the same as the new-born human wealth of the original endowment and Equation (11) says that the aggregate endowment of the alternative endowment is unchanged from the original aggregate endowment. Equations (10) and (11) constitute two equations in the three unknowns,  $\tilde{\omega}_1$ ,  $\tilde{\omega}_2$  and  $\tilde{\omega}_3$ .

We seek an alternative endowment pattern for which the autarkic steady-state of the original economy,  $R_{bal}$ , is also an equilibrium for the perturbed endowment. To accomplish this task, we parameterize the endowment profile by  $\tilde{\omega}_2$  and we solve equations(10) and (11) to find functions  $\tilde{\omega}_1(\tilde{\omega}_2)$  and  $\tilde{\omega}_3(\tilde{\omega}_2)$ . To further simplify the problem, we choose the special case of  $\lambda = \beta = 1$  to arrive at an aggregate savings function with just two parameters;  $\eta$  and  $\tilde{\omega}_2$ . We prove in Appendix A, that for these parameter values the slope of the aggregate savings function at the new endowment profile, evaluated at the steady state, changes sign, for values of

$$\eta < \frac{\tilde{\omega}_2 - 1}{4}, \quad (12)$$

and that this change in sign is associated with the creation of two new steady states. Since  $\tilde{\omega}_2$  is bounded above by  $1 + \lambda^\eta + \lambda^{2\eta}$  it follows from Inequality (12) that, when  $\lambda = 1$ , the intertemporal elasticity of substitution must be strictly less than 1/2 for multiple steady states to arise.

In the top panel of Figure 1 we graph the function  $f_{\tilde{\omega}}$  for  $\eta = 1/3.5$ ,  $\beta = 1$ ,  $\lambda = 1$  and an endowment profile  $\tilde{\omega} = \{0.335, 2.33, 0.335\}$ . For these parameters, the necessary condition for the existence of multiple steady state equilibria, Inequality (12), is satisfied. In the bottom panel we plot the excess demand for goods for these parameter values. This panel illustrates that when  $R = 1$ , the golden rule steady state and the middle balanced steady state coincide. This property is reflected in the fact that the excess demand for goods is tangent to the zero line at  $R = 1$ .<sup>7</sup>

## 6. THE DETERMINACY OF STEADY-STATE EQUILIBRIA

The fact that there exist multiple steady states does not imply anything about the determinacy properties of any one of them and for some parameter values there may exist other attracting sets including limit cycles and, possibly, chaotic attractors. However, we found computationally that, for a range of parameters, the golden rule steady state in our three-generation model displays second-degree indeterminacy for an active-passive policy combination. The following paragraphs

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<sup>7</sup>Varying  $\beta$  and or  $\lambda$  shifts both curves and leads to the separation of the golden rule steady-state equilibrium from the middle of the three balanced steady-state equilibria.

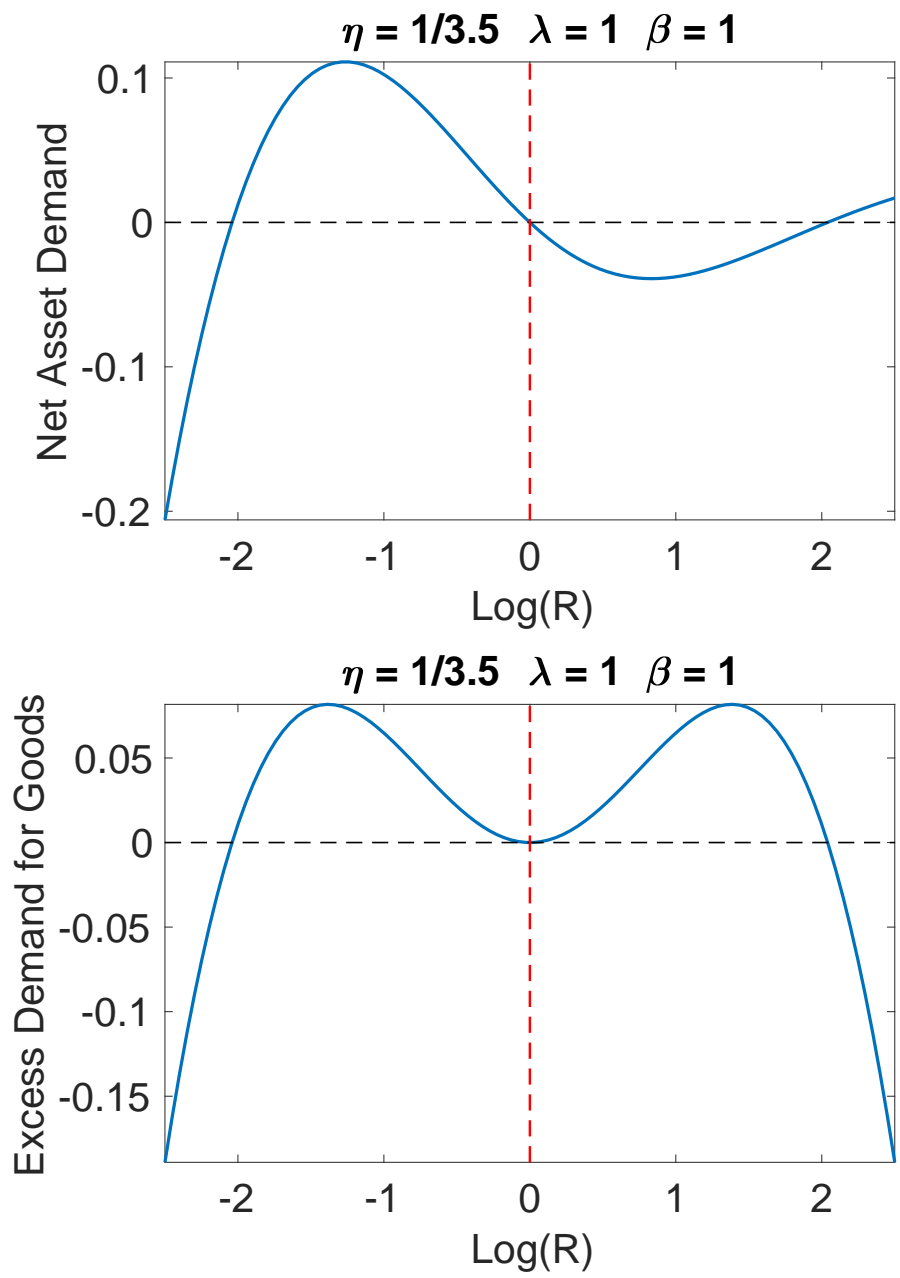


FIGURE 1. The Aggregate Savings Function for Parameter Values that Satisfy the Multiplicity Condition

define this concept and they explain its relevance for the theory of optimal fiscal and monetary policy.

Consider all pairs of initial values

$$b_1 \equiv \frac{B_0}{p_1}, \quad R_2,$$

8

that are close to the steady state values of  $b$  and  $R$  at the golden rule. If there is a unique pair,  $\{b_1, R_2\}$ , such that the trajectory that starts from this pair converges to the steady state, the golden rule steady state is said to be *locally determinate*.<sup>8</sup> If there is a one dimensional manifold of values – defined by a function  $b_1 = \phi(R_2)$  – such that all solutions to equations (6) and (7) that begin on this manifold converge to the steady state; the golden rule steady state is said to display one degree of indeterminacy. If there is a two dimensional manifold – containing the golden rule steady state – such that all solutions to equations (6) and (7) that begin on this manifold converge to the steady state; the steady state is said to display two degrees of indeterminacy.

The degree of indeterminacy of a steady state depends on the actions of the monetary and fiscal authorities. The assumption of a constant nominal interest rate implies that monetary policy is passive and the fact that  $d_t$  is not responsive to variations in the value of outstanding debt implies that fiscal policy is active. Arguably, this is the relevant policy mix for the recent policy environment in which the interest rate was at or near zero and unresponsive to realized inflation and where national treasuries were pursuing unrestrained spending programs that did not appear responsive to growing debt to GDP ratios. For this mix of an active fiscal and a passive monetary policy combination, our model displays *two-degrees of indeterminacy* at the golden rule steady state. In a representative agent economy, this policy mix would cause the steady state, targeted by the monetary authorities, to be determinate.<sup>9</sup>

## 7. THE $T$ -PERIOD PRODUCTION ECONOMY

This section describes the generalization of the 3-generation example to a production economy with  $T$ -generations. To handle this more general model we make two amendments to the exchange economy. First, we add a production sector and derive four functions that describe the dependence of the rental rate, the wage rate, the capital stock and output on the real interest rate. Second we explain how the addition of additional generations complicates the initial conditions. The main difference from our earlier 3-generation example is that in the  $T$ -generation model, the initial conditions depend on the nominal asset positions of  $T - 2$  non-generic generations and on the initial price level.

**7.1. Firms.** We assume that a unique commodity, denoted  $y_t$ , is produced from labor,  $L_t$  and capital  $k_t$  by a large number of competitive firms using a Cobb-Douglas technology,

$$y_t = L_t^{1-\theta} k_t^\theta, \tag{13}$$

where  $\theta$  is the elasticity of output with respect to capital. Labor is inelastically supplied and aggregate labor supply is fixed at

$$L_t = 1.$$

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<sup>8</sup> $R_2$  denotes the relative price of a good at date 2 for a good in period 1 and its value is known in the initial period. Although time begins at date 1, there is no  $R_1$ .

