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MONETARY AND FISCAL POLICY WHEN PEOPLE HAVE FINITE LIVES

Roger E.A. Farmer
Pawel Zabczyk

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ABSTRACT

We find a sufficient condition for a three-generation Overlapping Generations (OLG) model to possess four steady-state equilibria and we show that our result also holds in a 62- generation model where the income process is calibrated to U.S. data. In our calibrated model, there exists a dynamically efficient steady state equilibrium where money has positive value and where both the price level and the real interest rate are indeterminate. There are no frictions or rigidities of any kind other than the natural assumption that people are born and die at different dates. Our paper is the first to provide a long-lived example of an OLG model in which there exists an indeterminate steady state where money has value when both monetary and fiscal policy are active. Our work has implications for the question of which fiscal and monetary policies are welfare improving and which are not.

Roger E.A. Farmer
UCLA
Department of Economics
Box 951477
Los Angeles, CA 90095-1477
and CEPR
and also NBER
rfarmer@econ.ucla.edu

Pawel Zabczyk
International Monetary Fund
700 19th Street
N.W. Washington, D.C., 20431
pawel@zabczyk.com

a hump-shaped endowment, there is a critical value of the intertemporal elasticity of substitution, ies_c , such that for all values of $ies < ies_c$ there exist three autarkic steady states.

We build a 62-period endowment economy in which we calibrate the endowment process to U.S. data and we add a government that purchases goods and finances its expenditure with taxes and by borrowing in dollars. We show that, in our calibrated model, there are three autarkic equilibria and one golden rule equilibrium, just as in the three period example. We study the determinacy properties of these equilibria for a range of values of the discount rate and the intertemporal elasticity of substitution and we find that, for realistic values of these two parameters, the golden-rule equilibrium displays indeterminacy. When monetary policy is passive and fiscal policy is active in the sense of Leeper (1991), the golden rule displays two degrees of indeterminacy; neither the initial price level nor the initial real interest rate are pinned down by fundamentals. When monetary policy and fiscal policy are both active, the price level is determinate but the real interest rate is still a free initial condition.¹

Our work has implications for the question of which fiscal and monetary policies are welfare improving and which are not. Although our model is non-stochastic, it is well known that the existence of indeterminate steady-state equilibria is associated with the emergence of stationary stochastic equilibria driven by non-fundamental shocks.² In representative agent models, the combination of a monetary policy mix in which one policy is active and the other is passive leads to the existence of a unique equilibrium. That fact has led macroeconomists to advocate for an active/passive policy mix as a way of eliminating sunspot shocks. In our view, this advice is premature. In our calibrated 62-generation *relative prices* are indeterminate even when both monetary and fiscal policy are active.

2. THE RELATIONSHIP OF OUR WORK TO PREVIOUS LITERATURE

The generic existence of multiple indeterminate steady-state equilibria was established by Kehoe and Levine (1985) in the context of a two-period-lived model with multiple goods and multiple agents.³ With the exception of their paper, most previous examples of indeterminate equilibria in overlapping generations models have been restricted to two-generation or three-generation models in which indeterminacy was purely monetary (Samuelson, 1958), the model required negative money (Gale, 1973; Farmer, 1986), or was associated with unrealistic calibrations generally considered to be empirically irrelevant (Azariadis, 1981; Farmer and Woodford, 1997; Kehoe and Levine, 1983). In Kehoe and Levine (1983) there are three autarkic steady states. Our paper builds on the Kehoe-Levine example by finding sufficient conditions in a three-generation model for the emergence of multiple autarkic steady states and by constructing a calibrated example of a 62-generation economy in which this same phenomenon occurs. Our paper is the first to provide a long-lived example of an economy where the endowment profile is matched to U.S. micro data in which there

¹Although we study a pure exchange economy, similar results to those we develop below hold in economies with physical capital but inelastic labor supply. And the work of Reichlin (1986) suggests that our results can be extended to models with variable labor supply.

²Azariadis (1981); Cass and Shell (1983); Farmer and Woodford (1997); Farmer (1999); Farmer and Guo (1994).

³It follows from the results of Balasko and Shell (1981) that the two-period assumption is unrestrictive as long as there are multiple agents and multiple goods.

are multiple autarkic steady states and where the golden-rule equilibrium is indeterminate, when both monetary and fiscal policy are active.

In an influential paper, Aiyagari (1985) has demonstrated that, under some circumstances, the set of equilibria in the OLG model converges to that of the representative agent model as the length of life is increased. Importantly, Reichlin (1992) has shown that Aiyagari’s result requires the endowments of agents to be bounded away from zero and there are many interesting models where that property does not apply. Kubler and Schmedders (2011) present a variation on Aiyagari’s result. For the case of zero discounting, they explore the frequency of multiple equilibria in T -generation economies parameterized by endowments. They show that when endowments are chosen on a grid in the set $[0, 1]^T$ that, as T grows large, economies in which multiple equilibria emerge become increasingly rare. They conclude, as does Aiyagari, that the existence of multiple – possibly indeterminate – equilibria in the OLG model is a phenomenon that is unlikely to be observed in the real world.

We disagree with this conclusion. The life-cycle endowment pattern of a human-being is not well modeled as a set of random variables in which the endowment of a person of age t is uncorrelated with her endowment at age $t + 1$. Endowments follow a distinct hump-shaped pattern and that fact has implications for the likelihood of the emergence of multiple equilibria. It *is* true, as Kubler and Schmedders (2011) claim, that the emergence of multiple equilibria requires a relatively low intertemporal elasticity of substitution. But our robustness results in Section 9 show that multiplicity does not disappear as the length of life increases as long as endowments follow a hump-shape pattern of the same kind observed in the real world.

All *monetary* OLG models contain at least two steady-state equilibria and in the absence of deficit finance, money has value in only one of them. Researchers have typically ruled out one equilibrium by omitting government debt entirely, as in Ríos-Rull (1996), or by modeling debt as an obligation to repay real commodities, as in Auerbach and Kotlikoff (1987). Although the examples of Ríos-Rull (1996) and Auerbach and Kotlikoff (1987) are production economies that do not contain fiat money, the models developed in both of these papers allow for a hump-shaped income profile and the results we develop in Section 5 imply that there exist parameter values for which these models possess multiple steady state equilibria.⁴

A branch of the multiple equilibrium literature studies bubbles in the overlapping generations model. A non-exhaustive list of papers would include Tirole (1985), Martin and Ventura (2011, 2012), Miao and Wang (2012), Miao et al. (2012), Azariadis et al. (2015) and Miao (2016). The model we develop here contains what Tirole would call ‘bubbly equilibria’ but, unlike these papers, we develop our argument in the context of a complete markets overlapping generations model without the credit constraints introduced by these authors.⁵

⁴Although our results are developed in the context of an endowment economy, a simple extension of our analysis applies to the model with capital. It follows that the uniqueness finding of Ríos-Rull that

“In all the cases studied, these functions turned out to be monotone- decreasing, implying a unique steady state.” Ríos-Rull (1996, page 472),

is case specific and depends on the specific choice of parameters.

⁵Eggertsson et al. (2019) study steady-state equilibria in a fifty-six generation overlapping generations model with sticky prices. They use their model to discuss the idea that a negative real interest rate may be inconsistent

There is a separate but related literature that studies the emergence of indeterminacy in infinite horizon monetary models. In two recent working papers, Brunnermeier et al. (2022a,b) study the existence of bubbles in infinite horizon models in both continuous and discrete time and investigate the implications of the failure of the transversality condition for the fiscal theory of the price level. Miao and Su (2021) build a friction into a new Keynesian model with production and they study the emergence of debt as a bubble close to a low interest rate steady state. Aguiar et al. (2021) study Pareto improving policies in a model with idiosyncratic income risk and Reis (2021) investigates the role of liquidity effects in a model with aggregate shocks. Our paper is related to this literature but we do not consider models with either production or uncertainty of any kind.

Benhabib et al. (2001) exploit the global indeterminacy of equilibrium to point out that an active Taylor rule may lead to unexpected results. In contrast, our results are not driven by global or non-linear dynamics since we explicitly restrict attention to the properties of a linearized system of equilibrium conditions around a monetary steady state. Bassetto and Cui (2018) revisit the implications of fiscal policy for price level determination in models in which assets differ in characteristics because of risk, or because debt provides liquidity services. In contrast to their work, our results do not rely on dynamic inefficiency, risk premia or liquidity effects. We show that both the price level and the real interest rate are indeterminate in a model where the steady-state equilibrium is dynamically efficient and where there are no frictions or rigidities of any kind other than the natural assumption that people are born and die at different dates.

3. FISCAL AND MONETARY POLICY

In this section we explain the relationship between private and government trade in the asset markets. We construct a model in which the government purchases g_t units of a consumption good which it finances with dollar-denominated pure discount bonds and lump-sum taxes, τ_t . Let B_t be the quantity of pure-discount bonds, each of which promises to pay one dollar at date $t + 1$ and let Q_t be the date t dollar price of a discount bond. Further, let p_t be the date t dollar price of a consumption good. Using these definitions, government debt accumulation is represented by the following equation,

$$Q_t B_t + p_t \tau_t = B_{t-1} + p_t g_t.$$

Define i_t to be the net nominal interest rate from period t to period $t + 1$, and let Π_{t+1} , be the gross inflation rate. These variables are given by,

$$i_t \equiv \frac{1}{Q_t} - 1 \quad \text{and,} \quad \Pi_{t+1} \equiv \frac{p_{t+1}}{p_t}.$$

Further, let

$$b_t \equiv \frac{B_{t-1}}{p_t},$$

be the real value of government debt maturing in period t and define the *real primary deficit* as

$$d_t \equiv g_t - \tau_t,$$

with full employment, a concept that they refer to as ‘secular stagnation’. In one section of their paper, Eggertsson et al. study the transition path from one steady-state equilibrium to another. Their solution method assumes that this transition path is unique. This is an assumption that is called into question by the analysis in our paper.

where the negative of d_t is the *real primary surplus*. Let R_{t+1} represent the gross real return from t to $t + 1$, which from the Fisher-parity condition equals

$$R_{t+1} \equiv \frac{1 + i_t}{\Pi_{t+1}}. \quad (1)$$

We can combine these definitions to rewrite the government budget equation in purely real terms

$$b_{t+1} = R_{t+1}(b_t + d_t), \quad t = 1, \dots, \infty. \quad (2)$$

We can consolidate these equations into a single government budget constraint by defining \mathcal{Q}_t^k ,

$$\mathcal{Q}_t^k \equiv \prod_{j=t+1}^k \frac{1}{R_j}, \quad \mathcal{Q}_t^t = 1,$$

to be the relative price at date t of a commodity for delivery at date k . Using this definition and iterating Eq. (2) forwards,

$$\frac{B_0}{p_1} = - \sum_{t=1}^{\infty} \mathcal{Q}_1^t d_t + \lim_{T \rightarrow \infty} \mathcal{Q}_1^T b_T. \quad (3)$$

If $\lim_{T \rightarrow \infty} \mathcal{Q}_1^T b_T$ exists and is equal to zero, Eqn. (3) defines a function relating the initial price level to the real value of all future surpluses.

In New-Keynesian models in which the central bank sets an interest rate peg, the initial price level would be indeterminate if the government were constrained to balance its budget for all paths of $\{\mathcal{Q}_1^t\}_{t=1}^{\infty}$ and all initial price levels (McCallum, 2001). To resolve this apparent indeterminacy of the price level, advocates of the FTPL argue that the government should be treated differently from other agents in a general equilibrium model (Leeper, 1991; Woodford, 1994). Although Equation (2) is expressed in terms of real variables, the debt instrument issued by the treasury is nominal. It follows that, when Eq. (3) is interpreted as an equilibrium condition, the real value of debt in period 1 is determined by the period 1 price level through the definition

$$b_1 \equiv \frac{B_0}{p_1}.$$

This argument rests critically on the determinacy properties of the steady state and, as we show below, it does not hold in the OLG model when the *ies* is sufficiently low.

