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ABSTRACT

For at least three decades, the New-Keynesian representative agent model has been the preferred vehicle for analyzing the interaction of fiscal and monetary policy with the behavior of private agents in markets. We argue that the Overlapping Generations (OLG) model is an attractive alternative and that a number of features of real world economies arise naturally in a long-lived version of this model when it is calibrated to the U.S. income profile. These features include 1) the transmission of monetary shocks to real quantities, 2) excess volatility of real asset prices and 3) the existence of decade long periods with negative real interest rates. We provide an example of a sixty-two generation OLG model where all of these features arise in a dynamically efficient competitive equilibrium with complete markets in which money has positive value. Our results hold even in the case in which both monetary and fiscal policies are active in the sense of Leeper (1991).

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ABSTRACT. For at least three decades, the New-Keynesian representative agent model has been the preferred vehicle for analyzing the interaction of fiscal and monetary policy with the behavior of private agents in markets. We argue that the Overlapping Generations (OLG) model is an attractive alternative and that a number of features of real world economies arise naturally in a long-lived version of this model when it is calibrated to the U.S. income profile. These features include 1) the transmission of monetary shocks to real quantities, 2) excess volatility of real asset prices and 3) the existence of decade long periods with negative real interest rates. We provide an example of a sixty-two generation OLG model where all of these features arise in a dynamically efficient competitive equilibrium with complete markets in which money has positive value. Our results hold even in the case in which both monetary and fiscal policies are active in the sense of Leeper (1991).

1. INTRODUCTION

This paper is about the use of the overlapping generations model as a vehicle to understand how fiscal and monetary policy interact with the choices of private agents to determine prices and interest rates. We ask if well established results that hold in the workhorse representative agent model (Leeper and Leith, 2016) can be transferred to an Overlapping Generations (OLG) model that is calibrated with an income profile that matches real world data. Our main result is that they do not.

Although we study a pure exchange economy, similar results to those we develop below hold in economies with physical capital but inelastic labor supply. And the work of Reichlin (1986) suggests that our results can be extended to models with variable labor supply and

flexible prices. When a model with flexible prices displays multiple indeterminate equilibria, it is a small step to appeal to negligible menu costs to select the equilibrium in which nominal shocks have real effects in the short run. In this sense, we see our work as a complement to, rather than a substitute for, the menu-cost approach to nominal-real interactions.

An extensive macroeconomics literature classifies monetary and fiscal policy regimes into those that are *active* and those that are *passive*. The active-passive distinction was coined by Leeper (1991) who suggested the following classification. If the central bank raises the interest rate more than one-for-one in response to inflation, monetary policy is *active*. If the central bank raises the interest rate less than one-for-one in response to inflation, monetary policy is *passive*. If the fiscal authority borrows to finance an arbitrary path of expenditure and taxes, fiscal policy is *active*. And if the fiscal authority adjusts its expenditures and the tax rate to ensure fiscal solvency for all possible paths of the real interest rate, fiscal policy is *passive*.

The active-passive distinction leads to a simple policy prescription. Central banks and national treasuries should coordinate on a policy mix in which one policy is active and the other is passive. In simple representative agent models, policy mixes in this class lead to models in which the price level and the real interest rate are each fully determined by fundamentals. In overlapping generations model that is not the case and instead non-fundamental – in the terminology of Cass and Shell (1983) – *sunspot shocks* – may influence the allocation of resources between agents of different generations.

We are not the first to recognize that the overlapping generations model has very different characteristics from that of the representative agent model. Beginning with Samuelson (1958), the OLG model has been extensively used to analyze macroeconomic issues. The two-generation OLG model was the vehicle adopted by Lucas Jr. (1972) to introduce rational expectations into macroeconomics and multi-generation calibrated versions of the model have been studied by Auerbach and Kotlikoff (1987) to study fiscal policy, Ríos-Rull (1996) to study business cycles and Eggertsson et al. (2019) as a vehicle for understanding low-interest rate environments. All of the OLG policy-oriented literature of which we are aware has focused on models with at most two steady-state equilibria.

Monetary OLG models always contain *at least* two steady state equilibria. In one of these equilibria, money has value, and in the other, it does not. Our contribution in this paper is to to provide a robust example of an overlapping generations model in which the age-income profile is calibrated to U.S. data and where there there exists an indeterminate steady-state equilibrium where money has value. This equilibrium is dynamically efficient and yet the real interest rate

and the initial price level are both indeterminate. Our result holds even when either fiscal or monetary policy is active and the other policy is passive. When both policies are active, the price level and the real interest rate are still indeterminate, but they can no longer be chosen independently of each other.

Our example relies on a hump-shaped endowment profile with peak income occurring in middle age. The existence of a hump-shaped profile gives rise to a tension between the wealth effect and the substitution effect. At high rates of the real interest rate, there is a motive to transfer income from middle-age to old-age. At low levels of the real interest rate this motive acts in the opposite direction. We show that for values of the intertemporal elasticity of substitution (IES) less than or equal to 1/2, the tension between the wealth effect and the substitution effect leads to the emergence of multiple steady-state equilibria where money has no value and a single monetary equilibrium in which neither the price level nor the real interest rate are pinned down by fundamentals.

To explore the empirical relevance of our example, we construct a $62-$generation overlapping generations model in which we calibrate the endowment profile to U.S. micro data. We show that, in this calibrated example, the same phenomenon emerges. When monetary policy is passive and fiscal policy is active, there exists a steady-state equilibrium that displays two degrees of indeterminacy. It is not just the initial price level that remains unexplained by economic fundamentals; our calibrated model also fails to uniquely determine the real interest rate. If we assume that monetary and fiscal policy are *both* active, the degree of indeterminacy is reduced from two to one. In this case a linear combination of the real interest rate and the price level is determinate, but the model is still unable to uniquely determine the time path of either of these variables as a function of economic fundamentals.

Our work has implications for the question of which fiscal and monetary policies are welfare improving and which are not. Although our model is non-stochastic, it is well known that the existence of indeterminate steady-state equilibria is associated with the emergence of stationary stochastic equilibria driven by non-fundamental shocks.[1] And although our paper is about the existence of multiple equilibria in the overlapping generations model, similar results hold in models with credit constraints or borrowing limits.[2]

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[1]Azariadis (1981); Cass and Shell (1983); Farmer and Woodford (1997); Farmer (1999); Farmer and Guo (1994).

[2]Woodford (1988); Schmitt-Grohé and Uribe (2020).

Our results lead to novel explanations of a number of observed macroeconomic phenomena. First, a prominent feature of empirical data is the transmission of nominal shocks to real variables Sims (1989). This observation is explained in the New Keynesian model by the existence of price-adjustment costs. In our model sticky prices emerge endogenously as a property of equilibrium.[3]

Second, there is an extensive literature, initiated by Shiller (1981) and Leroy and Porter (1981) that examines excess volatility of asset prices. In a stochastic version of our model, excessive volatility may arise as a consequence of the indeterminacy of the real interest rate which permits fluctuations in asset prices driven by purely non-fundamental shocks.

Finally, the real interest rate has been declining for the past fifty years and for much of this period it has been lower than the growth rate of GDP. In our 62-generation calibrated model, adjustment dynamics are cyclical and extremely slow. As the real interest rate adjusts, following a shock, there are decade-long periods in which the interest rate is either greater than or less than the growth rate of GDP. And the adjustment path back to the steady state is associated with big swings in the age-distribution of wealth as the calendar birth date of an age cohort has a significant impact on its life-time earnings.

2. The Relationship of our Work to Previous Literature

The generic existence of multiple indeterminate steady-state equilibria was established by Kehoe and Levine (1985) in the context of a two-period-lived model with multiple goods and multiple agents.[4] With the exception of this paper, most previous examples of indeterminate equilibria in overlapping generations models have been restricted to two-generation or three-generation models in which indeterminacy was purely monetary (Samuelson, 1958), the model required negative money (Gale, 1973; Farmer, 1986), or was associated with unrealistic calibrations generally considered to be empirically irrelevant Azariadis (1981); Farmer and Woodford (1997); Kehoe and Levine (1983). Our paper is the first to provide a long-lived example of an economy where the endowment profile is matched to U.S. micro data, and where there exists an indeterminate steady-state equilibrium when both monetary and fiscal policy are active.

A branch of the multiple equilibrium literature studies bubbles in the overlapping generations model. A non-exhaustive list of papers, following Tirole's seminal contribution (Tirole, 1985), would include Martin and Ventura (2011, 2012), Miao and Wang (2012), Miao et al.

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[3]There is an extensive previous literature that makes this same point. See, for example, Farmer (2020) and the reference cited therein.

[4]It follows from the results of Balasko and Shell (1981) that the two-period assumption is unrestrictive as long as there are multiple agents and multiple goods.

(2012) Miao (2016) and Azariadis et al. (2015). The model we develop here contains what Tirole would call 'bubbly equilibria' but, unlike these papers, we develop our argument in the context of a complete markets overlapping generations model without the credit constraints introduced by these authors.

Benhabib et al. (2001) exploit the global indeterminacy of equilibrium to point out that an active Taylor rule, may lead to unexpected results. In contrast, our results are not driven by global or non-linear dynamics since we explicitly restrict attention to the properties of a linearized system of equilibrium conditions around a monetary steady state. A number of authors have studied the wealth distribution in stochastic overlapping generations models with and without complete securities markets. One branch of this literature includes papers by Ríos-Rull and Quadrini (1997) and Castañeda et al. (2003). Our work is peripherally related to that literature but we study a different question.

Bassetto and Cui (2018) revisit the implications of fiscal policy for price level determination in models in which assets differ in characteristics because of risk, or because debt provides liquidity services. In contrast to their work, our results do not rely on dynamic inefficiency, risk premia or liquidity effects. We show that both the price level and the real interest rate are indeterminate in a model where the steady-state equilibrium is dynamically efficient and where there are no frictions or rigidities of any kind other than the natural assumption that people are born and die at different dates.

Eggertsson et al. (2019) study steady-state equilibria in a fifty-six generation overlapping generations model with sticky prices. They use their model to discuss the idea that a negative real interest rate may be inconsistent with full employment, a concept that they refer to as 'secular stagnation'. In one section of their paper, Eggertsson et al. study the transition path from one steady-state equilibrium to another. Their solution method assumes that this transition path is unique. This is an assumption that is called into question by the analysis in our paper.

Given the extensive existing literature that applies the OLG model to real world data, a natural question is: why are we the first to draw attention to the possibility of multiple indeterminate equilibria? There are two answers to that question. The first is that our results hold only in monetary versions of the OLG model and the canonical calibrated models (Auerbach and Kotlikoff, 1987; Ríos-Rull, 1996) do not contain fiat money. The second is that multiple equilibria do exist in these models, but not for their authors' preferred calibrations.

All *monetary* OLG models contain at least two steady-state equilibria and in the absence of deficit finance, money has value in only one of them. Researchers have typically ruled out one equilibrium by omitting government debt entirely, as in Ríos-Rull (1996), or by modeling debt as an obligation to repay real commodities, as in Auerbach and Kotlikoff (1987).

Even allowing for the absence of money however, the models developed by both Auerbach and Kotlikoff (1987) and Ríos-Rull (1996) both allow for a hump-shaped income profile and the results we develop in Section 5 imply that there exist parameter values for which a purely real model possess three steady state equilibria.[5] It follows that the uniqueness finding of Rios-Rull that

> "In all the cases studied, these functions turned out to be monotone- decreasing, implying a unique steady state." Ríos-Rull (1996, page 472).

is case specific and depends on the specific choice of parameters.