<sup>9</sup>In Section 10 we relax the assumption of a passive monetary policy and we show that the golden rule equilibrium still displays one degree of indeterminacy, even for the case in which monetary and fiscal policy are both active. This finding means that, although there is a unique equilibrium price sequence for every initial real interest rate, the real interest rate itself is not pinned down by fundamentals.

Let  $r_t$  be the real rental rate,  $w_t$  the real wage rate and let  $\delta$  represent the rate of capital depreciation. We show in Appendix C that profit maximization leads to the following four functions which describe the real rental rate, the real wage rate, the capital stock and output at date  $t$  as functions of  $R_t$ .

$$r_t = F_r(R_t) \equiv R_t - 1 + \delta, \quad (14)$$

$$w_t = F_w(R_t) \equiv (1 - \theta) \left( \frac{\theta}{F_r(R_t)} \right)^{\frac{\theta}{1-\theta}}, \quad (15)$$

$$k_t = F_k(R_t) \equiv \left( \frac{\theta}{F_r(R_t)} \right)^{\frac{1}{1-\theta}}, \quad (16)$$

$$y_t = F_k(R_t)^\theta. \quad (17)$$

**7.2. Households.** Households are endowed with efficiency units of labor, distributed over the  $T$  periods of their lives according to the endowment profile  $\{\omega_1, \omega_2, \dots, \omega_T\}$  where

$$\sum_{t=1}^T \omega_t = 1.$$

In our calibrated example we fix the weights  $\omega_\tau$  for  $\tau = 1, \dots, T$  to mirror the U.S. income distribution by age.

A generation  $t$  household solves the problem

$$\max_{\{c_t^t, \dots, c_{t+T-1}^t\}} U^t(c_t^t, c_{t+1}^t, \dots, c_{t+T-1}^t),$$

such that

$$\begin{aligned} c_t^t + s_{t+1}^t &\leq \omega_1 F_w(R_t), & c_{t+1}^t + s_{t+2}^t &\leq R_{t+1} s_{t+1}^t + \omega_2 F_w(R_{t+1}), \\ \dots & & c_{t+T-1}^t &\leq R_{t+T-1} s_{t+T-1}^t + \omega_T F_w(R_{t+T-1}), \end{aligned} \quad (18)$$

where  $F_w(R_t)$  is the real wage at date  $t$  as a function of the gross interest rate. The solution to this problem is characterized by a set of  $T - 1$  savings functions, one for each of the first  $T - 1$  periods of life

$$s_k^t(R_t, R_{t+1}, \dots, R_{t+T-1}), \quad k = t, \dots, t + T - 2, \quad (19)$$

together with the requirement that the  $T$  budget constraints (18) hold with equality. In Appendix B, Section B.1, we characterize the solution to this problem for the case of CES preferences and we find an explicit formula for the aggregate asset demand function,

$$S(R_{t-T+2}, R_{t-T+3}, \dots, R_{t+T-1}) \equiv \sum_{\tau=t-T+2}^t s_t^\tau(R_\tau, R_{\tau+1}, \dots, R_{\tau+T-1}),$$

where  $S(\cdot)$  is the sum of the savings functions at date  $t$  for generations  $t - T + 2$  to  $t$ .

**7.3. Equilibrium.** Define the vector  $X_t$

$$X_t = [R_{t-T+2}, \dots, R_{t+T-1}]^\top,$$

and the private asset-demand function

$$F_A(X_t) \equiv S(R_{t-T+2}, \dots, R_{t+T-1}) - F_k(R_{t+1}). \quad (20)$$

A competitive equilibrium is a non-negative bounded sequence of real interest rates and a bounded sequence of net government bond demands that satisfies equations (21) and (22).

$$F_A(X_t) = b_t + d_t, \quad (21)$$

$$b_{t+1} = R_{t+1}(b_t + d_t). \quad (22)$$

Equation (21) characterizes sequences of real interest rates for which the net asset demand of the private sector is equal to the public sector borrowing requirement. Equation (22) describes the evolution of the public sector borrowing through time. These two equations differ from the representation of equilibrium in the 3-generation model in two ways. First, private savings may be held in the form of productive capital as well as in the form of government debt. This accounts for the appearance of the term  $F_k(R_{t+1})$  in Equation (20). Second, the savings function of generation  $t$  depends on  $R_t$  through the dependence of the date  $t$  wage on the capital stock.<sup>10</sup>

In Appendix C.2 we show that dynamic equilibria can be described by a difference equation

$$F(X_t, X_{t-1}) \equiv F_A(X_t) - R_t F_A(X_{t-1}) + d_t = 0, \quad (23)$$

and we find a linear approximation to that difference equation around a steady state of the form

$$J_1 \tilde{X}_t = J_2 \tilde{X}_{t-1}, \quad (24)$$

where  $\tilde{X}$  is a vector of deviations from a steady state and the matrices  $J_1$  and  $J_2$  are the Jacobians of  $F(\cdot)$  with respect to  $X_t$  and  $X_{t-1}$  evaluated at this steady state.

The analysis in Appendix C.2 establishes that  $X_t$  has  $2(T-1)$  elements and Appendix C.3 establishes that the non-generic equations place  $T-1$  restrictions on the elements of  $X_1$  and  $X_2$ . Using these results, in Appendix, D we prove the following proposition which is based on the work of Blanchard and Kahn (1980).

**Proposition 1** (Blanchard-Kahn). *Let  $K$  denote the number of generalized eigenvalues of  $(J_1, J_2)$  with modulus greater than 1.<sup>11</sup>*

- *If  $K > T - 1$  there are no bounded sequences that satisfy the equilibrium conditions in the neighbourhood of  $\bar{X}$ . In this case equilibrium does not exist.*
- *If  $K = T - 1$  there is a unique bounded sequence that satisfies the equilibrium equations. Further, this sequence converges asymptotically to the steady state  $(\bar{R}, \bar{b})$ . In this case the steady state equilibrium  $(\bar{R}, \bar{b})$  is determinate.*
- *If  $K \in \{0, \dots, T - 2\}$  there is a  $T - 1 - K$  dimensional subspace of initial conditions that satisfy the equilibrium equations. All of these initial conditions are associated with*

<sup>10</sup>In the exchange economy, the functions  $s_k^t(\cdot)$  for  $k = t, \dots, t + T - 2$  depend on  $\{R_{t+1}, \dots, R_{t+T-1}\}$ . In the model with production they depend on  $\{R_t, \dots, R_{t+T-1}\}$ . The extra term  $R_t$  appears because household income depends on  $w_t$  which is a function of  $R_t$  in equilibrium.

<sup>11</sup>The generalized eigenvalues of  $(A, B)$ , are values of  $\lambda \in \mathbb{C}$  that solve the equation  $\det(J_1 - \lambda J_2) = 0$ .

sequences that converge asymptotically to the steady state  $(\bar{R}, \bar{b})$ . In this case the steady state equilibrium  $(\bar{R}, \bar{b})$  is indeterminate with degree of indeterminacy equal to  $T - 1 - K$ .

It follows from Proposition 1 that we can compute the degrees of determinacy around a given steady-state equilibrium by comparing the number of generalized eigenvalues of  $(J_1, J_2)$  that lie outside the unit circle with  $T - 1$ , where  $T$  is the number of generations. In the simulations presented in Section 8, we use this proposition to compute these generalized eigenvalues in the neighbourhood of each of the four steady states and we simulate non-stationary paths by iterating a linear approximation to the function  $F(\cdot)$  around the golden-rule steady state.<sup>12</sup>

In our model, fiscal policy is active but monetary policy is passive. In a representative agent model, the FTPL dictates that this policy mix should lead to a unique initial price level. In Section 8 we provide an example of an OLG economy with a steady-state equilibrium where money has value and where the FTPL fails to hold. In this example, it is not only the initial price level that is indeterminate; it is also the initial real interest rate.

## 8. A SIXTY-TWO GENERATION EXAMPLE

In this section we construct a sixty-two generation model where each generation begins its economic life at age 18 and in which a period corresponds to one year. We calibrate the age-profile of the representative person's endowment to U.S. data and we show that – for low values of the intertemporal elasticity of substitution – there exists a steady state that displays *real* indeterminacy, even when monetary and fiscal policy are both active.

Our result is important because it casts doubt on the universal applicability of established results about the Fiscal Theory of the Price Level. In representative agent models, the combination of an active/passive monetary-fiscal policy pair can eliminate indeterminacy in a competitive equilibrium model. We show that this result does not extend to a realistically calibrated OLG model.

To extend our model to a  $T$ -generation production economy we assume that the members of generation  $t$  maximize the utility function,

$$u(c_t^t, \dots, c_{t+61}^t) = \sum_{i=1}^{62} \beta^{i-1} \left( \frac{[c_{t+i-1}^t]^\alpha - 1}{\alpha} \right),$$

and that the productivity of their labor varies over the lifecycle. All labor is inelastically supplied to a competitive production sector which combines labor and capital in a Cobb Douglas technology. Households save by holding productive capital and government debt which are perfect substitutes. We calibrate the income profile of a representative generation to U.S. data and we provide explicit formulas for the excess demand functions for this functional form in Appendix B.

We graph our calibrated income profile in Figure 2. Our representative generation enters the labour force at age 18, retires at age 66, and lives to age 79. We chose the lifespan to correspond to current U.S. life expectancy at birth and we chose the retirement age to correspond to the age

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<sup>12</sup>The code used to generate all of our results is available online. Our code also replicates the findings reported in Kehoe and Levine (1983).

at which a U.S. adult becomes eligible for social security benefits. For the working-age portion of this profile we use data from Guvenen et al. (2021) which is available for ages 25 to 60. The working-age income profiles for ages 18 to 24 and for ages 61 to 66, were extrapolated to earlier and later years using log-linear interpolation. For the retirement portion we used data from the U.S. Social Security Administration.