4. A THREE-GENERATION EXAMPLE

We introduce our main ideas in a 3-generation example. The preferences of people born in period 2 and later are given by the following utility function;

$$U^t = U(c_t^t, c_{t+1}^t, c_{t+2}^t). \quad (4)$$

We index generations by superscripts and calendar time by subscripts. Thus, c_τ^t is the consumption of generation t in period τ . We refer to the people born in period $T - 1$ and later as *generic generations* and we distinguish them from a set of *non-generic generations*. The non-generic generations are people alive in periods 1 through $T - 2$ who live for less than T periods. In the 3-generation model there are 2 non-generic generations; the initial middle-aged and the initial old.

The generic generations maximize utility subject to three budget constraints, one for each period of life,

$$c_t^t + s_{t+1}^t \leq \tilde{\omega}_1, \quad c_{t+1}^t + s_{t+2}^t \leq R_{t+1}s_{t+1}^t + \tilde{\omega}_2, \quad c_{t+2}^t \leq R_{t+2}s_{t+2}^t + \tilde{\omega}_3, \quad (5)$$

where $\{\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3\}$ is the after-tax endowment profile of a generic generation and s_τ^t is the demand for claims to $\tau + 1$ consumption goods by generation t in period τ . The subscript on the term $\tilde{\omega}_j$ indexes age and we assume throughout, that $\tilde{\omega}_j$ does not depend on calendar time.

The solution to this problem is fully characterized by a pair of asset demand functions

$$s_{t+1}^t(R_{t+1}, R_{t+2}), \quad s_{t+2}^t(R_{t+1}, R_{t+2}),$$

together with the requirement that the three budget constraints characterized in (5) hold with equality.

Let the aggregate demand for assets by all agents alive at date t be defined by the function

$$f(R_t, R_{t+1}, R_{t+2}) \equiv s_t^{t-1}(R_t, R_{t+1}) + s_t^t(R_{t+1}, R_{t+2}).$$

For this three-generation example, $f(\cdot)$ adds up the asset demand of the newborns, this is the term $s_t^t(\cdot)$ and the asset demands of the middle-aged, this is the term $s_t^{t-1}(\cdot)$. Equilibrium in the asset markets requires that

$$f(R_t, R_{t+1}, R_{t+2}) = R_{t+1}^{-1}b_{t+1}, \quad (6)$$

where $R_{t+1}^{-1}b_{t+1}$ is the public sector borrowing requirement in period t and the dynamics of public borrowing are given by the equation,

$$b_{t+1} = R_{t+1}(b_t + d_t). \quad (7)$$

We refer to equations (6) and (7) as the *generic market clearing equations*. Beginning with period 2, non-stationary equilibria are characterized by bounded sequences of real interest rates and debt that satisfy these equations and are consistent with a set of initial conditions that arise from the behavior of the initial generations.

To complete the description of equilibrium in the three-generation model, we must describe the equilibrium conditions in period 1. In this period there is one generic generation, the newborns, and two non-generic generations, the middle-aged and the old. We refer to the initial middle-aged as generation 0 and to the initial old as generation -1 . Generation -1 consume their wealth and they do not contribute to the demand for period 1 assets. Generation 0 maximize the function

$$U^0 = U(c_1^0, c_2^0), \quad (8)$$

subject to the two budget constraints,

$$c_1^0 + s_2^0 \leq \lambda_0 \frac{B_0}{p_1} + \tilde{\omega}_2, \quad c_2^0 \leq R_2 s_2^0 + \tilde{\omega}_3, \quad (9)$$

where B_0 is the dollar value of the initial government debt liability, p_1 is the dollar price of a commodity in period 1 and λ_0 is the share of the initial government debt liability held by generation 0. The remaining share, $1 - \lambda_0$, is held by the initial old. The solution to the problem of the initial

middle-aged is characterized by the asset demand function

$$s_2^0 \left(R_2, \lambda_0 \frac{B_0}{p_1} \right), \quad (10)$$

and the requirement that the two budget constraints, (9), hold with equality.

The asset market clearing condition in period 1 is given by the expression

$$s_2^0 \left(R_2, \lambda_0 \frac{B_0}{p_1} \right) + s_2^1(R_2, R_3) = R_2^{-1} b_2, \quad (11)$$

and the government debt equation is

$$b_2 = R_2 \frac{B_0}{p_1}. \quad (12)$$

We refer to equations (11) and (12) as the *non-generic market clearing equations*. For given values of λ_0 and B_0 , equations (11) and (12) determine R_3 and b_2 as functions of the initial conditions, R_2 and p_1 . In period 2 and later, equations (6) and (7) determine R_{t+2} and b_{t+1} as functions of b_t , R_{t+1} and R_t .

A steady-state equilibrium is a non-negative real number R and real number b such that

$$f(R, R, R) = R^{-1} b, \quad (13)$$

$$b = R(b + d). \quad (14)$$

When $d = 0$, inspection of Eqn. (14) shows that there are at least two steady-state equilibria; one in which $b = 0$ and one in which $R = 1$. We refer to an equilibrium in which $b = 0$ as an *autarkic steady state* and an equilibrium in which $R = 1$ is as a *golden rule steady state*. When $d \neq 0$, a continuity argument establishes that there is an open set $d \in (d_L, d_U)$, which contains $d = 0$, for which the number of steady-state equilibria and the determinacy properties of each steady-state equilibrium is the same as the case where $d = 0$. It follows that, as long as the primary budget deficit is not too large, our analysis of the properties of equilibria for the case of $d = 0$ carries over to the case where the treasury runs a primary deficit or a primary surplus.

5. MULTIPLICITY OF STEADY STATES

Overlapping generations models always possess at least two steady states. In the one-good model the golden rule is unique and there is always at least one autarkic steady state. In Section 5 we find a sufficient condition for the 3-generation OLG model to possess three autarkic steady-state equilibria in addition to the golden rule. Our result exploits a tension between the intertemporal elasticity of substitution and the peak of the income distribution.

In our example, the members of generation t maximizes the following constant elasticity of substitution utility function

$$U^t = \frac{(c_t^t)^\alpha + \beta(c_{t+1}^t)^\alpha + \beta^2(c_{t+2}^t)^\alpha}{\alpha}, \quad \alpha \leq 1, \quad \alpha \neq 0,$$

$$U^t = \log(c_t^t) + \beta \log(c_{t+1}^t) + \beta^2 \log(c_{t+2}^t), \quad \alpha = 0,$$

subject to the constraint

$$\frac{(c_t^t - 1)}{1} + \frac{(c_{t+1}^t - \lambda^\eta)}{R_{t+1}} + \frac{(c_{t+2}^t - \lambda^{2\eta})}{R_{t+1}R_{t+2}} \leq 0, \quad (15)$$

where $0 < \beta < 1$ and $0 < \lambda \leq 1$. The intertemporal elasticity of substitution, η , is equal to

$$\eta = \frac{1}{1 - \alpha},$$

and we focus here on the case where $\alpha \leq 0$ which implies an intertemporal elasticity of substitution between 0 and 1.

The solution to the consumer problem is given by the equations,

$$c_t^t = \frac{1}{\mu^\eta}, \quad c_{t+1}^t = \frac{\beta^\eta R_{t+1}^\eta}{\mu^\eta}, \quad c_{t+2}^t = \frac{\beta^{2\eta} (R_{t+1} R_{t+2})^{2\eta}}{\mu^\eta},$$

where μ is the multiplier on the budget constraint, Eq. (15). We chose an income profile of $\omega_\lambda \equiv \{1, \lambda^\eta, \lambda^{2\eta}\}$ in which the endowment process is related to the preference parameter η in order to simplify the numerical calculations of our example and nothing of substance hinges on this assumption. For this choice of the endowment process, the autarkic equilibrium is easy to compute and occurs at $R = \lambda/\beta$ and $\mu = 1$.⁶

We seek to redistribute the endowments between generations in a way that preserves two features of the equilibrium. First, we want to ensure that the aggregate endowment is unchanged. That condition requires that the following equation holds.

$$w_1 + w_2 + w_3 = 1 + \lambda^\eta + \lambda^{2\eta}, \quad (16)$$

where $\{w_1, w_2, w_3\}$ is the new endowment pattern and $\{1, \lambda^\eta, \lambda^{2\eta}\}$ is the original endowment pattern. Second, we want to ensure that the wealth of a newborn with the new endowment pattern is unchanged when $R = \frac{\lambda}{\beta}$. This requires that

$$w_1 + \frac{w_2\beta}{\lambda} + \frac{w_3\beta^2}{\lambda^2} = 1 + \frac{\lambda^\eta\beta}{\lambda} + \frac{\lambda^{2\eta}\beta^2}{\lambda^2}. \quad (17)$$

Equations (16) and (17) represent two equations in the three unknowns w_1 , w_2 and w_3 . We are free to choose a positive value for the middle-age endowment, w_2 , that satisfies the inequality,

$$w_2 \leq 1 + \lambda^\eta + \lambda^{2\eta},$$

and to solve equations (16) and (17). This procedure defines two functions $w_1 = w_1(w_2)$ and $w_3 = w_3(w_2)$ which redistribute the endowment in a way that preserves $R = \lambda/\beta$ as a steady state equilibrium.

An autarkic steady state equilibrium is characterized by a positive number \bar{R} and an aggregate savings function $f_\omega(R)$ such that

$$f_\omega(\bar{R}) \equiv s_{t+1}^t(\bar{R}; \omega) + s_{t+2}^t(\bar{R}; \omega) = 0.$$

⁶This is easily seen by substituting the expressions for c_t^t , c_{t+1}^t and c_{t+2}^t into Eq. (15), using the equilibrium conditions, $R = \frac{\lambda}{\beta}$, and $\mu = 1$.

Here the subscript ω on the function $f_\omega(R)$ signifies the dependence of the aggregate savings function on the endowment profile and $s_{t+1}^t(\bar{R}; \omega)$ and $s_{t+2}^t(\bar{R}; \omega)$ are the steady-state savings functions of the young and middle-aged that satisfy the utility maximization problem of a representative household with endowment profile $\omega = \{w_1, w_2, w_3\}$. For the endowment profile $\omega_\lambda = \{1, \lambda^\eta, \lambda^{2\eta}\}$, the autarkic equilibrium is given by $\bar{R} = \lambda/\beta$.

By picking a new endowment profile, $\tilde{\omega}$, that satisfies equations (16) and (17) we guarantee that

$$f_{\tilde{\omega}}\left(\frac{\lambda}{\beta}\right) = f_{\omega_\lambda}\left(\frac{\lambda}{\beta}\right) = 0.$$

This equality means that the interest rate $R = \lambda/\beta$ is a steady state equilibrium interest rate in the perturbed economy. But although the new savings profile is chosen so that f_{ω_λ} and $f_{\tilde{\omega}}$ coincide at the steady state, they will not in general coincide anywhere else.

To understand the mechanism that leads to multiple equilibria, consider the special case of the model in which $\lambda = \beta = 1$. For this case the income and consumption profiles are flat and the golden rule and autarkic steady states coincide. In this steady state equilibrium, $R = 1$ and every generation consumes its endowment in each period of life.

To explore the interplay between the endowment profile and the intertemporal elasticity of substitution, in Appendix A we derive an explicit expression for the slope of the aggregate savings function, evaluated at $\bar{R} = \lambda/\beta$, and show that when $\beta = \lambda = 1$ this expression is given by

$$\left. \frac{\partial f_{\tilde{\omega}}}{\partial R} \right|_{R=1} = \frac{1 - w_2 + 4\eta}{2}. \quad (18)$$

When $\eta = 1$, the utility function is logarithmic. And since $w_2 < 3$, it follows that for logarithmic preferences the aggregate savings function slopes up at the autarkic steady state. This is not true for other values of η . By rearranging Eq. (18) one can show that for values of

$$\eta < \frac{w_2 - 1}{4}, \quad (19)$$

the savings function changes sign from positive to negative at the autarkic steady state equilibrium. One can also show that $f_{\tilde{\omega}}$ is continuous, negative for low values of R and positive for high values. It follows from these three facts that if $f_{\tilde{\omega}}$ crosses zero with a negative slope at the autarkic equilibrium there must be at least two other steady state equilibria. The upper bound on w_2 is 3 and by inspecting Inequality (19) we see that the intertemporal elasticity of substitution must be strictly less than 1/2 for multiple steady states to arise.

Figure 1 plots the savings function for three different values of the intertemporal elasticity of substitution. In each case, $\tilde{\omega} = \{0.33, 2.33, 0.34\}$, and $\beta = \lambda = 1$. By rearranging the endowment across the age profile we induce a motive to move the endowment from middle age to youth. By lowering the intertemporal elasticity of substitution, we dampen this effect.