3. Fiscal and Monetary Policy

In this section we explain the relationship between private and government trade in the asset markets. We construct a model in which the government purchases g_t units of a consumption good which it finances with dollar-denominated pure discount bonds and lump-sum taxes, τ_t. Let B_t be the quantity of pure-discount bonds each of which promises to pay one dollar at date $t+1$ and let Q_t be the date t dollar price of a discount bond. Further, let p_t be the date t dollar price of a consumption good. Using these definitions, government debt accumulation is represented by the following equation,

$$Q_t B_t + p_t \tau_t = B_{t-1} + p_t g_t.$$

Define i_t to be the net nominal interest rate from period t to period $t+1$, and let Π_{t+1}, be the gross inflation rate. These variables are given by,

$$i_t \equiv \frac{1}{Q_t} - 1 \quad \text{and,} \quad \Pi_{t+1} \equiv \frac{p_{t+1}}{p_t}.$$

Further, let

$$b_t \equiv \frac{B_{t-1}}{p_t},$$

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[5]Although our results are developed in the context of an endowment economy, a simple extension of our analysis applies to the model with capital.

be the real value of government debt maturing in period t and define the *real primary deficit* as

$$d_t \equiv g_t - \tau_t,$$

where the negative of d_t is the *real primary surplus*. Let R_{t+1} represent the gross real return from t to $t+1$, which from the Fisher-parity condition equals

$$R_{t+1} \equiv \frac{1 + i_t}{\Pi_{t+1}}. \tag{1}$$

We can combine these definitions to rewrite the government budget equation in purely real terms

$$b_{t+1} = R_{t+1}(b_t + d_t), \quad t = 1, \ldots, \infty. \tag{2}$$

Although Equation (2) is expressed in terms of real variables, the debt instrument issued by the treasury is nominal. It follows that the real value of debt in period 1 is determined by the period 1 price level through the definition

$$b_1 \equiv \frac{B_0}{p_1}.$$

Equations (2), one for each future date, have been interpreted in two different ways in the literature on monetary and fiscal policy in representative agent models. According to advocates of the Fiscal Theory of the Price Level (FTPL) these are not budget equations in the usual sense; instead, when combined with a boundedness condition, they generate a debt valuation equation. To understand this argument, let $\mathcal{Q}_t^k$,

$$\mathcal{Q}_t^k \equiv \prod_{j=t+1}^{k} \frac{1}{R_j}, \qquad \mathcal{Q}_t^t = 1,$$

be the relative price at date t of a commodity for delivery at date k. Now, iterate Equation (2) forwards to write the current real value of debt outstanding as the present value of all future surpluses,

$$\frac{B_0}{p_1} = -\sum_{t=1}^{\infty} \mathcal{Q}_1^t d_t + \lim_{T \to \infty} \mathcal{Q}_1^T b_T. \tag{3}$$

The boundedness condition alluded to above, is the requirement that

$$\lim_{T \to \infty} \mathcal{Q}_1^T b_T \leq 0, \tag{4}$$

which turns Eq. (3) into a constraint,

$$\frac{B_0}{p_1} \leq -\sum_{t=1}^{\infty} \mathcal{Q}_1^t d_t. \tag{5}$$

Inequality (4), sometimes referred to as a no-Ponzi scheme condition, restricts the government from borrowing from the infinite future and it follows naturally in representative agent economies in which all agents are present at all points in time.[6]

If the government were to be treated in the same way as other agents, Inequality (5) would act as a constraint on feasible paths for the sequence of surpluses, $-\{d_t\}_{t=1}^{\infty}$, that would be required to hold for all paths of $\{\mathcal{Q}_1^t\}_{t=1}^{\infty}$ and all initial price levels, p_1. If the government behaves optimally, in the sense that all revenues are accounted for either to pay down debt or to fund expenditure, this inequality will hold with equality,

$$\frac{B_0}{p_1} = -\sum_{t=1}^{\infty} \mathcal{Q}_1^t d_t. \tag{6}$$

In New-Keynesian models in which the central bank sets an interest rate peg, the initial price level would be indeterminate if the government were constrained to balance its budget for all paths of $\{\mathcal{Q}_1^t\}_{t=1}^{\infty}$ and all initial price levels (McCallum, 2001). To resolve this apparent indeterminacy of the price level, advocates of the FTPL argue that the government should be treated differently from other agents in a general equilibrium model (Leeper, 1991; Woodford, 1994). When monetary policy is passive, Equation (6) should, they claim, be treated as a debt valuation equation that determines the value of p_1 as a function of the *specific* path of primary surpluses $-\{d_t\}_{t=1}^{\infty}$ chosen by the treasury. All initial price levels other than the specific value of p_1 that satisfies Equation (6) are infeasible since they lead to paths of government debt that eventually become unbounded. In contrast, in the OLG model neither the price level nor the real interest rate are pinned down uniquely and Equation (6) may hold for more than one value of the price level p_1. It follows that the logic behind the fiscal theory of the price level cannot be extended to the overlapping generations model.

4. A Three-Generation Example

Before introducing the $T-$period model, we introduce our main ideas in a 3-generation example. For this special case, the preferences of people born in period 2 and later are given

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[6]It does not, however, follow from the equilibrium conditions of an OLG economy and the failure of this condition is associated with the existence of *bubbles* (Tirole, 1985).

by the following utility function;

$$U^t = U(c_t^t, c_{t+1}^t, c_{t+2}^t). \tag{7}$$

We index generations by superscripts and calendar time by subscripts. Thus, c_τ^t is the consumption of generation t in period τ. We refer to the people born in period $T-1$ and later as *generic generations* and we distinguish them from a set of *non-generic generations*. The non-generic generations are people alive in periods 1 through $T-2$ who live for less than T periods. In the $3-$generation model there are 2 non-generic generations; the initial middle-aged and the initial old.

The generic generations maximize utility subject to three budget constraints, one for each period of life,

$$c_t^t + s_{t+1}^t \leq \tilde{\omega}_1, \quad c_{t+1}^t + s_{t+2}^t \leq R_{t+1}s_{t+1}^t + \tilde{\omega}_2, \quad c_{t+2}^t \leq R_{t+2}s_{t+2}^t + \tilde{\omega}_3, \tag{8}$$

where $\{\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3\}$ is the after-tax endowment profile of a generic generation and s_τ^t is the demand for claims to $\tau+1$ consumption goods by generation t in period τ. The subscript on the term $\tilde{\omega}_j$ indexes age and we assume throughout, that $\tilde{\omega}_j$ does not depend on calendar time.

The solution to this problem is fully characterized by a pair of asset demand functions

$$s_{t+1}^t(R_{t+1}, R_{t+2}), \quad s_{t+2}^t(R_{t+1}, R_{t+2}),$$

together with the requirement that the three budget constraints characterized in (8) hold with equality.

Let the aggregate demand for assets by all agents alive at date t be defined by the function

$$f(R_t, R_{t+1}, R_{t+2}) \equiv s_t^{t-1}(R_t, R_{t+1}) + s_t^t(R_{t+1}, R_{t+2}).$$

For this three-generation example, $f(\cdot)$ adds up the asset demand of the newborns, this is the term $s_t^t(\cdot)$ and the asset demands of the middle-aged, this is the term $s_t^{t-1}(\cdot)$. Equilibrium in the asset markets requires that

$$f(R_t, R_{t+1}, R_{t+2}) = R_{t+1}^{-1}b_{t+1}, \tag{9}$$

where $R_{t+1}^{-1}b_{t+1}$ is the public sector borrowing requirement in period t and the dynamics of public borrowing are given by the equation,

$$b_{t+1} = R_{t+1}(b_t + d_t). \tag{10}$$

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We refer to equations (9) and (10) as the *generic market clearing equations*. Beginning with period 2, non-stationary equilibria are characterized by bounded sequences of real interest rates and debt that satisfy these equations and are consistent with a set of initial conditions that arise from the behavior of the initial generations.

To complete the description of equilibrium in the three-generation model, we must describe the equilibrium conditions in period 1. In this period there is one generic generation, the newborns, and two non-generic generations, the middle-aged and the old. We refer to the initial middle-aged as generation 0 and to the initial old as generation -1. Generation -1 consume their wealth and they do not contribute to the demand for period 1 assets. Generation 0 maximize the function

$$U^0 = U(c_1^0, c_2^0), \tag{11}$$

subject to the two budget constraints,

$$c_1^0 + s_2^0 \leq \lambda_0 \frac{B_0}{p_1} + \tilde{\omega}_2, \quad c_2^0 \leq R_2 s_2^0 + \tilde{\omega}_3, \tag{12}$$

where B_0 is the dollar value of the initial government debt liability, p_1 is the dollar price of a commodity in period 1 and λ_0 is the share of the initial government debt liability held by generation 0. The remaining share, $1 - \lambda_0$, is held by the initial old. The solution to the problem of the initial middle-aged is characterized by the asset demand function

$$s_2^0 \left(R_2, \lambda_0 \frac{B_0}{p_1} \right), \tag{13}$$

and the requirement that the two budget constraints, (12), hold with equality.

The asset market clearing condition in period 1 is given by the expression

$$s_2^0 \left(R_2, \lambda_0 \frac{B_0}{p_1} \right) + s_2^1(R_2, R_3) = R_2^{-1} b_2, \tag{14}$$

and the government debt equation is

$$b_2 = R_2 \frac{B_0}{p_1}. \tag{15}$$

We refer to equations (14) and (15) as the *non-generic market clearing equations*. For given values of λ_0 and B_0, equations (14) and (15) determine R_3 and b_2 as functions of the initial conditions, R_2 and p_1. In period 2 and later, equations (9) and (10) determine R_{t+2} and b_{t+1} as functions of b_t, R_{t+1} and R_t.

A steady-state equilibrium is a non-negative real number R and real number b such that

$$f(R, R, R) = R^{-1}b, \tag{16}$$

$$b = R(b + d). \tag{17}$$

When $d = 0$, inspection of Eqn. (17) shows that there are at least two steady-state equilibria; one in which $b = 0$ and one in which $R = 1$. Following Gale (1973), an equilibrium in which $b = 0$ is called an *autarkic* steady-state and an equilibrium in which $R = 1$ is called a *golden rule* steady-state.[7]

Kehoe and Levine (1983) provide a calibrated three-generation example in which there is one golden rule and there are three autarkic steady-states. In their example, the golden rule is associated with a real value of government debt that is strictly positive and that displays two degrees of indeterminacy. The fact that the real value of nominal government liabilities is strictly positive at the golden-rule steady-state equilibrium implies that the sequence of money prices, defined by the equations

$$p_{t+1} = p_t \left(\frac{1 + \bar{i}}{R} \right), \quad p_{t=1} = p_1,$$

and the sequence of nominal debt obligations, defined by the equation,

$$B_t = bp_t,$$

remain strictly positive.

What does it mean for the golden rule steady state to be locally indeterminate? Consider all pairs of initial values

$$b_1 \equiv \frac{B_0}{p_1}, \quad R_2,$$

that are close to the steady state values of b and R at the golden rule. If there is a unique pair, $\{b_1, R_2\}$, such that the trajectory that starts from this pair converges to the steady state, the golden rule steady state is said to be *locally determinate*. If there is a one dimensional manifold of values – defined by a function $b_1 = \phi(R_2)$ – such that all solutions to equations (9) and (10) that begin on this manifold converge to the steady state; the golden rule steady state is said to display one degree of indeterminacy. If there is a two dimensional manifold – containing

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[7]When $d \neq 0$, a continuity argument establishes that there is an open set $d \in (d_L, d_U)$, which contains $d = 0$, for which the number of steady-state equilibria and the determinacy properties of each steady-state equilibrium is the same as the case where $d = 0$. It follows that, as long as the primary budget deficit is not too large, our analysis of the properties of equilibria for the case of $d = 0$ carries over to the case where the treasury runs a primary deficit or a primary surplus.

the golden rule steady state – such that all solutions to equations (9) and (10) that begin on this manifold converge to the steady state; the steady state is said to display two degrees of indeterminacy.

It is a remarkable feature of the Kehoe-Levine calibration that the golden rule steady state displays two degrees of indeterminacy and that the real value of government debt and the price level are both positive at this steady state. For the Kehoe-Levine calibration the length of life is 3, the endowment pattern is $\{3, 15, 2\}$, the preference weights on each period of life are equal to $\{2, 2, 1\}$ and the intertemporal elasticity of substitution is equal to $1/6$. We show in this paper, that the existence of a steady-state equilibrium with these properties is robust to increasing the length of life, the introduction of constant discounting, different values of the intertemporal elasticity of substitution and the calibration of the income profile to match U.S. micro data.

The degree of indeterminacy of equilibrium in the Kehoe-Levine example depends on the actions of the monetary and fiscal authorities. The assumption of a constant interest rate implies that monetary policy is passive and the fact that d_t is not responsive to variations in the value of outstanding debt implies that fiscal policy is active.[8] It follows that the finding of two-degrees of indeterminacy at the golden rule steady-state holds for a passive-active policy mix for which equilibrium, in a representative agent economy, would be determinate. In Section 9 we relax the assumption of a passive monetary policy and we show that the golden rule equilibrium still displays one degree of indeterminacy, even for the case in which monetary and fiscal policy are both active.