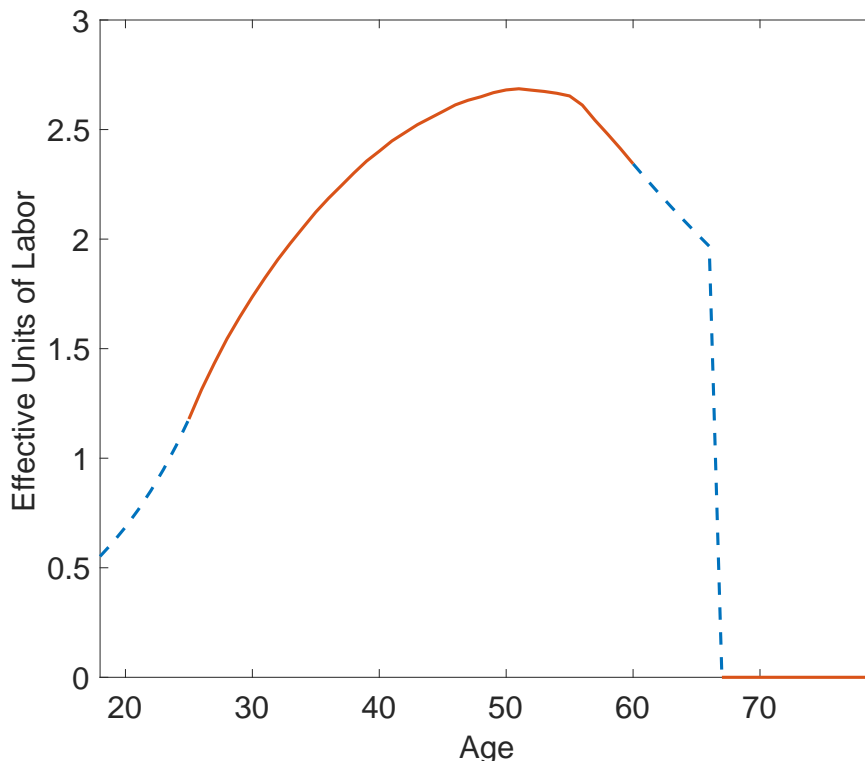


FIGURE 2. Normalized Endowment Profile. U.S. Data in Solid Red: Interpolated Data in Dashed Blue.

U.S. retirement income comes from three sources; private pensions, government social security benefits, and Supplemental Security Income. We treat private pensions and government social security benefits as perfect substitutes for private savings since the amount received in retirement is a function of the amount contributed while working. To calibrate the available retirement income that is independent of contributions, we used Supplemental Security Income which, for the U.S., we estimate at 0.137% of GDP.<sup>13</sup>

For the remaining parameters of our model we chose a primary budget deficit of  $d_t = 0$ , an annual discount rate of 0.995 and an elasticity of substitution of 0.034. The qualitative features of

<sup>13</sup>From Table 2 of the March 2018 Social Security Administration Monthly Statistical Snapshot we learn that the average monthly Supplemental Security Income for recipients aged 65 or older equalled \$447 (with 2,240,000 claimants), which implies that total monthly nominal expenditure on Supplemental Security Income equalled \$1,003 million. This compares to seasonally adjusted wage and salary disbursements (A576RC1 from FRED) in February 2018 of \$8,618,700 million per annum, or \$718,225 million per month. Back of the envelope calculations suggest that Supplemental Security Income in retirement equalled 0.137% of total labour income.

the equilibria are robust to the existence of a positive primary deficit with an upper bound that depends on the discount rate. For the calibrated income profile depicted in Figure 2 and for this choice of parameters, our model exhibits four steady-state equilibria. In Section 9 we explore the robustness of the properties of our model to alternative choices for the discount parameter and for the intertemporal elasticity of substitution.

The values and properties of all four steady-state equilibria are reported in Table 1. We refer to the balanced steady-state equilibria as Steady State A, Steady State C and Steady State D and to the golden-rule steady-state equilibrium as Steady State B. We see from this table that Steady States B, C and D are associated with a non-negative interest rate and are therefore dynamically efficient. Steady State A is associated with a negative interest rate of  $-9\%$  and is therefore dynamically inefficient.<sup>14</sup>

Type	Equilibrium Real Interest Rates				
	Interest Rate	Value of $\bar{b}$	# Unstable Roots	# Free Initial Conditions	Degree of Indeterminacy
Steady State A	$-9\%$	0	60	61	1
Steady State B	$0\%$	21 % of GDP	59	61	2
Steady State C	$.004\%$	0	60	61	1
Steady State D	$3.9\%$	0	61	61	0

TABLE 1. Steady States of the Sixty-Two Generation Model

The sixty-two generation production economy, with a calibrated income profile, is similar to the three generation endowment model from Section 5. In both models, the golden-rule steady-state equilibrium displays second degree indeterminacy. And in both models, the steady-state price level is positive and the initial price level is indeterminate even when fiscal policy is active. Importantly, because the monetary steady state is second-degree indeterminate, indeterminacy can hold even when both monetary and fiscal policy are active.

In Figure 3 we show the result of an experiment in which we perturb the initial value of the real interest rate by 1%, holding the price level, and the real wealth of all existing generations, fixed at their steady-state values. The upper panel of this Figure 3 plots the path by which the real interest rate returns to its steady-state value and the lower panel plots the return path of the value of physical capital expressed as a fraction of GDP. We refer to this perturbation as a 1% shock to the real interest rate.

This figure demonstrates that the return to the steady state following a relative price shock of this nature is extremely slow and that during the return the model displays prolonged periods of negative real interest rates.<sup>15</sup> This slow persistent return is generated by a pair of complex roots that are close to the unit circle and which *only* exist for calibrations of the model in which the steady-state equilibrium is indeterminate.

<sup>14</sup>See Cass (1972) for a definition and characterization of the conditions for dynamic efficiency.

<sup>15</sup>Extending these simulations to much longer time periods confirms that these oscillations do eventually converge back to the steady state.



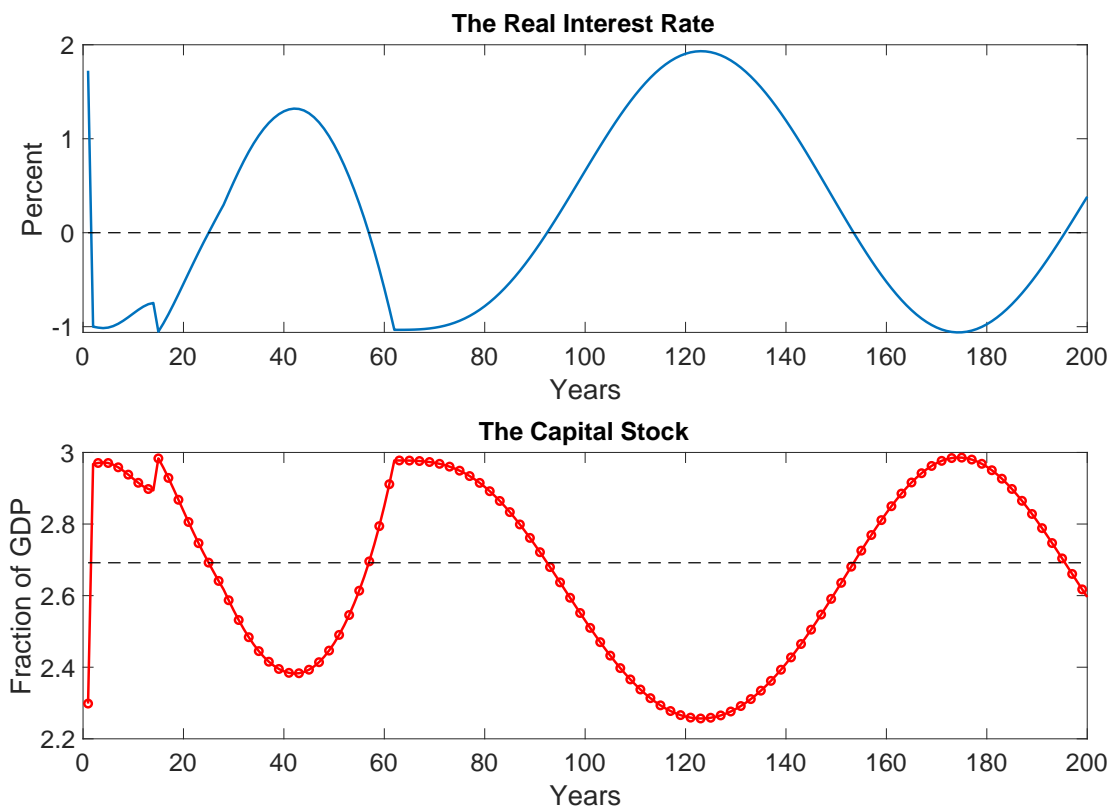


FIGURE 3. The Impact of a 1% Non-Fundamental Shock to the Initial Real Interest Rate

One may question whether the high degree of real interest rate persistence implied by our model is excessive. Have such long swings in real interest rates been observed in data? To address this question, Figure 4, reproduced from Yi and Zhang (2017), compares long run real interest rates in the G7 and documents that low-frequency real rate cycles, similar to those generated by our model, have characterized the evolution of real interest rates in all of these economies.<sup>16</sup>

## 9. ROBUSTNESS TO DIFFERENT CALIBRATIONS

To explore the robustness of our findings to alternative calibrations, in Table 2 we record the properties of our model for different values of the annual discount rate and the intertemporal elasticity of substitution. The example we featured in Section 8 had two degrees of indeterminacy and positive valued debt at the monetary steady state. Table 2 demonstrates that this property is not particularly special.