The three sub-panels in Figure 1 are drawn for decreasing values of the intertemporal elasticity of substitution. For the top panel, $\eta = 2/5$. For this value of η the savings function is increasing at the autarkic stationary equilibrium value, $R = \lambda/\beta$. For the middle panel, $\eta = 1/3$. At this value of η the slope of the savings function is zero at the autarkic steady-state equilibrium. For the lower

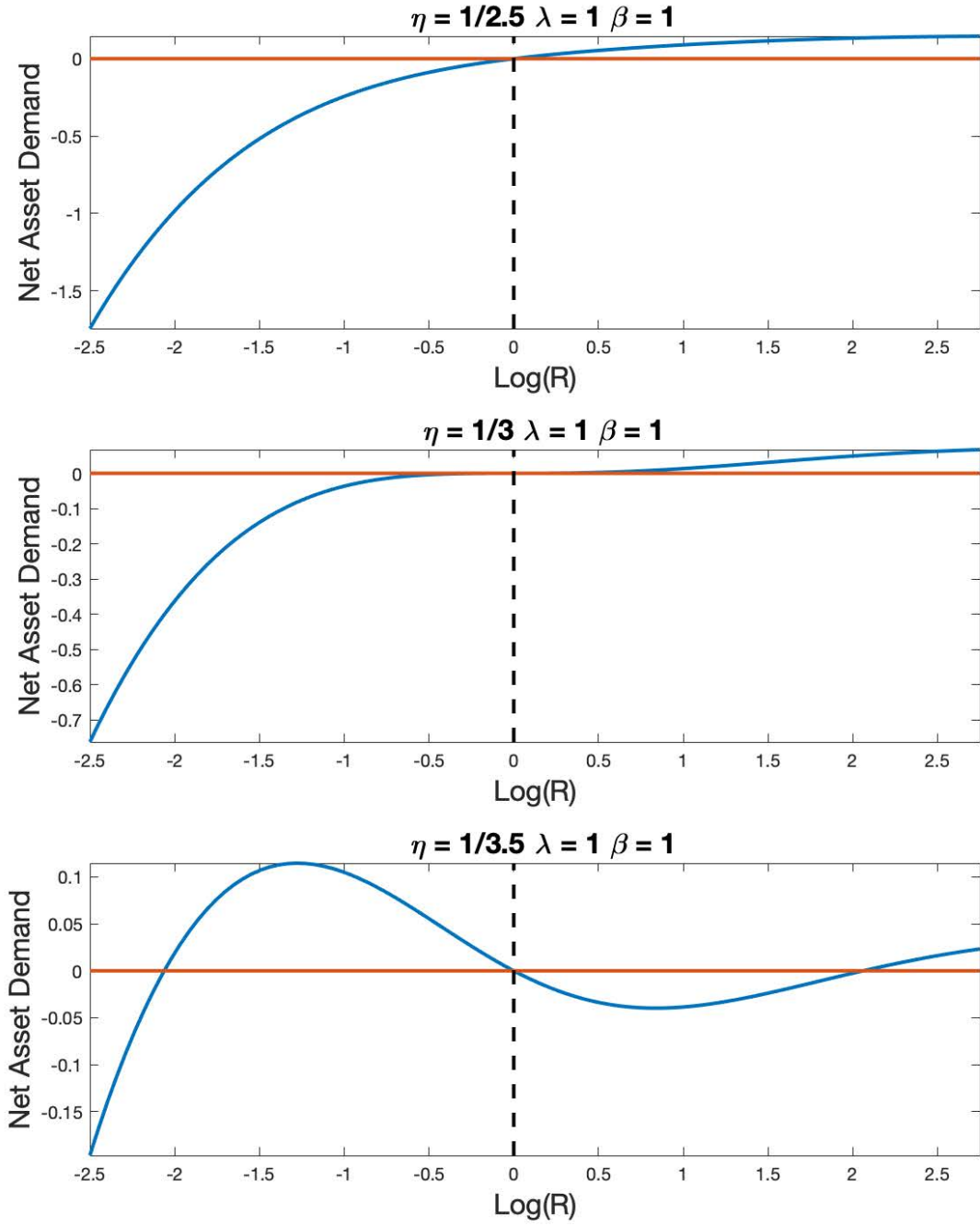


FIGURE 1. The Aggregate Savings Function for Three Different Values of η

panel, $\eta = 2/7$ and, for this value the function $f_{\hat{\omega}}(R)$ is decreasing at the autarkic steady state and two new steady states emerge, one greater than 1 and one smaller.

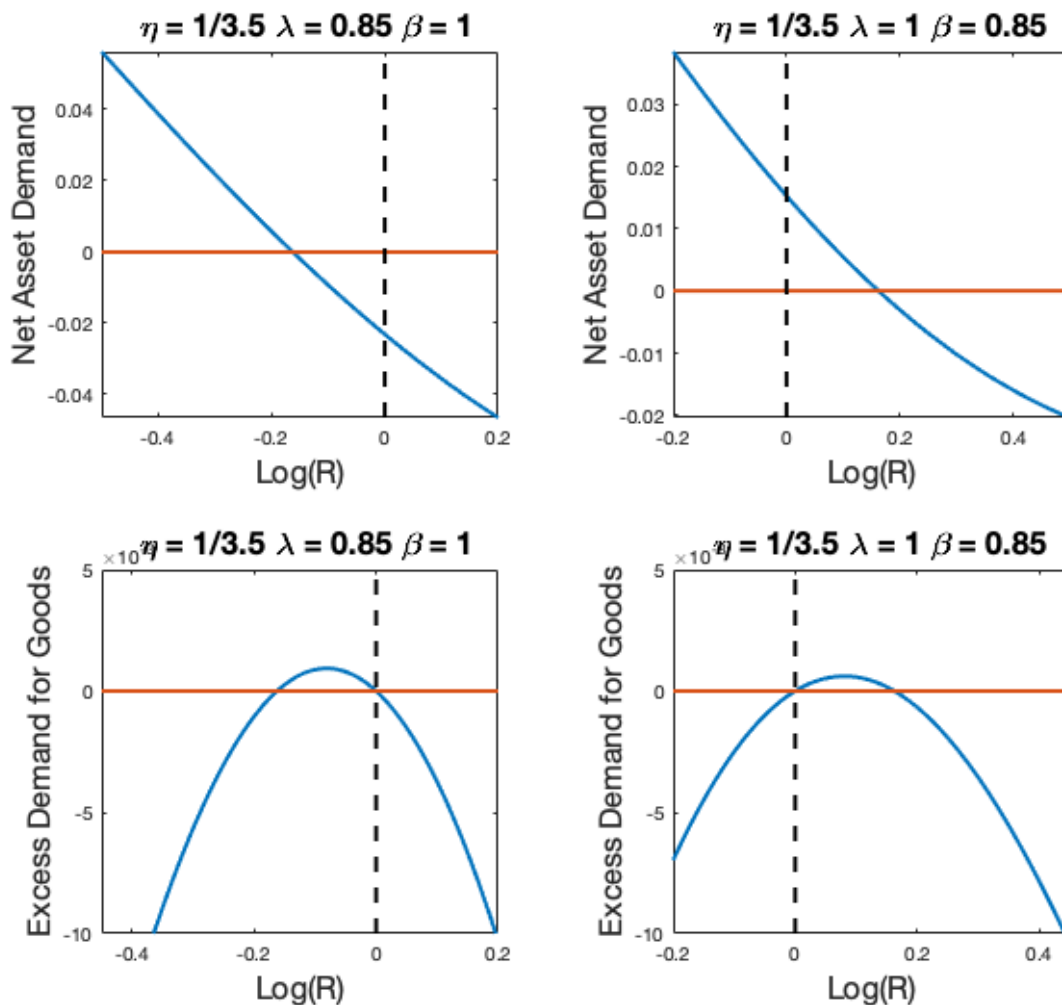


FIGURE 2. Aggregate Savings and the Excess Demand for Goods for Two Different Values of λ and β

Figure 2 explores the role of different values of λ and β . All four panels are drawn for the same endowment profile as Figure 1 and for a value of $\eta = 2/7$. The left two panels of Figure 2 are drawn for $\beta = 1$ and $\lambda = 0.85$. The right two panels depict the case when $\beta = 0.85$ and $\lambda = 1$. In each case, the upper panel depicts the aggregate savings function and the lower panel represents the aggregate excess demand for goods.

Although there are three intersections in each upper panel and four intersections in each lower panel, we have truncated the x-axis to depict only one intersection of the savings function and two intersections of the excess demand for goods in order to draw attention to the magnitude of asset holdings at the golden rule steady state. The golden rule in our example corresponds to a value

of $\log(R) = 0$ and from the two upper panels we see that as we vary the parameterization from $\lambda = 0.85$ and $\beta = 1$ to $\beta = 0.85$ and $\lambda = 1$, the sign of asset holdings in the golden-rule steady state changes from negative to positive. We infer from this analysis that a hump-shaped income profile and a low intertemporal elasticity of substitution provides a sufficient condition for the existence of three autarkic steady-state equilibria and that, when $\beta < \lambda$, money has positive value at the golden-rule steady state.

6. THE DETERMINACY OF STEADY-STATE EQUILIBRIA

The fact that there exist multiple steady states does not imply anything about the determinacy properties of any one of them and for some parameter values there may exist other attracting sets including limit cycles and, possibly, chaotic attractors. However, we found computationally that, for a range of parameters, the golden rule steady state in our three-generation model displays second-degree indeterminacy for an active-passive policy combination. What does it mean for the golden rule steady state to display second-degree indeterminacy?

Consider all pairs of initial values

$$b_1 \equiv \frac{B_0}{p_1}, \quad R_2,$$

that are close to the steady state values of b and R at the golden rule. If there is a unique pair, $\{b_1, R_2\}$, such that the trajectory that starts from this pair converges to the steady state, the golden rule steady state is said to be *locally determinate*. If there is a one dimensional manifold of values – defined by a function $b_1 = \phi(R_2)$ – such that all solutions to equations (6) and (7) that begin on this manifold converge to the steady state; the golden rule steady state is said to display one degree of indeterminacy. If there is a two dimensional manifold – containing the golden rule steady state – such that all solutions to equations (6) and (7) that begin on this manifold converge to the steady state; the steady state is said to display two degrees of indeterminacy.

The degree of indeterminacy of a steady state depends on the actions of the monetary and fiscal authorities. The assumption of a constant interest rate implies that monetary policy is passive and the fact that d_t is not responsive to variations in the value of outstanding debt implies that fiscal policy is active. Arguably, this is the relevant policy mix for the recent policy environment in which the interest rate has been at or near zero and unresponsive to realized inflation and where national treasuries have been, and are currently, pursuing unrestrained spending programs that, at least in the near future, do not appear responsive to growing debt to GDP ratios. It follows that the finding of two-degrees of indeterminacy at the golden rule steady state holds for a passive-active policy mix for which equilibrium, in a representative agent economy, would be determinate.

In Section 10 we relax the assumption of a passive monetary policy and we show that the golden rule equilibrium still displays one degree of indeterminacy, even for the case in which monetary and fiscal policy are both active. This finding means that, although there is a unique equilibrium price sequence for every initial real interest rate, the real interest rate itself is not pinned down by fundamentals.

7. THE GENERAL T -PERIOD CASE

This section describes the generalization of the 3-generation example to a model with T -generations. The main complication that arises for this case is associated with the existence of additional non-generic generations in the first $T - 2$ periods of the model and the technical arguments needed to deal with this complication are dealt with in Appendix C.2. The reader who is interested in the implications of our argument for the 62-generation calibrated example can skip ahead to Section 8.