5. Understanding the Mechanism

The three-generation example we provided in Section 4 relies on a hump-shaped income profile and a relatively low value of the elasticity of substitution. In this section, we provide analytical results for the three-generation case that illustrate the tension between the intertemporal elasticity of substitution and the peak of the income distribution. We begin with a base-line three-generation model for which the representative agent maximizes the following constant elasticity of substitution utility function

$$U^t = \frac{(c_t^t)^\alpha + \beta(c_{t+1}^t)^\alpha + \beta^2(c_{t+2}^t)^\alpha}{\alpha}, \qquad \alpha \leq 1, \quad \alpha \neq 0, \qquad (18)$$

$$U^t = \log(c_t^t) + \beta \log(c_{t+1}^t) + \beta^2 \log(c_{t+2}^t), \qquad \alpha = 0, \qquad (19)$$

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[8]Arguably, this is the relevant policy mix in the current environment in which the interest rate is at or near zero and is unresponsive to realized inflation and where national treasuries are pursuing unrestrained spending programs that, at least in the near future, do not appear responsive to growing debt to GDP ratios.

subject to the constraint

$$\frac{\left(c_t^t - 1\right)}{1} + \frac{\left(c_{t+1}^t - \lambda^\eta\right)}{R_{t+1}} + \frac{\left(c_{t+2}^t - \lambda^{2\eta}\right)}{R_{t+1}R_{t+2}} \leq 0, \tag{20}$$

where $0 < \beta < 1$ and $0 < \lambda \leq 1$. The intertemporal elasticity of substitution, η, is equal to

$$\eta = \frac{1}{1 - \alpha},$$

and we focus here on the case where $\alpha \leq 0$ which implies an intertemporal elasticity of substitution between 0 and 1. We chose an income profile of $\omega_\lambda \equiv \{1, \lambda^\eta, \lambda^{2\eta}\}$ in which the endowment process is related to the preference parameter η in order to simplify the numerical calculations of our example and nothing of substance hinges on this assumption. For this choice of the endowment process, the autarkic equilibrium interest factor is easy to compute and occurs at R $= \lambda/\beta$.

To explore the role of a humped shaped endowment, define a new income profile $\tilde{\omega} \equiv \{w_1, w_2, w_3\}$ that obeys the following two constraints,

$$w_1 + w_2 + w_3 = 1 + \lambda^\eta + \lambda^{2\eta}, \tag{21}$$

$$w_1 + \frac{w_2\beta}{\lambda} + \frac{w_3\beta^2}{\lambda^2} = 1 + \frac{\lambda^\eta\beta}{\lambda} + \frac{\lambda^{2\eta}\beta^2}{\lambda^2}. \tag{22}$$

Equation (21) ensures that the aggregate endowment is the same with the new hump-shaped endowment profile as with the initial flat endowment profile and Equation (22) ensures that the endowment is redistributed across the age-profile in a way that preserves the present value of the endowment of a newborn agent at the steady state interest rate $R = \lambda/\beta$. For any given value of,

$$w_2 \leq 1 + \lambda^\eta + \lambda^{2\eta},$$

we can solve equations (21) and (22) to find two functions $w_1 = w_1(w_2)$ and $w_3 = w_3(w_2)$ which redistribute the endowment in a way that preserves $R = \lambda/\beta$ as a steady state equilibrium.

An autarkic steady state equilibrium is characterized by a positive number $\bar{R}$ and an aggregate savings function $f_\omega(R)$ such that

$$f_\omega(\bar{R}) \equiv s_{t+1}^t(\bar{R}; \omega) + s_{t+2}^t(\bar{R}; \omega) = 0.$$

Here the subscript ω on the function $f_\omega(R)$ signifies the dependence of the aggregate savings function on the endowment profile and $s_{t+1}^t(\bar{R}; \omega)$ and $s_{t+2}^t(\bar{R}; \omega)$ are the steady-state savings functions of the young and middle-aged that satisfy the utility maximization problem of a

representative household with endowment profile $\omega = \{w_1, w_2, w_3\}$. For the endowment profile $\omega_\lambda = \{1, \lambda^\eta, \lambda^{2\eta}\}$, the autarkic equilibrium is given by $\bar{R} = \lambda/\beta$.

By picking a new endowment profile, $\tilde{\omega}$, that satisfies equations (21) and (22) we guarantee that

$$f_{\hat{\omega}}\left(\frac{\lambda}{\beta}\right) = f_{\omega_\lambda}\left(\frac{\lambda}{\beta}\right) = 0.$$

This equality means that the interest rate $R = \lambda/\beta$ is a steady state equilibrium interest rate in the perturbed economy. But although the new savings profile is chosen so that f_{ω_λ} and $f_{\hat{\omega}}$ coincide at the steady state, they will not in general coincide anywhere else.

To understand the mechanism that leads to multiple equilibria, consider the special case of the model in which $\lambda = \beta = 1$. For this case the income and consumption profiles are flat and the golden rule and autarkic steady states coincide. In this steady state equilibrium, $R = 1$ and every generation consumes its endowment in each period of life.

To explore the interplay between the endowment profile and the intertemporal elasticity of substitution, in Appendix A we derive an explicit expression for the slope of the aggregate savings function, evaluated at $\bar{R} = \lambda/\beta$, and show that when $\beta = \lambda = 1$ this expression is given by

$$\left.\frac{\partial f_{\hat{\omega}}}{\partial R}\right|_{R=1} = \frac{1 - w_2 + 4\eta}{2}. \tag{23}$$

When $\eta = 1$, the utility function is logarithmic. And since $w_2 < 3$, it follows that for logarithmic preferences the aggregate savings function slopes up at the autarkic steady-state. This is not true for other values of η. By rearranging Eq. (23) one can show that for values of

$$\eta < \frac{w_2 - 1}{4}, \tag{24}$$

the savings function changes sign from positive to negative at the autarkic steady steady equilibrium. One can also show that $f_{\tilde{\omega}}$ is continuous, negative for low values of R and positive for high values. It follows from these three facts that if $f_{\tilde{\omega}}$ crosses zero with a negative slope at the autarkic equilibrium there must be at least two other state state equilibria. The upper bound on w_2 is 3 and by inspecting Inequality (24) we see that the intertemporal elasticity of substitution must be strictly less than 1/2 for multiple steady states to arise.

Figure 1 plots the savings function for three different values of the intertemporal elasticity of substitution. In each case, $\tilde{\omega} = \{0.33, 2.33, 0.34\}$, and $\beta = \lambda = 1$, a parameterization for which the golden rule equilibrium and the autarkic equilibrium coincide. By rearranging the endowment across the age profile we highlight the wealth effect which provides an incentive to
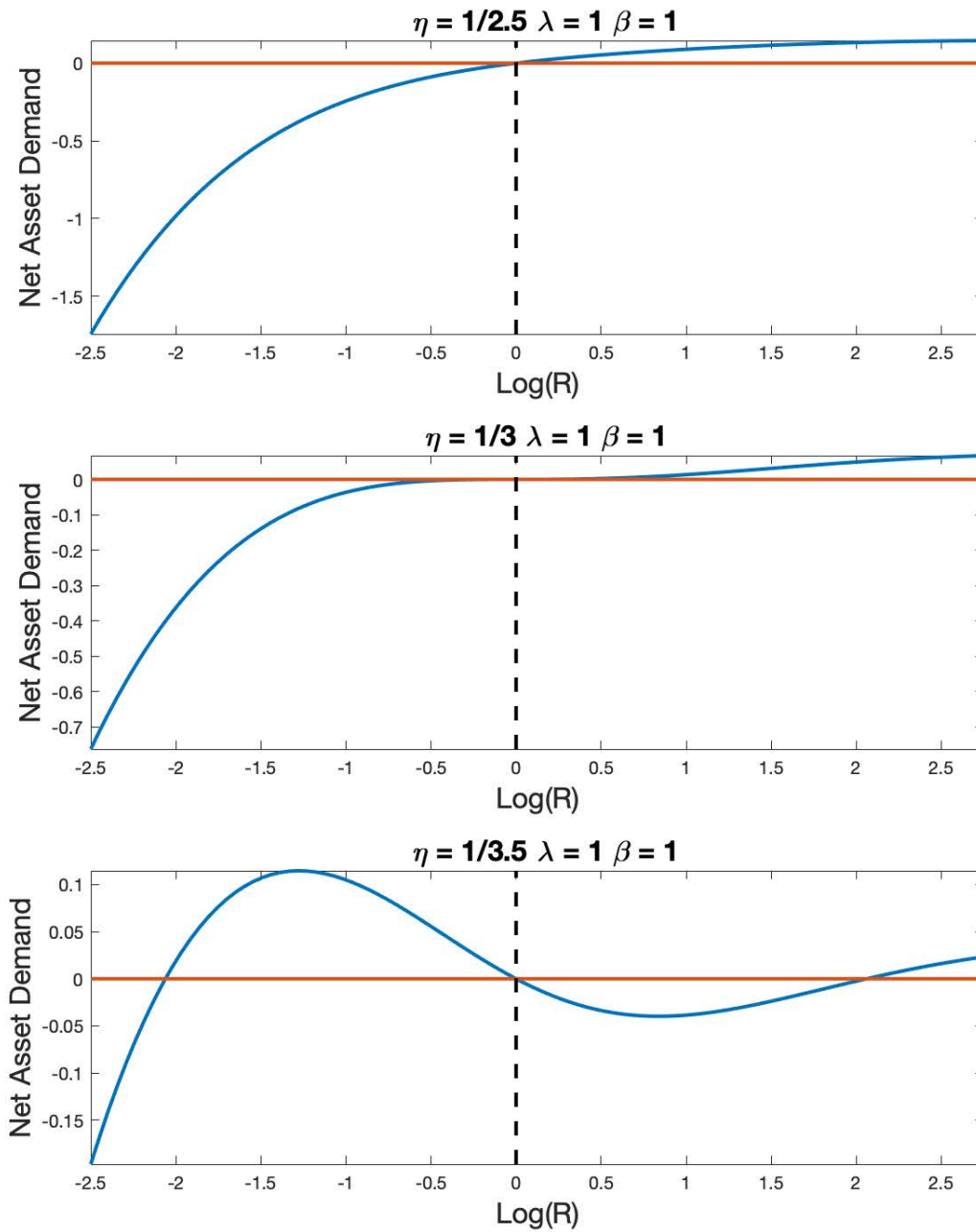
FIGURE 1. The Aggregate Savings Function for Three Different Values of η

move the endowment between periods to smooth consumption. By reducing the intertemporal elasticity of substitution, we dampen the substitution effect which provides an incentive to increase savings at the $R = \beta/\lambda$ steady state.

The three sub-panels in Figure 1 are drawn for decreasing values of the intertemporal elasticity of substitution. For the top panel, $\eta = 2/5$. For this value of η the savings function

is increasing at the autarkic stationary equilibrium value, $R = \lambda/\beta$. For the middle panel, $\eta = 1/3$. At this value of η the slope of the savings function is zero at the autarkic steady-state equilibrium. For the lower panel, $\eta = 2/7$ and, for this value the function $f_{\hat{\omega}}(R)$ is decreasing at the autarkic steady state and two new steady states emerge, one greater than 1 and one smaller.
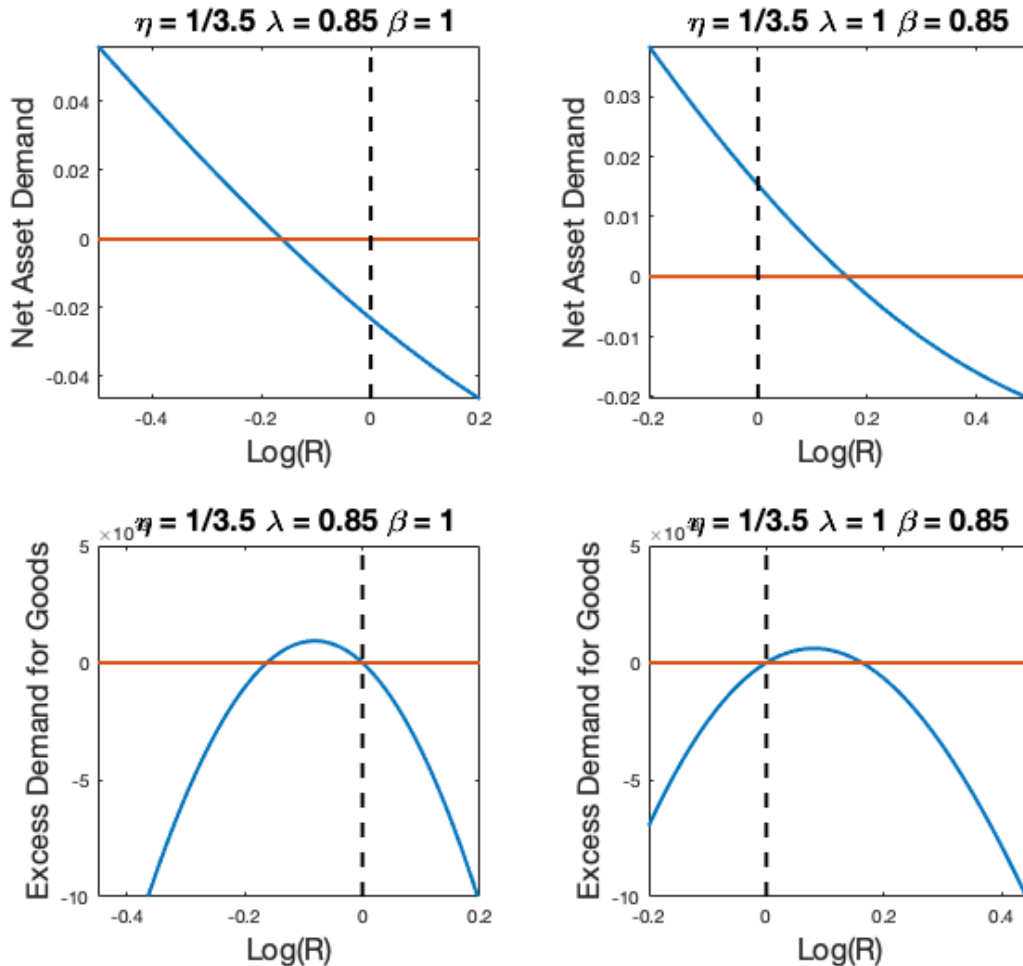
FIGURE 2. Aggregate Savings and the Excess Demand for Goods for Two Different Values of λ and β

Figure 2 explores the role of different values of λ and β. All four panels are drawn for the same endowment profile as Figure 1 and for a value of $\eta = 2/7$. The left two panels of Figure 2 are drawn for $\beta = 1$ and $\lambda = 0.85$. The right two panels depict the case when $\beta = 0.85$ and $\lambda = 1$. In each case, the upper panel depicts the aggregate savings function and the lower panel represents the aggregate excess demand for goods.