The table provides 40 different parameterizations of our model with intertemporal elasticity of substitution parameters ranging from .01 to .17 and discount rates ranging from 0.986 to 1. In all of these parameterizations we maintained the calibrated income profile illustrated in Figure

<sup>16</sup>See Yi and Zhang (2017) for a discussion of why long-run moving averages are likely to characterize trends in fundamental forces underlying real interest rates.

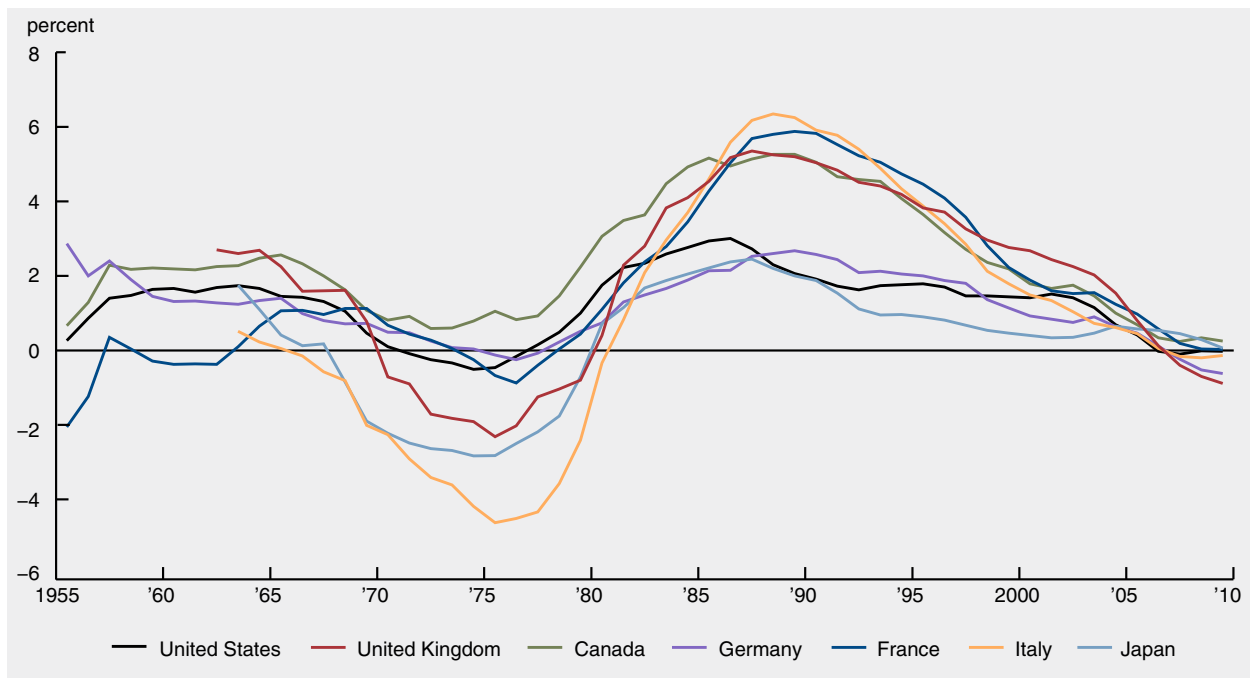


FIGURE 4. G7 Long-Run Real Interest Rates. Long-Run Real Interest Rates are 11-Year Centered Moving Averages of Annual Real Interest Rates.  
Source: Figure 1 in Yi and Zhang (2017)

		Annual Discount Factor							
		0.986	0.988	0.990	0.992	0.994	0.996	0.998	1.00
IES									
IES = 0.01	Degree of Indeterminacy	2	2	2	2	2	2	2	2
	Value of Debt	18	19	19	20	21	21	22	23
IES = 0.05	Degree of Indeterminacy	2	2	2	2	2	2	2	0
	Value of Debt	1	4	7	10	13	16	20	23
IES = 0.09	Degree of Indeterminacy	1	1	1	2	2	2	0	0
	Value of Debt	-16	-10	-5	1	6	12	17	23
IES = 0.13	Degree of Indeterminacy	1	1	1	1	1	2	0	0
	Value of Debt	-33	-25	-7	-9	-1	7	15	23
IES = 0.17	Degree of Indeterminacy	1	1	1	1	1	2	0	0
	Value of Debt	-50	-39	-29	-19	-8	2	12	23

TABLE 2. Robustness of Indeterminacy to Alternative Calibrations at the Golden Rule Steady State

2. For each calibration Table 2 displays the number of degrees of indeterminacy and the value of government debt at the golden-rule steady-state equilibrium. There are thirteen parameterizations in which the golden-rule steady state displays one degree of indeterminacy and twenty in which it displays two degrees of indeterminacy. In all twenty of these parameterizations, debt has positive value in the steady state.

## 10. FISCAL AND MONETARY POLICY

In this section we discuss what happens when we relax either the assumption that fiscal policy is active or the assumption that monetary policy is passive. We first show that passive fiscal policy makes indeterminacy more likely. We then demonstrate that ensuring bounded inflation under an active Taylor rule imposes an additional restriction on the set of equilibrium paths. This additional restriction reduces the degree of indeterminacy by one.

Consider first what happens when fiscal policy is passive. To model a passive fiscal policy we assume that the treasury raises taxes,  $\tau_t$ , in proportion to the real value of outstanding debt to ensure that the primary deficit  $d_t$  satisfies the equation

$$d_t = -\delta_b b_t,$$

where  $\delta_b \geq 0$  is a debt repayment parameter. Combining this assumption with the government debt accumulation equation leads to the following amended debt accumulation equation,

$$b_{t+1} = [R_{t+1} - \delta_b]b_t.$$

For values of  $[\bar{R} - \delta_b] < 1$  the effect of making fiscal policy passive is to introduce an additional stability mechanism that increases the degree of indeterminacy at each of the four steady states whenever  $\delta_b$  is large enough. Passive fiscal policy makes indeterminacy more likely.

We next assume that fiscal policy is active and the central bank follows a Taylor rule (Taylor, 1999),

$$1 + i_t = \left( \frac{\bar{R}}{\bar{\Pi}\phi_\pi} \right) \Pi_t^{1+\phi_\pi}, \quad t = 1, \dots, \infty. \quad (25)$$

Because this equation begins at date 1, the nominal interest rate in period 1 depends on  $p_0$  through the definition,  $\Pi_1 = p_1/p_0$ . We treat  $p_0$  as an initial condition that has the same status as the initial value of nominal debt,  $B_0$ . In Eq. (25),  $\bar{\Pi}$  is the inflation target,  $\bar{R}$  is the steady state real interest rate and  $\phi_\pi$  is the response coefficient of the policy rate to deviations of inflation from target. The Taylor Rule is passive if  $-1 \leq \phi_\pi \leq 0$  and active if  $\phi_\pi > 0$ .

When the central bank follows a Taylor Rule, the real interest rate and the real value of government debt continue to be determined by the bond market clearing equation and the debt accumulation equation. It follows that the conditions we have characterized in previous sections continue to ensure that the real interest rate and the real value of government debt remain bounded.

When the central bank follows a passive Taylor Rule, (see Appendix E.1) the following equation characterizes the asymptotic behaviour of the future inflation rate,

$$\lim_{T \rightarrow \infty} \tilde{\Pi}_{T+1} = \lim_{T \rightarrow \infty} (1 + \phi_\pi)^T \tilde{\Pi}_1 - \lim_{T \rightarrow \infty} \sum_{s=1}^T (1 + \phi_\pi)^{T-s} \tilde{R}_{T+1}, \quad (26)$$

where  $\kappa \equiv \bar{\Pi}/\bar{R}$  and the tilde denotes deviations from the steady state. The limit of the first term on the right side of Equation (E2) is zero because  $1 + \phi_\pi < 1$  and the second term is finite as a consequence of the boundedness of  $R_t$ . It follows that inflation is bounded whenever  $R_t$  is bounded. This is a generalization of the argument we made for the boundedness of the inflation rate when

the central bank follows an interest rate peg and it does not impose any additional restrictions on the equations of the model for an equilibrium to be determinate.

When the central bank follows an active Taylor Rule, (see Appendix E.2), the initial price level is determined by the forward-looking equation

$$p_1 = p_0 \left( \bar{\Pi} + \kappa \sum_{s=1}^{+\infty} \left( \frac{1}{1 + \phi_\pi} \right)^s (R_{1+s} - \bar{R}) \right). \quad (27)$$

Importantly, this restriction on the set of equilibrium paths is additional to the restriction

$$p_1 = \frac{B_0}{b_1},$$

that we used to generate the equilibrium sequence of interest rates. It follows that we are no longer free to pick  $R_2$  and  $p_1$  independently of each other. For any *given* choice of the initial interest rate,  $R_2$ , active monetary policy removes nominal indeterminacy. Crucially, however, it does not remove real indeterminacy and there continue to be many possible choices for the initial real interest rate, each of them associated with a different initial price level and a different equilibrium path for all future real interest rates and all future inflation rates.