In the T -generation case, generation t has a utility function defined over consumption in periods t through $t + T - 1$ and the members of generation t solve the problem

$$\max_{\{c_t^t, \dots, c_{t+T-1}^t\}} U^t(c_t^t, c_{t+1}^t, \dots, c_{t+T-1}^t),$$

such that

$$c_t^t + s_{t+1}^t \leq \tilde{\omega}_1, \quad c_{t+1}^t + s_{t+2}^t \leq R_{t+1}s_{t+1}^t + \tilde{\omega}_2, \quad \dots \quad c_{t+T-1}^t \leq R_{t+T-1}s_{t+T-1}^t + \tilde{\omega}_T. \quad (20)$$

The solution to this problem is characterized by a set of $T - 1$ savings functions, one for each of the first $T - 1$ periods of life

$$s_k^t(R_{t+1}, \dots, R_{t+T-2}, R_{t+T-1}), \quad k = t, \dots, t + T - 2, \quad (21)$$

together with the requirement that the T budget constraints (20) hold with equality. In Appendix B, Section B.1. we characterize the solution to this problem for the case of CES preferences and we find an explicit formula for the aggregate asset demand function,

$$f(R_{t-T+3}, R_{t-T+4}, \dots, R_{t+T-2}, R_{t+T-1}) = R_{t+1}^{-1} b_{t+1}, \quad (22)$$

where $f(\cdot)$ is the sum of the savings functions, defined in Equation (21). As in the three-period model, government borrowing follows the equation,

$$b_{t+1} = R_{t+1}(b_t + d_t). \quad (23)$$

In Appendix C.1 we show that dynamic equilibria can be described by a difference equation $F(X_t, X_{t-1})$ in a vector

$$X_t \equiv [R_{t+T-1}, R_{t+T-2}, \dots, R_{t-T+4}, b_t]^\top, \quad (24)$$

and we find a linear approximation to that difference equation around a steady state of the form

$$\tilde{X}_t = J\tilde{X}_{t-1} \quad (25)$$

where \tilde{X} is a vector of deviations from the steady state and J is constructed from the partial derivatives of $F(\cdot)$ evaluated at the steady state.

The analysis in Appendix C.1 establishes that the order of the difference equation that characterizes equilibrium sequences of real interest rates, Eq. (22), is equal to $2T - 3$. And the generalization of the non-generic equations in Appendix C.2 establishes that there are $T - 1$ non-generic equations which characterize equilibria in periods 1 through $T - 2$. Using these results, in Appendix, D we prove the following proposition which is based on the work of Blanchard and Kahn (1980).

Proposition 1 (Blanchard-Kahn). *Let K denote the number of eigenvalues of J with modulus greater than 1.*

- *If $K > T - 1$ there are no bounded sequences that satisfy the equilibrium conditions in the neighbourhood of \bar{X} . In this case equilibrium does not exist.*
- *If $K = T - 1$ there is a unique bounded sequence that satisfies the equilibrium equations. Further, this sequence converges asymptotically to the steady state (\bar{R}, \bar{b}) . In this case the steady state equilibrium (\bar{R}, \bar{b}) is determinate.*
- *If $K \in \{0, \dots, T - 2\}$ there is a $T - 1 - K$ dimensional subspace of initial conditions that satisfy the equilibrium equations. All of these initial conditions are associated with sequences that converge asymptotically to the steady state (\bar{R}, \bar{b}) . In this case the steady state equilibrium (\bar{R}, \bar{b}) is indeterminate with degree of indeterminacy equal to $T - 1 - K$.*

It follows from this proposition that we can compute the degrees of determinacy around a given steady-state equilibrium by comparing the roots of the matrix J , with $T - 1$, where T is the number of generations. In the simulations presented in Section 8, we use this proposition to compute the eigenvalues of J in the neighbourhood of each of the four steady states and we simulate non-stationary paths by iterating a linear approximation to the function $F(\cdot)$ around the golden-rule steady state.⁷

In our model fiscal policy is active but monetary policy is passive. According to the FTPL this policy mix should lead to a unique initial price level. In Section 8 we provide an example of an economy with a steady-state equilibrium where money has value and where the FTPL fails to hold. In this example, it is not only the initial price level that is indeterminate; it is also the initial real interest rate.

8. A SIXTY-TWO GENERATION EXAMPLE

In this section we construct a sixty-two generation model where each generation begins its economic life at age 18 and in which a period corresponds to one year. This extension is important in light of the work of Kubler and Schmedders (2010) who cast doubt on the ability of the long-lived OLG model to display multiple steady-state autarkic equilibria by studying the frequency with which multiplicity arises as the length of life increases. Kubler and Schmedders (2010) are correct that multiplicity becomes less frequent if income profiles are arbitrary. But in the data the income profile is not arbitrary; it has a particular age-profile with a distinct hump. We calibrate the age-profile of the representative person's endowment to U.S. data and we show that – for low values of the intertemporal elasticity of substitution – there exists a steady state that displays *real* indeterminacy, even when monetary and fiscal policy are both active.

Our result is important because it casts doubt on the universal applicability of established results about the Fiscal Theory of the Price Level. In representative agent models, the combination of an active/passive monetary-fiscal policy pair can eliminate indeterminacy in a competitive

⁷The code used to generate all of our results is available online and is documented in an accompanying online document “Numerical Recipes”. Our code also replicates the findings reported in Kehoe and Levine (1983).

equilibrium model. We show that this result does not extend to a realistically calibrated OLG model.

To our extend our model to T -period lives, we assume that the members of generation t maximize the utility function,

$$u(c_t^t, \dots, c_{t+61}^t) = \sum_{i=1}^{62} \beta^{i-1} \left(\frac{[c_{t+i-1}^t]^\alpha - 1}{\alpha} \right),$$

and we calibrate the income profile of a representative generation to U.S. data. We provide explicit formulas for the excess demand functions for this functional form in Appendix B.

We graph our calibrated income profile in Figure 3. Our representative generation enters the labour force at age 18, retires at age 66, and lives to age 79. We chose the lifespan to correspond to current U.S. life expectancy at birth and we chose the retirement age to correspond to the age at which a U.S. adult becomes eligible for social security benefits. For the working-age portion of this profile we use data from Guvenen et al. (2021) which is available for ages 25 to 60. The working-age income profiles for ages 18 to 24 and for ages 61 to 66, were extrapolated to earlier and later years using log-linear interpolation. For the retirement portion we used data from the U.S. Social Security Administration.

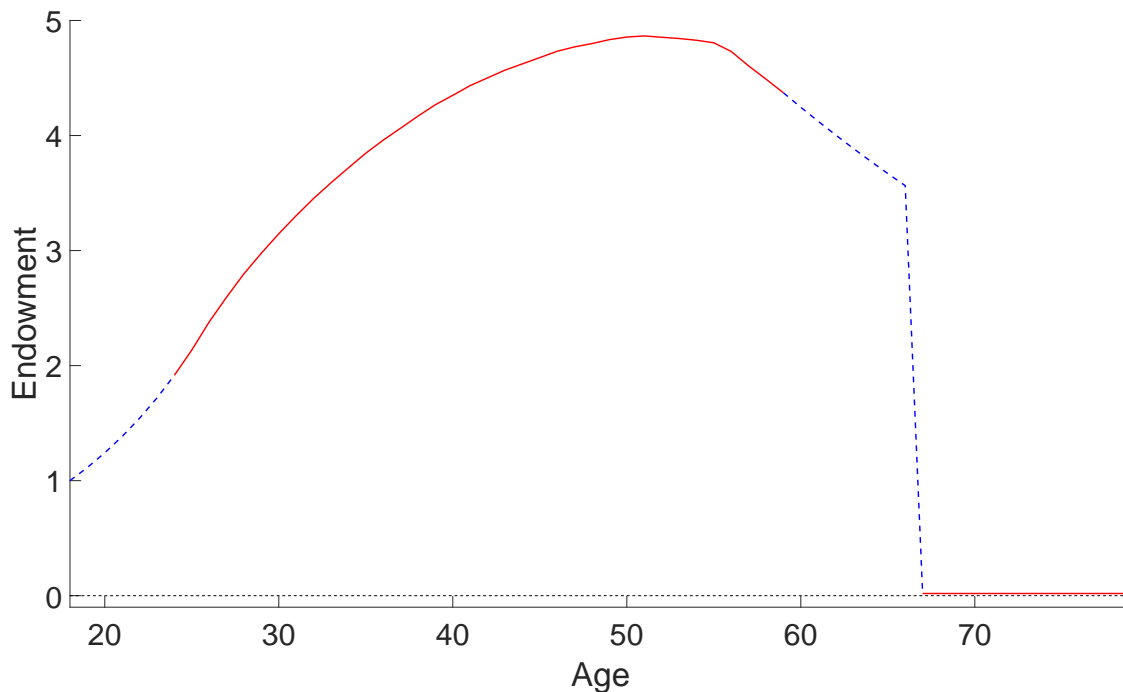


FIGURE 3. Normalized Endowment Profile. U.S. Data in Solid Red: Interpolated Data in Dashed Blue.

U.S. retirement income comes from three sources; private pensions, government social security benefits, and Supplemental Security Income. We treat private pensions and government social security benefits as perfect substitutes for private savings since the amount received in retirement is a function of the amount contributed while working. To calibrate the available retirement income

that is independent of contributions, we used Supplementary Security Income which, for the U.S., we estimate at 0.137% of GDP.⁸

For the remaining parameters of our model we chose a primary budget deficit of $d_t = 0$, an annual discount rate of 0.953 and an elasticity of substitution of 0.17. The qualitative features of the equilibria are robust to the existence of a positive primary deficit with an upper bound that depends on the discount rate. For the calibrated income profile depicted in Figure 3 and for this choice of parameters, our model exhibits four steady-state equilibria. In Section 9 we explore the robustness of the properties of our model to alternative choices for the discount parameter and for the intertemporal elasticity of substitution.

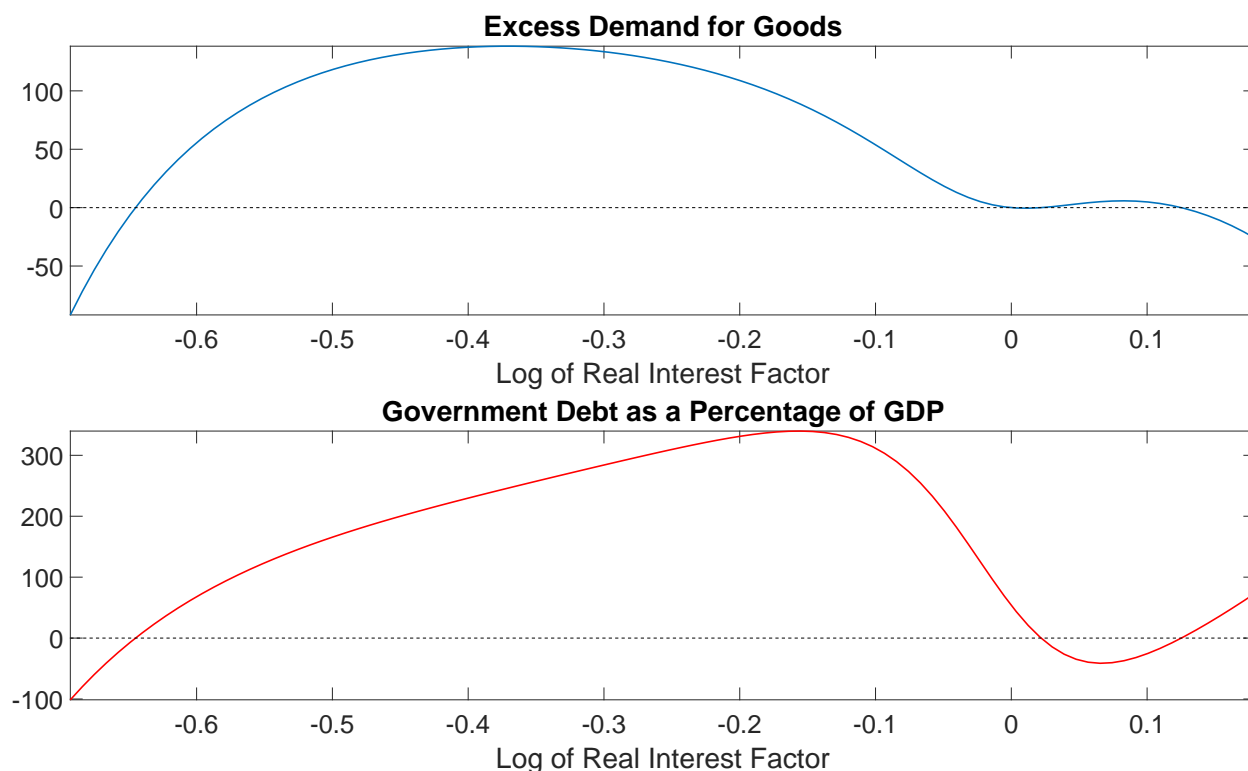


FIGURE 4. Steady States in the Sixty-Two Generation Model

In Figure 4 we graph the steady-state equilibria of our model. The upper panel of this figure plots the logarithm of the gross real interest rate on the horizontal axis and the steady-state excess demand for goods on the vertical axis. The lower panel plots government debt as a percentage of GDP at the steady state. We see from the upper panel that the excess demand function crosses the horizontal axis four times. And we see from the lower panel that three of these crossings are associated with steady-state equilibria in which steady-state government debt is equal to zero.