Although there are three intersections in each upper panel and four intersections in each lower panel, we have truncated the x-axis to depict only one intersection of the savings function

and two intersections of the excess demand for goods in order to draw attention to the magnitude of asset holdings at the golden rule steady state. The golden rule in our example corresponds to a value of $\log(R) = 0$ and from the two upper panels we see that as we vary the parameterization from $\lambda = 0.85$ and $\beta = 1$ to $\beta = 0.85$ and $\lambda = 1$, the sign of asset holdings in the golden-rule steady state changes from negative to positive.

Because a hump-shaped income profile is the norm in the real world, we infer from this analysis that the Kehoe-Levine example is not just a theoretical curiosity. It is a case that merits further analysis as a potential vehicle for understanding asset market equilibria in real world economies. To explore that possibility further, in Section 6 we extend the three-generation model to the case of $T-$period lives and in Section 7 we calibrate a $62-$period model using a realistic income profile that we fit to U.S. data.

6. The General $T-$Period Case

This section describes the generalization of the $3-$generation example to a model with $T-$generations. The main complication that arises for this case is associated with the existence of additional non-generic generations in the first $T - 2$ periods of the model and the technical arguments needed to deal with this complication are dealt with in Appendix C.2. The reader who is interested in the implications of our argument for the 62-generation calibrated example can skip ahead to Section 7.

In the $T-$generation case, generation t has a utility function defined over consumption in periods t through $t + T - 1$ and the members of generation t solve the problem

$$\max_{\{c_t^t, \ldots, c_{t+T-1}^t\}} U^t(c_t^t, c_{t+1}^t, \ldots, c_{t+T-1}^t),$$

such that

$$c_t^t + s_{t+1}^t \leq \tilde{\omega}_1, \quad c_{t+1}^t + s_{t+2}^t \leq R_{t+1}s_{t+1}^t + \tilde{\omega}_2, \quad \ldots \quad c_{t+T-1}^t \leq R_{t+T-1}s_{t+T-1}^t + \tilde{\omega}_T. \quad (25)$$

The solution to this problem is characterized by a set of $T - 1$ savings functions, one for each of the first $T - 1$ periods of life

$$s_k^t(R_{t+1}, \ldots, R_{t+T-2}, R_{t+T-1}), \quad k = t, \ldots, t + T - 2, \quad (26)$$

together with the requirement that the T budget constraints (25) hold with equality. In Appendix B, Section B.1. we characterize the solution to this problem for the case of CES preferences

and we find an explicit formula for the aggregate asset demand function,

$$f(R_{t-T+3}, R_{t-T+4}, \ldots, R_{t+T-2}, R_{t+T-1}) = R_{t+1}^{-1} b_{t+1}, \tag{27}$$

where $f(\cdot)$ is the sum of the savings functions, defined in Equation (26). As in the three-period model, government borrowing follows the equation,

$$b_{t+1} = R_{t+1}(b_t + d_t). \tag{28}$$

In Appendix C.1 we show that dynamic equilibria can be described by a difference equation $F(X_t, X_{t-1})$ in a vector

$$X_t \equiv [R_{t+T-1}, R_{t+T-2}, \ldots, R_{t-T+4}, b_t]^\top, \tag{29}$$

and we find a linear approximation to that difference equation around a steady state of the form

$$\tilde{X}_t = J\tilde{X}_{t-1} \tag{30}$$

where $\tilde{X}$ is a vector of deviations from the steady state and J is constructed from the partial derivatives of $F(\cdot)$ evaluated at the steady state.

The analysis in Appendix C.1 establishes that the order of the difference equation that characterizes equilibrium sequences of real interest rates, Eq. (27), is equal to $2T - 3$. And the generalization of the non-generic equations in Appendix C.2 establishes that there are $T-1$ non-generic equations which characterize equilibria in periods 1 through $T - 2$. Using these results, in Appendix, D we prove the following proposition which is based on the work of Blanchard and Kahn (1980).

Proposition 1 (Blanchard-Kahn). *Let K denote the number of eigenvalues of J with modulus greater than* 1.

- *If $K > T - 1$ there are no bounded sequences that satisfy the equilibrium conditions in the neighbourhood of $\bar{X}$. In this case equilibrium does not exist.*
- *If $K = T - 1$ there is a unique bounded sequence that satisfies the equilibrium equations. Further, this sequence converges asymptotically to the steady state $(\bar{R}, \bar{b})$. In this case the steady state equilibrium $(\bar{R}, \bar{b})$ is determinate.*
- *If $K \in \{0, \ldots, T - 2\}$ there is a $T - 1 - K$ dimensional subspace of initial conditions that satisfy the equilibrium equations. All of these initial conditions are associated with*

sequences that converge asymptotically to the steady state $(\bar{R}, \bar{b})$. In this case the steady state equilibrium $(\bar{R}, \bar{b})$ is indeterminate with degree of indeterminacy equal to $T - 1 - K$.

It follows from this proposition that we can compute the degrees of determinacy around a given steady-state equilibrium by comparing the roots of the matrix J, with $T - 1$, where T is the number of generations. In the simulations presented in Section 7, we use this proposition to compute the eigenvalues of J in the neighbourhood of each of the four steady states and we simulate non-stationary paths by iterating a linear approximation to the function $F(\cdot)$ around the golden-rule steady state.[9]

In our model fiscal policy is active but monetary policy is passive. According to the FTPL this policy mix should lead to a unique initial price level. In Section 7 we provide an example of an economy with a steady-state equilibrium where money has value and where the FTPL fails to hold. In this example, it is not only the initial price level that is indeterminate; it is also the initial real interest rate.

7. A Sixty-Two Generation Example

In this section we construct a sixty-two generation model where each generation begins its economic life at age 18 and in which a period corresponds to one year. To see if this model might provide a plausible explanation of a real-world economy we assume that the members of generation t maximize the utility function,

$$u\left(c_t^t, \ldots, c_{t+61}^t\right) = \sum_{i=1}^{62} \beta^{i-1} \left(\frac{[c_{t+i-1}^t]^\alpha - 1}{\alpha} \right),$$

and we calibrated the income profile of a representative generation to U.S. data. We provide explicit formulas for the excess demand functions for this functional form in Appendix B.

We graph our calibrated income profile in Figure 3. Our representative generation enters the labour force at age 18, retires at age 66, and lives to age 79. We chose the lifespan to correspond to current U.S. life expectancy at birth and we chose the retirement age to correspond to the age at which a U.S. adult becomes eligible for social security benefits. For the working-age portion of this profile we use data from Guvenen et al. (2021) which is available for ages 25 to 60. The working-age income profiles for ages 18 to 24 and for ages 61 to 66, were extrapolated to earlier and later years using log-linear interpolation. For the retirement portion we used data from the U.S. Social Security Administration.

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[9]The code used to generate all of our results is available online and is documented in an accompanying online document "Numerical Recipes". Our code also replicates the findings reported in Kehoe and Levine (1983).
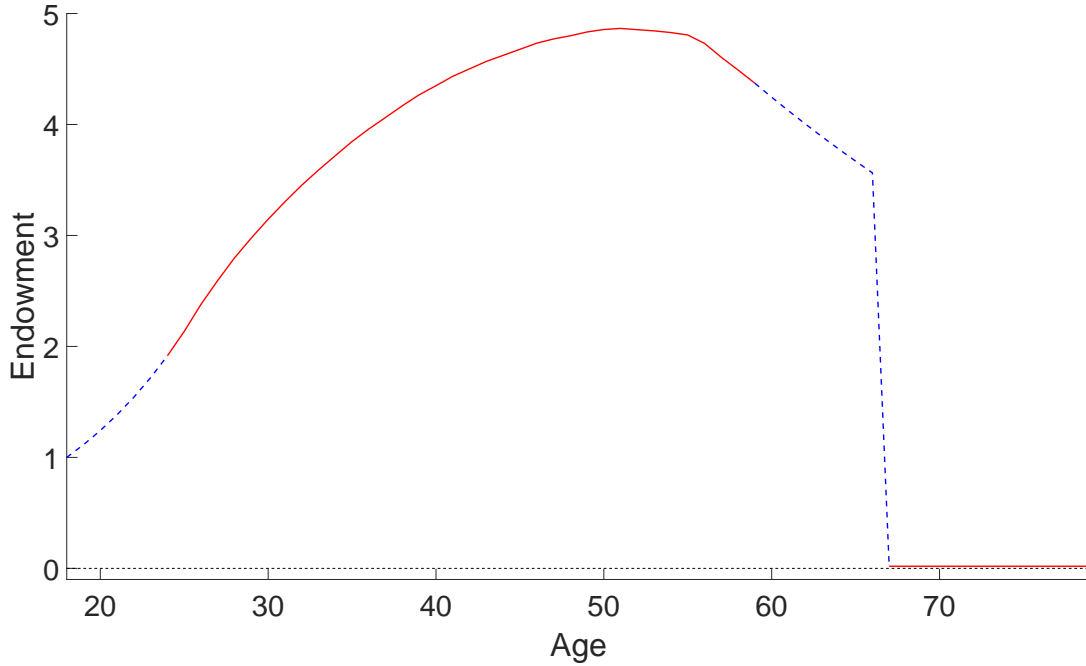
FIGURE 3. Normalized Endowment Profile. U.S. Data in Solid Red: Interpolated Data in Dashed Blue.

U.S. retirement income comes from three sources; private pensions, government security benefits, and Supplemental Security Income. We treat private pensions and government social security benefits as perfect substitutes for private savings since the amount received in retirement is a function of the amount contributed while working. To calibrate the available retirement income that is independent of contributions, we used Supplementary Security Income which, for the U.S., we estimate at 0.137% of GDP.[10]

For the remaining parameters of our model we chose a primary budget deficit of $d_t = 0$, an annual discount rate of 0.953 and an elasticity of substitution of 0.17. The qualitative features of the equilibria are robust to the existence of a positive primary deficit with an upper bound that depends on the discount rate. For the calibrated income profile depicted in Figure 3 and for this choice of parameters, our model exhibits four steady-state equilibria. In Section 8 we explore the robustness of the properties of our model to alternative choices for the discount parameter and for the elasticity of substitution parameter.