## 11. SO WHAT?

How should the reader react to our finding that a particular example of an OLG economy displays indeterminate steady state equilibria? One possible reaction is that the real world is demonstrably determinate in the sense that a general equilibrium theorist would use that term. An advocate of this position might claim that the profession has rejected the OLG model after careful consideration and that the new-Keynesian version of the representative agent model has been demonstrated through careful empirical work to be a much better fit to time series and cross-section data. We do not think this argument holds water. The OLG model fell from favor as the preferred vehicle for understanding monetary and fiscal policy for theoretical reasons, not because it failed a series of empirical tests.<sup>17</sup>

It might be argued that we have dealt with the finite-lived case and that in the real world people are connected by operative chains of bequests and that they effectively have an infinite horizon (Barro, 1974). To counter that argument, Weil (1987) takes on Barro's argument and considers an economy in which occasionally, family dynasties break apart and 'unloved children' build new families. Pietro Reichlin (1992) has shown that this model displays multiple sets of indeterminate steady-state equilibria even though each family may have an infinite horizon.

A further argument that might be levelled against indeterminate steady-state equilibria is that they are unlearnable. According to this argument, people are adaptive learners and a rational expectations equilibrium accurately describes the properties of an economy after learning has taken place. Proponents of this argument claim that determinate steady-state equilibria are often stable under learning and that this is a good reason to select these equilibria when a model has multiple

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<sup>17</sup>The model went out of favor, in part because a subset of influential macroeconomists considered the existence of indeterminacy in the model to be a shortcoming rather than a strength of the approach (Cherrier and Saidi, 2018). For summary of recent research that uses indeterminacy as a positive aspect of DSGE models see Farmer (2020).

steady state equilibria (McCallum, 2003, 2007). This would be a persuasive argument if it were always true that indeterminate steady states are unstable under learning, but exhaustive enquiries into the stability of adaptive learning schemes have found that both determinate and indeterminate steady-state equilibria may be stable under plausible adaptive learning schemes (Evans and Honkapohja, 2001). And the non-stationary equilibria of a monetary model in which artificially intelligent agents use deep reinforcement learning – an algorithm similar to the one that was used by Deep Blue (Hsu et al., 2018) to beat world chess grand masters – has been shown to converge to the Pareto superior equilibrium in a monetary model with two equilibria, even when that equilibrium is indeterminate (Chen et al., 2021).

We have established the *possibility* of the existence of multiple balanced indeterminate steady states: but are the parameter values that lead to indeterminacy consistent with empirical evidence? The calibrations in Table 2 demonstrate that our results require first, that the *ies* is low and second, that the discount factor is close to one. Are our calibrated values plausible? Consider first, our baseline calibration in which we choose  $ies = .034$ . In Thimme (2017), Julian Thimme reviews a range of micro and macro estimates of the *ies* and he concludes that in “... almost every subsection of this paper we list studies that report estimates not significantly different from 0, as well as studies that report estimates above 1”.<sup>18</sup> We conclude from this meta analysis that our calibrated value of the *ies* is consistent with some, but not all, existing estimates. Some of the studies cited by Thimme assume the existence of a representative agent, and others use micro-data sets. The main take away from his analysis is that there is no unique way to interpret ‘the *ies*’ and that the value assigned to the parameter in an empirical study is context dependent.

Consider second, our calibration of an annual discount factor of 0.995, which implies a much lower rate of time preference than those found in calibrated and estimated models based on the representative agent assumption. In these models there is a direct correspondence, between the rate of time preference and the rate of interest and a discount factor of 0.995 would be inconsistent with data. However, our model has an overlapping generations structure and in the OLG model there is no direct correspondence between the rate of time preference and the rate of interest.

The strongest evidence in favor of our calibration comes from parameter estimates of an empirical version of our model developed by Farmer and Farmer (2022). Building on the current paper, these authors construct a perpetual youth model in which agents have EZ preferences (Epstein and Zin, 1989) and in which asset prices are driven by both fundamental and non-fundamental uncertainty. They show that a hump-shaped income profile and a low *ies* are consistent with the existence of multiple indeterminate balanced growth paths. Further, the estimated values of the *ies* and the rate of time preference in their model are associated with an indeterminate steady state when both monetary and fiscal policy are active. We conclude from Farmer and Farmer’s empirical analysis, that the theoretical results we have developed in this paper can be used to construct a plausible alternative to the representative agent model in which non-fundamental shocks drive asset price movements.

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<sup>18</sup>Thimme (2017, pp 248).

## REFERENCES

- AIYAGARI, S. R. (1988): “Nonmonetary Steady States in Stationary Overlapping Generations Models with Long Lived Agents and Discounting: Multiplicity, Optimality and Consumption Smoothing,” *Journal of Economic Theory*, 45, 102–127.
- AUERBACH, A. J. AND L. KOTLIKOFF (1987): *Dynamic Fiscal Policy*, Cambridge: Cambridge University Press.
- AZARIADIS, C. (1981): “Self-fulfilling Prophecies,” *Journal of Economic Theory*, 25, 380–396.
- BALASKO, Y. AND K. SHELL (1981): “The overlapping-generations model. II. The case of pure exchange with money,” *Journal of Economic Theory*, 24, 112–142.
- BARRO, R. J. (1974): “Are Government Bonds Net Wealth?” *Journal of Political Economy*, 82, 1095–1117.
- BLANCHARD, O. J. (1985): “Debt, Deficits, and Finite Horizons,” *Journal of Political Economy*, 93, 223–247.
- BLANCHARD, O. J. AND C. M. KAHN (1980): “The Solution of Linear Difference Models Under Rational Expectations,” *Econometrica*, 48, 1305–1311.
- CASS, D. (1972): “On capital overaccumulation in the aggregative neoclassical model of economic growth: A complete characterization,” *Journal of Economic Theory*, 4, 200–223.
- CHALK, N. A. (2000): “The Sustainability of Bond-Financed Deficits: An Overlapping Generations Approach,” *Journal of Monetary Economics*, 45, 293–328.
- CHEN, M., A. JOSEPH, M. KUMHOF, X. PAN, R. SHI, AND X. ZHOU (2021): “Deep Reinforcement Learning in a Monetary Model,” *Rebuilding Macroeconomics Working Paper Series*.
- CHERRIER, B. AND A. SAIDI (2018): “The Indeterminate Fate of Sunspots in Economics: 1972–2014,” *History of Political Economy*, 50, 425–457.
- COCHRANE, J. H. (2023): *The Fiscal Theory of the Price Level*, Princeton, N.J.: Princeton University Press.
- EPSTEIN, L. AND S. ZIN (1989): “Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis.” *Journal of Political Economy*, 99, 263–286.
- EVANS, G. W. AND S. HONKAPOHJA (2001): *Learning and Expectations in Macroeconomics*, Princeton: Princeton University Press.
- FARMER, L. E. AND R. E. A. FARMER (2022): “Zoomers and Boomers: Asset Prices and Intergenerational Inequality,” *National Bureau of Economic Research*, 30419.
- FARMER, R. E. A. (1986): “Deficits and Cycles,” *Journal of Economic Theory*, 40, 77–88.
- (2020): “The Indeterminacy School in Macroeconomics,” *Oxford Research Encyclopedia of Economics and Finance*.
- FARMER, R. E. A. AND M. WOODFORD (1997): “Self-fulfilling Prophecies and the Business Cycle,” *Macroeconomic Dynamics*, 1, 740–769.
- GALE, D. (1973): “Pure Exchange Equilibrium of Dynamic Economic Models,” *Journal of Economic Theory*, 6, 12–26.

- GOLUB, G. H. AND C. F. V. LOAN (1996): *Matrix Computations*, John Hopkins University Press, third ed.
- GUVENEN, F., F. KARAHAN, S. OZKAN, AND J. SONG (2021): “What Do Data on Millions of U.S. Workers Reveal about Life-Cycle Earnings Risk?” *Econometrica*, 89, 2303–2339.
- HSU, F.-H., M. CAMPBELL, AND J. A. HOANE JR. (2018): “Deep Blue System Overview,” *Proceedings of the 9th International Conference on Supercomputing. 1995 International Conference on Supercomputing. Association for Computer Machinery*, 240–244.
- KEHOE, T. J. AND D. K. LEVINE (1983): “Indeterminacy of Relative Prices in Overlapping Generations Models,” *Unpublished MIT Working Paper no. 313*.
- (1985): “Comparative Statics and Perfect Foresight in Infinite Horizon Economies,” *Econometrica*, 53, 433–453.
- KUBLER, F. AND K. SCHMEDDERS (2011): “Lifecycle Portfolio Choice, the Wealth Distribution and Asset Prices,” *University of Zurich, mimeo*.
- LEEPER, E. M. (1991): “Equilibria Under ‘Active’ and ‘Passive’ Monetary and Fiscal Policies,” *Journal of Monetary Economics*, 27, 129–147.
- LEEPER, E. M. AND C. LEITH (2016): “Understanding Inflation as a Joint Monetary-Fiscal Phenomenon,” in *Handbook of Macroeconomics*, ed. by J. B. Taylor and H. Uhlig, Elsevier, vol. 2, chap. 30, 2305–2415.
- MCCALLUM, B. T. (2001): “Indeterminacy, Bubbles, and the Fiscal Theory of Price Level Determination,” *Journal of Monetary Economics*, 47, 19–30.
- (2003): “Multiple-Solution Indeterminacies in Monetary Policy Analysis,” *Journal of Monetary Economics*, 50, 1153–1175.
- (2007): “E-stability vis-à-vis Determinacy Results for a Broad Class of Linear Rational Expectations Models,” *Journal of Economic Dynamics and Control*, 31, 1376–1391.
- REICHLIN, P. (1992): “Endogenous Cycles with Long-Lived Agents,” *Journal of Economic Dynamics and Control*, 16, 243–266.
- RÍOS-RULL, J. V. (1996): “Life-Cycle Economies and Aggregate Fluctuations,” *Review of Economic Studies*, 63, 465–489.
- SAMUELSON, P. A. (1958): “An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money,” *Journal of Political Economy*, 66, 467–482.
- SPEAR, S. E. AND W. YOUNG (2023): *Overlapping Generations: Methods, Models and Morphology*, Bingley, UK: Emerald Publishing.
- TAYLOR, J. B. (1999): “An Historical Analysis of Monetary Policy Rules,” in *Monetary Policy Rules*, ed. by J. B. Taylor, Chicago: University of Chicago Press, 319–341.
- THIMME, J. (2017): “Intertemporal Substitution in Consumption: A Literature Review,” *Journal of Economic Surveys*, 31, 226–257.
- WEIL, P. (1987): “Love Thy Children: Reflections on the Barro Debt Neutrality Theorem,” *Journal of Monetary Economics*, 19, 377–391.
- WOODFORD, M. (1994): “Price-Level Determinacy Without Control of a Monetary Aggregate,” *Carnegie Rochester Series on Public Policy*, 43, 1–46.