⁸From Table 2 of the March 2018 Social Security Administration Monthly Statistical Snapshot we learn that the average monthly Supplemental Security Income for recipients aged 65 or older equalled \$447 (with 2,240,000 claimants), which implies that total monthly nominal expenditure on Supplemental Security Income equalled \$1,003 million. This compares to seasonally adjusted wage and salary disbursements (A576RC1 from FRED) in February 2018 of \$8,618,700 million per annum, or \$718,225 million per month. Back of the envelope calculations suggest that Supplemental Security Income in retirement equalled 0.137% of total labour income.

The three steady-state equilibria in which debt equals zero are autarkic. In these equilibria there is no trade with future unborn generations. The fourth steady-state equilibrium is the golden-rule. This steady-state equilibrium always exists in OLG models and in models with population growth it has the property that the real interest rate equals the rate of population growth. But although the golden-rule steady-state equilibrium always exists, *it is not true* that the golden-rule value of \bar{b} is always non-negative.

The golden-rule steady state occurs when the logarithm of the real interest factor equals zero. By inspecting the lower panel of Figure 4, it is apparent that government debt is positive at the golden rule steady state and, since debt is denominated in dollars, the price level is also positive in the golden-rule steady-state equilibrium. This is important because it is the empirically relevant case in most western democracies. For example, in the United States, government debt in the first quarter of 2019 exceeded \$22 trillion.

The values and properties of all four steady-state equilibria are reported in Table 1. We refer to the autarkic steady-state equilibria as Steady State A, Steady State C and Steady State D and to the golden-rule steady-state equilibrium as Steady State B. We see from this table that Steady States B, C and D are associated with a non-negative interest rate and are therefore dynamically efficient. Steady State A is associated with a negative interest rate of -47.5% and is therefore dynamically inefficient.⁹

Equilibrium Real Interest Rates					
Type	Value of R	Value of \bar{b}	# Unstable Roots	# Free Initial Conditions	Degree of Indeterminacy
Steady State A	0.525	0	60	61	1
Steady State B	1	53.7% of GDP	59	61	2
Steady State C	1.022	0	60	61	1
Steady State D	1.13	0	61	61	0

TABLE 1. Steady States of the Sixty-Two Generation Model

The sixty-two generation model with a calibrated income profile is similar to the three generation model from Section 5. In both models, the golden-rule steady-state equilibrium displays second degree indeterminacy. And in both models, the steady-state price level is positive and the initial price level is indeterminate even when fiscal policy is active. Importantly, because the monetary steady state is second-degree indeterminate, indeterminacy can hold even when both monetary and fiscal policy are active.

In Figure 5 we show the result of an experiment in which we perturb the initial value of b_1 by 3% and we perturb the real value of the initial wealth of all of the non-generic generations by the same amount. We refer to this perturbation as a 3% shock to the initial price level. We restrict R_2 to equal its steady state value but all other elements of the vector of initial conditions are allowed to respond to the shock to keep the path of interest rates and debt on a convergent path back to the

⁹See Cass (1972) for a definition and characterization of the conditions for dynamic efficiency.

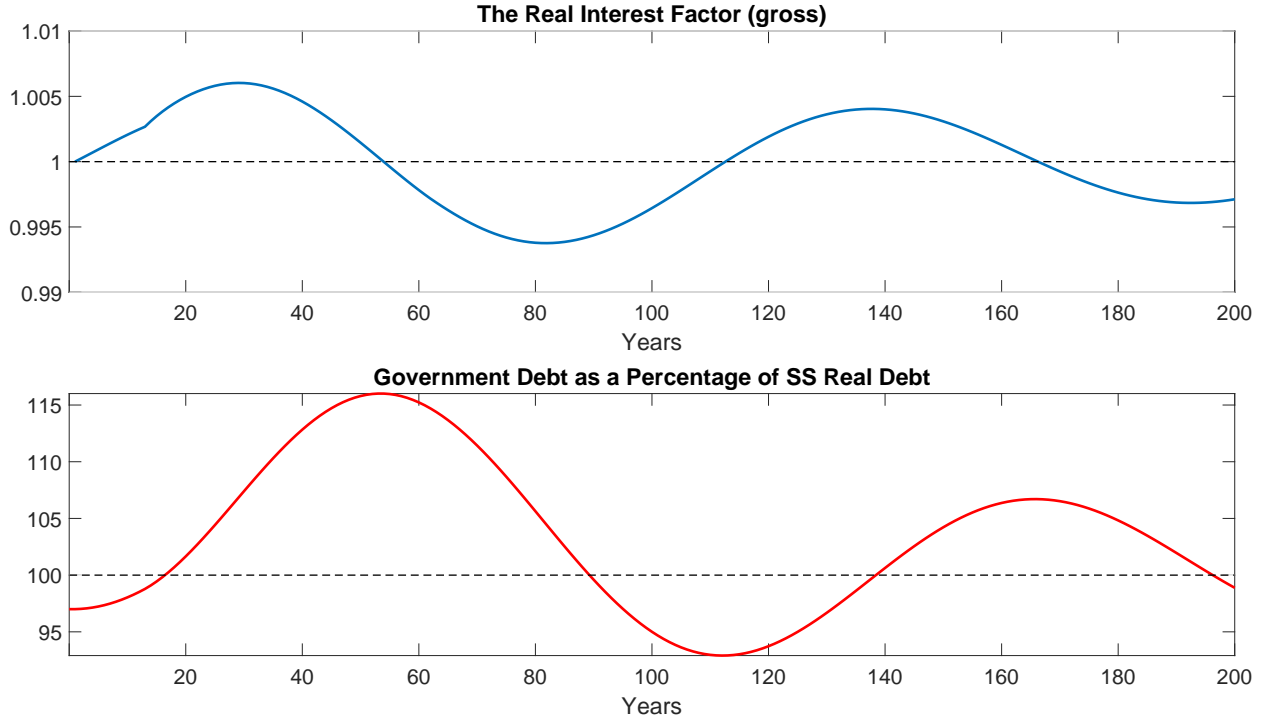


FIGURE 5. The Impact of the Initial Price Level Exceeding Its Steady State Value by 3%

steady state. Figure 5 demonstrates that the return to the steady state from an arbitrary initial condition is extremely slow.¹⁰

We also see from Figure 5 that our model can endogenously generate prolonged periods of negative real interest rates. The upper panel of this figure plots the path by which the real interest rate returns to its steady-state value and the lower panel plots the return path of the real value of government debt expressed as a percentage of GDP. The figure demonstrates that small deviations of initial conditions from the steady state can have long-lasting effects, and that during the convergence process the real interest rate may be negative for periods well in excess of ten years.

One may question whether the high degree of real interest rate persistence implied by our model is excessive. Have such long swings in real interest rates actually ever been observed? To address this question, Figure 6, reproduced from Yi and Zhang (2017), compares long run real interest rates in the G7 and documents that low-frequency real rate cycles, similar to those generated by our model, have characterized the evolution of real interest rates in all of these economies.¹¹

9. ROBUSTNESS TO DIFFERENT CALIBRATIONS

To explore the robustness of our findings to alternative calibrations, in Table 2 we record the properties of our model for different values of the annual discount rate and the intertemporal

¹⁰Note that our choice of 3% is entirely arbitrary and that this is one of many admissible equilibrium paths. In particular, since we endow the non-generic cohorts with steady state asset shares, therefore the steady state equilibrium with $\forall t \geq 1 : b_t \equiv \bar{b}$ and $R_{t+1} \equiv \bar{R}$ would have also been feasible.

¹¹See Yi and Zhang (2017) for a discussion of why long-run moving averages are likely to characterize trends in fundamental forces underlying real interest rates.

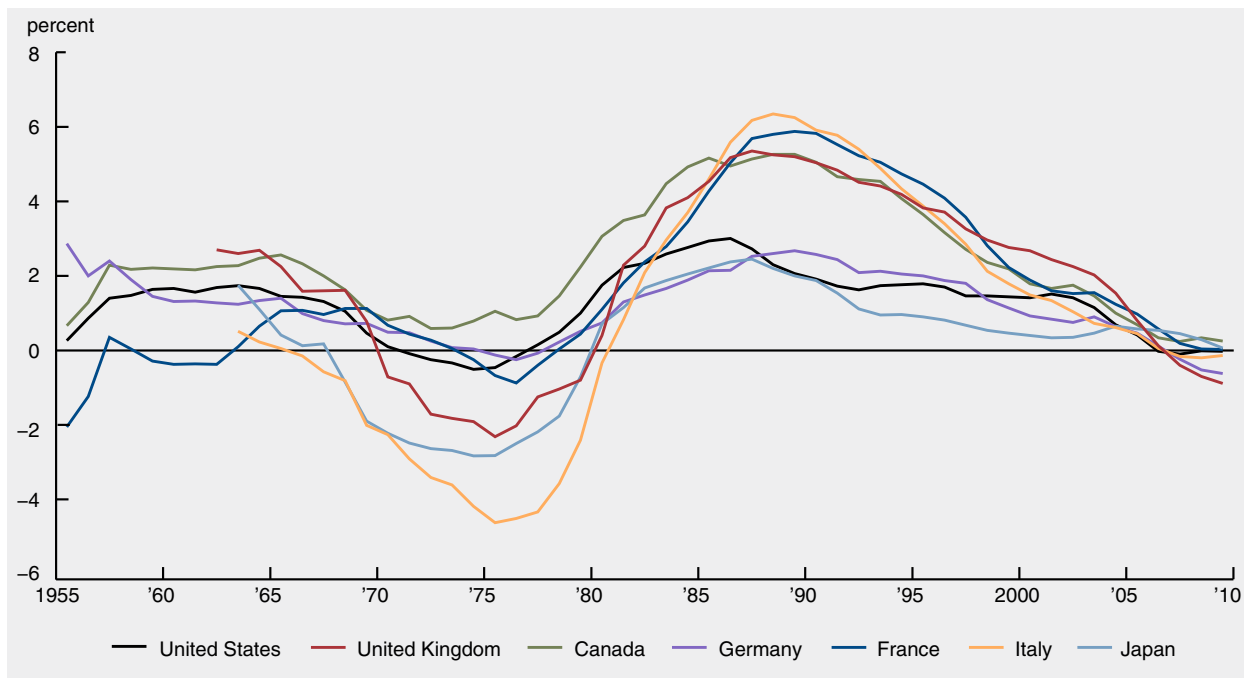


FIGURE 6. G7 Long-Run Real Interest Rates. Long-Run Real Interest Rates are 11-Year Centered Moving Averages of Annual Real Interest Rates. Source: Figure 1 in Yi and Zhang (2017)

elasticity of substitution. The example we featured in Section 8 had two degrees of indeterminacy and positive valued debt at the monetary steady state. Table 2 demonstrates that this property is not particularly special.

The table provides 40 different parameterizations of our model with intertemporal elasticity of substitution parameters ranging from $1/9$ to $1/5$ and discount rates ranging from 0.91 to 0.98. In all of these parameterizations we maintained the calibrated income profile illustrated in Figure 3. For each calibration Table 2 displays the number of steady-state equilibria in the interval $\bar{R} \in [0.5, 1.5]$, and the number of degrees of indeterminacy at the golden-rule steady-state equilibrium. There are fifteen parameterizations in which the golden-rule steady state displays one degree of indeterminacy and twelve in which it displays two degrees of indeterminacy. In all twelve of these parameterizations, debt has positive value in the steady state.