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[10]From Table 2 of the March 2018 Social Security Administration Monthly Statistical Snapshot we learn that the average monthly Supplemental Security Income for recipients aged 65 or older equalled $447 (with 2,240,000 claimants), which implies that total monthly nominal expenditure on Supplemental Security Income equalled $1,003 million. This compares to seasonally adjusted wage and salary disbursements (A576RC1 from FRED) in February 2018 of $8,618,700 million per annum, or $718,225 million per month. Back of the envelope calculations suggest that Supplemental Security Income in retirement equalled 0.137% of total labour income.
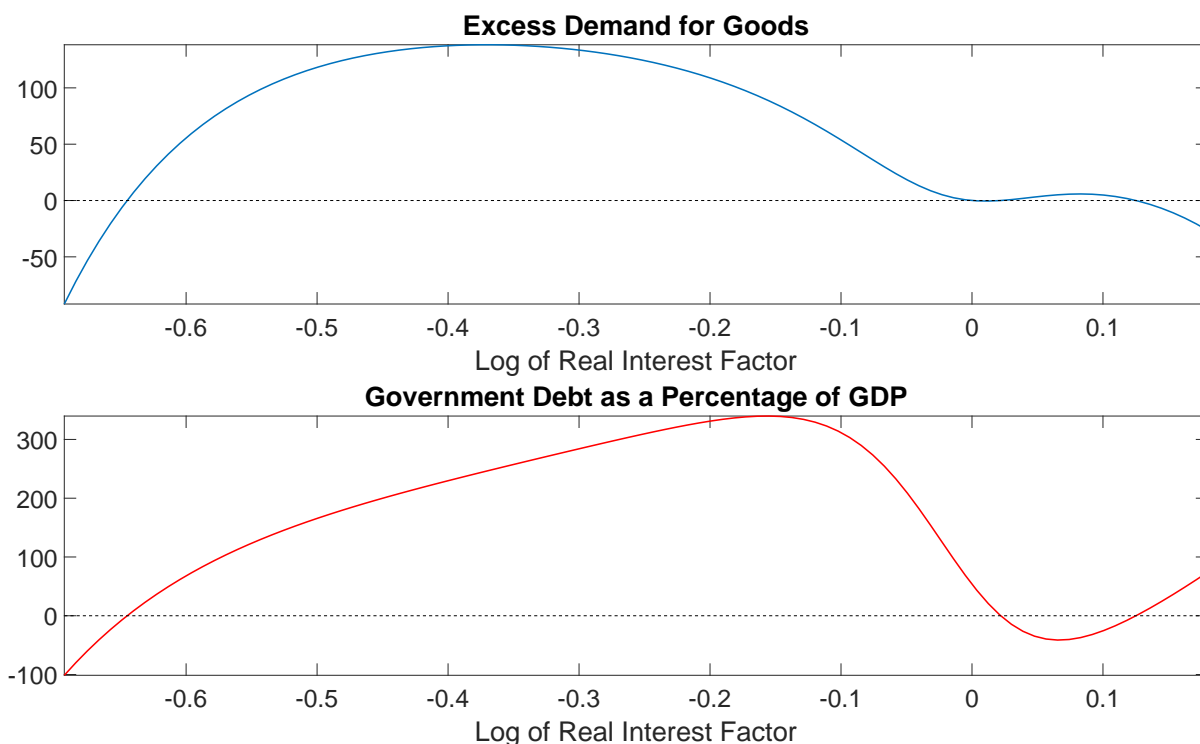
FIGURE 4. Steady States in the Sixty-Two Generation Model

In Figure 4 we graph the steady-state equilibria of our model. The upper panel of this figure plots the logarithm of the gross real interest rate on the horizontal axis and the steady-state excess demand for goods on the vertical axis. The lower panel plots government debt as a percentage of GDP at the steady state. We see from the upper panel that the excess demand function crosses the horizontal axis four times. And we see from the lower panel that three of these crossings are associated with steady-state equilibria in which steady-state government debt is equal to zero.

The three steady-state equilibria in which debt equals zero are autarkic. In these equilibria there is no trade with future unborn generations. The fourth steady-state equilibrium is the golden-rule. This steady-state equilibrium always exists in OLG models and in models with population growth it has the property that the real interest rate equals the rate of population growth. But although the golden-rule steady-state equilibrium always exists, *it is not true* that the golden-rule value of $\bar{b}$ is always non-negative.

The golden-rule steady state occurs when the logarithm of the real interest factor equals zero. By inspecting the lower panel of Figure 4, it is apparent that government debt is positive at the golden rule steady-state and, since debt is denominated in dollars, the price level is also positive in the golden-rule steady-state equilibrium. This is important because it is the

empirically relevant case in most western democracies. For example, in the United States, government debt in the first quarter of 2019 exceeded \$22 trillion.

The values and properties of all four steady-state equilibria are reported in Table 1. We refer to the autarkic steady-state equilibria as Steady-State A, Steady-State C and Steady-State D and to the golden-rule steady-state equilibrium as Steady-State B. We see from this table that Steady-States B, C and D are associated with a non-negative interest rate and are therefore dynamically efficient. Steady-State A is associated with a negative interest rate of -47.5% and is therefore dynamically inefficient.[11]

Equilibrium Real Interest Rates					
Type	Value of $\bar{R}$	Value of $\bar{b}$	# Unstable Roots	# Free Initial Conditions	Degree of Indeterminacy
Steady-State A	0.525	0	60	61	1
Steady-State B	1	53.7% of GDP	59	61	2
Steady-State C	1.022	0	60	61	1
Steady-State D	1.13	0	61	61	0

TABLE 1. Steady States of the Sixty-Two Generation Model

The sixty-two generation model with a calibrated income profile is similar in many respects to Kehoe-Levine's (1983) three generation model. In both examples, the golden-rule steady-state equilibrium displays second degree indeterminacy. And in both examples, the steady-state price level is positive and the initial price level is indeterminate even when fiscal policy is active. Importantly, because the monetary steady-state is second-degree indeterminate, indeterminacy of the price level can hold even when both monetary and fiscal policy are active.

In Figure 5 we show the result of an experiment in which we perturb the initial value of b_1 by 3% and we perturb the real value of the initial wealth of all of the non-generic generations by the same amount. We refer to this perturbation as a 3% shock to the initial price level. We restrict R_2 to equal its steady state value but all other elements of the vector of initial conditions are allowed to respond to the shock to keep the path of interest rates and debt on a convergent path back to the steady state. Figure 5 demonstrates that the return to the steady state from an arbitrary initial condition is extremely slow.[12]

We also see from Figure 5 that our model can endogenously generate prolonged periods of negative real interest rates. The upper panel of this figure plots the path by which the real interest rate returns to its steady-state value and the lower panel plots the return path of the

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[11]See Cass (1972) for a definition and characterization of the conditions for dynamic efficiency.

[12]Note that our choice of 3% is entirely arbitrary and that this is one of many admissible equilibrium paths. In particular, since we endow the non-generic cohorts with steady state asset shares, therefore the steady state equilibrium with $\forall t \geq 1 : b_t \equiv \bar{b}$ and $R_{t+1} \equiv \bar{R}$ would have also been feasible.
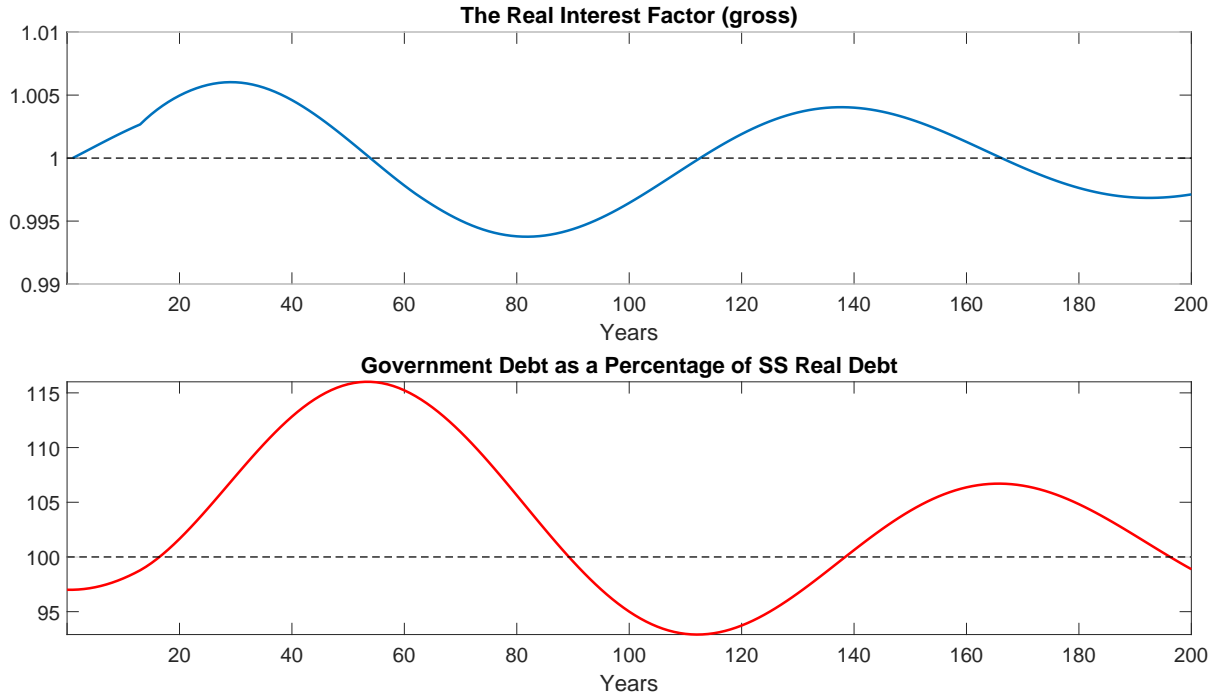
FIGURE 5. The Impact of the Initial Price Level Exceeding Its Steady State Value by 3%

real value of government debt expressed as a percentage of GDP. The figure demonstrates that small deviations of initial conditions from the steady state can have long-lasting effects, and that during the convergence process the real interest rate may be negative for periods well in excess of ten years.

One may question whether the high degree of real interest rate persistence implied by our model is excessive. Have such long swings in real interest rates actually ever been observed? To address this question, Figure 6, reproduced from Yi and Zhang (2017), compares long run real interest rates in the G7 and documents that low-frequency real rate cycles, similar to those generated by our model, have characterized the evolution of real interest rates in all of these economies.[13]

8. Robustness to Different Calibrations

To explore the robustness of our findings to alternative calibrations, in Table 2 we record the properties of our model for different values of the annual discount rate and the intertemporal elasticity of substitution. The example we featured in Section 7 had two degrees of indeterminacy and positive valued debt at the monetary steady state. Table 2 demonstrates that this property is not particularly special.

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[13]See Yi and Zhang (2017) for a discussion of why long-run moving averages are likely to characterize trends in fundamental forces underlying real interest rates.
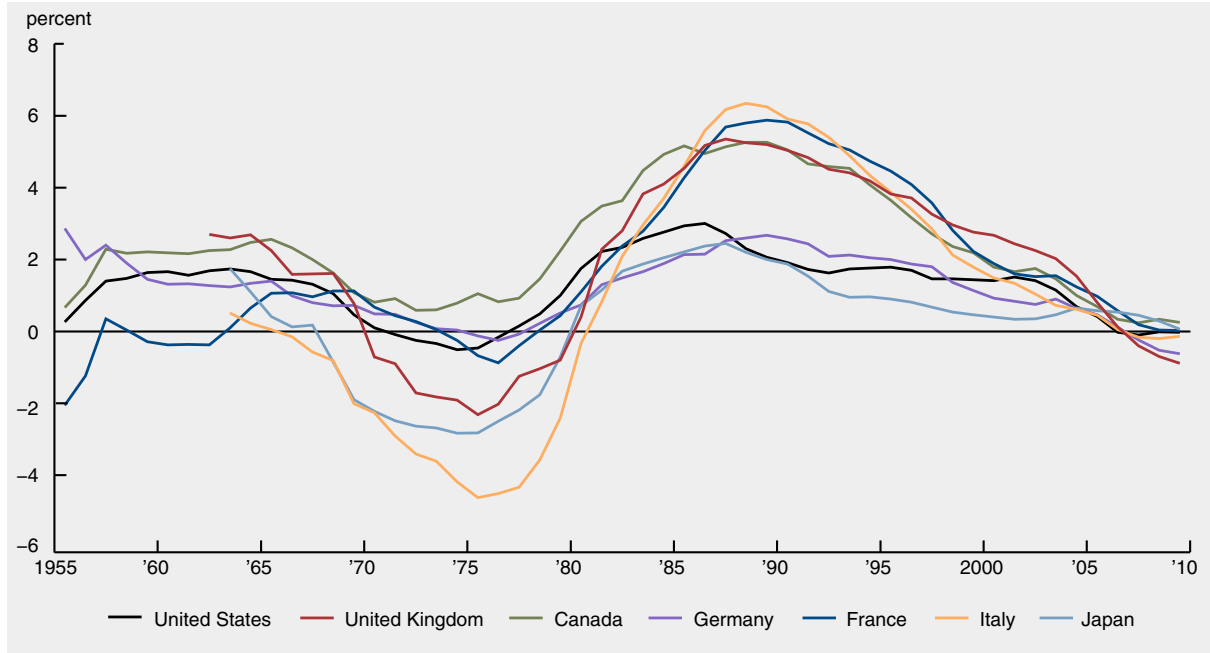
FIGURE 6. G7 Long-Run Real Interest Rates. Long-Run Real Interest Rates are 11-Year Centered Moving Averages of Annual Real Interest Rates. Source: Figure 1 in Yi and Zhang (2017)