——— (2001): “Fiscal Requirements for Price Stability,” *Journal of Money, Credit, and Banking*, 33, 669–728.

YI, K.-M. AND J. ZHANG (2017): “Understanding Global Trends in Long-run Real Interest Rates,” *Economic Perspectives*, 1–20.



APPENDIX A. THE HUMP-SHAPED PROFILE AND THE IES

In this Appendix we derive an expression for the slope of the steady state savings function evaluated at the steady state  $R = \lambda/\beta$  for the parameter values  $\lambda = \beta = 1$ . Define the functions  $W(R)$  and  $\phi(R)$

$$W(R) = \tilde{\omega}_1 + \frac{\tilde{\omega}_2}{R} + \frac{\tilde{\omega}_3}{R^2}, \quad (\text{A1})$$

$$\phi(R) = 1 + \frac{(\beta R)^\eta}{R} + \frac{(\beta R)^{2\eta}}{R^2}. \quad (\text{A2})$$

Applying the solution to the  $T$ -generation maximizing problem with CES preferences from Appendix B we have the following steady-state consumption demand functions

$$c_1(R) = \frac{W(R)}{\phi(R)}, \quad c_2(R) = (\beta R)^\eta \frac{W(R)}{\phi(R)}, \quad c_3(R) = (\beta R)^{2\eta} \frac{W(R)}{\phi(R)}, \quad (\text{A3})$$

where subscripts indicate age. Define the steady-state savings functions of the young and middle-aged as

$$s_1(R) = \tilde{\omega}_1 - c_1(R), \quad s_2(R) = R s_1(R) + \tilde{\omega}_2 - c_2(R). \quad (\text{A4})$$

Next, we seek expressions for the functions  $\tilde{\omega}_1(\tilde{\omega}_2)$  and  $\tilde{\omega}_3(\tilde{\omega}_2)$  which solve the equations

$$\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3 = 1 + \lambda^\eta + \lambda^{2\eta}, \quad (\text{A5})$$

$$\tilde{\omega}_1 + \frac{\tilde{\omega}_2 \beta}{\lambda} + \frac{\tilde{\omega}_3 \beta^2}{\lambda^2} = 1 + \frac{\lambda^\eta \beta}{\lambda} + \frac{\lambda^{2\eta} \beta^2}{\lambda^2}. \quad (\text{A6})$$

These are given by the expressions

$$\tilde{\omega}_1(\tilde{\omega}_2) = 1 + \frac{\frac{\beta}{\lambda} \lambda^\eta}{\left(1 + \frac{\beta}{\lambda}\right)} - \frac{\frac{\beta}{\lambda} \tilde{\omega}_2}{\left(1 + \frac{\beta}{\lambda}\right)}, \quad (\text{A7})$$

$$\tilde{\omega}_3(\tilde{\omega}_2) = \frac{\lambda^\eta}{\left(1 + \frac{\beta}{\lambda}\right)} + \lambda^{2\eta} - \frac{\tilde{\omega}_2}{\left(1 + \frac{\beta}{\lambda}\right)}. \quad (\text{A8})$$

Define the function

$$\psi(R) = \frac{W(R)}{\phi(R)}, \quad (\text{A9})$$

and note that aggregate savings in a steady state equilibrium,  $f_{\tilde{\omega}}(R)$ , defined as the sum of  $S_1(R; \tilde{\omega})$  and  $S_2(R; \tilde{\omega})$  is given by the expression,

$$f_{\tilde{\omega}}(R) = \left( \tilde{\omega}_1(\tilde{\omega}_2) - \psi(R) \right) + R \left( \tilde{\omega}_1(\tilde{\omega}_2) - \psi(R) \right) + \tilde{\omega}_2 - \psi(R) \left( \beta R \right)^\eta. \quad (\text{A10})$$

Rearranging terms, this leads to the equation

$$f_{\tilde{\omega}}(R) = \tilde{\omega}_1(\tilde{\omega}_2) \left( 1 + R \right) + \tilde{\omega}_2 - \psi(R) \left( 1 + R + (\beta R)^\eta \right). \quad (\text{A11})$$

We seek an expression for the derivative of  $f_{\tilde{\omega}}(R)$  evaluated at  $\lambda = \beta = 1$ . For these parameter values the functions  $\tilde{\omega}_1(\tilde{\omega}_2)$ ,  $W(R)$  and  $\phi(R)$  are given by the following formulae

$$\tilde{\omega}_1(\tilde{\omega}_2) = \frac{3 - \tilde{\omega}_2}{2}, \quad W(R) = 1 + \frac{1}{R} + \frac{1}{R^2}, \quad \phi(R) = 1 + R^{\eta-1} + R^{2(\eta-1)}. \quad (\text{A12})$$

Evaluating each term at  $R = \lambda/\beta = 1$  gives

$$W(1) = 3, \quad \phi(1) = 3, \quad \text{from which it follows that } \psi(1) = 1. \quad (\text{A13})$$

The partial derivatives of  $W(R)$  and  $\phi(R)$  are given by

$$\frac{\partial W}{\partial R} = -\frac{1}{R^2} - \frac{2}{R^3}, \quad \frac{\partial \phi}{\partial R} = (\eta - 1)R^{\eta-2} + 2(\eta - 1)R^{2\eta-3}, \quad (\text{A14})$$

which when evaluated at  $R = \lambda/\beta = 1$  gives

$$\left. \frac{\partial W}{\partial R} \right|_{R=1} = -3, \quad \left. \frac{\partial \phi}{\partial R} \right|_{R=1} = 3(\eta - 1). \quad (\text{A15})$$

We seek an expression for the partial derivative of  $f_{\tilde{\omega}}(R)$  evaluated at the steady state  $R = \lambda/\beta = 1$ .

Using the chain rule, this is equal to

$$\left. \frac{\partial f_{\tilde{\omega}}}{\partial R} \right|_{R=1} = \tilde{\omega}_1(\tilde{\omega}_2) - \psi(1) \left( 1 + \eta \right) - 3 \left. \frac{\partial \psi}{\partial R} \right|_{R=1}. \quad (\text{A16})$$

A further application of the chain rule to the function  $\psi(R)$  leads to the expression

$$\left. \frac{\partial \psi}{\partial R} \right|_{R=1} = \frac{\phi(1) \left. \frac{\partial W}{\partial R} \right|_{R=1} - W(1) \left. \frac{\partial \phi}{\partial R} \right|_{R=1}}{W(1)^2} = \frac{-9 - 9(\eta - 1)}{9} = -\eta. \quad (\text{A17})$$

Putting all these pieces together gives

$$\left. \frac{\partial f_{\tilde{\omega}}}{\partial R} \right|_{R=1} = \frac{3 - \tilde{\omega}_2}{2} - (1 + \eta) + 3\eta = \frac{1 - \tilde{\omega}_2 + 4\eta}{2}. \quad (\text{A18})$$

□

## APPENDIX B. ANALYTIC SOLUTIONS FOR EXCESS DEMAND

**B.1. The generic optimization problem.** Consider a person with CES preferences who lives for  $T$  periods and has perfect foresight of future prices. This person solves the problem,

**Problem 1.**

$$\max_{\{c_t^t, c_{t+1}^t, \dots, c_{t+T-1}^t\}} \frac{a_1(c_t^t)^\alpha + a_2(c_{t+1}^t)^\alpha + \dots + a_T(c_{t+T-1}^t)^\alpha}{\alpha}, \quad (\text{B1})$$

subject to the lifetime budget constraint

$$\sum_{i=1}^T Q_t^{t-1+i} c_{t-1+i}^t = \sum_{i=1}^T Q_t^{t-1+i} \omega_i F_w(R_i). \quad (\text{B2})$$

Here,  $c_s^t$  is consumption in period  $s$  of a person born in period  $t$ ,  $i \in 1, \dots, T$  is age, and  $\omega_i$  is the labor-endowment weight and  $F_w(R_i)$  is the real wage at date  $i$  as a function of the gross real interest rate between periods  $i - 1$  and  $i$ . The parameters  $a_i$  are utility weights and  $\alpha \leq 1$  is a curvature parameter which is related to intertemporal substitution,  $\eta$ , by the identity

$$\eta \equiv \frac{1}{1 - \alpha}. \quad (\text{B3})$$

The term  $Q_t^k$ , defined by the expression

$$Q_t^k \equiv \prod_{j=t+1}^k \frac{1}{R_j}, \quad Q_t^t = 1, \quad (\text{B4})$$

is the relative price at date  $t$  of a commodity for delivery at date  $k$ .