In Section 8, we showed that when $\beta = 0.953$ and $\eta = 1/6$, the golden-rule steady state displays two degrees of indeterminacy. An example with this property is interesting because it is not only the price level that is free to be determined by the beliefs of market participants; it is also the real rate of interest. We want to reiterate, however, that only one degree of indeterminacy is required for violations of the fiscal theory of the price level. And that occurs more frequently in our model than second degree indeterminacy.¹²

¹²If we hold constant the intertemporal elasticity of substitution and increase the discount rate, the number of unstable eigenvalues decreases initially from 60 to 59 and then changes abruptly to 61. We see this behaviour in Table 2 by moving along a typical row and observing that we pass from one degree of indeterminacy to two degrees of indeterminacy and then jump abruptly to 0 degrees of indeterminacy. At this last transition, a pair of complex

		Annual Discount Factor							
		0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98
IES									
IES = 1/5	# Steady States	4	4	4	4	4	2	2	2
	Degree of Indeterminacy	1	1	1	1	1	2	0	0
	Value of Debt	-2.8	-2.2	-1.5	-0.8	-0.2	0.5	1.1	1.8
IES = 1/6	# Steady States	4	4	4	4	4	4	2	2
	Degree of Indeterminacy	1	1	1	1	2	2	0	0
	Value of Debt	-1.9	-1.3	-0.7	-0.2	0.4	0.9	1.5	2.0
IES = 1/7	# Steady States	3	3	3	3	3	3	1	1
	Degree of Indeterminacy	1	1	1	2	2	0	0	0
	Value of Debt	-1.2	-0.7	-0.2	0.3	0.8	1.2	1.7	2.2
IES = 1/8	# Steady States	3	3	3	3	3	3	3	1
	Degree of Indeterminacy	1	1	2	2	2	0	0	0
	Value of Debt	-0.6	-0.2	0.2	0.6	1.0	1.5	1.9	2.3
IES = 1/9	# Steady States	3	3	3	3	3	3	3	3
	Degree of Indeterminacy	1	2	2	2	2	0	0	0
	Value of Debt	-0.2	0.1	0.5	0.9	1.3	1.6	2.0	2.4

TABLE 2. Robustness of Indeterminacy to Alternative Calibrations, Focusing Only on Steady States with $\bar{R} \in [0.5, 1.5]$

10. FISCAL AND MONETARY POLICY

In this section we discuss what happens when we relax either the assumption that fiscal policy is active or the assumption that monetary policy is passive. We first show that passive fiscal policy makes indeterminacy more likely. We then demonstrate that ensuring bounded inflation under an active Taylor rule imposes an additional restriction on the set of equilibrium paths. This additional restriction reduces the degree of indeterminacy by one.

Consider first what happens when fiscal policy is passive. To model a passive fiscal policy we assume that the treasury raises taxes, τ_t , in proportion to the real value of outstanding debt to ensure that the primary deficit d_t satisfies the equation

$$d_t = -\delta b_t,$$

where $\delta \geq 0$ is a debt repayment parameter. Combining this assumption with the definition of the government debt accumulation equation leads to the following amended debt accumulation equation,

$$b_{t+1} = [R_{t+1} - \delta]b_t.$$

roots crosses the unit circle, a phenomenon associated with a Hopf Bifurcation and the creation of a limit cycle. See Guckenheimer and Holmes (1983) for a discussion of the Hopf Bifurcation. We have not explored the phenomenon in this paper, but it is likely that for discount rates close to 1, this model displays endogenous limit cycles that are second-degree indeterminate.

For values of $[\bar{R} - \delta] < 1$ the effect of making fiscal policy passive is to introduce an additional stability mechanism that increases the degree of indeterminacy at each of the four steady states whenever δ is large enough. Passive fiscal policy makes indeterminacy more likely.

We next assume that fiscal policy is active and the central bank follows a Taylor rule (Taylor, 1999),

$$1 + i_t = \left(\frac{\bar{R}}{\bar{\Pi}^{\phi_\pi}} \right) \Pi_t^{1+\phi_\pi}, \quad t = 1, \dots, \infty. \quad (26)$$

Because this equation begins at date 1, the nominal interest rate in period 1 depends on p_0 through the definition, $\Pi_1 = p_1/p_0$. We treat p_0 as an initial condition that has the same status as the initial value of nominal debt, B_0 . In Eq. (26), $\bar{\Pi}$ is the inflation target, \bar{R} is the steady state real interest rate and ϕ_π is the response coefficient of the policy rate to deviations of inflation from target. The Taylor Rule is passive if $-1 \leq \phi_\pi \leq 0$ and active if $\phi_\pi > 0$.

When the central bank follows a Taylor Rule, the real interest rate and the real value of government debt continue to be determined by the bond market clearing equation and the debt accumulation equation. It follows that the conditions we have characterized in previous sections continue to ensure that the real interest rate and the real value of government debt remain bounded.

When the central bank follows a passive Taylor Rule, (see Appendix E.1) the following equation characterizes the asymptotic behaviour of the future inflation rate,

$$\lim_{T \rightarrow \infty} \tilde{\Pi}_{T+1} = \lim_{T \rightarrow \infty} (1 + \phi_\pi)^T \tilde{\Pi}_1 - \lim_{T \rightarrow \infty} \sum_{s=1}^T (1 + \phi_\pi)^{T-s} \tilde{R}_{T+1}, \quad (27)$$

where $\kappa \equiv \bar{\Pi}/\bar{R}$ and the tilde denotes deviations from the steady state. The limit of the first term on the right side of Equation (E2) is zero because $1 + \phi_\pi < 1$ and the second term is finite as a consequence of the boundedness of R_t . It follows that inflation is bounded whenever R_t is bounded. This is a generalization of the argument we made for the boundedness of the inflation rate when the central bank follows an interest rate peg and it does not impose any additional restrictions on the equations of the model for an equilibrium to be determinate.

When the central bank follows an active Taylor Rule, (see Appendix E.2), the initial price level is determined by the forward-looking equation

$$p_1 = p_0 \left(\bar{\Pi} + \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1 + \phi_\pi} \right)^s (R_{1+s} - \bar{R}) \right). \quad (28)$$

Importantly, this restriction on the set of equilibrium paths is additional to the restriction

$$p_1 = \frac{B_0}{b_1},$$

that we used to generate the equilibrium sequence of interest rates. It follows that we are no longer free to pick R_2 and p_1 independently of each other. This establishes that an active monetary policy eliminates one degree of indeterminacy.

If a model has one degree of indeterminacy when the policy combination is passive-active, an active-active policy combination would now admit a unique solution. When the steady-state

equilibrium displays second-degree indeterminacy, as in our sixty-two generation example, it is not just the initial price level that is indeterminate; it is also the initial real interest rate.

For any *given* choice of the initial interest rate, R_2 , active monetary policy removes nominal indeterminacy. Crucially, however, active monetary policy does not remove real indeterminacy and there continue to be many possible choices for the initial real interest rate, each of them associated with a different initial price level and a different equilibrium path for all future real interest rates and all future inflation rates.

11. SO WHAT?

How should the reader react to our finding that a particular example of an OLG economy displays indeterminate steady state equilibria? One possible reaction is that the real world is demonstrably determinate in the sense that a general equilibrium theorist would use that term. An advocate of this position might claim that the profession has rejected the OLG model after careful consideration and that the new-Keynesian version of the representative agent model has been demonstrated through careful empirical work to be a much better fit to time series and cross-section data. We do not think this argument holds water. The OLG model fell from favor as the preferred vehicle for understanding monetary and fiscal policy for theoretical reasons, not because it failed a series of empirical tests.¹³

Perhaps the analysis we have presented is only possible because of the restrictive assumption of an endowment economy. That turns out not to be the case. We have known since the work of Diamond (1965) that models with capital possess monetary equilibria as well as autarkic equilibria and a simple extension of the argument in Section 5 establishes that calibrated examples of the model with capital preserve the feature of the existence of a monetary steady state equilibrium with two degrees of indeterminacy at the steady state.¹⁴

It might be argued that we have dealt with the finite-lived case and that in the real world people are connected by operative chains of bequests and that they effectively have an infinite horizon (Barro, 1974). But Pietro Reichlin (1992) has shown that the perpetual youth model of Blanchard (1985) displays multiple sets of indeterminate steady-state equilibria even when people may live forever.

A further argument that might be levelled against indeterminate steady-state equilibria is that they are unlearnable. According to this argument, people are adaptive learners and a rational expectations equilibrium accurately describes the properties of an economy after learning has taken place. Proponents of this argument claim that determinate steady-state equilibria are often stable under learning and that this is a good reason to select these equilibria when a model has multiple steady state equilibria (McCallum, 2003, 2007). This would be a persuasive argument if it were always true that indeterminate steady states are unstable under learning, but exhaustive enquiries

¹³The model went out of favor, in part because a subset of influential macroeconomists considered the existence of indeterminacy in the model to be a shortcoming rather than a strength of the approach (Cherrier and Saidi, 2018). For summary of recent research that uses indeterminacy as a positive aspect of DSGE models see Farmer (2020).

¹⁴Because the model with capital raises additional questions, not least of which is the reason for employment fluctuations in a market economy, we have chosen not to include that analysis in this paper.

into the stability of adaptive learning schemes have found that both determinate and indeterminate steady-state equilibria may be stable under plausible adaptive learning schemes (Evans and Honkapohja, 2001). And the non-stationary equilibria of a monetary model in which artificially intelligent agents use deep reinforcement learning – an algorithm similar to the one that was used by Deep Blue (Hsu et al., 2018) to beat world chess grand masters – has been shown to converge to the Pareto superior equilibrium in a monetary model with two equilibria, even when that equilibrium is indeterminate (Chen et al., 2021).

So what impression do we hope to have left with the reader? We consider the example in our paper to be a proof of concept that, we hope, will encourage other researchers to explore further the properties of the OLG model as a vehicle for understanding how shocks are propagated through the asset markets. The indeterminacies we have showcased here are features, not just of overlapping generations models, but also of models with credit constraints and borrowing limits (Woodford, 1988; Schmitt-Grohé and Uribe, 2020). If, as we believe, the indeterminacies we have focused on are prevalent in the real world, the policy prescriptions of new-Keynesian representative agent economies, if followed by central banks and national fiscal authorities, may have unintended consequences. In our view, relative price indeterminacy is not just a theoretical curiosity; it is a feature of the real world.

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APPENDIX A. THE HUMP-SHAPED PROFILE AND THE IES

In this Appendix we derive an expression for the slope of the steady state savings function evaluated at the steady state $R = \lambda/\beta$ for the parameter values $\lambda = \beta = 1$. Define the functions $W(R)$ and $\phi(R)$

$$W(R) = \omega_1 + \frac{\omega_2}{R} + \frac{\omega_3}{R^2}, \quad (\text{A1})$$

$$\phi(R) = 1 + \frac{(\beta R)^\eta}{R} + \frac{(\beta R)^{2\eta}}{R^2}. \quad (\text{A2})$$

Applying the solution to the T -generation maximizing problem with CES preferences from Appendix B we have the following steady-state consumption demand functions

$$c_1(R) = \frac{W(R)}{\phi(R)}, \quad c_2(R) = (\beta R)^\eta \frac{W(R)}{\phi(R)}, \quad c_3(R) = (\beta R)^{2\eta} \frac{W(R)}{\phi(R)}, \quad (\text{A3})$$

where subscripts indicate age. Define the steady-state savings functions of the young and middle-aged as

$$s_1(R) = \omega_1 - c_1(R), \quad s_2(R) = R s_1(R) + \omega_2 - c_2(R). \quad (\text{A4})$$

Next, we seek expressions for the functions $\omega_1(\omega_2)$ and $\omega_3(\omega_2)$ which solve the equations

$$w_1 + w_2 + w_3 = 1 + \lambda^\eta + \lambda^{2\eta}, \quad (\text{A5})$$

$$w_1 + \frac{w_2 \beta}{\lambda} + \frac{w_3 \beta^2}{\lambda^2} = 1 + \frac{\lambda^\eta \beta}{\lambda} + \frac{\lambda^{2\eta} \beta^2}{\lambda^2}. \quad (\text{A6})$$