		Annual Discount Factor							
		0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98
IES									
IES = 1/5	# Steady States	4	4	4	4	4	2	2	2
	Degree of Indeterminacy	1	1	1	1	1	2	0	0
	Value of Debt	-2.8	-2.2	-1.5	-0.8	-0.2	0.5	1.1	1.8
IES = 1/6	# Steady States	4	4	4	4	4	4	2	2
	Degree of Indeterminacy	1	1	1	1	2	2	0	0
	Value of Debt	-1.9	-1.3	-0.7	-0.2	0.4	0.9	1.5	2.0
IES = 1/7	# Steady States	3	3	3	3	3	3	1	1
	Degree of Indeterminacy	1	1	1	2	2	0	0	0
	Value of Debt	-1.2	-0.7	-0.2	0.3	0.8	1.2	1.7	2.2
IES = 1/8	# Steady States	3	3	3	3	3	3	3	1
	Degree of Indeterminacy	1	1	2	2	2	0	0	0
	Value of Debt	-0.6	-0.2	0.2	0.6	1.0	1.5	1.9	2.3
IES = 1/9	# Steady States	3	3	3	3	3	3	3	3
	Degree of Indeterminacy	1	2	2	2	2	0	0	0
	Value of Debt	-0.2	0.1	0.5	0.9	1.3	1.6	2.0	2.4

TABLE 2. Robustness of Indeterminacy to Alternative Calibrations, Focusing Only on Steady States with $\bar{R} \in [0.5, 1.5]$

The table provides 40 different parameterizations of our model with intertemporal elasticity of substitution parameters ranging from 1/9 to 1/5 and discount rates ranging from 0.91 to 0.98. In all of these parameterizations we maintained the calibrated income profile illustrated in Figure 3. For each calibration Table 2 displays the number of steady-state equilibria in the interval

$\bar{R} \in [0.5, 1.5]$, and the number of degrees of indeterminacy at the golden-rule steady-state equilibrium. There are fifteen parameterizations in which the golden-rule steady state displays one degree of indeterminacy and twelve in which it displays two degrees of indeterminacy. In all twelve of these parameterizations, debt has positive value in the steady state.

In Section 7, we showed that when $\beta = 0.953$ and $\eta = 1/6$, the golden-rule steady state displays two degrees of indeterminacy. An example with this property is interesting because it is not only the price level that is free to be determined by the beliefs of market participants; it is also the real rate of interest. We want to reiterate, however, that only one degree of indeterminacy is required for violations of the fiscal theory of the price level. And that occurs more frequently in our model than second degree indeterminacy.[14]

9. Fiscal and Monetary Policy

In this section we discuss what happens when we relax either the assumption that fiscal policy is active or the assumption that monetary policy is passive. We first show that passive fiscal policy makes indeterminacy more likely. We then demonstrate that ensuring bounded inflation under an active Taylor rule imposes an additional restriction on the set of equilibrium paths. This additional restriction reduces the degree of indeterminacy by one.

Consider first what happens when fiscal policy is passive. To model a passive fiscal policy we assume that the treasury raises taxes, τ_t, in proportion to the real value of outstanding debt to ensure that the primary deficit d_t satisfies the equation

$$d_t = -\delta b_t,$$

where $\delta \geq 0$ is a debt repayment parameter. Combining this assumption with the definition of the government debt accumulation equation leads to the following amended debt accumulation equation,

$$b_{t+1} = [R_{t+1} - \delta] b_t.$$

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[14]If we hold constant the intertemporal elasticity of substitution and increase the discount rate, the number of unstable eigenvalues decreases initially from 60 to 59 and then changes abruptly to 61. We see this behaviour in Table 2 by moving along a typical row and observing that we pass from one degree of indeterminacy to two degrees of indeterminacy and then jump abruptly to 0 degrees of indeterminacy. At this last transition, a pair of complex roots crosses the unit circle, a phenomenon associated with a Hopf Bifurcation and the creation of a limit cycle. See Guckenheimer and Holmes (1983) for a discussion of the Hopf Bifurcation. We have not explored the phenomenon in this paper, but it is likely that for discount rates close to 1, this model displays endogenous limit cycles that are second-degree indeterminate.

For values of $[\bar{R} - \delta] < 1$ the effect of making fiscal policy passive is to introduce an additional stability mechanism that increases the degree of indeterminacy at each of the four steady states whenever δ is large enough. Passive fiscal policy makes indeterminacy more likely.

We next assume that fiscal policy is active and the central bank follows a Taylor rule (Taylor, 1999),

$$1 + i_t = \left(\frac{\bar{R}}{\bar{\Pi}^{\phi_\pi}} \right) \Pi_t^{1+\phi_\pi}, \quad t = 1, ...\infty. \tag{31}$$

Because this equation begins at date 1, the nominal interest rate in period 1 depends on p_0 through the definition, $\Pi_1 = p_1/p_0$. We treat p_0 as an initial condition that has the same status as the initial value of nominal debt, B_0. In Eq. (31), $\bar{\Pi}$ is the inflation target, $\bar{R}$ is the steady state real interest rate and ϕ_π is the response coefficient of the policy rate to deviations of inflation from target. The Taylor Rule is passive if $-1 \leq \phi_\pi \leq 0$ and active if $\phi_\pi > 0$.

When the central bank follows a Taylor Rule, the real interest rate and the real value of government debt continue to be determined by the bond market clearing equation and the debt accumulation equation. It follows that the conditions we have characterized in previous sections continue to ensure that the real interest rate and the real value of government debt remain bounded.

When the central bank follows a passive Taylor Rule, (see Appendix E.1) the following equation characterizes the asymptotic behaviour of the future inflation rate,

$$\lim_{T \to \infty} \tilde{\Pi}_{T+1} = \lim_{T \to \infty} (1 + \phi_\pi)^T \tilde{\Pi}_1 - \lim_{T \to \infty} \sum_{s=1}^{T} (1 + \phi_\pi)^{T-s} \tilde{R}_{T+1}, \tag{32}$$

where $\kappa \equiv \bar{\Pi}/\bar{R}$ and the tilde denotes deviations from the steady state. The limit of the first term on the right side of Equation (E2) is zero because $1 + \phi_\pi < 1$ and the second term is finite as a consequence of the boundedness of R_t. It follows that inflation is bounded whenever R_t is bounded. This is a generalization of the argument we made for the boundedness of the inflation rate when the central bank follows an interest rate peg and it does not impose any additional restrictions on the equations of the model for an equilibrium to be determinate.

When the central bank follows an active Taylor Rule, (see Appendix E.2), the initial price level is determined by the forward-looking equation

$$p_1 = p_0 \left(\bar{\Pi} + \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1 + \phi_\pi} \right)^s \left(R_{1+s} - \bar{R} \right) \right). \tag{33}$$

Importantly, this restriction on the set of equilibrium paths is additional to the restriction

$$p_1 = \frac{B_0}{b_1},$$

that we used to generate the equilibrium sequence of interest rates. It follows that we are no longer free to pick R_2 and p_1 independently of each other. This establishes that an active monetary policy eliminates one degree of indeterminacy.

If a model has one degree of indeterminacy when the policy combination is passive-active, an active-active policy combination would now admit a unique solution. When the steady-state equilibrium displays second-degree indeterminacy, as in our sixty-two generation example, it is not just the initial price level that is indeterminate; it is also the initial real interest rate.

For any *given* choice of the initial interest rate, R_2, active monetary policy removes nominal indeterminacy. Crucially, however, active monetary policy does not remove real indeterminacy and there continue to be many possible choices for the initial real interest rate, each of them associated with a different initial price level and a different equilibrium path for all future real interest rates and all future inflation rates.

10. So What?

How should the reader react to our finding that a particular example of an OLG economy displays indeterminate steady state equilibria? One possible reaction is that the real world is demonstrably determinate in the sense that a general equilibrium theorist would use that term. An advocate of this position might claim that the profession has rejected the OLG model after careful consideration and that the new-Keynesian version of the representative agent model has been demonstrated through careful empirical work to be a much better fit to time series and cross-section data. We do not think this argument holds water. The OLG model fell from favor as the preferred vehicle for understanding monetary and fiscal policy for theoretical reasons, not because it failed a series of empirical tests.[15]

Perhaps the analysis we have presented is only possible because of the restrictive assumption of an endowment economy. That turns out not to be the case. We have known since the work of Diamond (1965) that models with capital possess monetary equilibria as well as autarkic equilibria and a simple extension of the argument in Section 5 establishes that calibrated

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[15]The model went out of favor, in part because a subset of influential macroeconomists considered the existence of indeterminacy in the model to be a shortcoming rather than a strength of the approach (Cherrier and Saidi, 2018). For summary of recent research that uses indeterminacy as a positive aspect of DSGE models see Farmer (2020).

examples of the model with capital preserve the feature of the existence a monetary steady state equilibrium with two degrees of indeterminacy at the steady state.[16]

A second reason that macroeconomists abandoned the OLG model is because Aiyagari (1985) demonstrated that, under some circumstances, the set of equilibria in the OLG model converges to that of the representative agent model as the length of life is increased. Importantly, for our argument, the Aiyagari (1985) result requires the endowments of agents to be bounded away from zero and there are many interesting models where that property does not apply. It might be argued that we have dealt with the finite-lived case and that in the real world people are connected by operative chains of bequests and they effectively have an infinite horizon (Barro, 1974). But Pietro Reichlin (1992) has shown that the perpetual youth model of Blanchard (1985) displays multiple sets of indeterminate steady-state equilibria even when people may live forever.

A further argument that might be levelled against indeterminate steady-state equilibria is that they are unlearnable. According to this argument, people are adaptive learners and a rational expectations equilibrium accurately describes the properties of an economy after learning has taken place. Proponents of this argument claim that determinate steady-state equilibria are often stable under learning and that this is a good reason to select these equilibria when a model has multiple steady state equilibria (McCallum, 2003, 2007). This would be a persuasive argument if it were always true that indeterminate steady-states are unstable under learning, but exhaustive enquiries into the stability of adaptive learning schemes have found that both determinate and indeterminate steady-state equilibria may be stable under plausible adaptive learning schemes (Evans and Honkapohja, 2001). And the non-stationary equilibria of a monetary model in which artificially intelligent agents use deep reinforcement learning – an algorithm similar to the one that was used by Deep Blue (Hsu et al., 2018) to beat world chess grand masters – has been shown to converge to the Pareto superior equilbria in a monetary model with two equilibria, even when that equilibrium is indeterminate (Chen et al., 2021).

So what impression do we hope to have left with the reader? We consider the example in our paper to be a proof of concept that, we hope, will encourage other researchers to explore further the properties of the OLG model as a vehicle for understanding how shocks are propagated through the asset markets. The indeterminacies we have showcased here are features, not just of overlapping generations models, but also of models with credit constraints and borrowing

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[16]Because the model with capital raises additional questions, not least of which is the reason for employment fluctuations in a market economy, we have chosen not to include that analysis in this paper.

limits (Woodford, 1988). If, as we believe, the indeterminacies we have focused on are prevalent in the real world, the policy prescriptions of new-Keynesian representative agent economies, if followed by central banks and national fiscal authorities, may have unintended consequences. In our view, relative price indeterminacy is not just a theoretical curiosity; it is a feature of the real world.