This optimization problem includes the case of a constant discount factor  $\beta$  for which

$$[a_1, a_2, \dots, a_T] = [1, \beta, \dots, \beta^{T-1}] \quad (\text{B5})$$

and logarithmic preferences which is the limiting case when  $\alpha \rightarrow 0$ . We permit the discount factor to vary with age to nest the Kehoe and Levine (1983) example which we use to cross-check our results.

**Proposition 2.** *The solution to Problem 1 is given by*

$$\hat{c}_{t-1+k}^t = \frac{a_k^\eta \sum_{i=1}^T (\mathcal{Q}_t^{t-1+i} \omega_i F_w(R_i))}{\left(\mathcal{Q}_t^{t-1+k}\right)^\eta \sum_{i=1}^T \left(\mathcal{Q}_t^{t-1+i}\right)^{1-\eta} a_i^\eta}, \quad k = 1, \dots, T. \quad (\text{B6})$$

where  $\hat{c}_{t-1+k}^t$  denotes the consumption, at time  $t - 1 + k$ , of an agent born at time  $t$ .

*Proof.* The result follows directly from substituting the first-order conditions into the budget constraint and rearranging terms.  $\square$

**B.2. Non-generic optimization problems.** Let  $j$  be an index that runs from 1 to  $T-1$ . Consider a non-generic person born in period  $1 - j$  with real assets  $\frac{A_t^{1-j}}{p_1}$  who lives for  $T - j$  periods. This person solves Problem 2

**Problem 2.**

$$\max_{\{c_1^{1-j}, \dots, c_{1-j+T-1}^{1-j}\}} \frac{a_{T-j+1}(c_1^{1-j})^\alpha + a_{T-j+2}(c_2^{1-j})^\alpha + \dots + a_T(c_{1-j+T-1}^{1-j})^\alpha}{\alpha}, \quad j = 1, \dots, T-1 \quad (\text{B7})$$

subject to the lifetime budget constraint

$$\sum_{k=1}^{(1-j)+T-1} \mathcal{Q}_1^k \left( c_k^{1-j} - \omega_{k-(1-j)+1} F_w(R_{k-(1-j)+1}) \right) \leq \frac{A^{1-j}}{p_1}, \quad (\text{B8})$$

**Proposition 3.** *Let  $k \in \{1, \dots, T - j\}$ . The solution to Problem 2 is given by*

$$\hat{c}_k^{1-j} = \frac{a_{k+j}^\eta \left( \frac{A_t^{1-j}}{p_1} + \sum_{i=1}^{T-j} \mathcal{Q}_t^i \omega_{j+i} F_w(R_{j+i}) \right)}{\left(\mathcal{Q}_t^{t+k-1}\right)^\eta \sum_{i=1}^{T-j} \left(\mathcal{Q}_t^i\right)^{1-\eta} a_{j+i}^\eta}, \quad k = 1 \dots 1 - j + T - 1. \quad (\text{B9})$$

*Proof.* The problem above is identical to a generic one solved by an agent who has  $T - j$  periods to live, whose endowments are  $\left\{ \omega_{j+1} F_w(R_{j+1}) + \frac{A_t^{1-j}}{p_1}, \omega_{j+2} F_w(R_{j+2}), \dots, \omega_T F_w(R_T) \right\}$ , and whose preference parameters in the utility function are  $\{a_{j+1}, a_{j+2}, \dots, a_T\}$ .  $\square$

## APPENDIX C. EQUILIBRIUM AS THE SOLUTION TO A DIFFERENCE EQUATION

In Section 4 we showed that equilibria of the 3-generation model can be characterized as the solution to a difference equation, determined by the behaviour of the generic generations, together with a set of initial conditions determined by the behavior of the non-generic generations. In this Appendix we generalize our analysis to the  $T$ -generation model with capital.

**C.1. Production.** Let output  $y_t$  be produced by the function

$$y_t = k_t^\theta L_t^{1-\theta}, \quad (\text{C1})$$

and let capital depreciate at rate  $\delta$ . Profit maximization leads to the expressions

$$w_t L_t = (1 - \theta)y_t, \quad r_t k_t = \theta y_t, \quad (\text{C2})$$

where  $r_t$  is the real rental rate and  $w_t$  is the real wage. No arbitrage implies

$$r_t = F_r(R_t) \equiv R_t - 1 + \delta, \quad (\text{C3})$$

where  $R_t$  is the gross real rate of interest. By further rearranging equations (C1) – (C3) and imposing the labour supply equation,  $L_t = 1$ , we obtain the following expressions for  $w_t$ ,  $k_t$  and  $y_t$  as functions of  $R_t$ ;

$$w_t = F_w(R_t) \equiv (1 - \theta) \left( \frac{\theta}{F_r(R_t)} \right)^{\frac{\theta}{1-\theta}}, \quad (\text{C4})$$

$$k_t = F_k(R_t) \equiv \left( \frac{\theta}{F_r(R_t)} \right)^{\frac{1}{1-\theta}}, \quad (\text{C5})$$

$$y_t = F_k(R_t)^\theta. \quad (\text{C6})$$

**C.2. Equilibrium Difference Equation.** Define the vector  $X_t$  and the function  $F_A(X_t)$ ,

$$X_t = [R_{t-T+2} \dots, R_{t+T-1}]^\top, \quad (\text{C7})$$

$$F_A(X_t) \equiv S(R_{t-T+2}, \dots, R_{t+T-1}) - F_k(R_{t+1}), \quad (\text{C8})$$

where

$$S(R_{t-T+2}, \dots, R_{t+T-1}) \equiv \sum_{\tau=t-T+2}^t s_t^\tau(R_\tau, R_{\tau+1}, \dots, R_{\tau+T-1}), \quad (\text{C9})$$

and the functional forms of the functions  $s_t^r(\cdot)$  are derived by combining the solutions for the consumption functions from Appendix B with the fact that the sequence of budget constraints, (18), hold with equality.

Recall that a competitive equilibrium is characterized by a non-negative bounded sequence of real interest rates and a bounded sequence of net government bond demands that satisfies equations (C10) and (C11).

$$F_A(X_t) = b_t + d_t, \quad (\text{C10})$$

$$b_{t+1} = R_{t+1}(b_t + d_t), \quad (\text{C11})$$

and that a steady-state equilibrium is a non-negative real number  $\bar{R}$  and a (possibly negative) real number  $\bar{b}$  that solve the equations,

$$S(\bar{R}, \bar{R}, \dots, \bar{R}) - F_k(R) = \bar{b} + d, \quad \bar{b}(1 - \bar{R}) = \bar{R}d. \quad (\text{C12})$$

Let  $\{\bar{R}, \bar{b}\}$  be a steady state equilibrium and let

$$\tilde{R}_t \equiv R_t - \bar{R}, \quad \text{and} \quad \tilde{b}_t \equiv b_t - \bar{b}, \quad (\text{C13})$$

represent deviations of  $b_t$  and  $R_t$  from their steady state values. Define a function  $F(\cdot)$ ,

$$F(X_t, X_{t-1}) \equiv F_A(X_t) - R_t F_A(X_{t-1}) + d_t, \quad (\text{C14})$$

and let  $J_1$  and  $J_2$  represent the partial derivatives of this function with respect to  $X_t$  and  $X_{t-1}$ .

Using this notation, the local dynamics of equilibrium sequences close to the steady state can be approximated as solutions to the linear difference equation

$$J_1 \tilde{X}_t = J_2 \tilde{X}_{t-1}, \quad t = 2, \dots \quad (\text{C15})$$

with initial condition

$$\tilde{X}_1 = \bar{X}_1. \quad (\text{C16})$$

The local stability of these equations depends on the spectrum of the matrix pencil  $(A, B)$ , defined as solutions to the equation  $\det(J_1 - \lambda J_2)$ . We refer to the elements of the spectrum as generalized eigenvalues.

If one or more roots of  $\lambda(J_1, J_2)$  are outside of the unit circle there is no guarantee that sequences of interest factors and government debt generated by Equation (C15) will remain bounded. To ensure stability, we must choose initial conditions that place  $\tilde{X}_1$  in the linear subspace associated with the stable generalized eigenvalues of  $(J_1, J_2)$ . The initial conditions are determined by the non-generic equilibrium conditions which we turn to next.

**C.3. Initial Conditions.** Asset market equilibrium in periods 1 through  $T - 1$  is characterized by a family of aggregate net savings functions,  $G_{A_t}(\cdot)$ , for  $t = 1 \dots T - 1$  where  $G_{A_t}(\cdot)$  is aggregate private savings net of the period  $t + 1$  capital stock. These functions are non-generic analogues of the function  $F_A(X_t)$ . They are different at each date because the asset demand functions of the initial generations depend on the initial wealth distribution and the initial price level as well as on real interest rates.