These are given by the expressions

$$\omega_1(\omega_2) = 1 + \frac{\frac{\beta}{\lambda} \lambda^\eta}{\left(1 + \frac{\beta}{\lambda}\right)} - \frac{\frac{\beta}{\lambda} \omega_2}{\left(1 + \frac{\beta}{\lambda}\right)}, \quad (\text{A7})$$

$$\omega_3(\omega_2) = \frac{\lambda^\eta}{\left(1 + \frac{\beta}{\lambda}\right)} + \lambda^{2\eta} - \frac{\omega_2}{\left(1 + \frac{\beta}{\lambda}\right)}. \quad (\text{A8})$$

Define the function

$$\psi(R) = \frac{W(R)}{\phi(R)}, \quad (\text{A9})$$

and note that aggregate savings in a steady state equilibrium, $f_{\tilde{\omega}}(R)$, defined as the sum of $S_1(R; \tilde{\omega})$ and $S_2(R; \tilde{\omega})$ is given by the expression,

$$f_{\tilde{\omega}}(R) = \left(\omega_1(\omega_2) - \psi(R) \right) + R \left(\omega_1(\omega_2) - \psi(R) \right) + \omega_2 - \psi(R) \left(\beta R \right)^\eta. \quad (\text{A10})$$

Rearranging terms, this leads to the equation

$$f_{\tilde{\omega}}(R) = \omega_1(\omega_2) \left(1 + R \right) + \omega_2 - \psi(R) \left(1 + R + (\beta R)^\eta \right). \quad (\text{A11})$$

We seek an expression for the derivative of $f_{\tilde{\omega}}(R)$ evaluated at $\lambda = \beta = 1$. For these parameter values the functions $\omega_1(\omega_2)$, $W(R)$ and $\phi(R)$ are given by the following formulae

$$\omega_1(\omega_2) = \frac{3 - \omega_2}{2}, \quad W(R) = 1 + \frac{1}{R} + \frac{1}{R^2}, \quad \phi(R) = 1 + R^{\eta-1} + R^{2(\eta-1)}. \quad (\text{A12})$$

Evaluating each term at $R = \lambda/\beta = 1$ gives

$$W(1) = 3, \quad \phi(1) = 3, \quad \text{from which it follows that } \psi(1) = 1. \quad (\text{A13})$$

The partial derivatives of $W(R)$ and $\phi(R)$ are given by

$$\frac{\partial W}{\partial R} = -\frac{1}{R^2} - \frac{2}{R^3}, \quad \frac{\partial \phi}{\partial R} = (\eta - 1)R^{\eta-2} + 2(\eta - 1)R^{2\eta-3}, \quad (\text{A14})$$

which when evaluated at $R = \lambda/\beta = 1$ gives

$$\left. \frac{\partial W}{\partial R} \right|_{R=1} = -3, \quad \left. \frac{\partial \phi}{\partial R} \right|_{R=1} = 3(\eta - 1). \quad (\text{A15})$$

We seek an expression for the partial derivative of $f_{\tilde{\omega}}(R)$ evaluated at the steady state $R = \lambda/\beta = 1$.

Using the chain rule, this is equal to

$$\left. \frac{\partial f_{\tilde{\omega}}}{\partial R} \right|_{R=1} = \omega_1(\omega_2) - \psi(1) \left(1 + \eta \right) - 3 \left. \frac{\partial \psi}{\partial R} \right|_{R=1}. \quad (\text{A16})$$

A further application of the chain rule to the function $\psi(R)$ leads to the expression

$$\left. \frac{\partial \psi}{\partial R} \right|_{R=1} = \frac{\phi(1) \left. \frac{\partial W}{\partial R} \right|_{R=1} - W(1) \left. \frac{\partial \phi}{\partial R} \right|_{R=1}}{W(1)^2} = \frac{-9 - 9(\eta - 1)}{9} = -\eta. \quad (\text{A17})$$

Putting all these pieces together gives

$$\left. \frac{\partial f_{\tilde{\omega}}}{\partial R} \right|_{R=1} = \frac{3 - \omega_2}{2} - (1 + \eta) + 3\eta = \frac{1 - \omega_2 + 4\eta}{2}, \quad (\text{A18})$$

which is Eq. (18) in the body of the paper.

□

APPENDIX B. ANALYTIC SOLUTIONS FOR EXCESS DEMAND

B.1. The generic optimization problem. Consider a person with CES preferences who lives for T periods and has perfect foresight of future prices. This person solves the problem,

Problem 1.

$$\max_{\{c_t^t, c_{t+1}^t, \dots, c_{t+T-1}^t\}} \frac{a_1(c_t^t)^\alpha + a_2(c_{t+1}^t)^\alpha + \dots + a_T(c_{t+T-1}^t)^\alpha}{\alpha}, \quad (\text{B1})$$

subject to the lifetime budget constraint

$$\sum_{i=1}^T \mathcal{Q}_t^{t-1+i} c_{t-1+i} = \sum_{i=1}^T \mathcal{Q}_t^{t-1+i} \tilde{w}_i. \quad (\text{B2})$$

Here, c_s^t is consumption in period s of a person born in period t , $i \in 1, \dots, T$ is age, and \tilde{w}_i is after-tax endowment. The parameters a_i are utility weights and $\alpha \leq 1$ is a curvature parameter which is related to intertemporal substitution, η , by the identity

$$\eta \equiv \frac{1}{1 - \alpha}. \quad (\text{B3})$$

The term \mathcal{Q}_t^k , defined by the expression

$$\mathcal{Q}_t^k \equiv \prod_{j=t+1}^k \frac{1}{R_j}, \quad \mathcal{Q}_t^t = 1, \quad (\text{B4})$$

is the relative price at date t of a commodity for delivery at date k .

This optimization problem includes the case of a constant discount factor β for which

$$[a_1, a_2, \dots, a_T] = [1, \beta, \dots, \beta^{T-1}] \quad (\text{B5})$$

and logarithmic preferences which is the limiting case when $\alpha \rightarrow 0$. We permit the discount factor to vary with age to nest the Kehoe and Levine (1983) example which we use to cross-check our results.

Proposition 2. *The solution to Problem 1 is given by*

$$\hat{c}_{t-1+k}^t = \frac{a_k^\eta \sum_{i=1}^T (\mathcal{Q}_t^{t-1+i} \tilde{w}_i)}{\left(\mathcal{Q}_t^{t-1+k}\right)^\eta \sum_{i=1}^T (\mathcal{Q}_t^{t-1+i})^{1-\eta} a_i^\eta}, \quad k = 1, \dots, T. \quad (\text{B6})$$

where \hat{c}_{t-1+k}^t denotes the consumption, at time $t - 1 + k$, of an agent born at time t .

Proof. The result follows directly from substituting the first-order conditions into the budget constraint and rearranging terms. \square

B.2. Non-generic optimization problems. Let j be an index that runs from 1 to $T-1$. Consider a non-generic person born in period $1-j$ with real assets $\nu_{1-j} \equiv \lambda_{1-j} b_1$ who lives for $T-j$ periods. This person solves Problem 2

Problem 2.

$$\max_{\{c_1^{1-j}, \dots, c_{1-j+T-1}^{1-j}\}} \frac{a_{T-j+1} (c_1^{1-j})^\alpha + a_{T-j+2} (c_2^{1-j})^\alpha + \dots + a_T (c_{1-j+T-1}^{1-j})^\alpha}{\alpha}, \quad j = 1, \dots, T-1 \quad (\text{B7})$$

subject to the lifetime budget constraint

$$\sum_{k=1}^{(1-j)+T-1} \mathcal{Q}_1^k \left(c_k^{1-j} - \tilde{w}_{k-(1-j)+1} \right) \leq \lambda_{1-j} b_1, \quad (\text{B8})$$

Proposition 3. *Let $k \in \{1, \dots, T-j\}$. The solution to Problem 2 is given by*

$$\hat{c}_k^{1-j} = \frac{a_{k+j}^\eta \left(\nu_{1-j} + \sum_{i=1}^{T-j} \mathcal{Q}_t^i \tilde{w}_{j+i} \right)}{\left(\mathcal{Q}_t^{t+k-1}\right)^\eta \sum_{i=1}^{T-j} (\mathcal{Q}_t^i)^{1-\eta} a_{j+i}^\eta}, \quad k = 1 \dots 1-j+T-1. \quad (\text{B9})$$

Proof. The problem above is identical to a generic one solved by an agent who has $T-j$ periods to live, whose endowments are $\{\tilde{w}_{j+1} + \nu_{1-j}, \tilde{w}_{j+2}, \dots, \tilde{w}_T\}$, and whose preference parameters in the utility function are $\{a_{j+1}, a_{j+2}, \dots, a_T\}$. \square

APPENDIX C. EQUILIBRIUM AS THE SOLUTION TO A DIFFERENCE EQUATION

In Section 4 we showed that equilibria of the 3-generation model can be characterized as the solution to a difference equation, determined by the behaviour of the generic generations, together with set of initial conditions determined by the behavior of the on-generic generations. In this Appendix we generalize our analysis to the T -generation model.

C.1. Generic Equilibrium Conditions for the T -Generation Case. A steady-state equilibrium is a non-negative real number \bar{R} and a (possibly negative) real number \bar{b} that solve the equations,

$$f(\bar{R}, \bar{R}, \dots, \bar{R}) = \bar{b} + d, \quad \bar{b}(1 - \bar{R}) = \bar{R}d. \quad (\text{C1})$$

Let $\{\bar{R}, \bar{b}\}$ be a steady state equilibrium and let

$$\tilde{R}_t \equiv R_t - \bar{R}, \quad \text{and} \quad \tilde{b}_t \equiv b_t - \bar{b}, \quad (\text{C2})$$

represent deviations of b_t and R_t from their steady state values. Define a vector

$$X_t \equiv [R_{t+T-1}, R_{t+T-2}, \dots, R_{t-T+4}, b_t]^\top, \quad (\text{C3})$$

of length $2T - 3$ and a function $F(\cdot)$,

$$F(X_t, X_{t-1}) \equiv \begin{bmatrix} f(R_{t-T+3}, R_{t-T+4}, \dots, R_{t+T-2}, R_{t+T-1}) & - & b_t + d_t \\ & & b_t \\ & & - R_t(b_{t-1} + d_{t-1}) \end{bmatrix}, \quad (\text{C4})$$

and let J_1 and J_2 represent the partial derivatives of this function with respect to X_t and X_{t-1} . Using this notation, the local dynamics of equilibrium sequences close to the steady state can be approximated as solutions to the linear difference equation

$$J_1 \tilde{X}_t = J_2 \tilde{X}_{t-1}, \quad t = T - 1, \dots \quad (\text{C5})$$

with initial condition

$$\tilde{X}_{T-2} = \bar{X}_{T-2}. \quad (\text{C6})$$

The local stability of these equations depends on the eigenvalues of the matrix

$$J \equiv J_1^{-1} J_2. \quad (\text{C7})$$

If one or more roots of the matrix J is outside the unit circle there is no guarantee that sequences of interest factors and government debt generated by Equation (C5) will remain bounded. To ensure stability, we must choose initial conditions that place \tilde{X}_{T-2} in the linear subspace associated with

the stable eigenvalues of J . The initial conditions are determined by the non-generic equilibrium conditions which we turn to next.

C.2. Non-Generic Equilibrium Conditions for the T -Generation Case. Asset market equilibrium in periods 1 through $T - 2$ is characterized by a family of aggregate savings functions, $g_t^T(\cdot)$, one family for each value of T . These functions are different at each date t , because the asset demand functions of the non-generic generations depend on the initial wealth distribution and the initial price level as well as on real interest rates. To cut down on notation, we have suppressed the dependence of $g_t^T(\cdot)$ on the endowment profile which is parameterized by the multiple of initial debt owned or owed by the initial generations.

To understand the structure of non-generic asset demand equations it helps to build intuition by considering the case of $T = 4$. This is the simplest example where we must keep track of the debt-accumulation equations in the $T - 2$ initial periods. For the case of $T = 4$, there are two non-generic asset demand equations and one non-generic debt accumulation equation. These equations are described by the expression

$$G^4(Z_0^4) \equiv \begin{bmatrix} g_1^4(R_2, R_3, R_4, b_1) & - & (b_1 + d) \\ g_2^4(R_2, R_3, R_4, R_5, b_1) & - & (b_2 + d) \\ b_2 & - & R_2(b_1 + d) \end{bmatrix} = 0, \quad (\text{C8})$$

where the vector Z_0^4 , is a set of variables that are determined, in equilibrium, by asset market clearing and government asset accumulation in the initial $T - 2$ periods of the model. For the case of $T = 4$, the equation $G^4(\cdot) = 0$ places 3 restrictions on the 6 elements of Z_0^4 .

$$Z_0^4 \equiv [R_5, R_4, R_3, R_2, b_2, b_1]^\top \equiv [X_{T-2}, Y_{T-2}]^\top, \quad (\text{C9})$$

where Y_{T-2} is the vector

$$Y_{T-2} \equiv [b_1]^\top, \quad (\text{C10})$$

and X_{T-2} is a vector of initial conditions to the vector-valued difference equation characterized by equations (22) and (2).