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In this Appendix we derive an expression for the slope of the steady state savings function evaluated at the steady state $R = \lambda/\beta$ for the parameter values $\lambda = \beta = 1$. Define the functions $W(R)$ and $\phi(R)$

$$W(R) = \omega_1 + \frac{\omega_2}{R} + \frac{\omega_3}{R^2}, \tag{A1}$$

$$\phi(R) = 1 + \frac{(\beta R)^\eta}{R} + \frac{(\beta R)^{2\eta}}{R^2}. \tag{A2}$$

Applying the solution to the T-generation maximizing problem with CES preferences from Appendix B we have the following steady-state consumption demand functions

$$c_1(R) = \frac{W(R)}{\phi(R)}, \quad c_2(R) = (\beta R)^\eta \frac{W(R)}{\phi(R)}, \quad c_3(R) = (\beta R)^{2\eta} \frac{W(R)}{\phi(R)}, \tag{A3}$$

where subscripts indicate age. Define the steady-state savings functions of the young and middle-aged as

$$s_1(R) = \omega_1 - c_1(R), \quad s_2(R) = Rs_1(R) + \omega_2 - c_2(R). \tag{A4}$$

Next, we seek expressions for the functions $\omega_1(\omega_2)$ and $\omega_3(\omega_2)$ which solve the equations

$$w_1 + w_2 + w_3 = 1 + \lambda^\eta + \lambda^{2\eta}, \tag{A5}$$

$$w_1 + \frac{w_2 \beta}{\lambda} + \frac{w_3 \beta^2}{\lambda^2} = 1 + \frac{\lambda^\eta \beta}{\lambda} + \frac{\lambda^{2\eta} \beta^2}{\lambda^2}. \tag{A6}$$

These are given by the expressions

$$\omega_1(\omega_2) = 1 + \frac{\frac{\beta}{\lambda} \lambda^\eta}{\left(1 + \frac{\beta}{\lambda}\right)} - \frac{\frac{\beta}{\lambda} \omega_2}{\left(1 + \frac{\beta}{\lambda}\right)}, \tag{A7}$$

$$\omega_3(\omega_2) = \frac{\lambda^\eta}{\left(1 + \frac{\beta}{\lambda}\right)} + \lambda^{2\eta} - \frac{\omega_2}{\left(1 + \frac{\beta}{\lambda}\right)}. \tag{A8}$$

Define the function

$$\psi(R) = \frac{W(R)}{\phi(R)}, \tag{A9}$$

and note that aggregate savings in a steady state equilibrium, $f_{\tilde{\omega}}(R)$, defined as the sum of $S_1(R; \tilde{\omega})$ and $S_2(R; \tilde{\omega})$ is given by the expression,

$$f_{\tilde{\omega}}(R) = \left(\omega_1(\omega_2) - \psi(R) \right) + R\left(\omega_1(\omega_2) - \psi(R) \right) + \omega_2 - \psi(R)\left(\beta R \right)^{\eta}. \tag{A10}$$

Rearranging terms, this leads to the equation

$$f_{\tilde{\omega}}(R) = \omega_1(\omega_2)\left(1 + R \right) + \omega_2 - \psi(R)\left(1 + R + (\beta R)^{\eta} \right). \tag{A11}$$

We seek an expression for the derivative of $f_{\tilde{\omega}}(R)$ evaluated at $\lambda = \beta = 1$. For these parameter values the functions $\omega_1(\omega_2)$, $W(R)$ and $\phi(R)$ are given by the following formulae

$$\omega_1(\omega_2) = \frac{3 - \omega_2}{2}, \quad W(R) = 1 + \frac{1}{R} + \frac{1}{R^2}, \quad \phi(R) = 1 + R^{\eta-1} + R^{2(\eta-1)}. \tag{A12}$$

Evaluating each term at $R = \lambda/\beta = 1$ gives

$$W(1) = 3, \quad \phi(1) = 3, \quad \text{from which it follows that} \quad \psi(1) = 1. \tag{A13}$$

The partial derivatives of $W(R)$ and $\phi(R)$ are given by

$$\frac{\partial W}{\partial R} = -\frac{1}{R^2} - \frac{2}{R^3}, \quad \frac{\partial \phi}{\partial R} = (\eta - 1)R^{\eta-2} + 2(\eta - 1)R^{2\eta-3}, \tag{A14}$$

which when evaluated at $R = \lambda/\beta = 1$ gives

$$\left. \frac{\partial W}{\partial R} \right|_{R=1} = -3, \quad \left. \frac{\partial \phi}{\partial R} \right|_{R=1} = 3(\eta - 1). \tag{A15}$$

We seek an expression for the partial derivative of $f_{\tilde{\omega}}(R)$ evaluated at the steady state $R = \lambda/\beta = 1$. Using the chain rule, this is equal to

$$\left. \frac{\partial f_{\tilde{\omega}}}{\partial R} \right|_{R=1} = \omega_1(\omega_2) - \psi(1)\left(1 + \eta \right) - 3 \left. \frac{\partial \psi}{\partial R} \right|_{R=1}. \tag{A16}$$

A further application of the chain rule to the function $\psi(R)$ leads to the expression

$$\left. \frac{\partial \psi}{\partial R} \right|_{R=1} = \frac{\phi(1) \left. \frac{\partial W}{\partial R} \right|_{R=1} - W(1) \left. \frac{\partial \phi}{\partial R} \right|_{R=1}}{W(1)^2} = \frac{-9 - 9(\eta - 1)}{9} = -\eta. \tag{A17}$$

Putting all these pieces together gives

$$\left. \frac{\partial f_{\hat{\omega}}}{\partial R} \right|_{R=1} = \frac{3 - \omega_2}{2} - (1 + \eta) + 3\eta = \frac{1 - \omega_2 + 4\eta}{2}, \tag{A18}$$

which is Eq. (23) in the body of the paper.

$\square$

Appendix B. Analytic Solutions for Excess Demand

B.1. **The generic optimization problem.** Consider a person with CES preferences who lives for T periods and has perfect foresight of future prices. This person solves the problem,

Problem 1.

$$\max_{\{c_t^t, c_{t+1}^t, \ldots, c_{t+T-1}^t\}} \frac{a_1(c_t^t)^\alpha + a_2(c_{t+1}^t)^\alpha + \ldots + a_T(c_{t+T-1}^t)^\alpha}{\alpha}, \tag{B1}$$

subject to the lifetime budget constraint

$$\sum_{i=1}^T \mathcal{Q}_t^{t-1+i} c_{t-1+i} = \sum_{i=1}^T \mathcal{Q}_t^{t-1+i} \tilde{w}_i. \tag{B2}$$

Here, c_s^t is consumption in period s of a person born in period t, $i \in 1, \ldots T$ is age, and $\tilde{w}_i$ is after-tax endowment. The parameters a_i are utility weights and $\alpha \leq 1$ is a curvature parameter which is related to intertemporal substitution, η, by the identity

$$\eta \equiv \frac{1}{1-\alpha}. \tag{B3}$$

The term $\mathcal{Q}_t^k$, defined by the expression

$$\mathcal{Q}_t^k \equiv \prod_{j=t+1}^k \frac{1}{R_j}, \qquad \mathcal{Q}_t^t = 1, \tag{B4}$$

is the relative price at date t of a commodity for delivery at date k.

This optimization problem includes the case of a constant discount factor β for which

$$[a_1, a_2 \ldots, a_T] = \left[1, \beta, \ldots, \beta^{T-1}\right] \tag{B5}$$

and logarithmic preferences which is the limiting case when $\alpha \to 0$. We permit the discount factor to vary with age to nest the Kehoe and Levine (1983) example which we use to cross-check our results.

Proposition 2. *The solution to Problem 1 is given by*

$$\hat{c}_{t-1+k}^t = \frac{a_k^\eta \sum_{i=1}^T \left(\mathcal{Q}_t^{t-1+i} \tilde{w}_i\right)}{\left(\mathcal{Q}_t^{t-1+k}\right)^\eta \sum_{i=1}^T \left(\mathcal{Q}_t^{t-1+i}\right)^{1-\eta} a_i^\eta}, \quad k = 1, \ldots, T. \tag{B6}$$

where $\hat{c}_{t-1+k}^t$ denotes the consumption, at time $t-1+k$, of an agent born at time t.

Proof. The result follows directly from substituting the first-order conditions into the budget constraint and rearranging terms. □

B.2. **Non-generic optimization problems.** Let j be an index that runs from 1 to $T-1$. Consider a non-generic person born in period $1-j$ with real assets $\nu_{1-j} \equiv \lambda_{1-j} b_1$ who lives for $T-j$ periods. This person solves Problem 2

Problem 2.

$$
\max_{\{c_1^{1-j},\dots,c_{1-j+T-1}^{1-j}\}} \frac{a_{T-j+1}(c_1^{1-j})^\alpha + a_{T-j+2}(c_2^{1-j})^\alpha + \dots + a_T(c_{1-j+T-1}^{1-j})^\alpha}{\alpha}, \quad j=1\dots,T-1
$$

(B7)

subject to the lifetime budget constraint

$$
\sum_{k=1}^{(1-j)+T-1} \mathcal{Q}_1^k \left(c_k^{1-j} - \tilde{w}_{k-(1-j)+1} \right) \leq \lambda_{1-j} b_1,
$$

(B8)

Proposition 3. *Let* $k \in \{1,\dots,T-j\}$. *The solution to Problem 2 is given by*

$$
\hat{c}_k^{1-j} = \frac{a_{k+j}^\eta \left(\nu_{1-j} + \sum_{i=1}^{T-j} \mathcal{Q}_t^i \tilde{w}_{j+i} \right)}{\left(\mathcal{Q}_t^{t+k-1} \right)^\eta \sum_{i=1}^{T-j} \left(\mathcal{Q}_t^i \right)^{1-\eta} a_{j+i}^\eta}, \quad k=1\dots 1-j+T-1.
$$

(B9)

Proof. The problem above is identical to a generic one solved by an agent who has $T-j$ periods to live, whose endowments are $\{\tilde{w}_{j+1} + \nu_{1-j}, \tilde{w}_{j+2}, \dots, \tilde{w}_T\}$, and whose preference parameters in the utility function are $\{a_{j+1}, a_{j+2}, \dots, a_T\}$. □

APPENDIX C. EQUILIBRIUM AS THE SOLUTION TO A DIFFERENCE EQUATION

In Section 4 we showed that equilibria of the $3-$generation model can be characterized as the solution to a difference equation, determined by the behaviour of the generic generations, together with set of initial conditions determined by the behavior of the on-generic generations. In this Appendix we generalize our analysis to the $T-$generation model.

C.1. **Generic Equilibrium Conditions for the $T-$Generation Case.** A steady-state equilibrium is a non-negative real number $\bar{R}$ and a (possibly negative) real number $\bar{b}$ that solve the equations,

$$
f(\bar{R}, \bar{R}, \dots, \bar{R}) = \bar{b} + d, \quad \bar{b}(1-\bar{R}) = \bar{R}d.
$$

(C1)

Let $\{\bar{R}, \bar{b}\}$ be a steady state equilibrium and let

$$
\tilde{R}_t \equiv R_t - \bar{R}, \quad \text{and} \quad \tilde{b}_t \equiv b_t - \bar{b},
$$

(C2)

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represent deviations of b_t and R_t from their steady state values. Define a vector

$$X_t \equiv [R_{t+T-1}, R_{t+T-2}, \ldots, R_{t-T+4}, b_t]^\top, \tag{C3}$$

of length $2T - 3$ and a function $F(\cdot)$,

$$F(X_t, X_{t-1}) \equiv \begin{bmatrix} f(R_{t-T+3}, R_{t-T+4}, \ldots, R_{t+T-2}, R_{t+T-1}) & - & b_t + d_t \\ b_t & - & R_t(b_{t-1} + d_{t-1}) \end{bmatrix}, \tag{C4}$$

and let J_1 and J_2 represent the partial derivatives of this function with respect to X_t and X_{t-1}. Using this notation, the local dynamics of equilibrium sequences close to the steady state can be approximated as solutions to the linear difference equation

$$J_1 \tilde{X}_t = J_2 \tilde{X}_{t-1}, \quad t = T - 1, \ldots \tag{C5}$$

with initial condition

$$\tilde{X}_{T-2} = \bar{X}_{T-2}. \tag{C6}$$

The local stability of these equations depends on the eigenvalues of the matrix

$$J \equiv J_1^{-1} J_2. \tag{C7}$$

If one or more roots of the matrix J is outside the unit circle there is no guarantee that sequences of interest factors and government debt generated by Equation (C5) will remain bounded. To ensure stability, we must choose initial conditions that place $\tilde{X}_{T-2}$ in the linear subspace associated with the stable eigenvalues of J. The initial conditions are determined by the non-generic equilibrium conditions which we turn to next.

C.2. **Non-Generic Equilibrium Conditions for the $T-$Generation Case.** Asset market equilibrium in periods 1 through $T - 2$ is characterized by a family of functions, $g_t^T(\cdot)$, one family for each value of T. These functions are different at each date t, because the asset demand functions of the non-generic generations depend on the initial wealth distribution and the initial price level as well as on real interest rates. To cut down on notation, we have suppressed the dependence of $g_t^T(\cdot)$ on $\mathbb{L}$.

To understand the structure of non-generic asset demand equations it helps to build intuition by considering the case of $T = 4$. This is the simplest example where we must keep track

of the debt-accumulation equations in the $T-2$ initial periods. For the case of $T=4$, there are two non-generic asset demand equations and one non-generic debt accumulation equation. These equations are described by the expression

$$G^4(Z_0^4) \equiv \begin{bmatrix} g_1^4\left(R_2, R_3, R_4, b_1\right) & - & (b_1 + d) \\ g_2^4\left(R_2, R_3, R_4, R_5, b_1\right) & - & (b_2 + d) \\ b_2 & - & R_2\left(b_1 + d\right) \end{bmatrix} = 0, \tag{C8}$$

where the vector Z_0^4, is a set of variables that are determined, in equilibrium, by asset market clearing and government asset accumulation in the initial $T-2$ periods of the model. For the case of $T=4$, the equation $G^4(\cdot)=0$ places 3 restrictions on the 6 elements of Z_0^4.

$$Z_0^4 \equiv [R_5, R_4, R_3, R_2, b_2, b_1]^\top \equiv [X_{T-2}, Y_{T-2}]^\top, \tag{C9}$$

where Y_{T-2} is the vector

$$Y_{T-2} \equiv [b_1]^\top, \tag{C10}$$

and X_{T-2} is a vector of initial conditions to the vector-valued difference equation characterized by equations (27) and (2).