Consider the example of  $T = 4$  which leads to the following asset market equilibrium equations in periods 1 through 4,

$$G_{A_1} \left( \frac{A^{-1}}{p_1}, \frac{A^0}{p_1}, k_1, R_2, R_3, R_4 \right) - \frac{B_0}{p_1} - d_1 = 0, \quad (\text{C18})$$

$$G_{A_2} \left( \frac{A^0}{p_1}, k_1, R_2, R_3, R_4, R_5 \right) - R_2 G_{A_1} \left( \frac{A^{-1}}{p_1}, \frac{A^0}{p_1}, k_1, R_2, R_3, R_4 \right) - d_2 = 0, \quad (\text{C19})$$

$$G_{A_3} (k_1, R_2, R_3, R_4, R_5, R_6) - R_3 G_{A_2} \left( \frac{A^0}{p_1}, k_1, R_2, R_3, R_4, R_5 \right) - d_3 = 0, \quad (\text{C20})$$

$$F_A (R_2, R_3, R_4, R_5, R_6, R_7) - R_4 G_{A_3} (k_1, R_2, R_3, R_4, R_5, R_6) - d_4 = 0. \quad (\text{C21})$$

The function  $G_{A_1}$  determines the net demand for government bonds in period 1 which must equal the net supply,  $B_0/p_1 + d_1$ . The period 1 capital stock enters the function  $G_{A_1}$  as a state variable that determines the date 1 real wage. The nominal liabilities  $A_1^{-1}$  and  $A^0$  enter because generations  $-1$  and  $0$  participate in the date 1 asset market. In period 2 the term  $A^{-1}$  is dropped from the function  $G_{A_2}$  because generation  $-1$  does not enter the asset markets in their final period of life. The term  $R_5$  enters this function because it enters the budget constraint of generation 2. In writing equations (C19) through (C21) we have used the government budget rule to substitute out for  $b_t$ , using the equality of debt with net asset demand from the previous period.

An equilibrium for the 4-generation production economy is a first order non-linear vector-valued difference equation in the 6 variables  $X_t \equiv \{R_{t-2}, R_{t-1}, R_t, R_{t+1}, R_{t+2}, R_{t+3}\}$  with restrictions on

$X_1$  and  $X_2$  given by equations (C18) – (C21). These restrictions constitute a system of 4 equations in the 7 unknowns variables  $p_1, R_2, R_3, R_4, R_5, R_6$  and  $R_7$ , leaving 3 free initial conditions.

Adding one period of life adds one additional period and one additional variable to this system of equations. The general result is that if people live for  $T$  periods, there are  $T - 1$  free initial conditions. An equilibrium for the  $T$ -generation production economy is characterized by a first-order vector-valued difference equation in the  $2(T - 1)$  variables  $X_t \equiv \{R_{t-T+1}, \dots, R_{t+T-1}\}$  with  $T - 1$  free initial conditions.

#### APPENDIX D. PROOF OF PROPOSITION 1

Define the Jacobians  $J_1(X_t, X_{t-1})$  and  $J_2(X_t, X_{t-1})$

$$J_1(X_t, X_{t-1}) = \frac{\partial F(X_t, X_{t-1})}{\partial X_t}, \quad J_2(X_t, X_{t-1}) = \frac{\partial F(X_t, X_{t-1})}{\partial X_{t-1}}, \quad (\text{D1})$$

and let

$$J_1^k = J_1(\bar{X}^k, \bar{X}^k), \quad \text{and} \quad J_2^k = J_2(\bar{X}^k, \bar{X}^k), \quad (\text{D3})$$

be the values of the matrices  $J_1$  and  $J_2$  evaluated at steady state  $\bar{X}^k$ . In the following analysis, the dependence of  $J_1$  and  $J_2$  on  $k$  will be suppressed.

Using the generalized Schur decomposition, (Golub and Loan, 1996, page 377), define unitary matrices  $Q$ , and  $Z$  and upper triangular matrices  $S$  and  $T$  such that

$$J_1 = Q^\top S Z^\top, \quad \text{and} \quad J_2 = Q^\top T Z^\top. \quad (\text{D4})$$

The spectrum  $\lambda(J_1, J_2) \in \mathbb{C}$  is the set of solutions to the generalized eigenvalue problem  $\det(J_1 - \lambda J_2) = 0$ , and the values of  $\lambda$  are equal to the ratios of the diagonal elements of  $S$  and  $T$ .

Using Equation (D4) and the fact that  $Q^\top Q$  and  $Z^\top Z$  are identity matrices we can write the linear approximation to the function  $F(\cdot)$  close to a steady state, Equation (24), as

$$S Z' \tilde{X}_t = T Z' \tilde{X}_{t-1}, \quad (\text{D5})$$



which we break into stable and unstable blocks by ordering the Schur decomposition such that all of the elements of  $\lambda$  that are inside (outside) the unit circle appear in block 1 (block 2),

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} \tilde{Y}_t^1 \\ \tilde{Y}_t^2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} \tilde{Y}_{t-1}^1 \\ \tilde{Y}_{t-1}^2 \end{bmatrix} \quad (\text{D6})$$

where

$$\tilde{Y}_t^1 = Z^1 \tilde{X}_t, \quad \text{and} \quad \tilde{Y}_t^2 = Z^2 \tilde{X}_t, \quad (\text{D7})$$

and the matrices  $Z^1$  and  $Z^2$  are a conformable partition of  $Z$ . Let  $K$  be the number of generalized eigenvalues with modulus greater than 1. To eliminate the effect of the unstable generalized eigenvalues on the dynamics of  $\tilde{X}_t$  we set

$$\tilde{Y}_1^2 = 0, \quad (\text{D8})$$

and notice that, from the block diagonality of  $S$  and  $T$ , if  $\tilde{Y}_1^2 = 0$  then

$$\tilde{Y}_t^2 = (S_{22}^{-1}T_{22})^{t-1}\tilde{Y}_1^2 = 0 \quad \text{for all } t \geq 1. \quad (\text{D9})$$

The requirement that equilibrium sequences remain bounded places  $K$  linear restrictions on the elements of  $\tilde{X}$ .

To compute the values of  $\tilde{Y}_t^1$ , we use Equation (D6) and the fact that  $\tilde{Y}_t^2 = 0$  for all  $t$  to compute

$$\tilde{Y}_t^1 = (s_{11}^{-1}T_{11})^{t-1}\tilde{Y}_1^1. \quad (\text{D10})$$

We established in Section C.3 that the non-generic equilibrium conditions place  $T - 1$  linear restrictions on the  $2(T - 1)$  elements of  $X_t$  leaving  $T - 1$  free initial conditions. It follows that the above construction is feasible and unique whenever  $K = T - 1$ , infeasible if  $K > T - 1$  and that there are  $T - 1 - K$  feasible choices for the initial conditions whenever  $K < T - 1$ . This establishes Proposition 1.

APPENDIX E. INFLATION UNDER A TAYLOR RULE

In this Appendix we derive equations that characterize the behaviour of the inflation rate when the Taylor Rule is passive and when it is active.

**E.1. The case of a passive Taylor Rule.** Using the Taylor rule to substitute for  $1 + i_t$  in the Fisher parity condition yields the following difference equation for inflation

$$\Pi_{t+1} = \left( \frac{\bar{R}}{R_{t+1}} \right) \left( \frac{\Pi_t}{\bar{\Pi}} \right)^{\phi_\pi} \Pi_t, \quad \text{for all } t = 1, \dots, \infty \quad (\text{E1})$$

which we linearize around a steady state to obtain

$$\tilde{\Pi}_{t+1} = (1 + \phi_\pi) \tilde{\Pi}_t - \kappa \tilde{R}_{t+1}, \quad \text{for all } t = 1, \dots, \infty. \quad (\text{E2})$$

Here,  $\kappa \equiv \bar{\Pi}/\bar{R}$  and the tilde denotes deviations from the steady state. Iterating Equation (E2) we obtain

$$\lim_{T \rightarrow \infty} \tilde{\Pi}_{T+1} = \lim_{T \rightarrow \infty} (1 + \phi_\pi)^T \tilde{\Pi}_1 - \lim_{T \rightarrow \infty} \sum_{s=1}^T (1 + \phi_\pi)^{T-s} \tilde{R}_{T+1}. \quad (\text{E3})$$

This is Equation (26) in Section 10.

**E.2. The case of an active Taylor Rule.** To find conditions under which inflation is bounded when the Taylor Rule is active, we use Equation E2 to write the inflation rate at date  $t$  as a function of all future real interest rates and all future inflation rates,

$$\tilde{\Pi}_t = \kappa \sum_{s=1}^{+\infty} \left( \frac{1}{1 + \phi_\pi} \right)^s \tilde{R}_{t+s} + \lim_{T \rightarrow \infty} \left( \frac{1}{1 + \phi_\pi} \right)^T \tilde{\Pi}_{t+T}. \quad (\text{E4})$$

If inflation is bounded, and if the Taylor Rule is active, the second term on the right side of Equation (27) is zero. Evaluating Equation (E4) at  $t = 1$ , we arrive the following expression for the initial gross inflation rate.

$$\tilde{\Pi}_1 \equiv (\tilde{\Pi}_1 - \bar{\Pi}) = \kappa \sum_{s=1}^{+\infty} \left( \frac{1}{1 + \phi_\pi} \right)^s \tilde{R}_{1+s} \quad (\text{E5})$$

Using the definition of inflation in period 1, Equation (E5) places the following restriction on the initial price level,

$$p_1 = p_0 \left( \bar{\Pi} + \kappa \sum_{s=1}^{+\infty} \left( \frac{1}{1 + \phi_\pi} \right)^s (R_{1+s} - \bar{R}) \right). \quad (\text{E6})$$

This is Equation (27) in Section 10.

□