This example can be generalized. For the T -generation model, Z_0^T contains $3T - 6$ elements, equal to the union of the terms in the over-braces of the following expression,

$$Z_0^T \equiv \left[\overbrace{R_{2T-3}, R_{2T-4}, \dots, R_2, b_{T-2}}^{2T-3}, \overbrace{b_{T-3}, \dots, b_2, b_1}^{T-3} \right]^\top \equiv [X_{T-2}, Y_{T-2}]^\top. \quad (\text{C11})$$

The function $G(\cdot)$ contains $2T - 5$ rows,

$$G^T(Z_0^T) \equiv \begin{bmatrix} g_1^T(R_2, \dots, R_{1+T-2}, R_{1+T-1}, b_1) & - & (b_1 + d) \\ \vdots & & \vdots \\ g_{T-2}^T(R_2, \dots, R_{2T-4}, R_{2T-3}, b_1) & - & (b_{T-2} + d) \\ & b_2 & - & R_2(b_1 + d) \\ \vdots & & & \vdots \\ & b_{T-2} & - & R_{T-1}(b_{T-3} + d) \end{bmatrix} = 0, \quad (\text{C12})$$

and the equation $G^T(\cdot) = 0$ places $2T - 5$ restrictions on the $3T - 6$ elements of the unknown vector Z_0^T . Subtracting the number of these restrictions from the number of initial variables leaves $T - 1$ non-predetermined elements of X_{T-2} . In Appendix D we describe how the non-generic equilibrium equations may be combined with the assumption that equilibrium sequences must remain bounded to characterize the determinacy properties of equilibria.

□

APPENDIX D. PROOF OF PROPOSITION 1

To determine the conditions for local uniqueness of equilibrium close to a given steady-state equilibrium we seek a relationship between the roots of the matrix J and the number of initial conditions. We begin by describing the construction of J .

Proof. To find conditions under which equilibrium sequences remain bounded, we exploit the properties of the Jordan decomposition of J , which we write as

$$J = Q\Lambda Q^{-1}, \quad (\text{D1})$$

where the matrix $J = J_1^{-1}J_2$ and,¹⁵

$$J_1 X_t \equiv \begin{array}{c} \overbrace{\left[\begin{array}{ccccccc} f_{t+T-1} & f_{t+T-2} & \cdots & f_{t+1} & \cdots & f_{t-T+5} & f_{t-T+4} & -1 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array} \right]}^{J_1} \overbrace{\left[\begin{array}{c} \tilde{R}_{t+T-1} \\ \tilde{R}_{t+T-2} \\ \vdots \\ \tilde{R}_{t+1} \\ \vdots \\ \tilde{R}_{t-T+5} \\ \tilde{R}_{t-T+4} \\ \tilde{b}_t \end{array} \right]}^{\tilde{X}_t}, \end{array} \quad (\text{D2})$$

$$J_2 X_{t-1} \equiv \begin{array}{c} \overbrace{\left[\begin{array}{cccccc} 0 & \cdots & 0 & \cdots & 0 & -f_{t-T+3} & 0 \\ 1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \ddots & \vdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & \bar{b} + d & \cdots & 0 & 0 & \bar{R} \end{array} \right]}^{J_2} \overbrace{\left[\begin{array}{c} \tilde{R}_{t+T-2} \\ \tilde{R}_{t+T-3} \\ \vdots \\ \tilde{R}_t \\ \vdots \\ \tilde{R}_{t-T+4} \\ \tilde{R}_{t-T+3} \\ \tilde{b}_{t-1} \end{array} \right]}^{\tilde{X}_{t-1}}, \end{array} \quad (\text{D3})$$

where f_k is the partial derivative of the function f with respect to R_k evaluated at the steady state $\{\bar{R}, \bar{b}\}$.

The matrix Λ is an upper triangular matrix with the eigenvalues of J on the diagonal and Q is the matrix of left eigenvectors of J . The restrictions that prevent X_t from becoming unbounded are found by premultiplying \tilde{X}_{T-2} by the rows of Q^{-1} associated with the unstable eigenvalues of J and equating the product to zero. We refer to the matrix that contains these rows as Q_u^{-1} .¹⁶

¹⁵An updated solution method for rational expectations models that allows either J_1 or J_2 or both to be singular can be found in Sims (2001). Solution methods for models with indeterminacy are provided in Farmer et al. (2015).

¹⁶In practice, the Jordan decomposition is numerically unstable but there are several good computational methods to compute Q_u^{-1} that are implemented in all modern programming languages. The reader is referred to Golub and VanLoan (1996) for a description of the algorithms used to solve problems of this kind and to Farmer (1999) for an accessible introduction to solution methods for rational expectations models with indeterminate equilibria.

Let K be the number of eigenvalues of J that lie outside the unit circle. The requirement that equilibrium sequences remain bounded places K linear restrictions on the initial vector X_{T-2} which we express with the matrix equation

$$Q_u^{-1} X_{T-2} = 0, \quad (\text{D4})$$

where Q_u is a $K \times 2T - 3$ matrix.

Next, we turn to the non-generic generations which we index by their date of birth, measured by an index $1 - j$ where j runs from 1 to $T - 1$. We represent the financial wealth of generation $1 - j$ by a real number λ_{1-j} which may be positive or negative and that represents the share, owned by generation $1 - j$, of the period 1 dollar-valued government debt.

$$\nu_{1-j} \equiv \lambda_{1-j} \frac{B_0}{p_1}. \quad (\text{D5})$$

We refer to the vector of shares as $\mathbb{L} \equiv \{\lambda_0, \lambda_{-1}, \dots, \lambda_{2-T}\}$. These shares sum to 1 as a consequence of asset market clearing.¹⁷ In Appendix B

To determine the number of restrictions that come from the non-generic equilibrium conditions, we turn to the function $G^T(\cdot)$ derived in Appendix C.2. This function depends on Z_0^T and on the initial nominal wealth distribution \mathbb{L} . To ensure that our local analysis remains valid we first compute the steady state distribution of wealth from the equation¹⁸

$$\nu_{1-j} = \bar{R} s_1^{2-j}(\bar{R}, \bar{R}, \dots, \bar{R}), \quad j = 1, \dots, T - 1, \quad (\text{D6})$$

and we use the identity,

$$\lambda_{1-j} \equiv \frac{\nu_{1-j}}{\bar{b}}, \quad (\text{D7})$$

to recover the steady-state values of λ_{1-j} . In what follows we set the initial asset shares to these steady state values.

Let G_X, G_Y be the Jacobians of the function $G^T(\cdot)$ with respect to X_{T-2} and Y_{T-2} evaluated at the steady-state equilibrium $\{\bar{R}, \bar{b}\}$ and let $\tilde{X}_{T-2}, \tilde{Y}_{T-2}$ be deviations of these vectors from their steady-state values. Using this notation, the predetermined equilibrium conditions place the

¹⁷We require $B_0 \neq 0$ for the shares λ to be well-defined. See Niepelt (2004) for a discussion of a conceptual issue associated with non-zero values of B_0 .

¹⁸Here, s_1^{1-j} is the savings function defined in Equation (21) evaluated at the steady state sequence of real interest rates.

following $2T - 5$ restrictions on the $3T - 6$ unknowns $Z_0^T \equiv [X_{T-2}, Y_{T-2}]$.

$$G_X \tilde{X}_{T-2} + G_Y \tilde{Y}_{T-2} = 0. \quad (\text{D8})$$

The uniqueness of equilibrium then comes down to the question of the number of solutions to the equations

$$\begin{bmatrix} G_X & G_Y \\ Q_u^{-1} & 0 \end{bmatrix} \begin{bmatrix} \tilde{X}_{T-2} \\ \tilde{Y}_{T-2} \end{bmatrix} = \begin{bmatrix} \bar{G} \\ 0 \end{bmatrix}, \quad (\text{D9})$$

where $\bar{G} \neq 0$ represents the perturbations of the initial wealth distribution \mathbb{L} from its steady-state value. It follows immediately that a sufficient condition for the existence of a unique equilibrium is that the matrix

$$\mathcal{M} \equiv \begin{bmatrix} G_X & G_Y \\ Q_u^{-1} & 0 \end{bmatrix}, \quad (\text{D10})$$

is square and non singular.

If $K > T - 1$ then \mathcal{M} has more rows than columns and, except for the special case where the rows of \mathcal{M} are linearly dependent, Equation (D9) has no solution.

If $K = T - 1$ then \mathcal{M} is square. If \mathcal{M} also has full row rank, there exists a unique solution to this equation given by the expression

$$\begin{bmatrix} X_{T-2} \\ Y_{T-2} \end{bmatrix} = \mathcal{M}^{-1} \begin{bmatrix} \bar{G} \\ 0 \end{bmatrix}. \quad (\text{D11})$$

Finally, if $K \leq T - 1$ then \mathcal{M} has fewer rows than columns. In this case we are free to append a $(T - 1 - K) \times (3T - 6)$ matrix of linear restrictions to Equation (D9). For example, if $T - 1 - K = 1$ we can append a row that sets $b_1 = \bar{b}_1$. \square

APPENDIX E. INFLATION UNDER A TAYLOR RULE

In this Appendix we derive equations that characterize the behaviour of the inflation rate when the Taylor Rule is passive and when it is active.

E.1. The case of a passive Taylor Rule. Using the Taylor rule to substitute for $1 + i_t$ in the Fisher parity condition yields the following difference equation for inflation

$$\Pi_{t+1} = \left(\frac{\bar{R}}{R_{t+1}} \right) \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{\phi_\pi} \Pi_t, \quad \text{for all } t = 1, \dots, \infty \quad (\text{E1})$$

which we linearize around a steady state to obtain

$$\tilde{\Pi}_{t+1} = (1 + \phi_\pi) \tilde{\Pi}_t - \kappa \tilde{R}_{t+1}, \quad \text{for all } t = 1, \dots, \infty. \quad (\text{E2})$$

Here, $\kappa \equiv \bar{\Pi}/\bar{R}$ and the tilde denotes deviations from the steady state. Iterating Equation (E2) we obtain

$$\lim_{T \rightarrow \infty} \tilde{\Pi}_{T+1} = \lim_{T \rightarrow \infty} (1 + \phi_\pi)^T \tilde{\Pi}_1 - \lim_{T \rightarrow \infty} \sum_{s=1}^T (1 + \phi_\pi)^{T-s} \tilde{R}_{T+1}. \quad (\text{E3})$$

This is Equation (27) in Section 10.

E.2. The case of an active Taylor Rule. To find conditions under which inflation is bounded when the Taylor Rule is active, we use Equation E2 to write the inflation rate at date t as a function of all future real interest rates and all future inflation rates,

$$\tilde{\Pi}_t = \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1 + \phi_\pi} \right)^s \tilde{R}_{t+s} + \lim_{T \rightarrow \infty} \left(\frac{1}{1 + \phi_\pi} \right)^T \tilde{\Pi}_{t+T}. \quad (\text{E4})$$

If inflation is bounded, and if the Taylor Rule is active, the second term on the right side of Equation (28) is zero. Evaluating Equation (E4) at $t = 1$, we arrive the following expression for the initial gross inflation rate.

$$\tilde{\Pi}_1 \equiv (\tilde{\Pi}_1 - \bar{\Pi}) = \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1 + \phi_\pi} \right)^s \tilde{R}_{1+s} \quad (\text{E5})$$

Using the definition of inflation in period 1, Equation (E5) places the following restriction on the initial price level,

$$p_1 = p_0 \left(\bar{\Pi} + \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1 + \phi_\pi} \right)^s (R_{1+s} - \bar{R}) \right). \quad (\text{E6})$$

This is Equation (28) in Section 10.

□