This example can be generalized. For the T-generation model, Z_0^T contains $3T-6$ elements, equal to the union of the terms in the over-braces of the following expression,

$$Z_0^T \equiv [\overbrace{R_{2T-3}, R_{2T-4}, \ldots, R_2, b_{T-2}}^{2T-3}, \overbrace{b_{T-3}, \ldots, b_2, b_1}^{T-3}]^\top \equiv [X_{T-2}, Y_{T-2}]^\top. \tag{C11}$$

The function $G(\cdot)$ contains $2T-5$ rows,

$$G^T(Z_0^T) \equiv \begin{bmatrix} g_1^T\left(R_2, \ldots, R_{1+T-2}, R_{1+T-1}, b_1\right) & - & (b_1 + d) \\ \vdots & & \vdots \\ g_{T-2}^T\left(R_2, \ldots, R_{2T-4}, R_{2T-3}, b_1\right) & - & (b_{T-2} + d) \\ b_2 & - & R_2\left(b_1 + d\right) \\ \vdots & & \vdots \\ b_{T-2} & - & R_{T-1}\left(b_{T-3} + d\right) \end{bmatrix} = 0, \tag{C12}$$

and the equation $G^T(\cdot)=0$ places $2T-5$ restrictions on the $3T-6$ elements of the unknown vector Z_0^T. Subtracting the number of these restrictions from the number of initial variables leaves $T-1$ non-predetermined elements of X_{T-2}. In Appendix D we describe how the non-generic equilibrium equations may be combined with the assumption that equilibrium sequences

must remain bounded to characterize the determinacy properties of equilibria.

□

Appendix D. Proof of Proposition 1

To determine the conditions for local uniqueness of equilibrium close to a given steady-state equilibrium we seek a relationship between the roots of the matrix J and the number of initial conditions. We begin by describing the construction of J.

Proof. To find conditions under which equilibrium sequences remain bounded, we exploit the properties of the Jordan decomposition of J, which we write as

$$J = Q\Lambda Q^{-1}, \tag{D1}$$

where the matrix $J = J_1^{-1}J_2$ and,[17]

$$
J_1 X_t \equiv
\overbrace{
\begin{bmatrix}
f_{t+T-1} & f_{t+T-2} & \cdots & f_{t+1} & \cdots & f_{t-T+5} & f_{t-T+4} & -1 \\
0 & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1
\end{bmatrix}
}^{J_1}
\overbrace{
\begin{bmatrix}
\tilde{R}_{t+T-1} \\
\tilde{R}_{t+T-2} \\
\vdots \\
\tilde{R}_{t+1} \\
\vdots \\
\tilde{R}_{t-T+5} \\
\tilde{R}_{t-T+4} \\
\tilde{b}_t
\end{bmatrix}
}^{\tilde{X}_t},
\tag{D2}
$$

$$
J_2 X_{t-1} \equiv
\overbrace{
\begin{bmatrix}
0 & \cdots & 0 & \cdots & 0 & -f_{t-T+3} & 0 \\
1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \ddots & \vdots & 0 & 0 \\
0 & \cdots & 0 & \cdots & 1 & 0 & 0 \\
0 & \cdots & \bar{b}+d & \cdots & 0 & 0 & \bar{R}
\end{bmatrix}
}^{J_2}
\overbrace{
\begin{bmatrix}
\tilde{R}_{t+T-2} \\
\tilde{R}_{t+T-3} \\
\vdots \\
\tilde{R}_t \\
\vdots \\
\tilde{R}_{t-T+4} \\
\tilde{R}_{t-T+3} \\
\tilde{b}_{t-1}
\end{bmatrix}
}^{\tilde{X}_{t-1}},
\tag{D3}
$$

---

[17]An updated solution method for rational expectations models that allows either J_1 or J_2 or both to be singular can be found in Sims (2001). Solution methods for models with indeterminacy are provided in Farmer et al. (2015).

where f_k is the partial derivative of the function f with respect to R_k evaluated at the steady state $\{\bar{R}, \bar{b}\}$.

The matrix Λ is an upper triangular matrix with the eigenvalues of J on the diagonal and Q is the matrix of left eigenvectors of J. The restrictions that prevent X_t from becoming unbounded are found by premultiplying $\tilde{X}_{T-2}$ by the rows of Q^{-1} associated with the unstable eigenvalues of J and equating the product to zero. We refer to the matrix that contains these rows as Q_u^{-1}.[18]

Let K be the number of eigenvalues of J that lie outside the unit circle. The requirement that equilibrium sequences remain bounded places K linear restrictions on the initial vector X_{T-2} which we express with the matrix equation

$$Q_u^{-1} X_{T-2} = 0, \tag{D4}$$

where Q_u is a $K \times 2T - 3$ matrix.

Next, we turn to the non-generic generations which we index by their date of birth, measured by an index $1 - j$ where j runs from 1 to $T - 1$. We represent the financial wealth of generation $1 - j$ by a real number λ_{1-j} which may be positive or negative and that represents the share, owned by generation $1 - j$, of the period 1 dollar-valued government debt.

$$\nu_{1-j} \equiv \lambda_{1-j} \frac{B_0}{p_1}. \tag{D5}$$

We refer to the vector of shares as $\mathbb{L} \equiv \{\lambda_0, \lambda_{-1}, \ldots, \lambda_{2-T}\}$. These shares sum to 1 as a consequence of asset market clearing.[19] In Appendix B

To determine the number of restrictions that come from the non-generic equilibrium conditions, we turn to the function $G^T(\cdot)$ derived in Appendix C.2. This function depends on Z_0^T and on the initial nominal wealth distribution $\mathbb{L}$. To ensure that our local analysis remains valid we first compute the steady state distribution of wealth from the equation[20]

$$\nu_{1-j} = \bar{R} s_1^{2-j}(\bar{R}, \bar{R}, \ldots, \bar{R}), \quad j = 1, \ldots, T - 1, \tag{D6}$$

---

[18]In practice, the Jordan decomposition is numerically unstable but there are several good computational methods to compute Q_u^{-1} that are implemented in all modern programming languages. The reader is referred to Golub and VanLoan (1996) for a description of the algorithms used to solve problems of this kind and to Farmer (1999) for an accessible introduction to solution methods for rational expectations models with indeterminate equilibria.

[19]We require $B_0 \neq 0$ for the shares λ to be well-defined. See Niepelt (2004) for a discussion of a conceptual issue associated with non-zero values of B_0.

[20]Here, s_1^{1-j} is the savings function defined in Equation (26) evaluated at the steady state sequence of real interest rates.

and we use the identity,

$$\lambda_{1-j} \equiv \frac{\nu_{1-j}}{\bar{b}}, \tag{D7}$$

to recover the steady-state values of λ_{1-j}. In what follows we set the initial asset shares to these steady state values.

Let G_X, G_Y be the Jacobians of the function $G^T(\cdot)$ with respect to X_{T-2} and Y_{T-2} evaluated at the steady-state equilibrium $\{\bar{R}, \bar{b}\}$ and let $\tilde{X}_{T-2}$, $\tilde{Y}_{T-2}$ be deviations of these vectors from their steady-state values. Using this notation, the predetermined equilibrium conditions place the following $2T - 5$ restrictions on the $3T - 6$ unknowns $Z_0^T \equiv [X_{T-2}, Y_{T-2}]$.

$$G_X \tilde{X}_{T-2} + G_Y \tilde{Y}_{T-2} = 0. \tag{D8}$$

The uniqueness of equilibrium then comes down to the question of the number of solutions to the equations

$$
\overset{(2T-5+K)\times(3T-6)}{\begin{bmatrix} G_X & G_Y \\ Q_u^{-1} & 0 \end{bmatrix}}
\overset{(3T-6)\times 1}{\begin{bmatrix} \tilde{X}_{T-2} \\ \tilde{Y}_{T-2} \end{bmatrix}}
=
\overset{(2T-5+K)\times(3T-6)}{\begin{bmatrix} \bar{G} \\ 0 \end{bmatrix}}, \tag{D9}
$$

where $\bar{G} \neq 0$ represents the perturbations of the initial wealth distribution $\mathbb{L}$ from its steady-state value. It follows immediately that a sufficient condition for the existence of a unique equilibrium is that the matrix

$$\mathcal{M} \equiv \begin{bmatrix} G_X & G_Y \\ Q_u^{-1} & 0 \end{bmatrix}, \tag{D10}$$

is square and non singular.

If $K > T - 1$ then $\mathcal{M}$ has more rows than columns and, except for the special case where the rows of $\mathcal{M}$ are linearly dependent, Equation (D9) has no solution.

If $K = T - 1$ then $\mathcal{M}$ is square. If $\mathcal{M}$ also has full row rank, there exists a unique solution to this equation given by the expression

$$\begin{bmatrix} X_{T-2} \\ Y_{T-2} \end{bmatrix} = \mathcal{M}^{-1} \begin{bmatrix} \bar{G} \\ 0 \end{bmatrix}. \tag{D11}$$

Finally, if $K \leq T - 1$ then $\mathcal{M}$ has fewer rows than columns. In this case we are free to append a $(T - 1 - K) \times (3T - 6)$ matrix of linear restrictions to Equation (D9). For example, if $T - 1 - K = 1$ we can append a row that sets $b_1 = \bar{b}_1$. $\qquad\square$

Appendix E. Inflation Under a Taylor Rule

In this Appendix we derive equations that characterize the behaviour of the inflation rate when the Taylor Rule is passive and when it is active.

E.1. **The case of a passive Taylor Rule.** Using the Taylor rule to substitute for $1 + i_t$ in the Fisher parity condition yields the following difference equation for inflation

$$\Pi_{t+1} = \left(\frac{\bar{R}}{R_{t+1}}\right)\left(\frac{\Pi_t}{\bar{\bar{\Pi}}}\right)^{\phi_\pi}\Pi_t, \quad \text{for all} \quad t = 1, \ldots \infty \tag{E1}$$

which we linearize around a steady state to obtain

$$\tilde{\Pi}_{t+1} = (1 + \phi_\pi)\tilde{\Pi}_t - \kappa\tilde{R}_{t+1}, \quad \text{for all} \quad t = 1, \ldots, \infty. \tag{E2}$$

Here, $\kappa \equiv \bar{\Pi}/\bar{R}$ and the tilde denotes deviations from the steady state. Iterating Equation (E2) we obtain

$$\lim_{T \to \infty}\tilde{\Pi}_{T+1} = \lim_{T \to \infty}(1 + \phi_\pi)^T\tilde{\Pi}_1 - \lim_{T \to \infty}\sum_{s=1}^{T}(1 + \phi_\pi)^{T-s}\tilde{R}_{T+1}. \tag{E3}$$

This is Equation (32) in Section 9.

E.2. **The case of an active Taylor Rule.** To find conditions under which inflation is bounded when the Taylor Rule is active, we use Equation E2 to write the inflation rate at date t as a function of all future real interest rates and all future inflation rates,

$$\tilde{\Pi}_t = \kappa\sum_{s=1}^{+\infty}\left(\frac{1}{1 + \phi_\pi}\right)^s\tilde{R}_{t+s} + \lim_{T \to \infty}\left(\frac{1}{1 + \phi_\pi}\right)^T\tilde{\Pi}_{t+T}. \tag{E4}$$

If inflation is bounded, and if the Taylor Rule is active, the second term on the right side of Equation (33) is zero. Evaluating Equation (E4) at $t = 1$, we arrive the following expression for the initial gross inflation rate.

$$\tilde{\Pi}_1 \equiv \left(\tilde{\Pi}_1 - \bar{\Pi}\right) = \kappa\sum_{s=1}^{+\infty}\left(\frac{1}{1 + \phi_\pi}\right)^s\tilde{R}_{1+s} \tag{E5}$$

Using the definition of inflation in period 1, Equation (E5) places the following restriction on the initial price level,

$$p_1 = p_0\left(\bar{\Pi} + \kappa\sum_{s=1}^{+\infty}\left(\frac{1}{1 + \phi_\pi}\right)^s\left(R_{1+s} - \bar{R}\right)\right). \tag{E6}$$

This is Equation (33) in Section 9.

$\square$